Spanning forests in regular planar maps

Mireille Bousquet-Mélou, Julien Courtiel

SPANNING FORESTS IN REGULAR PLANAR MAPS

MIREILLE BOUSQUET-MÉLOU AND JULIEN COURTIEL

Abstract. We address the enumeration of p-valent planar maps equipped with a spanning forest, with a weight z per face and a weight u per connected component of the forest. Equivalently, we count p-valent maps equipped with a spanning *tree*, with a weight z per face and a weight $\mu := u + 1$ per *internally active* edge, in the sense of Tutte; or the (dual) p-angulations equipped with a *recurrent sandpile configuration*, with a weight z per vertex and a variable $\mu := u + 1$ that keeps track of the *level* of the configuration. This enumeration problem also corresponds to the limit $q \to 0$ of the q-state Potts model on p-angulations.

Our approach is purely combinatorial. The associated generating function, denoted $F(z, u)$, is expressed in terms of a pair of series defined implicitly by a system involving doubly hypergeometric series. We derive from this system that $F(z, u)$ is *differentially algebraic* in z, that is, satisfies a differential equation in z with polynomial coefficients in z and u. This has recently been proved to hold for the more general Potts model on 3-valent maps, but via a much more involved and less combinatorial proof.

For $u \geq -1$, we study the singularities of $F(z, u)$ and the corresponding asymptotic behaviour of its nth coefficient. For $u > 0$, we find the standard asymptotic behaviour of planar maps, with a subexponential term in $n^{-5/2}$. At $u = 0$ we witness a phase transition with a term n^{-3}. When $u \in [-1, 0)$, we obtain an extremely unusual behaviour in $n^{-3}(\ln n)^{-2}$. To our knowledge, this is a new "universality class" for planar maps.

1. Introduction

A *planar map* is a proper embedding of a connected graph in the sphere. The enumeration of planar maps has received a continuous attention in the past 60 years, first in combinatorics with the pionneering work of Tutte [45], then in theoretical physics [22], where maps are considered as random surfaces modelling the effect of *quantum gravity*, and more recently in probability theory [36, 38]. General planar maps have been studied, as well as sub-families obtained by imposing constraints of higher connectivity, or prescribing the degrees of vertices or faces (*e.g.*, triangulations). Precise definitions are given below.

Several robust enumeration methods have been designed, from Tutte's recursive approach (e.g. [44]), which leads to functional equations for the generating functions of maps, to the beautiful bijections initiated by Schaeffer [41], and further developed by physicists and combinatorics alike [11, 19], via the powerful approach based on matrix integrals [27]. See for instance [17] for a more complete (though non-exhaustive) bibliography.

Beyond the enumerative and asymptotic properties of planar maps, which are now well understood, the attention has also focussed on two more general questions: maps on higher genus surfaces [6, 24], and maps equipped with an additional structure. The latter question is particularly relevant in physics, where a surface on which nothing happens ("pure gravity") is of little interest. For instance, one has studied maps equipped with a polymer [29], with an Ising model [19, 34, 16, 18] or more generally a Potts model, with a proper colouring [46, 47], with loops models [14, 13], with a spanning tree [40], or percolation on planar maps [2, 10].

In particular, several papers have been devoted in the past 20 years to the study of the Potts model on families of planar maps [4, 15, 26, 30, 33, 49]. In combinatorial terms, this means counting maps equipped with a colouring in q colours, according to the size (*e.g.*, the

Both authors were supported by the French "Agence Nationale de la Recherche", project A3 ANR-08-BLAN-0190. MBM also acknowledges the hospitality of the Institute of Computer Science, Universität Leipzig, where part of this work was carried out.

number of edges) and the number of *monochromatic edges* (edges whose endpoints have the same colour). Up to a change of variables, this also means counting maps weighted by their *Tutte polynomial*, a bivariate combinatorial invariant which has numerous interesting specializations. By generalizing Tutte's formidable solution of properly coloured triangulations (1973-1982), it has recently been proved that the Potts generating function is *differentially algebraic*, that is, satisfies a (non-linear) differential equation[1] with polynomial coefficients [9, 8, 17]. This holds at least for general planar maps and for triangulations (or dualy, for cubic maps).

The method that yields these differential equations is extremely involved, and does not shed much light on the structure of q-coloured maps. Moreover, one has not been able, so far, to derive from these equations the asymptotic behaviour of the number of coloured maps, nor the location of phase transitions.

The aim of this paper is to remedy these problems — so far for a one-variable specialization of the Tutte polynomial. This specialization is obtained by setting to 1 one of the variables, or by taking (in an adequate way) the limit $q \to 0$ in the Potts model. Combinatorially, we are simply counting maps (in this paper, p-valent maps) equipped with a spanning forest. We call them *forested maps*. This problem has already been studied in [23] via a random matrix approach, but with no explicit solution. The generating function $F(z,u)$ that we obtain keeps track of the *size* of the map (the number of faces; variable z) and of the number of trees in the forest (minus one; variable u). The specialization $u=0$ thus counts maps equipped with a spanning *tree* and was determined a long time ago by Mullin [40].

Here is an outline of the paper. We begin in Section 2 with general definitions on maps, and on the Tutte polynomial. We recall some of its combinatorial descriptions, and underline in particular that the series $F(z,\mu-1)$, once expanded in powers of z and μ, has non-negative coefficients and admits several combinatorial interpretations. This important observation implies that the natural domain of the parameter u is $[-1,+\infty)$ rather than $[0,+\infty)$. In Section 3, we obtain in a purely combinatorial manner an expression of $F(z,u)$ in terms of the solution of a system of two functional equations. In Section 4 we derive from this system that $F(z,u)$ is differentially algebraic in z, and give explicit differential equations for cubic ($p=3$) and 4-valent ($p=4$) maps. Section 5 is a combinatorial interlude explaining why all series occurring in our equations, like $F(z,u)$ itself, still have non-negative coefficients when $u \in [-1,0]$.

The rest of the paper is devoted to asymptotic results, still for $p=3$ and $p=4$: when $u>0$, forested maps follow the standard asymptotic behaviour of planar maps ($\mu^n n^{-5/2}$) but then there is a phase transition at $u=0$ (where one counts maps equipped with a spanning tree), and a very unusual asymptotic behaviour in $\mu^n n^{-3}(\ln n)^{-2}$ holds when $u \in [-1,0)$. To our knowledge, this is the first time a class of planar maps exhibits this asymptotic behaviour. This proves in particular that $F(z,u)$ is not *D-finite*, that is, does not satisfy any *linear* differential equation in z for these values of u (nor for a generic value of u). This is in contrast with the case $u=0$, for which the generating function of maps equipped with a spanning forest is known to be D-finite.

Our key tool is the *singularity analysis* of [31]: its basic principle is to derive the asymptotic behaviour of the coefficients of a series $F(z)$ from the singular behaviour of F near its *dominant singularities* (*i.e.*, singularities of minimal modulus). The first case we study (4-valent maps with $u>0$) is simple: first, one of the two series involved in our system vanishes; the remaining one, denoted R, satisfies an inversion equation $\Omega(R(z))=z$ for which the (unique) dominant singularity ρ of R is such that $R(\rho)$ lies in the domain of analyticity of Ω. One obtains for R a "standard" square root singularity. This is well understood and almost routine. Two ingredients make the other cases significantly harder:

- when $u<0$, $R(\rho)$ is a singularity of Ω,

[1] with respect to the size variable

- when $p = 3$ (cubic maps) we have to deal with a system of two equations; the analysis of systems is delicate, even in the so-called *positive case*, which corresponds in our context to $u > 0$ (see [28, 5]).

These difficulties, which culminate when $p = 3$ and $u < 0$, are addressed in Sections 6 and 7. Section 6 establishes general results on implicitly defined series. Section 7 focusses on the inversion equation $\Omega(R(z)) = z$ in the case where (up to translation) Ω has a $z \ln z$ singularity at 0. One then applies these results to the asymptotic analysis of forested maps in Sections 8 (4-valent maps) and 10 (cubic maps). Section 9 exploits the results of Section 8 to study some properties of large *random* maps equipped with a spanning forest or a spanning tree.

We conclude in Section 11 with a few comments.

2. Preliminaries

2.1. Planar maps

A *planar map* is a proper embedding of a connected graph (possibly with loops and multiple edges) in the oriented sphere, considered up to continuous deformation. All maps in this paper are planar, and we often omit the term "planar". A *face* is a (topological) connected component of the complement of the embedded graph. Each edge consists of two *half-edges*, each incident to an endpoint of the edge. A *corner* is an ordered pair (e_1, e_2) of half-edges incident to the same vertex, such that e_2 immediately follows e_1 in counterclockwise order. The *degree* of a vertex or a face is the number of corners incident to it. A vertex of degree p is called *p-valent*. One-valent vertices are also called *leaves*. A map is *p-valent* if all vertices are p-valent. A *rooted map* is a map with a marked corner (e_1, e_2), called the *root* and indicated by an arrow in our figures. The *root vertex* is the vertex incident to the root. The *root half-edge* is e_2 and the *root edge* is the edge supporting e_2. This way of rooting maps is equivalent to the more standard way where one marks the root edge and orients it from e_2 to its other half-edge. All maps of the paper are rooted, and we often omit the term "rooted". The *dual* of a map M, denoted M^*, is the map obtained by placing a vertex of M^* in each face of M and an edge of M^* across each edge of M; see Figure 1(a). The dual of a p-valent map is a map with all faces of degree p, also called *p-angulation*.

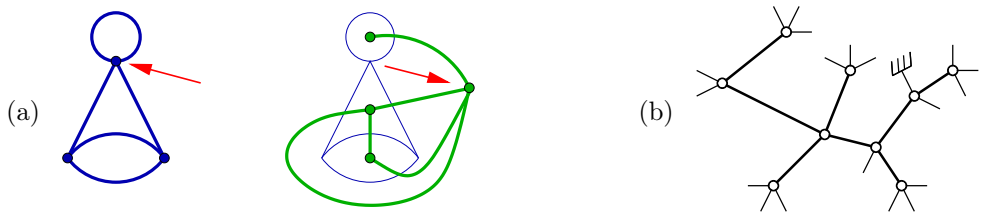

Figure 1. (a) A rooted planar map and its dual (rooted at the dual corner). (b) A 4-valent leaf-rooted tree.

A *(plane) tree* is a planar map with a unique face. A tree is p-valent if all non-leaf vertices have degree p. We consider the edges leading to the leaves as half-edges, as suggested by Figure 1(b). A *leaf-rooted* tree (resp. *corner-rooted*) is a tree with a marked leaf (resp. corner). The number of p-valent leaf-rooted (resp. corner-rooted) trees with k leaves is denoted by t_k (resp. t_k^c) (the notation should be $t_{k,p}$ and $t_{k,p}^c$, but we consider p as a fixed integer, $p \geq 3$). These numbers are well-known [42, Thm. 5.3.10]: they are 0 unless $k = (p-2)\ell + 2$ with $\ell \geq 1$, and in this case,

$$t_k = \frac{((p-1)\ell)!}{\ell!((p-2)\ell+1)!} \qquad \text{and} \qquad t_k^c = p\frac{((p-1)\ell)!}{(\ell-1)!((p-2)\ell+2)!}. \tag{1}$$

Let M be a rooted planar map with vertex set V. A *spanning forest* of M is a graph $F = (V, E)$ where E is a subset of edges of M forming no cycle. Each connected component of F is a tree,

and the *root component* is the tree containing the root vertex. We say that the pair (M, F) is a *forested map*. We denote by $F(z,u)$ the generating function of p-valent forested maps, counted by faces (variable z) and non-root components (variable u):

$$F(z,u) = \sum_{\substack{M\ p-\text{valent} \\ F\ \text{spanning forest}}} z^{\mathrm{f}(M)} u^{\mathrm{c}(F)-1}, \tag{2}$$

where f(.) denotes the number of faces and c(.) the number of components. When $p = 3$,

$$F(z,u) = (6+4\,u)\,z^3 + \left(140 + 234\,u + 144\,u^2 + 32\,u^3\right) z^4 + O(z^5). \tag{3}$$

The coefficient $(6+4u)$ means that there are 10 trivalent (or *cubic*) forested maps with 3 faces: 6 in which the forest is a tree, and 4 in which it has two components (Figure 2).

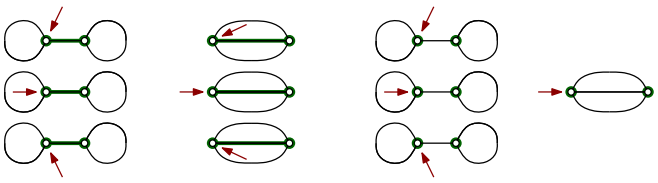

FIGURE 2. The 10 forested cubic maps with 3 faces.

2.2. Forest counting, the Tutte polynomial, and related models

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *Tutte polynomial* of G is the following polynomial in two indeterminates (see *e.g.* [12]):

$$\mathrm{T}_G(\mu,\nu) := \sum_{S \subseteq E} (\mu-1)^{\mathrm{c}(S)-\mathrm{c}(G)} (\nu-1)^{\mathrm{e}(S)+\mathrm{c}(S)-\mathrm{v}(G)}, \tag{4}$$

where the sum is over all spanning subgraphs of G (equivalently, over all subsets S of edges) and v(.), e(.) and c(.) denote respectively the number of vertices, edges and connected components. The quantity $\mathrm{e}(S) + \mathrm{c}(S) - \mathrm{v}(G)$ is the *cyclomatic number of* S, that is, the minimal number of edges one has to delete from S to obtain a forest.

When $\nu = 1$, the only subgraphs that contribute to (4) are the forests. Hence the generating function of forested maps defined by (2) can be written as

$$F(z,u) = \sum_{M\ p-\text{valent}} z^{\mathrm{f}(M)}\, \mathrm{T}_M(u+1, 1). \tag{5}$$

Note that we write T_M although the value of the Tutte polynomial only depends on the underlying graph of M, not on the embedding.

Even though this is not clear from (4), the polynomial $\mathrm{T}_G(\mu,\nu)$ has non-negative coefficients in μ and ν. This was proved combinatorially by Tutte [43], who showed that $\mathrm{T}_G(\mu,\nu)$ counts spanning *trees* of G according to two parameters, called *internal* and *external* activities. Other combinatorial descriptions of $\mathrm{T}_G(\mu,\nu)$, in terms of other notions of activity, were given later. Let us present the one due to Bernardi, which is nicely related to maps [7]. Following Mullin [40], we call *tree-rooted map* a map M equipped with a spanning tree T. Walking around T in counterclockwise order, starting from the root, defines a total order on the edges: the first edge that is met is the smallest one, and so on (Figure 3). An edge e is *internally active* if it belongs to T and is minimal in its cocycle; that is, all the edges $e' \neq e$ such that $(T \setminus \{e\}) \cup \{e'\}$ is a tree are larger than e. It is *externally active* if it does not belong to T and is minimal in the cycle created by adding e to T. Denoting by $\mathrm{int}(M,T)$ and $\mathrm{ext}(M,T)$ the numbers of internally and externally edges, one has:

$$\mathrm{T}_M(\mu,\nu) = \sum_{T\ \text{spanning tree}} \mu^{\mathrm{int}(M,T)} \nu^{\mathrm{ext}(M,T)}.$$

A non-obvious property of this description is that it only depends on the underlying graph of M.

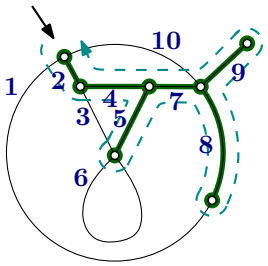

FIGURE 3. The edges of a tree-rooted map are naturally order by walking around the tree. The active edges are those labelled 1, 3, 6 and 9.

Returning to (5), we thus obtain a second description of $F(z,u)$:

$$F(z,u) = \sum_{\substack{M\ p-\text{valent} \\ T\ \text{spanning tree}}} z^{\mathrm{f}(M)}(u+1)^{\mathrm{int}(M,T)}. \tag{6}$$

In particular, it makes sense combinatorially to write $u = \mu - 1$ and take $u \in [-1, \infty)$.

We now give four more descriptions of $F(z,u)$ in terms of the dual p-angulations. For any planar map M, it is known that

$$\mathrm{T}_{M^*}(\mu,\nu) = \mathrm{T}_M(\nu,\mu).$$

Since

$$\mathrm{T}_M(1,\nu) = \sum_{S\subset E,\, S\,\text{connected}} (\nu-1)^{\mathrm{e}(S)+\mathrm{c}(S)-\mathrm{v}(M)},$$

we first derive from (5) that

$$\begin{aligned} F(z,u) &= \sum_{M\ p-\text{angulation}} z^{\mathrm{v}(M)}\,\mathrm{T}_M(1,u+1) \\ &= \sum_{\substack{M\ p-\text{angulation} \\ \mathrm{S}\ \text{connected subgraph}}} z^{\mathrm{v}(M)} u^{\mathrm{e}(S)+\mathrm{c}(S)-\mathrm{v}(M)} \end{aligned} \tag{7}$$

counts p-angulations M equipped with a connected (spanning) subgraph S, by the vertex number of M and the cyclomatic number of S. Also, the "dual" expression of (6) reads

$$F(z,u) = \sum_{\substack{M\ p-\text{angulation} \\ T\ \text{spanning tree}}} z^{\mathrm{v}(M)}(u+1)^{\mathrm{ext}(M,T)}. \tag{8}$$

Our next interpretation of $F(z,u)$, which we will not entirely detail, relies on the connection between $\mathrm{T}_M(1,\nu)$ and the *recurrent* (or: *critical*) configurations of the sandpile model on M. It is known [39, 25] that

$$\mathrm{T}_M(1,\nu) = \sum_{C\ \text{recurrent}} \nu^{\ell(C)},$$

where the sum runs over all recurrent configurations C, and $\ell(C)$ is the *level* of C. Hence

$$F(z,u) = \sum_{\substack{M\ p-\text{angulation} \\ C\ \text{recurrent}}} z^{\mathrm{v}(M)}(u+1)^{\ell(C)} \tag{9}$$

also counts p-angulations M equipped with a recurrent configuration C of the sandpile model, by the vertex number of M and the level of C.

Our final interpretation is in terms of the *Potts* model. Take $q \in \mathbb{N}$. A *q-colouring* of the vertices of $G = (V,E)$ is a map $c : V \to \{1,\dots,q\}$. An edge of G is *monochromatic* if its endpoints share the same colour. Every loop is thus monochromatic. The number of monochromatic edges

is denoted by $m(c)$. The *partition function of the Potts model* on G counts colourings by the number of monochromatic edges:

$$\mathrm{P}_G(q,\nu) = \sum_{c:V\to\{1,\dots,q\}} \nu^{m(c)}.$$

The Potts model is a classical magnetism model in statistical physics, which includes (for $q=2$) the famous Ising model (with no magnetic field) [48]. Of course, $\mathrm{P}_G(q,0)$ is the chromatic polynomial of G. More generally, it is not hard to see that $\mathrm{P}_G(q,\nu)$ is always a *polynomial* in q and ν, and a multiple of q. Let us define the *reduced Potts polynomial* $\tilde{\mathrm{P}}_G(q,\nu)$ by

$$\mathrm{P}_G(q,\nu) = q\,\tilde{\mathrm{P}}_G(q,\nu).$$

Fortuin and Kasteleyn established the equivalence of $\tilde{\mathrm{P}}_G$ with the Tutte polynomial [32]:

$$\tilde{\mathrm{P}}_G(q,\nu) \;=\; \sum_{S\subseteq E(G)} q^{\mathrm{c}(S)-1}(\nu-1)^{\mathrm{e}(S)} \;=\; (\mu-1)^{\mathrm{c}(G)-1}(\nu-1)^{\mathrm{v}(G)-1}\;\mathrm{T}_G(\mu,\nu),$$

for $q=(\mu-1)(\nu-1)$. Setting $\mu=1$, we obtain, for a *connected* graph G

$$\tilde{\mathrm{P}}_G(0,\nu) \;= (\nu-1)^{\mathrm{v}(G)-1}\;\mathrm{T}_G(1,\nu).$$

Returning to (7) finally gives

$$F(z,u) = u \sum_{M\ p-\text{angulation}} (z/u)^{\mathrm{v}(M)}\,\tilde{\mathrm{P}}_M(0,u+1). \tag{10}$$

2.3. Formal power series

Let $A=A(z)\in\mathbb{K}[[z]]$ be a power series in one variable with coefficients in a field $\mathbb{K}$. We say that A is *D-finite* if it satisfies a (non-trivial) linear differential equation with coefficients in $\mathbb{K}[z]$ (the ring of polynomials in z). More generally, it is *D-algebraic* if there exist a positive integer k and a non-trivial polynomial $P\in\mathbb{K}[z,x_0,\dots,x_k]$ such that $P\left(z,A,\frac{\partial A}{\partial z},\dots,\frac{\partial^k A}{\partial z^k}\right)=0$.

A k-variate power series $A=A(z_1,\dots,z_k)$ with coefficients in $\mathbb{K}$ is *D-finite* if its partial derivatives (of all orders) span a finite dimensional vector space over $\mathbb{K}(z_1,\dots,z_k)$.

3. Generating functions for forested maps

Fix $p\geq 3$. We give here a system of equations that defines the generating function $F(z,u)$ of p-valent forested maps, or, more precisely, the series $zF'_z(z,u)$ that counts forested maps with a marked face. We also give simpler systems for two variants of $F(z,u)$, involving no derivative.

3.1. p-Valent maps

Theorem 3.1. *Let θ, Φ_1 and Φ_2 be the following doubly hypergeometric series:*

$$\theta(x,y) = \sum_{i\geq 0}\sum_{j\geq 0} t^c_{2i+j}\binom{2i+j}{i,i,j}x^i y^j,$$

$$\Phi_1(x,y) = \sum_{i\geq 1}\sum_{j\geq 0} t_{2i+j}\binom{2i+j-1}{i-1,i,j}x^i y^j, \qquad \Phi_2(x,y) = \sum_{i\geq 0}\sum_{j\geq 0} t_{2i+j+1}\binom{2i+j}{i,i,j}x^i y^j, \tag{11}$$

where t_k and t^c_k are given by (1) and $\binom{a+b+c}{a,b,c}$ denotes the trinomial coefficient $(a+b+c)!/(a!b!c!)$.

There exists a unique pair (R,S) of power series in z with constant term 0 and coefficients in $\mathbb{Q}[u]$ that satisfy

$$R \;=\; z + u\,\Phi_1(R,S), \tag{12}$$

$$S \;=\; u\,\Phi_2(R,S). \tag{13}$$

The generating function $F(z,u)$ of p-valent forested maps is characterized by $F(0,u)=0$ and

$$F'_z(z,u) = \theta(R,S). \tag{14}$$

Remarks
1. These equations allow us to compute the first terms in the expansion of $F(z,u)$, for any fixed $p \geq 3$. This is how we obtained (3).
2. When p is even, then $t_{2i+1} = 0$ for all i. In particular, all terms occurring in the definition (11) of Φ_2 are multiples of y, so that $S = 0$. The simplified system reads:

$$F'_z(z,u) = \theta(R) \qquad \text{and} \qquad R = z + u\,\Phi(R), \tag{15}$$

with

$$\theta(x) = \sum_{i\geq 0} t^c_{2i} \binom{2i}{i} x^i \qquad \text{and} \qquad \Phi(x) = \sum_{i\geq 1} t_{2i} \binom{2i-1}{i} x^i.$$

3. When $u = 0$, an even more drastic simplification follows from (12-13): not only $S = 0$, but also $R = z$, so that (14) becomes an explicit expression of F'_z:

$$F'_z(z,0) = \sum_{i\geq 0} t^c_{2i} \binom{2i}{i} z^i,$$

or equivalently,

$$F(z,0) = \sum_{i\geq 0} t^c_{2i} \binom{2i}{i} \frac{z^{i+1}}{i+1} = \sum_{\ell\geq 1} \frac{p((p-1)\ell)!}{(\ell-1)!(1+(p-2)\ell/2)!(2+(p-2)\ell/2)!} z^{2+(p-2)\ell/2}, \tag{16}$$

where we require ℓ to be even if p is odd. This series counts p-valent maps equipped with a spanning tree, and this expression was already proved by Mullin [40].
4. The series θ and Φ_i are explicited when $p = 4$ and $p = 3$ in Sections 4.2 and 4.3, respectively.

In order to prove Theorem 3.1, we first relate $F(z,u)$ to the generating function of planar maps counted by the distribution of their vertex degrees. More precisely, let $\bar{M} \equiv \bar{M}(z,u; g_1, g_2, \ldots; h_1, h_2, \ldots)$ be the generating function of rooted planar maps, with a weight z per face, ug_k per *non-root* vertex of degree k and h_k if the root vertex has degree k.

Lemma 3.2. *The series $F(z,u)$ is related to $\bar{M}$ through:*

$$F(z,u) = \bar{M}(z,u; t_1, t_2, \ldots; t^c_1, t^c_2, \ldots).$$

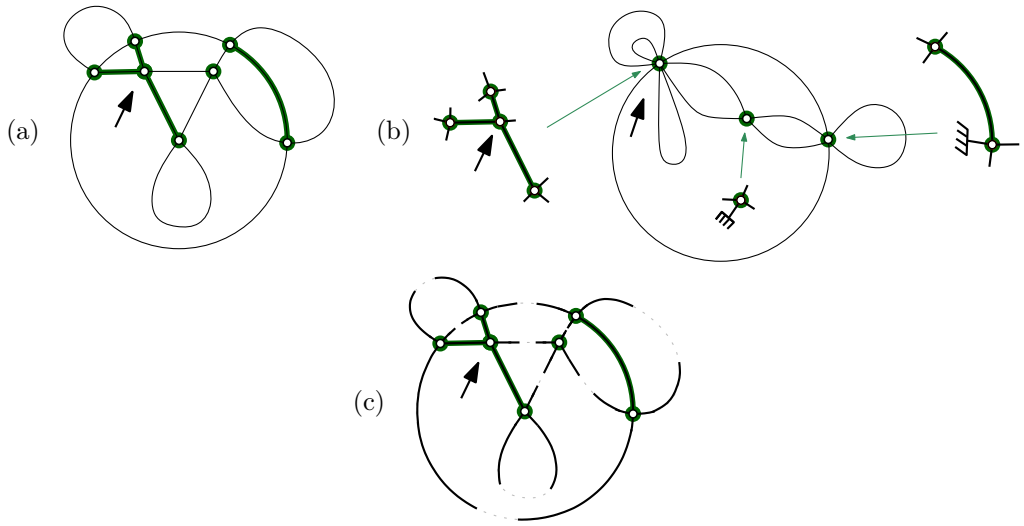

FIGURE 4. (a) A 4-valent forested map with 9 faces and 2 non-root components. (b) The same map, after contraction of the forest. (c) Assembling the 3 trees gives the original forested map.

Proof. The idea is to contract each tree of a spanning forest, incident to k half-edges, into a k-valent vertex. It is adapted from [19, Appendix A], where the authors study 4-valent forested maps for which the root edge is not in the forest. It can also be seen as an extension of Mullin's construction for maps equipped with a spanning tree [40]. Finally, it also appears in [23].

Let us now get into the details. First, let us recall that rooted maps have no symmetries: all vertices, edges and half-edges are distinguishable. In particular, one can fix, for every rooted planar map M' (with arbitrary valences) a total order on its half-edges. This order may have a combinatorial significance — a good choice is the order in which half-edges are visited when applying the construction of [20] — but can also be arbitrary.

We now describe a bijection Φ, illustrated in Figure 4, between forested p-valent maps (M, F) and pairs formed of a map M' and a collection $(T_v, v \in V(M'))$ of p-valent trees associated with the vertices of M', such that the tree associated with the root vertex of M' is corner-rooted, the others are leaf-rooted, and the number of leaves of T_v is the degree of v in M'.

The map M' is obtained by contracting all edges of the forest F (Figure 4(b)). The arrow that marks the root corner remains at the same place. Now split into two half-edges each edge of M that is not in F: this gives a collection of p-valent trees, each of them being naturally associated with a vertex v of M'. The half-edges of these trees form together the edges of M' (Figure 4(c)). If v is the root vertex of M', then T_v inherits the corner-rooting of M. Otherwise, we root T_v at the smallest of its half-edges, for the total order on half-edges of M'.

The following properties are readily checked:

- T_v has k leaves if v has degree k in M',
- M and M' have the same number of faces,
- the number of vertices of M' is the number of components of F.

Let us now prove that Φ is bijective. To recover the forested map (M, F) from the contracted map M' and the associated collection of trees, we inflate each vertex v of M' into the corresponding tree T_v. If v is the root vertex of M', the root corner of T_v must coincide with the root corner of M'. Otherwise, the root half-edge of T_v is put on the smallest of the half-edges incident to v in M'. This proves the injectivity of Φ. Since this reverse construction can be applied to any map M' with a corresponding collection of trees, Φ is also surjective. □

Proof of Theorem 3.1. In a recent paper, Bouttier and Guitter [21] have expressed the series $\bar{M}$ via a system of equations, established bijectively[2]. Their expression is actually fairly complicated [21, Eq. (1.4)], but the series $z\bar{M}'_z$, which counts maps with a marked face, has a much simpler expression [21, Eq. (2.6)]:

$$\bar{M}'_z = \sum_{i\geq 0}\sum_{j\geq 0} h_{2i+j}\binom{2i+j}{i,i,j} R^i S^j, \tag{17}$$

where $h_0 = 0$ and, by [21, Eq. (2.5)],

$$R = z + u\sum_{i\geq 1}\sum_{j\geq 0} g_{2i+j}\binom{2i+j-1}{i-1,i,j} R^i S^j, \qquad S = u\sum_{i\geq 0}\sum_{j\geq 0} g_{2i+j+1}\binom{2i+j}{i,i,j} R^i S^j. \tag{18}$$

Theorem 3.1 follows by specialization, using Lemma 3.2.

It remains to check that (12–13) defines a unique pair of series R and S in z with constant terms 0. This is readily proved by observing that (12) determines R up to order n if we know R and S up to order $n-1$; and that (13) determines S up to order n if we know R up to order n and S up to order $n-1$. □

Remark. The expression of $\bar{M}$ given in [21, Eq. (1.4)] leads to an explicit expression of $F(z, u)$ in terms of R and S. However, this expression involves a triple sum (a double sum when p is

[2]Strictly speaking, they do not take the vertex or face number into account, but both are prescribed by the distribution of vertex degrees.

even, see for instance (96)). This is why we prefer handling the expression of F'. We discuss this further in the final section.

3.2. Quasi-p-valent maps (p odd)

A map is said to be *quasi-p-valent* if all its vertices have degree p, apart from one vertex which is a leaf. Such maps exist only when p is odd. They are naturally rooted at their leaf: the root corner is the unique corner incident to the leaf and the root edge is the unique edge incident to the leaf. Let $G(z,u)$ denote the generating function of quasi-p-valent forested maps counted by faces (z) and non-root components (u) (see Figure 5).

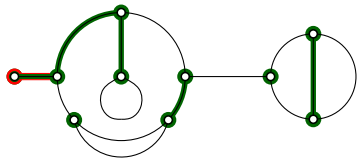

FIGURE 5. A quasi-cubic forested map with 6 faces and 4 non-root components.

Proposition 3.3. *The generating function of quasi-p-valent forested maps is*

$$G(z,u) = (1+\bar{u})\left(zS - u\sum_{i\geq 2}\sum_{j\geq 0} t_{2i+j-1}\binom{2i+j-2}{i-2,i,j} R^i S^j \right), \tag{19}$$

where $\bar{u} = 1/u$, the series R and S are defined by (12-13) and the numbers t_k by (1). Also,

$$G'_z(z,u) = (1+\bar{u})S.$$

As in the previous subsection, the key of this result is to relate $G(z,u)$ to a well-understood generating function of maps — here, the generating function $\Gamma_1 \equiv \Gamma_1(z,u;g_1,g_2,\ldots)$ that counts planar maps rooted at leaf, with a weight z per face and ug_k per k-valent *non-root* vertex.

Lemma 3.4. *The following analogue of Lemma* 3.2 *holds for quasi-p-valent forested maps:*

$$G(z,u) = (1+\bar{u})\,\Gamma_1(z,u;t_1,t_2,\ldots)$$

with $\bar{u} = 1/u$.

Proof. The bijection used in the proof of Lemma 3.2 shows that the series $\Gamma_1(z,u;t_1,t_2,\ldots)$ counts quasi-p-valent forested maps such that the root edge *is not* in the forest. (With the notation used in that proof, the root vertex of M', of degree 1, remains a trivial tree during the inflation step). To each such forested map, we can add the root edge to the forest. The resulting forested map has one less component, hence the factor $\bar{u} = 1/u$. □

Proof of Proposition 3.3. The series Γ_1 has also been expressed by Bouttier *et al.* in terms of the series R and S of (18) (see [20, Eq. (2.6)]):

$$\Gamma_1 = zS - u\sum_{i\geq 2}\sum_{j\geq 0} g_{2i+j-1}\binom{2i+j-2}{i-2,i,j} R^i S^j. \tag{20}$$

This gives the first part of Proposition 3.3. For the second part, we observe that Γ_1 is by definition the coefficient of h_1 in the series $\bar{M}(z,u;g_1,\ldots;h_1,\ldots)$ defined above Lemma 3.2. Hence it follows from (17) that $\Gamma_1' = S$ (this can also be derived combinatorially from [20]). □

3.3. When the root edge is outside the forest

We now focus on forested maps such that the root edge is outside the forest. Let $H(z,u)$ denote the associated generating function.

Proposition 3.5. *The generating function of p-valent forested maps where the root edge is outside the forest is*

$$H(z,u) = \bar{u}zR + \bar{u}zS^2 - \bar{u}z^2 \\ - 2S\sum_{i\geq 2}\sum_{j\geq 0} t_{2i+j-1}\binom{2i+j-2}{i-2,i,j}R^iS^j - \sum_{i\geq 3}\sum_{j\geq 0} t_{2i+j-2}\binom{2i+j-3}{i-3,i,j}R^iS^j, \quad (21)$$

where $\bar{u} = 1/u$, the series R and S are defined by (12-13) *and the numbers t_k by* (1).

When p is even, then $S = 0$ and the first double sum disappears. In this case, we also have a very simple expression of $H'_z(z,u)$:

$$H'_z(z,u) = 2\bar{u}(R-z). \quad (22)$$

Again, the key of this result is to relate $H(z,u)$ to a well-understood generating function of maps — here, the generating function $M \equiv M(z,u;g_1,g_2,\ldots)$ that counts rooted planar maps with a weight z per face and ug_k per vertex of degree k.

Lemma 3.6. *The following analogue of Lemma* 3.2 *holds:*

$$H(z,u) = \bar{u}\, M(z,u;t_1,t_2,\ldots).$$

Proof. Let us consider again the bijection used in the proof of Lemma 3.2: the fact that the root edge of M is not in the forest F means that, in the corner-rooted tree associated with the root vertex of M', the root half-edge is a leaf. It is then equivalent to root this tree at this leaf. □

Proof of Proposition 3.5. The first part of the proposition follows from the known characterization of M (see [20, Eq. (2.1)]):

$$M = \frac{\Gamma_1^2 + \Gamma_2}{z} - z^2,$$

where Γ_1 is given by (20) and

$$\Gamma_2 = z^2R - uz\sum_{i\geq 3}\sum_{j\geq 0}\binom{2i+j-3}{i-3,i,j}R^iS^j - u^2\left(\sum_{i\geq 2}\sum_{j\geq 0} g_{2i+j-1}\binom{2i+j-2}{i-2,i,j}R^iS^j\right)^2,$$

with R and S satisfying (18). This gives the first part of the proposition.

Observe that $M(z,u;g_1,g_2,\ldots) = u\bar{M}(z,u;g_1,g_2,\ldots;g_1,g_2,\ldots)$ where $\bar{M}$ is defined just above Lemma 3.2. When p is even, the maps obtained by contracting forests have even degrees ($g_{2k+1} = 0$ for all k), the series S given by (18) vanishes, and the combination of (17) and (18) gives $\bar{M}'_z(z,u;g_1,g_2,\ldots;g_1,g_2,\ldots) = 2\bar{u}(R-z)$. Thus $H'_z = \bar{u}M'_z = \bar{M}'_z = 2\bar{u}(R-z)$, as stated in (22). □

4. Differential equations

The equations established in the previous section imply that series counting regular forested maps are D-algebraic. We compute explicitly a few differential equations.

4.1. The general case

Theorem 4.1. *The generating function $F(z,u)$ of p-valent forested maps is D-algebraic (with respect to z). The same holds for the series $G(z,u)$ and $H(z,u)$ of Propositions 3.3 and 3.5.*

Proof. We start from the expression (14) of $F'(z,u)$ (as we always differentiate with respect to z, we simply denote $F'(z,u)$ for $F'_z(z,u)$). We first observe that the doubly hypergeometric series θ, Φ_1, Φ_2 are D-finite (this follows from the closure properties of D-finite power series [37]).

Then, by differentiating (12) and (13) with respect to z, we obtain rational expressions of R' and S' in terms of u and the partial derivatives $\partial\Phi_\ell/\partial x$ and $\partial\Phi_\ell/\partial y$, evaluated at (R,S), for $\ell = 1, 2$. (Indeed, differentiating (12) and (13) gives a linear system in R' and S'. Its determinant is a power series in z with coefficients in $\mathbb{Q}[u]$. It is non-zero, since it equals 1 at $u = 0$.)

Let $\mathbb{K}$ be the field $\mathbb{Q}(u)$. Using (14) and the previous point, it is now easy to prove by induction that for all $k \geq 1$, there exists a rational expression of $F^{(k)}(z,u)$ in terms of

$$\left\{ \frac{\partial^{i+j}\Phi_\ell}{\partial x^i \partial y^j}(R,S), \frac{\partial^{i+j}\theta}{\partial x^i \partial y^j}(R,S) \right\}_{i\geq 0, j\geq 0, \ell\in\{1,2\}}$$

with coefficients in $\mathbb{K}$. But since θ, Φ_1 and Φ_2 are D-finite, the above set of series spans a vector space of finite dimension d over $\mathbb{K}(R,S)$. Therefore there exist d elements $\varphi_1, \ldots, \varphi_d$ in this space, and rational functions $A_k \in \mathbb{K}(x, y, x_1, \ldots, x_d)$, such that $F^{(k)}(z,u) = A_k(R, S, \varphi_1, \ldots, \varphi_d)$ for all $k \geq 1$.

Since the transcendance degree [35, p. 254] of $\mathbb{K}(R, S, \varphi_1, \ldots, \varphi_d)$ over $\mathbb{K}$ is (at most) $d + 2$, the $d + 3$ series $F^{(k)}(z,u)$, for $1 \leq k \leq d + 3$, are algebraically dependent, so that F' (and thus F) is D-algebraic.

The proof is similar for the series $G(z,u)$ and $H(z,u)$, with θ replaced by the adequate D-finite series derived from (19) and (21). Moreover, since these two expressions involve z explicitly, the field $\mathbb{Q}(u)$ used in the above argument must be replaced by $\mathbb{Q}(z,u)$. □

4.2. The 4-valent case

We specialize the above argument to the case $p = 4$. As explained in the second remark following Theorem 3.1, the series S vanishes and $F'(z,u)$ is given by the system (15), with

$$\theta(x) = 4\sum_{i\geq 2} \frac{(3i-3)!}{(i-2)!i!^2} x^i \quad \text{and} \quad \Phi(x) = \sum_{i\geq 2} \frac{(3i-3)!}{(i-1)!^2 i!} x^i. \tag{23}$$

The series $\theta(x)$, $\Phi(x)$ and their derivatives lie in a 3-dimensional vector space over $\mathbb{Q}(x)$ spanned (for instance) by 1, $\Phi(x)$ and $\Phi'(x)$. This follows from the following equations, which are easily checked:

$$x(27x-1)\Phi''(x) + 6\Phi(x) + 6x = 0, \tag{24}$$

$$3\theta(x) = 2(27x-1)\Phi'(x) - 42\Phi(x) + 12x. \tag{25}$$

By the argument described above, we can now express first R', and then F' and all its derivatives as rational functions of u, R, $\Phi(R)$ and $\Phi'(R)$. But since $R = z + u\Phi(R)$, this means a rational function of u, z, R and $\Phi'(R)$. We compute the explicit expressions of F', F'' and F''', eliminate R and $\Phi'(R)$ from these three equations, and this gives a differential equation of order 2 and degree 7 satisfied by F', the details of which are not particularly illuminating:

$$\begin{aligned}
&9\,F'^2F''^5u^6+36\,F'^2F''^3F'''\,u^5z+144\,F'^2F''^4u^5-12\,(21\,z-1)F'\,F''^5u^5+432\,F'^2F''^2F'''\,u^4z-48\,(24\,z-1)F'\,F''^3F'''\,u^4z\\
&+864\,F'^2F''^3u^4-96\,(27\,z-2)F'\,F''^4u^4+4\,(27\,z-1)(15\,z-1)F''^5u^4+1728\,F'^2F''\,F'''\,u^3z-288\,(21\,z-2)F'\,F''^2F'''\,u^3z\\
&+10368\,F'\,F'''^2u^2z^3+16\,(27\,z-1)(21\,z-1)F''^3F'''\,u^3z+2304\,F'^2F''^2u^3-288\,(31\,z-4)F'\,F''^3u^3\\
&-64\left(6\,uz-162\,z^2+33\,z-1\right)F''^4u^3+2304\,F'^2F'''\,u^2z-2304\,(6\,z-1)F'\,F''\,F'''\,u^2z\\
&-192\left(8\,uz-54\,z^2+29\,z-1\right)F''^2F'''\,u^2z-768\,(2\,u+189\,z-7)F'''^2uz^3+2304\,F'^2F''\,u^2-3072\,(3\,z-1)F'\,F''^2u^2\\
&-192\left(24\,uz-27\,z^2+55\,z-2\right)F''^3u^2-1536\,(21\,z-2)F'\,F'''\,uz-768\left(12\,uz+81\,z^2+24\,z-1\right)F''\,F'''\,uz+1536\,(9\,z+2)F'\,F''\,u\\
&-512\left(39\,uz+81\,z^2+51\,z-2\right)F''^2u+36864\,F'\,z-1024\left(12\,uz-162\,z^2+33\,z-1\right)F'''\,z-1024\,(36\,uz+27\,z-1)F''-24576\,z=0.
\end{aligned}$$

As discussed in Section 11, we conjecture that F does not satisfy a differential equation of order 2.

We have applied the same method to the series H of Proposition 3.5:

$$H(z,u) = \bar{u}zR - \bar{u}z^2 - \Lambda(R)$$

where

$$\Lambda(x) = \sum_{i\geq 3} \frac{(3i-6)!}{(i-3)!(i-2)!i!} x^i$$

satisfies

$$30\Lambda(x) = x(27x-1)\Phi'(x) + (1-24x)\Phi(x) + 3x^2.$$

This gives for H an equation of order 2 and degree 3:

$$3\,(u+1)u^2H'^2H'' + 12\,u^2zH'H'' + 6\,(u-8)uH'^2 + 240\,H \\ + 4\,(6\,uz - 54\,z + 1)H' + 4\,(3\,uz^2 + 30\,uH + 27\,z^2 - z)H'' + 24\,z^2 = 0.$$

One reason explaining this more modest size is the simplicity of the expression (22) of H'.

4.3. The cubic case

We start from the expression of F' given in Theorem 3.1. We now have to deal with series θ, Φ_1 and Φ_2 in two variables:

$$\theta(x,y) = 3\sum_{i\geq 0} \sum_{\substack{j\geq 0 \\ 2i+j\geq 3}} \frac{(4\,i+2\,j-4)!}{(2i+j-3)!\,i!^2j!} x^i y^j,$$

$$\Phi_1(x,y) = \sum_{i\geq 1} \sum_{\substack{j\geq 0 \\ 2i+j\geq 3}} \frac{(4i+2j-4)!}{(2i+j-2)!\,(i-1)!\,i!\,j!} x^i y^j, \tag{26}$$

$$\Phi_2(x,y) = \sum_{i\geq 0} \sum_{\substack{j\geq 0 \\ 2i+j\geq 2}} \frac{(4i+2j-2)!}{(2i+j-1)!i!^2j!} x^i y^j. \tag{27}$$

Let us first observe that

$$\theta(x,y) = -2\Phi_1(x,y) + (1-y)\Phi_2(x,y) - 2x - y^2.$$

Consequently, Theorem 3.1 gives:

$$F' = 2z\bar{u} + \bar{u}S - (1+\bar{u})(2R+S^2). \tag{28}$$

Then, the summations over the variable j that occur in Φ_1 and Φ_2 can be performed explicitly, which gives to the cubic case a one-variable flavour. Indeed,

$$\Phi_1(x,y) = (1-4y)^{3/2}\ \Psi_1(t) - x, \tag{29}$$

$$\Phi_2(x,y) = \sqrt{1-4y}\,\Psi_2(t) + \frac{1}{4}\left(1-\sqrt{1-4y}\right)^2, \tag{30}$$

where $t = x/(1-4y)^2$ and

$$\Psi_1(z) = \sum_{i\geq 1} \frac{(4\,i-4)!}{(2\,i-2)!\,i!\,(i-1)!}\, z^i, \quad \Psi_2(z) = \sum_{i\geq 1} \frac{(4\,i-2)!}{(2\,i-1)!\,i!^2}\, z^i.$$

Our system thus reads:

$$R = z + u\,(1-4\,S)^{3/2}\ \Psi_1(T) - uR, \tag{31}$$

$$S = u\sqrt{1-4S}\,\Psi_2(T) + \frac{u}{4}\left(1-\sqrt{1-4S}\right)^2, \tag{32}$$

with $T = R/(1-4S)^2$.

The series $\Psi_1(z)$, $\Psi_2(z)$ and their derivatives lie in a 3-dimensional vector space over $\mathbb{Q}(z)$ spanned (for instance) by 1, $\Psi_1(z)$ and $\Psi_2(z)$. This follows from the following identities, which are easily checked:

$$(1-64z)\Psi_1'(z) + 48\Psi_1(z) + 2\Psi_2(z) = 1, \qquad z(1-64z)\Psi_2'(z) + 6\Psi_1(z) + 16z\Psi_2(z) = 8z. \tag{33}$$

By the argument of Section 4.1, we can now express R' and S' as rational functions of u, R, S, $\Psi_1(T)$ and $\Psi_2(T)$. But $\Psi_1(T)$ and $\Psi_2(T)$ can be expressed rationally in terms of z, u, R and $\sqrt{1-4S}$ using (31) and (32). Hence we obtain rational expressions in u, z, R and $\sqrt{1-4S}$. In fact no square root occurs:

$$\begin{aligned} R' &= \frac{R(48z-1+16(u+1)R+2(3+u)S-8(u+1)S^2)}{D}, \\ S' &= \frac{2(3z+(u-3)R-12zS+4(u+1)RS)}{D}, \end{aligned}$$

with

$$D = 36z^2+(24z-1+24uz)R+4(u+1)RS-4(u+1)^2RS^2+4(u+1)^2R^2.$$

Combining these two equations with (28), one can now express F', F'' and F''' in terms of u, z, R and S, and then eliminate R and S to obtain a differential equation of order 2 satisfied by F' (of degree 17). For the generating function $G(z,u)$ of quasi-cubic forested maps (Proposition 3.3), we replace (28) by

$$10G = (1+\bar{u})\,(z-R+6zS+2(u+1)RS)\,,$$

and obtain a differential equation of order 2 and degree 5. It becomes a bit simpler when we rewrite $G = (W+z\bar{u})/2$:

$$\begin{aligned} 0 = &\left(3\,u^4zW'^4-u^3(5\,Wu-uz+z)W'^3+4\,(u+1)(5\,Wu-uz+z)^2\right)W'' \\ &-48\,u^2z(u+1)W'^3+8\,u(u+1)(5\,Wu-uz+z)W'^2+4\,(u^2-1)(5\,Wu-uz+z)W'. \end{aligned}$$

Introducing the series W is natural in the solution of the Potts model presented in [8], where the above equation was first obtained.

5. Combinatorics of forested trees

Equation (5), and the positivity of the Tutte coefficients, show that the series $F(z,u)$ that counts p-valent forested maps has non-negative coefficients when expanded in $(1+u)$. We say that it is $(u+1)$-*positive*. Section 2.2 presents several combinatorial descriptions of $F(z,\mu-1)$ (see (6), (8), (9), (10)). This will lead us to study the asymptotic behaviour of the coefficient of z^n in $F(z,u)$ not only for $u \geq 0$, but for $u \geq -1$. In this study, we will need to know that several other series are also $(u+1)$-positive. We prove this thanks to a combinatorial argument that applies to certain classes of *forested trees*.

5.1. Positivity in $(1+u)$

Let T be a tree having at least one edge, and $\mathcal{F}$ a set of spanning forests of T. We define a property of $\mathcal{F}$ that guarantees that the generating function $A_{\mathcal{F}}(u)$ that counts forests of $\mathcal{F}$ by the number of components is $(u+1)$-positive (after division by u).

Let $F \in \mathcal{F}$, and let e be an edge of T. By *flipping* e in the forest F, we mean adding e to F if it is not in F, and removing it from F otherwise. This gives a new forest F' of T. We say that e is *flippable* for F if F' still belongs to $\mathcal{F}$. We say that $\mathcal{F}$ is *stable* if for each $F \in \mathcal{F}$,

(i) every edge of T not belonging to F is flippable,
(ii) flipping a flippable edge gives a new forest with the same set of flippable edges.

We say that a set $\mathcal{E}$ of forested trees is *stable* if, for each tree T, the set of forests F such that $(T,F) \in \mathcal{E}$ is stable. We consider below a generating function $E(z,u)$ of $\mathcal{E}$, where each forested tree (T,F) is weighted by $z^n u^k$, where n is the size of T (number of edges, of leaves...) and k the number of components of F, minus 1.

Lemma 5.1. *With the above notation, assume $\mathcal{F}$ is stable. Then all elements of $\mathcal{F}$ have the same number, denoted by f, of flippable edges. The generating function of forests of $\mathcal{F}$, counted by components, is $A_{\mathcal{F}}(u) = u(1+u)^f$. Consequently, if $\mathcal{E}$ is a stable set of forested trees, then $E(z,u)$ is $(u+1)$-positive.*

Proof. Condition (i) implies that the forest $F_{\max}$ consisting of all edges of T belongs to $\mathcal{F}$. Moreover, we can obtain $F_{\max}$ from any forest F of $\mathcal{F}$ by adding iteratively flippable edges. By Condition (ii), this implies that any forest F of $\mathcal{F}$ has the same set of flippable edges as $F_{\max}$. It also implies that, to construct a forest F of $\mathcal{F}$, it suffices to choose, for each flippable edge of $F_{\max}$, whether it belongs to F or not. Since $F_{\max}$ has a unique component, and since deleting an edge from a forest increases by 1 the number of components, the expression of $A_{\mathcal{F}}(u)$ follows. □

5.2. Enriched blossoming trees

We now apply the above principle to establish $(u+1)$-positivity properties for the series R and S given by (12–13), and for the series $\tilde{S} \equiv \tilde{S}(z,u)$ defined by

$$\tilde{S}(0,u) = 0, \quad \tilde{S} = u\,\Phi_2(z,\tilde{S}), \tag{34}$$

where Φ_2 is given by (11).

We consider plane trees rooted at a half-edge, which we draw hanging from their root as in Figure 6. A vertex of degree d is thus seen as the parent of $d-1$ children. A *subtree* consists of a vertex v and all its descendants. It is naturally rooted at the half-edge incident to v and located just above it. A *blossoming tree* is a leaf-rooted plane tree with two kinds of childless vertices: *leaves*, represented by white arrows, and *buds*, represented by black arrows. The edges that carry leaves and buds are considered as half-edges. (This means that leaves and buds are not actual nodes of the tree, so that a spanning forest of a blossoming tree does not contain any of its half-edges.) The root half-edge does not carry any leaf or bud. Each leaf is assigned a *charge* $+1$ while each bud is assigned a charge -1. The *charge* of a blossoming tree is the difference between the number of leaves and buds that it contains. This definition is extended to subtrees.

Definition 5.2. *Let $p \geq 3$. A p-valent blossoming tree T equipped with a spanning forest F is an* enriched R- (resp. S-) tree *if*

(i) *its charge is* 1 *(resp.* 0*),*
(ii) *any subtree rooted at an edge* not in F *has charge* 0 *or* 1.

We also consider as an enriched R-tree a single root half-edge carrying at a leaf (Figure 6, left).

The pair (T,F) is an enriched $\tilde{\text{S}}$-tree *if each component of F is incident to as many leaves as buds. In this case it is also an enriched S-tree.*

An enriched R-tree is shown in Figure 6. The readers who are familiar with the R- and S-trees of [20] will recognize that our enriched R- and S-trees are obtained from them by inflating each vertex of degree k into a (leaf rooted) p-valent tree with k leaves. The following proposition should not come as a surprise for them.

Proposition 5.3. *Let $p \geq 3$. The series R, S and $\tilde{S}$ defined by* (12), (13) *and* (34) *count respectively enriched R-, S- and $\tilde{S}$-trees, by the number of leaves (z) and the number of components in the forest (u).*

Proof. The equations follow from a recursive decomposition of enriched trees. For instance, an enriched R-tree is either reduced to a single leaf, with no forest at all (contribution: z), or contains a root node. This node belongs to a component of the forest. This component is incident to several half-edges (not belonging to the forest), one of them being the root half-edge. Each of the other incident half-edges can be a leaf, a bud, or the root of a non-trivial subtree. In this case, the definition of enriched R-trees implies that this subtree is itself an enriched R-tree (of charge 1) or an enriched S-tree (of charge 0). Since a single leave is considered as an R-tree, we can say that every half-edge incident to the root component of the forest carries either a bud, or an R- or S-tree. If there are i attached R-trees, we must have $i-1$ buds for the tree charge to be 1, and an arbitrary number j of S-trees. The root component of the forest is then a leaf-rooted tree with $k = 2i + j$ leaves. This gives (12), where the multinomial coefficient

occurring in Φ_1 describes the order in which the i R-trees, the $i-1$ buds and the j S-trees are organized.

The proof of (13) is similar, but now as many buds as R-trees must be attached to the root component of the forest to make the charge 0.

Finally, an $\tilde{\text{S}}$-tree is an S-tree in which all attached R-trees are actually leaves. This explains that (34) is obtained from (13) by replacing each occurrence of R by z. □

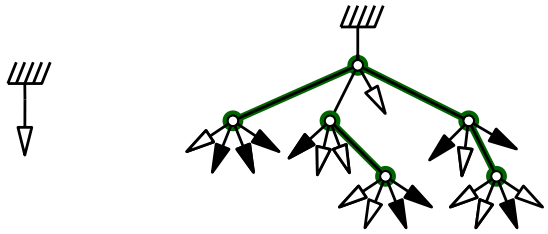

FIGURE 6. *Left:* The smallest enriched R-tree. *Right:* An enriched 5-valent R-tree having 10 leaves (white; charge $+1$) and 9 buds (black; charge -1).

We now come back to $(u+1)$-positivity.

Proposition 5.4. *The set of p-valent enriched R- (resp. S-, $\tilde{S}$-) trees having at least one edge is stable, in the sense of Section* 5.1.

Proof. For enriched R- and S-trees, an edge is flippable if and only if the attached subtree has charge 0 or 1, and this condition is independent of the forest.

For enriched $\tilde{\text{S}}$-trees, an edge is flippable if and only if the attached subtree is incident to as many leaves as buds, and this condition is again independent of the forest. □

By combining this proposition with Lemma 5.1 and Proposition 5.3, we obtain:

Corollary 5.5. *The series $\bar{u}(R-z)$, $\bar{u}S$ and $\bar{u}\tilde{S}$ are $(u+1)$-positive. When $u=\mu-1$, they count respectively (non-empty) enriched R-, S- and $\tilde{S}$-trees, by the number of leaves (z) and the number of flippable edges (μ).*

When $p=3$ for instance, we have, writing $\mu=u+1$,

$$\begin{aligned}\bar{u}(R-z) &= 2(2\mu+1)z^2+4(10\mu^3+12\mu^2+9\mu+4)z^3+O(z^4),\\ \bar{u}S &= 2z+6(2\mu^2+2\mu+1)z^2+8(16\mu^4+28\mu^3+30\mu^2+22\mu+9)z^3+O(z^4),\\ \bar{u}\tilde{S} &= 2z+2(2\mu^2+8\mu+5)z^2+8(2\mu^4+12\mu^3+33\mu^2+40\mu+18)z^3+O(z^4).\end{aligned}$$

We will also need the following variant of these results.

Lemma 5.6. *Define Φ_2 by* (11) *and $\tilde{S}$ by* (34). *The series $\frac{\partial\Phi_2}{\partial y}(z,\tilde{S})$ is $(u+1)$-positive.*

Proof. Let us extend the definition of $\tilde{\text{S}}$-enriched trees to p-valent blossoming trees that, in addition to leaves and buds, contain also one dangling half-edge, having no charge (Figure 7). Using the arguments of Proposition 5.3, one can prove that the series $u\frac{\partial\Phi_2}{\partial y}(z,\tilde{S})$ counts such $\tilde{\text{S}}$-enriched trees for which the half-edge is incident to the root component (as before, z counts leaves and u components).

The set of such trees is again stable: indeed, an edge is flippable if it is flippable in the $\tilde{\text{S}}$-sense, and is not on the path from the root to the dangling half-edge. □

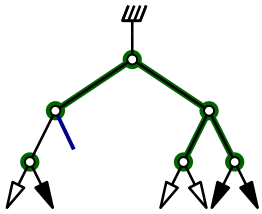

FIGURE 7. A cubic enriched S̃-tree with a dangling half-edge incident to the root component.

6. Implicit functions: some general results

The singular behaviour of a series $Y(z)$ defined by an implicit equation $H(z, Y(z)) = 0$ is well-understood when the singularities of Y occur at a point z such that H is analytic at $(z, Y(z))$, but $H'_y(z, Y(z)) = 0$. A typical situation is the so-called *smooth implicit schema* of [31, Sec. VII.4], which leads to square root singularities in Y.

However, in our asymptotic analysis of 4-valent and cubic forested maps, we will have to deal with implicit functions Y that become singular at a point z such that H ceases to be analytic at $(z, Y(z))$. Our series Y have non-negative real coefficients, which implies that their radius is also a dominant singularity, and leads us to pay a special attention to the behaviour of Y along the positive real axis.

In this section, we thus examine how far a real series Y defined by an implicit equation can be extended along the positive real axis. We first establish a general result for equations of the form $H(z, Y(z)) = 0$ (Proposition 6.1), which will apply for instance to the series $\tilde{S}$ defined by (34). We then specialize this proposition to an *inversion equation* of the form $\Omega(Y(z)) = z$ (Corollary 6.3). This corollary will apply in particular to the series R defined, in the 4-valent case, by $R = z + u\Phi(R)$ (see (15)).

Proposition 6.1. *Let $H(x, y)$ be a real bivariate power series, analytic in a neighbourhood of $(0, 0)$, satisfying $H(0, 0) = 0$ and $H'_y(0, 0) > 0$. Let $Y \equiv Y(z)$ be the unique power series satisfying $Y(0) = 0$ and $H(z, Y(z)) = 0$. Then Y has a non-zero radius of convergence. Moreover, there exists $\rho > 0$ such that:*

- (a) *Y has an analytic continuation, still denoted by Y, in a neighbourhood of $[0, \rho]$, which is real valued,*
- (b) *H has an analytic continuation, still denoted by H, in a neighbourhood of $\{(z, Y(z)), z \in [0, \rho]\}$,*
- (c) *$H(z, Y(z)) = 0$ for $z \in [0, \rho]$,*
- (d) *$H'_y(z, Y(z)) > 0$ for $z \in [0, \rho]$.*

Moreover, if $\tilde{\rho}$ is the supremum (in $\mathbb{R} \cup \{+\infty\}$) of these values ρ, at least one of the following properties holds:

- (i) *$\tilde{\rho} = +\infty$,*
- (ii) *$\liminf_{z \to \tilde{\rho}^-} H'_y(z, Y(z)) = 0$,*
- (iii) *for each $y \in [\liminf_{z \to \tilde{\rho}^-} Y(z), \limsup_{z \to \tilde{\rho}^-} Y(z)]$, H is singular at $(\tilde{\rho}, y)$,*
- (iv) *$\limsup_{z \to \tilde{\rho}^-} |Y(z)| = +\infty$.*

We begin with a simple lemma.

Lemma 6.2. *Let $a < 0 < b$ and let Y be a function analytic in a neighbourhood of $[a, b]$, whose Taylor expansion at 0 has real coefficients. Then Y takes real values on $[a, b]$.*

Proof. The functions $z \mapsto Y(z)$ and $z \mapsto \overline{Y(\bar{z})}$ are analytic and coincide in the neighbourhood of 0 where $Y(z)$ is given by its Taylor expansion. Hence they coincide everywhere, and $Y(z)$ is real when z is real. □

Proof of Proposition 6.1. The uniqueness of Y comes from the fact that its coefficients can be computed by induction using the equation $H(z, Y(z)) = 0$ and the initial condition $Y(0) = 0$ (the assumption $H'_y(0,0) \neq 0$ is crucial here). Note that these coefficients are real, so that Lemma 6.2 will apply. But let us first prove that Y has a positive radius of convergence. Since $H'_y(0,0) > 0$, the analytic implicit function theorem at $z = 0$ implies the existence of a locally analytic solution $\hat{Y}$ to the implicit equation $H(z, \hat{Y}(z)) = 0$ satisfying $\hat{Y}(0) = 0$. The expansion of $\hat{Y}$ around 0 must satisfy this equation as well (in the world of formal power series), and thus coincides with Y. Hence Y has a positive radius.

Now consider the set

$$I = \{\rho > 0 \mid \rho \text{ satisfies conditions (a), (b), (c), (d)}\}.$$

This is clearly an open interval of the form $(0, \tilde{\rho})$, and it is non-empty since (a), (b), (c) and (d) hold in the neighbourhood of 0. Assume that none of the properties (i), (ii), (iii) and (iv) hold at $\tilde{\rho}$. In particular, $\tilde{\rho}$ is finite. We will reach a contradiction by proving that $\tilde{\rho} \in I$.

Since (iv) does not hold, Y is bounded on $[0, \tilde{\rho})$. By continuity, the set of accumulation points of $\{Y(z), z \in [0, \tilde{\rho})\}$ is an interval, which coincides with $[\liminf_{z\to\tilde{\rho}^-} Y(z), \limsup_{z\to\tilde{\rho}^-} Y(z)]$. For each y in this interval, the point $(\tilde{\rho}, y)$ is in the closure of the set $\{(z, Y(z)), z \in [0, \tilde{\rho})\}$ where H is known to be analytic. Since (iii) does not hold, there exists an element $\tilde{y}$ in this interval such that H is analytic at $(\tilde{\rho}, \tilde{y})$. In particular, it is continuous at this point, and (c) implies that $H(\tilde{\rho}, \tilde{y}) = 0$. Finally, since (d) holds, but (ii) does not, $H'_y(\tilde{\rho}, \tilde{y}) > 0$.

These three properties allow us to apply the analytic implicit function theorem: there exists an analytic function $\tilde{Y}$ defined in a neighbourhood of $\tilde{\rho}$ such that $H(z, \tilde{Y}(z)) = 0$ and $\tilde{Y}(\tilde{\rho}) = \tilde{y}$. We want to prove that $\tilde{Y}$ is an analytic continuation of Y at $\tilde{\rho}$, so that, in particular, the interval $[\liminf_{z\to\tilde{\rho}^-} Y(z), \limsup_{z\to\tilde{\rho}^-} Y(z)]$ is reduced to the point $\tilde{y}$.

Since $H'_y(\tilde{\rho}, \tilde{y}) > 0$, there exists $\delta > 0$ and a complex neighbourhood V of $(\tilde{\rho}, \tilde{y})$ such that for (x, y) and (x, y') in V,

$$|H(x, y) - H(x, y')| \geq \delta |y - y'|.$$

We can also assume that $\tilde{Y}(x)$ is well-defined for $(x, y) \in V$.

Since $(\tilde{\rho}, \tilde{y})$ is an accumulation point of $\{(z, Y(z)), z \in (0, \tilde{\rho})\}$, and Y is continuous, there exists an interval $[z_0, z_1] \subset (0, \tilde{\rho})$ such that $(z, Y(z)) \in V$ for $z \in [z_0, z_1]$. Then for z in this interval,

$$0 = |H(z, Y(z)) - H(z, \tilde{Y}(z))| \geq \delta |Y(z) - \tilde{Y}(z)|,$$

which shows that the analytic functions Y and $\tilde{Y}$ coincide on $[z_0, z_1]$. So they coincide where they are both defined, and $\tilde{Y}$ is an analytic continuation of Y at $\tilde{\rho}$. This tells us that (a) holds at $\tilde{\rho}$. Now (b) also holds by the choice of $\tilde{y}$, (c) holds by construction of $\tilde{Y}$, and (d) holds as well, as argued above. Thus $\tilde{\rho}$ belongs to I, which cannot be true since it is the supremum of the open interval I. Hence one of the properties (i), (ii), (iii) and (iv) must hold. □

Corollary 6.3. *Let $\Omega(y)$ be a real power series such that $\Omega(0) = 0$ and $\Omega'(0) > 0$. Let $\omega \in (0, +\infty]$ be the first singularity of Ω on the positive real axis, if it exists, and $+\infty$ otherwise. Let $Y \equiv Y(z)$ be the unique power series satisfying $Y(0) = 0$ and $\Omega(Y(z)) = z$. Then Y has a non-zero radius of convergence. Moreover, there exists $\rho \in (0, \infty]$ such that:*

1. *Y has an analytic continuation, still denoted by Y, in a neighbourhood of $[0, \rho)$, which is real valued,*
2. *Y is increasing on $[0, \rho)$,*
3. *$Y(z) \in [0, \omega)$ for $z \in [0, \rho)$,*
4. *$\Omega(Y(z)) = z$ for $z \in [0, \rho)$,*
5. *$\lim_{z\to\rho^-} Y(z) = \tau$ and $\lim_{y\to\tau^-} \Omega(y) = \rho$,*

where

$$\tau = \min\{y \in [0, \omega) \mid \Omega'(y) = 0\}$$

if this set is non-empty, and $\tau = \omega$ otherwise.

This result is stated as an existence result for ρ, but (5) actually *determines* the value of ρ.

Proof. We specialize Proposition 6.1 to $H(x,y) = \Omega(y) - x$. Clearly, H is analytic around $(0,0)$, $H(0,0) = 0$ and $H'_y(0,0) = \Omega'(0) > 0$. We take for ρ the value $\tilde{\rho}$ of Proposition 6.1. Then (1) follows from (a). Conditions (b) and (c) tell us that Ω has an analytic continuation on $\{Y(z), z \in [0,\rho)\}$, such that $\Omega(Y(z)) = z$ for $z \in [0,\rho)$. By differentiating this identity, we obtain $Y'(z)\Omega'(Y(z)) = 1$, so that (2) now follows from (d). Thus the existence of an analytic continuation of Ω on $\{Y(z), z \in [0,\rho)\}$ now translates into (3). The monotonicity of Y also allows us to define $Y(\rho) := \lim_{z\to\rho^-} Y(z)$, which is not necessarily finite.

Let us now derive (5) from the second series of properties of Proposition 6.1. We have already seen (this is (3)) that $Y(\rho) \leq \omega$. By Condition (d) of Proposition 6.1, and by definition of τ, the value $Y(\rho)$ is also less than or equal to τ. Assume $Y(\rho) < \tau$. Then Ω is analytic at $Y(\rho)$, and by continuity of Ω and Y, $\rho = \Omega(Y(\rho)) < +\infty$, so that (i) cannot hold. By definition of τ, we cannot have (ii). It is easy to see that Conditions (iii) and (iv) do not hold either. So we have reached a contradiction, and $Y(\rho) = \tau$. Returning to (4) gives $\rho = \Omega(Y(\rho)) = \Omega(\tau)$. □

7. Inversion of functions with a $z \ln z$ singularity

The inversion of a locally injective analytic function is a well-understood topic: if Ψ is analytic in the disk C_s of radius s centered at 0 and $\Psi(z) \sim z$ as $z \to 0$, then there exist $\rho \in (0,s)$ and $\rho' > 0$, and a function Υ analytic on C'_ρ, taking its values in C_ρ, such that

$$\forall (y,z) \in C'_\rho \times C_\rho, \qquad \Psi(z) = y \Longleftrightarrow z = \Upsilon(y).$$

The aim of this section is to see to what extent this can be generalized to a function $\Psi(z)$ having a singularity in $z \ln z$ around 0. Of course we cannot consider disks anymore, and our local domains will be of the following form:

$$D_{\rho,\alpha} := \{z = re^{i\theta} : r \in (0,\rho) \text{ and } |\theta| < \alpha\}.$$

Theorem 7.1 (Log-Inversion). *Let Ψ be analytic on $D_{s,\pi}$ for some $s > 0$. Assume that as z tends to 0 in this domain,*

$$\Psi(z) \sim -cz \ln z$$

with $c > 0$. Then for each $\alpha \in (0,\pi)$, there exist $\rho \in (0,s)$ and $\rho' > 0$, and a function Υ analytic in $D_{\rho',\alpha}$, taking its values in $D_{\rho,\pi}$, that satisfies

$$\forall (y,z) \in D_{\rho',\alpha} \times D_{\rho,\pi}, \qquad \Psi(z) = y \Longleftrightarrow z = \Upsilon(y).$$

Moreover, as $y \to 0$ in $D_{\rho',\alpha}$,

$$\Upsilon(y) \sim -\frac{y}{c \ln y}.$$

The proof is rather long. The most difficult part is to prove the existence of a unique preimage of y under Ψ in $D_{\rho,\pi}$, for each $y \in D_{\rho',\alpha}$ (Lemma 7.5). This preimage is of course $\Upsilon(y)$. Proving the analyticity of Υ is then a simple application of the analytic implicit function theorem. In order to prove Lemma 7.5, we first study the injectivity and surjectivity of the function $H : z \mapsto -z \ln z$ around 0 (Section 7.1), before transferring them to the function Ψ (Section 7.2).

7.1. The function $z \mapsto -z \ln z$

Consider the following function

$$\begin{array}{rccc} H: & \mathbb{C} \setminus \mathbb{R}^- & \to & \mathbb{C} \\ & z & \mapsto & -z \ln z, \end{array}$$

where ln denotes the principal value of logarithm: if $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi,\pi)$, then $\ln z = \ln r + i\theta$. We also define $\operatorname{Arg} z := \theta$. Let us begin with a few elementary properties of H.

Lemma 7.2. *The function H satisfies*

$$H(\bar{z}) = \overline{H(z)}. \tag{35}$$

For $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi)$,

$$|H(z)| = r\sqrt{\ln^2 r + \theta^2}. \tag{36}$$

The arguments of z and $-\ln z$ have opposite signs. If in addition $r < 1$, then $\operatorname{Arg}(-\ln z) \in (-\pi/2, \pi/2)$. Hence

$$\operatorname{Arg} H(z) = \operatorname{Arg} z + \operatorname{Arg}(-\ln z) = \theta + \arctan\left(\frac{\theta}{\ln r}\right). \tag{37}$$

If in addition $r \le 1/\sqrt{e}$, then

$$|\operatorname{Arg} H(z)| \le \theta. \tag{38}$$

In particular, $H(z) \notin \mathbb{R}^-$.

Proof. The first two identities are straightforward. The first part of (37) follows from the fact that the arguments of z and $-\ln z$ have opposite signs. The second part follows from $\operatorname{Arg}(-\ln z) \in (-\pi/2, \pi/2)$. Let us now prove (38). Assume $\theta \ge 0$. Then $\operatorname{Arg}(-\ln z) \le 0$ and the first part of (37) gives $\operatorname{Arg} H(z) \le \theta$. Moreover, $\arctan x \ge x$ if $x \le 0$, and thus by the second part of (37),

$$\operatorname{Arg} H(z) \ge \theta + \frac{\theta}{\ln r} \ge \left(1 - \frac{1}{\ln(1/\sqrt{e})}\right)\theta = -\theta.$$

The case where $\theta \le 0$ now follows using (35). □

Observe that H is not injective on $\mathbb{C}$: for instance, $H(i) = \pi/2 = H(-i)$. However, H is injective in a (slit) neighbourhood of 0.

Proposition 7.3. *The function $H : z \mapsto -z \ln z$ is injective on $D_{e^{-1},\pi}$.*

Proof. Assume there exist z_1 and z_2 in $D_{e^{-1},\pi}$ such that $H(z_1) = H(z_2)$. By Lemma 7.2, the value $H(z_1)$ is not real and negative, and thus $\ln H(z_1) = \ln H(z_2)$.

This lemma also implies that for $z \in D_{e^{-1},\pi}$, we have

$$\ln H(z) = \ln z + \ln(-\ln z). \tag{39}$$

Hence

$$|\ln z_1 - \ln z_2| = |\ln(-\ln z_1) - \ln(-\ln z_2)|. \tag{40}$$

Let $\kappa = -\max(\ln|z_1|, \ln|z_2|) > 1$. Then $-\ln z_1$ and $-\ln z_2$ lie in $\{z \mid \operatorname{Re}(z) \ge \kappa\}$. This set is convex, so the (vectorial) mean value inequality gives

$$|\ln(-\ln z_1) - \ln(-\ln z_2)| \le |\ln z_1 - \ln z_2| \sup_{z \in [-\ln z_1, -\ln z_2]} |\ln'(z)| \le \frac{1}{\kappa}|\ln z_1 - \ln z_2|.$$

Combining this with (40) gives $|\ln z_1 - \ln z_2| = 0$, so that $z_1 = z_2$. □

We now address the surjectivity of the map H.

Proposition 7.4. *For $0 < \alpha < \pi$ and ρ small enough (depending on α), we have*

$$D_{-\rho \ln \rho, \alpha} \subset H(D_{\rho,\pi}).$$

Proof. We are going to prove that the inclusion holds for every $\rho \in (0, 1/e)$ satisfying

$$\arctan\left(\frac{\pi}{|\ln \rho|}\right) \le \pi - \alpha. \tag{41}$$

Let us fix a complex number $se^{i\gamma}$ with $0 < s < -\rho \ln \rho$ and $|\gamma| < \alpha$. We want to prove the existence of $r < \rho$ and $\theta \in (-\pi, \pi)$ such that $H(re^{i\theta}) = se^{i\gamma}$. We proceed in two steps.

(1) There exists a continuous function $\theta : (0, \rho) \to (-\pi, \pi)$ such that $\forall r \in (0, \rho)$,

$$\operatorname{Arg} H(r\, e^{i\theta(r)}) = \gamma.$$

Proof. Fix $r \in (0,\rho)$. For $\theta \in (-\pi,\pi)$, Lemma 7.2 gives

$$f(r,\theta) := \operatorname{Arg} H(re^{i\theta}) = \theta + \arctan\left(\frac{\theta}{\ln r}\right).$$

Differentiating with respect to θ gives

$$f'_\theta(r,\theta) = 1 + \frac{1}{\left(1+\frac{\theta^2}{\ln^2 r}\right)\ln r} \geq 1 + \frac{1}{\ln r} > 0.$$

Hence $f(r,\theta)$ is a continuous increasing function of θ, sending $(-\pi,\pi)$ onto $(-\pi-\arctan(\pi/\ln r), \pi+\arctan(\pi/\ln r))$. Since $r<\rho$ and ρ satisfies (41), this interval contains $(-\alpha,\alpha)$, and thus the value γ. This proves the existence, and uniqueness (since f increases), of $\theta(r)$.

Now in the neighbourhood of $(r,\theta(r))$, we can apply the implicit function theorem to the equation $f(r,\theta)=\gamma$, and this shows that θ is continuous on $(0,\rho)$.

(2) There exists $r \in (0,\rho)$ such that $|H(r\,e^{i\theta(r)})| = s$.

Proof. The function

$$r \mapsto |H(r\,e^{i\theta(r)})| = r\sqrt{\ln^2 r + \theta(r)^2}$$

is continuous on $(0,\rho)$. It tends to 0 as r tends to 0, and to a value at least equal to $-\rho\ln\rho$ as r tends to ρ. Since $\alpha < -\rho\ln\rho$, the intermediate value theorem implies that there exists $r\in(0,\rho)$ such that $|H(re^{i\theta(r)})| = s$.

This completes the proof of the proposition. □

7.2. The Log-Inversion Theorem

By combining Propositions 7.3 and 7.4, we see that for $\alpha \in (0,\pi)$ and ρ small enough, every point of $D_{-\rho\ln\rho,\alpha}$ has a unique preimage under H in $D_{\rho,\pi}$. We now want to adapt this result to functions Ψ that behave like H in the neighbourhood of the origin.

Lemma 7.5. *Let Ψ be analytic on $D_{s,\pi}$ for some $s>0$. Assume that as z tends to 0 in this domain,*

$$\Psi(z) \sim H(z) = -z\ln z.$$

For all $\alpha\in(0,\pi)$, there exist $\rho\in(0,s)$ and $\rho'>0$ such that every point of $D_{\rho',\alpha}$ has a unique preimage under Ψ in $D_{\rho,\pi}$.

Proof. By assumption, $\Psi(z)-H(z) = o(z\ln z) = o(-|z|\ln|z|)$ as z tends to 0. Let $\rho\in(0,s)$ be small enough for every $z\in D_{\rho,\pi}$ to satisfy

$$|\Psi(z)-H(z)| < -\min\left(\frac{1}{2}, \sin\left(\frac{\pi-\alpha}{4}\right)\right)|z|\ln|z|, \tag{42}$$

$$1+\frac{1}{\ln|z|} > \frac{1}{2}+\frac{\alpha}{\pi+\alpha}, \tag{43}$$

$$|z\ln z| \leq -2|z|\ln|z|, \tag{44}$$

$$-\ln\frac{|z|}{8} \leq -2\ln|z|, \tag{45}$$

and assume moreover than ρ is also small enough for the following property to hold:

$$D_{-\frac{\rho}{8}\ln\frac{\rho}{8},\alpha} \subset H(D_{\frac{\rho}{8},\pi}). \tag{46}$$

This inclusion is made possible by Proposition 7.4. Several of the above listed conditions can be described by an explicit upper bound on ρ (for instance, (45) just means that $\rho \leq e^{-8}$), but we will use them in the above form and find convenient to write them so.

Now fix $y_0 \in D_{\rho',\alpha}$ with $\rho' = -\frac{\rho}{8}\ln\frac{\rho}{8}$. We want to prove that y_0 has a unique preimage under Ψ in $D_{\rho,\pi}$. By (46) and Proposition 7.3, it has a unique preimage under H, denoted by z_0, in $D_{\rho,\pi}$ (in fact, $|z_0|<\rho/8$). We thus want to prove that the functions $\Psi-y_0$ and $H-y_0$ have the same number of roots in $D_{\rho,\pi}$, and we will do so using Rouché's theorem.

For $\varepsilon \in (0, |z_0|/8)$, let $\Gamma \equiv \Gamma^{(\varepsilon)}$ be the contour shown in Figure 8. The interior of Γ converges to $D_{\rho,\pi}$ as $\varepsilon \to 0$. Hence for ε small enough, it contains the point z_0, and we just need to prove that $\Psi - y_0$ and $H - y_0$ have the same number of roots inside $\Gamma^{(\varepsilon)}$ for every small enough ε.

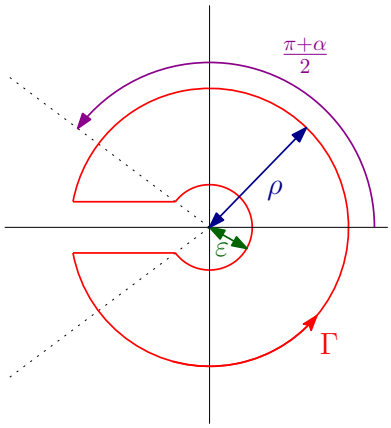

FIGURE 8. The contour $\Gamma^{(\varepsilon)}$.

By Rouché's theorem, it suffices to show that $|\Psi - H| < |H - y_0|$ on Γ. Let us decompose Γ into three (non-disjoint) parts :

$$\Gamma_1 = \Gamma \cap \{z : |z| = \rho\}, \quad \Gamma_2 = \Gamma \cap \{z : |z| = \varepsilon\} \quad \text{and} \quad \Gamma_3 = \Gamma \cap \left\{z : |\operatorname{Arg} z| > \frac{\pi + \alpha}{2}\right\}.$$

We will use in the study of Γ_1 and Γ_2 the following elementary result.

Lemma 7.6. *If $\rho \geq |z| \geq 8\,|z'|$ with $z, z' \in \mathbb{C} \setminus (-\infty, 0]$, then*

$$|z \ln z - z' \ln z'| \geq -\frac{1}{2}|z| \ln |z|.$$

Proof. We have the following lower bounds:

$$\begin{aligned} |z \ln z - z' \ln z'| &\geq |z \ln z| - |z' \ln z'| \\ &\geq -|z| \ln |z| + 2|z'| \ln |z'| && \text{by (36) and (44),} \\ &\geq -|z| \ln |z| + \frac{|z|}{4} \ln \frac{|z|}{8} && \text{because } |z'| \leq |z|/8, \\ &\geq -\frac{1}{2}|z| \ln |z| && \text{by (45).} \end{aligned}$$

□

Since $|z_0| < \rho/8$, we can apply this lemma to $z \in \Gamma_1$ and $z' = z_0$. This gives, using (42):

$$|H(z) - y_0| \geq -\frac{1}{2}|z| \ln |z| > |\Psi(z) - H(z)|.$$

Since $\varepsilon < |z_0|/8$, applying Lemma 7.6 to z_0 and $z \in \Gamma_2$ gives:

$$|y_0 - H(z)| \geq -\frac{1}{2}|z_0| \ln |z_0| \geq -\frac{1}{2}|z| \ln |z| > |\Psi(z) - H(z)|.$$

We are left with the contour Γ_3. If $z \in \Gamma_3$, we claim that

$$|\operatorname{Arg} H(z)| \geq \alpha + \frac{\pi - \alpha}{4}. \tag{47}$$

By (35), it suffices to prove this when $\operatorname{Arg} z \geq 0$. In this case,

$$\begin{aligned}
\operatorname{Arg} H(z) &= \operatorname{Arg} z + \arctan\left(\frac{\operatorname{Arg} z}{\ln|z|}\right) && \text{by (37)},\\
&\geq \left(1+\frac{1}{\ln|z|}\right)\operatorname{Arg} z && \text{since } \arctan x \geq x,\\
&> \left(\frac{1}{2}+\frac{\alpha}{\pi+\alpha}\right)\operatorname{Arg} z && \text{by (43)},\\
&\geq \alpha + \frac{\pi-\alpha}{4} && \text{since } \operatorname{Arg} z \geq \frac{\pi+\alpha}{2}.
\end{aligned}$$

Hence (47) holds on Γ_3. But since $|\operatorname{Arg} H(z_0)| = |\operatorname{Arg} y_0| < \alpha$, we have

$$|\operatorname{Arg} H(z) - \operatorname{Arg} H(z_0)| > \frac{\pi-\alpha}{4}. \tag{48}$$

We still need one more result to conclude.

Lemma 7.7. *For $\beta > 0$ and complex numbers a and b in $\mathbb{C}\setminus(-\infty,0]$,*

$$|\operatorname{Arg} a - \operatorname{Arg} b| \geq \beta \implies |a-b| \geq |a|\sin\beta.$$

Proof. (1) If $|\operatorname{Arg} a - \operatorname{Arg} b| \leq \frac{\pi}{2}$, then

$$|a-b| = \left||a|e^{i(\operatorname{Arg} a - \operatorname{Arg} b)} - |b|\right| \geq \left|\operatorname{Im}\left(|a|e^{i(\operatorname{Arg} a - \operatorname{Arg} b)}\right)\right| \geq |a|\sin\beta.$$

(2) If $|\operatorname{Arg} a - \operatorname{Arg} b| \geq \frac{\pi}{2}$, then

$$\begin{aligned}
|a-b| = \left||a| - |b|e^{i(\operatorname{Arg} b - \operatorname{Arg} a)}\right| &\geq \left|\operatorname{Re}\left(|a| - |b|e^{i(\operatorname{Arg} b - \operatorname{Arg} a)}\right)\right|\\
&= |a| - |b|\cos(\operatorname{Arg} b - \operatorname{Arg} a) \geq |a| \geq |a|\sin\beta.
\end{aligned}$$

□

By applying this lemma to (48) with $\beta = (\pi-\alpha)/4$, we obtain

$$\begin{aligned}
|H(z) - y_0| &\geq |H(z)|\sin\left(\frac{\pi-\alpha}{4}\right)\\
&\geq -\sin\left(\frac{\pi-\alpha}{4}\right)|z|\ln|z| && \text{by (36)}\\
&> |\Psi(z) - H(z)| && \text{by (42)}.
\end{aligned}$$

We have finally proved that $|\Psi(z) - H(z)| < |H(z) - y_0|$ everywhere on the contour $\Gamma \equiv \Gamma^{(\varepsilon)}$, and we can now conclude that $\Psi - y_0$ has, like $H - y_0$, a unique root in $D_{\rho,\pi}$. □

We are finally ready to prove the log-inversion theorem (Theorem 7.1).

Proof of Theorem 7.1. Upon writing $\Psi = c\Psi_1$ and $\Upsilon(y) = \Upsilon_1(y/c)$, we can assume without loss of generality that $c = 1$. We then choose ρ and ρ' as in Lemma 7.5. For $y_0 \in D_{\rho',\alpha}$, we define $\Upsilon(y_0)$ as the unique point z_0 of $D_{\rho,\pi}$ such that $\Psi(z_0) = y_0$. We now apply the analytic implicit function theorem to the equation $\Psi(\Upsilon(y)) = y$, in the neighbourhood of (y_0, z_0). The function Ψ is analytic at z_0 and locally injective by Lemma 7.5. Therefore $\Psi'(z_0) \neq 0$, and there exists an analytic function $\bar{\Upsilon}$ defined in the neighbourhood of y_0 such that $\bar{\Upsilon}(y_0) = z_0$ and $\Psi(\bar{\Upsilon}(y)) = y$ in this neighbourhood.

This forces $\Upsilon(y)$ and $\bar{\Upsilon}(y)$ to coincide in a neighbourhood of y_0, and implies that Υ is analytic at y_0 — and hence in the domain $D_{\rho',\alpha}$.

Let us conclude with the singular behaviour of Υ near 0. The equation $\Psi(\Upsilon(y)) = y$, combined with $\Psi(z) \sim -z\ln z$, implies that $\Upsilon(y) \to 0$ as $y \to 0$. Thus

$$y \sim -\Upsilon(y)\ln(\Upsilon(y))$$

as $y \to 0$. Upon taking logarithms, and using (39), this gives

$$\ln y \sim \ln(\Upsilon(y)) + \ln(-\ln(\Upsilon(y))) \sim \ln(\Upsilon(y)).$$

Combining the last two equations finally gives $\Upsilon(y) \sim -y/\ln y$. □

8. Asymptotics for 4-valent forested maps

Let $F(z,u) = \sum_n f_n(u) z^n$ be the generating function of 4-valent forested maps, given by Theorem 3.1. That is, $f_n(u)$ counts forested 4-valent maps with n faces by the number of non-root components. As recalled in Section 2.2, the polynomial $f_n(\mu - 1)$ has several interesting combinatorial descriptions in terms of maps equipped with an additional structure, and we will study the asymptotic behaviour of $f_n(u)$ for any $u \geq -1$.

Recall that $F(z,u)$ is characterized by (15) where θ and Φ are given by (23). As discussed after Theorem 3.1, $F(z,0)$ is explicit and given by (16):

$$F(z,0) = \int \theta(z) dz = 4 \sum_{i \geq 2} \frac{(3i-3)!}{(i-2)!i!(i+1)!} z^{i+1},$$

which makes the case $u = 0$ of the following theorem a simple application of Stirling's formula.

Theorem 8.1. *Let $p = 4$, and take $u \geq -1$. The radius of convergence of $F(z,u)$ is*

$$\rho_u = \tau - u\Phi(\tau) \tag{49}$$

where Φ is given by (23) and

$$\begin{cases} \tau = 1/27 & \text{if } u \leq 0, \\ 1 - u\Phi'(\tau) = 0 & \text{if } u > 0. \end{cases}$$

The later condition determines a unique $\tau \equiv \tau_u$ in $(0, 1/27)$.

In particular, ρ_u is an affine function of u on $[-1, 0]$:

$$\rho_u = \frac{1}{27} - u\Phi\left(\frac{1}{27}\right) = \frac{1+u}{27} - u\frac{\sqrt{3}}{12\pi}. \tag{50}$$

The function ρ_u is decreasing, real-analytic everywhere except at 0, where it is still infinitely differentiable: as $u \to 0^+$,

$$\rho_u = \frac{1}{27} - u\Phi\left(\frac{1}{27}\right) + O\left(\exp\left(-\frac{2\pi}{\sqrt{3}u}\right)\right). \tag{51}$$

Let $f_n(u)$ be the coefficient of z^n in $F(z,u)$. There exists a positive constant c_u such that

$$f_n(u) \sim \begin{cases} c_u\, \rho_u^{-n} n^{-3} (\ln n)^{-2} & \text{if } u \in [-1, 0), \\ c_u\, \rho_u^{-n} n^{-3} & \text{if } u = 0, \\ c_u\, \rho_u^{-n} n^{-5/2} & \text{if } u > 0. \end{cases}$$

The constant c_u is given explicitly in Propositions 8.3 (for $u > 0$) and 8.4 (for $u < 0$), and $c_0 = 2/(9\sqrt{3}\pi)$.

The exponent $-5/2$ found for $u > 0$ is standard for planar maps (see for instance Tables 1 and 2 in [3]). The behaviour for $u < 0$ is much more surprising, and, to our knowledge, it is the first time that it is observed in the world of maps. A plot of ρ_u is shown in Figure 9. Note that $\rho_{-1} = \sqrt{3}/(12\pi)$, a transcendental radius for the series counting 4-valent maps equipped with an internally inactive spanning tree.

The proof of the theorem uses the *singularity analysis* of [31, Ch. VI]. We thus need to locate the *dominant* singularities of the series F' (that is, those of minimal modulus), and to find how F' behave in their vicinity. In order to do this, we begin with the series R, defined by $R = z + u\Phi(R)$, and then move to $F' = \theta(R)$. We will find that both series have the same radius ρ_u. Moreover, since F' and $\bar{u}(R - z)$ have non-negative coefficients in z, this radius is a singularity of each (by Pringsheim's theorem). We will prove that neither F' nor R have other

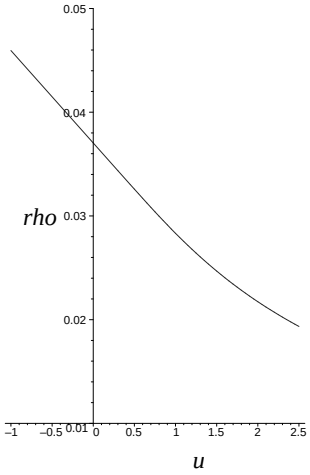

FIGURE 9. The radius ρ_u of $F(z,u)$, as a function of $u \geq -1$.

dominant singularities, and obtain estimates of these functions near ρ_u (the same estimate, up to a multiplicative factor).

Now the location of ρ_u, and its nature as a singularity, depend on whether $u > 0$ or $u < 0$ (Figure 10). For $u > 0$, the series R will be shown to satisfy the *smooth implicit schema* of [31, Sec. VII.4]. In brief, the dominant singularity ρ_u of R comes from the failure of the assumption $u\Phi'(R(z)) \neq 1$ in the implicit function theorem. The value $R(\rho_u)$ lies in the analyticity domain of Φ and θ, and the singularities of these series play no role. Both R and F' will be proved to have a square root dominant singularity. If $u < 0$ however, the series R reaches at ρ_u the dominant singularity of Φ and θ, and the singular behaviours of R and F' at ρ_u depend on the singular behaviours of Φ and θ. In particular, we find that, around $\rho \equiv \rho_u$, the function $F''(z,u)$ behaves like $1/\ln(1 - z/\rho)$, up to a multiplicative constant. Since this cannot be the singular behaviour of a D-finite series [31, p. 520 and 582], we have the following corollary.

Corollary 8.2. *For $u \in [-1, 0)$, the generating function $F(z,u)$ of 4-valent forested maps is not D-finite. The same holds when u is an indeterminate.*

Recall that $F(z,u)$ is, however, differentially algebraic (Theorem 4.1).

8.1. The series Φ and θ

Recall the definition (23) of these series. The ith coefficient of θ is asymptotic to $27^i/i^2$, up to a multiplicative constant, and the same holds for Φ. Hence both series have radius of convergence $1/27$, converge at this point, but their derivatives diverge.

This is as much information as we need to obtain the asymptotic behaviour of $f_n(u)$ when $u > 0$. When $u < 0$, we will need to know singular expansions of Φ and θ near $1/27$. Let us first observe that

$$\Phi(x) = x \left({}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 2; 27x\right) - 1 \right) \tag{52}$$

where ${}_2F_1(a, b; c; x)$ denotes the standard hypergeometric function with parameters a, b and c:

$${}_2F_1(a, b; c; x) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

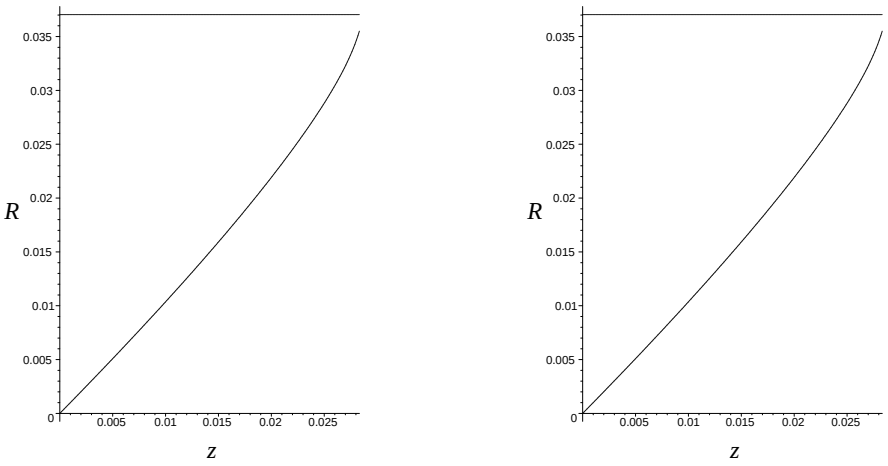

FIGURE 10. Plot of $R(z,u)$, for $z \in [0,\rho_u]$. *Left:* when $u=1$, and more generally $u>0$, R does not reach the dominant singularity of Φ (which is $1/27 \simeq 0.037$). *Right:* When $u=-1/2$, and more generally when $u \in [-1,0]$, we have $R(\rho_u)=1/27$.

with $(a)_n$ the rising factorial $a(a+1)\cdots(a+n-1)$. The series ${}_2F_1\left(\frac{1}{3},\frac{2}{3};2;27x\right)$ can be analytically continued in $\mathbb{C}\setminus[1/27,+\infty)$, and its behaviour as z approaches $1/27$ in this domain is given by [1, Eq. (15.3.11)]. Translated it terms of Φ, this gives, as $\varepsilon \to 0$,

$$\Phi\left(\frac{1}{27}-\varepsilon\right) = \frac{\sqrt{3}}{12\pi} - \frac{1}{27} + \frac{\sqrt{3}}{2\pi}\,\varepsilon\ln\varepsilon + \left(1-\frac{\sqrt{3}}{2\pi}\right)\varepsilon + O(\varepsilon^2\ln\varepsilon). \tag{53}$$

One also has:

$$\Phi'\left(\frac{1}{27}-\varepsilon\right) = -\frac{\sqrt{3}}{2\pi}\,\ln\varepsilon - 1 + O(\varepsilon\ln\varepsilon). \tag{54}$$

The series θ is related to Φ by (25). It has the same analyticity domain as Φ, with local expansion at $1/27$:

$$\theta\left(\frac{1}{27}-\varepsilon\right) = \frac{2}{3} - \frac{7\sqrt{3}}{6\pi} + \frac{2\sqrt{3}}{\pi}\varepsilon\ln\varepsilon + \frac{7\sqrt{3}}{\pi}\varepsilon + O(\varepsilon^2\ln\varepsilon). \tag{55}$$

Also,

$$\theta'\left(\frac{1}{27}-\varepsilon\right) = -\frac{2\sqrt{3}}{\pi}\ln\varepsilon - \frac{9\sqrt{3}}{\pi} + O(\varepsilon\ln\varepsilon). \tag{56}$$

8.2. **WHEN** $u>0$

As in [31, Def. VI.1, p. 389], we call Δ-*domain of radius* ρ any domain of the form

$$\{z : |z|<r, z\neq\rho \text{ and } |\operatorname{Arg}(z-\rho)|>\phi\}$$

for some $r>\rho$ and $\phi \in (0,\pi/2)$.

Proposition 8.3. *Assume $u>0$. Then the series $R(z,u)$ is aperiodic and satisfies the smooth implicit schema of* [31, Def. VII.4, p. 467]. *Its radius is given by* (49)*, and satisfies* (51)*. The series R is analytic in a Δ-domain of radius $\rho\equiv\rho_u$, with a square root singularity at ρ:*

$$R(z,u) = \tau - \gamma\sqrt{1-z/\rho} + O(1-z/\rho), \tag{57}$$

where τ is defined as in Theorem 8.1, and $\gamma = \sqrt{\frac{2\rho}{u\Phi''(\tau)}}$ with Φ given by (23).

The series $F'(z,u)$ is also analytic in a Δ-domain of radius ρ, with a square root singularity at ρ:

$$F'(z,u) = \theta(\tau) - \gamma\theta'(\tau)\sqrt{1-z/\rho} + O(1-z/\rho), \tag{58}$$

where γ is given above and θ is defined by (23). *Consequently, the nth coefficient of F satisfies, as $n \to \infty$,*

$$f_n(u) \sim \theta'(\tau)\sqrt{\frac{\rho^3}{2\pi u\Phi''(\tau)}}\rho^{-n}n^{-5/2}.$$

This proposition establishes the case $u > 0$ of Theorem 8.1.

Proof. The results that deal with R are a straightforward application of Definition VII.4 and Theorem VII.3 of [31, p. 467-468]. Using the notation of this book, $G(z,w) = z + u\Phi(w)$ is analytic for $(z,w) \in \mathbb{C} \times \{|w| < 1/27\}$. The so-called *characteristic system* holds at (ρ,τ) where τ is the unique element of $(0,1/27)$ such that $G_w(\rho,\tau) = u\Phi'(\tau) = 1$, and $\rho := \tau - u\Phi(\tau)$. The existence and uniqueness of τ is guaranteed by the fact that $\Phi'(w)$ increases (strictly) from 0 to $+\infty$ as w goes from 0 to $1/27$. The aperiodicity of R is obvious from the first terms of its expansion: $R = z + 3z^2u + 6u(3u+5)z^3 + O(z^4)$.

We now move to $F' = \theta(R)$. Since $R(\rho,u) = \tau < 1/27$, and R has non-negative coefficients, there exists a Δ-domain of radius ρ in which R is analytic and strictly bounded (in modulus) by $1/27$. Since θ has radius $1/27$, the series $F' = \theta(R)$ is also analytic in this domain, and its singular behaviour around ρ follows from a Taylor expansion. One then applies the Transfer Theorem VI.4 from [31, p. 393] to obtain the behaviour of the nth coefficient of F', which is $(n+1)f_{n+1}(u)$. The estimate of $f_n(u)$ follows.

It remains to find an estimate of ρ_u as $u \to 0^+$. Recall that $u\Phi'(\tau) = 1$. Thus $\tau \equiv \tau_u$ approaches $1/27$ as $u \to 0$, and (54) gives

$$\ln(1/27 - \tau) = -\frac{2\pi(1+\bar{u})}{\sqrt{3}} + o(1)$$

with $\bar{u} = 1/u$, so that

$$\tau - \frac{1}{27} \sim -\exp\left(-\frac{2\pi(1+\bar{u})}{\sqrt{3}}\right). \tag{59}$$

Since $\rho = \tau - u\Phi(\tau)$, this gives (51) in view of the expansion (53) of Φ. □

8.3. When $u < 0$

Proposition 8.4. *Let $u \in [-1,0)$. The series R and F' have radius $\rho \equiv \rho_u$ given by* (50). *They are analytic in a Δ-domain of radius ρ, and the following estimates hold in this domain, as $z \to \rho$:*

$$R(z) - \frac{1}{27} \sim -\frac{2\pi\rho}{\sqrt{3}u}\frac{1-z/\rho}{\ln(1-z/\rho)}, \tag{60}$$

$$F''(z) + 4\bar{u} \sim \frac{72\sqrt{3}\pi\bar{u}^2\rho}{\ln(1-z/\rho)}. \tag{61}$$

Consequently, the nth coefficient of F satisfies, as $n \to \infty$,

$$f_n(u) \sim 72\sqrt{3}\pi\bar{u}^2\frac{\rho^{-n+3}}{n^3\ln^2 n}.$$

Since (61) cannot be the singular behaviour of a D-finite series [31, p. 520 and 582], this proves Corollary 8.2. This proposition also establishes the case $u < 0$ of Theorem 8.1.

Proof. We begin as before with the series R. The equation $R = z + u\Phi(R)$ reads $\Omega(R) = z$ with $\Omega(y) = y - u\Phi(y)$. Clearly $\Omega(0) = 0$ and $\Omega'(0) = 1 > 0$, so that we can apply Corollary 6.3, in which the role of Y is played by R. Let ω, τ and ρ be defined as in this corollary. It follows from Section 8.1 that $\omega = 1/27$. Since $u < 0$, $\Omega'(y) = 1 - u\Phi'(y)$ does not vanish on $[0, 1/27)$. Hence $\tau = 1/27$ as well. By Property (5) of Corollary 6.3,

$$\rho = \Omega\left(\frac{1}{27}\right) = \frac{1}{27} - u\Phi\left(\frac{1}{27}\right),$$

which, combined with (53), gives (50).

Corollary 6.3 tells us that R has an analytic continuation along $[0, \rho)$. Moreover, $R(z)$ increases from 0 to $1/27$ on $[0, \rho)$, and the equation

$$R = z + u\Phi(R) \tag{62}$$

holds in the whole interval $[0, \rho)$.

By Corollary 5.5, the series $\bar{u}(R - z)$ has non-negative coefficients. As R itself, it is analytic along $[0, \rho)$. By Pringsheim's theorem, its radius is at least ρ, and this holds for R as well. We will now study the behaviour of R in the neighbourhood of ρ, and prove that it is singular at this point, so that ρ is indeed the radius of R.

For $z \in \mathbb{C} \setminus \mathbb{R}^-$, let us define

$$\Psi(z) := \rho + \frac{z-1}{27} + u\,\Phi\left(\frac{1-z}{27}\right).$$

As explained above, $1 - 27\,R(\rho - y)$ increases from 0 to 1 as y goes from 0 to ρ, and the functional equation (62) satisfied by R reads, for $y \in [0, \rho)$,

$$\Psi(1 - 27\,R(\rho - y)) = y. \tag{63}$$

By (53), we have $\Psi(z) \sim -cz \ln z$ where

$$c = -\frac{\sqrt{3}u}{54\pi} > 0.$$

Let us apply the log-inversion theorem (Theorem 7.1) to Ψ, with $\alpha = 3\pi/4$ (we now denote r and r' the numbers ρ and ρ' of Theorem 7.1): There exists $r > 0$ and $r' > 0$, and a function Υ analytic on $D_{r',\alpha} = \{|z| < r' \text{ and } |\operatorname{Arg} z| < 3\pi/4\}$, such that $\Psi(\Upsilon(y)) = y$. Furthermore, $\Upsilon(y)$ is the only preimage of y under Ψ that can be found in $D_{r,\pi} = \{|z| < r \text{ and } |\operatorname{Arg} z| < \pi\}$. Comparing with (63) shows that for y small enough and positive, one has $\Upsilon(y) = 1 - 27\,R(\rho - y)$. Returning to the original variables, this means that, for z real and close to ρ^-,

$$R(z) = \frac{1}{27}\left(1 - \Upsilon(\rho - z)\right),$$

so that R can be analytically continued on $\{|z - \rho| < r \text{ and } |\operatorname{Arg}(z - \rho)| > \pi/4\}$. Moreover, the final statement of Theorem 7.1 gives (60). This shows that R is singular at ρ, which is thus the radius of R.

In order to prove that R is analytic in a Δ-domain of radius ρ, we now have to prove that it has no singularity other than ρ on its circle of convergence. So let $\mu \neq \rho$ have modulus ρ. Since $\mathcal{R} := \bar{u}(R - z)$ has positive coefficients and $|\mathcal{R}(\rho)| < +\infty$, the series $\mathcal{R}$ converges at μ, and so does R. Recall that Φ is analytic in $\mathbb{C} \setminus [1/27, +\infty)$. Hence (62), which holds in a neighbourhood of 0, will hold in the closed disk of center ρ if we can prove the following lemma.

Lemma 8.5. *For $|z| \leq \rho$ and $z \neq \rho$, we have $R(z) \notin [1/27, +\infty)$.*

Proof. We have already seen that the property holds (since R is increasing) on the interval $[0, \rho)$. On the interval $[-\rho, 0]$, the function R is real (Lemma 6.2) and continuous. Hence, if R exits $(-\infty, 1/27)$ on this interval, there exists $t \in [-\rho, 0]$ such that $R(t) = 1/27$. Let t be maximal for

this property. Then $R(z) \in \mathbb{C} \setminus [1/27, +\infty)$ on a complex neighbourhood of $(t, 0]$, and (62) holds there. By differentiating it, we obtain

$$R'(z) = \frac{1}{1 - u\Phi'(R(z))} \neq 0. \tag{64}$$

In particular, $R'(0) = 1$. But since $R(t) = 1/27 > R(0) = 0$, the function $R'(z)$ must vanish in $(t, 0)$, which is impossible in view of its expression above.

Assume now that z is not real, and let us prove that $R(z)$ is not real either. First,

$$|\operatorname{Im} R(z)| = |\operatorname{Im}(z + u\mathcal{R}(z))| \geq |\operatorname{Im} z| + u\,|\operatorname{Im} \mathcal{R}(z)|. \tag{65}$$

Then:

$$\begin{aligned} |\operatorname{Im} \mathcal{R}(z)| = |\operatorname{Im}(\mathcal{R}(z) - \mathcal{R}(\operatorname{Re} z))| &\leq |\mathcal{R}(z) - \mathcal{R}(\operatorname{Re} z)| \\ &< |z - \operatorname{Re} z| \max_{y \in [\operatorname{Re} z, z]} |\mathcal{R}'(y)| \leq |\operatorname{Im} z| \max_{|y| \leq \rho} |\mathcal{R}'(y)|. \end{aligned} \tag{66}$$

The strict inequality comes from the fact that $\mathcal{R}'$ is not constant over $[\operatorname{Re} z, z]$. But $\mathcal{R}'$ is a power series with positive coefficients, and thus for $|y| \leq \rho$,

$$|\mathcal{R}'(y)| \leq \mathcal{R}'(\rho) = \bar{u}\,(R'(\rho) - 1) = \bar{u}\left(\lim_{t \to \rho} \frac{1}{1 - u\Phi'(R(t))} - 1\right) = -\bar{u}, \tag{67}$$

because $\Phi'(z)$ tends to $+\infty$ as $z \to 1/27$. Returning to (66) gives $|\operatorname{Im} \mathcal{R}(z)| < -\bar{u}|\operatorname{Im} z|$, and this inequality, combined with (65), gives $|\operatorname{Im} R(z)| > 0$. □

So we now know that (62) holds everywhere in the disk of radius ρ, with R only reaching the critical value $1/27$ at ρ. By differentiation, (64) holds as well. Let us return to our point $\mu \neq \rho$, of modulus ρ. We now want to apply the analytic implicit function theorem to (62) at the point $(\mu, R(\mu))$. We know that Φ is analytic around $R(\mu)$. Could it be that $u\Phi'(R(\mu)) = 1$? By (64), this would imply that $|R'(z)|$, and thus $|\mathcal{R}'(z)|$, is not bounded as z approaches μ in the disk. However, $\mathcal{R}'$ has non-negative coefficients and $\mathcal{R}'(\rho)$ has been shown to converge (see (67)). Thus $\mathcal{R}'(z)$ remains bounded in the disk of radius ρ, and in particular $u\Phi'(R(\mu)) \neq 1$. The analytic implicit function theorem then implies that R is analytic at μ.

In conclusion, we have proved that there exists a Δ-domain of radius ρ where R is analytic and avoids the half-line $[1/27, +\infty)$.

Let us now turn our attention to $F' = \theta(R)$. Since θ is analytic in $\mathbb{C} \setminus [1/27, +\infty)$, the series F' is analytic in the same Δ-domain as R. The estimate (60) of R, combined with the expansion (55) of θ, does not give immediately the singular behaviour of F'. Another route would be possible, but it is more direct to work with F'' instead. Indeed,

$$F''(z) = R'(z)\theta'(R(z)) = \frac{\theta'(R(z))}{1 - u\Phi'(R(z))}. \tag{68}$$

By (54) and (56),

$$\begin{aligned} \frac{\theta'(1/27 - \varepsilon)}{1 - u\Phi'(1/27 - \varepsilon)} &= -4\bar{u} - 2\bar{u}\left(9 - \frac{4\pi(1 + \bar{u})}{\sqrt{3}}\right)\frac{1}{\ln \varepsilon} + O(1/\ln^2 \varepsilon) \\ &= -4\bar{u} + \frac{72\sqrt{3}\pi\bar{u}^2\rho}{\ln \varepsilon} + O(1/\ln^2 \varepsilon), \end{aligned}$$

in view of (50). This, combined with (68) and the estimate (60) of $R(z)$, gives (61). One finally applies the Transfer Theorem VI.4 from [31, p. 393] to obtain the behaviour of the nth coefficient of F'', which is $(n + 2)(n + 1)f_{n+2}(u)$. The estimate of $f_n(u)$ follows. □

9. Large random maps equipped with a forest or a tree

We still focus in this section on 4-valent maps, equipped either with a spanning forest or with a spanning tree. In each case, we define a Boltzmann probability distribution on maps of size n, which involves a parameter u and takes into account the number of components of the spanning forest, or the number of internally active edges of the spanning tree (equivalently, the level of a recurrent sandpile configuration, as explained in Section 2.2). We observe on several random variables the effect of the phase transition found at $u = 0$ in the previous section.

9.1. Forested maps: Number and size of components

Fix $n \in \mathbb{N}$ and $u \in [0, +\infty)$. Consider the following probability distribution on 4-valent forested maps (M, F) having n faces :

$$\mathbb{P}_c(M, F) = \frac{u^{c(F)-1}}{f_n(u)},$$

where $c(F)$ is the number of components of F, and $f_n(u)$ counts 4-valent forested maps by the number of non-root components. Under this distribution, let C_n be the number of components of F, and S_n the size (number of vertices) of the root component. When $u = 0$, only tree-rooted maps have a positive probability, $C_n = 1$ and $S_n = n - 2$, the total number of vertices in the map. Let us examine how this changes when $u > 0$.

Proposition 9.1. *Assume $u > 0$. Under the distribution $\mathbb{P}_c$, we have, as $n \to \infty$:*

$$\mathbb{E}_c(C_n) \sim \frac{u\Phi(\tau)}{\tau - u\Phi(\tau)}\, n,$$

where Φ is given by (23) and $\tau \equiv \tau_u$ is the unique solution in $(0, 1/27)$ of $u\Phi'(y) = 1$.

The size S_n of the root component admits a discrete limit law: for $k \geq 1$,

$$\lim_{n\to+\infty} \mathbb{P}_c(S_n = k) = \frac{4\,(3\,k)!}{(k-1)!\,k!\,(k+1)!}\frac{\tau^k}{\theta'(\tau)} \tag{69}$$

with θ defined by (23).

Proof. We have

$$\mathbb{E}_c(C_n - 1) = \sum_{(M,F)} (c(F) - 1)\,\frac{u^{c(F)-1}}{f_n(u)} = u\frac{f_n'(u)}{f_n(u)} = u\frac{[z^{n-1}]F''_{zu}(z,u)}{[z^{n-1}]F'_z(z,u)}. \tag{70}$$

It follows from the definition (15) of R and F that

$$F''_{zu}(z,u) = \frac{\Phi(R)\theta'(R)}{1 - u\Phi'(R)}. \tag{71}$$

We now use singularity analysis. The functions Φ and θ are analytic at $\tau = R(\rho, u)$, the number τ satisfies $1 = u\Phi'(\tau)$, and a singular estimate of $R - \tau$ is given by (57). This gives, as $z \to \rho$,

$$F''_{zu}(z,u) \sim \frac{\Phi(\tau)\theta'(\tau)}{u\Phi''(\tau)\gamma\sqrt{1 - z/\rho}}$$

where γ is as in Proposition 8.3. An estimate of $F'_z(z,u)$ is given by (58). Our estimate of $\mathbb{E}_c(C_n)$ then follows from a transfer theorem, and the fact that $\rho = \tau - u\Phi(\tau)$.

To study S_n, we add to our generating function $F(z,u)$ a weight x for each vertex belonging to the root component. Lemma 3.2 becomes

$$F(z,u,x) = \bar{M}(z,u;\, 0,0,0,t_4,0,t_6,\ldots;0,0,0,x\,t_4^c,0,x^2\,t_6^c,\ldots).$$

(Recall that $t_2 = t_{2k+1} = t_2^c = t_{2k+1}^c = 0$ for every $k \geq 0$ when $p = 4$.) Thanks to (17), the first equation of (15) becomes

$$xF'_z(z,u,x) = \theta(x\,R),$$

where $R = R(z,u)$ is as before. We can express $\mathbb{P}_c(S_n = k)$ is terms of F'_z:

$$\mathbb{P}_c(S_n = k) = \frac{[z^{n-1}x^k]F'_z(z,u,x)}{[z^{n-1}]F'_z(z,u,1)} = \frac{[z^{n-1}x^{k+1}]\theta(x\,R)}{[z^{n-1}]\theta(R)}.$$

We can now apply Proposition IX.1 from [31, p. 629]. Proposition 8.3 guarantees that its hypotheses are indeed satisfied, and this gives (69), using the expression (23) of θ. □

9.2. Tree-rooted maps: Number of internally active edges

Fix $n \in \mathbb{N}$ and $u \in [-1, +\infty)$. Consider the following probability distribution on 4-valent tree-rooted maps (M,T) having n faces :

$$\mathbb{P}_i(M,T) = \frac{(u+1)^{i(M,T)}}{f_n(u)},$$

where $i(M,T)$ is the number of internally active edges in (M,T). Eq. (6) shows that this is indeed a probability distribution. Under this distribution, let I_n denote the number of internally active edges. As shown by (9), I_n can also be described as the level $\ell(C)$ of a recurrent sandpile configuration C of an n-vertex quadrangulation M, drawn according to the distribution

$$\mathbb{P}_s(M,C) = \frac{(u+1)^{\ell(C)}}{f_n(u)}.$$

Proposition 9.2. *The expected number of internally active edges undergoes a (very smooth) phase transition at $u = 0$: as $n \to \infty$,*

$$\mathbb{E}_i(I_n) \sim \kappa_u\, n, \tag{72}$$

with

$$\kappa_u = \frac{(1+u)\Phi(\tau)}{\tau - u\Phi(\tau)}$$

where Φ is given by (23) and $\tau \equiv \tau_u$ is defined in Proposition 8.1. The function κ_u is real-analytic everywhere except at 0, where it is still infinitely differentiable: as $u \to 0^+$,

$$\kappa_u = \frac{(1+u)\Phi(1/27)}{1/27 - u\Phi(1/27)} + O\left(\exp\left(-\frac{2\pi}{\sqrt{3u}}\right)\right).$$

Proof. We have

$$\mathbb{E}_i(I_n) = \sum_{(M,T)} i(M,T)\frac{(u+1)^{i(M,T)}}{f_n(u)} = (u+1)\frac{f'_n(u)}{f_n(u)} = (u+1)\frac{[z^{n-1}]F''_{zu}(z,u)}{[z^{n-1}]F'_z(z,u)}. \tag{73}$$

Comparing with (70), we see that for $u > 0$, we have $\mathbb{E}_i(I_n) = (1+\bar{u})\mathbb{E}_c(C_n)$. Thus (72) follows from Proposition 9.1 when $u > 0$. The expansion of κ_u near 0^+ follows from the estimate (59) of τ and the expansion (53) of Φ.

Let us now take $u \in [-1, 0)$. The series F''_{zu} is still given by (71), which can also be written $\Phi(R)F''_{zz}$ (by (68)), or $\bar{u}(R-z)F''_{zz}$. In view of the estimates (60) and (61) of R and F''_{zz}, we find

$$[z^{n-1}]F''_{zu}(z,u) \sim \bar{u}(1/27 - \rho)[z^{n-1}]F''_{zz}(z,u).$$

Returning to (73) gives (72) by singularity analysis, since $\rho = 1/27 - u\Phi(1/27)$.

When $u = 0$, we have $R = z$. Hence (71) reads $F''_{zu}(z,0) = \Phi(z)\theta'(z)$, while $F'_z(z,0) = \theta(z)$. As above, (72) follows from (73) by singularity analysis, using (53), (55) and (56). □

10. Asymptotics for cubic forested maps

We study in this section the singular behaviour of the series $F(z,u)$ that counts cubic forested maps by the number of components, and the asymptotic behaviour of its nth coefficient $f_n(u)$. As expected, we observe a "universality" phenomenon: our results are qualitatively the same as for 4-valent maps (Theorem 8.1). However, the cubic case is more difficult since we now have to deal with a pair of equations:

$$R = z + u\Phi_1(R,S), \quad S = u\Phi_2(R,S),$$

where Φ_1 and Φ_2 are given by (26) and (27). Our results are less complete than in the 4-valent case: when $u<0$, we only determine the singular behaviour of $F'(z,u)$ as z approaches the radius of F' on the real axis. We do not know if F' has dominant singularities other than its radius. Consequently, we have not obtained the asymptotic behaviour of $f_n(u)$ when $u<0$.

Theorem 10.1. *Let $p=3$, and take $u \geq -1$. The radius of convergence of $F(z,u)$ reads*

$$\rho_u = \tau - u\Phi_1(\tau,\sigma)$$

where the pair (τ,σ) satisfies

$$\sigma = u\Phi_2(\tau,\sigma)$$

and

$$\begin{cases} 64\tau = (1-4\sigma)^2 & \text{if } u \leq 0, \\ \\ (1-u\Phi_1^x(\tau,\sigma))\,(1-u\Phi_2^y(\tau,\sigma)) = u^2\Phi_1^y(\tau,\sigma)\Phi_2^x(\tau,\sigma) & \text{if } u>0. \end{cases}$$

The series Φ_1 and Φ_2 are given by (26) and (27), and Φ_i^x (resp. Φ_i^y) denotes the derivative of Φ_i with respect to its first (resp. second) variable.
In particular, ρ_u is an algebraic function of u on $[-1,0]$:

$$\rho_u = \frac{3(1-u^2)^2\pi^4 + 96u^2\pi^2(1-u^2) + 512u^4 + 16u\sqrt{2}\left(\pi^2(1-u^2)+8u^2\right)^{3/2}}{192\pi^4(1+u)^3}. \tag{74}$$

Let $f_n(u)$ be the coefficient in z^n in $F(z,u)$. There exists a positive constant c_u such that

$$f_n(u) \sim \begin{cases} c_u\,\rho_u^{-n} n^{-3} & \text{if } u=0, \\ c_u\,\rho_u^{-n} n^{-5/2} & \text{if } u>0. \end{cases}$$

For $u \in [-1,0]$, the series $F'(z) \equiv F'(z,u)$ has the following singular expansion as $z \to \rho_u^-$:

$$F'(z) = F'(\rho_u) + \alpha(\rho_u - z) + \beta\,\frac{\rho_u - z}{\ln(\rho_u - z)}\,(1+o(1)), \tag{75}$$

where

$$\beta = \frac{4u - 3\sqrt{2}\sqrt{\pi^2(1-u^2)+8u^2}}{2u^2} < 0.$$

Remarks
1. As in the 4-valent case, the singular behaviour of F' obtained when $u<0$ is incompatible with D-finiteness [31, p. 520 and 582].

Corollary 10.2. *For $u \in [-1,0)$, the generating function $F(z,u)$ of cubic forested maps is not D-finite. The same holds when u is an indeterminate.*

2. The series $F(z,0)$ has a simple explicit expression given by (16):

$$F(z,0) = 3\sum_{\ell \geq 1} \frac{(4\ell)!}{(2\ell-1)!(\ell+1)!(\ell+2)!} z^{\ell+2}.$$

The above theorem follows in this case from Stirling's formula. One has $\sigma = 0$ and $\rho_0 = \tau = 1/64$. We will thus focus below on the cases $u>0$ and $u<0$.
3. At $u=-1$, one finds $\rho_{-1} = \pi^2/384$, a beautiful transcendental radius of convergence for the series counting cubic maps equipped with an internally inactive spanning tree.

10.1. The series Φ_1, Φ_2, Ψ_1 and Ψ_2

We have performed in Section 4.3 a useful reduction by showing that the bivariate series $\Phi_1(x,y)$ and $\Phi_2(x,y)$ can be expressed in terms of the univariate hypergeometric series Ψ_1 and Ψ_2 (see (29–30)). The ith coefficient of Ψ_1 is asymptotic to $64^i/i^2$, up to a multiplicative constant, and the same holds for Ψ_2. Hence both series have radius of convergence $1/64$, converge at this point, but their derivatives diverge. In fact,

$$\Psi_1(z)/z = {}_2F_1(1/4, 3/4; 2; 64z),$$

so that Ψ_1 can be analytically defined on $\mathbb{C} \setminus [1/64, +\infty)$. The same holds for $\Psi_2(z)$ in view of (33). It follows from [1, Eq. (15.3.11)] that, as $\varepsilon \to 0$ in $\mathbb{C} \setminus \mathbb{R}^-$,

$$\Psi_1\left(\frac{1}{64} - \varepsilon\right) = \frac{\sqrt{2}}{24\,\pi} + \frac{\sqrt{2}}{2\,\pi}\,\varepsilon\,\ln\varepsilon - \frac{\sqrt{2}}{2\,\pi}\varepsilon + O\left(\varepsilon^2 \ln\varepsilon\right). \tag{76}$$

By (33), we also have

$$\Psi_2\left(\frac{1}{64} - \varepsilon\right) = \frac{1}{2} - \frac{\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\,\varepsilon\,\ln\varepsilon + \frac{12\,\sqrt{2}}{\pi}\varepsilon + O\left(\varepsilon^2 \ln\varepsilon\right). \tag{77}$$

Let us now return to Φ_1 and Φ_2. The series $\sqrt{1-4y}$ has radius $1/4$, the series Ψ_1 and Ψ_2 have radius $1/64$, and thus $\Phi_1(x,y)$ and $\Phi_2(x,y)$ converge absolutely for $|y| < 1/4$ and $64|x| < (1-4|y|)^2$ (Figure 11, left). The expressions (29) and (30) show that Φ_1 and Φ_2 have an analytic continuation for $y \in \mathbb{C}\setminus[1/4, +\infty)$ and $x/(1-4y)^2 \in \mathbb{C}\setminus[1/64, +\infty)$ (Figure 11, right). As $\Psi_1'(t)$ and $\Psi_2'(t)$ tend to $+\infty$ when $t \to 1/64$, there is no way to extended analytically Φ_1 or Φ_2 at a point of the *critical parabola* $\{64x = (1-4y)^2\}$.

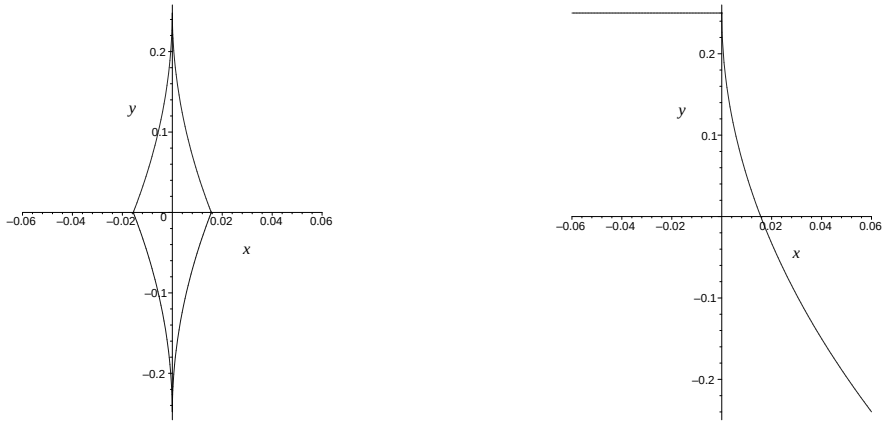

FIGURE 11. *Left:* The domain of absolute convergence of the series Φ_1 and Φ_2, in the real plane. *Right:* A domain where an analytic continuation exists. No analytic continuation exists at a point of the parabola.

10.2. **When** $u > 0$

Proposition 10.3. *Assume $u > 0$. The series R, S and F' have the same radius of convergence, denoted ρ_u, which satisfies the conditions stated in Theorem* 10.1. *The three series are analytic in a Δ-domain of radius ρ_u, with a square root singularity at ρ_u. In particular,*

$$f_n(u) \sim c_u \rho_u^{-n} n^{-5/2}$$

for some positive constant c_u.

Proof. Recall that these three series are defined by the system (28), (31), (32). The analysis of systems of functional equations can be a tricky exercise, even in the *positive case*[3] and with 2 equations only. In particular, the connection between the location of the radius and the solution(s) of the so-called *characteristic system* is subtle (see [28, 5]). In our case however, the equation that defines S does not involve the variable z explicitely, and this allows us to proceed safely in two steps. As in Section 5.2, we first define $\tilde{S} \equiv \tilde{S}(z,u)$ as the unique power series in z satisfying $\tilde{S}(0,u) = 0$ and

$$\tilde{S} = u\,\Phi_2(z,\tilde{S}) \tag{78}$$

$$= u\sqrt{1-4\tilde{S}}\;\Psi_2\left(\frac{z}{(1-4\tilde{S})^2}\right) + \frac{u}{4}\left(1-\sqrt{1-4\tilde{S}}\right)^2. \tag{79}$$

We will first study $\tilde{S}$, and then move to R, which is now defined by the following equation:

$$R = z + u\Phi_1(R,\tilde{S}(R)) \tag{80}$$

$$= z + u(1-4\tilde{S}(R))^{3/2}\Psi_1\left(\frac{R}{(1-4\tilde{S}(R))^2}\right) - uR, \tag{81}$$

where we have denoted for short $\tilde{S}(z) = \tilde{S}(z,u)$. Of course, $S = \tilde{S}(R)$.

So let us begin with $\tilde{S}$. One can prove that (78) fits in the smooth implicit function schema of [31, Def. VII.4], but we can actually content ourselves with an application of Proposition 6.1, where $\tilde{S}$ plays the role of Y. The series $H(x,y) = y - u\Phi_2(x,y)$ satisfies the assumptions of this proposition. Define $\tilde{\rho}$ as in the proposition. Since $\tilde{S}$ has non-negative coefficients, the points $(z,\tilde{S}(z))$ form, as z goes from 0 to $\tilde{\rho}$, an increasing curve starting from $(0,0)$ in the plane $\mathbb{R}^2$. Condition (b), together with the properties of Φ_2 described in Section 10.1, implies that this curve cannot go beyond the parabola $64x = (1-4y)^2$. This rules out the possibilities (i) and (iv). Now $H'_y(x,y) = 1 - u\Phi'_2(x,y)$ approaches $-\infty$ as (x,y) approach the parabola, and thus Condition (d) rules out the possibility (iii). The curve $(z,\tilde{S}(z))$ thus ends (at $z = \tilde{\rho}$) before reaching the parabola. Moreover (ii) holds: $H'_y(\tilde{\rho},\tilde{S}(\tilde{\rho})) = 0$, or equivalently,

$$1 = u\frac{\partial\Phi_2}{\partial y}(\tilde{\rho},\tilde{S}(\tilde{\rho})). \tag{82}$$

(The lim inf of (ii) becomes here a true limit because of the positivity of the coefficients of Φ_2 and $\tilde{S}$.) By (a), the radius of $\tilde{S}$ is at least $\tilde{\rho}$. Finally, it follows from (78) that for $z \in [0,\tilde{\rho})$,

$$\tilde{S}'(z) = u\,\frac{\frac{\partial\Phi_2}{\partial x}(z,\tilde{S}(z))}{1 - u\,\frac{\partial\Phi_2}{\partial y}(z,\tilde{S}(z))}. \tag{83}$$

By (82), this derivative tends to $+\infty$ as $z \to \tilde{\rho}$. Hence $\tilde{S}$ has radius $\tilde{\rho}$. Figure 12 (left) illustrates the behaviour of $\tilde{S}$ on $[0,\tilde{\rho}]$.

Let us now consider the equation (80) that defines R, and prove that it fits in the smooth implicit function schema of [31, Def. VII.4, p. 467-468]. With the notation of this definition,

[3]By this, we mean a system given by equations of the form $R_i = F_i(R_1,\ldots,R_m)$ where the series F_i have non-negative coefficients.

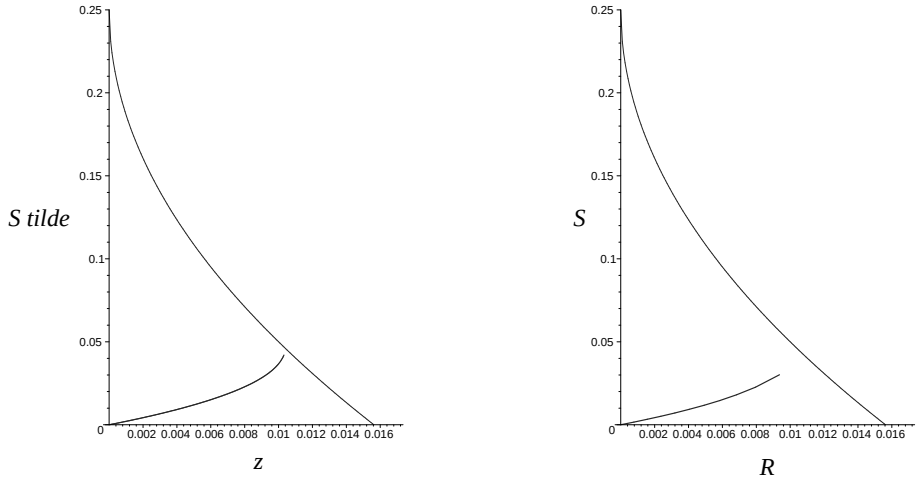

FIGURE 12. *Left:* Plot of $\tilde{S}(z)$ for $u = 1$ and $z \in [0, \tilde{\rho}]$. The points $(z, \tilde{S}(z))$ remain below the parabola $64z = (1 - 4\tilde{S})^2$. The plot was obtained using the expansion of $\tilde{S}(z)$ at order 80 (this is why the divergence of $\tilde{S}'$ at $\tilde{\rho}$ is not very clear on the picture), and the estimate $\tilde{\rho} \simeq 0.01032$. *Right:* Plot of $(R(z), S(z))$ for $u = 1$ and $z \in [0, \rho]$, with $\rho \simeq 0.0098$. This curve follows the plot of $\tilde{S}$, but stops at the point $(R(\rho), S(\rho))$, for which $R(\rho) < \tilde{\rho}$.

$G(z, w) = z + u\Phi_1(w, \tilde{S}(w))$. The properties of $\tilde{S}$ established above show that G is analytic in $\mathbb{C} \times \{w : |w| < \tilde{\rho}\}$. The characteristic equation $1 = G'_w(\rho, \tau)$ does not involve ρ and reads

$$1 = u \left(\frac{\partial \Phi_1}{\partial x}(\tau, \tilde{S}(\tau)) + \tilde{S}'(\tau) \frac{\partial \Phi_1}{\partial y}(\tau, \tilde{S}(\tau)) \right). \tag{84}$$

The right-hand side of this equation increases from 0 to $+\infty$ as τ goes from 0 to r (because, as observed above, $\tilde{S}'(\tilde{\rho}) = +\infty$). Hence (84) determines a unique value of τ in $(0, \tilde{\rho})$. The equation $\tau = G(\rho, \tau)$ gives the value of ρ:

$$\rho = \tau - u\Phi_1(\tau, \tilde{S}(\tau)). \tag{85}$$

Let $\sigma = \tilde{S}(\tau)$. The combination of (85), (78), (84) and (83) proves the properties of ρ, τ and σ stated in Theorem 10.1.

The rest of the argument is analogous to the end of the proof of Proposition 8.3. First, R is irreducible as shown by the first terms of its expansion at 0:

$$R = z + 2u(2u + 3)z^2 + 4u(42u^2 + 63u + 10u^3 + 35)z^3 + O(z^4).$$

By Theorem VII.3 of [31, p. 468], it has radius ρ, and is analytic in a Δ-domain of radius ρ. It takes the value τ at ρ, with a square root singularity there. By composition with the series $\tilde{S}$, which has radius $\tilde{\rho} > \tau$, the same properties hold for $S = \tilde{S}(R)$, and finally for the series F' given by (14) (since $\theta(x, y)$ is analytic in $\mathbb{R}^2$ for $64x < (1 - 4y)^2$).

The behaviour of R and S is illustrated in Figure 12 (right). □

10.3. WHEN $u < 0$

Proposition 10.4. *Let $u \in [-1, 0)$. The series R, S and F' have radius $\rho \equiv \rho_u$ given by (74). As $z \to \rho_u^-$, these three series admit an expansion of the form (75), with $\beta > 0$.*

Proof. As a preliminary remark, recall that $F(z,u)$ is $(u+1)$-positive, with several combinatorial interpretations described in Section 2.2. By Pringsheim's theorem, the radius of F is also its smallest real positive singularity. By Corollary 5.5, the same holds for R, S and $\tilde{S}$. This will be used frequently in the proof, without further reference to Pringsheim's theorem.

As in the case $u > 0$, we proceed in two steps, and study first the series $\tilde{S}$ defined by (78), and then the series R defined by (80). Let us begin with $\tilde{S}$, and apply Proposition 6.1 with $H(x,y) = y - u\Phi_2(x,y)$. Let us rule out the possibilities (i), (ii) and (iv).
(i) Could $\tilde{S} \equiv \tilde{S}(z,u)$ have an analytic continuation on $(0,+\infty)$? That is, an infinite radius of convergence? Corollary 5.5 implies that the radius of $\tilde{S}$ is at most the radius of $\tilde{S}(z,-1)$, which counts (by the number of leaves) enriched $\tilde{\text{S}}$-trees with no flippable edge. Since these trees can have arbitrary large size (Figure 13), $\tilde{S}(z,-1)$ is not a polynomial. Its coefficients are non-negative integers, and hence its radius is at most 1. The same thus holds for $\tilde{S}(z,u)$.

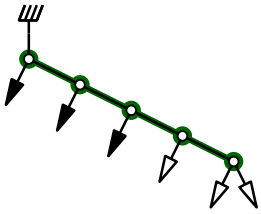

FIGURE 13. A cubic enriched $\tilde{\text{S}}$-tree with no flippable edge.

(ii) By Lemma 5.6, the series $\frac{\partial\Phi_2}{\partial y}(z,\tilde{S}(z))$ has non-negative coefficients. Since its constant term is 0, the function $1 - u\frac{\partial\Phi_2}{\partial y}(z,\tilde{S}(z))$ is increasing on $[0,\tilde{\rho})$, with initial value 1: this rules out (ii).
(iv) By Corollary 5.5, $\tilde{S}$ is negative and decreases on $[0,\tilde{\rho})$. Assume that it tends to $-\infty$. Since $\tilde{\rho}$ is finite, this implies that

$$\lim_{z\to\tilde{\rho}^-} \Psi_2\left(\frac{z}{(1-4\,\tilde{S}(z))^2}\right) = \Psi_2(0) = 0.$$

But then (79) gives

$$(1+\bar{u})\,\tilde{S} = -u\sqrt{1-4\tilde{S}}/2 + o\left(\sqrt{1-4\tilde{S}}\right),$$

which is impossible if $\tilde{S} \to -\infty$.

We conclude that (iii) holds, so that Φ_2 has no analytic continuation at $(\tilde{\rho},\tilde{S}(\tilde{\rho}))$. Given the properties of Φ_2 described in Section 10.1, this means that

$$64\tilde{\rho} = (1-4\tilde{S}(\tilde{\rho}))^2.$$

The radius of $\tilde{S}$ is at least $\tilde{\rho}$, the value of which we will determine explicitely later. Figure 14 shows a plot of $\tilde{S}$ for $u = -1/2$. One can in fact prove that $\tilde{\rho}$ *is* the radius of $\tilde{S}$, but we will not use that.

Let us now consider the equation (80) that defines R, and apply Corollary 6.3 with $\Omega(y) = y - u\Phi_1(y,\tilde{S}(y))$. We have just seen that, as y goes from 0 to $\tilde{\rho}$, the pair $(y,\tilde{S}(y))$ reaches for the first time the critical parabola at $\tilde{\rho}$. Hence, with the notation of Corollary 6.3, the first singularity of Ω on the positive real axis satisfies $\omega \geq \tilde{\rho}$. Let us define τ and ρ as in Corollary 6.3.

Could it be that $\Omega'(\tau) = 0$? By Corollary 6.3, $R(z)$ increases on $(0,\rho)$ and $\Omega'(R(z)) = 1/R'(z) \geq 0$. So could it be that $R'(z)$ tends to $+\infty$ as z tends to ρ? No: by Corollary 5.5, $\bar{u}(R'(z)-1)$ has non-negative coefficients, and thus is always larger that its value at $z = 0$, which is 0. Since $u < 0$, this gives $R'(z) \leq 1$ on $(0,\rho)$, and we conclude that $\Omega'(\tau) > 0$. Hence $\tau = \omega \geq \tilde{\rho}$.

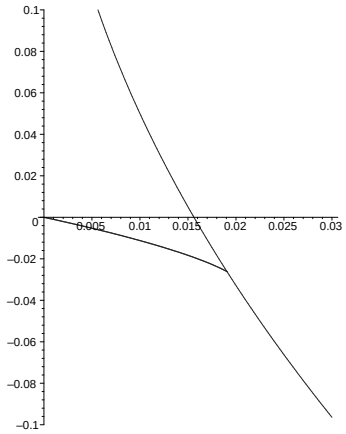

FIGURE 14. A plot of $\tilde{S}(z)$ for $u = -1/2$ and $z \in [0, \tilde{\rho}]$. The curve reaches the parabola $64z = (1 - 4\tilde{S})^2$ at $\tilde{\rho}$. The plot was obtained using the expansion of $\tilde{S}(z)$ up to order 25. Plotting the pairs $(R(z), S(z))$ for $z \in [0, \rho)$ gives the same curve.

Since $R(z)$ increases from 0 to ω on $[0, \rho]$, there exists a unique $\hat{\rho}$ such that $R(\hat{\rho}) = \tilde{\rho}$. Since $\tilde{S}$ has radius at least $\tilde{\rho}$, the series $S = \tilde{S}(R)$ has also radius at least $\hat{\rho}$. The plot of the pairs $(R(z), S(z))$ for $z \in [0, \hat{\rho}]$ coincides with the plot of $(z, \tilde{S}(z))$ for $z \in [0, \tilde{\rho}]$ shown in Figure 14.

We will now use the system (31–32) defining R and S to obtain expansions of R and S near $\hat{\rho}$. These expansions will be found to be singular at $\hat{\rho}$: this implies that $\hat{\rho} = \rho$ is the radius of R and S.

We adopt the following notation: $z = \hat{\rho} - x$, $R(z) = \tilde{\rho} - r$, $S(z) = S(\hat{\rho}) - s$ and

$$\frac{R(z)}{(1 - 4S(z))^2} = \frac{1}{64} - \varepsilon. \tag{86}$$

The quantities x, r, s and ε tend to 0 as z tends to $\hat{\rho}$. Let us begin by expanding (32) for z close to $\hat{\rho}$. Using the expansion (77) of Ψ_2 near $1/64$, we obtain

$$a_1 + b_1 s + c_1 \varepsilon \ln \varepsilon + d_1 \varepsilon = O(\varepsilon^2 \ln \varepsilon) + O(s^2) + O(s\, \varepsilon \ln \varepsilon), \tag{87}$$

with

$$a_1 = \frac{1+u}{4}\delta^2 - \frac{u\sqrt{2}}{\pi}\delta + \frac{u-1}{4},$$

$$b_1 = -\frac{2u\sqrt{2}}{\pi\delta} + 1 + u, \quad c_1 = \frac{4\sqrt{2}}{\pi}u\delta, \quad d_1 = 3c_1,$$

and $\delta = \sqrt{1 - 4\tilde{S}(\tilde{\rho})}$. In particular, a_1 must vanish, which gives the value of δ:

$$\delta = \sqrt{1 - 4\tilde{S}(\tilde{\rho})} = \frac{2\sqrt{2}u + \sqrt{\pi^2(1-u^2) + 8u^2}}{\pi(1+u)}. \tag{88}$$

(The choice of a minus sign before a square root would give a negative value, which is impossible for $\delta = \sqrt{1 - 4\tilde{S}(\tilde{\rho})}$.) Note that u has a rational expression in terms of δ:

$$u = -\frac{\pi(\delta^2 - 1)}{\pi\delta^2 - 4\sqrt{2}\delta + \pi}.$$

We will replace all occurrences of u by this expression, to avoid handling algebraic coefficients.

Let us now return to the expansion (87). Given that $\delta > 0$, we have $b_1 > 0$ for $u \in [-1, 0)$. Hence $s = O(\varepsilon \ln \varepsilon)$, and (87) can be rewritten as

$$b_1 s + c_1 \varepsilon \ln \varepsilon + d_1 \varepsilon = O(\varepsilon^2 \ln^2 \varepsilon). \tag{89}$$

Let us now move to (86). Using $\tilde{\rho} = \delta^4/64$, it gives

$$b_2 s + d_2 \varepsilon + e_2 r = O(\varepsilon^2 \ln^2 \varepsilon) \tag{90}$$

with $b_2 = -8\delta^2, d_2 = 64\delta^4, e_2 = -64$. Finally, the equation (32) that defines R gives

$$a_3 + b_3 s + c_3 \varepsilon \ln \varepsilon + d_3 \varepsilon + e_3 r + f_3 x = O(\varepsilon^2 \ln^2 \varepsilon), \tag{91}$$

where, in particular,

$$a_3 = 96(\pi\delta^2 - 4\sqrt{2}\delta + \pi)\hat{\rho} + \delta^3(2\sqrt{2}\delta^2 - 3\pi\delta + 4\sqrt{2}).$$

Since a_3 must vanish, we obtain a rational expression of $\hat{\rho}$ in terms of δ, and then, using (88), an explicit expression which coincides with (74). We do not give here the expressions of b_3, c_3, d_3, e_3 and f_3, which are rational in δ. They are easy to compute. Let us just mention that $f_3 \neq 0$.

Now, using (89), (90) and (91) in this order, we obtain for s, r and finally x expansions in ε of the form

$$\begin{aligned} s &= c_4\, \varepsilon \ln \varepsilon + d_4\, \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon), && (92)\\ r &= c_5\, \varepsilon \ln \varepsilon + d_5\, \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon), && (93)\\ x &= c_6\, \varepsilon \ln \varepsilon + d_6\, \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon). && (94) \end{aligned}$$

In particular, $c_6 \neq 0$ for $u \in [-1, 0)$ and the latter equation gives $x \sim c_6\, \varepsilon \ln \varepsilon$, so that $\ln x \sim \ln \varepsilon$ and thus

$$\varepsilon = \frac{x}{c_6 \ln x}\,(1 + o(1))\,. \tag{95}$$

To conclude, we use (94) to express $\varepsilon \ln \varepsilon$ as a linear combination of x and ε (plus $O()$ terms), and (95) to express ε in terms of x. This replaces (92) and (93) by

$$\begin{aligned} s &= \frac{c_4}{c_6} x + \frac{d_4 c_6 - c_4 d_6}{c_6^2} \frac{x}{\ln x}(1 + o(1)),\\ r &= \frac{c_5}{c_6} x + \frac{d_5 c_6 - c_5 d_6}{c_6^2} \frac{x}{\ln x}(1 + o(1)). \end{aligned}$$

These equations, written explicitely, read

$$\begin{aligned} S(z) &= \frac{1 - \delta^2}{4} + \frac{4\pi}{\delta\sqrt{\pi^2(1-u^2) + 8u^2}}(\hat{\rho} - z) - \frac{2\sqrt{2}\pi}{u\delta} \frac{\hat{\rho} - z}{\ln(\hat{\rho} - z)}\,(1 + o(1))\,,\\ R(z) &= \tilde{\rho} - \frac{\pi\delta}{2\sqrt{\pi^2(1-u^2) + 8u^2}}(\hat{\rho} - z) - \frac{\sqrt{2}\pi\delta}{4u} \frac{\hat{\rho} - z}{\ln(\hat{\rho} - z)}\,(1 + o(1))\,, \end{aligned}$$

as $z \to \hat{\rho}$. In particular, R and S are singular at $\hat{\rho}$, so that $\hat{\rho} = \rho$ is their common radius.

Using (28), we finally compute an expansion of $F'(z)$ near ρ, which gives (75). The coefficient β of $(\rho - z)/\ln(\rho - z)$ does not vanish on $[-1, 0)$, and F' has radius ρ as well. □

11. Final comments

11.1. Universality

Our asymptotic results remain incomplete when $p = 3$, as we have not been able to obtain the asymptotic behaviour of $f_n(u)$ for negative values of u (but only the singular behaviour of $F'(z,u)$). We still expect $f_n(u)$ to behave like $c_u \rho_u^{-n} n^{-3} (\ln n)^{-2}$, as in the 4-valent case.

We have also examined general even values of p. As explained below Theorem 3.1, the series S vanishes, so that we only deal with one equation (in R). When $p = 6$ for instance, it reads:

$$R = z + u\Phi(R) = z + u \sum_{\ell \geq 1} \frac{(5\ell)!}{\ell!(4\ell+1)!} R^{2\ell+1}.$$

New difficulties arise from the periodicity of Φ and R, but we still expect the same behaviour for the numbers $f_n(u)$, even though R and F will have multiple singularities on their circle of convergence.

We also plan the study of general (non-regular) forested maps.

11.2. A differential equation involving F, rather than F'?

The two differential equations (DEs) obtained for the series F in Section 4, for the 4-valent, and then for the cubic case, are in fact equations of order 2 satisfied by F'. It is natural to ask if F itself satisfies a DE of order 2. Let us examine in detail the case $p = 4$.

Returning to Lemma 3.2, we first need an expression of $\bar{M}$. Since $t_{2i+1} = t^c_{2i+1} = 0$ when $p = 4$, we can content ourselves with an expression of $\bar{M}$ valid when $g_{2i+1} = 0$ for all i. Such an expression is easily obtained from the expression (17) of $\bar{M}'_z$. Indeed, $S = 0$ in the even case, and the equations (17) and (18), written as

$$\bar{M}'_z = \bar{\theta}(R), \quad R = z + u\bar{\Phi}(R),$$

imply at once

$$\bar{M} = \bar{\Psi}(R)$$

where

$$\begin{aligned} \bar{\Psi}(x) &= \int \bar{\theta}(x) \left(1 - u\bar{\Phi}'(x)\right) dx \\ &= \sum_{i \geq 1} h_{2i} \binom{2i}{i} \frac{x^{i+1}}{i+1} - u \sum_{i \geq 1, j \geq 0} h_{2i} g_{2j+2} (2j+1) \binom{2i}{i} \binom{2j}{j} \frac{x^{i+j+1}}{i+j+1}. \end{aligned}$$

This should be compared to Eq. (1.4)in [21], which reads, in the even case,

$$\bar{M} = \sum_{n \geq 1} h_{2n} \binom{2n}{n} \frac{R^{n+1}}{n+1} - u \sum_{n \geq 1, q \geq 0, k > q} h_{2n} g_{2k} \binom{2n+2q}{n+q} \binom{2k-2q-2}{k-q-1} \frac{R^{n+k}}{n+q+1}.$$

Our (simpler) expression is obtained by summing over q.

Hence for $p = 4$, Lemma 3.2 gives

$$F(z,u) = 4 \sum_{i \geq 2} \frac{(3i-3)!}{(i-2)!i!^2} \frac{R^{i+1}}{i+1} - u \sum_{i \geq 2, j \geq 1} \frac{(3i-3)!}{(i-2)!i!^2} \frac{(3j)!}{j!^3} \frac{R^{i+j+1}}{i+j+1} = \Psi(R), \tag{96}$$

where $\Psi(x) = \Psi_1(x) - u\Psi_2(x)$,

$$\Psi_1(x) \int \theta(x), \qquad \Psi_2(x) = \int \theta(x)\Phi'(x)dx,$$

and now R is defined by $R = z + u\Phi(R)$, where Φ is given by (23).

Now assume that F is differentially algebraic of order 2: there exists a non-zero polynomial P such that

$$P(F, F', F'', z, u) = 0.$$

Equivalently,

$$P(\Psi(R), \theta(R), R'\theta'(R), z, u) = 0.$$

Using $z = R - u\Phi(R)$, $R' = (1 - u\Phi'(R))^{-1}$ and the equations (24) and (25) that relate θ, Φ, and their derivatives, we conclude that either $\Psi(x)$ is algebraic over $\mathbb{Q}(x, u, \Phi(x), \Phi'(x))$, or Φ and Φ' are algebraically related over $\mathbb{Q}(x)$. Let us examine these two possibilities.

1. Can $\Psi(x)$ be algebraic over $\mathbb{Q}(x, u, \Phi(x), \Phi'(x))$? Given that

$$15\Psi_1(x) = 15 \int \theta(x)dx = 54x^2 - 2(1 + 81x)\Phi(x) + 8x(27x - 1)\Phi'(x)$$

and

$$3\Psi_2(x) = 3 \int \theta(x)\Phi'(x)dx = 12x\Phi(x) - 2(1 - 27x)\Phi(x)\Phi'(x) - 48\Phi^2(x) + 12 \int \frac{\Phi^2(x)}{x}dx,$$

this is equivalent to saying that $\int \Phi(x)^2/xdx$ is algebraic over $\mathbb{Q}(x, \Phi(x), \Phi'(x))$. Or, using (52), that the hypergeometric function

$$f(x) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 2; x\right)$$

is such that $g(x) := \int xf^2(x)dx$ is algebraic over $\mathbb{Q}(x, f(x), f'(x))$ (here, we use the fact that

$$20 \int xf(x)dx = 9x^2 f(x) + 9x^2(1 - x)f'(x).\Big)$$

A related question is whether g is a linear combination of $f^2, ff', (f')^2$. Given that

$$2f(x) + 18(x - 1)f'(x) + 9(x - 1)f''(x) = 0,$$

the vector space spanned over $\mathbb{Q}(x)$ by these 3 series contains all products $f^{(i)}f^{(j)}$ and is closed by differentiation. This would imply that g satisfies a linear DE of order 4 with coefficients in $\mathbb{Q}(x)$. Starting from the order 4 DE satisfied by g',

$$-4g'(x) + 8x(x - 1)g''(x) + 27x(x - 1)^2 g^{(3)}(x) + 9x^2(x - 1)^2 g^{(4)}(x) = 0,$$

the Maple command `ode_int_y` tells us that g satisfies no linear DE of order 4. Following discussions with Alin Bostan and Bruno Salvy, this seems to imply actually that g is not algebraic over $\mathbb{Q}(x, f, f')$.

2. Now could it be that F' satisfies a DE of order 1? This would imply that Φ and Φ' are algebraically linked over $\mathbb{Q}(x)$, or, equivalently, that f and f' are algebraically linked over $\mathbb{Q}(x)$. One can prove that this is not the case, by combining the fact that $f'(x)$ diverges at 1 like $\ln(1 - x)$, while $f(1) = \sqrt{3}/(12\pi)$ is finite and transcendental.

These considerations lead us to believe that we have found in Section 4.2 the equation of minimal order satisfied by F.

Acknowledgements. We are grateful to Yvan Le Borgne and Andrea Sportiello for interesting discussions on this problem, and to Alin Bostan and Bruno Salvy for their help with hypergeometric series.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] O. Angel. Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.*, 13(5):935–974, 2003.

[3] C. Banderier, P. Flajolet, G. Schaeffer, and M. Soria. Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures Algorithms*, 19(3-4):194–246, 2001.

[4] R. J. Baxter. Dichromatic polynomials and Potts models summed over rooted maps. *Ann. Comb.*, 5(1):17–36, 2001.

[5] J. P. Bell, S. N. Burris, and K. A. Yeats. Characteristic points of recursive systems. *Electron. J. Combin.*, 17(1):Research Paper 121, 34, 2010.
[6] E. A. Bender and E. R. Canfield. The asymptotic number of rooted maps on a surface. *J. Combin. Theory Ser. A*, 43(2):244–257, 1986.
[7] O. Bernardi. A characterization of the Tutte polynomial via combinatorial embeddings. *Ann. Comb.*, 12(2):139–153, 2008.
[8] O. Bernardi and M. Bousquet-Mélou. The Potts model on planar maps. In preparation.
[9] O. Bernardi and M. Bousquet-Mélou. Counting colored planar maps: algebraicity results. *J. Combin. Theory Ser. B*, 101(5):315–377, 2011.
[10] O. Bernardi, N. Curien, and G. Miermont. A nested loop approach to percolation on random triangulations. In preparation.
[11] O. Bernardi and É. Fusy. A bijection for triangulations, quadrangulations, pentagulations, etc. *J. Combin. Theory Ser. A*, 119(1):218–244, 2012.
[12] B. Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.
[13] G. Borot, J. Bouttier, and E. Guitter. More on the $O(n)$ model on random maps via nested loops: loops with bending energy. *J. Phys. A: Math. Theor.*, 45:275206, 2012. ArXiv:1202.5521.
[14] G. Borot, J. Bouttier, and E. Guitter. A recursive approach to the $O(n)$ model on random maps via nested loops. *J. Phys. A: Math. Theor.*, 45:045002, 2012. ArXiv:1106.0153.
[15] G. Borot, J. Bouttier, and E. Guitter. Loop models on random maps via nested loops: case of domain symmetry breaking and application to the Potts model. *J. Phys. A*, to appear. ArXiv:1207.4878.
[16] D. V. Boulatov and V. A. Kazakov. The Ising model on a random planar lattice: the structure of the phase transition and the exact critical exponents. *Phys. Lett. B*, 186(3-4):379–384, 1987.
[17] M. Bousquet-Mélou. Counting planar maps, coloured or uncoloured. In *Surveys in combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 1–49. Cambridge Univ. Press, Cambridge, 2011.
[18] M. Bousquet-Mélou and G. Schaeffer. The degree distribution of bipartite planar maps: applications to the Ising model. In K. Eriksson and S. Linusson, editors, *Formal Power Series and Algebraic Combinatorics*, pages 312–323, Vadstena, Sweden, 2003. ArXiv math.CO/0211070.
[19] J. Bouttier, P. Di Francesco, and E. Guitter. Blocked edges on Eulerian maps and mobiles: application to spanning trees, hard particles and the Ising model. *J. Phys. A*, 40(27):7411–7440, 2007.
[20] J. Bouttier, P. Di Francesco, and E. Guitter. Census of planar maps: from the one-matrix model solution to a combinatorial proof. *Nuclear Physics B*, 645(3):477 – 499, 2002.
[21] J. Bouttier and E. Guitter. Planar maps and continued fractions. *Comm. Math. Phys.*, 309(3):623–662, 2012.
[22] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Comm. Math. Phys.*, 59(1):35–51, 1978.
[23] S. Caracciolo and A. Sportiello. Spanning forests on random planar lattices. *J. Stat. Phys.*, 135(5-6):1063–1104, 2009.
[24] G. Chapuy, M. Marcus, and G. Schaeffer. A bijection for rooted maps on orientable surfaces. *SIAM J. Discrete Math.*, 23(3):1587–1611, 2009.
[25] R. Cori and Y. Le Borgne. The sand-pile model and Tutte polynomials. *Adv. in Appl. Math.*, 30(1-2):44–52, 2003. Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001).
[26] J.-M. Daul. q-State Potts model on a random planar lattice. ArXiv:hep-th/9502014, 1995.
[27] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin. 2D gravity and random matrices. *Phys. Rep.*, 254(1-2), 1995. 133 pp.
[28] M. Drmota. Systems of functional equations. *Random Structures Algorithms*, 10(1-2):103–124, 1997.
[29] B. Duplantier and I. Kostov. Conformal spectra of polymers on a random surface. *Phys. Rev. Lett.*, 61(13):1433–1437, 1988.
[30] B. Eynard and G. Bonnet. The Potts-q random matrix model: loop equations, critical exponents, and rational case. *Phys. Lett. B*, 463(2-4):273–279, 1999.
[31] P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.
[32] C. M. Fortuin and P. W. Kasteleyn. On the random cluster model: I. Introduction and relation to other models. *Physica*, 57:536–564, 1972.
[33] A. Guionnet, V. F. R. Jones, D. Shlyakhtenko, and P. Zinn-Justin. Loop Models, Random Matrices and Planar Algebras. *Comm. Math. Phys.*, 316(1):45–97, 2012.
[34] V. A. Kazakov. Ising model on a dynamical planar random lattice: exact solution. *Phys. Lett. A*, 119(3):140–144, 1986.
[35] S. Lang. *Algebra*. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965.
[36] J.-F. Le Gall and G. Miermont. Scaling limits of random planar maps with large faces. *Ann. Probab.*, 39(1):1–69, 2011.
[37] L. Lipshitz. D-finite power series. *J. Algebra*, 122:353–373, 1989.
[38] J.-F. Marckert and G. Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.*, 35(5):1642–1705, 2007.
[39] C. Merino López. Chip firing and the Tutte polynomial. *Ann. Comb.*, 1(3):253–259, 1997.

[40] R. C. Mullin. On the enumeration of tree-rooted maps. *Canad. J. Math.*, 19:174–183, 1967.
[41] G. Schaeffer. Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees. *Electron. J. Combin.*, 4(1):Research Paper 20, 14 pp. (electronic), 1997.
[42] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.
[43] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Canadian J. Math.*, 6:80–91, 1954.
[44] W. T. Tutte. A census of planar triangulations. *Canad. J. Math.*, 14:21–38, 1962.
[45] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.
[46] W. T. Tutte. Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$. *Canad. J. Math.*, 25:426–447, 1973.
[47] W. T. Tutte. Map-colourings and differential equations. In *Progress in graph theory (Waterloo, Ont., 1982)*, pages 477–485. Academic Press, Toronto, ON, 1984.
[48] D. J. A. Welsh and C. Merino. The Potts model and the Tutte polynomial. *J. Math. Phys.*, 41(3):1127–1152, 2000.
[49] P. Zinn-Justin. The dilute Potts model on random surfaces. *J. Statist. Phys.*, 98(1-2):j10–264, 2000.

MBM & JC: CNRS, LaBRI, UMR 5800, Université de Bordeaux, 351 cours de la Libération, 33405 Talence Cedex, France

E-mail address: bousquet@labri.fr, jcourtie@labri.fr