On the bending and tension of thermoelastic shells undergoing phase transitions

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Applying the general non-linear theory of shells undergoing phase transitions, we derive the balance equations along the singular surface curve modelling the phase interface in the shell. From the integral forms of balance laws of linear momentum, angular momentum, and energy as well as the entropy inequality we obtain the local static balance equations along the curvilinear phase interface. We also derive the thermodynamic condition allowing one to determine the interface position on the deformed shell midsurface. The theoretical model is illustrated by the example of thin circular cylindrical shell made of two-phase material subjected to tensile forces and bending couples at the shell boundary. The elastic solution reveals the existence of the hysteresis loop whose size depends upon values of several loading parameters.

1 Introduction

Phase transition (PT) phenomenon in continuous media originally described by Gibbs in 1875–1878, see [1], was developed in a number of papers summarised in several recent books, for example in [2–5]. In this approach one assumes existence of the sharp phase interface being a sufficiently regular surface dividing different material phases. The position and motion of the phase interface itself is among the most discussed issues in the field. In the literature many model one-dimensional (1D) problems were analysed theoretically, numerically and experimentally which adequately described behaviour of bars, rods, and beams made of martensitic materials.

The non-linear equilibrium conditions of elastic shells undergoing PT of martensitic type were formulated in [6–8] within the dynamically and kinematically exact theory of shells presented in [9–11]. In this shell theory the translation vector $\boldsymbol{u}$ and rotation tensor $\boldsymbol{Q}$ fields are the only independent variables. By analogy to the 3D case, the two-phase shell was regarded as the Cosserat surface consisting of two material phases divided by a sufficiently smooth surface curve. Existence of such a curve was confirmed by several experiments on thin-walled samples.

2 Basic relations

The two-dimensional (2D) local laws of shell thermomechanics can be derived by direct and exact through-the-thickness integration of global 3D balances of forces, moments, energy and the entropy inequality, see [6–8]. After appropriate transformations the resulting 2D local Lagrangian laws in $M\backslash C$ become

$$\begin{aligned}
&Div_s\boldsymbol{N}+\boldsymbol{f}=0\,,\quad Div_s\boldsymbol{M}+ax(\boldsymbol{N}\boldsymbol{F}^T-\boldsymbol{F}\boldsymbol{N}^T)+\boldsymbol{c}=0\,,\\
&\rho\frac{d\varepsilon}{dt}=\rho(q^+ + q^- + q_\Pi) - Div_s\,\boldsymbol{q}+\boldsymbol{N}\bullet\boldsymbol{E}^\circ+\boldsymbol{M}\bullet\boldsymbol{K}^\circ\,,\\
&\rho\frac{d\psi}{dt}\le\rho\eta\frac{dT}{dt}+\boldsymbol{N}\bullet\boldsymbol{E}^\circ+\boldsymbol{M}\bullet\boldsymbol{K}^\circ+Grad_s\left(\frac{1}{T}\right)\cdot\boldsymbol{q}+\rho q^+\left(1-\frac{T}{T^+_{ext}}\right)+\rho q^-\left(1-\frac{T}{T^-_{ext}}\right),
\end{aligned}\tag{1}$$

where $\boldsymbol{f}$, $\boldsymbol{c}$ are the resultant surface force and couple vector fields acting on $N\backslash\mathcal{D}$, but measured per unit area of $M\backslash\mathcal{C}$, M and N are the shell midsurfaces in the undeformed and deformed placements, respectively, $\mathcal{C}\subset M$ and $\mathcal{D}\subset N$ are the curvilinear phase interfaces, $(\boldsymbol{N},\boldsymbol{M})\in E\otimes T_xM$ the surface tangential stress resultant and stress couple tensors of the first Piola-Kirchhoff type, $\boldsymbol{F}=Grad_s\,\boldsymbol{y}$ the surface deformation gradient, $\boldsymbol{F}\in E\otimes T_xM$, $ax(\ldots)$ the axial vector associated with the skew tensor $(\ldots)$, $(\boldsymbol{E}^\circ,\boldsymbol{K}^\circ)\in E\otimes T_xM$ the corotational derivatives of the shell strain measures work-conjugate to $(\boldsymbol{N},\boldsymbol{M})$, and Div_s the surface divergence operator on M. Additionally, ε and η are the surface internal energy and entropy densities, ρ the undeformed surface mass density, $q^\pm$ the heat influx densities through the upper $(+)$ and lower $(-)$ shell faces, q_Π the internal surface heat supply density, $\boldsymbol{q}$ the surface heat influx vector, T the through-the-thickness average temperature, T^+_{ext} and T^-_{ext} temperatures of the external media surrounding the shell from above and below, and $\psi=\varepsilon-T\eta$ the surface free energy density. For constitutive equations of thermoelastic and thermoviscoelastic shells see [8].

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We use the kinetic equation, describing motion of the phase interface for all quasistatic processes, in the form

$$V = -\mathcal{F}\left(\boldsymbol{\nu}\cdot[\![\boldsymbol{C}]\!]\boldsymbol{\nu}\right), \quad \boldsymbol{C} = \rho\psi\mathbf{A} - \boldsymbol{N}^T\boldsymbol{F} - \boldsymbol{M}^T\boldsymbol{K}, \tag{2}$$

where $\mathcal{F}$ is the non-negative definite kinetic function depending on the jump of $\boldsymbol{C}$ at $\mathcal{C}$, i.e. $\mathcal{F}(\varsigma) \geq 0$ for $\varsigma > 0$, the expression $[\![\ldots]\!] = (\ldots)_B - (\ldots)_A$ means the jump at $\mathcal{C}$, $\boldsymbol{\nu}$ the surface unit vector externally normal to ∂M, V the exterior normal velocity of the phase curve $\mathcal{C}$, $\mathbf{A} = \mathbf{1} - \boldsymbol{n}\otimes\boldsymbol{n}$, and $\mathbf{1}$ the 3D unit tensor.

We assume $\mathcal{F}(\varsigma)$ in the form

$$\mathcal{F}(\varsigma) = \begin{cases} \dfrac{k(\varsigma-\varsigma_0)}{1+a(\varsigma-\varsigma_0)} & \varsigma \geq \varsigma_0, \\ 0 & -\varsigma_0 < \varsigma < \varsigma_0, \\ \dfrac{k(\varsigma+\varsigma_0)}{1-a(\varsigma+\varsigma_0)} & \varsigma \leq -\varsigma_0. \end{cases} \tag{3}$$

Here ς_0 describes the effects associated with nucleation of the new phase and action of the surface tension, a is a parameter describing the limit value of the phase transition velocity, and k is a positive kinetic factor.

3 Bending and tension of elastic thin-walled tube

As an example, we discuss the thin circular cylindrical shell of length L, radius R, and thickness h made of material undergoing phase transition. The phase interface $\mathcal{C}$ is given by $z = \ell$. The tube is extended by forces P and bent by couples m uniformly distributed at the right shell boundary. The two-phase solution is presented in Fig. 1.

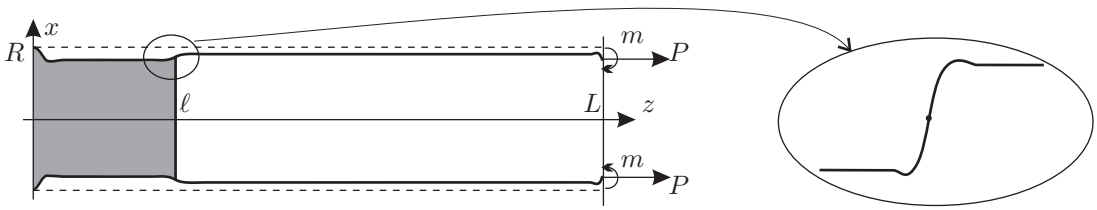

Fig. 1 Shape of the thin-walled two-phase tube after phase transition (magnified).

The proposed 2D model allows one to take into account several additional factors such as solutions of the boundary layer type or more differentiated ways of loading and unloading.

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