Boundary layers, Rellich estimates and extrapolation of solvability for elliptic systems

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ABSTRACT. The purpose of this article is to study extrapolation of solvability for boundary value problems of elliptic systems in divergence form on the upper half-space assuming De Giorgi type conditions. We develop a method allowing to treat each boundary value problem independently of the others. We shall base our study on solvability for energy solutions, estimates for boundary layers, equivalence of certain boundary estimates with interior control so that solvability reduces to a one-sided Rellich inequality. Our method then amounts to extrapolating this Rellich inequality using atomic Hardy spaces, interpolation and duality. In the way, we reprove the Regularity-Dirichlet duality principle between dual systems and extend it to H^1-BMO. We also exhibit and use a similar Neumann-Neumann duality principle.

CONTENTS

1. INTRODUCTION

Boundary value problems for second order elliptic equations have a long history. The breakthroughs of Dahlberg [Da] for the Laplace equation on Lipschitz domains and the boundedness of the corresponding layer potentials by Coifman, McIntosh

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and Meyer [CMcM] opened the door to a thorough study of such problems, generalizing domains or operators. By flattening the boundary, one instead looks at equations with measurable coefficients and considers two types of domains, either the upper-half space as a prototype for unbounded domains or the unit ball as a prototype for a bounded domain. There one can study boundary value problems with different types of data spaces. All of this is well explained in the book by Kenig [Ke]. Solving these boundary value problems can be a difficult task; there is no comprehensive nor unified treatment of this issue at this time. Let us just mention that the solution of the Kato conjecture [AHLMcT] and its developments gave rise to new estimates and new methods so that progress in the area is rather impressive as of now.

The purpose of this article is to study extrapolation of boundary value problems for elliptic systems in divergence form on the upper half-space $\mathbb{R}^{1+n}_+$, $1+n \geq 2$. Extrapolation means that, assuming the problem can be solved for some space X of data, one can push the solvability range to some other spaces. For Regularity and Neumann problems, X is an L^r space, $r > 1$ and one extrapolates to L^p for $1 < p < r$ and H^p data for some range of p below 1. For Dirichlet problems, one starts from L^q for some $q < \infty$ and extrapolates to L^p for $q < p < \infty$, BMO and Hölder spaces up to some exponent. In fact, one can see the Dirichlet problem as a Regularity problem in spaces of data with regularity exponent -1. One can also formulate Neumann problems in spaces of data with regularity -1.

We do not treat here the openness property of extrapolation, that is that solvability at one space of data can be perturbed to nearby spaces in the given scale. This will be treated in [AS] using further developments.

These types of extrapolation results are not new, at least for equations, starting from the seminal works of [DaK] for the Laplace equation on Lipschitz domains and [KP] for real symmetric equations. Further contributions are in [Br] for the Laplace equation looking at H^p data for $p < 1$, in [Di] in the context of the Laplace equation on smooth domains of Riemannian manifolds and in [DK] for real equations on bounded Lipschitz domains. See also some comments in [HKMP2] outlining a strategy using Kalton-Mitrea extrapolation [KM] when layer potentials associated to the operators are invertible. Of course, we are just mentioning the works related to extrapolation in this subject and not the numerous ones on solvability for second order elliptic operators under various assumptions. In some sense, we are after a sort of extrapolation reminiscent to the Calderón and Zygmund extrapolation for singular integrals because the operators under considerations can be thought of as generalized singular integrals.

To do so, we introduce a new method which allows to treat each boundary value problem independently of the other ones and to consider systems and not just equations, assuming De Giorgi-Nash type local Hölder regularity, in the interior and for reflections across the boundary. For the Neumann problem, this is completely new: in [KP], which is the closest antecedent to our results here, extrapolation of solvability for the Neumann problem was linked to that of the Regularity problem. Our method will clarify the Regularity-Dirichlet duality principle for solvability obtained in [HKMP2], extend its range to $H^1 - BMO$ and we will also formulate and use a new duality principle for Neumann problems. Also our exponents are explicitly determined by the ones in the De Giorgi conditions. Our strategy can be summarized as follows: try to distinguish as much as possible interior and boundary estimates

so as to use *a priori* estimates most of the time. To do so, we have to reverse the order in which we use some tools compared to other works.

Our divergence operators will be precisely defined in Section 2 and ellipticity will be taken in the sense of some Gårding inequality. The boundary value problems are treated for operators whose coefficients do not depend on t, the transverse variable to the boundary, but some results do not need this. For the purpose of the introduction, it is best to assume t-independence.

We shall rely on energy solutions. Indeed, Regularity and Neumann problems are always well-posed (modulo constants) in the energy class without any further information. We deviate here from the treatment done in [KR] or [HKMP2] by using the "natural" energy space given by the Dirichlet integral $\int |\nabla u|^2$. Even in the unbounded situation of the upper half-space, things turn out to work rather well with this space. Solvability of Regularity and Neumann problems means here that energy solutions satisfy the required inequality respectively: control of $\|\widetilde{N}_*(\nabla u)\|_p$, the L^p norm of (a modified) non-tangential maximal function of ∇u, by $\|\nabla_x u|_{t=0}\|_p$, where $\nabla_x u|_{t=0}$ is the tangential gradient at the boundary or by $\|\partial_{\nu_A} u|_{t=0}\|_p$, where $\partial_{\nu_A} u|_{t=0}$ is the conormal derivative at the boundary. For the boundary estimates, the norm $\|\ \|_p$ denotes an L^p norm if $p > 1$ and a Hardy H^p norm if $p \leq 1$.

One of the main results here is the following. Assuming interior De Giorgi type conditions, there is an *a priori* equivalence between $\|\widetilde{N}_*(\nabla u)\|_p$ and the sum $\|\partial_{\nu_A} u|_{t=0}\|_p + \|\nabla_x u|_{t=0}\|_p$ in a range $1-\varepsilon < p \leq 2$ for energy solutions (and for other types of solutions as well) with ε specified by our assumptions.[1] For $p = 2$, this was one of the key result in [AA]. The bound from below holds for any weak solution: it was known in the range $1 < p < \infty$ from [KP] and has been proved recently in a range $1 - \varepsilon < p \leq 1$ in [HMiMo]. The bound from above has been addressed in the range $1 < p < 2 + \varepsilon'$ in [HKMP2] for a class of solutions u which can be represented by the layer potentials built in [AAAHK] using Green's representation formula

$$u(t,x) = S_t(\partial_{\nu_A} u|_{t=0})(x) - D_t(u|_{t=0})(x) \tag{1}$$

where S_t and D_t are respectively the single and double layer potentials associated to $\mathrm{div} A\nabla$ on $\mathbb{R}^{1+n}$. We shall prove (Theorem 9.1) that the bound from above holds in the range $1-\varepsilon < p \leq 2$ for any solution in the energy class and other classes. Here, we use the newly discovered relation by A. Rosén [R1] between the layer potentials and the first order formalism of [AAMc], which gives L^2 boundedness of the double layer potential and $L^2 \to \dot{W}^{1,2}$ boundedness of the single layer potential in full generality. It allows to use instead the differentiated form of (1) which actually comes before in the analysis (and one does not care about the constant of integration at this stage)

$$\nabla u(t,x) = \nabla S_t(\partial_{\nu_A} u|_{t=0})(x) - \nabla D_t(u|_{t=0})(x) \tag{2}$$

and one only needs to have two *a priori* bounds for the layer potentials. The first one is $\|\widetilde{N}_*(\nabla D_t h)\|_p \lesssim \|\nabla_x h\|_p$ that we obtain in the range $1 - \varepsilon < p < 2$ (again, recall that boundary norms are H^p norms when $p \leq 1$). We note this was proved for $1 < p < 2 + \varepsilon'$ for complex equations and $1 + n \geq 3$ in [HKMP2, Proposition 5.9] and there is an interesting comment to make as an illustration of reversing the order in which we use tools. The argument there uses an estimate for what is called

[1] We shall not attempt to treat here the range $2 < p < 2 + \varepsilon'$. In fact, it can be shown to hold without the De Giorgi condition, at the expense of some further work which shall be presented in [AS].

L-harmonic conjugates (such an estimate is a one-sided Rellich inequality in disguise (see below)) and seems therefore to be limited to $p > 1$. Instead, our argument for this inequality does not require such an estimate; it only uses atomic theory and interpolation. This is made possible because we use (2) and not (1); the Rellich inequality is used later. The other needed *a priori* bound $\|\widetilde{N}_*(\nabla S_t h)\|_p \lesssim \|h\|_p$ in the range $1-\varepsilon < p < 2+\varepsilon'$ was known from [HMiMo] at least in the equation case and $1+n \geq 3$. With this in hand, solvability for a given p in this range is equivalent to a boundary estimate of Rellich type (again for energy solutions) which is

$$\|\partial_{\nu_A} u|_{t=0}\|_p \lesssim \|\nabla_x u|_{t=0}\|_p \tag{3}$$

for the Regularity problem and

$$\|\nabla_x u|_{t=0}\|_p \lesssim \|\partial_{\nu_A} u|_{t=0}\|_p \tag{4}$$

for the Neumann problem.

The outcome of this is that in order to extrapolate solvability, it suffices to extrapolate a one-sided Rellich inequality. This step, therefore, completely happens at the boundary. Basically, in the spirit of the ideas in [DaK] and [KP], we can use Hardy space atomic theory on the boundary and interpolation. But the difference is that we only have to prove (3) or (4) with given data a 2-atom (Section 4) and we do this by showing that the missing data is a molecule (Section 10) without going back to non-tangential maximal estimates. Aside from some pointwise estimates on solutions shown in Section 6 relying on some form of boundary regularity, this step uses, of course, the initial solvability assumption even to get the molecular decay. Note also that harmonic measure techniques are forbidden to us as we work with systems. The way it works is that we use in fact the dual formulation of the inequalities (3) or (4) when $p > 1$. We were therefore led to investigate this further (Section 3). The dual of (3) is an inequality akin to the one needed to solve the adjoint Dirichlet problem in $L^{p'}$. The dual formulation of (4) is new. What is also new is that these dual formulations do not require any assumption on the operator, not even t-independence, but the ellipticity, because we use duality brackets and not integrals. We also use the integrated layer potential representation (1) for solutions of the dual system: the De Giorgi condition comes into play to show that (1) holds whenever $u|_{t=0} \in L^p$ and $\partial_{\nu_A} u|_{t=0} \in \dot{W}^{-1,p}$ for $p > 2$.

As for the Dirichlet problem, we can basically treat it with the duality principle that Regularity solvability with L^p data is equivalent to Dirichlet solvability for $L^{p'}$ data of the dual system. While the Regularity to Dirichlet direction has been known since [KP] for real symmetric equations, the converse is fairly recent for general systems (some partial results for real symmetric equations in Lipschitz domains are in [S]) and requires to incorporate square functions in the formulation of the Dirichlet problem. This was proved in full generality in [AR] for $p = 2$ and then in [HKMP2] for equations and $1 + n \geq 3$ and $p \neq 2$ (Both articles allow some $t$dependence as well). We reprove and strengthen it even with the hypotheses there and also extend it to H^1 for Regularity vs BMO (or VMO) for Dirichlet. The Dirichlet problem is stated only with a square function estimate and no non-tangential maximal control which in fact comes as *a priori* information. We shall use in this part a recent result obtained by one of us together with S. Stahlhut [AS].

We only discuss the case of the upper half-space, but of course, the analogous results hold for systems in the lower half-space. Also by change of variable, one can

treat the case of special Lipschitz domains with operators that do not depend on the vertical variable. The principal example is the Laplacian for which extrapolation results were proved in the seminal paper [DaK] and pursued in [Br]. We also mention that the same strategy can certainly be developed in the unit ball with radially independent coefficients instead, that is, the framework of [KP], using the first order formalism developed in [AR]. This would require writing out some details on layer potentials. We leave this to further developments.

All our estimates in this article depend only on ellipticity constants $\|A\|_\infty$ and the largest λ in the specified ellipticity inequality, and on the constants in the De Giorgi condition when assumed.

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2. General theory and energy solutions

If $E(\Omega)$ is a normed space of $\mathbb{C}$-valued functions on a set Ω and F a normed space, then $E(\Omega;F)$ is the space of F-valued functions with $\||f|_F\|_{E(\Omega)}<\infty$.

Denote points in $\mathbb{R}^{1+n}$ by boldface letter $\mathbf{x},\mathbf{y},\dots$ and in coordinates in $\mathbb{R}\times\mathbb{R}^n$ by (t,x) etc. We set $\mathbb{R}^{1+n}_+=(0,\infty)\times\mathbb{R}^n$. Consider the system of m equations given by

$$\sum_{i,j=0}^{n}\sum_{\beta=1}^{m}\partial_i\Big(A_{i,j}^{\alpha,\beta}(\mathbf{x})\partial_j u^\beta(\mathbf{x})\Big)=0,\qquad \alpha=1,\dots,m \tag{5}$$

in $\mathbb{R}^{1+n}_+$, where $\partial_0=\frac{\partial}{\partial t}$ and $\partial_i=\frac{\partial}{\partial x_i}$ if $i=1,\dots,n$. For short, we write $\mathrm{div}A\nabla u=0$ to mean (5), where we always assume that the matrix

$$A(\mathbf{x})=(A_{i,j}^{\alpha,\beta}(\mathbf{x}))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m}\in L^\infty(\mathbb{R}^{1+n}_+;\mathcal{L}(\mathbb{C}^{m(1+n)})), \tag{6}$$

is bounded and measurable and satisfies some ellipticity. For systems, we use several forms of ellipticity. One is the Gårding inequality

$$\int_{\mathbb{R}^{1+n}_+}\mathrm{Re}(A(\mathbf{x})\nabla g(\mathbf{x})\cdot\overline{\nabla g(\mathbf{x})})\,d\mathbf{x}\geq\lambda\sum_{i=0}^{n}\sum_{\alpha=1}^{m}\int_{\mathbb{R}^{1+n}_+}|\partial_i g^\alpha(\mathbf{x})|^2d\mathbf{x} \tag{7}$$

for all $g\in C^1_0(\mathbb{R}^{1+n}_+;\mathbb{C}^m)$ (C^1 functions with compact support) and some $\lambda>0$, and sometimes one needs the stronger Gårding inequality

$$\int_{\mathbb{R}^{1+n}_+}\mathrm{Re}(A(\mathbf{x})\nabla g(\mathbf{x})\cdot\overline{\nabla g(\mathbf{x})})\,d\mathbf{x}\geq\lambda\sum_{i=0}^{n}\sum_{\alpha=1}^{m}\int_{\mathbb{R}^{1+n}_+}|\partial_i g^\alpha(\mathbf{x})|^2d\mathbf{x} \tag{8}$$

for all $g\in C^1_0(\mathbb{R}^{1+n};\mathbb{C}^m)$ and some $\lambda>0$. We have set

$$A(\mathbf{x})\xi\cdot\overline{\eta}=\sum_{i,j=0}^{n}\sum_{\alpha,\beta=1}^{m}A_{i,j}^{\alpha,\beta}(\mathbf{x})\xi_j^\beta\,\overline{\eta_i^\alpha}.$$

Note that the integrals are on the upper-half space. For systems, an elementary computation shows that (8) is equivalent to the Gårding inequality (17) on $\mathbb{R}^{1+n}$ (see below) for the extended matrix $A^\sharp$ to $\mathbb{R}^{1+n}$ obtained by changing the sign when t changes sign of the coefficients for the mixed t,x_i derivatives in (5). The scalar case

corresponding to equations is when $m = 1$. In this case, the accretivity condition above are equivalent to the usual pointwise accretivity condition

$$\mathrm{Re}(A(\mathbf{x})\xi \cdot \overline{\xi}) \geq \lambda|\xi|^2,\ \xi \in \mathbb{C}^{1+n},\ a.e.\ on\ \mathbb{R}^{1+n}_+. \tag{9}$$

Alternately, scalar can mean a diagonal system in the sense that $A^{\alpha,\beta}_{i,j} = A_{i,j}\delta^{a,\beta}$ using the Kronecker symbol. When A has t-independent coefficients, that is $A(t,x) = A(x)$, (8) is implied by the strict accretivity of A on the subspace $\mathcal{H}^0$ of $L^2(\mathbb{R}^n;\mathbb{C}^{m(1+n)})$ defined by $(f^\alpha_j)_{j=1,\dots,n}$ is curl free in $\mathbb{R}^n$ for all α, that is, for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \mathrm{Re}(A(x)f(x)\cdot\overline{f(x)})\,dx \geq \lambda \sum_{i=0}^{n}\sum_{\alpha=1}^{m}\int_{\mathbb{R}^n}|f^\alpha_i(x)|^2dx,\ \forall\ f \in \mathcal{H}_0. \tag{10}$$

Even when A is t-independent, (9) is stronger than (10) when $m \geq 2$ except when $n = 1$. See [AAMc] and [AR] for details. Such conditions are stable under taking adjoint of A.

The system (5) is always considered in the sense of distributions with weak solutions, that is $H^1_{loc}(\mathbb{R}^{1+n}_+;\mathbb{C}^m)$ solutions.

There is an important space for the theory of energy (or variational) solutions in $\mathbb{R}^{1+n}_+$. We shall use the homogeneous space of energy solutions

$$\mathcal{E} := \dot{H}^1(\mathbb{R}^{1+n}_+;\mathbb{C}^m),$$

which is different from the one used in [KR] and [HKMP2]. Recall that $\dot{H}^1(\mathbb{R}^{1+n}_+)$ is the space of $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ functions u with finite energy $\int_{\mathbb{R}^{1+n}_+}|\nabla u(\mathbf{x})|^2\,d\mathbf{x} < \infty$. The fact that we assume local square integrability is of particular help (in fact, it suffices to even assume u to be a distribution as u can then be identified with an L^2_{loc} function), even if the "norm" is defined modulo constant. Indeed, the proof of Lemma 3.1 in [AMcM] shows that $\dot{H}^1(\mathbb{R}^{1+n}_+)$ imbeds into $C([0,\infty);L^2_{\mathrm{loc}}(\mathbb{R}^n))$, where $C(\Omega)$ stands for the space of continuous functions on Ω, (up to identification of measurable functions on null sets) and that the restriction to $\mathbb{R}^{1+n}_+$ of $C^\infty_0(\mathbb{R}^{1+n})$ is dense in $\dot{H}^1(\mathbb{R}^{1+n}_+)$. In particular, the trace of $\dot{H}^1(\mathbb{R}^{1+n}_+)$, identifying $\partial\mathbb{R}^{1+n}_+$ with $\mathbb{R}^n$, is the space of $f \in L^2_{loc}(\mathbb{R}^n)$ such that $f \in \dot{H}^{1/2}(\mathbb{R}^n)$ and has $C^\infty_0(\mathbb{R}^n)$ as dense subspace. Thus we can interpret boundary equalities also in L^2_{loc} and measure size only with the "homogeneous" norm in $\dot{H}^{1/2}(\mathbb{R}^n)$. Recall that $\dot{H}^1_0(\mathbb{R}^{1+n}_+)$ is the subspace of $\dot{H}^1(\mathbb{R}^{1+n}_+)$ consisting of functions with constant trace: it is the closure of $C^\infty_0(\mathbb{R}^{1+n}_+)$ for the semi-norm above. Note also that $\dot{H}^1(\mathbb{R}^{1+n}_+)$ is stable (as a set) under multiplication by $C^\infty_0(\mathbb{R}^{1+n})$ functions restricted to $\mathbb{R}^{1+n}_+$.

In what follows, we denote by $\dot{H}^s(\mathbb{R}^n)$ the homogeneous Sobolev space with exponent $s \in \mathbb{R}$ defined as the completion of $L^2(\mathbb{R}^n)$ for the semi-norm $\|(-\Delta)^{s/2}f\|_2$, where Δ is the self-adjoint Laplace operator on $L^2(\mathbb{R}^n)$. For $s > 0$, it is the closure of $C^\infty_0(\mathbb{R}^n)$ (= limits of Cauchy sequences for the homogeneous semi-norms) and can be realized as a subset of $L^2_{loc}(\mathbb{R}^n)$, and it becomes a Banach space when moding out polynomials of some order. For $s < 0$, it is a space of tempered distributions, identified with the dual of $\dot{H}^{-s}(\mathbb{R}^n)$ in the usual sesquilinear pairing. It is convenient to introduce the space $\dot{H}^s_\nabla(\mathbb{R}^n;\mathbb{C}^n) := \nabla\dot{H}^{s+1}(\mathbb{R}^n) = \nabla(-\Delta)^{-1/2}\dot{H}^s(\mathbb{R}^n)$. Using the boundedness of the Riesz transforms on $\dot{H}^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, it is the subspace of the curl free elements in $\dot{H}^s(\mathbb{R}^n;\mathbb{C}^n)$. We will use them for $s = -1/2$ and $s = 1/2$,

in which case they are dual spaces for the usual duality and notice that they both have $\mathcal{D}_\nabla(\mathbb{R}^n;\mathbb{C}^n) := \nabla C_0^\infty(\mathbb{R}^n)$ as a dense subspace.

Let us continue with the definition of the conormal derivative and the abstract Green's formula.

Lemma 2.1. *Let $A(\mathbf{x})$ be any bounded measurable matrix in $\mathbb{R}^{1+n}_+$. Let $u \in \mathcal{E}$ such that $\mathrm{div} A\nabla u = 0$ in $\mathbb{R}^{1+n}_+$. Then, there exists a distribution in $\dot{H}^{-1/2}(\mathbb{R}^n;\mathbb{C}^m)$, denoted by $\partial_{\nu_A} u|_{t=0}$ or $\partial_{\nu_A} u_0$[2] and called the conormal derivative of u at the boundary, such that for any $\phi \in \mathcal{E}$ with $\phi_0 = \phi|_{t=0}$,*

$$\int_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi} = -\langle \partial_{\nu_A} u_0, \phi_0 \rangle. \tag{11}$$

In particular, for $u, w \in \mathcal{E}$ with $\mathrm{div} A\nabla u = 0$ and $\mathrm{div} A^*\nabla w = 0$ in $\mathbb{R}^{1+n}_+$, one has the abstract Green's formula

$$\langle u_0, \partial_{\nu_{A^*}} w_0 \rangle = \langle \partial_{\nu_A} u_0, w_0 \rangle. \tag{12}$$

The brackets are interpreted in the $\dot{H}^{-1/2}, \dot{H}^{1/2}$ sesquilinear duality, but, by abuse, in no definite order for the factors so as to make the formula look like the Green's formula obtained by integration by parts (when feasible). The conormal derivative agrees with $\nu \cdot (A\nabla u)|_{t=0}$ whenever this makes sense, where ν is the upward unit vector in the t-direction (hence the inward normal for $\mathbb{R}^{1+n}_+$). This convention for conormal derivatives will be useful later. This explains the negative sign in the defining formula.

Proof. The definition of the conormal derivative is a consequence of the facts that (11) is 0 when $\phi \in \mathcal{E}$ with constant trace (because $C_0^\infty(\mathbb{R}^{1+n}_+;\mathbb{C}^m)$ is dense in it) and that the trace is bounded from $\mathcal{E}$ onto $\dot{H}^{1/2}(\mathbb{R}^n)$. The details are left to the reader. The abstract Green's formula follows immediately from definition of the conormal derivatives. □

Remark that the theory of energy solutions (that is, solutions of $\mathrm{div} A\nabla u = 0$ in $\mathcal{E}$) done in [AMcM, Section 3] for t-independent systems satisfying (10) extends immediately to t-dependent systems satisfying the appropriate Gårding inequality allowing to use the Lax-Milgram lemma. We state the well-posedness results for convenience. Note that by density, (7) and (8) extend to all g in $\dot{H}^1_0(\mathbb{R}^{1+n}_+;\mathbb{C}^m)$ and $\dot{H}^1(\mathbb{R}^{1+n}_+;\mathbb{C}^m) = \mathcal{E}$ respectively.

Lemma 2.2. *Let $A(\mathbf{x})$ be bounded measurable with the stronger Gårding inequality (8). Let $g \in \dot{H}^{-1/2}(\mathbb{R}^n;\mathbb{C}^m)$. Then, there is an energy solution $u \in \mathcal{E}$, unique modulo constants in $\mathbb{C}^m$, of the system $\mathrm{div} A\nabla u = 0$ in $\mathbb{R}^{1+n}_+$ with $\partial_{\nu_A} u|_{t=0} = g$ in $\dot{H}^{-1/2}(\mathbb{R}^n;\mathbb{C}^m)$.*

This uses the Lax-Milgram lemma in $\mathcal{E}/\mathbb{C}^m$. One can define the *Neumann to Dirichlet operator* as the bounded linear operator

$$\Gamma_{ND} : \dot{H}^{-1/2}(\mathbb{R}^n;\mathbb{C}^m) \to \dot{H}^{-1/2}_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$$

in such a way that $\Gamma_{ND}(\partial_{\nu_A} u|_{t=0}) = \nabla_x u|_{t=0}$, if u is one of the energy solution with given Neumann datum $\partial_{\nu_A} u|_{t=0}$.

[2]We shall use both notations.

Lemma 2.3. *Let $A(\mathbf{x})$ be bounded measurable with the Gårding inequality (7). Let $f \in L^2_{\rm loc}(\mathbb{R}^n;\mathbb{C}^m) \cap \dot H^{1/2}(\mathbb{R}^n;\mathbb{C}^m)$. Then, there is a unique energy solution $u \in \mathcal{E}$ of the equation ${\rm div}A\nabla u = 0$ where $u|_{t=0} = f$ holds in $\dot H^{1/2}(\mathbb{R}^n;\mathbb{C}^m) \cap L^2_{\rm loc}(\mathbb{R}^n;\mathbb{C}^m)$.*

Proof. Given an extension ϕ of f in $\mathcal{E}$, by the Lax-Milgram lemma applied in $\dot H^1_0(\mathbb{R}^{1+n}_+;\mathbb{C}^m)$, there exists, unique modulo $\mathbb{C}^m$, a solution $w \in \mathcal{E}$ to ${\rm div}A\nabla w = -{\rm div}A\nabla\phi$ with $w|_{t=0} = 0$ with equality in $\dot H^{1/2}(\mathbb{R}^n;\mathbb{C}^m)$. Thus $u = w + \phi$ solves ${\rm div}A\nabla u = 0$ with $u|_{t=0} = f$. Since $f \in L^2_{\rm loc}(\mathbb{R}^n;\mathbb{C}^m)$, we can fix the constant by imposing the equality in $L^2_{\rm loc}(\mathbb{R}^n;\mathbb{C}^m)$. Thus u is uniquely defined. □

Similarly, one can define the *Dirichlet to Neumann operator* as the bounded linear operator

$$\Gamma_{DN} : \dot H^{-1/2}_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n) \to \dot H^{-1/2}(\mathbb{R}^n;\mathbb{C}^m)$$

in such a way that $\Gamma_{DN}(\nabla_x u|_{t=0}) = \partial_{\nu_A} u|_{t=0}$, if u is the energy solution with given Dirichlet datum $u|_{t=0}$ (or alternately, any of the energy solution with given regularity datum $\nabla_x u|_{t=0}$).

Let us come to some local inequalities. We use the notation $B(\mathbf{x},r)$ to denote the open ball in $\mathbb{R}^{n+1}$, centred at $\mathbf{x}$, of radius r. Given such a ball $B = B(\mathbf{x},r)$, we let κB denote the concentric dilate of B by a factor of κ. For $x \in \mathbb{R}^n$, we let $\Delta = \Delta(x,r) := B((0,x),r) \cap (\{0\}\times\mathbb{R}^n)$ denote the "surface ball" on $\mathbb{R}^n$ centred at x and with radius r and $B_+(x,r) = B((0,x),r) \cap \mathbb{R}^{1+n}_+$ the half-ball.

If $A(\mathbf{x})$ is bounded measurable with the Gårding inequality (7), any weak solution u in a ball $B = B(\mathbf{x},r)$ with $B \subset \mathbb{R}^{1+n}_+$ of ${\rm div}A\nabla u = 0$ enjoys the Caccioppoli inequality for any $0 < \alpha < \beta < 1$ and some C depending on the ellipticity constants, n, m, α and β,

$$\int_{\alpha B} |\nabla u|^2 \le Cr^{-2} \int_{\beta B} |u|^2, \tag{13}$$

and any weak solution $u \in W^{1,2}(B_+;\mathbb{C}^m) = H^1(B_+;\mathbb{C}^m)$ of ${\rm div}A\nabla u = 0$ on $B_+ = B_+(x,r)$ with $u|_{t=0} = 0$ on $\Delta(x,r)$ enjoys the boundary Caccioppoli inequality for any $0 < \alpha < \beta < 1$ and some C depending on the ellipticity constants, n, m, α and β,

$$\int_{\alpha B_+} |\nabla u|^2 \le Cr^{-2} \int_{\beta B_+} |u|^2. \tag{14}$$

If $A(\mathbf{x})$ satisfies the stronger boundary Gårding inequality (8), then any weak solution $u \in H^1(B_+;\mathbb{C}^m)$ of ${\rm div}A\nabla u = 0$ on $B_+ = B_+(x,r)$ with $\partial_{\nu_A} u|_{t=0} = 0$[3] on $\Delta(x,r)$ enjoys the boundary Caccioppoli inequality (14). The proofs are standard.

This gives for example the following kind of local boundary estimates.

Proposition 2.4. *Let $A(\mathbf{x})$ be bounded measurable with the Gårding inequality (7). Let $u, w \in \mathcal{E}$ with ${\rm div}A\nabla u = 0$ and ${\rm div}A^*\nabla w = 0$ in $\mathbb{R}^{1+n}_+$. Assume that u_0 is supported in a surface ball $\Delta_0 = \Delta(x_0,\rho)$ and w_0 is supported in a surface ball $\Delta = \Delta(x,r)$ with $4\Delta \cap \Delta_0 = \emptyset$. Then*

$$|\langle \partial_{\nu_A} u_0, w_0\rangle| \le Cr^{-2} \left(\int_{\Omega_+} |u|^2\right)^{1/2} \left(\int_{\Omega_+} |w|^2\right)^{1/2} \tag{15}$$

with $\Omega_+ = 3B_+ \setminus 2B_+$, $B_+ = B_+(x,r)$.

[3]It can be defined locally.

Proof. Fix $\varphi \in C_0^\infty(\mathbb{R}^n)$ supported in $3B$, $\varphi = 1$ on $2B$ where $B = B((0,x),r)$ with $\|\varphi\|_\infty \leq 1$ and $\|\nabla\varphi\|_\infty \lesssim r^{-1}$. Let φ_0 be its restriction to $\mathbb{R}^n$. Remark that $\varphi w \in \mathcal{E}$ so its trace $\varphi_0 w_0$ is well-defined in $L^2_{\rm loc} \cap \dot{H}^{1/2}$ and $\varphi_0 w_0 = w_0$ using that $\varphi_0 = 1$ on the support of w_0. Thus $\langle \partial_{\nu_A} u_0, w_0\rangle = \langle \partial_{\nu_A} u_0, \varphi_0 w_0\rangle$. Next,

$$\langle \partial_{\nu_A} u_0, \varphi_0 w_0\rangle = -\int A\nabla u \cdot \overline{\nabla(\varphi w)} = +\int uA\nabla\varphi \cdot \overline{\nabla w} - \int A\nabla u \cdot \overline{w}\nabla\varphi$$

where the last equality uses the fact that $\mathrm{div} A^*\nabla w = 0$ and that $\varphi u \in \mathcal{E}$ with $\varphi_0 u_0 = 0$ so that $-\int A\nabla(\varphi u)\cdot\overline{\nabla w} = \langle \varphi_0 u_0, \partial_{\nu_{A^*}} w_0\rangle = 0$. We conclude for both terms by using Cauchy-Schwarz inequality, Caccioppoli and boundary Caccioppoli inequalities, and that the support of $\nabla\varphi$ is contained in Ω_+. □

Remark 2.5. There are variants for the right hand side. As u vanishes on $3\Delta \setminus 2\Delta$, one can show $r^{-1}\left(\int_{\Omega_+}|u|^2\right)^{1/2} \lesssim \left(\int_{\Omega_+}|\nabla u|^2\right)^{1/2}$ by using variants of Poincaré's inequality. The similar observation applies to w.

There is a similar statement for disjointly supported conormal derivatives.

Proposition 2.6. *Let $A(\mathbf{x})$ be bounded measurable with the stronger Gårding inequality (8). Let $u, w \in \mathcal{E}$ with $\mathrm{div} A\nabla u = 0$ and $\mathrm{div} A^*\nabla w = 0$ in $\mathbb{R}^{1+n}_+$. Assume that $\partial_{\nu_A} u_0$ is supported in a surface ball $\Delta_0 = \Delta(x_0,\rho)$ and $\partial_{\nu_{A^*}} w_0$ is supported in a surface ball $\Delta = \Delta(x,r)$ with $4\Delta \cap \Delta_0 = \emptyset$. Then*

$$|\langle u_0, \partial_{\nu_{A^*}} w_0\rangle| \leq Cr^{-2}\left(\int_{\Omega_+}|u|^2\right)^{1/2}\left(\int_{\Omega_+}|w|^2\right)^{1/2} \tag{16}$$

with $\Omega_+ = 3B_+ \setminus 2B_+$, $B_+ = B_+(x,r)$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be as above. Again $\varphi u \in \mathcal{E}$ so its trace $\varphi_0 u_0$ is well-defined in $L^2_{\rm loc} \cap \dot{H}^{1/2}$. Thus $\langle u_0, \partial_{\nu_{A^*}} w_0\rangle = \langle \varphi_0 u_0, \partial_{\nu_{A^*}} w_0\rangle$ using that $\varphi_0 = 1$ on the support of $\partial_{\nu_{A^*}} w_0$. We conclude exactly as in the previous argument. We skip details. □

Remark 2.7. Note that one can replace u by $u - c$ in this argument as they have the same conormal derivative. Thus one can choose c to our like. For example, if we choose the solution u whose average on Ω_+ equals 0, then $r^{-1}\left(\int_{\Omega_+}|u|^2\right)^{1/2} \lesssim \left(\int_{\Omega_+}|\nabla u|^2\right)^{1/2}$ by Poincaré's inequality. One can do similarly with w. In our applications, we shall need decay estimates for u if Δ_0 and Δ are far apart and some control on w. See Theorem 10.6.

3. Rellich estimates and duality principles for $1 < p < \infty$

We next want to shed a new light on duality principles for global boundary estimates of Rellich type. Recall that we will not assume anything but ellipticity on the coefficients at this point.

For $1 < p < \infty$, let $\dot{W}^{1,p}(\mathbb{R}^n) = \{f \in L^1_{\rm loc}(\mathbb{R}^n)\,;\, \nabla f \in L^p(\mathbb{R}^n;\mathbb{C}^n)\}$ (one can show that this is the same space, upon identification, assuming instead $f \in \mathcal{D}'(\mathbb{R}^n)$) and set $\|f\|_{\dot{W}^{1,p}} = \|\nabla f\|_p$. For $p = 2$, this is also $\dot{H}^1(\mathbb{R}^n)$. Some well-known properties are summarized here.

Proposition 3.1. (1) *$C_0^\infty(\mathbb{R}^n)$ is dense in $\dot{W}^{1,p}(\mathbb{R}^n)$.*

(2) $\dot W^{-1,p'}(\mathbb{R}^n)$, *the dual of* $\dot W^{1,p}(\mathbb{R}^n)$, *is the space of distributions* $\mathrm{div} g$ *for some* $g\in L^{p'}(\mathbb{R}^n;\mathbb{C}^n)$ *with norm* $\inf\|g\|_{p'}$ *taken oven all choices of* g.

We note the importance of the $L^1_{\rm loc}$ requirement to get the density. The following well-known lemma will be useful.

Lemma 3.2. *For* $f\in L^1_{\rm loc}(\mathbb{R}^n)$, $\|\nabla f\|_{\dot W^{-1,p'}}\sim \inf\{\|f+c\|_{p'}\,;\,c\in\mathbb{C}\}$.

The left hand side is the norm in $\dot W^{-1,p'}(\mathbb{R}^n;\mathbb{C}^n)$. In other words, the left hand side is finite if and only if there exists one (and only one since constants are not in $L^{p'}(\mathbb{R}^n)$) $c\in\mathbb{C}$ such that $f+c\in L^{p'}(\mathbb{R}^n)$.

As we identify $\partial\mathbb{R}^{1+n}_+$ with $\mathbb{R}^n$, we use here the subscript 0 to indicate the restriction to the boundary. Thus ∇u_0 is short notation for $\nabla_x u_0$.

Theorem 3.3. *Let* $A(\mathbf{x})$ *be a bounded measurable matrix with the Gårding inequality* (7). *Let* $1<p<\infty$. *The following are equivalent.*

(1) *There exists* $C_p<\infty$ *such that for any* $u\in\mathcal{E}$ *solution of* $\mathrm{div}A\nabla u=0$, $\|\partial_{\nu_A}u_0\|_p\le C_p\|\nabla u_0\|_p$.
(2) *There exists* $C_{p'}<\infty$ *such that for any* $w\in\mathcal{E}$ *solution of* $\mathrm{div}A^*\nabla w=0$, $\|\partial_{\nu_{A^*}}w_0\|_{\dot W^{-1,p'}}\le C_{p'}\|\nabla w_0\|_{\dot W^{-1,p'}}$.

Theorem 3.4. *Let* $A(\mathbf{x})$ *be a bounded measurable matrix with the stronger Gårding inequality* (8). *Let* $1<p<\infty$. *The following are equivalent.*

(1) *There exists* $C_p<\infty$ *such that for any* $u\in\mathcal{E}$ *solution of* $\mathrm{div}A\nabla u=0$, $\|\nabla u_0\|_p\le C_p\|\partial_{\nu_A}u_0\|_p$.
(2) *There exists* $C_{p'}<\infty$ *such that for any* $w\in\mathcal{E}$ *solution of* $\mathrm{div}A^*\nabla w=0$, $\|\nabla w_0\|_{\dot W^{-1,p'}}\le C_{p'}\|\partial_{\nu_{A^*}}w_0\|_{\dot W^{-1,p'}}$.

Some remarks are necessary. The tangential gradient and conormal derivative at the boundary of an energy solution are distributions in $\mathbb{R}^n$ (in $\dot H^{-1/2}$). Thus, finiteness of any of the norms above means that the distribution is identified with an element in the considered space which is also embedded in the space of distributions. Theorem 3.3 concerns boundary inequalities needed for solving the regularity problem for $\mathrm{div}A\nabla$ in L^p and the Dirichlet problem for $\mathrm{div}A^*\nabla w=0$ in $L^{p'}$, or rather a regularity problem in $\dot W^{-1,p'}$. For $p=2$, this is akin to a result of [AR]. It can be compared with Theorem 3.1 of [HKMP2], stated only for t-independent equations with De Giorgi condition and a restriction on p. In contrast, our result here is independent of any kind of interior control on solutions besides the energy estimate and this is why it holds for any p. The energy class is used here as an existence and uniqueness class. Any other such class would do a similar job. Theorem 3.4 is new and relates the Neumann problem for $\mathrm{div}A\nabla$ in L^p to a Neumann problem for $\mathrm{div}A^*\nabla$ in $\dot W^{-1,p'}$, which has not been studied up to our knowledge. A related statement appears in [R2] for $p=2$.

Proof of Theorem 3.3. Assume (1) and let $w\in\mathcal{E}$ be a solution of $\mathrm{div}A^*\nabla w=0$. Assume also $\|\nabla w_0\|_{\dot W^{-1,p'}}<\infty$ otherwise there is nothing to prove. By lemma 3.2 and the fact that for any $c\in\mathbb{C}^m$, $w+c$ is also a solution with same conormal derivative as w, we may assume $\|w_0\|_{p'}<\infty$. By Proposition 3.1, it is enough to estimate $\langle\partial_{\nu_{A^*}}w_0,g\rangle$ for any $g\in C_0^\infty(\mathbb{R}^n\,;\,\mathbb{C}^m)$ with $\|\nabla g\|_p\le 1$. Let $u\in\mathcal{E}$ be the solution of $\mathrm{div}A\nabla u=0$ with $u_0=g$ (Lemma 2.3). By Lemma 2.1, $\langle\partial_{\nu_{A^*}}w_0,g\rangle=\langle w_0,\partial_{\nu_A}u_0\rangle$.

Now $w_0 \in L^{p'}$ and by (1), $\|\partial_{\nu_A} u_0\|_p \leq C_p \|\nabla g\|_p \leq C_p$. Hence, reinterpreting the last bracket in the usual $L^{p'}, L^p$ duality and using Hölder's inequality, we obtain

$$|\langle \partial_{\nu_{A^*}} w_0, g\rangle| \leq \|w_0\|_{p'} \|\partial_{\nu_A} u_0\|_p \leq C_p \|w_0\|_{p'}$$

and we conclude for (2).

Conversely assume (2) and let $u \in \mathcal{E}$ solution of $\mathrm{div} A\nabla u = 0$. Assume also $u_0 \in \dot{W}^{1,p}$ and $\|\nabla u_0\|_p < \infty$ otherwise there is nothing to prove. It is enough to estimate $\langle \partial_{\nu_A} u_0, g\rangle$ for any $g \in C_0^\infty(\mathbb{R}^n\,;\mathbb{C}^m)$ with $\|g\|_{p'} \leq 1$. Let $w \in \mathcal{E}$ be the solution of $\mathrm{div} A^*\nabla w = 0$ with $w_0 = g$ (Lemma 2.3). By Lemma 2.1, $\langle \partial_{\nu_A} u_0, g\rangle = \langle u_0, \partial_{\nu_{A^*}} w_0\rangle$. Now $u_0 \in \dot{W}^{1,p}$ and, using (2) and Lemma 3.2, $\|\partial_{\nu_{A^*}} w_0\|_{\dot{W}^{-1,p'}} \leq C_{p'} \|\nabla g\|_{\dot{W}^{-1,p'}} \lesssim \|g\|_{p'} \leq 1$. Thus reinterpreting the last bracket in the $\dot{W}^{1,p}, \dot{W}^{-1,p'}$ duality, we obtain

$$|\langle \partial_{\nu_A} u_0, g\rangle| \leq \|u_0\|_{\dot{W}^{1,p}} \|\partial_{\nu_{A^*}} w_0\|_{\dot{W}^{-1,p'}} \lesssim \|\nabla u_0\|_p$$

and we conclude for (1) by density. □

Proof of Theorem 3.4. Assume (1) and let $w \in \mathcal{E}$ be a solution of $\mathrm{div} A^*\nabla w = 0$. Assume also $\|\partial_{\nu_{A^*}} w_0\|_{\dot{W}^{-1,p'}} < \infty$ otherwise there is nothing to prove. By Proposition 3.1 (in a vector-valued form), it is enough to estimate $\langle \nabla w_0, g\rangle$ for any $g \in C_0^\infty(\mathbb{R}^n\,;(\mathbb{C}^m)^n)$ with $\|g\|_{\dot{W}^{1,p}} = \|\nabla g\|_p \leq 1$. Let $u \in \mathcal{E}$ be a solution of $\mathrm{div} A\nabla u = 0$ with $\partial_{\nu_A} u_0 = -\mathrm{div} g$ (Lemma 2.2). By Lemma 2.1, $\langle \nabla w_0, g\rangle = \langle w_0, -\mathrm{div} g\rangle = \langle \partial_{\nu_{A^*}} w_0, u_0\rangle$. By (1), $\|u_0\|_{\dot{W}^{1,p}} \leq C_p \|\partial_{\nu_A} u_0\|_p \leq C_p \|\mathrm{div} g\|_p \lesssim 1$. Hence, reinterpreting the last bracket in the $\dot{W}^{-1,p'}, \dot{W}^{1,p}$ duality we obtain

$$|\langle \nabla w_0, g\rangle| \leq \|\partial_{\nu_{A^*}} w_0\|_{\dot{W}^{-1,p'}} \|u_0\|_{\dot{W}^{1,p}} \lesssim \|\partial_{\nu_{A^*}} w_0\|_{\dot{W}^{-1,p'}}$$

and we conclude for (2).

Conversely, assume (2) and let $u \in \mathcal{E}$ solution of $\mathrm{div} A\nabla u = 0$. Assume also $\|\partial_{\nu_A} u_0\|_p < \infty$ otherwise there is nothing to prove. It is enough to estimate $\langle \nabla u_0, g\rangle$ for any $g \in C_0^\infty(\mathbb{R}^n\,;(\mathbb{C}^m)^n)$ with $\|g\|_{p'} \leq 1$. Let $w \in \mathcal{E}$ be a solution of $\mathrm{div} A^*\nabla w = 0$ with $\partial_{\nu_{A^*}} w_0 = -\mathrm{div} g$ (Lemma 2.2). By (2), any such w satisfies $\|\nabla w_0\|_{\dot{W}^{-1,p'}} \leq C_{p'} \|\partial_{\nu_{A^*}} w_0\|_{\dot{W}^{-1,p'}} \leq C_{p'} \|\mathrm{div} g\|_{\dot{W}^{-1,p'}} \leq C_{p'} \|g\|_{p'} \leq C_{p'}$. By Lemma 3.2, there exists $c \in \mathbb{C}^m$ such that $w_0 + c \in L^{p'}$ with $\|w_0 + c\|_{p'} \lesssim \|\nabla w_0\|_{\dot{W}^{-1,p'}}$. Since $w + c$ is also a solution of the same problem, we may select w by imposing $w_0 \in L^{p'}$ which we do. By Lemma 2.1, $\langle \nabla u_0, g\rangle = \langle u_0, -\mathrm{div} g\rangle = \langle \partial_{\nu_A} u_0, w_0\rangle$. As $w_0 \in L^{p'}$ and $\|\partial_{\nu_A} u_0\|_p < \infty$, it follows by reinterpreting the last bracket in the $L^p, L^{p'}$ duality that

$$|\langle \nabla u_0, g\rangle| \leq C_{p'} \|\partial_{\nu_A} u_0\|_p \|w_0\|_{p'} \lesssim \|\partial_{\nu_A} u_0\|_p$$

and we conclude for (1). □

A consequence of the proofs is the following self-improvement of each of the 4 boundary inequalities in the above statements.

We say that an energy solution of $\mathrm{div} A\nabla u = 0$ has smooth Dirichlet data if $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ and has smooth Neumann data whenever $\partial_{\nu_A} u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ (necessarily with mean value 0).

Theorem 3.5. *Let $A(\mathbf{x})$ be a bounded measurable matrix with the Gårding inequality (7). Let $1 < p < \infty$. The following holds.*

(i) *If there exists $C_p < \infty$ such that for any energy solution u of $\mathrm{div} A\nabla u = 0$ with smooth Dirichlet data, one has $\|\partial_{\nu_A} u_0\|_p \leq C_p \|\nabla u_0\|_p$, then this holds for any energy solution u of $\mathrm{div} A\nabla u = 0$, possibly with a different constant.*

(ii) *If there exists* $C_p < \infty$ *such that for any energy solution of* $\mathrm{div} A\nabla u = 0$ *with smooth Dirichlet data one has* $\|\partial_{\nu_A} u_0\|_{\dot{W}^{-1,p}} \leq C_p \|\nabla u_0\|_{\dot{W}^{-1,p}}$, *then this holds for any energy solution* u *of* $\mathrm{div} A\nabla u = 0$, *possibly with a different constant.*

Proof. For (i), we remark that to prove (1) implies (2) in Theorem 3.3, we use (1) with smooth data. Thus the assumption of (i) implies (2) in Theorem 3.3 and we conclude using the converse (2) implies (1) in the same theorem. The proof of (ii) is similar starting from (2) for A and p instead of A^* and p' in Theorem 3.3. □

For Neumann problems we have,

Theorem 3.6. *Let* $A(\mathbf{x})$ *be a bounded measurable matrix with the stronger Gårding inequality* (8). *Let* $1 < p < \infty$. *The following holds.*

(i) *If there exists* $C_p < \infty$ *such that for any energy solution* u *of* $\mathrm{div} A\nabla u = 0$, *with smooth Neumann data one has* $\|\nabla u_0\|_p \leq C_p \|\partial_{\nu_A} u_0\|_p$, *then this holds for any energy solution* u *of* $\mathrm{div} A\nabla u = 0$, *possibly with a different constant.*

(ii) *If there exists* $C_p < \infty$ *such that for any energy solution of* $\mathrm{div} A\nabla u = 0$ *with smooth Neumann data one has* $\|\nabla u_0\|_{\dot{W}^{-1,p}} \leq C_{p'} \|\partial_{\nu_A} u_0\|_{\dot{W}^{-1,p}}$, *then this holds for any energy solution* u *of* $\mathrm{div} A\nabla u = 0$, *possibly with a different constant.*

The proof is similar noting that we use smooth data of the form $-\mathrm{div} g$ in the arguments. Details are left to the reader.

4. Rellich estimates: the case $\frac{n}{n+1} < p \leq 1$

Here, the duality equivalence is a subtle issue for $p < 1$ but remains for $p = 1$. We prove this first. Then we consider the problem of extension from estimates on atoms to global estimates.

Let $H^p(\mathbb{R}^n)$ denote the real Hardy space if $\frac{n}{n+1} < p \leq 1$. We have that $H^p(\mathbb{R}^n)$ are distributions spaces and, in this range, $C_0^\infty(\mathbb{R}^n)$ functions with mean value 0 form a dense subspace. For $\frac{n}{n+1} < p \leq 1$, let $\dot{H}^{1,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n); \partial_{x_i} f \in H^p(\mathbb{R}^n), i = 1, \dots, n\}$ with norm $\|f\|_{\dot{H}^{1,p}(\mathbb{R}^n)} = \|\nabla f\|_{H^p(\mathbb{R}^n;\mathbb{C}^n)}$. This is the homogeneous Hardy-Sobolev space which has been studied in many places ([Str], [Mi], [ART], [BB], [BG], [KS], [LMc]...). In particular, elements in these spaces are known to be locally integrable functions and $C_0^\infty(\mathbb{R}^n)$ is a dense subspace.

Let us turn to recalling duality. For all of them, we use the standard hermitian duality on functions, extended appropriately. Recall that if $\alpha = n(1/p - 1) \in [0, 1)$, the dual of $H^p(\mathbb{R}^n)$ is identified with $\dot{\Lambda}^0(\mathbb{R}^n) := BMO(\mathbb{R}^n)$ is $p = 1$ and with the homogeneous Hölder space $\dot{\Lambda}^\alpha(\mathbb{R}^n)$ of those continuous functions such that $|u(x) - u(y)| \leq C|x - y|^\alpha$ for all $x, y \in \mathbb{R}^n$, the smallest C defining the semi-norm. These spaces can also be seen within $\mathcal{D}_0'(\mathbb{R}^n)$, the space of distributions modulo constants, in which they are Banach. Recall also that $H^1(\mathbb{R}^n)$ is the dual space of $\mathrm{VMO}(\mathbb{R}^n)$ (sometimes called CMO), the closure of $C_0^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$. The dual of $\dot{H}^{1,p}(\mathbb{R}^n)$ is identified with $\dot{\Lambda}^{\alpha-1}(\mathbb{R}^n)$ defined as the space of distributions $\mathrm{div} f$, $f \in \dot{\Lambda}^{\alpha-1}(\mathbb{R}^n)$, equipped with the quotient norm.

Let us call $X = H^p(\mathbb{R}^n; \mathbb{C}^d)$ with $d = m$ or mn indifferently. Let $Y = \dot{\Lambda}^\alpha$ be the dual space and $Y^{-1} = \dot{\Lambda}^{\alpha-1}$.

First we complete Theorems 3.3 and 3.4 by the following results.

Theorem 4.1. *Let $A(\mathbf{x})$ be a bounded measurable matrix with the Gårding inequality (7). Let $\frac{n}{n+1} < p \leq 1$, $0 \leq \alpha = n(\frac{1}{p} - 1) < 1$, and X and Y^{-1} be the corresponding boundary spaces. Then (1) implies (2), where*

(1) *There exists $C_X < \infty$ such that for any $u \in \mathcal{E}$ solution of $\operatorname{div} A\nabla u = 0$, $\|\partial_{\nu_A} u_0\|_X \leq C_X \|\nabla u_0\|_X$.*
(2) *There exists $C_{Y^{-1}} < \infty$ such that for any $w \in \mathcal{E}$ solution of $\operatorname{div} A^*\nabla w = 0$, $\|\partial_{\nu_{A^*}} w_0\|_{Y^{-1}} \leq C_{Y^{-1}} \|\nabla w_0\|_{Y^{-1}}$.*

The converse holds in the case $p = 1$.

Theorem 4.2. *Let $A(\mathbf{x})$ be a bounded measurable matrix with the stronger Gårding inequality (8). Let $\frac{n}{n+1} < p \leq 1$, $0 \leq \alpha = n(\frac{1}{p} - 1) < 1$, and X and Y^{-1} be the corresponding boundary spaces. Then (1) implies (2), where*

(1) *There exists $C_X < \infty$ such that for any $u \in \mathcal{E}$ solution of $\operatorname{div} A\nabla u = 0$, $\|\nabla u_0\|_X \leq C_X \|\partial_{\nu_A} u_0\|_X$.*
(2) *There exists $C_{Y^{-1}} < \infty$ such that for any $w \in \mathcal{E}$ solution of $\operatorname{div} A^*\nabla w = 0$, $\|\nabla w_0\|_{Y^{-1}} \leq C_{Y^{-1}} \|\partial_{\nu_{A^*}} w_0\|_{Y^{-1}}$.*

The converse holds if $p = 1$.

The proofs are *mutatis mutandi* the same as when $1 < p < \infty$ using C_0^∞ functions with mean value 0 as test functions in H^p. The converses at $p = 1$ use the fact that H^1 is the dual space of VMO in which test functions are dense and also that $\|\nabla f\|_{BMO^{-1}} \sim \|f\|_{BMO}$ for $f \in L^1_{loc}$. Details are left to the reader.

We now turn to the extension problem. Recall that a 2-atom for $H^p(\mathbb{R}^n)$ is a function $a \in L^2(\mathbb{R}^n)$ such that

(1) the support of a is contained in a ball $\Delta(x_0, r)$,
(2) $\|a\|_2 \leq r^{-n(1/p-1/2)}$,
(3) $\int a = 0$.

A 2-atom for $H^p(\mathbb{R}^n)$ is smooth if it is $C_0^\infty(\mathbb{R}^n)$. Set $\mathcal{D}_0(\mathbb{R}^n)$ the subspace of $C_0^\infty(\mathbb{R}^n)$ of functions with mean 0. For our purpose here, observe that 2-atoms for $H^p(\mathbb{R}^n)$ are elements of $\dot{H}^{-1/2}(\mathbb{R}^n)$. In fact, if a is such a function, a classical result of Nečas [N] asserts that there exists a function $b \in W^{1,2}(\mathbb{R}^n; \mathbb{C}^n)$ (inhomogeneous Sobolev space) with support in the ball supporting a such that $a = \operatorname{div} b$ on $\mathbb{R}^n$. Thus, if $f \in \dot{H}^{1/2}(\mathbb{R}^n)$, $\langle a, f\rangle = -\langle b, \nabla f\rangle$ and we remark that by interpolation $\|b\|_{\dot{H}^{1/2}(\mathbb{R}^n, \mathbb{C}^n)} \leq C(\|b\|_2 \|\nabla b\|_2)^{1/2} < \infty$, while $\nabla f \in \dot{H}^{-1/2}(\mathbb{R}^n; \mathbb{C}^n)$.

Let

$$H^p_\nabla(\mathbb{R}^n; \mathbb{C}^n) = \{g \in H^p(\mathbb{R}^n; \mathbb{C}^n); \operatorname{curl} g = 0\} = \{\nabla f; f \in \dot{H}^{1,p}(\mathbb{R}^n)\}$$

and $\mathcal{D}_\nabla(\mathbb{R}^n; \mathbb{C}^n) := \nabla(C_0^\infty(\mathbb{R}^n))$. It is easy to see using $\dot{H}^{1,p}$ spaces that $\mathcal{D}_\nabla(\mathbb{R}^n; \mathbb{C}^n)$ is dense in $H^p_\nabla(\mathbb{R}^n; \mathbb{C}^n)$. As for the duality, one can see that the dual (for the same duality as the other spaces) of $H^p_\nabla(\mathbb{R}^n; \mathbb{C}^n)$ is $\dot{\Lambda}^\alpha_\nabla(\mathbb{R}^n; \mathbb{C}^n)$ identified as the subspace of $\dot{\Lambda}^\alpha(\mathbb{R}^n; \mathbb{C}^n)$ with curl free elements. The identification is easy. For the duality, if $\mathcal{R} = \nabla(-\Delta)^{-1/2}$ is the array of Riesz transforms, then the self-adjoint operator $\mathcal{R}\mathcal{R}^*$ extends to a bounded projection from $H^p(\mathbb{R}^n; \mathbb{C}^n)$ onto $H^p_\nabla(\mathbb{R}^n; \mathbb{C}^n)$ and similarly from $\dot{\Lambda}^\alpha(\mathbb{R}^n; \mathbb{C}^n)$ onto $\dot{\Lambda}^\alpha_\nabla(\mathbb{R}^n; \mathbb{C}^n)$. From here, the duality for the ranges of the projection follows from that of the source spaces.

For $H^p_\nabla(\mathbb{R}^n; \mathbb{C}^n)$, the 2-atoms in [LMc] for differential forms on $\mathbb{R}^n$, identifying ∇ with the exterior derivative on functions, suit our needs. It was done for $p = 1$ there

(Definition 6.1), but careful inspection shows it extends to $\frac{n}{n+1} < p \leq 1$ with the following definition.

Definition 4.3. Let $\frac{n}{n+1} < p \leq 1$. A 2-atom for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ is a function $a \in L^2(\mathbb{R}^n;\mathbb{C}^n)$ such that

(1) there exists $b \in L^2(\mathbb{R}^n)$ such that $a = \nabla b$ in $\mathcal{D}'(\mathbb{R}^n)$,
(2) the supports of a and b are contained in a ball $\Delta(x_0, r)$,
(3) $\|a\|_2 \leq r^{-n(1/p-1/2)}$,
(4) $\|b\|_2 \leq r^{1-n(1/p-1/2)}$.

Note that 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ are in particular 2-atoms for $H^p(\mathbb{R}^n;\mathbb{C}^n)$ since they satisfy $\int a = 0$. A 2-atom for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ is smooth when $b \in C^\infty(\mathbb{R}^n)$.

It is easily seen from the definition that 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ belong to the space $\dot H^{-1/2}_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. We shall require the following result.

Proposition 4.4. (1) *Let T be a linear operator defined on $\mathcal{D}_0(\mathbb{R}^n)$ such that $\sup \|Ta\|_{H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)} < \infty$, where the supremum is taken over all smooth 2-atoms for $H^p(\mathbb{R}^n)$. Then T has a bounded extension from $H^p(\mathbb{R}^n)$ into $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$.*

Suppose, in addition, that T was originally a bounded linear operator from $\dot H^{-1/2}(\mathbb{R}^n)$ into $\dot H^{-1/2}_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. Then T and the above extension coincide on $\dot H^{-1/2}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$.

(2) *Let T be a linear operator defined on $\mathcal{D}_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ such that $\sup \|Ta\|_{H^p(\mathbb{R}^n)} < \infty$, where the supremum is taken over all smooth 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. Then T has a bounded extension from $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ into $H^p(\mathbb{R}^n)$.*

Suppose, in addition, that T was originally a bounded linear operator from $\dot H^{-1/2}_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ into $\dot H^{-1/2}(\mathbb{R}^n)$. Then T and the above extension coincide on $\dot H^{-1/2}_\nabla(\mathbb{R}^n;\mathbb{C}^n) \cap H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$.

Of course the statement applies with $\mathbb{C}^m$-valued functions instead of $\mathbb{C}$-valued functions.

Proof. The first part of (1) is a special case of Theorem 1.1 in [YZ]. For the second part, we adapt a classical procedure found, for example as Proposition 4.2 of [MSV], which is also reminiscent of the method of proof of Theorems 3.6 and 3.5. Call $\tilde T$ the extension defined above. First if $f \in \dot H^{1/2}_\nabla(\mathbb{R}^n;\mathbb{C}^n) \cap \dot\Lambda^\alpha_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ and $g \in \mathcal{D}_0(\mathbb{R}^n)$,

$$\langle g, T^*f\rangle = \langle Tg, f\rangle = \langle \tilde T g, f\rangle = \langle g, \tilde T^* f\rangle.$$

The first two brackets are interpreted in the $\dot H^{-1/2}, \dot H^{1/2}$ duality, then we use that $Tg = \tilde T g$ as g can be seen as a multiple of a 2-atom for $H^p(\mathbb{R}^n)$. This allows us to reinterpret the last two brackets in the $H^p, \dot\Lambda^\alpha$ duality. We conclude that $T^*f = \tilde T^* f$ in $\mathcal{D}'_0(\mathbb{R}^n)$, hence they both belong to $\dot H^{1/2}_\nabla(\mathbb{R}^n;\mathbb{C}^n) \cap \dot\Lambda^\alpha_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ and differ by a constant. Next, let $f \in \mathcal{D}_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ (contained in both $\dot H^{1/2}_\nabla(\mathbb{R}^n;\mathbb{C}^n), \dot\Lambda^\alpha_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ and dense in the first) and $g \in \dot H^{-1/2}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$. Then

$$\langle Tg, f\rangle = \langle g, T^*f\rangle = \langle g, \tilde T^* f\rangle = \langle \tilde T g, f\rangle.$$

Here, the first two brackets are interpreted in the $\dot H^{-1/2}, \dot H^{1/2}$ duality. The second can be reinterpreted in the $H^p, \dot\Lambda^\alpha$ duality. In the second equality, we then use $T^*f = \tilde T^* f$ up to a constant, which is annihilated. In particular, we obtain that

$|\langle \tilde{T}g, f\rangle| \leq \|Tg\|_{\dot{H}^{-1/2}}\|f\|_{\dot{H}^{1/2}}$. Thus, $\tilde{T}g \in \dot{H}^{-1/2}_{\nabla}(\mathbb{R}^n;\mathbb{C}^n)$ and we conclude that $\tilde{T}g = Tg$.

The proof of (2) is the same, once we make the following observation. The proof of Theorem 1.1 in [YZ] depends only on having a Calderón reproducing formula with smooth and compactly supported convolution kernels and the characterisation of the Hardy space by the Lusin functional based on the kernels involved. Now, the atomic decomposition of [LMc] is exactly obtained via the same strategy with further algebraic constraints on the kernels to obtain the gradient form of the 2-atoms. Thus the analysis in [YZ] applies and their Theorem 1.1 extends to our situation. This provides us with the extension. The second part of the argument is *mutatis mutandi* the same. □

We can now state the results we are after.

Theorem 4.5. *Let $A(\mathbf{x})$ be a bounded measurable matrix with the Gårding inequality (7). Let $\frac{n}{n+1} < p \leq 1$. If $\sup \|\partial_{\nu_A} u_0\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)} \leq C_p$ taken over all energy solutions u of $\mathrm{div}A\nabla u = 0$ with (smooth) 2-atoms for $H^p_{\nabla}(\mathbb{R}^n;(\mathbb{C}^m)^n)$ as regularity data, then $\|\partial_{\nu_A} u_0\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)} \leq C_p\|\nabla u_0\|_{H^p_{\nabla}(\mathbb{R}^n;(\mathbb{C}^m)^n)}$ for any energy solution u of $\mathrm{div}A\nabla u = 0$, possibly with a different constant.*

Theorem 4.6. *Let $A(\mathbf{x})$ be a bounded measurable matrix with the stronger Gårding inequality (8). Let $\frac{n}{n+1} < p \leq 1$. If $\sup \|\nabla u_0\|_{H^p_{\nabla}(\mathbb{R}^n;(\mathbb{C}^m)^n)} \leq C_p$ taken over all energy solutions u of $\mathrm{div}A\nabla u = 0$ with (smooth) 2-atoms for $H^p(\mathbb{R}^n;\mathbb{C}^m)$ as Neumann data, then $\|\nabla u_0\|_{H^p_{\nabla}(\mathbb{R}^n;(\mathbb{C}^m)^n)} \leq C_p\|\partial_{\nu_A} u_0\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)}$ for any energy solution u of $\mathrm{div}A\nabla u = 0$, possibly with a different constant.*

The proof of the first theorem follows on applying (2) of the above proposition to the Dirichlet to Neumann operator Γ_{DN} and of the second on applying (1) of the above proposition to the Neumann to Dirichlet operator Γ_{ND}.

5. Fundamental solutions

Assuming the De Giorgi condition for the operators $\mathrm{div}A\nabla$ and $\mathrm{div}A^*\nabla$ in $\mathbb{R}^{1+n}$, these operators have fundamental solutions which have the expected estimates. It is convenient to state the relevant statements and references. We use the notation of section 2 for points and balls in $\mathbb{R}^{1+n}$.

Consider an elliptic system $\mathrm{div}A\nabla$ in $\mathbb{R}^{1+n}$, with bounded measurable matrix $A(\mathbf{x})$ depending on all variables. Ellipticity is taken in the sense the Gårding inequality

$$\int_{\mathbb{R}^{1+n}} \mathrm{Re}(A(\mathbf{x})\nabla g(\mathbf{x}) \cdot \overline{\nabla g(\mathbf{x})})\, d\mathbf{x} \geq \lambda \sum_{i=0}^{n}\sum_{\alpha=1}^{m} \int_{\mathbb{R}^{1+n}} |\partial_i g^\alpha(\mathbf{x})|^2 d\mathbf{x}, \tag{17}$$

for all $g \in \dot{H}^1(\mathbb{R}^{1+n};\mathbb{C}^m)$ and some $\lambda > 0$. We say that $\mathrm{div}A\nabla$ satisfies the De Giorgi condition if

$$\int_{B(\mathbf{x},r)} |\nabla u|^2 \lesssim (r/R)^{n-1+2\mu} \int_{B(\mathbf{x},R)} |\nabla u|^2 \tag{18}$$

holds for all weak solutions u to $\mathrm{div}A\nabla u = 0$ in $B(\mathbf{x},2R) \subset \mathbb{R}^{1+n}$ and all $\mathbf{x} \in \mathbb{R}^{1+n}$ and $0 < r < R$, for some $\mu \in (0,1]$. It is known that (18) is equivalent to the Hölder

estimate of Nash

$$\text{ess sup}_{\mathbf{y},\mathbf{z}\in B(\mathbf{x},R),\mathbf{y}\neq\mathbf{z}}\frac{|u(\mathbf{y})-u(\mathbf{z})|}{|\mathbf{y}-\mathbf{z}|^\alpha}\lesssim R^{-\alpha-(1+n)/2}\left(\int_{B(\mathbf{x},2R)}|u|^2\right)^{1/2} \tag{19}$$

whenever u is a weak solution to $\mathrm{div}A\nabla u=0$ in $B(\mathbf{x},3R)\subset\mathbb{R}^{1+n}$, for any $\mathbf{x}\in\mathbb{R}^{1+n}$ and $0<r<R$, for some $\alpha\in(0,1]$. Furthermore, the upper bounds of μ's in (18) and α's in (19) are equal, which we set μ^A_{DG} and call the De Giorgi exponent of $\mathrm{div}A\nabla$.

De Giorgi's theorem [DeG] states that (18), or equivalently (19) of Nash [Na], holds for all divergence form equations ($m=1$) $\mathrm{div}A\nabla u=0$ when A is real. It also holds for any system if dimension $1+n=2$ [Mor]. [AAAHK], Section 11, shows it is also the case in dimension $1+n=3$ (the argument presented for equations, works for our systems as it relies on Meyers' [Me] and Caccioppoli estimates which holds for such systems) when, in addition, A has t-independent coefficients. Finally, in [A], it is shown that (18) is a stable property under L^∞ perturbations of A (again, this is shown for equations but it holds for our systems).

Estimates (18) and (19) also imply the *Moser local boundedness estimate* [Mo]

$$\text{ess sup}_{\mathbf{y}\in B(\mathbf{x},R)}|u(\mathbf{y})|\lesssim R^{-(1+n)/2}\left(\int_{B(\mathbf{x},2R)}|u|^2\right)^{1/2} \tag{20}$$

whenever $\mathrm{div}A\nabla u=0$ in $B(\mathbf{x},3R)\subset\mathbb{R}^{1+n}$ for all $\mathbf{x}\in\mathbb{R}^{1+n}$ and $0<R<\infty$. We refer to [HK, Sec. 2] for details.

Proposition 5.1. *Let $n+1\geq 2$ and assume that $\mathrm{div}A\nabla$ and $\mathrm{div}A^*\nabla$ satisfy the De Giorgi condition or equivalently, the Nash local regularity condition. Then $\mathrm{div}A\nabla$ and $\mathrm{div}A^*\nabla$ have a fundamental solution $\Gamma^A(\mathbf{x};\mathbf{y})=\Gamma^A_{\mathbf{y}}(\mathbf{x})\in W^{1,1}_{loc}(\mathbb{R}^{1+n};\mathcal{L}(\mathbb{C}^m))$ at pole $\mathbf{y}\in\mathbb{R}^{1+n}$ and $\Gamma^{A^*}(\mathbf{y};\mathbf{x})=\Gamma^{A^*}_{\mathbf{x}}(\mathbf{y})\in W^{1,1}_{loc}(\mathbb{R}^{1+n};\mathcal{L}(\mathbb{C}^m))$ at pole $\mathbf{x}\in\mathbb{R}^{1+n}$ (ie, $\mathrm{div}_{\mathbf{x}}A(\mathbf{x})\nabla_{\mathbf{x}}\Gamma^A_{\mathbf{y}}(\mathbf{x})=\delta_{\mathbf{y}}(\mathbf{x})$ and $\mathrm{div}_{\mathbf{y}}A^*(\mathbf{y})\nabla_{\mathbf{y}}\Gamma^{A*}_{\mathbf{x}}(\mathbf{y})=\delta_{\mathbf{x}}(\mathbf{y})$) with for some $0<\mu<\inf(\mu^A_{DG},\mu^{A^*}_{DG})$,*

$$|\Gamma^A(\mathbf{x};\mathbf{y})|\lesssim|\mathbf{x}-\mathbf{y}|^{1-n},\ \text{if } 1+n\geq 3,\ \text{and}\ \lesssim 1+|\ln|\mathbf{x}-\mathbf{y}||\ \text{if } 1+n=2, \tag{21}$$

$$|\Gamma^A(\mathbf{x};\mathbf{y})-\Gamma^A(\mathbf{x};\mathbf{y}')|\lesssim\left(\frac{|\mathbf{y}-\mathbf{y}'|}{|\mathbf{x}-\mathbf{y}|}\right)^\mu|\mathbf{x}-\mathbf{y}|^{1-n},\ \text{if } |\mathbf{y}-\mathbf{y}'|\leq|\mathbf{x}-\mathbf{y}|/2, \tag{22}$$

and

$$\int_{B(\mathbf{z},\rho)}|\nabla_{\mathbf{y}}\Gamma^{A^*}(\mathbf{y};\mathbf{x})|^2\,d\mathbf{y}\leq C\frac{\rho^{n-1+2\mu}}{|\mathbf{x}-\mathbf{z}|^{2n-2+2\mu}}\ \text{if } \rho>0 \text{ and } |\mathbf{x}-\mathbf{z}|\geq 2\rho, \tag{23}$$

and symmetrically by exchanging the roles of Γ^A and Γ^{A^*}.

Proof. This result is in [R1], Theorem 1.2. Note that this result is stated slightly differently there but all what is used is the De Giorgi condition. Note also that the estimate (22) is stated with an extra multiplicative log factor when $1+n=2$, but the proof there does give what we state. □

Remark 5.2. 1) If $\rho\sim|\mathbf{x}-\mathbf{z}|/2$, then the right hand side of (23), $|\mathbf{x}-\mathbf{z}|^{1-n}$, is obtained during the construction. The gain μ comes from use of the De Giorgi condition (18) with the balls $B(\mathbf{z},\rho)\subset B(\mathbf{z},|\mathbf{z}-\mathbf{x}|/2)$.

2) Assume $1+n\geq 3$. There is a previous construction in [HK] under the stronger pointwise ellipticity assumption on A. But examination shows that only (17) is

required. More estimates are obtained there. These are the only ones we need here. In particular, uniqueness of the fundamental solution is proved together with the symmetry relation $\Gamma^{A^*}(\mathbf{y};\mathbf{x}) = \Gamma^A(\mathbf{x};\mathbf{y})^*$, where the latter is the hermitian adjoint of $\Gamma^A(\mathbf{x};\mathbf{y})$ as an $m \times m$ matrix.

3) Assume $1+n = 2$. The first construction for complex coefficients is in [AMcT] for scalar operators ($m = 1$). An analogous estimate was obtained in [DoK], Theorem 2.21, for systems but was only carried out explicitly assuming strong ellipticity. See also [CDoK]. [B, Chapter 4] used the construction in [AMcT] and showed uniqueness and also that it is possible to choose the constant of integration in such a way the symmetry relation holds. This construction extends *mutatis mutandi* to systems and does give the above estimates, with possible exception of uniqueness as the argument relies on properties of harmonic functions.

6. Decay estimates for energy solutions

In this section, we consider without mention systems with $A(\mathbf{x})$ bounded, measurable, non necessarily t-independent, with the stronger Gårding inequality (8) and we assume that the reflected matrix $A^\sharp$ and its adjoint satisfy the De Giorgi condition on $\mathbb{R}^{1+n}$.[4] The number $\mu > 0$ in the statements below will be any number less than the De Giorgi exponents for $A^\sharp$ and its adjoint.

This situation covers dimension $1+n = 2$ or dimensions $1+n \geq 3$ with A close in L^∞ to a real and scalar matrix (for systems, scalar means diagonal). In particular we cover the case of real equations. In this respect, our first result extends Lemma 4.9 of [HKMP1].

Lemma 6.1. *Let $x_0 \in \mathbb{R}^n$, $r > 0$, and set $\mathbf{x}_0 := (0, x_0)$, $B := B(\mathbf{x}_0, r)$, $\Delta := \Delta(x_0, r)$. Suppose that $w \in L^2_{\rm loc}(\mathbb{R}^{1+n}_+ \setminus \overline{B}; \mathbb{C}^m)$ with $\nabla w \in L^2(\mathbb{R}^{1+n}_+ \setminus \overline{B}; (\mathbb{C}^m)^{1+n})$ is a weak solution of $\mathrm{div} A\nabla w = 0$ in $\mathbb{R}^{1+n}_+ \setminus \overline{B}$, and that $w|_{\mathbb{R}^n \setminus \overline{\Delta}} \equiv 0$. Then w is (identified to) a bounded and continuous function on $\overline{\mathbb{R}^{1+n}_+} \setminus 3B$, and for some constants C and $\mu > 0$, depending only upon the assumption on A,*

$$|w(\mathbf{x})| \leq C \frac{r^{\frac{n+1}{2}+\mu-2}}{|\mathbf{x}-\mathbf{x}_0|^{n-1+\mu}} \left(\int_{\Omega_+} |w|^2 \right)^{1/2}, \qquad |\mathbf{x}-\mathbf{x}_0| \geq 3r\,.$$

Here, $\Omega_+ = 3B_+ \setminus 2B_+$ and $B_+ = \mathbb{R}^{1+n}_+ \cap B$. In particular, $w \to 0$ at infinity.

Proof. Let us drop the dependence on m in the notation to simplify the exposition. First, the assumption $w \in L^2_{\rm loc}(\mathbb{R}^{1+n}_+ \setminus \overline{B})$ with $\nabla w \in L^2(\mathbb{R}^{1+n}_+ \setminus \overline{B})$ implies that $w \in C([0,\infty); L^2_{\rm loc}(\mathbb{R}^n \setminus \overline{\Delta}))$. See the argument in [AMcM]. In particular, the equation $w|_{\mathbb{R}^n \setminus \overline{\Delta}} \equiv 0$ holds in $L^2_{\rm loc}$. Set $v = w\chi$ where χ is a smooth real-valued function supported on $\mathbb{R}^{1+n} \setminus (11/5)B$, which is 1 on $\mathbb{R}^{1+n} \setminus (14/5)B$ with $\|\chi\|_\infty \leq 1$ and $\|\nabla\chi\|_\infty \leq C/r$. One has that $v \in \dot{H}^1(\mathbb{R}^{1+n}_+)$, $v|_{\mathbb{R}^n} \equiv 0$ holds in $L^2_{\rm loc}$ and $\mathrm{div} A\nabla v = f + \mathrm{div}\mathbf{g}$ weakly in $\mathbb{R}^{1+n}_+$, with $f = A\nabla\chi.\nabla w$ and $\mathbf{g} = A\nabla\chi w$. Note that

$$\|\mathbf{g}\|_2 \lesssim r^{-1} \left(\int_{\Omega_+} |w|^2 \right)^{1/2},$$

[4] This is a way of saying that A and its adjoint satisfy both interior and boundary De Giorgi condition on the upper half-space. Some variants in the hypotheses are certainly possible here.

the implicit constant depending on the L^∞ bound for A and dimension. Also

$$\|f\|_2 \leq r^{-1}\left(\int_{\mathbb{R}^{1+n}_+\cap((14/5)B\setminus(11/5)B)}|\nabla w|^2\right)^{1/2} \lesssim r^{-2}\left(\int_{\Omega_+}|w|^2\right)^{1/2},$$

where the last inequality uses boundary and interior Caccioppoli inequalities.

One can represent v using the method of reflection. Let $v^\sharp$, $f^\sharp$ be the odd extensions of v, f and $\mathbf{g}^\sharp$ is the extension of $\mathbf{g}$ defined by $\mathbf{g}^\sharp(\mathbf{y}) = N\mathbf{g}(N\mathbf{y})$, with $N(t,y) = (t,y)^\sharp = (-t,y)$. Remark that since $f^\sharp \in L^2$, with support in $3B$ and mean value condition $\int f^\sharp = 0$, then $f^\sharp \in \dot H^{-1}(\mathbb{R}^{1+n})$ with $\|f^\sharp\|_{\dot H^{-1}(\mathbb{R}^{1+n})} \lesssim r\|f^\sharp\|_2$. Thus $v^\sharp \in \dot H^1(\mathbb{R}^{1+n})$ with $\mathrm{div}A^\sharp\nabla v^\sharp = f^\sharp + \mathrm{div}\mathbf{g}^\sharp$. As $\dot H^1(\mathbb{R}^{1+n})$ is a uniqueness class modulo constants for this equation (since we have (17) for $A^\sharp$), it follows that $v^\sharp$ is the unique odd (with respect to N) element in $\dot H^1(\mathbb{R}^{1+n})$ solving this equation. If $v_1^\sharp$ is the unique odd solution obtained from $f^\sharp$ and $v_2^\sharp$ is the unique odd solution obtained from $-\mathrm{div}g^\sharp$, one has $v^\sharp = v_1^\sharp - v_2^\sharp$ (in L^2_{loc}). Now using the fundamental solution $\Gamma^{A^\sharp}$, $v_1^\sharp(\mathbf{x})$ and $v_2^\sharp(\mathbf{x})$ have the respective integral representations for $\mathbf{x}\in\mathbb{R}^{1+n}$ away from the supports of $f^\sharp$ and $g^\sharp$,

$$v_1^\sharp(\mathbf{x}) = \int_{\mathbb{R}^{1+n}} \Gamma^{A^\sharp}(\mathbf{x};\mathbf{y})f^\sharp(\mathbf{y})\,d\mathbf{y},$$

$$v_2^\sharp(\mathbf{x}) = \int_{\mathbb{R}^{1+n}} (\nabla_{\mathbf{y}}\Gamma^{A^\sharp})(\mathbf{x};\mathbf{y})\mathbf{g}^\sharp(\mathbf{y})\,d\mathbf{y}.$$

One can check that changing $\mathbf{x}$ to $\mathbf{x}^\sharp$ change the signs of both integrals. That is, both integrals are odd with respect to N. It follows that $v_1^\sharp$ and $v_2^\sharp$ agree with these integrals in L^2_{loc} away of the supports of $f^\sharp$ and $g^\sharp$. Next, restricting to $\mathbf{x}\in\mathbb{R}^{1+n}_+$, still away from the supports of f and g, we can rewrite the integrals as

$$v_1^\sharp(\mathbf{x}) = \int_{\mathbb{R}^{1+n}_+} (\Gamma^{A^\sharp}(\mathbf{x};\mathbf{y}) - \Gamma^{A^\sharp}(\mathbf{x};\mathbf{y}^\sharp))f(\mathbf{y})\,d\mathbf{y},$$

$$v_2^\sharp(\mathbf{x}) = \int_{\mathbb{R}^{1+n}_+} ((\nabla_{\mathbf{y}}\Gamma^{A^\sharp})(\mathbf{x};\mathbf{y}) - (\nabla_{\mathbf{y}}\Gamma^{A^\sharp})(\mathbf{x},\mathbf{y}^\sharp))\mathbf{g}(\mathbf{y})\,d\mathbf{y}.$$

We have shown that v is the difference of these 2 integrals in L^2_{loc} away from the supports of f and g. As they have the desired pointwise bounds using Proposition 5.1 applied with $A^\sharp$, the conclusion follows from the fact that $v = w$ on the range where these pointwise inequalities hold. □

Lemma 6.2. *Let $f\in L^2(\mathbb{R}^n;\mathbb{C}^m)\cap\dot H^{1/2}(\mathbb{R}^n;\mathbb{C}^m)$ with compact support in the surface ball $\Delta = \Delta(x_0,r)$. Then the solution of $\mathrm{div}A\nabla u = 0$ where $u|_{t=0} = f$ given by Lemma 2.3 is locally Hölder continuous on $\mathbb{R}^{1+n}_+$, continuous up the boundary away from $B(\mathbf{x}_0,3r)$, tends to 0 at ∞ and, has the estimate for some $C,\mu>0$,*

$$|u(\mathbf{x})| \leq C\frac{r^{\frac{n+1}{2}+\mu-1}}{|\mathbf{x}-\mathbf{x}_0|^{n-1+\mu}}\,\|f\|_{\dot H^{1/2}(\mathbb{R}^n;\mathbb{C}^m)}, \qquad |\mathbf{x}-\mathbf{x}_0|\geq 3r\,.$$

Proof. By Remark 2.5, one can change $\left(\int_{\Omega_+}|u|^2\right)^{1/2}$ by $r\left(\int_{\Omega_+}|\nabla u|^2\right)^{1/2}$ in the right hand side of the estimate of Lemma 6.1. The latter is controlled by $r\|f\|_{\dot H^{1/2}(\mathbb{R}^n;\mathbb{C}^m)}$ by the existence theory for energy solutions. □

Here $3r$ is for convenience of the statements and can be changed to $(1+\varepsilon)r$ for any $\varepsilon > 0$.

It is worth relating the above results to solutions constructed by harmonic measure, even if we do not use this estimate.

Lemma 6.3. *Assume $1+n \geq 2$ and $A(\mathbf{x})$ has scalar and real (not necessarily t-independent) coefficients and ellipticity is taken in the usual pointwise sense. Then for all Lispchitz functions f with bounded support in a surface ball $\Delta(x_0, r)$, the solution u with boundary data f given by harmonic measure for $\mathrm{div} A\nabla$ is an energy solution. Hence it agrees with the solution given in Lemma 6.2. In particular, it has further the estimate for any $\rho > r$*

$$|u(\mathbf{x})| \leq C \frac{r^{n+\mu}}{|\mathbf{x}-\mathbf{x}_0|^{n-1+\mu}} \|\nabla f\|_\infty, \qquad |\mathbf{x}-\mathbf{x}_0| \geq \rho, \tag{24}$$

so that $u \to 0$ at infinity.

Proof. First notice that writing $f = f_+ - f_-$, the positive and negative parts both satisfy the same assumptions as f. Hence we may assume $f \geq 0$.

Let $R > 2r$ and $\Omega_R = \mathbb{R}^{1+n}_+ \cap B(\mathbf{x}_0, R)$. Now let us recall the construction of the solution given by harmonic measure on $\mathbb{R}^{1+n}_+$ taken from granted the construction on bounded domains (See [Ke]). Let $\omega^{\mathbf{x}}_R$ be the harmonic measure for $\mathrm{div} A\nabla$ on Ω_R at pole $\mathbf{x}$. Hence $\mathbf{x} \mapsto u_R(\mathbf{x}) = \int_{\partial\Omega_R} f\, d\omega^{\mathbf{x}}_R$ is the unique continuous function on $\overline{\Omega_R}$, solution of the classical Dirichlet problem $\mathrm{div} A\nabla u_R = 0$ with $u|_{\partial\Omega_R} = f$, where we have naturally extended f by 0 on $\partial\Omega_R \cap \mathbb{R}^{1+n}_+$. It is also an energy solution on Ω_R, $\int_{\Omega_R} |\nabla u_R(\mathbf{x})|^2\, d\mathbf{x}$ bounded by a uniform constant. Indeed, it is constructed as $u_R = \phi_R + F$ where F is a fixed Lipschitz extension of f and ϕ_R solves $\mathrm{div} A\nabla \phi_R = -\mathrm{div} A\nabla F$ with $\phi_R \in W^{1,2}_0(\Omega_R)$, so that the constant in the energy inequality depend on the Lipschitz norm of f and the ellipticity constants of A.

Using the maximum principle of Stampacchia and the positivity of f, we have $0 \leq u_R \leq u_{R'} \leq \sup_{\mathbb{R}^n} f$ in $\overline{\Omega_R}$ when $R < R'$. Thus for any $\mathbf{x} \in \overline{\mathbb{R}^{1+n}_+}$, $u_R(\mathbf{x})$ converges to a finite number $u(\mathbf{x})$ as $R \to \infty$ (with $u(0,\cdot) = f$ on $\mathbb{R}^n$ since $u_R(0,\cdot) = f$ on $\Delta(x_0, R)$). Already, this and the density of the space of compactly supported Lipschitz continuous functions on $\mathbb{R}^n$ into the space of compactly supported continuous functions imply that $\omega^{\mathbf{x}}_R|_{\mathbb{R}^n}$ converges weakly to a finite positive measure on $\mathbb{R}^n$, denoted $\omega^{\mathbf{x}}$, and that $u(\mathbf{x}) = \int_{\mathbb{R}^n} f\, d\omega^{\mathbf{x}}$. Also, by Harnack's principle and Ascoli's theorem, u_R (naturally extended by 0 outside $\overline{\Omega_R}$) converges locally uniformly to u on $\mathbb{R}^{1+n}_+$. Next, this extension of u_R is an element of $\dot{H}^1(\mathbb{R}^{1+n}_+)$, form a bounded family in that space. It easily follows that u is an energy solution of $\mathrm{div} A\nabla u = 0$ in $\mathbb{R}^{1+n}_+$ by a weak limit argument with $u|_{t=0} = f$. By uniqueness in Lemma 2.3, u is the only one. The rest of the proof is left to the reader. □

We turn to decay estimates useful for Neumann solutions.

Lemma 6.4. *Let $u \in \mathcal{E}$ be an energy solution of $\mathrm{div} A\nabla u = 0$ whose conormal derivative at the boundary is further integrable and supported in some boundary ball $\Delta(x_0, r)$. After a suitable choice of the constant of integration, we have*

$$|u(\mathbf{x})| \leq C \frac{r^{\mu}}{|\mathbf{x}-\mathbf{x}_0|^{n-1+\mu}} \|\partial_{\nu_A} u|_{t=0}\|_1$$

whenever $|\mathbf{x}-\mathbf{x}_0|\geq 2r$ *for some* C *depending on the assumptions on* A*, with* $\mathbf{x}_0=(0,x_0)$*. In particular,* $u\to 0$ *at infinity in any direction.*

Proof. Let $\alpha=\partial_{\nu_A}u|_{t=0}$. We assumed u belongs to the energy class, so it is determined up to a constant. We shall select one in a moment. Using the reflection principle, we see that the even extension of u across the boundary is a solution of the equation

$$\int_{\mathbb{R}^{1+n}} A^\sharp\nabla u^\sharp\cdot\overline{\nabla\phi}\,d\mathbf{x}=-2\int_{\mathbb{R}^n}\alpha(x)\overline{\phi(0,x)}\,dx=-2\langle\alpha\delta|_{t=0},\phi\rangle$$

for all $\phi\in C_0^1(\mathbb{R}^{1+n};\mathbb{C}^m)$ where $A^\sharp$ is the reflected matrix of A (the $-$ sign in the formula comes from our convention for ∂_{ν_A}). Observe that the bracket is the $\dot H^{-1}(\mathbb{R}^{1+n};\mathbb{C}^m)$, $\dot H^1(\mathbb{R}^{1+n};\mathbb{C}^m)$ duality by seeing $\alpha\delta|_{t=0}\in\dot H^{-1}(\mathbb{R}^{1+n};\mathbb{C}^m)$ from trace theory. Let $L^\sharp=\mathrm{div}A^\sharp\nabla$ on $\mathbb{R}^{1+n}$. By invertibility of $L^\sharp$ we have $u^\sharp=-2L^{\sharp\,-1}(\alpha\delta|_{t=0})$ in $\dot H^1(\mathbb{R}^{1+n};\mathbb{C}^m)$, which means that the two agree up to a constant. Under the assumption of the lemma, Proposition 5.1 applies to $A^\sharp$ and let $\Gamma^{A^\sharp}$ be the fundamental solution of $\mathrm{div}A^\sharp\nabla$. Using the fact that $\alpha\in L^1$ with support in the surface ball $\Delta(x_0,r)$, we have up to a constant for $|\mathbf{x}-\mathbf{x}_0|\geq 2r$,

$$2L^{\sharp\,-1}(\alpha\delta|_{t=0})(\mathbf{x})=2\int_{\mathbb{R}^n}\Gamma^{A^\sharp}(\mathbf{x};0,y)\alpha(y)\,dy$$

as the integral converges from the size condition (21). We now choose the constant of integration so as $u^\sharp(\mathbf{x})$ agrees with this integral when $|\mathbf{x}-\mathbf{x}_0|\geq 2r$. As α is the conormal derivative of u and is integrable, we have necessarily $\int\alpha=0$. Thus

$$u^\sharp(\mathbf{x})=-2\int_{\mathbb{R}^n}(\Gamma^{A^\sharp}(\mathbf{x};0,y)-\Gamma^{A^\sharp}(\mathbf{x};0,x_0))\alpha(y)\,dy.$$

Then (22) readily gives the desired estimate using, in addition, the support of α. □

7. Short review of the first order formalism

In this section, we assume that the matrix $A(\mathbf{x})$ is bounded, measurable, t-independent (*i.e.*, $A(\mathbf{x})=A(x)$ when $\mathbf{x}=(t,x)$) and satisfies the accretivity assumption (10) on $\mathbb{R}^n$. It is convenient to write A in a 2×2 block form. Identifying $\mathbb{C}^{(1+n)m}=(\mathbb{C}^m)^{1+n}=\mathbb{C}^m\times(\mathbb{C}^m)^n$, $A(x)$ takes the form of a 2×2 matrix

$$A(x)=\begin{bmatrix}a(x) & b(x)\\ c(x) & d(x)\end{bmatrix},$$

where $a(x)\in\mathcal{L}(\mathbb{C}^m)$, etc... Call $\mathcal{A}$ the set of such 2×2 block matrices A.

Following [AAMc] and [AA], one can characterize weak solutions u to the divergence form equation (5), by replacing u by its conormal gradient $\nabla_A u$ as the unknown function. More precisely (5) for u is replaced by (26) for

$$F(t,x)=\nabla_A u(t,x)=\begin{bmatrix}\partial_{\nu_A}u(t,x)\\ \nabla_x u(t,x)\end{bmatrix},$$

and $\partial_{\nu_A}u(t,x):=(A\nabla_{t,x}u)_\perp$ denotes the upward conormal derivative of u, that is the first component of $A\nabla_{t,x}u$, consistently with earlier notation. Here we use the notation $v=\begin{bmatrix}v_\perp\\ v_\parallel\end{bmatrix}$ for vectors in $(\mathbb{C}^m)^{1+n}$ and $v_\perp\in\mathbb{C}^m$ is called the scalar

part and $v_\parallel \in (\mathbb{C}^m)^n$ the tangential part of v. For example, $\partial_t u = (\nabla_{t,x} u)_\perp$ and $\nabla_x u = (\nabla_{t,x} u)_\parallel$.

We remark that there is the pointwise comparison $|\nabla u| \sim |\nabla_A u|$.

Proposition 7.1. *The pointwise transformation*

$$A \mapsto \hat{A} := \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix} \tag{25}$$

is a self-inverse bijective transformation of the set of matrices in $\mathcal{A}$.

For a pair of coefficient matrices $A = \hat{B}$ and $B = \hat{A}$, the pointwise map $\nabla_{t,x} u \mapsto F = \nabla_A u$ gives a one-one correspondence, with inverse $F \mapsto \nabla_{t,x} u = \begin{bmatrix} (BF)_\perp \\ F_\parallel \end{bmatrix}$, between gradients of weak solutions $u \in H^1_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^m)$ to (5) and solutions $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; (\mathbb{C}^m)^{1+n})$ of the generalized Cauchy–Riemann equations

$$\partial_t F + \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix} BF = 0 \quad \text{and} \quad \mathrm{curl}_x F_\parallel = 0, \tag{26}$$

where the derivatives are taken in the $\mathbb{R}^{1+n}_+$ distributional sense.

This originates from [AAMc] and is proved in this generality in [AA]. Denote by $\mathbf{D}$ the self-adjoint operator on $\mathcal{H} = L^2(\mathbb{R}^n; (\mathbb{C}^m)^{1+n})$ defined by

$$\mathbf{D} := \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix} \quad \text{with} \quad \mathsf{D}(\mathbf{D}) = \begin{bmatrix} \mathsf{D}(\nabla) \\ \mathsf{D}(\mathrm{div}) \end{bmatrix}.$$

The closure of the range of $\mathbf{D}$ is the set of $F \in \mathcal{H}$ such that $\mathrm{curl}_x F_\parallel = 0$, that is $\overline{\mathsf{R}(\mathbf{D})} = \mathcal{H}^0$. It is shown in [AKMc] that the operators $\mathbf{D}B$ and $B\mathbf{D}$ with respective domains $B^{-1}\mathsf{D}(\mathbf{D})$ and $\mathsf{D}(\mathbf{D})$ are bisectorial operators with bounded holomorphic functional calculi on the closure of their range $\mathcal{H}^0$ and $B\mathcal{H}^0$ respectively. Observe the similarity relation

$$B(\mathbf{D}B) = (B\mathbf{D})B \quad \text{on } \mathsf{D}(\mathbf{D}B) \tag{27}$$

that allows to transfer functional properties between $\mathbf{D}B$ and $B\mathbf{D}$. In particular, if $\mathrm{sgn}(z) = 1$ for $\mathrm{Re}\, z > 0$ and -1 for $\mathrm{Re}\, z < 0$, the operators $\mathrm{sgn}(\mathbf{D}B)$ and $\mathrm{sgn}(B\mathbf{D})$ are well-defined bounded involutions on $\mathcal{H}^0$ and $B\mathcal{H}^0$ respectively. One defines the spectral spaces $\mathcal{H}^{0,\pm}_{\mathbf{D}B} = \mathsf{N}(\mathrm{sgn}(\mathbf{D}B) \mp I)$ and $\mathcal{H}^{0,\pm}_{B\mathbf{D}} = \mathsf{N}(\mathrm{sgn}(B\mathbf{D}) \mp I)$. They topologically split $\mathcal{H}^0$ and $B\mathcal{H}^0$ respectively. The restriction of $\mathbf{D}B$ to the invariant space $\mathcal{H}^{0,+}_{\mathbf{D}B}$ is sectorial of type less than $\pi/2$, hence it generates an analytic semi-group $e^{-t\mathbf{D}B}$, $t \geq 0$, on it. Similarly, the restriction of $B\mathbf{D}$ to the invariant space $\mathcal{H}^{0,+}_{B\mathbf{D}}$ is sectorial of type less than $\pi/2$, hence it generates an analytic semi-group $e^{-tB\mathbf{D}}$, $t \geq 0$, on $\mathcal{H}^{0,+}_{B\mathbf{D}}$.

Theorem 7.2. *Let $u \in H^1_{loc}(\mathbb{R}^{1+n}_+; \mathbb{C}^m)$. The function u is a weak solution of $\mathrm{div} A\nabla u = 0$ with $\|\widetilde{N}_*(\nabla u)\|_2 < \infty$ if and only if there exists $F_0 \in \mathcal{H}^{0,+}_{\mathbf{D}B}$ such that $\nabla_A u = e^{-t\mathbf{D}B} F_0$. Moreover, F_0 is unique and $\|F_0\|_2 \approx \|\widetilde{N}_*(\nabla u)\|_2$. We set $\nabla_A u|_{t=0} := F_0$.*

The if part was obtained in [AAMc] and the only if part in [AA, Theorems 8.2]. Here $\widetilde{N}_*(g)$ is the Kenig-Pipher modified non-tangential function where

$$\widetilde{N}_*(g)(x) := \sup_{t>0} t^{-(1+n)/2} \|g\|_{L_2(W(t,x))}, \qquad x \in \mathbb{R}^n,$$

with $W(t,x) := (c_0^{-1}t, c_0 t) \times \Delta(x, c_1 t)$, for some fixed constants $c_0 > 1$, $c_1 > 0$. A remark is that the same proof shows when coefficients are t-independent that for the equivalence to hold one could replace $\|\widetilde{N}_*(\nabla u)\|_2$ by $\sup_{t>0}(\frac{1}{t}\int_t^{2t} \|\nabla_{t,x}u\|_2^2\, ds)^{1/2}$ or the stronger $\sup_{t>0}\|\nabla_{t,x}u\|_2$ or even the square function $(\int_{\mathbb{R}^{1+n}_+} t|\partial_t \nabla_{t,x}u|^2 dtdx)^{1/2}$, so that in the end all these quantities are *a priori* equivalent for weak solutions.

Let us pursue further the discussion by extending this to Sobolev spaces with negative order. Say that $u \in \mathcal{E}_s$ with $s < 0$ if $\int_{\mathbb{R}^{1+n}_+} t^{-2s-1}|\nabla_{t,x}u|^2 dtdx < \infty$ while $u \in \mathcal{E}_0$ if $\|\widetilde{N}_*(\nabla u)\|_2 < \infty$. With this notation $\mathcal{E}_{-1/2} = \mathcal{E}$.

Proposition 7.3. *Let $s \in [-1, 0)$.*

(1) *The operator $\mathbf{D}B|_{\overline{R(\mathbf{D})}}$ can be extended to a bi-sectorial operator on the homogeneous Sobolev space $\dot{\mathcal{H}}^s$ which is the closure of $\overline{R(\mathbf{D})} = \dot{\mathcal{H}}^0$ for the homogeneous Sobolev norm $\|(-\Delta)^{-s/2}f\|_2$. This operator, which we keep writing $\mathbf{D}B$ for simplicity, has a bounded holomorphic functional calculus on $\dot{\mathcal{H}}^s$. In particular, the operators $\operatorname{sgn}(\mathbf{D}B) \mp I$ are well-defined projections on $\dot{\mathcal{H}}^s$ and their ranges $\dot{\mathcal{H}}^{s,\pm}_{\mathbf{D}B}$ form a splitting of $\dot{\mathcal{H}}^s$.*

(2) *Let $u \in H^1_{loc}(\mathbb{R}^{1+n}_+)$. Then u is a weak solution of $\operatorname{div} A\nabla u = 0$ in $\mathbb{R}^{1+n}_+$ with $u \in \mathcal{E}_s$ if and only if there exists $F_0 \in \dot{\mathcal{H}}^{s,+}_{\mathbf{D}B}$ such that $\nabla_A u = e^{-t\mathbf{D}B}F_0$. Moreover, F_0 is unique and*

$$\|F_0\|_{\dot{\mathcal{H}}^s} \approx \left(\int_{\mathbb{R}^{1+n}_+} t^{-2s-1}|\nabla_{t,x}u|^2 dtdx\right)^{1/2}.$$

We set $\nabla_A u|_{t=0} := F_0$.

Here Δ is the self-adjoint Laplacian acting componentwise on $L^2(\mathbb{R}^n; (\mathbb{C}^m)^{1+n})$. It agrees with $-D^2$ on $\overline{\mathsf{R}(\mathbf{D})}$.

Proof. Item (1) is in Proposition 4.5 of [AMcM] where $\mathbf{D}B|_{\overline{\mathsf{R}(\mathbf{D})}}$ is called $\underline{T}$ there. Item 2 for $s = -1$ is Corollary 4.5 of [AMcM], for $s = -1/2$ is Proposition 4.7 of [AMcM]. The other cases are treated in [R2]. □

Remark 7.4. We have introduced a notion of conormal gradient at the boundary $\nabla_A u|_{t=0}$ for solutions in $\mathcal{E}_s$. Strictly speaking this notion depends on s as well and in particular for $s = -1/2$, we recover the notions already defined for energy solutions. What allows us not to distinguish s in the notation is that it is a consistent notion for two different values of s. More precisely, if $u \in \mathcal{E}_s \cap \mathcal{E}_{s'}$ with $s, s' \in [-1, 0]$, then the convergence of $\nabla_A u(t,\cdot)$ as $t \to 0$ is both in $\dot{\mathcal{H}}^s$ and $\dot{\mathcal{H}}^{s'}$, hence the limits agree in the space of distributions.

8. Boundary layer operators

In this section, we assume that A is bounded measurable t-independent matrix which is strictly accretive on $\mathcal{H}^0$, that is satisfying (10).

It has been proved recently in [R1] using the functional calculi for $\mathbf{D}B$ and $B\mathbf{D}$ that the classical single and double layer operators for $\operatorname{div} A\nabla$, ∇S_t and D_t, can be defined as L^2 bounded operators, uniformly with respect to $t > 0$, with limits at

$t = 0$. More precisely, for $t > 0$, define $\nabla_A \mathcal{S}_t$ and $\mathcal{D}_t$ for $h \in L^2(\mathbb{R}^n; \mathbb{C}^m)$ by

$$(\nabla_A \mathcal{S}_t h)(x) := \left(e^{-t\mathbf{D}B} X_+(\mathbf{D}B) \begin{bmatrix} h \\ 0 \end{bmatrix} \right)(x) \tag{28}$$

and

$$(\mathcal{D}_t h)(x) := -\left(e^{-tB\mathbf{D}} X_+(B\mathbf{D}) \begin{bmatrix} h \\ 0 \end{bmatrix} \right)_\perp (x), \tag{29}$$

where $X_+(z) = 1$ if $\operatorname{Re} z > 0$ and 0 if $\operatorname{Re} z < 0$ so that $X_+(z) = \frac{1}{2}(\operatorname{sgn}(z) + 1)$. Remark that at this general level, there is an abuse of language as the operator $\mathcal{S}_t$ is not defined (although it will in $\dot{H}^1(\mathbb{R}^n; \mathbb{C}^m)$), only $\nabla_A \mathcal{S}_t$ is. It follows from the bounded holomorphic functional calculus for $\mathbf{D}B$ and $B\mathbf{D}$ that the right hand sides are L^2-bounded operators and have strong limits when $t \to 0$.

Lemma 8.1. *Whenever $h \in W^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$,*

$$\nabla_A \mathcal{D}_t h = -e^{-t\mathbf{D}B} X_+(\mathbf{D}B) \begin{bmatrix} 0 \\ \nabla h \end{bmatrix}. \tag{30}$$

Proof. From the calculations in [AA], we have

$$\begin{aligned} \nabla_A \left(e^{-tB\mathbf{D}} X_+(B\mathbf{D}) \begin{bmatrix} h \\ 0 \end{bmatrix} \right)_\perp &= -\mathbf{D} e^{-tB\mathbf{D}} X_+(B\mathbf{D}) \begin{bmatrix} h \\ 0 \end{bmatrix} \\ &= -\mathbf{D} e^{-tB\mathbf{D}} X_+(B\mathbf{D}) \begin{bmatrix} h \\ 0 \end{bmatrix} \\ &= -e^{-t\mathbf{D}B} X_+(\mathbf{D}B) \mathbf{D} \begin{bmatrix} h \\ 0 \end{bmatrix} \\ &= +e^{-t\mathbf{D}B} X_+(\mathbf{D}B) \begin{bmatrix} 0 \\ \nabla h \end{bmatrix}. \end{aligned}$$

□

The right-hand side in (30) makes sense for any distribution h such that $\nabla h \in L^2(\mathbb{R}^n; (\mathbb{C}^m)^n)$, that is, $h \in \dot{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$.

Lemma 8.2 (Boundary layer representation)**.** *Assume that $u \in \mathcal{E}_0$, i.e., $\|\widetilde{N}_*(\nabla u)\|_2 < \infty$. Then*

$$\nabla_A u(t, \cdot) = \nabla_A \mathcal{S}_t(\partial_{\nu_A} u|_{t=0}) - \nabla_A \mathcal{D}_t(u|_{t=0})$$

where $\nabla_A \mathcal{D}_t(u|_{t=0})$ is interpreted as the right hand side of (30). The equality holds in $\mathcal{E}_0 \cap C([0, +\infty); \dot{\mathcal{H}}^{0,+}_{\mathbf{D}B})$.

Proof. Using Theorem 7.2, if $\|\widetilde{N}_*(\nabla u)\|_2 < \infty$ then $\nabla_A u = e^{-t\mathbf{D}B} F_0$, with $F_0 \in \mathcal{H}^{0,+}_{\mathbf{D}B}$ and $F_0 = \nabla_A u|_{t=0}$. As $X_+(\mathbf{D}B)$ is a projection on $\mathcal{H}^{0,+}_{\mathbf{D}B}$, we have $X_+(\mathbf{D}B)F = F$

when $F \in \mathcal{H}^{0,+}_{\mathbf{D}B}$, so that

$$\begin{aligned}\nabla_A u &= e^{-t\mathbf{D}B}\begin{bmatrix}\partial_{\nu_A} u|_{t=0}\\ \nabla_x u|_{t=0}\end{bmatrix}\\ &= e^{-t\mathbf{D}B}X_+(\mathbf{D}B)\begin{bmatrix}\partial_{\nu_A} u|_{t=0}\\ \nabla_x u|_{t=0}\end{bmatrix}\\ &= e^{-t\mathbf{D}B}X_+(\mathbf{D}B)\begin{bmatrix}\partial_{\nu_A} u|_{t=0}\\ 0\end{bmatrix} + e^{-t\mathbf{D}B}X_+(\mathbf{D}B)\begin{bmatrix}0\\ \nabla_x u|_{t=0}\end{bmatrix}\\ &= \nabla_A \mathcal{S}_t(\partial_{\nu_A} u|_{t=0}) - \nabla_A \mathcal{D}_t(u|_{t=0}).\end{aligned}$$

□

Remark 8.3. If one can make sense of both $S_t(\partial_{\nu_A} u|_{t=0})$ and $\mathcal{D}_t(u|_{t=0})$ as distributions and fixing the constants of integration, one has the representation

$$u = \mathcal{S}_t(\partial_{\nu_A} u|_{t=0}) - \mathcal{D}_t(u|_{t=0}).^{5}$$

This, of course, is the classical formula obtained from Green's theorem if one can write $\mathcal{S}_t$ and $\mathcal{D}_t$ in integral form using the fundamental solution of L^*. We come to this below.

Corollary 8.4 (Generalized boundary layer representation)**.** *Let $s \in [-1, 0]$, and $u \in \mathcal{E}_s$ be a weak solution of $Lu = 0$ in $\mathbb{R}^{1+n}_+$. Then*

$$\nabla_A u(t,\cdot) = \nabla_A \mathcal{S}_t(\partial_{\nu_A} u|_{t=0}) - \nabla_A \mathcal{D}_t(u|_{t=0})$$

where $\nabla_A \mathcal{D}_t(u|_{t=0})$ is interpreted as the right hand side of (30) with $\nabla h \in \dot{H}^s_\nabla(\mathbb{R}^n; (\mathbb{C}^m)^n)$ and $\nabla_A u|_{t=0} = F_0$ given by Theorem 7.2 for $s = 0$ and Proposition 7.3 for $s < 0$. The equality holds in $\mathcal{E}_s \cap C([0,+\infty); \dot{\mathcal{H}}^{s,+}_{\mathbf{D}B})$.

Proof. For $s = 0$, this is Lemma 8.2. For $s < 0$, using the extension of functional calculus of $\mathbf{D}B|_{\overline{\mathsf{R}(\mathbf{D})}}$ on $\dot{\mathcal{H}}^s$ in Proposition 7.3, one defines $\nabla_A \mathcal{S}_t$ on the scalar Sobolev space $\dot{H}^s(\mathbb{R}^n; \mathbb{C}^m)$ and $\nabla_A \mathcal{D}_t$ by (30) with $\nabla h \in \dot{H}^s_\nabla(\mathbb{R}^n; (\mathbb{C}^m)^n)$. The proof is now the same as for $s = 0$. □

Remark 8.5. Let $Lu = 0$ with $u \in \mathcal{E}_s$ and $s < 0$. We know that the semigroup equation $\nabla_A u(t,\cdot) = e^{-t\mathbf{D}B}\nabla_A u(0,\cdot)$ holds in $C([0,+\infty); \dot{\mathcal{H}}^{s,+}_{\mathbf{D}B})$. Thus for all $\varepsilon > 0$ and $t > 0$, $\nabla_A u(t+\varepsilon,\cdot) = e^{-t\mathbf{D}B}\nabla_A u(\varepsilon,\cdot)$. From $\int_{\mathbb{R}^{1+n}_+} t^{-2s-1}|\nabla_{t,x} u|^2 dtdx < \infty$, for almost every $\varepsilon > 0$, $\nabla_A u(\varepsilon,\cdot) \in L^2(\mathbb{R}^n; (\mathbb{C}^m)^{1+n})$, hence to $\nabla_A u(\varepsilon,\cdot) \in \overline{\mathsf{R}(\mathbf{D})}$ as any L^2-conormal gradient. Set $u_\varepsilon(t,x) = u(t+\varepsilon, x)$. By Theorem 7.2, the semigroup equation implies that $u_\varepsilon \in \mathcal{E}_0$, that is $\|\widetilde{N}_*(\nabla_A u_\varepsilon)\|_2 < \infty$ and $\nabla_A u_\varepsilon \in C([0,\infty); \dot{\mathcal{H}}^{0,+}_{\mathbf{D}B})$. An easy argument shows that this must hold for all $\varepsilon > 0$. In particular, the generalized boundary layer representation in the statement above holds in $C((0,\infty); L^2(\mathbb{R}^n, (\mathbb{C}^m)^{1+n})$ (not at the boundary $t = 0$) as well, and even in $C^\infty((0,\infty); L^2(\mathbb{R}^n, (\mathbb{C}^m)^{1+n})$ by semigroup theory. Thus there is instantaneous regularisation of solutions in the upper half-space.

[5]Since we have upward convention for conormal derivatives and fundamental solutions for $\mathrm{div} A\nabla$ (usually it is taken for $-\mathrm{div} A\nabla$), we obtain the same sign rule as in the usual Green's formula, due to the cancellation of two minus signs. Had we been working in the lower-half space though, the upward normal is the outward normal and the sign rule would be opposite.

Proposition 8.6 ([R1]). *Let A be as in the beginning of this section. Assume further that* $\mathrm{div}A\nabla$ *and* $\mathrm{div}A^*\nabla$ *satisfy the De Giorgi condition or equivalently, the Nash local regularity condition. Let Γ^A and Γ^{A^*} be the fundamental solutions constructed in Proposition 5.1.*

i) *For $t \in \mathbb{R}$, $t \neq 0$, let D_t be the operator given by the double layer integral (when it converges for suitable h)*

$$D_t h(x) = \int_{\mathbb{R}^n} \langle h(y), \partial_{\nu_{A^*}} \Gamma^{A^*}(s,y;t,x)_{|s=0} \rangle \, dy \tag{31}$$
$$= \int_{\mathbb{R}^n} \langle h(y), (A^*(y)\nabla_{s,y}\Gamma^{A^*}(0,y;t,x))_\perp \rangle \, dy, \ t>0, x \in \mathbb{R}^n.$$

Here, $\langle\ ,\ \rangle$ stands for the canonical complex inner product on $\mathbb{C}^m$. Then, the abstract operator $\mathcal{D}_t$ agrees with the usual double layer potential in the sense that one has for $h \in L^2(\mathbb{R}^n;\mathbb{C}^m)$ with compact support and $t>0$, $\mathcal{D}_t h = D_t h$ and thus showing that D_t extends to a bounded map on $L^2(\mathbb{R}^n;\mathbb{C}^m)$, uniformly in $t>0$, with strong limit as $t \to 0$.

ii) *For $t \in \mathbb{R}$, $t \neq 0$, let S_t be the operator defined by the single layer integral (when it converges for suitable h)*

$$S_t h(x) = \int_{\mathbb{R}^n} \Gamma^A(t,x;0,y) h(y) \, dy. \tag{32}$$

Then for $h \in L^2(\mathbb{R}^n;\mathbb{C}^m)$ with compact support and $t>0$, $\nabla_A \mathcal{S}_t h = \nabla_A S_t h$, thus allowing to define $\mathcal{S}_t h(x)$ by the single layer integral $S_t h(x)$ and showing that S_t extends to a bounded map, uniformly in t, from $L^2(\mathbb{R}^n;\mathbb{C}^m)$ into $\dot{W}^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$, with strong limit as $t \to 0$.

We next recall estimates on the layer potentials.

Lemma 8.7. *Let A be as in the beginning of this section and assume that* $\mathrm{div}A\nabla$ *and* $\mathrm{div}A^*\nabla$ *satisfy the De Giorgi condition or equivalently, the Nash local regularity condition. Then*

(1) *The single layer operator S_t maps $L^p(\mathbb{R}^n;\mathbb{C}^m)$ to $\dot{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^m)$ for $1<p\leq 2$ uniformly in $t>0$, and converges when $t \to 0$ for the weak operator topology.*
(2) *The single layer operator S_t maps $\dot{W}^{-1,p}(\mathbb{R}^n;\mathbb{C}^m)$ to $L^p(\mathbb{R}^n;\mathbb{C}^m)$ for $2 \leq p < \infty$ uniformly in $t>0$, and converges when $t \to 0$ for the weak operator topology.*
(3) *The double layer operator is bounded on $L^p(\mathbb{R}^n;\mathbb{C}^m)$ for $2 \leq p < \infty$, uniformly in $t>0$, and converges when $t \to 0$ for the weak operator topology.*

Proof. The proof of (1) and (3) is given for equations and $1+n \geq 3$ in [HMiMo], but the arguments using the De Giorgi conditions are applicable here. We skip details. (2) is the dual statement of (1) as the adjoint of the single layer for A is the single layer for A^* and we use De Giorgi condition for both. □

The next result was observed in a special case as part of the proof of Theorem 5.35 in [HKMP2]. It receives a much simpler proof here.

Corollary 8.8. *Let A be as in the beginning of this section and assume that* $\mathrm{div}A\nabla$ *and* $\mathrm{div}A^*\nabla$ *satisfy the De Giorgi condition or equivalently, the Nash local regularity condition. Let u be an energy solution to* $\mathrm{div}A\nabla u = 0$ *in $\mathbb{R}^{1+n}_+$. Assume that for*

some $2 \leq p < \infty$, $u|_{t=0} \in L^p(\mathbb{R}^n;\mathbb{C}^m)$ *and* $\partial_{\nu_A} u|_{t=0} \in \dot{W}^{-1,p}(\mathbb{R}^n;\mathbb{C}^m)$. *Then the abstract boundary layer representation*

$$u(t,x) = \mathcal{S}_t(\partial_{\nu_A} u|_{t=0})(x) - \mathcal{D}_t(u|_{t=0})(x)$$

holds for all $t \geq 0$ *in* $L^2_{loc}(\mathbb{R}^n;\mathbb{C}^m)$. *In particular,* $\sup_{t\geq 0} \|u(t,\cdot)\|_{L^p(\mathbb{R}^n;\mathbb{C}^m)} < \infty$.

Proof. By Corollary 8.4, the equality holds up to a constant, that is

$$u(t,x) = \mathcal{S}_t(\partial_{\nu_A} u|_{t=0})(x) - \mathcal{D}_t(u|_{t=0})(x) + c, \quad t > 0,$$

in $L^2_{\rm loc}(\mathbb{R}^{1+n}_+;\mathbb{C}^m)$, but also in $L^2_{loc}(\mathbb{R}^n;\mathbb{C}^m)$ for each $t > 0$ as $p \geq 2$ and the right hand side belongs to $L^p(\mathbb{R}^n;\mathbb{C}^m) + \mathbb{C}^m$ by the previous lemma and the left hand side is in $L^2_{\rm loc}(\mathbb{R}^n;\mathbb{C}^m)$. One can pass to the limit in $t \to 0$, after testing against a $C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ function. For the right hand side, we use the previous lemma and for the left hand side, this is because $t \to u(t,\cdot)$ is continuous at 0 in $L^2_{\rm loc}(\mathbb{R}^n;\mathbb{C}^m)$ as u is an energy solution. One obtains $u|_{t=0}(x) = \mathcal{S}_0(\partial_{\nu_A} u|_{t=0})(x) - \mathcal{D}_{0^+}(u|_{t=0})(x) + c$. As all the functions belong to $L^p(\mathbb{R}^n;\mathbb{C}^m)$, we conclude that $c = 0$. □

Remark 8.9. The same statement holds for solutions in the classes $\mathcal{E}_s$ for all $s \in [-1,0]$. For $s > -1$, $\mathcal{E}_s$ can be shown to imbed into $C([0,\infty);L^2_{loc}(\mathbb{R}^n;\mathbb{C}^m))$, so the proof is the same. For $s = -1$, it follows from [AA] that any solution of $\mathrm{div} A\nabla u = 0$ in the class $\mathcal{E}_{-1}$, that is with the square function bound $\iint t|\nabla u|^2\, dtdx < \infty$, belongs in fact to $C([0,\infty);L^2(\mathbb{R}^n;\mathbb{C}^m)) + \mathbb{C}^m \subset C([0,\infty);L^2_{loc}(\mathbb{R}^n;\mathbb{C}^m))$. This is enough to finish the argument. It can be shown that the boundary layer representation also holds in the space of continuous functions valued in $L^p(\mathbb{R}^n;\mathbb{C}^m)$ equipped with the weak topology.

9. Interior non-tangential maximal estimates

We prove here the following *a priori* inequality.[6] See the introduction for history and differences in approach for this result.

Theorem 9.1. *Let* $\mathrm{div} A\nabla$ *be a uniformly elliptic system with* $A(x)$ *measurable, bounded, t-independent, complex coefficients on* $\mathbb{R}^{1+n}$ *with the strict Gårding inequality on* $\mathcal{H}^0$, *namely* (10). *Assume that* $\mathrm{div} A\nabla$ *and* $\mathrm{div} A^*\nabla$ *satisfy the De Giorgi condition and call* $0 < \mu_{DG}$ *the exponent that works for both. Then for all* $\frac{n}{n+\mu_{DG}} < p \leq 2$ *and for any weak solution of* $Lu = 0$ *on the upper half-space* $\mathbb{R}^{1+n}_+$, $1+n \geq 2$, *in any of the classes* $\mathcal{E}_s$, $-1 \leq s \leq 0$, *we have*

$$\|\widetilde{N}_*(\nabla u)\|_p \lesssim \|\partial_{\nu_A} u|_{t=0}\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)} + \|\nabla_x u|_{t=0}\|_{H^p(\mathbb{R}^n;(\mathbb{C}^m)^n)}, \tag{33}$$

where $H^p(\mathbb{R}^n)$ *denotes the real Hardy space if* $p \leq 1$ *and* $L^p(\mathbb{R}^n)$ *for* $p > 1$.

Recall that for $h \in L^2(\mathbb{R}^n;\mathbb{C}^m)$ and $t > 0$,

$$\nabla_A \mathcal{S}_t h = e^{-t\mathbf{D}B} X_+(\mathbf{D}B) \begin{bmatrix} h \\ 0 \end{bmatrix}$$

and for $h \in W^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$ and $t > 0$,

$$\nabla_A \mathcal{D}_t h = -e^{-t\mathbf{D}B} X_+(\mathbf{D}B) \begin{bmatrix} 0 \\ \nabla h \end{bmatrix}.$$

[6] This inequality will be proved in larger generality in [AS] with a different argument.

Remark that ∇ means here the tangential gradient ∇_x, while ∇_A still means the full conormal gradient. Recall also that the size of the full conormal gradient is pointwise comparable to that of the full gradient $\nabla_{t,x}$.

It is convenient to set $H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n) = \nabla \dot{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^m)$ (or again, those L^p curl-free functions) for $p > 1$ (for $p \leq 1$ it was defined in Section 4) and to define the operator V_t on $H^2_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$ by

$$V_t g = -e^{-t\mathbf{D}B} X_+(\mathbf{D}B) \begin{bmatrix} 0 \\ g \end{bmatrix}$$

for any $g \in \dot{H}^2_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$.

Theorem 9.1 follows immediately from the next *a priori* boundedness result, together with Proposition 8.4.

Theorem 9.2. *Let L be as in Theorem 9.1. Then for $\frac{n}{n+\mu_{DG}} < p \leq 2$,*

$$\|\widetilde{N}_*(\nabla_A \mathcal{S}_t h)\|_p \lesssim \|h\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)} \tag{34}$$

$$\|\widetilde{N}_*(V_t g)\|_p \lesssim \|g\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)} \tag{35}$$

This means that there is a linear extension of the map $h \mapsto (\nabla_A \mathcal{S}_t h)_{t>0}$ defined on $L^2(\mathbb{R}^n;\mathbb{C}^m) \cap H^p(\mathbb{R}^n;\mathbb{C}^m)$ to $H^p(\mathbb{R}^n;\mathbb{C}^m)$ with such an estimate, and of the map $g \mapsto (V_t g)_{t>0}$ from $H^2_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n) \cap H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$ to all of $H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$ with such an estimate.

In particular, this yields

$$\|\widetilde{N}_*(\nabla_A \mathcal{D}_t h)\|_p \lesssim \|\nabla h\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)} \tag{36}$$

whenever $h \in L^2(\mathbb{R}^n;\mathbb{C}^m)$ and $\nabla h \in L^2(\mathbb{R}^n;(\mathbb{C}^m)^n)$ as well (in fact more general h can be used provided one makes sense of the various objects).

We use the notation $N^p_2(\mathbb{R}^n)$ to denote the (quasi-)Banach space of all $L^2_{loc}(\mathbb{R}^{1+n}_+)$ functions such that $\|\widetilde{N}_*(f)\|_p < \infty$, $0 < p < \infty$. These spaces are further studied in [HR]. See also [Hu] for a more systematic approach.

For this purpose, we use again the 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ but in a slightly different way.

Lemma 9.3. *Let $\frac{n}{n+1} < p \leq 1$.*

1. *$H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ has the following atomic characterization: Let $g \in \mathcal{D}'(\mathbb{R}^n;\mathbb{C}^n)$. Then $g \in H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ if and only if $g = \sum \lambda_j a_j$ in $\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^n)$ with $\sum |\lambda_j|^p < \infty$ and a_j are 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. Moreover, $\|g\|_{H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)} \sim \inf \|(\lambda_j)\|_{\ell^p}$ with the infimum taken over all such decompositions.*
2. *$H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n) \cap H^2_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ is the subspace of $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ of those g having an atomic decomposition with $\|g\|_{H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)} \sim \|(\lambda_j)\|_{\ell^p}$ and which converges also in $H^2_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. It is dense in $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$.*
3. *A bounded linear operator $T : H^2_\nabla(\mathbb{R}^n;\mathbb{C}^n) \to N^2_2(\mathbb{R}^n)$ with $\sup \|\widetilde{N}_*(Ta)\|_p < \infty$, where the supremum is taken over all 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$, extends to a bounded map from $H^q_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ to $N^q_2(\mathbb{R}^n)$, for $p \leq q \leq 2$.*
4. *A bounded linear operator $T : H^r_\nabla(\mathbb{R}^n;\mathbb{C}^n) \to N^r_2(\mathbb{R}^n)$ for some $1 < r < 2$ with $\sup \|\widetilde{N}_*(Ta)\|_p < \infty$, where the supremum is taken over all 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$, extends to a bounded map from $H^q_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ to $N^q_2(\mathbb{R}^n)$, for $p \leq q \leq r$.*

All this extends straightforwardly to $H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$ spaces.

Proof. The proof of (1) is done in [LMc] when $p=1$. As already mentioned, the method in [LMc] is to construct a Calderón reproducing formula that allows to see that $H^p_\nabla(\mathbb{R}^n)$ is a retract of the tent space T^p_2 of [CMS] : $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ is isomorphic to closed and complemented subspace of T^p_2 and one can use the atomic decomposition of T^p_2 which, given the particular form of the retract mappings in [LMc], gives (1). Their method extends to the range $\frac{n}{n+1}<p\leq 1$ without difficulty. We skip details.

The proof of (2) is as follows. The retract mappings are of Littlewood-Paley type with smooth and compactly supported convolution kernels with mean 0 so they work simultaneously and boundedly for all $\frac{n}{n+1}<p<\infty$. Denoting by S the mapping from T^p_2 to $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ of the retract diagram, we have $S(T^p_2\cap T^2_2)=H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)\cap H^2_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. Thus, it suffices to show that $T^p_2\cap T^2_2$ is the subspace of T^p_2 of those elements having a T^p_2 atomic decomposition that converges also in T^2_2. This fact is implicit in the proof of [AMcR, Theorem 4.9, step 3] for $p=1$ and the very same argument applies when $p<1$. Again we skip details. A different and explicit method is in [JY], Proposition 3.1. The density follows from the density of $T^p_2\cap T^2_2$ in T^p_2.

The proof of (3) is now simple using (2). To prove the boundedness at $q=p$, choose an atomic decomposition $\sum\lambda_j a_j$ for $g\in H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)\cap H^2_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ that converges also in $H^2_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. Then this convergence and boundedness of T imply $Tg=\sum\lambda_j Ta_j$ and it follows that $\|Tg\|_{N^p_2(\mathbb{R}^n)}\lesssim\|g\|_{H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)}$ using $\sup\|\widetilde{N}_*(Ta)\|_p<\infty$. It remains to extend by density. The boundedness when $p<q<2$ follows by interpolation. The spaces $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ for $\frac{n}{n+1}<p<\infty$ interpolate by the retract property and the interpolation property of the tent spaces ([CMS], [Be] and [CV] for $p<1$). The result follows by using real interpolation for the sublinear operator $g\mapsto\widetilde{N}_*(Tg)$.

We finish with the proof of (4). Remark that 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ are elements of $H^r_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ as $r<2$, so the statement is meaningful. It is enough to prove the boundedness at $q=p$ as interpolation takes care of the other values of q. Choose an atomic decomposition $\sum\lambda_j a_j$ for $g\in H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)\cap H^2_\nabla(\mathbb{R}^n;\mathbb{C}^n)$ that converges also in $H^2_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. Of course, one has also the convergence in $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. Interpolation implies that it converges also in $H^r_\nabla(\mathbb{R}^n;\mathbb{C}^n)$. Thus $Tg=\sum\lambda_j Ta_j$ by boundedness of T at exponent r and it follows that $\|Tg\|_{N^p_2(\mathbb{R}^n)}\lesssim\|g\|_{H^p_\nabla(\mathbb{R}^n;\mathbb{C}^n)}$ using $\sup\|\widetilde{N}_*(Ta)\|_p<\infty$. It remains to extend by density. □

Proof of Theorem 9.2. We prove (35). By lemma 9.3 it is enough to prove the bound for 2-atoms for $H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$ when $\frac{n}{n+\mu}=p>p_0=\frac{n}{n+\mu_0}$ with $0<\mu_0<\mu_{DG}$. Fix such a p. The argument follows a method of Kenig-Pipher [KP]. Let $a=\nabla b$ be a 2-atom for $H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$, with a,b supported in a surface ball $\Delta(x_0,r)$. We note that in this case $V_t a=\nabla_A\mathcal{D}_t b$ as both a,b are L^2 functions. As our techniques are scale invariant, we assume that $x_0=0$ and $r=1$ to simplify the exposition. We let $\Delta_k=\Delta(0,2^k)$ and $C_k=\Delta_{k+1}\setminus\Delta_k$ for $k\in\mathbb{N}$. We have

$$\begin{aligned}\|\widetilde{N}_*(\nabla_A\mathcal{D}_t b)\|_{L^p(\Delta_2)}&\leq|\Delta_2|^{1/p-1/2}\|\widetilde{N}_*(\nabla_A\mathcal{D}_t b)\|_{L^2(\Delta_2)}\\&\leq C\|\nabla b\|_2|\Delta_2|^{1/p-1/2}\leq C4^{n(1/p-1/2)}.\end{aligned}$$

It remains to show $\|\widetilde{N}_*(\nabla_A \mathcal{D}_t b)\|_{L^2(C_k)} \leq C2^{-k(n/2+\mu_0)} = C2^{-k(n/p_0-n/2)}$ when $k \geq 2$, which implies $\|\widetilde{N}_*(\nabla_A \mathcal{D}_t b)\|^p_{L^p(C_k)} \leq C2^{-k(n/p_0-n/p)p}$. Indeed, summing all these estimates for $k \geq 1$, yields $\|\widetilde{N}_*(\nabla_A \mathcal{D}_t b)\|^p_p \lesssim 1$.

Set $u(t,x) = D_t b(x)$ be the solution of $\mathrm{div} A\nabla$ for $(t,x) \in \mathbb{R}^{1+n}$ away from the support of b (identifying $\mathbb{R}^n$ with $\{0\} \times \mathbb{R}^n$) given by the double layer integral in (31) (which will be shown to converge under the De Giorgi assumption on $\mathrm{div} A\nabla$ and its adjoint). Under these assumptions, we know that $u(t,x) = \mathcal{D}_t b(x)$ for $t > 0$ (Proposition 8.6) where $\mathcal{D}_t$ is the abstract double layer operator. We claim that

$$|u(t,x)| \lesssim |(t,x)|^{-n+1-\mu_0}, \ |(t,x)| \geq 2. \tag{37}$$

Indeed, using (31), Proposition 2.1 in [AAAHK] for the solution $(s,y) \mapsto \Gamma^{A^*}(s,y;t,x)$ for L^* in $(-2,2) \times \Delta(0,2)$, as $\mathrm{div} A^*\nabla$ is t-independent (this results extends *mutatis mutandi* to systems), and then (23), we have

$$\begin{aligned} |u(t,x)| &\leq \|A^*\|_\infty \|b\|_2 \left(\int_{\Delta(0,1)} |(\nabla_{s,y}\Gamma^{A^*})(0,y;t,x)|^2 \, dy \right)^{1/2} \\ &\lesssim \left(\int_{\Delta(0,1)} \int_{-1}^{1} |(\nabla_{s,y}\Gamma^{A^*})(s,y;t,x)|^2 \, dsdy \right)^{1/2} \\ &\lesssim |(t,x)|^{-n+1-\mu_0}. \end{aligned}$$

Remark that a similar strategy gives (bad but finite) pointwise bounds for $D_t b(x)$ for (t,x) not in the support of b, showing that u is well-defined. As we shall see, (37) is all we need to run the Kenig-Pipher method.

Fix $k \geq 2$, $x \in C_k$. We estimate $t^{-(1+n)} \int_{W(t,x)} |\nabla u|^2$. If $t \geq 2^k$, by Caccioppoli inequality and (37)

$$t^{-(1+n)} \int_{W(t,x)} |\nabla_A u|^2 \leq C t^{-(3+n)} \int_{\widetilde{W}(t,x)} |u|^2 \lesssim 2^{-2k(n+\mu_0)},$$

with $\widetilde{W}(t,x)$ a slightly enlarged version of $W(t,x)$.

It remains to consider the case $t < 2^k$. The argument of [KP], Lemma 8.10, p. 494, yields the estimate for some C depending only on ellipticity and dimension,

$$\sup_{t<2^k} t^{-(1+n)} \int_{W(t,x)} |\nabla_A u|^2 \leq C \sup_{t \leq (1+c_0)2^k, x \in \widetilde{C}_k} |\partial_t u(t,x)|^2 + CM(|\nabla_x u(0,\cdot)|^q 1_{\widetilde{C}_k})(x)^{2/q},$$

for some $q < 2$ (coming from usage of Poincaré inequalities in $\mathbb{R}^n$), where M is the Hardy-Littlewood maximal operator and $\widetilde{C}_k$ is the union of all $\Delta(x, c_1 t)$ for $x \in C_k$ and $t < 2^k$. Thus if c_1 in the definition of $W(t,x)$ is chosen small to start with, $\widetilde{C}_k$ is an annulus at distance proportional to 2^k from the support of b of the form $c_3 2^k \leq |x| \leq c_4 2^k$. As A is t-independent, we have that $\partial_t u$ is also a solution. Moser's local estimate (20), Caccioppoli inequality and (37) imply that

$$\begin{aligned} \sup_{t \leq (1+c_0)2^k, x \in \widetilde{C}_k} |\partial_t u(t,x)|^2 &\lesssim 2^{-k(n+1)} \int_{\check{C}_k} \int_{-(1+c_0)2^{k+1/2}}^{(1+c_0)2^{k+1/2}} |\partial_t u(t,x)|^2 \, dtdx \\ &\lesssim 2^{-k(n+3)} \int_{\check{C}_k} \int_{-(1+c_0)2^{k+1}}^{(1+c_0)2^{k+1}} |u(t,x)|^2 \, dtdx \\ &\lesssim 2^{-2k(n+\mu_0)}, \end{aligned}$$

where $\widetilde{C}_k \subsetneq \check{C}_k \subsetneq \hat{C}_k$ are annuli of the form $|x| \sim 2^k$.

For the last term, we have by the Hardy-Littlewood theorem,

$$\|M(|\nabla_x u(0,\cdot)|^q 1_{\widetilde{C}_k})(x)^{2/q}\|^2_{L^2(C_k)} \leq C \int_{\widetilde{C}_k} |\nabla_x u(0,\cdot)|^2 \, dx,$$

and since u is a weak solution of L away from the support of b and A has t-independent coefficients, we have by Proposition 2.1 in [AAAHK] and then Caccioppoli inequality

$$\begin{aligned}\int_{\widetilde{C}_k} |\nabla_x u(0,x)|^2 \, dx &\lesssim 2^{-k} \int_{\widetilde{C}_k} \int_{-2^k}^{2^k} |\nabla_{t,x} u(t,x)|^2 \, dtdx \\ &\leq 2^{-3k} \int_{\check{C}_k} \int_{-2^{k+1}}^{2^{k+1}} |u(t,x)|^2 \, dtdx \\ &\lesssim 2^{-k(n+2\mu_0)},\end{aligned}$$

where $\check{C}_k$ is again a slightly larger version of $\widetilde{C}_k$.

Gathering all the estimates we have obtained that $\|\widetilde{N}_*(\nabla_A \mathcal{D}_t b)\|_{L^2(C_k)} \leq C 2^{-k(n/2+\mu_0)}$ as desired.

Let us present the proof for the single layer. By [MSV], Theorem 4.1, if $p = 1$, and [YZ] if $p \leq 1$, and interpolation, it is enough to prove the bound for 2-atoms for $H^p(\mathbb{R}^n; \mathbb{C}^m)$ for p as above.

Let a be a 2-atom for $H^p(\mathbb{R}^n; \mathbb{C}^m)$, with a supported in a surface ball $\Delta(x_0, r)$. Again, we assume that $x_0 = 0$ and $r = 1$ to simplify the exposition. We let $\Delta_k = \Delta(0, 2^k)$ and $C_k = \Delta_{k+1} \setminus \Delta_k$ for $k \in \mathbb{N}$. We have

$$\begin{aligned}\|\widetilde{N}_*(\nabla_A \mathcal{S}_t a)\|_{L^p(\Delta_2)} &\leq |\Delta_2|^{1/p-1/2} \|\widetilde{N}_*(\nabla_A \mathcal{S}_t a)\|_{L^2(B_2)} \\ &\leq C\|a\|_2 |\Delta_2|^{1/p-1/2} \leq C 4^{n(1/p-1/2)}.\end{aligned}$$

As above, it is enough to show for $u(t,x) = S_t a(x)$, which is is a weak solution of $\mathrm{div} A\nabla$ for $(t,x) \in \mathbb{R}^{1+n}$ away from the support of a, the estimate

$$|u(t,x)| \lesssim |(t,x)|^{-n+1-\mu_0}, \ |(t,x)| \geq 2. \tag{38}$$

Indeed, we know from Proposition 8.6 that $\nabla_A \mathcal{S}_t a = \nabla_A S_t a$ for $t > 0$. By the mean value of a we can write

$$u(t,x) = \int_{\Delta(0,1)} (\Gamma^A(t,x;0,y) - \Gamma^A(t,x;0,0)) a(y) \, dy$$

and conclude using (22). □

10. Extrapolation of solvability for regularity and Neumann problems

We are now ready to attack the extrapolation for solvability by gathering all pieces of information obtained so far.

Let $\mathrm{div} A\nabla$ be a uniformly complex elliptic system with $A(\mathbf{x})$ measurable, bounded, t-independent on $\mathbb{R}^{1+n}$ with the strict Gårding inequality on $\mathcal{H}^0$, namely (10).

In addition, assume that $\mathrm{div} A\nabla$ and its adjoint satisfy the De Giorgi condition. Consider the reflected matrix $A^\sharp$. It is no longer with t-independent coefficients. However, it is easy to see that it does satisfy the Gårding inequality (17) on $\mathbb{R}^{1+n}$. We also assume that the second order system with matrix $A^\sharp$ and its adjoint satisfy

the De Giorgi condition. We call $\mu_{DG} \in (0,1]$ the best exponent that works for all 4 operators.

With these conditions, all results in prior sections apply. Again, this situation covers dimension $1+n=2$ or dimensions $1+n \geq 3$ with A close in L^∞ to a real and scalar matrix.

Here is a fact we are going to use. Let $\frac{n}{n+1} < p \leq 2$. It is shown in [HMiMo] (Lemma 6.1) that any weak solution u of $\mathrm{div} A\nabla u = 0$ with $\|\widetilde{N}_*(\nabla_A u)\|_p < \infty$ admits a conormal gradient at the boundary in $H^p(\mathbb{R}^n; (\mathbb{C}^m)^{1+n})$ with $\|\nabla_A u|_{t=0}\|_{H^p} \lesssim \|\widetilde{N}_*(\nabla_A u)\|_p$ and that $\nabla_A u(t,\cdot)$ converges in the sense of distributions to $\nabla_A u|_{t=0}$ as $t \to 0$. The implicit constant depends only on the L^∞ bound for A and dimension. In particular, if u is also an energy solution, then the two notions of conormal gradients at the boundary must coincide from the convergence in the sense of distributions.

As mentioned in the introduction, we shall restrict our attention to solvability exponents not exceeding 2. See [HKMP2] for the Regularity problem for exponents exceeding 2. See also the forthcoming [AS].

10.1. **Regularity problem.** Slightly modifying the original approach of [KP], we say that the Regularity problem (R^p_A) is solvable if there exists $C_p < \infty$ such that for any $f \in H^p_\nabla(\mathbb{R}^n; (\mathbb{C}^m)^n) \cap \dot{H}^{-1/2}_\nabla(\mathbb{R}^n; (\mathbb{C}^m)^n)$ the energy solution u of $\mathrm{div} A\nabla u = 0$ with regularity data $\nabla_x u|_{t=0} = f$ satisfies

$$\|\widetilde{N}_*(\nabla_A u)\|_p \leq C_p \|f\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)}.$$

There is a difference between solvability and well-posedness in the class where $\|\widetilde{N}_*(\nabla_A u)\|_p < \infty$ given the data f. See the discussion in [HKMP2] about uniqueness. Axelsson [Ax] also showed by an explicit example for a real equation in dimension $1+n=2$ that there might be solutions not in the energy class, even for very smooth data, while the energy solution does not satisfy this bound.

Solvability implies well-posedness of the following restricted problem: given $f \in H^p_\nabla(\mathbb{R}^n; (\mathbb{C}^m)^n)$, there exists a unique solution u of $\mathrm{div} A\nabla u = 0$ with $\|\widetilde{N}_*(\nabla_A u)\|_p \leq C_p \|f\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)}$, $\nabla_x u|_{t=0} = f$ and such that there exists a sequence of energy solutions u_k with $\|\widetilde{N}_*(\nabla_A u - \nabla_A u_k)\|_p \to 0$. The constant C_p is the one specified by solvability assumption. This follows from density of $H^p_\nabla(\mathbb{R}^n; (\mathbb{C}^m)^n) \cap \dot{H}^{-1/2}_\Delta(\mathbb{R}^n; (\mathbb{C}^m)^n)$ in $H^p_\nabla(\mathbb{R}^n; (\mathbb{C}^m)^n)$. This fact, which is just a reformulation of the extension by continuity for linear maps, is left to the reader.

Theorem 10.1. *Assume that A is as specified at the beginning of the section with $\mu_{DG} \in (0,1]$. Let $p_{DG} = \frac{n}{n+\mu_{DG}}$. Let $1 < r \leq 2$. Assume that the Regularity problem (R^r_A) is solvable. Then the Regularity problem (R^p_A) is solvable for $p_{DG} < p < r$.*

Corollary 10.2. *This theorem applies to the following situations for A in addition to having the assumption at the beginning of the section (t-independence, ellipticity and De Giorgi conditions):*

1. *A is constant plus t-independent L^∞ perturbation*
2. *A is hermitian plus t-independent L^∞ perturbation*
3. *A is block upper-triangular plus t-independent L^∞ perturbation*
4. *A real (non necessarily symmetric) and scalar plus t-independent L^∞ perturbation*

Proof. We know from [A] that De Giorgi assumption is stable under L^∞ perturbations. It suffices to show that (R_A^r) is solvable in the four items for some $1 < r \leq 2$. From [AAMc], we also know that (R_A^2) is stable under t-independent L^∞ perturbation of A and is verified for A constant or hermitian (for real symmetric scalar A, this was done in [KP]), while [AMcM] proves (R_A^2) for A block upper-triangular (the block diagonal case is a direct consequence of [AHLMcT]). Hence, the first three items satisfy (R_A^2). The fourth item is shown on combining [KKPT, KR] and [B] (who also shows (R_A^1)) if $n+1=2$ and [HKMP2] for $n+1 \geq 3$. □

Remark 10.3. The block upper-triangular case can be slightly relaxed. Instead of the lower coefficient c to be 0, we may only assume $\mathrm{div} c = 0$. See [AMcM, Remark 6.7].

Lemma 10.4. *Let $p_{DG} < p \leq 2$. Then (R_A^p) is solvable if and only if there exists $C_p < \infty$ such that for any $u \in \mathcal{E}$ solution of* $\mathrm{div} A\nabla u = 0$,

$$\|\partial_{\nu_A} u|_{t=0}\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)} \leq C_p \|\nabla_x u|_{t=0}\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)}. \tag{39}$$

Proof. Let u be the energy solution with regularity data $f = \nabla_x u|_{t=0}$. Let $\alpha = \partial_{\nu_A} u|_{t=0}$. The solvability of (R_A^p), together with $\|\nabla_A u|_{t=0}\|_{H^p(\mathbb{R}^n;(\mathbb{C}^m)^{1+n})} \lesssim \|\widetilde{N}_*(\nabla_A u)\|_p$, implies the desired inequality. Conversely, assuming this estimate for any energy solution, we can use Theorem 9.1 to conclude that $\|\widetilde{N}_*(\nabla_A u)\|_p \lesssim C_p \|\nabla_x u|_{t=0}\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)}$, hence (R_A^p) is solvable. □

Proof of Theorem 10.1. Let us begin with the case $p_{DG} < p \leq 1$. By Lemma 10.4 and Theorem 4.5, it suffices to show that if a is a 2-atom for $H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$, then we obtain a uniform estimate $\|\alpha\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)} \leq C$ where α is the conormal derivative of the energy solution u produced by the Dirichlet datum b with $a = \nabla b$ as in the definition of 2-atoms for $H^p_\nabla(\mathbb{R}^n;\mathbb{C}^m)$. By scale invariance of our assumptions, we assume that a and b are supported in the surface ball $\Delta(0,1)$. We shall show that α is a r-molecule for $H^p(\mathbb{R}^n;\mathbb{C}^m)$ (see below) with bound independent of a. Hence there is a constant C independent of a such that $\|\alpha\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)} \leq C$ as desired.

We now prove the r-molecule property for α. As in the proof of Theorem 9.2, set $\Delta_k = \Delta(0,2^k)$ and $C_k = \Delta_{k+1} \setminus \Delta_k$ for $k \in \mathbb{N}$. It suffices to show that $\|\alpha\|_{L^r(\Delta_2)} \lesssim 1$ and that $\|\alpha\|_{L^r(C_k)} \lesssim 2^{-k\mu} 2^{-nk/r'}$ for some $0 < \mu < \mu_{DG}$ with $p > \frac{n}{n+\mu}$ with r' the conjugate exponent to r. Indeed, $2^{-k\mu}2^{-nk/r'} = 2^{-k\varepsilon}2^{-nk(1/p-1/r)}$ for $\varepsilon = -\mu + n(1/p-1) > 0$ which is the right decay for being in the Hardy space H^p. The local estimate $\|\alpha\|_{L^r(\Delta_2)} \lesssim 1$ follows from the global bound $\|\alpha\|_r \lesssim \|a\|_r \lesssim \|a\|_2 \lesssim 1$ (here we use that the support of a is contained in $\Delta_0 = \Delta(0,1)$). The main task is therefore to obtain the decay on C_k. Note that C_k can be covered by boundedly (in k) many surface balls Δ with radius proportional to 2^k and with distance to Δ_0 proportional to 2^k and with $4\Delta \cap \Delta_0 = \emptyset$. Thus it is enough to work on one of those. Let $g \in C_0^\infty(\Delta;\mathbb{C}^m)$ with $\|g\|_{r'} \leq 1$. It suffices to estimate $\langle \alpha, g\rangle$. Let w be the energy solution of $\mathrm{div} A^*\nabla w = 0$ on $\mathbb{R}^{1+n}_+$ with $w|_{t=0} = g$ (Lemma 2.3). Using Theorem 3.3, we deduce from (R_A^r) that

$$\|\partial_{\nu_{A^*}} w|_{t=0}\|_{\dot{W}^{-1,r'}} \leq C_{p'} \|\nabla g\|_{\dot{W}^{-1,r'}} \lesssim \|g\|_{r'} \leq 1.$$

We deduce from the representation of Corollary 8.8 that $\sup_{t>0}\|w(t,\cdot)\|_{r'}\lesssim 1$. We now invoke (15), which tells

$$|\langle\alpha,g\rangle|\leq C2^{-2k}\left(\int_{\Omega_+}|u|^2\right)^{1/2}\left(\int_{\Omega_+}|w|^2\right)^{1/2},$$

with Ω_+ contained in a box $[0,c2^k]\times 3\Delta$. Hölder's inequality using $r'\geq 2$ yields

$$\left(\int_{\Omega_+}|w|^2\right)^{1/2}\leq|\Omega_+|^{1/2-1/r'}\left(\int_0^{c2^k}\|w(t,\cdot)\|^{r'}\right)^{1/r'}\lesssim 2^{k/r'}2^{(1+n)k(1/2-1/r')}.$$

Now for u we use the decay estimate from Lemma 6.2 together with the observation that $\|b\|_{\dot H^{1/2}}\lesssim(\|b\|_2\|\nabla b\|_2)^{1/2}\lesssim 1$, to get $|u|\leq 2^{-k(n-1+\mu)}$ on Ω_+. Working out the powers of 2^k we obtain the desired bound for $|\langle\alpha,g\rangle|$.

We now continue with the case $1<p<r$. Another way to reformulate Lemma 10.4 is to say that the Dirichlet to Neumann operator satisfies $\|\Gamma_{DN}f\|_{H^r(\mathbb{R}^n;\mathbb{C}^m)}\leq C_r\|f\|_{H^r_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)}$ for all $f\in H^r_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)\cap\dot H^{-1/2}_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$. Let T_r be the continuous extension from $H^r_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$ into $H^r(\mathbb{R}^n;\mathbb{C}^n)$. We just showed a uniform estimate for $\|\Gamma_{DN}a\|_{H^1(\mathbb{R}^n;\mathbb{C}^m)}$ when a is a 2-atom for $H^1_\nabla(\mathbb{R}^n;\mathbb{C}^m)$, which are elements in $H^r_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)\cap\dot H^{-1/2}_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$. Hence $T_ra=\Gamma_{DN}a$. We can apply the same interpolation procedure as for (4) of Proposition 9.3. Hence, for $1<p<r$, we obtain T_r bounded from $H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$ into $H^p(\mathbb{R}^n;\mathbb{C}^m)$. In particular, we obtain $\|\Gamma_{DN}f\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)}\leq C_p\|f\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)}$ for all $f\in\mathcal{D}_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$, that is $\|\partial_{\nu_A}u|_{t=0}\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)}\leq C_p\|\nabla_x u|_{t=0}\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)}$ for all energy solutions with smooth Dirichlet data. We conclude using (1) in Theorem 3.5 to waive the restriction on the data and then Lemma 10.4 again. □

Remark 10.5. When $p<1$, the solvability information is used to obtain the decay for α but in a dual way, not on the solution u attached to α. Note that this argument has the flavor of many of the different steps for Theorem 5.2 of [KP]. But the order in which they are invoked is completely different trying to use a priori estimates as much as possible. In particular, we avoid the localization technique there and the recourse to solvability of dual Dirichlet problem *per se*. We only use available *a priori* estimates. This last point will be important later.

10.2. **Neumann problem.** Slightly modifying the original approach of [KP], we say that the Neumann problem (N^p_A) is solvable if there exists $C_p<\infty$ such that for any $g\in H^p(\mathbb{R}^n;\mathbb{C}^m)\cap\dot H^{-1/2}(\mathbb{R}^n;\mathbb{C}^m)$ the (modulo constants) energy solution u of $\mathrm{div}A\nabla u=0$ with conormal derivative $\partial_{\nu_A}u|_{t=0}=h$ satisfies

$$\|\widetilde N_*(\nabla_A u)\|_p\leq C_p\|h\|_{H^p(\mathbb{R}^n;\mathbb{C}^m)}.$$

Here too, there is a difference between solvability and well-posedness in the class where $\|\widetilde N_*(\nabla_A u)\|_p<\infty$ given the data f. An explicit example for a real equation in dimension $1+n=2$ in [Ax] shows that there might be solutions not in the energy class, even for very smooth data, while the energy solution does not satisfy this bound. Note that in 2 dimensions, Neumann and Regularity problems are the same up to taking conjugates.

Solvability implies well-posedness of the following restricted problem: given $h\in H^p(\mathbb{R}^n;\mathbb{C}^m)$, there exists a unique solution u of $\mathrm{div}A\nabla u=0$ with $\|\widetilde N_*(\nabla_A u)\|_p\leq C_p\|h\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)}$, $\partial_{\nu_A}u|_{t=0}=h$ and such that there exists a sequence of energy

solutions u_k with $\|\widetilde{N}_*(\nabla_A u - \nabla_A u_k)\|_p \to 0$. The constant C_p is the one specified by solvability assumption. This follows from density of $H^p(\mathbb{R}^n;\mathbb{C}^m) \cap \dot{H}^{-1/2}(\mathbb{R}^n;\mathbb{C}^m)$ in $H^p(\mathbb{R}^n;\mathbb{C}^n)$. This fact is left to the reader.

Theorem 10.6. *Assume that A is as specified at the beginning of the section with $\mu_{DG} \in (0,1]$. Let $p_{DG} = \frac{n}{n+\mu_{DG}}$. Let $1 < r \leq 2$. Assume that the Neumann problem (N_A^r) is solvable. Then the Neumann problem (N_A^p) is solvable for $p_{DG} < p < r$.*

Corollary 10.7. *This theorem applies to the following situations for A in addition to having the assumption at the beginning of the section (t-independence, ellipticity and De Giorgi conditions):*

1. *A is constant plus t-independent L^∞ perturbation*
2. *A is hermitian plus t-independent L^∞ perturbation*
3. *A is block lower-triangular plus t-independent L^∞ perturbation*
4. *A real (non necessarily symmetric) and scalar if $1+n = 2$ plus t-independent L^∞ perturbation*

Proof. It suffices to show that (N_A^r) is solvable in the four items for some $1 < r \leq 2$. We know from[A] that De Giorgi assumption is stable under L^∞ perturbations. From [AAMc], we also know that (N_A^2) is stable under t-independent L^∞ perturbation of A and is verified for A constant or hermitian (for real symmetric scalar A, this was done in KP), while [AMcM] proves (N_A^2) for A block upper-triangular (the block diagonal case is a direct consequence of [AHLMcT]). Hence, the first three items satisfy (N_A^2). The fourth item is shown on combining [KR] and [B] (who also shows (N_A^1)) as $n+1 = 2$. □

We note that solvability of the Neumann problems for real (non-symmetric) equations is still open in dimensions $1 + n \geq 3$.

Lemma 10.8. *Let $p_{DG} < p \leq 2$. Then (N_A^p) is solvable if and only if there exists $C_p < \infty$ such that for any $u \in \mathcal{E}$ solution of $\operatorname{div} A\nabla u = 0$, $\|\nabla_x u|_{t=0}\|_{H^p(\mathbb{R}^n;(\mathbb{C}^m)^n)} \leq C_p \|\partial_{\nu_A} u|_{t=0}\|_{H^p_\nabla(\mathbb{R}^n;\mathbb{C}^m)}$.*

Same proof as Lemma 10.4.

Proof of Theorem 10.6. The case $1 < p < r$ is exactly as in the proof of Theorem 10.1 once we have done the case $p_{DG} < p \leq 1$.

Let us assume $p_{DG} < p \leq 1$. By Lemma 10.4 and Theorem 4.5, it suffices to show if a is a 2-atom for $H^p(\mathbb{R}^n;\mathbb{C}^n)$, then we obtain a uniform estimate $\|f\|_{H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)} \leq C$ where f is the tangential derivative of any energy solution u produced by the Neumann datum a. By scale invariance of our assumptions, we assume that a is supported in the surface ball $\Delta(0,1)$. We shall show that f is a r-molecule for $H^p(\mathbb{R}^n;(\mathbb{C}^m)^n)$ with bound independent of a. Hence there is a constant C independent of a such that $\|f\|_{H^p(\mathbb{R}^n;(\mathbb{C}^m)^n} \leq C$ as desired. Since f is of a gradient form, it automatically fulfills the $H^p_\nabla(\mathbb{R}^n;(\mathbb{C}^m)^n)$ estimate.

We now prove the r-molecule property for f. As in the proof of of Theorem 9.2, set $\Delta_k = \Delta(0,2^k)$ and $C_k = \Delta_{k+1} \setminus \Delta_k$ for $k \in \mathbb{N}$. As before, it suffices to show that $\|f\|_{L^r(\Delta_2)} \lesssim 1$ and that $\|f\|_{L^r(C_k)} \lesssim 2^{-k\mu} 2^{-nk/r'}$ for some $0 < \mu < \mu_{DG}$ with $p > \frac{n}{n+\mu}$ with r' the conjugate exponent to r. The local estimate $\|f\|_{L^r(\Delta_2)} \lesssim 1$ follows from the global bound $\|f\|_r \lesssim \|a\|_r \lesssim \|a\|_2 \lesssim 1$ (here we use that the support of a is contained in $\Delta_0 = \Delta(0,1)$). The main task is therefore to obtain the decay on C_k.

Again it is enough to work on a surface ball Δ with radius proportional to 2^k and with distance to Δ_0 proportional to 2^k and with $4\Delta\cap\Delta_0=\emptyset$. Let $g\in C_0^\infty(\Delta;(\mathbb{C}^m)^n)$ with $\|g\|_{r'}\leq 1$. It suffices to estimate $\langle f,g\rangle$. Write $f=\nabla_x u|_{t=0}$ with $u|_{t=0}$ being the still unspecified Dirichlet data as we have not yet chosen the constant of integration. Let w be one of the energy solutions of $\mathrm{div}A^*\nabla w=0$ on $\mathbb{R}^{1+n}_+$ with $\partial_{\nu_{A^*}}w|_{t=0}=-\mathrm{div}g$ (Lemma 2.2). We have, therefore, $\langle f,g\rangle=\langle u|_{t=0},\partial_{\nu_{A^*}}w|_{t=0}\rangle$. Using Theorem 3.4, we deduce from (N_A^r) that

$$\|\nabla_x w|_{t=0}\|_{\dot W^{-1,r'}}\leq C_{r'}\|-\mathrm{div}g\|_{\dot W^{-1,r'}}\sim\|g\|_{r'}\leq 1.$$

Using Lemma 3.2, we now choose w so that $w|_{t=0}\in L^{r'}$. We can deduce from the boundary layer representation of Corollary 8.8 that $\sup_{t>0}\|w(t,\cdot)\|_{r'}\lesssim 1$. Next, invoke (16), which tells

$$|\langle u|_{t=0},\partial_{\nu_{A^*}}w|_{t=0}\rangle|\leq C2^{-2k}\left(\int_{\Omega_+}|u|^2\right)^{1/2}\left(\int_{\Omega_+}|w|^2\right)^{1/2},$$

with Ω_+ contained in a box $[0,c2^k]\times 3\Delta$. Hölder's inequality using $r'\geq 2$ yields

$$\left(\int_{\Omega_+}|w|^2\right)^{1/2}\leq|\Omega_+|^{1/2-1/r'}\left(\int_0^{c2^k}\|w(t,\cdot)\|^{r'}\right)^{1/r'}\lesssim 2^{k/r'}2^{(1+n)k(1/2-1/r')}.$$

Remark that we have not yet specified the constant of integration for u. We choose it now so as to use the decay estimate from Lemma 6.4 to get $|u|\leq 2^{-k(n-1+\mu)}$ on Ω_+ since $\|a\|_1\lesssim 1$. Working out the powers of 2^k we obtain the desired bound for $|\langle f,g\rangle|$. □

Remark 10.9. In the block lower-triangular case, when the upper coefficient b to be 0, or equivalently that the conormal vector field is proportional to the transversal vector field, one can obtain an L^2-molecular decay for the tangential gradient by a direct integration by parts which does not use at all the initial L^2 solvability information. This one is only used for the local estimate. This means that the difficulty in the study of Neumann problems lies in the upper coefficient of A. It remains to understand its exact role when it is not 0 or when A is not hermitian in dimensions $1+n\geq 3$.

11. Extrapolation of solvability for Dirichlet problems and other Neumann problems

We gather in this section the needed results to prove extrapolation of Dirichlet problems and of a new type of problems, namely Neumann problems with data in negative Sobolev spaces.

It is convenient to introduce the following correspondences of spaces to be read line by line.

exponents	Y	Y^{-1}	$\mathcal{T}$	X^1	X
$1<p,q<\infty$, $q=p'$	L^q	$\dot W^{-1,q}$	T_2^q	$\dot W^{1,p}$	L^p
$\alpha=0$, $p=1$	BMO	$\dot{\mathrm{BMO}}^{-1}$	T_2^∞	$\dot H^{1,1}$	H^1
$0<\alpha=n(\frac{1}{p}-1)<1$	$\dot\Lambda^\alpha$	$\dot\Lambda^{\alpha-1}$	$T_{2,\alpha}^\infty$	$\dot H^{1,p}$	H^p

Here Y, Y^{-1} are the dual spaces of X, X^1 respectively. They are spaces on the boundary. Next, $\mathcal{T}$ are tent spaces on $\mathbb{R}^{1+n}_+$. For $1 < q \leq \infty$, T^q_2 is the tent space of [CMS]. For $q = \infty$, this is defined via Carleson measures:

$$\iint_{(0,r)\times\Delta} |f(t,x)|^2 \frac{dtdx}{t} \leq \|f\|^2_{T^\infty_2} |\Delta|.$$

For $0 < \alpha < 1$,

$$\iint_{(0,r)\times\Delta} |f(t,x)|^2 \frac{dtdx}{t} \leq \|f\|^2_{T^\infty_{2,\alpha}} |\Delta|^{1+\frac{2\alpha}{n}}.$$

Here Δ are balls in $\mathbb{R}^n$ and r is the radius of Δ.

We next turn to equivalence of boundary norms with interior estimates of tent space nature.

Theorem 11.1. *Let* $\mathrm{div}A\nabla$ *be a uniformly elliptic system with* $A(x)$ *measurable, bounded, t-independent, complex coefficients on* $\mathbb{R}^{1+n}$ *with the strict Gårding inequality on* $\mathcal{H}^0$*, namely* (10)*. Assume that* $\mathrm{div}A\nabla$ *and* $\mathrm{div}A^*\nabla$ *satisfy the De Giorgi condition and call* $0 < \mu_{DG}$ *the exponent that works for both. Then for all spaces in the table with* $2 \leq q$ *and* $\alpha < \mu_{DG}$ *and for any weak solution of* $Lu = 0$ *on the upper half-space* $\mathbb{R}^{1+n}_+, 1 + n \geq 2$*, in any of the classes* $\mathcal{E}_s$*,* $-1 \leq s \leq 0$*, we have*

$$\|t\nabla u\|_{\mathcal{T}} \approx \|\partial_{\nu_A} u|_{t=0}\|_{Y^{-1}} + \|\nabla_x u|_{t=0}\|_{Y^{-1}}. \tag{40}$$

Again, we do not consider the case $2 - \varepsilon < q < 2$ which can be handled without the De Giorgi condition (See the forthcoming [AS]).

Proof. The inequality $\lesssim$ follows from the generalized boundary layer representation of Corollary 8.4 together with the estimates proved in [HMaMo] for the single and double layer potentials (again in the case of equations and $1 + n \geq 3$ but with immediate extension to our situation). The converse inequality is a result from [AS], where the other direction is proved as well in this generality. □

Remark 11.2. Remark that in the case $2 < q < \infty$, the inequality $\gtrsim$ is akin to the inequality (3.9) in [HKMP2]. It is more precise though as it does not contain any non-tangential maximal function. In fact, [AS] will show under the above assumptions that the non-tangential maximal function of u is controlled in L^q by the T^q_2 norm of $t\nabla u$. This was proved for real equations in [HKMP1].

11.1. **The Dirichlet problem.** Let $Y = Y(\mathbb{R}^n; \mathbb{C}^m)$ be one of the spaces from the above table. We say that the Dirichlet problem (D^Y_A) is solvable if there exists $C_Y < \infty$ such that for any $f \in Y \cap \dot{H}^{1/2}(\mathbb{R}^n; \mathbb{C}^m)$ the energy solution u of $\mathrm{div}A\nabla u = 0$ with Dirichlet data $u|_{t=0} = f$ satisfies

$$\|t\nabla u\|_{\mathcal{T}} \leq C_Y \|f\|_Y.$$

We remark that we formulate here the Dirichlet problem uniquely in term of the tent space estimate. From the remark above, the non tangential maximal estimate comes as a bonus.

Corollary 11.3. *For the spaces* Y *considered in Theorem 11.1, we have that* (D^Y_A) *is solvable if and only if there exists* $C_{Y^{-1}} < \infty$ *such that for any* $u \in \mathcal{E}$ *solution of* $\mathrm{div}A\nabla u = 0$*,* $\|\partial_{\nu_A} u|_{t=0}\|_{Y^{-1}} \leq C_{Y^{-1}} \|\nabla_x u|_{t=0}\|_{Y^{-1}}$.

The proof is a direct consequence of Theorem 11.1. As before, the solvability is reduced to a boundary estimate. We obtain a refined version of the main result of [HKMP2], which treats only the cases $X = L^p, Y = L^{p'}, 1 < p < 2 + \varepsilon$, but not the endpoint spaces.

Corollary 11.4. *Consider the spaces Y of Theorem 11.1 and their preduals X. If (R_A^X) is solvable then $(D_{A^*}^Y)$ is solvable. The converse holds when $X = L^p$ and $Y = L^q$, $q = p'$ and when $X = H^1$ and $Y = BMO$.*

Proof. The equivalence in the range $X = L^p$, $Y = L^q$ follows from Corollary 11.3, Lemma 10.4 and Theorem 3.3. The implication in the other cases and the equivalence when $p = 1$ follow from Theorem 3.3, Theorem 4.1 and Corollary 11.3. □

This applies when the conditions of Corollary 10.2 are satisfied for A^*. Details are left to the reader.

Remark that the back and forth proof allows to replace $Y = BMO$ by $Y = VMO$ in the statement. This fact that the Dirichlet problems for VMO or BMO data are equivalent was also observed in [DKP] for real equations.

To finish we can state the extrapolation result for the Dirichlet problem. This is only here that we use more assumptions on A.

Theorem 11.5. *Consider an elliptic system with all the assumptions at the beginning of Section 10. Let $2 \leq q < \infty$ and assume (D_A^Y) is solvable for $Y = L^q(\mathbb{R}^n; \mathbb{C}^m)$. Then (D_A^Y) is solvable for $Y = L^p(\mathbb{R}^n; \mathbb{C}^m)$, $q < p < \infty$, $BMO(\mathbb{R}^n; \mathbb{C}^m)$, and $\dot{\Lambda}^\alpha(\mathbb{R}^n; \mathbb{C}^m)$ with $0 < \alpha < \mu_{DG}$.*

Proof. It suffices to combine Corollary 11.4 with Theorem 10.1. □

11.2. **The Neumann problem in negative Sobolev spaces.** Let $Y = Y(\mathbb{R}^n; \mathbb{C}^m)$ be one of the spaces from the above table. We say that the Neumann problem $(N_A^{Y^{-1}})$ is solvable if there exists $C_{Y^{-1}} < \infty$ such that for any $f \in Y^{-1} \cap \dot{H}^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ the energy solution u of $\mathrm{div} A\nabla u = 0$ with Neumann data $\partial_{\nu_A} u|_{t=0} = f$ satisfies

$$\|t\nabla u\|_{\mathcal{T}} \leq C_{Y^{-1}} \|f\|_{Y^{-1}}.$$

Corollary 11.6. *For the spaces Y considered in Theorem 11.1, we have that $(N_A^{Y^{-1}})$ is solvable if and only if there exists $C_{Y^{-1}} < \infty$ such that for any $u \in \mathcal{E}$ solution of $\mathrm{div} A\nabla u = 0$, $\|\nabla_x u|_{t=0}\|_{Y^{-1}} \leq C_{Y^{-1}} \|\partial_{\nu_A} u|_{t=0}\|_{Y^{-1}}$.*

The proof is a direct consequence of Theorem 11.1. As before, the solvability is reduced to a boundary estimate.

Corollary 11.7. *Consider the spaces Y of Theorem 11.1 and their preduals X. If (N_A^X) is solvable then $(N_{A^*}^{Y^{-1}})$ is solvable. The converse holds when $X = L^p$ and $Y^{-1} = \dot{W}^{-1,q}$, $q = p'$ and when $X = H^1$ and $Y^{-1} = BMO^{-1}$.*

Proof. The equivalence in the range $X = L^p$, $Y = L^q$ follows from Corollary 11.3, Lemma 10.4 and Theorem 3.4. The implication in the other cases and the equivalence when $p = 1$ follow from Theorem 3.4, Theorem 4.2 and Corollary 11.3. □

This applies when the conditions of Corollary 10.7 are satisfied for A^*. Details are left to the reader.

Remark that the back and forth proof allows to replace $Y = BMO$ by $Y = VMO$ in the statement.

To finish we can state the extrapolation result for the Neumann problem in negative Sobolev spaces. This is only here that we use more assumptions on A.

Theorem 11.8. *Consider an elliptic system with all the assumptions at the beginning of Section 10. Let $2 \leq q < \infty$ and assume $(N_A^{Y^{-1}})$ is solvable for $Y^{-1} = \dot{W}^{-1,q}(\mathbb{R}^n;\mathbb{C}^m)$. Then $(N_A^{Y^{-1}})$ is solvable for $Y^{-1} = \dot{W}^{-1,p}(\mathbb{R}^n;\mathbb{C}^m)$, $q < p < \infty$, $BMO^{-1}(\mathbb{R}^n;\mathbb{C}^m)$, and $\dot{\Lambda}^{\alpha-1}(\mathbb{R}^n;\mathbb{C}^m)$ with $0 < \alpha < \mu_{DG}$.*

Proof. It suffices to combine Corollary 11.7 with Theorem 10.6. □

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