HAL
open science

Existence of steady states for the Maxwell-Schrödinger-Poisson system: exploring the applicability of the concentration-compactness principle

Isabelle Catto, Jean Dolbeault, Óscar Sánchez, Juan Soler

To cite this version:

Isabelle Catto, Jean Dolbeault, Óscar Sánchez, Juan Soler. Existence of steady states for the Maxwell-Schrödinger-Poisson system: exploring the applicability of the concentration-compactness principle. Mathematical Models and Methods in Applied Sciences, 2013, 23 (10), pp.1915-1938. hal-00834551

HAL Id: hal-00834551

https://hal.science/hal-00834551v1

Submitted on 16 Jun 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

EXISTENCE OF STEADY STATES FOR THE MAXWELL-SCHRÖDINGER-POISSON SYSTEM: EXPLORING THE APPLICABILITY OF THE CONCENTRATION-COMPACTNESS PRINCIPLE*

I. CATTO(1), J. DOLBEAULT(1), O. SÁNCHEZ(2) and J. SOLER(2)

(1) *Ceremade (UMR CNRS no. 7534), Université Paris-Dauphine*
Place de Lattre de Tassigny, F-75775 Paris Cedex 16, France
E-mail: catto@ceremade.dauphine.fr, dolbeaul@ceremade.dauphine.fr

(2) *Departamento de Matemática Aplicada,*
University of Granada, 18071–Granada, Spain
E-mail: ossanche@ugr.es, jsoler@ugr.es

This paper is intended to review recent results and open problems concerning the existence of steady states to the Maxwell-Schrödinger system. A combination of tools, proofs and results are presented in the framework of the concentration–compactness method.

Keywords: Steady states; variational methods; subadditivity inequality; constrained minimization; Sharp nonexistence. L^2-norm constraint; Schrödinger-Poisson; Maxwell-Schrödinger; Schrödinger-Poisson-X^α; Concentration-Compactness; Standing waves; Semiconductors; Plasma physics

1. Introduction

The concentration–compactness method is nowadays a basic tool in applied mathematics for the analysis of variational problems with lack of compactness or more specifically for proving existence of solutions of non-linear partial differential equations which are invariant under a group of transformations. In this review we explore the applicability of the concentration–compactness method on the X^α-Schrödinger-Poisson model. We will also highlight some related questions, which raise a number of open issues.

Our purpose is to study the existence of steady states of the so-called X^α-Schrödinger-Poisson (X^α-SP) model or Maxwell-Schrödinger-Poisson system:

$$\begin{aligned} &i\,\frac{\partial\psi}{\partial t} = -\Delta_x\psi + V(x,t)\,\psi - C\,|\psi(x,t)|^{2\alpha}\,\psi\,,\\ &-\Delta_x V = \epsilon\,4\pi\,|\psi|^2,\\ &\psi(x,t=0) = \phi(x)\,, \end{aligned} \tag{1.1}$$

* This work has been partially supported by MINECO (Spain), Project MTM2011-23384, and ANR projects NoNAP and STAB (France).

with $\phi \in \mathrm{L}^2(\mathbb{R}^3)$, $x \in \mathbb{R}^3$, $t \geq 0$. The self-consistent Poisson potential V is explicitly given by $V(x,t) = \epsilon\, |\psi(x,t)|^2 \star |x|^{-1}$, where $\star$ refers to the convolution with respect to x on $\mathbb{R}^3$ and where ϵ takes the value $+1$ or -1, depending whether the interaction between the particles is repulsive or attractive. The system (1.1) can therefore be reduced to a single non-linear and non-local Schrödinger-type equation

$$\begin{aligned} & i\,\frac{\partial \psi}{\partial t} = -\Delta_x \psi + \epsilon \left(|\psi|^2 \star |x|^{-1} \right) \psi - C\, |\psi|^{2\alpha}\, \psi\,, \\ & \psi(x, t=0) = \phi(x)\,. \end{aligned} \tag{1.2}$$

Such a model appears in various frameworks, such as black holes in gravitation ($\epsilon = -1$) [31], one-dimensional reduction of electron density in plasma physics ($\epsilon = +1$), as well as in semiconductor theory ($\epsilon = +1$), as a correction to the Schrödinger-Poisson system (which is X^α-SP with $C = 0$): see [6,22,29] and references therein.

In the plasma physics case, the X^α-SP correction takes into account a non-linear, although local, correction to the Poisson potential of opposite sign given by $-\,C\,|\psi|^{2\alpha}$, where C is a positive constant and where the parameter α, responsible for the name of the model, takes values in the range $0 < \alpha \leq \frac{2}{3}$. Some relevant values are for example $\alpha = \frac{1}{3}$, which is called the Slater correction, or $\alpha = \frac{2}{3}$, which gives rise to the so-called Dirac correction. The idea is to balance the Poisson potential (also called Coulombian potential in the electrostatic case) with a local potential term of opposite sign. This generates a competition between the two potential energies and the kinetic energy that, depending on the values of the constant C, can modify the typically dispersive dynamics of the Schrödinger-Poisson system [18,32] in the plasma physics case. The local nonlinear term also modifies the properties of the solutions in the gravitational case, thus leading to a richer behaviour [5]. Note that the physical constants have been normalized to unity here for the sake of simplicity.

Throughout the paper we focus our attention on the plasma physical case. Similar techniques can be used for extending our results to the gravitational case. Notice that when $\epsilon = -1$ (gravitational case), the sign of the energy associated to the Poisson potential (also called Newtonian potential) allows to introduce symmetric rearrangements that contribute to simplify some computations [20,21]. In this paper, we shall therefore assume that

$$\epsilon = +1\,.$$

We will be concerned with the existence of standing waves, that is, solutions to (1.2) of the form

$$\psi(x,t) = e^{i\ell_M t}\, \varphi(x)$$

with $\ell_M > 0$ and φ in $\mathrm{L}^2(\mathbb{R}^3)$ solving

$$-\,\Delta\varphi + \epsilon \left(|\varphi|^2 \star |x|^{-1} \right) \varphi - C\, |\varphi|^{2\,\alpha}\, \varphi + \ell_M\, \varphi = 0\,. \tag{1.3}$$

Equation (1.3) is a special case of Schrödinger-Maxwell equations [13].

The existence and stability analysis of such solutions relies on some preserved physical quantities. The total *mass* (which is also the total electronic charge in the repulsive case, when $\epsilon = +1$)

$$M[\psi] := \int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx$$

and the *energy* functional

$$\mathrm{E}[\psi] := \mathrm{E}_{\mathrm{kin}}[\psi] + \mathrm{E}_{\mathrm{pot}}[\psi]$$

are invariant quantities for any solution of X^α-SP along the time evolution, where the *kinetic* and *potential* energies are defined by

$$\mathrm{E}_{\mathrm{kin}}[\psi] := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla\psi(x,t)|^2 \, dx \,, \quad \mathrm{E}_{\mathrm{pot}}[\psi] := \frac{\epsilon}{4} \, \mathrm{D}[\psi] - \frac{C}{2\alpha+2} \int_{\mathbb{R}^3} |\psi(x,t)|^{2\alpha+2} \, dx$$

and

$$\mathrm{D}[\psi] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\psi(x,t)|^2 \, |\psi(x',t)|^2}{|x-x'|} \, dx \, dx' \,.$$

The existence of standing waves has been carried out from various perspectives in the vast mathematical literature devoted to this topic. Either one investigates the existence of critical points of the functional $\mathrm{E}[\varphi] + \ell_M \, M[\varphi]$ on the whole space $\mathrm{H}^1(\mathbb{R}^3)$, with the parameter ℓ_M being given and fixed, and in that case the $\mathrm{L}^2(\mathbb{R}^3)$ norm of the solution is not prescribed (see for instance [30] and references therein); or one looks for critical points of the energy functional $\mathrm{E}[\varphi]$ with prescribed $\mathrm{L}^2(\mathbb{R}^3)$ norm, and then the parameter ℓ_M enters into the game as a Lagrange multiplier of the constrained minimization problem. From a physical point of view, the most interesting critical points, the so-called *steady states,* are the minimizers of the problem

$$I_M := \inf \big\{ \mathrm{E}[\varphi] \, : \, \varphi \in \Sigma_M \big\} \,, \quad \Sigma_M := \big\{ \varphi \in \mathrm{H}^1(\mathbb{R}^3) \, : \, \|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = M \big\} \,. \tag{1.4}$$

Their interest lies in *stability* properties stated in terms of the energy and the mass. Such a feature is of course well known in the literature, see for instance [11], and it provides an easier approach than other methods, which are anyway needed when elaborate variational methods are required like in [2]. The energy functional is not bounded from below when $\alpha > \frac{2}{3}$. When $\alpha > 2$, the exponent $2\alpha + 2$ lies outside of the interval $(2, 6)$ and then $\mathrm{H}^1(\mathbb{R}^3)$ is not embedded in $L^{2\alpha+2}(\mathbb{R}^3)$. We therefore restrict our analysis to the range α in $(0, 2)$.

Concerning the existence of steady states, let us make the following observations. First of all, the energy and mass functionals are translation invariant that is, for every $y \in \mathbb{R}^3$,

$$\mathrm{E}[\varphi(\cdot + y)] = \mathrm{E}[\varphi] \,, \quad M[\varphi(\cdot + y)] = M[\varphi] \,.$$

Therefore the concentration–compactness method [23,24,25] is the natural framework for the study of the existence of a minimizer and for the analysis of the behavior of the minimizing sequences to (1.4) and their possible lack of compactness. According

to the terminology of the concentration–compactness principle, from any minimizing sequence $\{\varphi_n\}_{n\geq 1}$ in Σ_M we can extract a subsequence (denoted in the same way for simplicity) that either *vanishes,* that is,

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{y+B_R} \varphi_n^2 \, dx = 0 \quad \forall\, R > 0\,, \tag{1.5}$$

or satisfies the property

$$\exists\, R_0 > 0\,,\ \exists\, \varepsilon_0 > 0\,,\ \exists\, \{y_n\}_{n\geq 1} \subset \mathbb{R}^3 \quad \text{such that} \quad \int_{y_n+B_{R_0}} \varphi_n^2 \, dx \geq \varepsilon_0\,. \tag{1.6}$$

In the first case, for any sequence $\{y_n\}_{n\geq 1}$ in $\mathbb{R}^3$, $\{\varphi_n(\cdot + y_n)\}_{n\geq 1}$ converges to zero weakly in $\mathrm{H}^1(\mathbb{R}^3)$. In the second case, up to the extraction of a subsequence, the sequence $\{\varphi_n(\cdot + y_n)\}_{n\geq 1}$ converges weakly towards a nonzero function φ_* such that

$$\int_{\mathbb{R}^3} \varphi_*^2 \, dx = \mu > 0\,.$$

If $\mu = M$, then compactness (*i.e.,* the strong convergence of subsequences) holds. In the opposite case, $\mu < M$, then *dichotomy* occurs, that is, the splitting of the functions in at least two parts that are going away from each other: see [23,24,25] for more details.

The concentrated–compactness method yields the strict inequalities

$$I_M < I_{M'} + I_{M-M'} \quad \forall\, M\,,\ M' \quad \text{such that} \quad 0 < M' < M \tag{1.7}$$

as necessary and sufficient conditions for the *relative compactness* up to translations of all minimizing sequences. In this case, we deduce the existence of a minimizer and its orbital stability under the flow (1.1). The proof of this equivalence is based on the fact that the only possible loss of compactness for minimizing sequences occurs either from vanishing or from dichotomy. Note that the so-called large inequalities

$$I_M \leq I_{M'} + I_{M-M'} \quad \forall\, M\,,\ M' \quad \text{such that} \quad 0 < M' < M \tag{1.8}$$

always hold true due to the translation invariance. For any $\varepsilon > 0$, one may indeed find C^∞ functions $\phi_\varepsilon \in \Sigma_{M'}$ and $\psi_\varepsilon \in \Sigma_{M-M'}$, both with compact supports, such that $I_{M'} \leq \mathrm{E}[\phi_\varepsilon] \leq I_{M'} + \varepsilon$ and $I_{M-M'} \leq \mathrm{E}[\psi_\varepsilon] \leq I_{M-M'} + \varepsilon$. Then, for any unit vector e in $\mathbb{R}^3$ and for $n \in \mathbb{N}$ large enough such that ϕ_ε and $\psi_\varepsilon(\cdot + n\,e)$ have disjoint supports, we have $\phi_\varepsilon + \psi_\varepsilon(\cdot + n\,e) \in \Sigma_M$ and

$$I_M \leq \limsup_{n\to+\infty} \mathrm{E}[\phi_\varepsilon + \psi_\varepsilon(\cdot + n\,e)] \leq I_{M'} + I_{M-M'} + 2\varepsilon\,.$$

The conclusion follows since ε can be made arbitrarily small. For our particular problem, it can be easily proved that vanishing cannot hold for any minimizing sequence of (1.4) if $I_M < 0$, although it might hold when $I_M = 0$. This is based on Lemma I.1 in [25] that ensures that vanishing minimizing sequences converge to zero strongly in $L^{2\alpha+2}(\mathbb{R}^3)$. When $I_M = 0$, vanishing has to be avoided by considering particular sequences.

Furthermore, when relative compactness up to translations can be proved for any minimizing sequence, it can also be stated that the minimizing steady state solution is orbitally stable in the sense developed in [11], thanks to the fact that mass and energy are time preserved quantities for solutions to (1.1). In this sense, let us mention that the well-posedness of the X^α-SP system was proved in [10] (Remark 6.5.3) for $\alpha \in (0, \frac{2}{3})$. For the case $\alpha = \frac{2}{3}$, the existence of global solutions was proved [10] only for initial data with $\|\phi\|_{\mathrm{H}^1(\mathbb{R}^3)}$ small enough. A theory of existence of $\mathrm{L}^2(\mathbb{R}^3)$ mixed-state solutions was developed in [6] for the Slater case, $\alpha = \frac{1}{3}$. Stability properties have been proved to be false for other kind of standing waves, see for instance [2].

Our aim is to discuss the applicability of the concentration–compactness method to the problem (1.4) for proving the existence of X^α-SP *steady states.* Recall that such solutions are minimizers of the energy functional under mass constraint. Let us summarize the results presented in this work in Table 1, with some references for previously known results. In this table, the constant C_α denotes the

α	Energy infimum	Existence of steady states	Ref.
0	$I_M < 0$	No	18,32
$(0, \frac{1}{2})$	$I_M < 0$	Yes, for small M Open for large M	9,33,3,4
$\frac{1}{2}$	$I_M = 0$ if $C < \frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}}$ $I_M = 0$ if $C = \frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}}$ $I_M < 0$ if $C > \frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}}$	No Open Yes	19
$(\frac{1}{2}, \frac{2}{3})$	$I_M = 0$ if $C\,M^{4\alpha-2} < V_c(\alpha)$ $I_M = 0$ if $C\,M^{4\alpha-2} = V_c(\alpha)$ $I_M < 0$ if $C\,M^{4\alpha-2} > V_c(\alpha)$	No Yes Yes	19 4
$\frac{2}{3}$	$I_M = 0$ if $C\,M^{\frac{2}{3}} \le \frac{5}{3\,\mathrm{C}_{2/3}}$ $I_M = -\infty$ if $C\,M^{\frac{2}{3}} > \frac{5}{3\,\mathrm{C}_{2/3}}$	No No	
$(\frac{2}{3}, 2)$	$I_M = -\infty$	No	2

Table 1. Table of existence results of steady states and related references.

optimal constant in the inequality

$$\|u\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \le \mathrm{C}_\alpha \, \|u\|^{8\alpha-4}_{\mathrm{L}^2(\mathbb{R}^3)} \, \mathrm{D}[u]^{2-3\alpha} \, \|\nabla u\|^{6\alpha-2}_{\mathrm{L}^2(\mathbb{R}^3)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^3)\,,$$

with $\mathrm{D}[u] = 4\pi \int_{\mathbb{R}^3} u^2\,(-\Delta)^{-1}\,u^2\,dx$. The constant

$$V_c(\alpha) := \frac{\alpha+1}{\mathrm{C}_\alpha} \left(\frac{1}{3\alpha-1}\right)^{3\alpha-1} \left(\frac{1}{2\,(2-3\alpha)}\right)^{2-3\alpha} \tag{1.9}$$

will appear in Proposition 2.2.

In this review, we emphasize that many partial results can been found in various papers and, concerning variational approaches, particularly in [4,3,19,2]. For other existence and non-existence results with the Lagrange parameter taken as a parameter, we refer to [12,13,14,30,34]. For solutions satisfying a *Pohozaev constraint* (see Proposition 2.3) and in particular the so-called *ground state* solutions, we refer to [1,13,30]. Our contribution mostly lies in a unified framework based on the concentration-compactness method. Results corresponding to the ranges $0 < \alpha < \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\frac{1}{2} < \alpha < \frac{2}{3}$ have been collected respectively in Propositions 3.1, 3.2, and 3.4. Our main original contribution deals with the threshold case $\alpha = \frac{1}{2}$. We also invite the reader to pay attention to the remarks of Section 3 and to Proposition 3.3 for a some open problems.

In the range $\alpha \in (0, \frac{1}{2})$, we are going to prove that the strict inequalities (1.7) hold at least for M small enough. The strategy of proof is inspired by [9] (Appendix 3) and is reproduced here for the reader's convenience. The same result has been derived in [4,33] for $\alpha = \frac{1}{3}$ and $0 < M < M_c$, and in [3,4,9,33] for any $\alpha \in (0, \frac{1}{2})$ and any small positive M. As far as the authors know, the critical case ($\alpha = \frac{1}{2}$) has been treated only in [19] in the specific case $C = 1$, where $I_M = 0$; in that case the non-existence of a minimizer has been established. We will show here that there exists a critical value for C, which is $3/(\sqrt{2}\,\mathrm{C}_{1/2})$, such that for larger values of C the minimizers exists but not for smaller values. The existence of minimizers for the critical value of C is still an open problem, equivalent to the existence of optimal functions for the above inequality with $\alpha = \frac{1}{2}$. When $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$, existence holds if and only if M is large enough. The result of existence of steady states was previously obtained in [4]. No steady states exist in the cases $\alpha = 0$ or $\alpha \in \left[\frac{2}{3}, 2\right)$. The result for $\alpha = 0$ is in agreement with the general dispersion property verified by the solutions to the repulsive Schrödinger-Poisson system proved in [18,32]. It is also one of the motivations for introducing the local, nonlinear correction to the model. Although the existence of minimizers cannot be expected in the case $\alpha \in \left(\frac{2}{3}, 2\right)$ because $I_M = -\infty$, the existence and instability of other standing waves has recently been proved in [2]. Also see [1,13,30] for *ground state* solutions.

For completeness, let us mention that symmetry breaking issues are not completely understood [28,16]. In this direction, new approaches could be useful like those developed in [15] and subsequent papers. Stability of minimizers with null energy also raises a number of open questions.

2. *A priori* estimates and consequences

Before tackling the existence of steady states, we have to make sure that the minimization problem is well-posed for $\alpha \in [0, \frac{2}{3})$, and for small masses M in the case $\alpha = \frac{2}{3}$. Let us first recall the Gagliardo-Nirenberg inequality

$$\|u\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \leq \mathrm{C}_{\mathrm{GN}}(\alpha)\, \|\nabla u\|^{3\alpha}_{\mathrm{L}^2(\mathbb{R}^3)}\, \|u\|^{2-\alpha}_{\mathrm{L}^2(\mathbb{R}^3)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^3) \tag{2.1}$$

where $\mathrm{C}_{\mathrm{GN}}(\alpha)$ is the optimal constant, depending only on $\alpha \in [0,2]$.

Lemma 2.1. *For any $\alpha \in [0, \frac{1}{2}]$, there is a positive constant K_α such that, for any $u \in \mathrm{H}^1(\mathbb{R}^3)$, we have*

$$\|u\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \leq \mathrm{K}_\alpha \, \|u\|^{2-4\alpha}_{\mathrm{L}^2(\mathbb{R}^3)} \, \mathrm{D}[u]^\alpha \, \|\nabla u\|^\alpha_{\mathrm{L}^2(\mathbb{R}^3)} \tag{2.2}$$

and for any $\alpha \in [\frac{1}{2}, \frac{2}{3}]$, there is a positive constant C_α such that, for any $u \in \mathrm{H}^1(\mathbb{R}^3)$, we have

$$\|u\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \leq \mathrm{C}_\alpha \, \|u\|^{8\alpha-4}_{\mathrm{L}^2(\mathbb{R}^3)} \, \mathrm{D}[u]^{2-3\alpha} \, \|\nabla u\|^{6\alpha-2}_{\mathrm{L}^2(\mathbb{R}^3)} \, . \tag{2.3}$$

The case $\alpha = \frac{1}{2}$ has been established by P.-L. Lions [26] in Formula (55) page 54 and is common to the two inequalities, with $\mathrm{K}_{1/2} = \mathrm{C}_{1/2}$. The case $\alpha = \frac{2}{3}$ is a special case of (2.1), with $\mathrm{C}_{2/3} = \mathrm{C}_{\mathrm{GN}}(2/3)$. For completeness, let us give a proof.

Proof. We recall that $\mathrm{D}[u] = 4\pi \int_{\mathbb{R}^3} u^2 \, (-\Delta)^{-1} \, u^2 \, dx$. By expanding the square and integrating by parts, we get that

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^3} |\nabla u - a \, \nabla (-\Delta)^{-1} \, u^2|^2 \, dx \\ = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + a^2 \int_{\mathbb{R}^3} u^2 \, (-\Delta)^{-1} \, u^2 \, dx - 2a \int_{\mathbb{R}^3} u^3 \, dx \, , \end{aligned}$$

that is, for an arbitrary positive parameter a,

$$\int_{\mathbb{R}^3} u^3 \, dx \leq \frac{1}{2a} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{a}{2} \int_{\mathbb{R}^3} u^2 \, (-\Delta)^{-1} \, u^2 \, dx \, .$$

After optimizing on a, we obtain that

$$\|u\|^6_{\mathrm{L}^3(\mathbb{R}^3)} \leq \frac{1}{4\pi} \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3)} \, \mathrm{D}[u] \, . \tag{2.4}$$

This proves (2.2) and (2.3) when $\alpha = \frac{1}{2}$. The range $\alpha \in [0, \frac{1}{2}]$ is then covered by Hölder's inequality $\|u\|_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \leq \|u\|^{2-4\alpha}_{\mathrm{L}^2(\mathbb{R}^3)} \, \|u\|^{6\alpha}_{\mathrm{L}^3(\mathbb{R}^3)}$.

For $\alpha = \frac{2}{3}$, (2.3) coincides with (2.1), namely

$$\|u\|^{10/3}_{\mathrm{L}^{10/3}(\mathbb{R}^3)} \leq \mathrm{C}_{\mathrm{GN}}(\tfrac{2}{3}) \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^3)} \, \|u\|^{4/3}_{\mathrm{L}^2(\mathbb{R}^3)} \, .$$

Hence the case $\alpha \in [\frac{1}{2}, \frac{2}{3}]$ is covered by Hölder's inequality

$$\|u\|^{\alpha+1}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \leq \|u\|^{3(2-3\alpha)}_{\mathrm{L}^3(\mathbb{R}^3)} \, \|u\|^{5(2\alpha-1)}_{\mathrm{L}^{10/3}(\mathbb{R}^3)} \, . \qquad \square$$

Notice that from (2.4) we know that

$$\mathrm{C}_{1/2} \leq \frac{1}{2\sqrt{\pi}} \, .$$

Lemma 2.2. *The energy functional E is bounded from below in Σ_M, if either $\alpha \in [0, \frac{2}{3})$ or $\alpha = \frac{2}{3}$ and $C \, \mathrm{C}_{\mathrm{GN}}(\frac{2}{3}) \, M^{2/3} \leq \frac{5}{3}$. If either $\alpha \in [0, \frac{2}{3})$ or $\alpha = \frac{2}{3}$ and*

$C\,\mathrm{C}_{\mathrm{GN}}(\frac{2}{3})\,M^{2/3} < \frac{5}{3}$, *any minimizing sequence for* I_M *is uniformly bounded in* $\mathrm{H}^1(\mathbb{R}^3)$.

Proof. As a direct consequence of (2.1), for every $\varphi \in \Sigma_M$ we have the estimate

$$\mathrm{E}[\varphi] \geq \frac{1}{2}\,\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^3)} - \frac{C\,\mathrm{C}_{\mathrm{GN}}(\alpha)}{2\alpha+2}\,M^{\frac{2-\alpha}{2}}\,\|\nabla\varphi\|^{3\alpha}_{\mathrm{L}^2(\mathbb{R}^3)}\,. \qquad \square$$

One of the main ingredients in our analysis is the scaling properties of the terms involved in the functional E.

Lemma 2.3. *Let* $\varphi \in \mathrm{H}^1(\mathbb{R}^3)$. *Assume that* $\lambda > 0$, *let* p *and* q *be real numbers and define* $\varphi_\lambda^{p,q}(x) := \lambda^p\,\varphi(\lambda^q\,x)$. *Then we have*

$$\int_{\mathbb{R}^3} |\varphi_\lambda^{p,q}(x)|^2\,dx = \lambda^{2p-3q}\int_{\mathbb{R}^3}|\varphi(x)|^2\,dx\,,$$
$$\mathrm{E}[\varphi_\lambda^{p,q}] = \tfrac{1}{2}\,\lambda^{2p-q}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx + \tfrac{1}{4}\,\lambda^{4p-5q}\,\mathrm{D}[\varphi] - \tfrac{\lambda^{(2\alpha+2)p-3q}}{2\alpha+2}\,C\int_{\mathbb{R}^3}|\varphi|^{2\alpha+2}\,dx\,.$$

In the particular case $\varphi_\lambda(x) := \lambda^{\frac{3}{2}}\,\varphi(\lambda\,x)$, *the mass is preserved,*

$$\int_{\mathbb{R}^3}|\nabla\varphi_\lambda|^2\,dx = \lambda^2\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx\,,\quad \mathrm{D}[\varphi_\lambda] = \lambda\,\mathrm{D}[\varphi]\,,$$
$$\text{and}\quad \int_{\mathbb{R}^3}|\varphi_\lambda|^{2\alpha+2}\,dx = \lambda^{3\alpha}\int_{\mathbb{R}^3}|\varphi|^{2\alpha+2}\,dx\,.$$

As a consequence, we have that $M \mapsto I_M$ *is non increasing and*

$$I_M \leq 0 \quad \forall\, M \geq 0\,,$$

with $I_M = -\infty$ *when* $\alpha > \frac{2}{3}$, *for every* $M > 0$.

Proof. The reader is invited to check the changes of variables. Let φ be any function in Σ_M. Then, we have

$$I_M \leq \mathrm{E}[\varphi_\lambda] = \frac{\lambda^2}{2}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx + \frac{\lambda}{4}\,\mathrm{D}[\varphi] - \frac{\lambda^{3\alpha}\,C}{2\alpha+2}\int_{\mathbb{R}^3}|\varphi|^{2\alpha+2}\,dx$$

for all $\lambda > 0$, and one concludes by letting the scaling parameter λ go to zero that $I_M \leq 0$. As a consequence of (1.8), the function $M \mapsto I_M$ is non-increasing. The last claim follows by assuming that $\alpha > \frac{2}{3}$ and by letting λ go to infinity. $\square$

Remark 2.1. If $I_M = 0$ for some $M > 0$, we may built a minimizing sequence that converges to zero weakly in $\mathrm{H}^1(\mathbb{R}^3)$ by using the scaling properties. In fact, Lemma I.1 in [25] can be applied to any minimizing sequence in order to prove that vanishing cannot hold in the opposite case, $I_M < 0$. Therefore, the condition $I_M < 0$ is necessary to ensure the relative compactness up to translations of any minimizing sequence. This is the motivation for characterizing the situations in which E reaches negative values.

Lemma 2.4. *Let $M > 0$ and $\alpha \in [\frac{1}{3}, \frac{2}{3}]$. Then* E *takes negative values in Σ_M if and only if the functional*

$$\varphi \mapsto \left(\frac{1}{3\alpha-1}\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx\right)^{3\alpha-1}\left(\frac{\mathrm{D}[\varphi]}{2\,(2-3\alpha)}\right)^{2-3\alpha} - \frac{C}{\alpha+1}\int_{\mathbb{R}^3}|\varphi|^{2\alpha+2}\,dx$$

also takes negative values in Σ_M. Moreover, if $\alpha \in (\frac{1}{3}, \frac{2}{3})$, then

$$\mathrm{E}[\varphi] \geq \frac{1}{4}\,\lambda[\varphi]\,\mathrm{D}[\varphi]\left[1-\left(\frac{C\,M^{4\alpha-2}}{V_c(\alpha)}\right)^{\frac{1}{3-2\alpha}}\right] \qquad \forall\,\varphi\in\Sigma_M \tag{2.5}$$

with $\lambda[\varphi] := \left(\frac{3\alpha-1}{\alpha+1}\,C\,\frac{\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx}{\int_{\mathbb{R}^3}\varphi^{2\alpha+2}\,dx}\right)^{\frac{1}{2-3\alpha}}$ and $V_c(\alpha)$ given by (1.9).

Here we adopt the convention that $x^x = 1$ whenever $x = 0$, in order to include the endpoints of the interval.

Proof. Let $\varphi \in \Sigma_M$. Consider the family $\{\varphi_\lambda\}_{\lambda>0}$ associated with φ, such that $\|\varphi_\lambda\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = M$ for any $\lambda > 0$, as in Lemma 2.3. We are interested in the sign of

$$\frac{1}{\lambda}\,\mathrm{E}[\varphi_\lambda] = \frac{\lambda}{2}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx + \frac{1}{4}\,\mathrm{D}[\varphi] - \lambda^{3\alpha-1}\frac{C}{2\alpha+2}\int_{\mathbb{R}^3}|\varphi|^{2\alpha+2}\,dx\,.$$

In the case $\alpha = \frac{1}{3}$, both potential terms in the r.h.s. are scale invariant and we obviously have that E reaches negative values if and only if

$$\frac{1}{4}\,\mathrm{D}[\varphi] - \frac{3\,C}{8}\int_{\mathbb{R}^3}|\varphi|^{\frac{8}{3}}\,dx < 0\,.$$

If $\alpha \in (\frac{1}{3}, \frac{2}{3})$, the minimum of the r.h.s. with respect to λ is achieved by $\lambda = \lambda[\varphi]$ and it is negative when

$$-(2-3\alpha)\left(\frac{C}{2\alpha+2}\int_{\mathbb{R}^3}|\varphi|^{2\alpha+2}\,dx\right)^{\frac{1}{2-3\alpha}}\left(\frac{1}{2(3\alpha-1)}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx\right)^{\frac{1-3\alpha}{2-3\alpha}} + \frac{1}{4}\,\mathrm{D}[\varphi] < 0\,.$$

Inequality (2.5) is then a consequence of the definition of $V_c(\alpha)$. Finally, for $\alpha = \frac{2}{3}$ we have that

$$\mathrm{E}[\varphi_\lambda] = \lambda^2\left(\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx - \frac{3\,C}{10}\int_{\mathbb{R}^3}|\varphi|^{\frac{10}{3}}\,dx\right) + \lambda\,\frac{1}{4}\,\mathrm{D}[\varphi] \tag{2.6}$$

takes negative values if and only if the leading order coefficient w.r.t. λ,

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx - \frac{3\,C}{10}\int_{\mathbb{R}^3}|\varphi|^{\frac{10}{3}}\,dx\,,$$

is negative. We conclude the proof by observing that the three different conditions obtained above correspond to the precise statement of the lemma. □

Remark 2.2. In the case $\alpha = \frac{2}{3}$, the functional (2.6) is not bounded from below in Σ_M when the leading order coefficient w.r.t. λ takes negative values. This remark shows the optimality of the condition on the mass stated in Lemma 2.2 for $\alpha = \frac{2}{3}$.

In the range $\frac{1}{2} < \alpha < \frac{2}{3}$, we will need an additional estimate to handle the critical case corresponding to $C\,M^{4\alpha-2} = V_c(\alpha)$, that goes as follows.

Corollary 2.1. *Let $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$. Then, for any $\varphi \in \Sigma_M$,*

$$\|\varphi\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \leq \mathrm{C}_{1/2}^{2-2\alpha}\,\mathrm{C}_{\mathrm{GN}}(1)^{2\alpha-1}\,M^{\alpha-\frac{1}{2}}\,\|\nabla\varphi\|^{4\alpha-1}_{\mathrm{L}^2(\mathbb{R}^3)}\,\mathrm{D}[\varphi]^{1-\alpha}\,.$$

Proof. Let $\varphi \in \Sigma_M$. If $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$, then we have that $3 < 2\alpha + 2 < \frac{10}{3} < 4$. Using Hölder's inequality we get

$$\|\varphi\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \leq \|\varphi\|^{3(2-2\alpha)}_{\mathrm{L}^3(\mathbb{R}^3)}\,\|\varphi\|^{4(2\alpha-1)}_{\mathrm{L}^4(\mathbb{R}^3)}\,.$$

From (2.3) written for $\alpha = \frac{1}{2}$, we know that

$$\|\varphi\|^3_{\mathrm{L}^3(\mathbb{R}^3)} \leq \mathrm{C}_{1/2}\,\mathrm{D}[\varphi]^{\frac{1}{2}}\,\|\nabla\varphi\|_{\mathrm{L}^2(\mathbb{R}^3)}\,.$$

On the other hand, (2.1) with $\alpha = 1$ gives

$$\|\varphi\|^4_{\mathrm{L}^4(\mathbb{R}^3)} \leq \mathrm{C}_{\mathrm{GN}}(1)\,\|\nabla\varphi\|^3_{\mathrm{L}^2(\mathbb{R}^3)}\,M^{\frac{1}{2}}$$

Altogether, these estimates provide the result. □

We split the analysis of the strict negativity of I_M into two results, from which we will conclude that this property depends on α and in some cases also on the mass. Let us start with $\alpha < \frac{1}{2}$.

Proposition 2.1. *Let $M > 0$. If $\alpha \in [0, \frac{1}{2})$, then the functional* E *always reaches negative values in Σ_M. As a consequence, $I_M < 0$ for all $M > 0$ if $\alpha \in [0, \frac{1}{2})$.*

Proof. For $\alpha \in [0, \frac{1}{3})$ the result is a trivial consequence of the mass-preserving scaling in Lemma 2.3, since we have that

$$\lambda^{-3\alpha}\,\mathrm{E}[\varphi_\lambda] = \frac{1}{2}\,\lambda^{2-3\alpha}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx + \frac{1}{4}\,\lambda^{1-3\alpha}\,\mathrm{D}[\varphi] - \frac{C}{2\alpha+2}\int_{\mathbb{R}^3}|\varphi|^{2\alpha+2}\,dx$$

is negative for any non-trivial $\varphi \in \mathrm{H}^1(\mathbb{R}^3)$ if $\lambda > 0$ is chosen small enough.

To complete the proof for $\alpha \in [\frac{1}{3}, \frac{1}{2})$, it remains to find a particular test function $\varphi \in \Sigma_M$ with negative energy for any $M > 0$. We follow a classical approach in the literature on the concentration–compactness method, see for instance [27]. Consider $M > 0$ and $\eta \in \Sigma_M$ such that $\mathrm{supp}(\eta) \subset B(0,1)$, where $B(0,1)$ denotes the unit sphere centered at 0. For any positive integer n, define $\eta_n(x) := \eta(n^{\frac{1}{3}}x)$. Then the support of η_n is contained in $B(0,1)$ and by direct calculations we have

$$\|\eta_n\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = \tfrac{1}{n}\,\|\eta\|^2_{\mathrm{L}^2(\mathbb{R}^3)}\,, \quad \mathrm{D}[\eta_n] = \tfrac{1}{n^{5/3}}\,\mathrm{D}[\eta]\,,$$
$$\|\eta_n\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} = \tfrac{1}{n}\,\|\eta\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)}\,, \quad \textstyle\int_{\mathbb{R}^3}|\nabla\eta_n|^2\,dx = \frac{1}{n^{1/3}}\int_{\mathbb{R}^3}|\nabla\eta|^2\,dx\,.$$

Let n be a given integer bigger than 1 and let us consider the test function $\varphi(x) := \sum_{i=1}^n \eta_n(x - x_i)$, where the points $x_i \in \mathbb{R}^3$, $i = 1, \ldots n$ are chosen such that

$$|x_i - x_j| \geq \frac{M^2}{\mathrm{D}[\eta]}\,n^{2/3} + 2 \quad \forall\, i \neq j\,.$$

By definition φ verifies $\|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = \|\eta\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = M$, $\|\varphi\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} = \|\eta\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)}$ and $\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx = n^{2/3}\int_{\mathbb{R}^3} |\nabla\eta|^2\,dx$. Now, we estimate $\mathrm{D}[\varphi]$ as follows:

$$\begin{aligned}
\mathrm{D}[\varphi] &= \sum_{i,\,j=1}^{n} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |\eta_n(x-x_i)|^2\,|\eta_n(x'-x_j)|^2 \frac{dx\,dx'}{|x-x'|} \\
&= n\,\mathrm{D}[\eta_n] + \sum_{j\neq i} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\eta_n(x)|^2\,|\eta_n(x')|^2}{|x+x_i-x'-x_j|}\,dx\,dx' \\
&\leq \frac{\mathrm{D}[\eta]}{n^{2/3}} + \sum_{j\neq i} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\eta_n(x)|^2\,|\eta_n(x')|^2}{|x_i-x_j|-2}\,dx\,dx' \\
&\leq \frac{\mathrm{D}[\eta]}{n^{2/3}} + \frac{\mathrm{D}[\eta]}{M^2\,n^{2/3}}\,\frac{M^2\,n(n-1)}{2\,n^2} = \frac{2\,\mathrm{D}[\eta]}{n^{2/3}}\,.
\end{aligned}$$

Combining these estimates and Lemma 2.4 with the fact that $(3\alpha-1)-(2-3\alpha)<0$ if $\alpha<\frac{1}{2}$, we are done with the proof. □

If $\alpha\in(\frac{1}{2},\frac{2}{3}]$, the functional E might not reach negative values depending on the value of the mass M and the constant C, as stated in the following result.

Proposition 2.2. *In the case $\alpha\in[\frac{1}{2},\frac{2}{3}]$, $I_M=0$ if and only if*

$$C\,M^{4\alpha-2} \leq V_c(\alpha) \tag{2.7}$$

holds, where the constant $V_c(\alpha)$ is given in (1.9)*. On the contrary, if* (2.7) *does not hold, then I_M is negative.*

We recall that $V_c(\alpha) = \frac{\alpha+1}{\mathrm{C}_\alpha}\left(\frac{1}{3\alpha-1}\right)^{3\alpha-1}\left(\frac{1}{2\,(2-3\alpha)}\right)^{2-3\alpha}$ where C_α is the optimal constant in (2.3).

Proof. According to Lemma 2.4, $I_M=0$ for $\alpha\in[\frac{1}{2},\frac{2}{3}]$ if and only if

$$\|\varphi\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \leq \frac{\alpha+1}{C}\left(\frac{\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^3)}}{3\alpha-1}\right)^{3\alpha-1}\left(\frac{\mathrm{D}[\varphi]}{2(2-3\alpha)}\right)^{2-3\alpha}$$

for all $\varphi\in\Sigma_M$. Comparing with the definition of C_α in (2.3), this clearly entails that $I_M=0$ if and only if (2.7) holds. According to Lemma 2.3, I_M is negative (and eventually $-\infty$) otherwise. □

Although our problem is originally set in the framework of complex valued functions, we finally observe that we can reduce it to non-negative real valued functions.

Lemma 2.5. *Consider a complex valued minimizer ψ to the problem* (1.4)*. Then, the real function $|\psi|$ is also a minimizer for* (1.4)*.*

Proof. It is well known that if $\psi \in \Sigma_M$, then $|\psi|$ also belongs to Σ_M. Since the potential energy only depends on $|\psi|^2$, it takes the same value on ψ and $|\psi|$. On the other hand, the kinetic enegy verifies

$$\int_{\mathbb{R}^3} \big|\nabla|\psi|\,\big|^2\,dx \le \int_{\mathbb{R}^3} \Big(\nabla|\mathrm{Re}\,\psi|^2 + \nabla|\mathrm{Im}\,\psi|^2\Big)\,dx = \int_{\mathbb{R}^3} |\nabla\psi|^2\,dx$$

as a consequence of the convexity inequality for gradients [21], where equality holds if and only if $|\mathrm{Re}\,\psi(x)| = c\,|\mathrm{Im}\,\psi(x)|$ for some constant c. Hence, $|\psi|$ is also a minimizer. □

If I_M is achieved, we can then prove the *Virial Theorem* relation for the terms of the energy functional by using their scaling properties.

Proposition 2.3. *Assume that* $0 < \alpha < \frac{2}{3}$. *Any minimizer* φ_M *of* I_M *satisfies*

$$\int_{\mathbb{R}^3} |\nabla\varphi_M|^2\,dx + \frac{1}{4}\,\mathrm{D}[\varphi_M] - \frac{3\,\alpha\,C}{2\alpha+2}\int_{\mathbb{R}^3} |\varphi_M|^{2\alpha+2}\,dx = 0\,. \tag{2.8}$$

Proof. Let us assume that there exists a minimizer $\varphi_M \in \Sigma_M$ of I_M. According to Lemma 2.3, for every $\lambda > 0$ the rescaled function $\varphi_{M,\lambda} = \lambda^{3/2}\,\varphi_M(\lambda\,\cdot)$ also lies in Σ_M. The function $\lambda \mapsto \mathrm{E}[\varphi_{M,\lambda}]$ attains its minimal value at $\lambda = 1$. Since

$$\mathrm{E}[\varphi_{M,\lambda}] = \frac{1}{2}\,\lambda^2 \int_{\mathbb{R}^3} |\nabla\varphi_M|^2\,dx + \lambda\,\frac{1}{4}\,\mathrm{D}[\varphi_M] - \lambda^{3\alpha}\,\frac{C}{2\alpha+2}\int_{\mathbb{R}^3} |\varphi_M|^{2\alpha+2}\,dx\,,$$

the cancellation of the derivative with respect to λ at $\lambda = 1$ provides with (2.8). □

At this stage, we can write down the Euler-Lagrange equation corresponding to the minimization problem I_M and deduce an energy identity.

Lemma 2.6. *Assume that* $0 < \alpha < \frac{2}{3}$. *Any minimizer* φ_M *of* I_M *satisfies* (1.3) *and*

$$\int_{\mathbb{R}^3} |\nabla\varphi_M|^2\,dx + \mathrm{D}[\varphi_M] - C\int_{\mathbb{R}^3} |\varphi_M|^{2\alpha+2}\,dx + \ell_M\,M = 0\,. \tag{2.9}$$

In particular, at least for $\alpha \in (0, \frac{1}{5}] \cup (\frac{1}{2}, \frac{2}{3})$, *we have* $\ell_M > 0$. *If* $\alpha = \frac{1}{2}$, *then* $\ell_M = -\frac{6}{M}\,I_M \ge 0$.

Proof. Identity (2.9) is obtained by multiplying the Euler-Lagrange equation (1.3) by φ_M and integrating by parts. If we eliminate $\mathrm{D}[\varphi_M]$ and $\|\varphi_M\|_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)}$ from (2.8), (2.9) and use

$$E[\varphi_M] = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi_M|^2\,dx + \frac{1}{4}\mathrm{D}[\varphi_M] - \frac{C}{2\alpha+2}\int_{\mathbb{R}^3} |\varphi_M|^{2\alpha+2}\,dx = -|I_M|\,,$$

we complete the proof using

$$\ell_M = \frac{2}{M}\left(\frac{2\alpha-1}{3\alpha-1}\int_{\mathbb{R}^3} |\nabla\varphi_M|^2\,dx + \frac{5\alpha-1}{3\alpha-1}\,|I_M|\right)\,.$$

□

Corollary 2.2. *Assume that* $\alpha \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{2}{3})$. *Any minimizer* φ_M *of* I_M *is such that*

$$\int_{\mathbb{R}^3} |\nabla \varphi_M|^2 \, dx = \frac{1}{2} (3\alpha - 1) \, \varepsilon_M - (5\alpha - 1) \, \eta_M$$
$$\mathrm{D}[\varphi_M] = (2 - 3\alpha) \, \varepsilon_M - 2 \, (2 - \alpha) \, \eta_M$$
$$\int_{\mathbb{R}^3} |\varphi_M|^{2\alpha+2} \, dx = \frac{1}{4} \, \varepsilon_M - \frac{3}{2} \, \eta_M$$

where

$$\varepsilon_M := \frac{M \, \ell_M}{2\alpha - 1} \quad \textit{and} \quad \eta_M := \frac{I_M}{1 - 2\alpha} \, .$$

Proof. The proof is a straightforward consequence of $\mathrm{E}[\varphi_M] = I_M$, (2.8) and (2.9). □

Lemma 2.6 has interesting consequences concerning the decay of the minimizers, that can be derived from Lemma 19 and Theorem 6 in [7], as shown in the following result. Also see Theorem 1.3 in [2] and Theorem 6.1 in [17] for related results.

Lemma 2.7. *Consider a nonnegative solution to* (1.3) *such that*

$$\frac{1}{2} \int_{\mathbb{R}^3} |\nabla \varphi_M|^2 \, dx + \frac{1}{4} \, \mathrm{D}[\varphi_M] + \ell_M \int_{\mathbb{R}^3} \varphi_M^2 \, dx < \infty \, , \ \textit{with } \ell_M \geq 0 \, .$$

Then, there exist positive constants K *and* δ *such that*

$$\varphi_M(x) \leq K \, e^{-\delta \sqrt{1+|x|}} \quad \forall \, x \in \mathbb{R}^3 \, .$$

In the case $\ell_M = 0$, this result ensures that the above solution belongs to $\mathrm{H}^1(\mathbb{R}^3)$, since the exponential decay also guarantees that the minimizer is in $\mathrm{L}^2(\mathbb{R}^3)$.

The rescaled problem. Given that our main tool in proving the existence of minimizers will consist in checking the strict inequalities (1.7), we are going to study the infimum value I_M as a function of the mass M. To this purpose, we fix a function $\varphi_1 \in \Sigma_1$ and apply the scaling properties in Lemma 2.3 with $2p - 3q = 1$ and $\lambda = M$. We denote by $\varphi_{M,p}$ be the corresponding rescaled function. Then, according to Lemma 2.3 we have that $\varphi_{M,p} \in \Sigma_M$ and

$$\mathrm{E}[\varphi_{M,p}] = \tfrac{1}{2} \, M^{\frac{4p+1}{3}} \, \|\nabla \varphi_1\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \tfrac{1}{4} \, M^{\frac{2p+5}{3}} \, \mathrm{D}[\varphi_1] - \tfrac{C}{2\alpha+2} \, M^{2\alpha p+1} \, \|\varphi_1\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} \, , \tag{2.10}$$

for any real number p.

3. Existence and non-existence of steady states

In this section we analyze the existence of minimizers for the variational problem (1.4).

3.1. *Non-existence results when $\alpha = 0$ or $\alpha = 2/3$*

In the case $\alpha = 0$, the minimization problem reduces to

$$\begin{aligned} I_M &= \inf\left\{\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx + \frac{1}{4}\,\mathrm{D}[\varphi] - \frac{C}{2}\int_{\mathbb{R}^3} |\varphi|^2\,dx \,:\, \varphi\in\Sigma_M\right\} \\ &= \inf\left\{\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx + \frac{1}{4}\,\mathrm{D}[\varphi] \,:\, \varphi\in\Sigma_M\right\} - \frac{C}{2}\,M = -\frac{C}{2}\,M\,, \end{aligned}$$

by a scaling argument. Therefore, I_M is never achieved when $M > 0$ (despite it is always negative) since any possible minimizer would make the gradient term vanish, and then should vanish itself in $\mathbb{R}^3$.

In the case $\alpha = \frac{2}{3}$, either $I_M = 0$ or $I_M = -\infty$, and in both cases there are no minimizers. Actually, $I_M = 0$ if and only if

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx - \frac{3\,C}{10}\int_{\mathbb{R}^3} |\varphi|^{\frac{10}{3}}\,dx \geq 0\,.$$

See Lemma 2.4 and its proof for details. Hence, the minimum cannot be attained, otherwise $\mathrm{D}[\varphi] = 0$, which is absurd.

From now on we shall assume that $0 < \alpha < \frac{2}{3}$. We first examine the range $0 < \alpha < \frac{1}{2}$ in Subsection 3.2. Subsection 3.3 is devoted to the special limiting case $\alpha = \frac{1}{2}$. Finally, the range $\frac{1}{2} < \alpha < \frac{2}{3}$ is analyzed in Subsection 3.4.

3.2. *The interval $0 < \alpha < \frac{1}{2}$*

We prove the following :

Proposition 3.1. *Let $0 < \alpha < \frac{1}{2}$. Then, for $M > 0$ small enough, the strict inequalities*

$$I_M < I_{M'} + I_{M-M'}$$

hold for every M' such that $0 < M' < M$. In particular, all minimizing sequences are compact in $\mathrm{H}^1(\mathbb{R}^3)$ up to translations and the extraction of a subsequence. Therefore, I_M is attained for M small enough.

Proof. In Proposition 2.1 we have proved that $I_M < 0$ for every $M > 0$. In the range $\alpha \in (0, \frac{1}{2})$, we may choose the parameter p in the rescaled problem (2.10) such that

$$0 \leq \frac{4p+1}{3} = 2\alpha p + 1 < \frac{2p+5}{3}\,,$$

i.e., such that the gradient and the power term are of the same order for small M and dominate the Poisson energy in this regime. With this choice we can deduce

$$I_M = M^{\frac{2-\alpha}{2-3\alpha}}\,J_1^M \tag{3.1}$$

where, for every $\mu > 0$,

$$J_\mu^M = \inf\left\{\frac{1}{2}\,\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^3)} - \frac{C}{2\alpha+2}\,\|\varphi\|^{2\alpha+2}_{\mathrm{L}^{2\alpha+2}(\mathbb{R}^3)} + M^{\frac{2(1-2\alpha)}{2-3\alpha}}\,\frac{1}{4}\,\mathrm{D}[\varphi] \,:\, \varphi\in\Sigma_\mu\right\}.$$

Note that the same scaling argument shows that

$$J_\mu^M = \mu^{\frac{2-\alpha}{2-3\alpha}}\,J_1^{\mu M}\,. \tag{3.2}$$

With $\mu = \frac{M'}{M}$ and using (3.1) and (3.2), it is easily proved that the strict inequalities of Proposition 3.1 are equivalent to

$$J_1^M < J_\mu^M + J_{1-\mu}^M \quad \forall\,\mu\in(0,1)\,. \tag{3.3}$$

We are going to prove that the above strict inequalities hold for M small enough.

Observe now that $M^{\frac{2(1-2\alpha)}{2-3\alpha}}$ goes to zero as M does, and $\lim_{M\to 0} J_1^M = J_1^0$. The key point is that for every $\lambda > 0$, J_λ^0 satisfies the strict inequalities of the concentration–compactness principle, namely

$$J_\lambda^0 < J_\mu^0 + J_{\lambda-\mu}^0\,, \quad \forall\,\mu\in(0,\lambda)\,. \tag{3.4}$$

This is an immediate consequence of the fact that $J_\lambda^0 = \lambda^{\frac{2-\alpha}{2-3\alpha}}\,J_1^0$, with $J_1^0 < 0$ and $\frac{2-\alpha}{2-3\alpha} > 1$. The sign of J_1^0 is deduced from the scaling argument in Lemma 2.3, by observing that the negative term dominates the gradient contributions for $3\alpha < 2$. We now prove that J_1^M satisfies the strict inequalities (3.3) for M small enough. We argue by contradiction assuming that this is not the case. Then, there exist a sequence $\{M_n\}_{n\geq 1}$ going to 0 and a sequence $\{\lambda_n\}_{n\geq 1}$ in $(0,1)$ such that

$$J_1^{M_n} = J_{\lambda_n}^{M_n} + J_{1-\lambda_n}^{M_n}\,. \tag{3.5}$$

Assume that $\frac{1}{2} \leq \lambda_n < 1$ (if $\lambda_n \in (0,\frac{1}{2})$, we may exchange the roles of λ_n and $1-\lambda_n$). By continuity with respect to M we conclude that $\lambda_n \to 1$, otherwise we get a contradiction with (3.4). In addition, we may choose as λ_n the infimum of the set $\{\lambda\in[\frac{1}{2},1) \,:\, J_1^{M_n} = J_\lambda^{M_n} + J_{1-\lambda}^{M_n}\}$.

We now claim that, for n large enough, $J_{\lambda_n}^{M_n}$ satisfies the strict inequalities of the concentration–compactness principle

$$J_{\lambda_n}^{M_n} < J_\mu^{M_n} + J_{\lambda_n-\mu}^{M_n} \quad \forall\,\mu\in(0,\lambda_n)\,. \tag{3.6}$$

If not, there exists a sequence $\{\mu_n\}_{n\geq 1}$ with $\mu_n \in (\frac{1}{2}\,\lambda_n, \lambda_n)$ such that

$$J_{\lambda_n}^{M_n} = J_{\mu_n}^{M_n} + J_{\lambda_n-\mu_n}^{M_n}\,. \tag{3.7}$$

Then, from (3.5) and (3.7) we find

$$\begin{aligned} J_{\mu_n}^{M_n} + J_{1-\mu_n}^{M_n} \geq J_1^{M_n} &= J_{\mu_n}^{M_n} + J_{\lambda_n-\mu_n}^{M_n} + J_{1-\lambda_n}^{M_n} \\ &\geq J_{\mu_n}^{M_n} + J_{1-\mu_n}^{M_n}\,, \end{aligned}$$

for the reverse large inequalities $J_{1-\mu_n}^{M_n} \leq J_{1-\lambda_n}^{M_n} + J_{1-\mu_n-(1-\lambda_n)}^{M_n}$ always hold true. Hence, the equality

$$J_{\mu_n}^{M_n} + J_{1-\mu_n}^{M_n} = J_1^{M_n} \tag{3.8}$$

is verified. By definition of λ_n and since $\mu_n \geq \frac{\lambda_n}{2}$ with $\lambda_n \geq \frac{1}{2}$, we must have $\frac{1}{4} \leq \mu_n < \frac{1}{2}$. Extracting a subsequence if necessary, we may assume that μ_n converges to μ with $\frac{1}{4} \leq \mu \leq \frac{1}{2}$. Passing to the limit in (3.8), we get $J^0_\mu + J^0_{1-\mu} = J^0_1$ with $\mu \in (0,1)$ thereby reaching a contradiction with (3.4). So far we have proved that the strict inequalities (3.6) hold.

In particular, for n large enough, there exists a minimizer φ_n of $J^{M_n}_{\lambda_n}$ such that $\{(\lambda_n)^{-1/2}\varphi_n\}_{n\geq 1}$ is a minimizing sequence for J^0_1. Since (3.4) holds, this sequence converges strongly in $\mathrm{H}^1(\mathbb{R}^3)$ up to translations to a minimizer φ_∞ of J^0_1. The same holds for $\{\varphi_n\}_{n\geq 1}$, given that $\lambda_n \to 1$. Without loss of generality we may assume that φ_n and $\varphi_\infty > 0$ satisfy the respective Euler–Lagrange equations in $\mathbb{R}^3$

$$-\Delta\varphi_n - C\,{\varphi_n}^{2\alpha+1} + M_n^{\frac{2(1-2\alpha)}{2-3\alpha}} \Big(\varphi_n^2 * \frac{1}{|x|}\Big)\varphi_n + \theta_n\,\varphi_n = 0\,,$$

with $\|\varphi_n\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = \lambda_n$ and

$$-\Delta\varphi_\infty - C\,\varphi_\infty^{2\alpha+1} + \theta_1\,\varphi_\infty = 0\,,$$

with $\|\varphi_\infty\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = 1$ and $\theta_1 > 0$. Having in mind to contradict (3.5) we argue as follows. We first write

$$\frac{J^{M_n}_1 - J^{M_n}_{\lambda_n}}{1-\lambda_n} = \frac{J^{M_n}_{1-\lambda_n}}{1-\lambda_n}\,.$$

As λ_n goes to 1, the left-hand side can be bounded from above by $-\theta_1$, while from (3.2) the quotient

$$\frac{J^{M_n}_{1-\lambda_n}}{1-\lambda_n} = (1-\lambda_n)^{\frac{2\alpha}{2-3\alpha}}\, J_1^{(1-\lambda_n)M_n}$$

converges to 0 because $J_1^{(1-\lambda_n)M_n}$ converges to J^0_1 as $\lambda_n \to 1$, and $\frac{2\alpha}{2-3\alpha}$ is positive. □

Remark 3.1. The general case for any M is still an open problem. The possibility of dichotomy is the delicate case to be analyzed since vanishing is easily ruled out by the fact that I_M is negative.

3.3. *The limiting case* $\alpha = \frac{1}{2}$

Our main result is the following.

Proposition 3.2. *Let* $\mathrm{C}_{1/2}$ *be the best constant in* (2.3) *with* $\alpha = \frac{1}{2}$.

(i) If $\frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}} > C$, *then* $I_M = 0$ *and* I_M *is not achieved for any* $M > 0$.

(ii) If $\frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}} < C$, *then* $I_M < 0$ *and* I_M *is achieved for every* $M > 0$. *In addition, all minimizing sequences are relatively compact in* $\mathrm{H}^1(\mathbb{R}^3)$ *up to translations.*

Remark 3.2. The remaining case $\frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}} = C$, where $I_1 = 0$, might be attained if and only if P.-L. Lions' inequality (2.4) has an optimal function in Σ_1. This is, for the moment, an open question.

Proof. As an immediate consequence of the scaling formulae of Lemma 2.3, by taking $p = 2$ in (2.10), we have that

$$I_M = M^3\, I_1\,, \tag{3.9}$$

for every $M > 0$, and I_M is achieved if and only if I_1 is also achieved. This is the only case in which all powers of M appearing in the right-hand side of (2.10) are identical. When $I_1 < 0$, it is a well-known fact [23,24] that the relation (3.9) implies the strict inequalities (1.7), hence the result. Indeed, the strict inequalities (1.7) hold as a consequence of the convexity of $M \mapsto M^3$.

Assume now that $I_1 = 0$, so that $I_M = 0$ for every $M > 0$. We assume that I_1 is achieved by some function φ_1 in $\mathrm{H}^1(\mathbb{R}^3)$. Then φ_1 satisfies the Euler–Lagrange equation (1.3) with a zero Lagrange multiplier (since it is also a minimizer without any constraint on the $\mathrm{L}^2(\mathbb{R}^3)$ norm). Also see Lemma 2.6 for a direct proof. If we apply the corresponding equation to φ_1, integrate over $\mathbb{R}^3$ and use the information $I_1 = \mathrm{E}[\varphi_1] = 0$, we deduce

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi_1|^2\,dx = \frac{1}{4}\,\mathrm{D}[\varphi_1] = \frac{C}{6}\int_{\mathbb{R}^3} |\varphi_1|^3\,dx\,.$$

Hence, by definition of $\mathrm{C}_{1/2}$ we obtain

$$\frac{1}{\mathrm{C}_{1/2}} \le \frac{C\sqrt{2}}{3}\,,$$

or equivalently

$$\frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}} \le C\,.$$

Therefore, using (2.7), the equality $I_1 = 0$ can be achieved only when

$$\frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}} = C\,.$$

As a consequence, I_1 (and, up to a scaling, I_M) is attained if and only if the optimal constant in (2.4) is attained by a minimizer in $\mathrm{L}^2(\mathbb{R}^3)$. □

We conclude this section by examining the critical case $\alpha = \frac{1}{2}$ with the limiting constant $C = \frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}}$. The problem is open, but we can prove that lack of compactness may occur only by vanishing as shown by the following result.

Proposition 3.3. *Assume that $\alpha = \frac{1}{2}$ and $C = \frac{3}{\sqrt{2}\,\mathrm{C}_{1/2}}$. Let $\{\phi_n\}_{n\ge1}$ be a minimizing sequence for I_1. If* vanishing does not occur, *that is, if* (1.6) *holds, then there exists a minimizer for I_1.*

By Lemma 2.2, $\{\phi_n\}_{n\geq 1}$ is bounded in $\mathrm{H}^1(\mathbb{R}^3)$. Since I_1 is invariant by translation, relative compactness in $\mathrm{H}^1(\mathbb{R}^3)$ may only be expected up to translations. Also, since $I_\lambda = 0$ for every $\lambda > 0$, concentration–compactness type inequalities turn into equalities. In particular, there exist minimizing sequences that are not relatively compact in $\mathrm{H}^1(\mathbb{R}^3)$, up to any translations. According to the concentration–compactness terminology [23,24,25], either $\{\phi_n\}_{n\geq 1}$ fulfills (1.5) and *vanishing* occurs, or (1.6) holds. If there exists some minimizing sequence for which vanishing does not occur, we will now prove that existence of a minimizer is guaranteed.

Proof. We first show that (1.6) ensures the existence of a minimizer. Indeed, the new minimizing sequence $\{\phi_n(\cdot + y_n)\}_{n\geq 1}$ converges (up to a subsequence) to a function ϕ in $\mathrm{H}^1(\mathbb{R}^3)$, weakly in $\mathrm{H}^1(\mathbb{R}^3)$ and in $\mathrm{L}^p(\mathbb{R}^3)$ for every $2 \leq p \leq 6$, strongly in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^3)$ for every $1 \leq p < 6$ (by the Rellich-Kondrachov theorem); consequently, it also converges almost everywhere in $\mathbb{R}^3$. The condition (1.6) guarantees that $\phi \neq 0$ since $\int_{B_{R_0}} \phi^2\, dx \geq \varepsilon_0$ by passing to the limit as n goes to infinity. Let $\mu = \int_{\mathbb{R}^3} \phi^2\, dx$ with $0 < \mu \leq 1$.

If $\mu = 1$, we are done : $\{\phi_n(\cdot + y_n)\}_{n\geq 1}$ converges to ϕ strongly in $\mathrm{L}^2(\mathbb{R}^3)$, and therefore in $\mathrm{L}^p(\mathbb{R}^3)$ for every $2 \leq p < 6$ by Hölder's inequality. In particular, the convergence is also strong in $\mathrm{L}^3(\mathbb{R}^3)$ and $0 = \liminf_{n\to+\infty} \mathrm{E}[\phi_n] \geq \mathrm{E}[\phi)] \geq I_1$. Hence, $\mathrm{E}[\phi] = 0$ and ϕ is a minimizer of I_1. In addition, the convergence is strong in $\mathrm{H}^1(\mathbb{R}^3)$ since all above inequalities turn into equalities.

If $\mu < 1$, we are in the so-called dichotomy case. We shall prove that ϕ is a minimizer of I_μ. Then, according to Lemma 3.2, I_1 is also achieved. Let us define $r_n := \phi_n(\cdot + y_n) - \phi$. Then, $\{r_n\}_{n\geq 1}$ is bounded in $\mathrm{H}^1(\mathbb{R}^3)$. Up to a subsequence, it converges to 0 weakly in $\mathrm{H}^1(\mathbb{R}^3)$ and in $\mathrm{L}^p(\mathbb{R}^3)$ for every $2 \leq p < 6$, strongly in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^3)$ for every $1 \leq p < 6$, and almost everywhere in $\mathbb{R}^3$. In addition, by taking weak limits we find

$$\begin{aligned} &\lim_{n\to+\infty} \int_{\mathbb{R}^3} r_n^2\, dx = 1 - \mu\,, \\ &\int_{\mathbb{R}^3} |\nabla \phi_n|^2\, dx = \int_{\mathbb{R}^3} |\nabla r_n|^2\, dx + \int_{\mathbb{R}^3} |\nabla \phi|^2\, dx + o_n(1)\,, \end{aligned} \tag{3.10}$$

where $o_n(1)$ is a shorthand for a quantity that goes to 0 when n goes to infinity. Using Theorem 1 in [8], we have

$$\int_{\mathbb{R}^3} |\phi_n|^3\, dx = \int_{\mathbb{R}^3} |r_n|^3\, dx + \int_{\mathbb{R}^3} |\phi|^3\, dx + o_n(1)\,. \tag{3.11}$$

We first check as in [9] that

$$\lim_{n\to+\infty} \|\phi\, r_n\|_{\mathrm{L}^p(\mathbb{R}^3)} = 0 \quad \forall\, p \in [1,3)\,. \tag{3.12}$$

We just argue for $p = 1$, as the analysis for the other powers follows by interpolation. Since $\{r_n\}_{n\geq 1}$ converges strongly to 0 in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\{\phi\, r_n\}_{n\geq 1}$ converges strongly to 0

in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3)$ as $n \to \infty$. Next, for every $R > 0$ we have

$$\int_{|x|\ge R} |\phi\, r_n|\, dx \le \Big(\int_{|x|\ge R} |\phi|^2\, dx\Big)^{1/2} \Big(\int_{|x|\ge R} |r_n|^2\, dx\Big)^{1/2}.$$

The first term in the right-hand side may be taken arbitrarily small for R large enough since $\phi \in \mathrm{L}^2(\mathbb{R}^3)$, while the second one is bounded independently of n and R since $\{r_n\}_{n\ge1}$ is bounded in $\mathrm{L}^2(\mathbb{R}^3)$. Writing $\int_{\mathbb{R}^3} |\phi\, r_n|\, dx = \int_{|x|\le R} |\phi\, r_n|\, dx + \int_{|x|\ge R} |\phi\, r_n|\, dx$ we get the result. By writing down

$$\int_{\mathbb{R}^3} |\phi_n|^3\, dx - \int_{\mathbb{R}^3} |r_n|^3\, dx - \int_{\mathbb{R}^3} |\phi|^3\, dx = 3 \int_{\mathbb{R}^3} |\phi\, r_n|\, (|r_n| + |\phi|)\, dx\,,$$

we obtain (3.11) since $\{|r_n|+|\phi|\}_{n\ge1}$ is bounded in $\mathrm{L}^2(\mathbb{R}^3)$ and $\{\phi\, r_n\}_{n\ge1}$ converges to 0 in $\mathrm{L}^2(\mathbb{R}^3)$. Finally, we check that

$$\liminf_{n\to+\infty} \mathrm{D}[\phi_n] \ge \mathrm{D}[\phi] + \liminf_{n\to+\infty} \mathrm{D}[r_n]\,. \tag{3.13}$$

On the one hand, since $\{\phi_n\}_{n\ge1}$ is bounded in $\mathrm{H}^1(\mathbb{R}^3)$, $\{\phi_n^2 \star \frac{1}{|x|}\}_{n\ge1}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^3)$ thanks to

$$\begin{aligned}\Big\|\phi^2 \star \frac{1}{|x|}\Big\|_{\mathrm{L}^\infty(\mathbb{R}^3)} = \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x-y|}\, dy &\le \sup_{x\in\mathbb{R}^3} \Big(\int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x-y|^2}\, dy\Big)^{1/2} \|\phi\|_{\mathrm{L}^2(\mathbb{R}^3)} \\ &\le 2\, \|\nabla\phi\|_{\mathrm{L}^2(\mathbb{R}^3)}\, \|\phi\|_{\mathrm{L}^2(\mathbb{R}^3)}\,,\end{aligned}$$

where we have used Cauchy-Schwarz' inequality and Hardy's inequality. Then, we have

$$\Big|\int_{\mathbb{R}^3} \Big(\phi_n \star \frac{1}{|x|}\Big)\, (\phi\, r_n)\, dx\Big| \le \Big\|\phi_n^2 \star \frac{1}{|x|}\Big\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \|\phi\, r_n\|_{\mathrm{L}^1(\mathbb{R}^3)}\,,$$

and hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \Big(\phi_n \star \frac{1}{|x|}\Big)\, (\phi\, r_n)\, dx = 0\,,$$

because of (3.12). On the other hand, $\int_{\mathbb{R}^3} \big((\phi\, r_n) \star \frac{1}{|x|}\big)\, (\phi\, r_n)\, dx \ge 0$. Actually it is also converging to 0 as $n \to \infty$, and (3.13) follows. Gathering together (3.10), (3.11) and (3.13), we obtain

$$\begin{aligned}0 = \limsup_{n\to+\infty} \mathrm{E}[\phi_n] = \liminf_{n\to+\infty} \mathrm{E}[\phi_n] &= \mathrm{E}[\phi] + \liminf_{n\to+\infty} \mathrm{E}[r_n] \\ &\ge \mathrm{E}[\phi] \ge I_\mu = 0\end{aligned}$$

since $\mathrm{E}[r_n]$ is nonnegative for every $n \ge 1$. Then, all above inequalities turn into equalities. In particular, $I_\mu = \mathrm{E}[\phi] = 0$ is attained. By Lemma 3.2, I_1 also is attained. This concludes the proof of Proposition 3.3. □

3.4. *The region* $\frac{1}{2} < \alpha < \frac{2}{3}$

We recall from Proposition 2.2 the existence of a critical value V_c such that $I_M = 0$ if and only if $C\, M^{4\alpha-2} \le V_c$, and $I_M < 0$ otherwise. Let us define

$$M_c := \left(\frac{V_c}{C}\right)^{\frac{1}{4\alpha-2}}$$

and notice that $4\alpha - 2$ is positive if $\frac{1}{2} < \alpha < \frac{2}{3}$. The main result in this region is stated in the following proposition.

Proposition 3.4. *Assume that $\alpha \in (\frac{1}{2}, \frac{2}{3})$. The following assertions hold true:*

(i) If $M < M_c$, then I_M is not achieved.
(ii) If $M = M_c$, then there exists a minimizer.
(iii) If $M > M_c$, then the strict inequalities (1.7) *always hold, and in particular there exists a minimizer.*

In the critical case $M = M_c$, the strict inequalities (1.7) do not hold. As consequence, the stability of such a solution cannot be ensured by usual arguments.

Proof. We first assume that $M < M_c$, so that $I_M = 0$ by Proposition 2.2. Define

$$\mathrm{E}_M[\varphi] := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi|^2\, dx + \frac{1}{4}\,\mathrm{D}[\varphi] - M^{4\alpha-2}\,\frac{C}{2\alpha+2}\int_{\mathbb{R}^3} |\varphi|^{2\alpha+2}\, dx\,.$$

We may observe that $\mathrm{E}_1 = \mathrm{E}$. By applying Lemma 2.3 with $p = 2$ and $q = 1$ (or, equivalently, (2.10) with $p = 2$), we get

$$\mathrm{E}\big[M^2\,\varphi(M\cdot)\big] = M^3\,\mathrm{E}_M[\varphi] \quad \forall\,\varphi \in \Sigma_1\,.$$

We argue by contradiction. Assume that I_M is achieved. Then, there exists a minimizer φ_M of

$$\inf\{\mathrm{E}_M[\varphi] \,:\, \varphi \in \Sigma_1\} = M^{-3}\, I_M\,.$$

In this way, φ_M is a test function for $M_c^{-3}\, I_{M_c}$ and $\mathrm{E}_{M_c}[\varphi_M] < \mathrm{E}_M[\varphi_M] = 0$ since $M_c > M$. We contradict the fact that $I_{M_c} = 0$, thus proving (i).

Next, we assume that $M > M_c$. In order to prove the strict inequalities (1.7) and establish (iii) in Proposition 3.4, the key point is the following.

Lemma 3.1. *If $M > M_c$, then we have*

$$I_{M'} \le \left(\frac{M'}{M}\right)^3 I_M \quad \forall\, M' > M\,. \tag{3.14}$$

In particular, the function $M \mapsto I_M$ is decreasing on $[M_c, +\infty)$. Furthermore,

$$I_M < I_m + I_{M-m} \quad \forall\, m \in (0, M)\,. \tag{3.15}$$

Proof. Consider $\varphi \in \Sigma_M$ and let $\tilde{\varphi} := \left(\frac{M'}{M}\right)^2 \varphi\left(\frac{M'}{M}\,\cdot\right)$. We notice that $\tilde{\varphi} \in \Sigma_{M'}$ and

$$I_{M'} \le \mathrm{E}[\tilde{\varphi}] = \left(\frac{M'}{M}\right)^3 \left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx + \frac{1}{2}\,\mathrm{D}[\varphi] - \frac{C}{2\alpha+2}\left(\frac{M'}{M}\right)^{4\alpha-2}\int_{\mathbb{R}^3} |\varphi|^{2\alpha+2}\,dx\right] \le \left(\frac{M'}{M}\right)^3 \mathrm{E}[\varphi]\,.$$

We deduce (3.14) by taking the infimum of the right-hand side over all functions φ in Σ_M and the monotonicity of $M \mapsto I_M$ on $[M_c, +\infty)$ follows.

We now turn our attention to the proof of (3.15). If $I_m = I_{M-m} = 0$, inequalities (3.15) obviously holds for any $M > M_c$, since I_M is negative. If $I_m < 0$ but $I_{M-m} = 0$ (so that $M_c < m$ and $M - m \le M_c$), then (3.15) reduces to $I_M < I_m$, which is again guaranteed by (3.14). If both I_m and I_{M-m} are negative (this is equivalent to $m > M_c$ and $M - m > M_c$, and therefore it may occur only if $M > 2M_c$), then we have

$$I_M \le \left(\frac{M}{m}\right)^3 I_M < \frac{M}{m}\,I_m \quad \text{and} \quad I_M \le \left(\frac{M}{M-m}\right)^3 I_M < \frac{M}{M-m}\,I_{M-m}$$

by using (3.14). Hence, $I_M = \frac{m}{M}\,I_M + \frac{M-m}{M}\,I_M < I_m + I_{M-m}$. This concludes the proof of Lemma 3.1. □

Let us come back to the proof of Proposition 3.4. In order to prove the existence of minimizers in the limiting case $C\,M^{4\alpha-2} = V_c$, that is $M = M_c$, we follow the arguments in [19], where a proof for the case $C = 1$ is given. As noted in Remark 2.1, relative compactness (up to translations) of all minimizing sequences cannot be proved in this case, since $I_{M_c} = 0$. We build a particular minimizing sequence as follows.

Let $M_n = M_c + \frac{1}{n}$, for every positive integer n, and assume that φ_n is a minimizer of I_{M_n} in Σ_{M_n}, which is already known to exist since $M_n > M_c$ and therefore $I_{M_n} < 0$ for any $n \ge 1$. Since $\{M_n\}_{n\ge1}$ converges towards M_c, it can be deduced that $\lim_{n\to\infty} \mathrm{E}[\varphi_n] = \lim_{n\to\infty} I_{M_n} = I_{M_c} = 0$. With the notations of Corollary 2.2, this means that $\lim_{n\to\infty} \eta_{M_n} = 0$. If we combine the results of Corollary 2.1 and Corollary 2.2, then we obtain

$$\frac{1}{4}\,\varepsilon_{M_n} - \frac{3}{2}\,\eta_{M_n} \le \mathrm{C}_{1/2}^{2-2\alpha}\,\mathrm{C}_{\mathrm{GN}}(1)^{2\alpha-1}\,M^{\alpha-\frac{1}{2}}\left[\frac{1}{2}\,(3\alpha-1)\,\varepsilon_{M_n} - (5\alpha-1)\,\eta_{M_n}\right]^{4\alpha-1} \left[(2-3\alpha)\,\varepsilon_{M_n} - 2\,(2-\alpha)\,\eta_{M_n}\right]^{1-\alpha}.$$

By passing to the limit as $n \to \infty$, we find that

$$\frac{1}{4} \le \mathrm{C}_{1/2}^{2-2\alpha}\,\mathrm{C}_{\mathrm{GN}}(1)^{2\alpha-1}\,M^{\alpha-\frac{1}{2}}\left[\frac{1}{2}\,(3\alpha-1)\right]^{4\alpha-1}\left[(2-3\alpha)\right]^{1-\alpha}\liminf_{n\to\infty}\varepsilon_{M_n}^{3\alpha-1}\,,$$

thus proving that $\liminf_{n\to\infty}\varepsilon_{M_n} > 0$ and hence

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3} |\varphi_n|^{2\alpha+2}\,dx > 0\,.$$

Then, by Lemma I.1 in [25] the sequence $\{\varphi_n\}_{n\geq 1}$ satisfies the non-vanishing condition (1.6). Consequently, up to translations, there exists a subsequence that converges weakly in $\mathrm{H}^1(\mathbb{R}^3)$, strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ and pointwise almost everywhere, towards a nonzero function φ_∞. This sequence can be also assumed to be strongly convergent in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ and pointwise convergent almost everywhere. Thanks to [8] and Lemma 2.2 in [35], we get

$$0 = \lim_{n\to\infty} I_{M_n} = \lim_{n\to\infty} \mathrm{E}[\varphi_n] = \mathrm{E}[\varphi_\infty] + \lim_{n\to\infty} \mathrm{E}[\varphi_n - \varphi_\infty]\,.$$

Since $0 < \|\varphi_\infty\|^2_{\mathrm{L}^2(\mathbb{R}^3)} \leq M_c$, we have $\mathrm{E}[\varphi_\infty] \geq 0$ and

$$\lim_{n\to\infty} \|\varphi_n - \varphi_\infty\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = M_c - \|\varphi_\infty\|^2_{\mathrm{L}^2(\mathbb{R}^3)} < M_c\,,$$

then $\lim_{n\to\infty} \mathrm{E}[\varphi_n - \varphi_\infty] \geq 0$. Therefore, $\mathrm{E}[\varphi_\infty] = 0$. To conclude the proof of Proposition 3.4 we observe that $\|\varphi_\infty\|^2_{\mathrm{L}^2(\mathbb{R}^3)} = M_c$. Otherwise, φ_∞ is a minimizer of I_M, for some $M < M_c$, and we reach a contradiction with the first statement in Proposition 3.4. □

© 2013 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes.

Bibliography

1. A. Azzollini and A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl., 345 (2008), pp. 90–108.
2. J. Bellazzini, L. Jeanjean, and T. Luo, *Existence and instability of standing waves with prescribed norm for a class of Schrödinger–Poisson equations*, Proceedings of the London Mathematical Society, (2013).
3. J. Bellazzini and G. Siciliano, *Scaling properties of functionals and existence of constrained minimizers*, J. Funct. Anal., 261 (2011), pp. 2486–2507.
4. ———, *Stable standing waves for a class of nonlinear Schrödinger-Poisson equations*, Z. Angew. Math. Phys., 62 (2011), pp. 267–280.
5. O. Bokanowski, J. L. López, Ó. Sánchez, and J. Soler, *Long time behaviour to the Schrödinger-Poisson-X^α systems*, in Mathematical physics of quantum mechanics, vol. 690 of Lecture Notes in Phys., Springer, Berlin, 2006, pp. 217–232.
6. O. Bokanowski, J. L. López, and J. Soler, *On an exchange interaction model for quantum transport: the Schrödinger-Poisson-Slater system*, Math. Models Methods Appl. Sci., 13 (2003), pp. 1397–1412.
7. D. Bonheure and C. Mercuri, *Embedding theorems and existence results for nonlinear Schrödinger-Poisson systems with unbounded and vanishing potentials*, J. Differential Equations, 251 (2011), pp. 1056–1085.
8. H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., 88 (1983), pp. 486–490.
9. I. Catto and P.-L. Lions, *Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories. I. A necessary and sufficient condition for the stability of general molecular systems*, Comm. Partial Differential Equations, 17 (1992), pp. 1051–1110.
10. T. Cazenave, *An introduction to nonlinear Schrödinger equations*, vol. 26, Instituto de Matemática, UFRJ, 1993.
11. T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys., 85 (1982), pp. 549–561.

12. G. M. Coclite, *A multiplicity result for the nonlinear Schrödinger-Maxwell equations*, Commun. Appl. Anal., 7 (2003), pp. 417–423.
13. T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004), pp. 893–906.
14. P. d'Avenia, *Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations*, Adv. Nonlinear Stud., 2 (2002), pp. 177–192.
15. V. Felli and M. Schneider, *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type*, J. Differential Equations, 191 (2003), pp. 121–142.
16. V. Georgiev, F. Prinari, and N. Visciglia, *On the radiality of constrained minimizers to the Schrödinger-Poisson-Slater energy*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 29 (2012), pp. 369–376.
17. I. Ianni and D. Ruiz, *Ground and bound states for a static Schrödinger-Poisson-Slater problem*, Communications in Contemporary Mathematics, 14 (2012), p. 1250003.
18. R. Illner, P. F. Zweifel, and H. Lange, *Global existence, uniqueness and asymptotic behaviour of solutions of the Wigner-Poisson and Schrödinger-Poisson systems*, Math. Methods Appl. Sci., 17 (1994), pp. 349–376.
19. L. Jeanjean and T. Luo, *Sharp nonexistence results of prescribed L^2-norm solutions for some class of Schrödinger–Poisson and quasi-linear equations*, Zeitschrift für angewandte Mathematik und Physik, (2012), pp. 1–18.
20. E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math., 57 (1976/77), pp. 93–105.
21. E. H. Lieb and M. Loss, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2001.
22. E. H. Lieb and B. Simon, *The Thomas-Fermi theory of atoms, molecules and solids*, Advances in Math., 23 (1977), pp. 22–116.
23. P.-L. Lions, *Principe de concentration-compacité en calcul des variations*, C. R. Acad. Sci. Paris Sér. I Math., 294 (1982), pp. 261–264.
24. P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 109–145.
25. ———, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 223–283.
26. ———, *Solutions of Hartree-Fock equations for Coulomb systems*, Comm. Math. Phys., 109 (1987), pp. 33–97.
27. P.-L. Lions, *On some minimization problems in mathematical physics: how to check strict subadditivity conditions ?*, Progress in variational methods in Hamiltonian systems and elliptic equations (L'Aquila, 1990), Pitman Res. Notes Math. Ser., 243, Longman Sci. Tech., Harlow, 1992, pp. 119–126.
28. O. Lopes and M. Mariş, *Symmetry of minimizers for some nonlocal variational problems*, J. Funct. Anal., 254 (2008), pp. 535–592.
29. N. J. Mauser, *The Schrödinger-Poisson-X^α equation*, Appl. Math. Lett., 14 (2001), pp. 759–763.
30. D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal., 237 (2006), pp. 655–674.
31. E. Ruíz Arriola and J. Soler, *A variational approach to the Schrödinger-Poisson system: asymptotic behaviour, breathers, and stability*, J. Statist. Phys., 103 (2001), pp. 1069–1106.

32. Ó. Sánchez and J. Soler, *Asymptotic decay estimates for the repulsive Schrödinger-Poisson system*, Math. Methods Appl. Sci., 27 (2004), pp. 371–380.
33. ———, *Long-time dynamics of the Schrödinger-Poisson-Slater system*, J. Statist. Phys., 114 (2004), pp. 179–204.
34. Z. Wang and H.-S. Zhou, *Positive solution for a nonlinear stationary Schrödinger-Poisson system in* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst., 18 (2007), pp. 809–816.
35. L. Zhao and F. Zhao, *On the existence of solutions for the Schrödinger-Poisson equations*, J. Math. Anal. Appl., 346 (2008), pp. 155–169.