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Nadine Guillotin-Plantard, Julien Poisat, Renato Soares dos Santos. A Quenched Functional Central Limit Theorem for Planar Random Walks in Random Sceneries. 2013. hal-00834297v3

HAL Id: hal-00834297

<https://hal.science/hal-00834297v3>

Preprint submitted on 6 Sep 2013 (v3), last revised 18 Sep 2013 (v4)

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A Quenched Functional Central Limit Theorem for Planar Random Walks in Random Sceneries

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September 6, 2013

Abstract

Random walks in random sceneries (RWRS) are simple examples of stochastic processes in disordered media. They were introduced at the end of the 70's by Kesten-Spitzer and Borodin, motivated by the construction of new self-similar processes with stationary increments. Two sources of randomness enter in their definition: a random field $\xi = (\xi(x))_{x \in \mathbb{Z}^d}$ of i.i.d. random variables, which is called the *random scenery*, and a random walk $S = (S_n)_{n \in \mathbb{N}}$ evolving in \mathbb{Z}^d , independent of the scenery. The RWRS $Z = (Z_n)_{n \in \mathbb{N}}$ is then defined as the accumulated scenery along the trajectory of the random walk, i.e., $Z_n := \sum_{k=1}^n \xi(S_k)$. The law of Z under the joint law of ξ and S is called “annealed”, and the conditional law given ξ is called “quenched”. Recently, functional central limit theorems under the quenched law were proved for Z by the first two authors for a class of transient random walks including walks with finite variance in dimension $d \geq 3$. In this paper we extend their results to dimension $d = 2$.

1 Introduction

Let $d \geq 1$ and $(\xi(x))_{x \in \mathbb{Z}^d}$ be a collection of independent and identically distributed (i.i.d.) real random variables, further referred to as *scenery*, and $(S_n)_{n \geq 0}$ a random walk evolving in \mathbb{Z}^d , independent of the scenery. The random walk in random scenery (RWRS) is the process obtained by adding up the values of the scenery seen by the random walk along its trajectory, that is, $Z_n =$

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Key words: Random walk in random scenery; Limit theorem; Local time; Associated Random Variables.

AMS Subject Classification: 60F05, 60G52.

This work was supported by the french ANR project MEMEMO2 10-BLAN-0125-03 and ERC Advanced Grant 267356 VARIS.

$\xi(S_1) + \dots + \xi(S_n)$, $n \geq 1$. This model was introduced independently by Kesten and Spitzer [22] and by Borodin [6, 7].

RWRS appears naturally in a variety of contexts, for instance (i) in the energy function of statistical mechanics models of polymers interacting with a random medium, (ii) in Bouchaud's trap model via the clock process, see [2], (iii) in the study of random walks in randomly oriented lattices, as in [8, 11]. The last example is related to the phenomenon of anomalous diffusion in layered random media, see Le Doussal [16] and Matheron and de Marsily [25] on this matter. Indeed, Kesten and Spitzer's original motivation was to build a new class of self-similar stochastic processes with non-standard normalizations.

Results were first established under the *annealed* measure, that is when one averages here that the random walk increment and the scenery at the origin are in the domains of attraction of different stable laws with index a and $b \in (0, 2]$, respectively. In the case $d = 1 < a$, Kesten and Spitzer [22] proved that the process $(n^{-\delta} Z_{\lfloor nt \rfloor})_{t \geq 0}$ converges weakly, as $n \rightarrow \infty$, to a continuous δ -self-similar process, where $\delta = 1 - a^{-1} + (ab)^{-1}$. Later on, Bolthausen [4] proved a functional central limit for $(\sqrt{n \log n}^{-1} Z_{\lfloor nt \rfloor})_{t \geq 0}$ in the case $d = a = b = 2$, and his result also covers the case $d = a = 1$, $b = 2$. More recently, Castell, Guillotin-Plantard and Pène [10] proved that, for $d = a \in \{1, 2\}$ and $0 < b < 2$, $(Z_{\lfloor nt \rfloor})_{t \geq 0}$ has to be normalized by $n^{1/b} (\log n)^{1-1/b}$ so that it converges to a limiting process, which is stable of index b . The case of a transient random walk (i.e., $a < d$) has also been treated in [10] (see also [28, 22, 7]): rescaling by $n^{1/b}$ one obtains as limit a stable process of index b . Other results on RWRS include strong approximation results and laws of the iterated logarithm [14, 15, 23], limit theorems for correlated sceneries or walks [13, 21], large and moderate deviations results [1, 9, 12, 18, 19], ergodic and mixing properties [17].

Distributional limit theorems for *quenched* sceneries (that is, conditionally given the scenery) are more recent. The first result in this direction that we are aware of was obtained by Ben Arous and Černý [2], in the case of a heavy-tailed scenery and planar random walk. Recently, the first two authors proved in [20] that a quenched functional central limit theorem (with the usual \sqrt{n} -scaling and Gaussian law in the limit) holds for a class of transient random walks. Moreover, with one of the methods used there, namely convergence of moments, they could prove convergence along a subsequence for sceneries having finite moments of all orders and planar random walks with finite non-singular covariance matrices, after a non-standard scaling by $\sqrt{n \log n}$. The question was raised whether the convergence takes place along the full sequence. In this paper we are able to answer this question in the positive when the scenery has slightly more than a second moment.

2 Notation, assumptions and results

Let us start with a few words about notation. We will denote by $\mathbb{N} := \{0, 1, 2, \dots\}$ the set of non-negative integers and put $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. We will write C to denote a generic positive constant that may change from expression to expression.

We now proceed to define the model. Let $S = (S_n)_{n \geq 0}$ be a random walk in \mathbb{Z}^2 starting at 0, i.e., $S_0 = 0$ and

$$(S_n - S_{n-1})_{n \geq 1} \text{ is a sequence of i.i.d. } \mathbb{Z}^2\text{-valued random variables.} \quad (2.1)$$

We denote the local times of the random walk by

$$N_n(x) := \sum_{1 \leq k \leq n} \mathbf{1}_{\{S_k = x\}}, \quad x \in \mathbb{Z}^2. \quad (2.2)$$

Let $\xi = (\xi(x))_{x \in \mathbb{Z}^2}$ be a field of i.i.d. real random variables independent of S . The field ξ is called the *random scenery*.

The random walk in random scenery (RWRS) $Z = (Z_n)_{n \geq 0}$ is defined by setting $Z_0 := 0$ and, for $n \in \mathbb{N}^*$,

$$Z_n := \sum_{i=1}^n \xi(S_i) = \sum_{x \in \mathbb{Z}^2} \xi(x) N_n(x). \quad (2.3)$$

We will denote by \mathbb{P} the joint law of S and ξ , and by P the marginal of S . The law \mathbb{P} is called the *annealed* law, while the conditional law $\mathbb{P}(\cdot | \xi)$ is called the *quenched* law.

We will make the following two assumptions on the random walk and on the random scenery:

(A1) The random walk increment S_1 has a centered law with a finite and non-singular covariance matrix Σ . We further suppose that the random walk is aperiodic in the sense of Spitzer [28], which means that S is not confined to a proper subgroup of \mathbb{Z}^2 .

(A2) $\mathbb{E}[\xi(0)] = 0$, $\mathbb{E}[|\xi(0)|^2] = 1$ and there exists a $\gamma > 1$ such that

$$\mathbb{E} [|\xi(0)|^2 (\log^+ |\xi(0)|)^\gamma] < \infty, \quad (2.4)$$

where $\log^+ x := \max(0, \log x)$.

The aim of this paper is to prove the following quenched functional central limit theorem.

Theorem 2.1. *Under assumptions (A1) and (A2), for \mathbb{P} -a.e. ξ , the process*

$$W^{(n)} = \left(W_t^{(n)} \right)_{t \geq 0} := \left(\frac{Z_{\lfloor nt \rfloor}}{\sqrt{n \log n}} \right)_{t \geq 0} \quad (2.5)$$

converges weakly as $n \rightarrow \infty$ under $\mathbb{P}(\cdot | \xi)$ in the Skorohod topology to a Brownian motion with variance $\sigma^2 = (\pi \sqrt{\det \Sigma})^{-1}$.

Remark: The conclusion of this theorem still holds if, alternatively, the assumption (A1) is replaced by the following one:

(A1') The sequence $S = (S_n)_{n \geq 0}$ is an aperiodic random walk in \mathbb{Z} starting from 0 such that $\left(\frac{S_n}{n}\right)_n$ converges in distribution to a random variable with characteristic function given by $t \mapsto \exp(-a|t|)$, $a > 0$. In this case, $\sigma^2 = 2(\pi a)^{-1}$.

Indeed, the proof of Theorem 2.1 depends on the random walk S only through certain local time properties which are known to be the same under assumptions (A1) or (A1'). These properties are listed in Section 4.

3 Outline of the proof of Theorem 2.1

We will use a method introduced by Bolthausen and Sznitman in [5]. The idea is to pass the functional CLT from the annealed to the quenched law using concentration of quenched expectations of Lipschitz functionals of the rescaled process. In our setting, the annealed version of Theorem 2.1 was proved by Bolthausen in [4], and his proof also works under (A1').

To describe the method more precisely, let $\mathcal{W}^{(n)}$ be the polygonal interpolation of $W^{(n)}$, that is,

$$\mathcal{W}_t^{(n)} := \frac{Z_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)(Z_{\lfloor nt \rfloor + 1} - Z_{\lfloor nt \rfloor})}{\sqrt{n \log n}}. \quad (3.1)$$

For $T > 0$, consider the space $C([0, T], \mathbb{R})$ of continuous functions from $[0, T]$ to \mathbb{R} equipped with the sup norm. We abuse notation by writing $\mathcal{W}^{(n)}$ to mean also the restriction of this process to the interval $[0, T]$, depending on context.

Following the reasoning in Lemma 4.1 of [5], we see that Theorem 2.1 will follow from the annealed functional CLT in [4] if we show that, for any $T > 0$, $b \in (1, 2]$ and any bounded Lipschitz function $F : C([0, T], \mathbb{R}) \rightarrow \mathbb{C}$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[F \left(\mathcal{W}^{(\lfloor b^n \rfloor)} \right) \mid \xi \right] - \mathbb{E} \left[F \left(\mathcal{W}^{(\lfloor b^n \rfloor)} \right) \right] = 0 \quad \mathbb{P}\text{-a.s.} \quad (3.2)$$

To prove (3.2) we will use a martingale decomposition, in a similar fashion as in Bolthausen and Sznitman [5] (proof of Theorem 4.2), Berger and Zeitouni [3] (proof of Theorem 4.1) and Rassoul-Agha and Seppäläinen [27] (proof of Proposition 6.1). In order to control the martingale via exponential inequalities, we introduce first as a technical step a truncation of the scenery, from which the restriction $\gamma > 1$ originates.

The rest of the paper is organized as follows. In Section 4 we collect two facts about two-dimensional random walks that we will need. Section 5 contains the proof of Theorem 2.1, given in two steps: in Section 5.1 we define a truncation of the RWRS and reduce the problem to showing (3.2) for the truncated version, and this last step is carried out in Section 5.2.

4 Two-dimensional random walks

We mention here two facts about two-dimensional random walks satisfying (A1) that will be needed in the sequel. Analogous statements are valid under (A1').

Lemma 4.1. *There exists a $K \in (0, \infty)$ such that*

$$(i) \quad \sup_{x \in \mathbb{Z}^2} E[N_n(x)] \leq K \log n \quad \forall n \geq 2. \quad (4.1)$$

$$(ii) \quad \sum_{x \in \mathbb{Z}^2} E[N_n(x)]^2 \leq Kn \quad \forall n \in \mathbb{N}^*. \quad (4.2)$$

Proof. Item (i) can be found e.g. in the proof of Lemma 2.5 in [4]. Item (ii) follows from the proof of Corollary 3.2 in [24]; note that the l.h.s. of (4.2) is the expectation of the mutual intersection local time of two independent copies of S , denoted by J_n in [24]. \square

5 Proof of Theorem 2.1

The proof consists of two steps: first we define a truncation of the RWRS that approximates well the original process, and then we prove (3.2) for the truncated version.

5.1 Truncation

For $n \geq 2$, set $M_n := \sqrt{n/(\log n)^\gamma}$, define $\xi_n, \widehat{\xi}_n \in \mathbb{R}^{\mathbb{Z}^2}$ by

$$\begin{aligned} \xi_n(x) &:= \xi(x) \mathbf{1}_{\{|\xi(x)| \leq M_n\}} \\ \widehat{\xi}_n(x) &:= \xi_n(x) - \mathbb{E}[\xi_n(x)] \end{aligned} \quad \text{for } x \in \mathbb{Z}^2, \quad (5.1)$$

and let $Z^{(n)}$ and $\widehat{Z}^{(n)}$ be defined by

$$\begin{aligned} Z_k^{(n)} &:= \sum_{i=1}^k \xi_n(S_i) = \sum_{x \in \mathbb{Z}^2} \xi_n(x) N_k(x) \\ \widehat{Z}_k^{(n)} &:= \sum_{i=1}^k \widehat{\xi}_n(S_i) = \sum_{x \in \mathbb{Z}^2} \widehat{\xi}_n(x) N_k(x) \end{aligned} \quad \text{for } k \in \mathbb{N}^*. \quad (5.2)$$

The following two propositions show that, in order to prove Theorem 2.1, it is enough to prove the same statement for $\widehat{W}_t^{(n)} := (n \log n)^{-\frac{1}{2}} \widehat{Z}_{[nt]}^{(n)}$, $t \geq 0$.

Proposition 5.1. (Comparison between Z and $Z^{(n)}$)

Fix $T > 0$. There exists \mathbb{P} -a.s. a random time $T_0 \in \mathbb{N}^$ such that, if $n \geq T_0$, then $Z_k^{(n)} = Z_k$ for all $1 \leq k \leq [nT]$.*

Proof. Let

$$R_k := \{x \in \mathbb{Z}^2 : N_k(x) > 0\} \quad (5.3)$$

be the range of the random walk S up to time k , and set

$$\mathcal{D}_n := \{x \in R_{[nT]} : \xi_n(x) \neq \xi(x)\}. \quad (5.4)$$

We have

$$\mathcal{D}_n \setminus \mathcal{D}_{n-1} = \{x \in R_{\lfloor nT \rfloor} \setminus R_{\lfloor (n-1)T \rfloor} : |\xi(x)| > M_n\}. \quad (5.5)$$

Therefore,

$$\begin{aligned} \mathbb{P}(\mathcal{D}_n \setminus \mathcal{D}_{n-1} \neq \emptyset) &\leq \mathbb{P}(\exists \lfloor (n-1)T \rfloor < k \leq \lfloor nT \rfloor : |\xi(S_k)| > M_n) \\ &\leq (T+1)\mathbb{P}(|\xi(0)| > M_n). \end{aligned} \quad (5.6)$$

Since $f(x) := x^2(\log^+ x)^\gamma$ is non-decreasing on $(0, \infty)$ and $f(M_n) \geq Cn$ for some $C > 0$ and all $n \geq 2$, we have

$$\mathbb{P}(|\xi(0)| > M_n) \leq \mathbb{P}(|\xi(0)|^2(\log^+ |\xi(0)|)^\gamma \geq Cn) \quad (5.7)$$

which is summable by (A2), so by the Borel-Cantelli lemma there exists a random index $N_0 \in \mathbb{N}^*$ such that a.s. $\mathcal{D}_n \subset \mathcal{D}_{N_0}$ for all $n \geq N_0$. Therefore, setting

$$T_0 := \inf \left\{ n \geq N_0 : M_n > \sup_{x \in \mathcal{D}_{N_0}} |\xi(x)| \right\}, \quad (5.8)$$

we have $\mathcal{D}_n = \emptyset$ for $n \geq T_0$. \square

Proposition 5.2. (Comparison between $Z^{(n)}$ and $\widehat{Z}^{(n)}$)

$$\lim_{n \rightarrow \infty} \sup_{1 \leq k \leq \lfloor nT \rfloor} \frac{|\widehat{Z}_k^{(n)} - Z_k^{(n)}|}{\sqrt{n \log n}} = 0 \quad \mathbb{P}\text{-a.s. for any } T > 0. \quad (5.9)$$

Proof. Since ξ is centered,

$$\begin{aligned} |\mathbb{E}[\xi(0)\mathbf{1}_{\{|\xi(0)| \leq M_n\}}]| &= |\mathbb{E}[\xi(0)\mathbf{1}_{\{|\xi(0)| > M_n\}}]| \leq \frac{\mathbb{E}[|\xi(0)|^2(\log^+ |\xi(0)|)^\gamma]}{M_n(\log M_n)^\gamma} \\ &\leq \frac{C}{\sqrt{n(\log n)^\gamma}}. \end{aligned} \quad (5.10)$$

Therefore, for $1 \leq k \leq \lfloor nT \rfloor$,

$$\frac{|Z_k^{(n)} - \widehat{Z}_k^{(n)}|}{\sqrt{n \log n}} = \frac{k |\mathbb{E}[\xi(0)\mathbf{1}_{\{|\xi(0)| \leq M_n\}}]|}{\sqrt{n \log n}} \leq \frac{CT}{(\log n)^{\frac{\gamma+1}{2}}}. \quad (5.11)$$

\square

5.2 Control of the truncated version

From now on we will work with the truncated and recentered version $\widehat{Z}^{(n)}$ of the RWRS. Let $\widehat{\mathcal{W}}^{(n)}$ be the analogue of $\mathcal{W}^{(n)}$ in (3.1) for $\widehat{Z}^{(n)}$, i.e.,

$$\widehat{\mathcal{W}}_t^{(n)} := \frac{\widehat{Z}_{\lfloor nt \rfloor}^{(n)} + (nt - \lfloor nt \rfloor) \left(\widehat{Z}_{\lfloor nt \rfloor + 1}^{(n)} - \widehat{Z}_{\lfloor nt \rfloor}^{(n)} \right)}{\sqrt{n \log n}}, \quad t \geq 0. \quad (5.12)$$

Fix $T > 0$, $b \in (1, 2]$ and $F : C([0, T], \mathbb{R}) \rightarrow \mathbb{C}$ bounded and Lipschitz. By Propositions 5.1–5.2, weak convergence of either $W^{(n)}$ or $\widehat{W}^{(n)}$ implies the same convergence for the other, under both the quenched and annealed laws; therefore our work will be done once we show that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[F \left(\widehat{W}^{(b^n \cdot)} \right) \middle| \xi \right] - \mathbb{E} \left[F \left(\widehat{W}^{(b^n \cdot)} \right) \right] = 0 \quad \mathbb{P}\text{-a.s.} \quad (5.13)$$

Proof of (5.13). Fix an arbitrary enumeration of $\mathbb{Z}^2 := \{x_1, x_2, \dots\}$, define

$$\mathcal{G}_k := \sigma(\xi(x_i) : i \leq k), \quad k \in \mathbb{N}^*, \quad (5.14)$$

and let

$$\Delta_k^{(n)} := \mathbb{E} \left[F \left(\widehat{W}^{(n)} \right) \middle| \mathcal{G}_k \right] - \mathbb{E} \left[F \left(\widehat{W}^{(n)} \right) \middle| \mathcal{G}_{k-1} \right], \quad (5.15)$$

where \mathcal{G}_0 is the trivial σ -algebra. The latter are increments of a bounded martingale. By the martingale convergence theorem,

$$\mathbb{E} \left[F \left(\widehat{W}^{(n)} \right) \middle| \xi \right] - \mathbb{E} \left[F \left(\widehat{W}^{(n)} \right) \right] = \sum_{k=1}^{\infty} \Delta_k^{(n)}. \quad (5.16)$$

To control the $\Delta_k^{(n)}$, we introduce a coupling. Let ξ' be an independent copy of ξ , set

$$\widehat{\xi}_n^{(k)}(x) := \begin{cases} \widehat{\xi}_n'(x) & \text{if } x = x_k, \\ \widehat{\xi}_n(x) & \text{otherwise,} \end{cases} \quad (5.17)$$

and let $\widehat{Z}^{(n,k)}, \widehat{W}^{(n,k)}$ be the analogues of $\widehat{Z}^{(n)}, \widehat{W}^{(n)}$, but defined from $\widehat{\xi}_n^{(k)}$ and the same random walk S . Let \mathbb{P}' denote the joint law of ξ', ξ and S . Then

$$\Delta_k^{(n)} = \mathbb{E}' \left[F \left(\widehat{W}^{(n)} \right) - F \left(\widehat{W}^{(n,k)} \right) \middle| \mathcal{G}_k \right] \quad \mathbb{P}\text{-a.s.} \quad (5.18)$$

Recalling (5.12) and (2.3), we see that

$$\begin{aligned} \sup_{t \in [0, T]} |\widehat{W}_t^{(n)} - \widehat{W}_t^{(n,k)}| &\leq \sqrt{n \log n}^{-1} \sup_{1 \leq m \leq \lfloor nT \rfloor + 1} |\widehat{\xi}_n(x_k) - \widehat{\xi}_n'(x_k)| N_m(x_k) \\ &= \sqrt{n \log n}^{-1} |\widehat{\xi}_n(x_k) - \widehat{\xi}_n'(x_k)| N_{\lfloor nT \rfloor + 1}(x_k). \end{aligned} \quad (5.19)$$

Therefore, by (5.18), the Lipschitz property of F and Lemma 4.1(i), we have

$$\begin{aligned} |\Delta_k^{(n)}| &\leq C \frac{M_n E [N_{\lfloor nT \rfloor + 1}(x_k)]}{\sqrt{n \log n}} \\ &\leq C \frac{\log(nT + 1)}{(\log n)^{\frac{\gamma+1}{2}}} \leq \frac{C}{(\log n)^\delta} \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (5.20)$$

where $\delta := (\gamma - 1)/2 > 0$, and also

$$\begin{aligned} \mathbb{E} \left[|\Delta_k^{(n)}|^2 \middle| \mathcal{G}_{k-1} \right] &\leq C \frac{\mathbb{E}' \left[\mathbb{E}' \left[|\widehat{\xi}_n(x_k) - \widehat{\xi}'_n(x_k)| \middle| \mathcal{G}_k \right]^2 \middle| \mathcal{G}_{k-1} \right] E \left[N_{\lfloor nT \rfloor + 1}(x_k) \right]^2}{n \log n} \\ &\leq C \frac{\mathbb{E}' \left[|\widehat{\xi}_n(x_k) - \widehat{\xi}'_n(x_k)|^2 \right] E \left[N_{\lfloor nT \rfloor + 1}(x_k) \right]^2}{n \log n} \\ &\leq C \frac{E \left[N_{\lfloor nT \rfloor + 1}(x_k) \right]^2}{n \log n} \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (5.21)$$

where for the second line we used the Cauchy-Schwarz inequality. By (5.21) and Lemma 4.1(ii), we have

$$\sum_{k=1}^{\infty} \mathbb{E} \left[|\Delta_k^{(n)}|^2 \middle| \mathcal{G}_{k-1} \right] \leq C \frac{\lfloor nT \rfloor + 1}{n \log n} \leq \frac{C}{\log n} \quad \mathbb{P}\text{-a.s.} \quad (5.22)$$

Therefore, by Bernstein's inequality for martingales (see e.g. Theorem 1.2A in [26]), for any $\epsilon > 0$,

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{k=1}^{\infty} \Delta_k^{(n)} \right| > \epsilon \right) &\leq \exp \left\{ -C \frac{\epsilon^2}{(\log n)^{-1} + \epsilon (\log n)^{-\delta}} \right\} \\ &\leq \exp \left\{ -C (\log n)^{\delta \wedge 1} \right\}, \end{aligned} \quad (5.23)$$

which is summable along b^n for any $b > 1$; thus, by the Borel-Cantelli lemma, (5.13) holds. \square

Acknowledgments:

The authors are grateful to Mohamed El Machkouri and to Christophe Sabot for helpful and stimulating discussions.

References

- [1] A. Asselah and F. Castell, Random walk in random scenery and self-intersection local times in dimensions $d \geq 5$, *Probab. Theory Related Fields* 138 (1-2) (2007) 1–32
- [2] G. Ben Arous and J. Černý, Scaling limit for trap models on \mathbb{Z}^d , *Ann. Probab.* 35 (6) (2007) 2356–2384.
- [3] N. Berger and O. Zeitouni, A quenched invariance principle for certain ballistic random walks in i.i.d. environments, in: *In and Out of Equilibrium 2*, *Progress in Probability* 60 137–160, Birkhäuser, Basel.
- [4] E. Bolthausen, A central limit theorem for two-dimensional random walks in random sceneries, *Ann. Probab.* 17 (1989) 108–115.

- [5] E. Bolthausen and A-S. Sznitman, On the static and dynamic points of view for certain random walks in random environment, *Methods Appl. Anal.* 9 (2002) 345–375.
- [6] A. N. Borodin, A limit theorem for sums of independent random variables defined on a recurrent random walk, *Dokl. Akad. nauk SSSR* 246 (4) (1979) 786-787
- [7] A. N. Borodin, Limit theorems for sums of independent random variables defined in a transient random walk, in *Investigations in the Theory of Probability Distributions, IV*, Zap, Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 85 (1979) 17-29. 237, 244.
- [8] M. Campanino and D. Pétritis, Random walks in randomly oriented lattices, *Markov Proces. Related Fields* 9 (2003), no. 3, 391-412.
- [9] F. Castell, Moderate deviations for diffusions in a random Gaussian shear flow drift, *Ann. Inst. H. Poincaré Probab. Statist.* 40 (3) (2004) 337–366.
- [10] F. Castell, N. Guillotin-Plantard, and F. Pène, Limit theorems for one and two-dimensional random walks in random scenery, *Ann. Inst. Henri Poincaré Probab. Statist.* 49 (2) (2013) 506–528.
- [11] F. Castell, N. Guillotin-Plantard, F. Pène and B. Schapira A local limit theorem for random walks in random scenery and on randomly oriented lattices, *Ann. Probab.* 39 (6) (2011) 2079-2118.
- [12] F. Castell and F. Pradeilles, Annealed large deviations for diffusions in a random Gaussian shear flow drift, *Stochastic Process. Appl.*, 94 (2) (2001) 171–197.
- [13] S. Cohen and C. Dombry, Convergence of dependent walks in a random scenery to fBm-local time fractional stable motions, *J. Math. Kyoto Univ.* 49 (2) (2009) 267–286.
- [14] E. Csáki, W. König and Z. Shi, An embedding for the Kesten-Spitzer random walk in random scenery, *Stochastic Process. Appl.* 82 (2) (1999) 283-292.
- [15] E. Csáki and P. Révész, Strong invariance for local times, *Z. Wahrsch. Verw. Gebiete* 62 (2) (1983) 263-278.
- [16] P. Le Doussal, Diffusion in layered random flows, polymers, electrons in random potentials, and spin depolarization in random fields, *J. Statist. Phys.* 69 (1992), no. 5-6, 917-954.
- [17] F. den Hollander and J.E. Steif, Random walk in random scenery: a survey of some recent results, *Dynamics and stochastics*, 53–65, IMS Lecture Notes Monogr. Ser., 48, Inst. Math. Statist., Beachwood, OH, 2006.

- [18] K. Fleishmann, P. Möerters and V. Wachtel, Moderate deviations for random walk in random scenery, *Stoch. Proc. Appl.* 118 (2008) 1768–1802.
- [19] N. Gantert, W. König and Z. Shi, Annealed deviations of random walk in random scenery, *Ann. Inst. H. Poincaré Probab. Statist.* 43 (1) (2007) 47–76.
- [20] N. Guillotin-Plantard and J. Poisat, Quenched central limit theorems for random walks in random scenery, *Stochastic Process. Appl.* 123 (4) (2013) 1348–1367.
- [21] N. Guillotin-Plantard and C. Prieur, Limit theorem for random walk in weakly dependent random scenery, *Ann. Inst. Henri Poincaré Probab. Stat.* 46 (4) (2010) 1178–1194
- [22] H. Kesten and F. Spitzer, A limit theorem related to a new class of self-similar processes, *Z. Wahrsch. Verw. Gebiete* 50 (1) (1979) 5–25.
- [23] D. Khoshnevisan and T.M. Lewis, A law of the iterated logarithm for stable processes in random scenery, *Stochastic Process. Appl.* 74 (1) (1998) 89–121.
- [24] J.-F. Le Gall and J. Rosen, The range of stable random walks, *Ann. Probab.* 19 (2) (1991) 650–705.
- [25] G. Matheron and G. de Marsily, Is Transport in Porous Media Always Diffusive? A Counterexample, *Water Resources Research*, vol. 16, no. 5 (1980) 901-917.
- [26] V.H. de la Peña, A general class of exponential inequalities for martingales and ratios, *Ann. Probab.* 27 (1999) 537–564.
- [27] F. Rassoul-Agha and T. Seppäläinen, Almost sure functional central limit theorem for ballistic random walk in random environment, *Ann. Inst. H. Poincaré Probab. Statist.* 45 (2009) 373–420.
- [28] F. Spitzer, Principles of Random Walks, second ed., in: Graduate Texts in Mathematics, vol. 34, Springer-Verlag, New-York, 1976.