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Abstract

Random walks in random sceneries (RWRS) are simple examples of stochastic processes in disordered media. They were introduced at the end of the 70's by Kesten-Spitzer and Borodin, motivated by the construction of new self-similar processes with stationary increments. Two sources of randomness enter in their definition: a random field $\xi = (\xi_x)_{x\in\mathbb{Z}^d}$ of i.i.d. random variables, which is called the *random scenery*, and a random walk $S = (S_n)_{n\in\mathbb{N}}$ evolving in $\mathbb{Z}^d$, independent of the scenery. The RWRS $Z = (Z_n)_{n\in\mathbb{N}}$ is then defined as the accumulated scenery along the trajectory of the random walk, i.e., $Z_n := \sum_{k=1}^{n} \xi_{S_k}$. The law of Z under the joint law of ξ and S is called "annealed", and the conditional law given ξ is called "quenched". Recently, central limit theorems under the quenched law were proved for Z by the first two authors for a class of transient random walks including walks with finite variance in dimension $d \geq 3$. In this paper we extend their results to dimension $d = 2$.

1 Introduction

Let $d \geq 1$ and $(\xi_x, x \in \mathbb{Z}^d)$ be a collection of independent and identically distributed (i.i.d.) real random variables, further referred to as *scenery*, and $(S_n)_{n\geq 0}$ a random walk evolving in $\mathbb{Z}^d$, independent of the scenery. The random walk in random scenery (RWRS) is the process obtained by adding up the values of the scenery seen by the random walk along its trajectory, that is

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$Z_n = \xi_{S_1} + \ldots + \xi_{S_n}$, for all $n \geq 1$. This model was introduced independently by Kesten and Spitzer [23] and by Borodin [4, 5].

RWRS appears naturally in a variety of contexts, for instance (i) in the energy function of statistical mechanics models of polymers interacting with a random medium, (ii) in Bouchaud's trap model via the clock process, see [2], (iii) in the study of random walks in randomly oriented lattices, as in [6, 9]. The last example is related to the phenomenon of anomalous diffusion in layered random media, see Le Doussal [17] and Matheron and de Marsily [26] on this matter. Indeed, Kesten and Spitzer's original motivation was to build a new class of self-similar sochastic processes with non-standard normalizations.

Results were first established under the *annealed* measure, that is when one averages at the same time over the scenery and the random walk. Let us suppose here that the random walk increment and the scenery at the origin are in the domains of attraction of different stable laws with index a and $b \in (0, 2]$, respectively. In the case $d = 1 < a$, Kesten and Spitzer [23] proved that the process $(n^{-\delta} Z_{\lceil nt \rceil})_{t \geq 0}$ converges weakly, as $n \to \infty$, to a continuous δ-self-similar process, where $\delta = 1 - a^{-1} + (ab)^{-1}$. Later on, Bolthausen [3] proved a functional central limit for $(\sqrt{n \log n}^{-1} Z_{\lceil nt \rceil})_{t \geq 0}$ in the case $d = a = b = 2$, and his result also covers the case $d = a = 1$, $b = 2$. More recently, Castell, Guillotin-Plantard and Pène [8] proved that, for $d = a \in \{1, 2\}$ and $0 < b < 2$, $(Z_{\lceil nt \rceil})_{t \geq 0}$ has to be normalized by $n^{1/b}(\log n)^{1-1/b}$ so that it converges to a limiting process, which is stable of index b. The case of a transient random walk (i.e., $a < d$) has also been dealt in [8] (see also [28, 23, 5]): rescaling by $n^{1/b}$ one obtains as limit a stable process of index b. Other results on RWRS include strong approximation results and laws of the iterated logarithm [15, 16, 24], limit theorems for correlated sceneries or walks [14, 22], large and moderate deviations results [1, 7, 10, 20], ergodic and mixing properties [18].

Distributional limit theorems for *quenched* sceneries (that is, conditionally given the scenery) are more recent. The first result in this direction that we are aware of was obtained by Ben Arous and Černý [2], in the case of a heavy-tailed scenery and planar random walk. Recently, the first two authors proved in [21] that a quenched central limit theorem (with the usual $\sqrt{n}$-scaling and Gaussian law in the limit) holds for a class of transient random walks. With one of the methods used there, namely convergence of moments, they could prove convergence along a subsequence for sceneries having finite moments of all orders and planar random walks with finite non-singular covariance matrices, after a non-standard scaling by $\sqrt{n \log n}$. The question was raised whether the convergence takes place along the full sequence. In this paper we are able to answer this question in the positive when the scenery has slightly more than a second moment.

2 Notation, assumptions and results

Let us start with a few words about notation. We will denote by $\mathbb{N} := \{0, 1, 2, \ldots\}$ the set of non-negative integers and put $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

For two sequences a_n, b_n of positive real numbers, we will write $a_n \sim b_n$ to denote that $\lim_{n\to\infty} a_n/b_n = 1$.

We will write K and C to denote generic positive constants that may change from expression to expression; we will consistently use K in statements of lemmas, etc, and C inside proofs.

We now proceed to define the model. Let $S = (S_n)_{n\geq 0}$ be a random walk in $\mathbb{Z}^2$ starting at 0, i.e., $S_0 = 0$ and

$$(S_n - S_{n-1})_{n\geq 1} \text{ is a sequence of i.i.d. } \mathbb{Z}^2\text{-valued random variables.} \tag{2.1}$$

We denote the local times of the random walk by

$$N_n(x) := \sum_{1\leq k\leq n} \mathbf{1}_{\{S_k = x\}}, \quad \forall x \in \mathbb{Z}^2. \tag{2.2}$$

Let $\xi = (\xi_x)_{x\in\mathbb{Z}^2}$ be a field of i.i.d. real random variables independent of S. The field ξ is called *the random scenery*.

The random walk in random scenery (RWRS) $Z = (Z_n)_{n\geq 0}$ is defined by setting $Z_0 := 0$ and, for $n \in \mathbb{N}^*$,

$$Z_n := \sum_{i=1}^{n} \xi_{S_i} = \sum_{x\in\mathbb{Z}^2} \xi_x N_n(x). \tag{2.3}$$

We will denote by $\mathbb{P}$ the joint law of S and ξ, and by P the marginal of S. The law $\mathbb{P}$ is called the *annealed* law, while the conditional law $\mathbb{P}(\cdot|\xi)$ is called the *quenched* law.

We will make the following two assumptions on the random walk and on the random scenery:

(A1) The random walk increment S_1 has a centered law with a finite and non-singular covariance matrix Σ. We further suppose that the random walk is aperiodic in the sense of Spitzer [28], which means that S is not confined to a proper subgroup of $\mathbb{Z}^2$.

(A2) $\mathbb{E}[\xi_0] = 0$, $\mathbb{E}[\xi_0^2] = 1$ and there exists a $\gamma > 3$ such that

$$\mathbb{E}\left[|\xi_0|^2 (\log^+ |\xi_0|)^\gamma\right] < \infty, \tag{2.4}$$

where $\log^+ x := \max(0, \log x)$.

The aim of this paper is to prove the following quenched central limit theorem.

Theorem 2.1. *Under assumptions (A1) and (A2), for $\mathbb{P}$-a.e. ξ, for all $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(Z_n \geq x\sqrt{n\log n}\,\middle|\,\xi\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_x^\infty e^{-u^2/2\sigma^2}du, \tag{2.5}$$

where $\sigma^2 = (\pi\sqrt{\det\Sigma})^{-1}$.

Remark: The conclusion of this theorem still holds if, alternatively, the assumption (A1) is replaced by the following one:

(A1') The sequence $S = (S_n)_{n\geq 0}$ is an aperiodic random walk in $\mathbb{Z}$ starting from 0 such that $\left(\frac{S_n}{n}\right)_n$ converges in distribution to a random variable with characteristic function given by $t \mapsto \exp(-a|t|)$, $a > 0$. In this case, $\sigma^2 = 2(\pi a)^{-1}$.

Indeed, the proof of Theorem 2.1 depends on the random walk S only through certain properties of its local times (including self and mutual intersection local times), which are known to be the same under assumptions (A1) or (A1'). These properties are listed in Section 4.

3 Outline of the proof of Theorem 2.1

In [21], the result for $d = 2$ is obtained by applying the method of moments under the quenched law. More precisely, it is shown that, under the assumption that the scenery has moments of all orders, the random variables $\mathbb{E}[Z_n^k|\xi]$, $k \in \mathbb{N}^*$, when properly normalized converge a.s. to the corresponding moments of a normal random variable along a certain subsequence. This subsequence however grows too fast (super-exponentially) to allow for a comparison with the process along the integers, preventing the conclusion of the quenched CLT for Z_n.

Here we will use a similar approach, but with a number of differences. First of all, the quenched moments are not all well defined under assumptions (A1)–(A2) alone, so in order to apply the method of moments we will first need to truncate the scenery in an appropriate way. For the truncated version, we are able to prove convergence of quenched moments along the subsequence

$$\tau_n := \lceil \exp n^\alpha \rceil, \quad \alpha \in (1/2, 1)\,, \tag{3.1}$$

which grows slowly enough to allow for a comparison with the original process at the end. The convergence along τ_n is obtained by carefully decomposing the quenched moments and then showing sufficient concentration of the terms around their means, which are in turn shown to converge to the right limits.

The following two propositions directly imply Theorem 2.1.

Proposition 3.1. *Under (A1)–(A2), for $\mathbb{P}$-a.e. ξ, for all $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(Z_{\tau_n} \geq x\sqrt{\tau_n\log\tau_n}\,\middle|\,\xi\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_x^\infty e^{-u^2/2\sigma^2}du. \tag{3.2}$$

Proposition 3.2. *Define $i(n) \in \mathbb{N}$ by $\tau_{i(n)} \leq n < \tau_{i(n)+1}$. Then, for $\mathbb{P}$-a.e. ξ, the difference*

$$\frac{Z_n}{\sqrt{n \log n}} - \frac{Z_{\tau_{i(n)}}}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \tag{3.3}$$

converges in probability to 0 as $n \to \infty$ under $\mathbb{P}(\cdot|\xi)$.

The rest of the paper is organized as follows. In Section 4 we collect all results concerning two-dimensional random walks that will be used in the other sections. In Section 5, we prove Proposition 3.1, while Proposition 3.2 is proved in Section 6.

4 Two-dimensional random walks

We gather here some useful facts concerning the local times of two-dimensional random walks. In the following we always assume (A1). We note that all the results below hold also under the alternative assumption (A1').

4.1 Maximal local time

Let

$$N_n^* := \sup_{x \in \mathbb{Z}^2} N_n(x) \tag{4.1}$$

be the maximal local time of the random walk S up to time n.

Lemma 4.1.

(i) For all $k \in \mathbb{N}$, there exists a $K > 0$ such that

$$\sup_{x \in \mathbb{Z}^2} E\left[N_n^k(x)\right] \leq K(\log n)^k \ \ \forall\, n \geq 2. \tag{4.2}$$

(ii) There exists a $K > 0$ such that

$$P\left(N_n^* > K(\log n)^2\right) \leq n^{-2} \ \ \forall\, n \in \mathbb{N}^*. \tag{4.3}$$

Proof. Item *(i)* follows from the proof of Lemma 2.5 in [3]. Item *(ii)* follows from Lemma 18 in [19]. □

4.2 Self and mutual intersection local times

For $p \geq 1$, the p-fold self-intersection local time $I_n^{[p]}$ of S up to time n is defined by

$$I_n^{[p]} := \sum_{x \in \mathbb{Z}^2} N_n^p(x). \tag{4.4}$$

When $p = 2$ we will omit the superscript and write I_n. Now let $\widehat{S}$ be an independent copy of S. We will denote by $P^{\otimes 2}$ the joint law of S and $\widehat{S}$. The p-fold mutual intersection local time $Q_n^{[p]}$ of S and $\widehat{S}$ up to time n is defined by

$$Q_n^{[p]} := \sum_{x \in \mathbb{Z}^2} N_n^p(x) \widehat{N}_n^p(x), \tag{4.5}$$

where $\widehat{N}_n(x) = \sum_{i=1}^n \mathbf{1}_{\{\widehat{S}_i = x\}}$ is the local time of $\widehat{S}$ at x up to time n. When $p = 1$ we will omit the superscript and write Q_n.

Lemma 4.2.

(i) For all $p \geq 2$ and $k \in \mathbb{N}$, there exists a $K > 0$ such that

$$E\left[(I_n^{[p]})^k\right] \leq K n^k (\log n)^{k(p-1)} \ \ \forall\, n \geq 2. \tag{4.6}$$

(ii) For all $k \in \mathbb{N}$,

$$\lim_{n \to \infty} \frac{E\left[I_n^k\right]}{(n \log n)^k} = \sigma^{2k}. \tag{4.7}$$

(iii) For all $k \in \mathbb{N}$, there exists a $K > 0$ such that

$$E^{\otimes 2}\left[Q_n^k\right] \leq K n^k \ \ \forall\, n \in \mathbb{N}^*. \tag{4.8}$$

Proof. Statement (i) can be found in [21] (Proposition 2.3) and statement (ii) comes as a combination of statement (i) and Theorem 1 in [11]. The last statement follows from the following facts:
1- (Proof of Lemma 5.2 in [12]) For all $k \in \mathbb{N}$, there exists some $C > 0$ such that

$$E^{\otimes 2}\left[Q_n^k\right] \leq C E^{\otimes 2}\left[Q_n\right]^k \ \ \forall\, n \in \mathbb{N}^*.$$

2- (Proof of Corollary 3.2 in [25]) There exists some $C > 0$ such that

$$E^{\otimes 2}\left[Q_n\right] \leq C n \ \ \forall\, n \in \mathbb{N}^*.$$

□

5 Proof of Proposition 3.1

The proof is split into several steps. First, we define in Section 5.1 a truncation of the scenery that will allow us to apply the method of moments. This truncation will be again useful in the proof of Proposition 3.2 below. Next we show in Section 5.2 how to decompose the quenched moments into a sum of terms that can be more easily analysed. This analysis is carried out in Section 5.3. Finally in Section 5.4 these results are combined to conclude the proof of Proposition 3.1.

5.1 Truncation

For $n \geq 2$, set $b_n := \sqrt{n/(\log n)^\gamma}$, define $\xi^{(n)}$, $\widehat{\xi}^{(n)} \in \mathbb{R}^{\mathbb{Z}^2}$ by

$$\begin{array}{rcl} \xi_x^{(n)} & := & \xi_x \mathbf{1}_{\{|\xi_x| \leq b_n\}} \\ \widehat{\xi}_x^{(n)} & := & \xi_x^{(n)} - \mathbb{E}\left[\xi_x^{(n)}\right] \end{array} \quad \text{for } x \in \mathbb{Z}^2, \tag{5.1}$$

and let $Z^{(n)}$ and $\widehat{Z}^{(n)}$ be defined by

$$\begin{array}{rcccl} Z_k^{(n)} & := & \sum_{i=1}^k \xi_{S_i}^{(n)} & = & \sum_{x \in \mathbb{Z}^2} \xi_x^{(n)} N_k(x) \\ \widehat{Z}_k^{(n)} & := & \sum_{i=1}^k \widehat{\xi}_{S_i}^{(n)} & = & \sum_{x \in \mathbb{Z}^2} \widehat{\xi}_x^{(n)} N_k(x) \end{array} \quad \text{for } k \in \mathbb{N}^*. \tag{5.2}$$

The following two propositions show that, in order to prove Proposition 3.1 for Z_n, it is enough to prove it for $\widehat{Z}_n^{(n)}$.

Proposition 5.1. (Comparison between Z and $Z^{(n)}$)
There exists $\mathbb{P}$-a.s. a random time $T_0 \in \mathbb{N}^*$ such that, if $n \geq T_0$, then $Z_k^{(n)} = Z_k$ for all $1 \leq k \leq n$.

Proof. Let

$$R_n := \{x \in \mathbb{Z}^2 \colon\ N_n(x) > 0\} \tag{5.3}$$

be the range of the random walk S up to time n, and set

$$\mathcal{W}_n := \{x \in R_n \colon\ \xi_x^{(n)} \neq \xi_x\}. \tag{5.4}$$

We have

$$\mathcal{W}_n \setminus \mathcal{W}_{n-1} = \begin{cases} \{S_n\} & \text{if } S_n \notin R_{n-1} \text{ and } |\xi_{S_n}| > b_n, \\ \emptyset & \text{otherwise.} \end{cases} \tag{5.5}$$

Since $f(x) := x^2(\log^+ x)^\gamma$ is non-decreasing on $(0, \infty)$ and $f(b_n) \geq Cn$ for some $C > 0$ and all $n \geq 2$, we have

$$\mathbb{P}\left(|\xi_{S_n}| > b_n\right) \leq \mathbb{P}\left(|\xi_0|^2 (\log^+ |\xi_0|)^\gamma \geq Cn\right) \tag{5.6}$$

which is summable by (A2), so by the Borel-Cantelli lemma there exists a random index $N_0 \in \mathbb{N}^*$ such that a.s. $\mathcal{W}_n \subset \mathcal{W}_{N_0}$ for all $n \geq N_0$. Therefore, setting

$$T_0 := \inf\left\{ n \geq N_0 \colon\ b_n > \sup_{x \in \mathcal{W}_{N_0}} |\xi_x| \right\}, \tag{5.7}$$

we have $\mathcal{W}_n = \emptyset$ for $n \geq T_0$. □

Proposition 5.2. (Comparison between $Z^{(n)}$ and $\widehat{Z}^{(n)}$)

$$\lim_{n \to \infty} \frac{|\widehat{Z}_n^{(n)} - Z_n^{(n)}|}{\sqrt{n \log n}} = 0 \quad \mathbb{P}\text{-a.s.} \tag{5.8}$$

Proof. Since ξ is centered,

$$\left|\mathbb{E}\left[\xi_0 \mathbf{1}_{\{|\xi_0|\leq b_n\}}\right]\right| = \left|\mathbb{E}\left[\xi_0 \mathbf{1}_{\{|\xi_0|> b_n\}}\right]\right| \leq \frac{\mathbb{E}\left[|\xi_0|^2(\log^+|\xi_0|)^\gamma\right]}{b_n(\log b_n)^\gamma} \leq \frac{C}{\sqrt{n(\log n)^\gamma}}. \tag{5.9}$$

Therefore,

$$\frac{|Z_n^{(n)} - \widehat{Z}_n^{(n)}|}{\sqrt{n\log n}} = \frac{n\left|\mathbb{E}\left[\xi_0 \mathbf{1}_{\{|\xi_0|\leq b_n\}}\right]\right|}{\sqrt{n\log n}} \leq \frac{C}{(\log n)^{\frac{\gamma+1}{2}}}. \tag{5.10}$$

□

5.2 Decomposition of quenched moments

From now on, we will work with the truncated and recentered version $\widehat{Z}^{(n)}$ of the RWRS. Since the $\widehat{\xi}_x^{(n)}$ are bounded, the quenched moments

$$\widehat{m}_n^{(k)} := \sum_{x_1,\dots,x_k} \prod_{i=1}^k \widehat{\xi}_{x_i}^{(n)}\, E\left[\prod_{i=1}^k N_n(x_i)\right] \tag{5.11}$$

are all well defined and satisfy

$$\widehat{m}_n^{(k)} = \mathbb{E}\left[(\widehat{Z}_n^{(n)})^k \,\middle|\, \xi\right] \quad \mathbb{P}\text{-a.s.} \tag{5.12}$$

We aim to prove that the $\widehat{m}_n^{(k)}$ when properly normalized converge a.s. along τ_n to the corresponding moments of a Gaussian random variable. In order to do that, we will first show how they can be decomposed into sums of terms that are easier to control.

The first step is to note that $\widehat{m}_n^{(k)}$ may be decomposed as follows:

$$\widehat{m}_n^{(k)} = \sum_{j=1}^k \sum_{\substack{l=(l_1,\dots,l_j)\in(\mathbb{N}^*)^j \\ l_1+\dots+l_j=k}} \binom{k}{l_1,\dots,l_j} \widehat{m}_n^{(k)}(j,l), \tag{5.13}$$

where

$$\binom{k}{l_1,\dots,l_j} = \frac{k!}{l_1!\dots l_j!} \tag{5.14}$$

are multinomial coefficients and

$$\widehat{m}_n^{(k)}(j,l) := \sum_{x_1<\dots<x_j} \prod_{i=1}^j (\widehat{\xi}_{x_i}^{(n)})^{l_i} E\left[\prod_{i=1}^j N_n^{l_i}(x_i)\right]. \tag{5.15}$$

By "$<$" in (5.15) we mean the usual lexicographical order of $\mathbb{Z}^2$; we will see in the following that taking ordered sums will be convenient for us.

Next, using the identity

$$\prod_{i\in\mathcal{I}}(a_i+b_i)=\sum_{A\subset\mathcal{I}}\prod_{i\in A}a_i\prod_{i\notin A}b_i, \tag{5.16}$$

we see that we may further decompose $\widehat{m}_n^{(k)}(j,l)$ as

$$\widehat{m}_n^{(k)}(j,l)=\sum_{A\subset\{1,\ldots,j\}}\widehat{m}_n^{(k)}(j,l,A), \tag{5.17}$$

where $\widehat{m}_n^{(k)}(j,l,A):=$

$$\sum_{x_1<\cdots<x_j}\prod_{i\in A}\mathbb{E}\left[(\widehat{\xi}_{x_i}^{(n)})^{l_i}\right]\prod_{i\notin A}\left((\widehat{\xi}_{x_i}^{(n)})^{l_i}-\mathbb{E}\left[(\widehat{\xi}_{x_i}^{(n)})^{l_i}\right]\right)E\left[\prod_{i=1}^{j}N_n^{l_i}(x_i)\right]. \tag{5.18}$$

For fixed j and l, let

$$\mathcal{A}:=\{i\in\{1,\ldots,j\}\colon l_i>1\}. \tag{5.19}$$

Noting that $\widehat{m}_n^{(k)}(j,l,A)=0$ if $A\cap\mathcal{A}^c\neq\emptyset$, and that $\widehat{m}_n^{(k)}(j,l,\{1,\ldots,j\})=\mathbb{E}\left[\widehat{m}_n^{(k)}(j,l)\right]$, we have

$$\widehat{m}_n^{(k)}(j,l)-\mathbb{E}\left[\widehat{m}_n^{(k)}(j,l)\right]=\sum_{A\subset\mathcal{A}\colon\,A^c\neq\emptyset}\widehat{m}_n^{(k)}(j,l,A). \tag{5.20}$$

Moreover, when $A^c\neq\emptyset$ we may write

$$\widehat{m}_n^{(k)}(j,l,A)=\sum_{x_i\colon\,i\notin A}\prod_{i\notin A}\left\{(\widehat{\xi}_{x_i}^{(n)})^{l_i}-\mathbb{E}\left[(\widehat{\xi}_{x_i}^{(n)})^{l_i}\right]\right\}\widehat{a}_n\big((x_i)_{i\notin A},l,A\big), \tag{5.21}$$

where

$$\widehat{a}_n\big((x_i)_{i\notin A},l,A\big):=\sum_{x_i\colon\,i\in A}\prod_{i\in A}\mathbb{E}\left[(\widehat{\xi}_{x_i}^{(n)})^{l_i}\right]E\left[\prod_{i=1}^{j}N_n^{l_i}(x_i)\right]\mathbf{1}_{\{x_1<\cdots<x_j\}}. \tag{5.22}$$

Finally, using the i.i.d. structure of ξ and the fact that we are taking ordered sums, we may write

$$\begin{aligned}&\left\|\widehat{m}_n^{(k)}(j,l,A)\right\|_2^2\\&\qquad=\prod_{i\notin A}\left\|(\widehat{\xi}_0^{(n)})^{l_i}-\mathbb{E}\left[(\widehat{\xi}_0^{(n)})^{l_i}\right]\right\|_2^2\sum_{x_i\colon\,i\notin A}\widehat{a}_n^2\left((x_i)_{i\notin A},l,A\right).\end{aligned} \tag{5.23}$$

5.3 Analysis of the terms

We begin with the terms in which $A^c = \emptyset$, i.e., the ones corresponding to $\mathbb{E}[\widehat{m}_n^{(k)}(j,l)]$.

Proposition 5.3.

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\widehat{m}_n^{(k)}(j,l)\right]}{(n\log n)^{k/2}} = \begin{cases} \frac{1}{j!}\sigma^{2j} & \text{if } 2j = k \text{ and } l_i = 2 \ \ \forall\, 1 \le i \le j, \\ 0 & \text{otherwise.} \end{cases} \tag{5.24}$$

Proof. Integrating (5.15), we get

$$\begin{aligned} \mathbb{E}\left[\widehat{m}_n^{(k)}(j,l)\right] &= \sum_{x_1<\cdots<x_j} \prod_{i=1}^{j} \mathbb{E}\left[(\widehat{\xi}_{x_i}^{(n)})^{l_i}\right] E\left[\prod_{i=1}^{j} N_n^{l_i}(x_i)\right] \\ &= \prod_{i=1}^{j} \mathbb{E}\left[(\widehat{\xi}_0^{(n)})^{l_i}\right] \sum_{x_1<\cdots<x_j} E\left[\prod_{i=1}^{j} N_n^{l_i}(x_i)\right]. \end{aligned} \tag{5.25}$$

If $2j > k$, then at least one of the l_i's is equal to one; therefore, by (5.1), $\mathbb{E}[\widehat{m}_n^{(k)}(j,l)] = 0$ for all $n \in \mathbb{N}^*$.

If $2j = k$ and all l_i's are larger than 1, then $l_i = 2$ for all $1 \le i \le j$, and

$$\begin{aligned} \mathbb{E}\left[\widehat{m}_n^{(k)}(j,l)\right] &= \mathbb{E}\left[(\widehat{\xi}_0^{(n)})^2\right]^j \sum_{x_1<\cdots<x_j} E\left[\prod_{i=1}^{j} N_n^2(x_i)\right] \\ &= \mathbb{E}\left[(\widehat{\xi}_0^{(n)})^2\right]^j \frac{1}{j!} \sum_{\substack{x_1,\ldots,x_j \\ \text{all distinct}}} E\left[\prod_{i=1}^{j} N_n^2(x_i)\right]. \end{aligned} \tag{5.26}$$

Since $\lim_{n\to\infty} \mathbb{E}\left[(\widehat{\xi}_0^{(n)})^2\right] = 1$, the claim will follow from Lemma 4.2(ii) once we show that

$$\sum_{\substack{x_1,\ldots,x_j \\ \text{all distinct}}} E\left[\prod_{i=1}^{j} N_n^2(x_i)\right] \sim E\left[I_n^j\right] = \sum_{x_1,\ldots,x_j} E\left[\prod_{i=1}^{j} N_n^2(x_i)\right]. \tag{5.27}$$

But

$$\begin{aligned} \sum_{\substack{x_1,\ldots,x_j \\ \exists\, i_1\neq i_2 :\, x_{i_1}=x_{i_2}}} E\left[\prod_{i=1}^{j} N_n^2(x_i)\right] &\le j^2 \sum_{x_2,\ldots,x_j} E\left[N_n^4(x_2)\prod_{i=3}^{j} N_n^2(x_i)\right] \\ &= j^2 E\left[I_n^{[4]} I_n^{j-2}\right] \\ &\le C n^{j-1} (\log n)^{j+1}, \end{aligned} \tag{5.28}$$

where we used the symmetry of the summands, Hölder's inequality and Lemma 4.2(i). Since (5.28) divided by $(n\log n)^j$ goes to 0 as $n \to \infty$, (5.27) follows.

Consider now the case $2j < k$. We may assume that $l_i \geq 2$ for all $1 \leq i \leq j$. Estimating

$$\mathbb{E}\left[|\widehat{\xi}_0^{(n)}|^{l_i}\right] \leq 2^{l_i}\mathbb{E}\left[|\xi_0^{(n)}|^{l_i}\right] = 2^{l_i}\mathbb{E}\left[|\xi_0^{(n)}|^{l_i-2}|\xi_0^{(n)}|^2\right] \leq 2^{l_i} b_n^{l_i-2}, \tag{5.29}$$

we obtain

$$\begin{aligned}\mathbb{E}\left[\widehat{m}_n^{(k)}(j,l)\right] &\leq 2^k b_n^{k-2j} E\left[\prod_{i=1}^{j} I_n^{[l_i]}\right] \\ &\leq C b_n^{k-2j} n^j (\log n)^{k-j},\end{aligned} \tag{5.30}$$

where we used the multidimensional version of Hölder's inequality and Lemma 4.2(i). Hence

$$\frac{\mathbb{E}\left[\widehat{m}_n^{(k)}(j,l)\right]}{(n\log n)^{k/2}} \leq C\left(\frac{b_n^2 \log n}{n}\right)^{\frac{k}{2}-j} = \frac{C}{(\log n)^{(\gamma-1)(\frac{k}{2}-j)}} \tag{5.31}$$

which goes to 0 as $n \to \infty$ since $\gamma > 1$. □

The rest of the analysis consists in showing that all other terms with $A^c \neq \emptyset$ converge to zero a.s. along τ_n when normalized.

Proposition 5.4. *For any fixed choice of k, j, l, if $A^c \neq \emptyset$ then*

$$\lim_{n\to\infty} \frac{\widehat{m}_{\tau_n}^{(k)}(j,l,A)}{(\tau_n \log \tau_n)^{k/2}} = 0 \ \ \mathbb{P}\text{-}a.s. \tag{5.32}$$

Proof. We may suppose that $A \subset \mathcal{A}$. Recall that the second moment of $\widehat{m}_n^{(k)}(j,l,A)$ is given by (5.23). Ignoring the indicator function in the definition of $\widehat{a}_n$ given in (5.22), we may estimate

$$\begin{aligned}\left\|\widehat{m}_n^{(k)}(j,l,A)\right\|_2^2 \leq \\ \prod_{i\notin A}\left\|(\widehat{\xi}_0^{(n)})^{l_i} - \mathbb{E}\left[(\widehat{\xi}_0^{(n)})^{l_i}\right]\right\|_2^2 \prod_{i\in A}\mathbb{E}\left[(\widehat{\xi}_0^{(n)})^{l_i}\right]^2 B_n(l,A)\end{aligned} \tag{5.33}$$

where

$$\begin{aligned}B_n(l,A) &:= \sum_{x_i:\, i\notin A}\left(\sum_{x_i:\, i\in A} E\left[\prod_{i=1}^{j} N_n^{l_i}(x_i)\right]\right)^2 \\ &= \sum_{x_i:\, i\notin A}\left(E\left[\prod_{i\in A} I_n^{[l_i]} \prod_{i\notin A} N_n^{l_i}(x_i)\right]\right)^2.\end{aligned} \tag{5.34}$$

We proceed to bound $B_n(l,A)$. Denoting by $\widehat{N}_n(x)$ and $\widehat{I}_n^{[p]}$ the analogues of $N_n(x)$ and $I_n^{[p]}$ for an independent copy $\widehat{S}$ of S, we can rewrite (5.34) as

$$\begin{aligned}
&\sum_{x_i:\, i\notin A} E^{\otimes 2}\left[\prod_{i\in A} I_n^{[l_i]}\widehat{I}_n^{[l_i]}\prod_{i\notin A} N_n^{l_i}(x_i)\widehat{N}_n^{l_i}(x_i)\right]\\
&= E^{\otimes 2}\left[\prod_{i\in A} I_n^{[l_i]}\widehat{I}_n^{[l_i]}\prod_{i\notin A} Q_n^{[l_i]}\right]\\
&\le \prod_{i\in A} E\left[(I_n^{[l_i]})^j\right]^{2/j}\prod_{i\notin A} E^{\otimes 2}\left[(Q_n^{[l_i]})^j\right]^{1/j}\\
&\le Cn^{2|A|}(\log n)^{2\sum_{i\in A}(l_i-1)}\prod_{i\notin A} E^{\otimes 2}\left[(Q_n^{[l_i]})^j\right]^{1/j}, \qquad (5.35)
\end{aligned}$$

where, for the next-to-last inequality, we use the multidimensional version of Hölder's inequality and, for the last one, Lemma 4.2(i).

Since $A\subset\mathcal{A}$, we have

$$2|A|+|A^c|\le \sum_{i\in A} l_i+\sum_{i\notin A} l_i = k. \qquad (5.36)$$

There are two cases. If the inequality in (5.36) is strict, we estimate

$$\begin{aligned}
E^{\otimes 2}\left[(Q_n^{[p]})^j\right] &= \sum_{x_1,\dots,x_j} E\left[\prod_{i=1}^j N_n^p(x_i)\right]^2\\
&\le \sum_{x_1,\dots,x_j} E\left[\prod_{i=1}^j N_n^{2p}(x_i)\right]\\
&= E\left[(I_n^{[2p]})^j\right]\le C(n(\log n)^{2p-1})^j, \qquad (5.37)
\end{aligned}$$

where we used the Cauchy-Schwarz' inequality and Lemma 4.2(i). From (5.35) we obtain that, in this case,

$$B_n(l,A)\le Cn^{2|A|+|A^c|}(\log n)^{2k-2|A|-|A^c|}. \qquad (5.38)$$

If there is equality in (5.36), then $l_i=2$ for all $i\in A$ and $l_i=1$ for all $i\notin A$. Thus we may estimate, using (5.35) and Lemma 4.2(iii),

$$B_n(l,A)\le Cn^{2|A|+|A^c|}(\log n)^{2\sum_{i\in A} l_i-2|A|} = Cn^k(\log n)^{k-|A^c|}. \qquad (5.39)$$

Summarizing, we have

$$B_n(l,A)\le \begin{cases} Cn^k(\log n)^{k-|A^c|} & \text{if } 2|A|+|A^c|=k,\\ Cn^{2|A|+|A^c|}(\log n)^{2k-2|A|-|A^c|} & \text{if } 2|A|+|A^c|<k. \end{cases} \qquad (5.40)$$

For convenience we define

$$t := k-2|A|-|A^c|\ge 0. \qquad (5.41)$$

We will consider three cases separately:

1. $t \geq 1$;
2. $t = 0$ and $|A^c| \geq 2$;
3. $t = 0$ and $|A^c| = 1$.

For each of these cases we will show that, for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|\widehat{m}_{\tau_n}^{(k)}(j,l,A)| > \epsilon\sqrt{\tau_n \log \tau_n}\right) < \infty. \tag{5.42}$$

In the first two cases this will be done via Markov's inequality together with (5.33) and (5.40). In the third case, the bound thus obtained is not good enough, but we will get a better estimate using Bernstein's inequality.

Case 1: $(t \geq 1)$ For $i \in A$, we have

$$\mathbb{E}\left[(\widehat{\xi}_0^{(n)})^{l_i}\right]^2 \leq 2^{2l_i} b_n^{2(l_i-2)} \tag{5.43}$$

as in (5.29) and, for $i \notin A$, we can estimate in a similar fashion

$$\begin{aligned}\left\|(\widehat{\xi}_0^{(n)})^{l_i} - \mathbb{E}\left[(\widehat{\xi}_0^{(n)})^{l_i}\right]\right\|_2^2 &\leq 2^{2l_i}\mathbb{E}\left[(\xi_0^{(n)})^{2l_i}\right] \\ &= 2^{2l_i}\mathbb{E}\left[(\xi_0^{(n)})^{2(l_i-1)}(\xi_0^{(n)})^2\right] \\ &\leq 2^{2l_i} b_n^{2(l_i-1)}. \end{aligned} \tag{5.44}$$

Using (5.33) and (5.40) we get

$$\frac{\left\|\widehat{m}_n^{(k)}(j,l,A)\right\|_2^2}{(n\log n)^k} \leq C\left(\frac{b_n^2 \log n}{n}\right)^t = C(\log n)^{-t(\gamma-1)}, \tag{5.45}$$

which is summable along τ_n since $\alpha > 1/2 > (\gamma-1)^{-1}$.

Case 2: $(t = 0, |A^c| \geq 2)$ As mentioned above, in this case $l_i = 2$ for $i \in A$ and $l_i = 1$ for $i \notin A$. Using $\mathbb{E}[|\widehat{\xi}_0^{(n)}|^2] \leq \mathbb{E}\left[|\xi_0|^2\right]$, we get from (5.33) and (5.40) that

$$\frac{\left\|\widehat{m}_n^{(k)}(j,l,A)\right\|_2^2}{(n\log n)^k} \leq C(\log n)^{-2} \tag{5.46}$$

which is summable along τ_n since $\alpha > 1/2$.

Case 3: $(t = 0, |A^c| = 1)$ In this case, (5.33) is not enough to prove (5.42). However, since $|A^c| = 1$ and $t = 0$, these terms are of the form

$$\widehat{m}_n^{(k)}(j,l,A) = \sum_{x\in\mathbb{Z}^2} \widehat{\xi}_x^{(n)} a_n(x) \tag{5.47}$$

where

$$a_n(x) = \sum_{x_i\,:\,i\neq i_*} \mathbb{E}\left[(\widehat{\xi}_0^{(n)})^2\right]^{(j-1)} \mathbb{E}\left[N_n(x)\prod_{i\neq i_*} N_n^2(x_i)\right] \mathbf{1}_{\{x_1<\cdots<x_j\}} \tag{5.48}$$

for some $i_* \in \{1,\dots,j\}$. Thus, the term is a sum of independent, centered and uniformly bounded random variables, and we may apply Bernstein's inequality (see e.g. [13], Exercise 4.3.14.).

Set $Y_x := \widehat{\xi}_x^{(n)} a_n(x)$. Ignoring the indicator function in (5.48) and using Hölder's inequality and Lemmas 4.1–4.2, we get that

$$\sup_{x\in\mathbb{Z}^2} |Y_x| \leq C b_n n^{j-1} (\log n)^j = C n^{j-\frac{1}{2}} (\log n)^{j-\gamma/2}. \tag{5.49}$$

Furthermore,

$$\mathrm{Var}\left(\sum_{x\in\mathbb{Z}^2} Y_x\right) = \mathbb{E}\left[(\widehat{\xi}_0^{(n)})^2\right] \sum_{x\in\mathbb{Z}^2} a_n^2(x) \leq C B_n(l,A) \leq C n^k (\log n)^{k-1} \tag{5.50}$$

by (5.40). Summing Y_x first in a finite subset B of $\mathbb{Z}^2$ and then taking $B \uparrow \mathbb{Z}^2$, we get from Bernstein's inequality that (observe that $j = (k+1)/2$))

$$\begin{aligned}
&\mathbb{P}\left(|\widehat{m}_n^{(k)}(j,l,A)| > \epsilon (n\log n)^{k/2}\right) \\
&\qquad\qquad \leq \exp\left\{\frac{-C(n\log n)^k}{n^k(\log n)^{k-1} + n^k(\log n)^{k-\frac{(\gamma-1)}{2}}}\right\} \\
&\qquad\qquad \leq \exp\left\{-\frac{C}{2}\log n\right\}
\end{aligned} \tag{5.51}$$

since $\gamma > 3$, and (5.51) is summable along τ_n.

□

5.4 Conclusion

From the results of Section 5.3 we obtain the following two propositions. Together with Propositions 5.1 and 5.2, they will allow us to finish the proof of Proposition 3.1.

Proposition 5.5. (Convergence of annealed moments)
For every $k \in \mathbb{N}^*$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mathbb{E}\left[\widehat{m}_n^{(k)}\right]}{(n\log n)^{k/2}} &= \frac{1}{\sqrt{2\pi}\sigma} \int x^k e^{-x^2/2\sigma^2} dx \\
&= \begin{cases} \sigma^k (k-1)!! & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}
\end{aligned} \tag{5.52}$$

Proof. Taking expectations in (5.13) and applying Proposition 5.3, we see that all the terms converge to zero if k is odd, while, if $k = 2j$, then the only surviving term is the one with $l_i = 2$ for all $1 \le i \le j$. The corresponding combinatorial factor in (5.13) is

$$\binom{k}{2,\dots,2} = \frac{(2j)!}{2^j} = 1 \cdot 3 \cdots (2j-1) j! = (k-1)!!\, j!, \tag{5.53}$$

so (5.52) follows from Proposition 5.3. □

Proposition 5.6. (Convergence of quenched moments)
For every $k \in \mathbb{N}^*$,

$$\lim_{n\to\infty} \frac{\widehat{m}^{(k)}_{\tau_n} - \mathbb{E}\left[\widehat{m}^{(k)}_{\tau_n}\right]}{(\tau_n \log \tau_n)^{k/2}} = 0 \ \ \mathbb{P}\text{-}a.s. \tag{5.54}$$

Proof. Combining (5.13) and (5.20), we see that $\widehat{m}^{(k)}_n - \mathbb{E}\left[\widehat{m}^{(k)}_n\right]$ is a sum of terms $\widehat{m}^{(k)}_n(j,l,A)$ with $A \subset \mathcal{A}$, $A^c \neq \emptyset$, so the result follows from Proposition 5.4. □

Proof of Proposition 3.1. The conclusion is now straightforward: Propositions 5.5–5.6 give us (3.2) with $\widehat{Z}^{(n)}_n$ in place of Z_n by the method of moments, and this is passed to Z_n by Propositions 5.1–5.2. □

6 Proof of Proposition 3.2

Before we start, we note some properties of the subsequence τ_n that will be used in the sequel: there exist positive constants K_1, K_2 such that

$$\begin{array}{ll} \text{(p1)} & \lim_{n\to\infty} \tau_{n+1}/\tau_n = 1; \\ \text{(p2)} & K_1 \exp\left(n^\alpha/2\right) \le \tau_{n+1} - \tau_n \le K_2 \tau_n / n^{1-\alpha} \ \ \forall\, n \in \mathbb{N}^*; \\ \text{(p3)} & \tau_n \le K_2 \exp\left(n^\alpha\right) \ \ \forall\, n \in \mathbb{N}^*. \end{array} \tag{6.1}$$

Proof. For $b > a \in \mathbb{N}$, let

$$Z_{a,b} := Z_b - Z_a = \sum_{j=a+1}^{b} \xi_{S_j} = \sum_{x\in\mathbb{Z}^2} \xi_x N_{a,b}(x), \tag{6.2}$$

where $N_{a,b}(x) := \sum_{j=a+1}^{b} \mathbf{1}_{\{S_j = x\}}$. Once we show that

$$\lim_{n\to\infty} \sup_{\tau_n < k \le \tau_{n+1}} \frac{|Z_{\tau_n,k}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.}, \tag{6.3}$$

Proposition 3.2 will follow by noting that

$$\left|\frac{Z_n}{\sqrt{n\log n}}-\frac{Z_{\tau_{i(n)}}}{\sqrt{\tau_{i(n)}\log\tau_{i(n)}}}\right|\le\frac{|Z_n-Z_{\tau_{i(n)}}|}{\sqrt{n\log n}} + \frac{|Z_{\tau_{i(n)}}|}{\sqrt{\tau_{i(n)}\log\tau_{i(n)}}}\left(1-\sqrt{\frac{\tau_{i(n)}\log\tau_{i(n)}}{n\log n}}\right). \quad (6.4)$$

Then, by (6.3) and since $\lim_{n\to\infty} n^{-1}\tau_{i(n)} = 1$, the first term in the r.h.s. of (6.4) converges a.s. to 0. Moreover, the second term converges for $\mathbb{P}$-a.e. ξ in $\mathbb{P}(\cdot|\xi)$-probability to 0 since, by Proposition 3.1, $Z_{\tau_n}/\sqrt{\tau_n\log\tau_n}$ is a.s. tight under $\mathbb{P}(\cdot|\xi)$. Therefore, we only need to show (6.3). For this end, we will again make use of a truncation argument.

Analogously to (5.2), let

$$\begin{aligned} Z^{(n)}_{a,b} &:= \sum_{i=a+1}^{b}\xi^{(n)}_{S_i} = \sum_{x\in\mathbb{Z}^2}\xi^{(n)}_x N_{a,b}(x),\\ \widehat{Z}^{(n)}_{a,b} &:= \sum_{i=a+1}^{b}\widehat{\xi}^{(n)}_{S_i} = \sum_{x\in\mathbb{Z}^2}\widehat{\xi}^{(n)}_x N_{a,b}(x). \end{aligned} \quad (6.5)$$

We will now show that it is enough to prove (6.3) for $\widehat{Z}^{(\tau_{n+1})}_{\tau_n,k}$ in place of $Z_{\tau_n,k}$. First, a direct consequence of Proposition 5.1 is

$$\lim_{n\to\infty}\sup_{\tau_n<k\le\tau_{n+1}}\frac{\left|Z_{\tau_n,k}-Z^{(\tau_{n+1})}_{\tau_n,k}\right|}{\sqrt{\tau_n\log\tau_n}}=0 \quad \mathbb{P}\text{-a.s.} \quad (6.6)$$

On the other hand,

$$\left|Z^{(n)}_{a,b}-\widehat{Z}^{(n)}_{a,b}\right| = \left|\mathbb{E}\left[\xi^{(n)}_0\right]\right|(b-a)\le\frac{C(b-a)}{\sqrt{n(\log n)^\gamma}} \quad (6.7)$$

as in (5.9); thus for $\tau_n < k \le \tau_{n+1}$ we have

$$\left|Z^{(\tau_{n+1})}_{\tau_n,k}-\widehat{Z}^{(\tau_{n+1})}_{\tau_n,k}\right|\le C\frac{\tau_{n+1}-\tau_n}{\sqrt{\tau_{n+1}(\log\tau_{n+1})^\gamma}}\le C\frac{\sqrt{\tau_n}}{n^{1-\alpha}} \quad (6.8)$$

by properties of τ_n, and we obtain

$$\lim_{n\to\infty}\sup_{\tau_n<k\le\tau_{n+1}}\frac{\left|Z^{(\tau_{n+1})}_{\tau_n,k}-\widehat{Z}^{(\tau_{n+1})}_{\tau_n,k}\right|}{\sqrt{\tau_n\log\tau_n}}=0. \quad (6.9)$$

Therefore by (6.6) and (6.9) it is enough to show that

$$\lim_{n\to\infty}\sup_{\tau_n<k\le\tau_{n+1}}\frac{\left|\widehat{Z}^{(\tau_{n+1})}_{\tau_n,k}\right|}{\sqrt{\tau_n\log\tau_n}}=0 \quad \mathbb{P}\text{-a.s.} \quad (6.10)$$

To prove (6.10), we will use a maximal inequality for demimartingales proved by Newman and Wright in [27], and Bernstein's inequality. We first discuss the maximal inequality.

For fixed a and n, we can see using (6.5) that the process $(\widehat{Z}^{(n)}_{a,k})_{k>a}$ has positively associated increments under $\mathbb{P}(\cdot|S)$, as they are all increasing functions of $\widehat{\xi}^{(n)}$ which is an i.i.d. random field. By Proposition 2 in [27], $\widehat{Z}^{(n)}_{a,k}$ is a demimartingale under $\mathbb{P}(\cdot|S)$, which implies, by the definition of a demimartingale, that it is also a demimartingale under $\mathbb{P}$. Therefore, by Corollary 6 of [27], for all $u > 0$,

$$\mathbb{P}\left(\sup_{a<k\leq b} |\widehat{Z}^{(n)}_{a,k}| > 2u\right) \leq \frac{\sqrt{2\mathbb{E}\left[|\widehat{Z}^{(n)}_{a,b}|^2\right]}}{u}\mathbb{P}\left(|\widehat{Z}^{(n)}_{a,b}| > u\right)^{\frac{1}{2}}. \tag{6.11}$$

Note that

$$\mathbb{E}\left[(\widehat{Z}^{(n)}_{a,b})^2\Big|S\right] \leq CI_{a,b} \tag{6.12}$$

where $I_{a,b} := \sum_x N^2_{a,b}(x)$ has the same distribution as I_{b-a}. Using Lemma 4.2(i), (6.11) and the properties of τ_n we see that for all $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{\tau_n<k\leq\tau_{n+1}} |\widehat{Z}^{(\tau_{n+1})}_{\tau_n,k}| > 2\epsilon\sqrt{\tau_n \log \tau_n}\right) \leq \frac{C}{\epsilon}\mathbb{P}\left(|\widehat{Z}^{(\tau_{n+1})}_{\tau_n,\tau_{n+1}}| > \epsilon\sqrt{\tau_n \log \tau_n}\right)^{\frac{1}{2}}. \tag{6.13}$$

We proceed to bound the r.h.s. of (6.13) with the help of Bernstein's inequality. From (6.5), we can see that $\widehat{Z}^{(n)}_{a,b}$ is under $\mathbb{P}(\cdot|S)$ a finite sum of independent and centered random variables, which are uniformly bounded by $b_n N^*_{a,b}$, where $N^*_{a,b} := \sup_x N_{a,b}(x)$, and with variance bounded by (6.12). Using Bernstein's inequality applied under $\mathbb{P}(\cdot|S)$ and then integrating, we obtain

$$\mathbb{P}\left(|\widehat{Z}^{(n)}_{a,b}| > u\right) \leq E\left[\exp\left\{-C\frac{u^2}{I_{a,b} + b_n N^*_{a,b} u}\right\}\right]. \tag{6.14}$$

Since the pair $I_{a,b}, N^*_{a,b}$ has the same law as I_{b-a}, N^*_{b-a}, we may write

$$\mathbb{P}\left(|\widehat{Z}^{(\tau_{n+1})}_{\tau_n,\tau_{n+1}}| > \epsilon\sqrt{\tau_n \log \tau_n}\right) \leq E\left[\exp\left\{-C\frac{\tau_n \log \tau_n}{I_{\tau_{n+1}-\tau_n} + b_{\tau_{n+1}} N^*_{\tau_{n+1}-\tau_n}\sqrt{\tau_n \log \tau_n}}\right\}\right]. \tag{6.15}$$

Recall that, by Lemma 4.1(ii), there exists $C > 0$ such that

$$P\left(N^*_k > C(\log k)^2\right) \leq k^{-2} \ \ \forall\, k \in \mathbb{N}^*. \tag{6.16}$$

Now fix $0 < \delta < \alpha^{-1} - 1$ and an integer $\theta > 2/(\alpha\delta)$. By Markov's inequality and Lemma 4.2(i) we have

$$P\left(I_k > k(\log k)^{1+\delta}\right) \leq \frac{E\left[I_k^\theta\right]}{k^\theta (\log k)^{(1+\delta)\theta}} \leq \frac{C}{(\log k)^{\theta\delta}} \quad \forall\, k \geq 2. \tag{6.17}$$

By (6.15), (6.16), (6.17), the subadditivity of $\sqrt{\cdot}$, and the fact that $e^{-2x/(y+z)} \leq e^{-x/y} + e^{-x/z}$ for any $x, y, z > 0$, we see that (6.13) is at most

$$C_1(\tau_{n+1} - \tau_n)^{-1} + C_2(\log(\tau_{n+1} - \tau_n))^{-\frac{\theta\delta}{2}} + e^{-C_3 a_n/b_n} + e^{-C_3 a_n/c_n}, \tag{6.18}$$

where C_1, C_2 and C_3 are positive constants and

$$\begin{aligned} a_n &:= \tau_n \log \tau_n, \\ b_n &:= (\tau_{n+1} - \tau_n)\left[\log(\tau_{n+1} - \tau_n)\right]^{1+\delta}, \\ c_n &:= b_{\tau_{n+1}}[\log(\tau_{n+1} - \tau_n)]^2 \sqrt{\tau_n \log \tau_n}. \end{aligned} \tag{6.19}$$

Using the properties of τ_n, we see that the first term of (6.18) is summable; by our choice of θ, so is the second. Furthermore,

$$a_n/b_n \geq Cn^{1-\alpha}/(\log \tau_n)^\delta \geq Cn^{1-\alpha(1+\delta)}, \tag{6.20}$$

so the third term is summable by our choice of δ. As for the last term, note that

$$c_n \leq C\tau_n (\log \tau_n)^{2-\frac{(\gamma-1)}{2}} \tag{6.21}$$

and so

$$a_n/c_n \geq C(\log \tau_n)^{\frac{(\gamma-1)}{2}-1}. \tag{6.22}$$

Since $\gamma > 3$, the last term is again summable. Therefore, by the Borel-Cantelli lemma, (6.10) holds, and the proof is complete. □

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