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# A Quenched Central Limit Theorem for Planar Random Walks in Random Sceneries

Nadine Guillotin-Plantard <sup>\*</sup>, Julien Poisat <sup>†</sup>  
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## Abstract

Random walks in random sceneries (RWRS) are simple examples of stochastic processes in disordered media. They were introduced at the end of the 70's by Kesten-Spitzer and Borodin, motivated by the construction of new self-similar processes with stationary increments. Two sources of randomness enter in their definition: a random field  $\xi = (\xi_x)_{x \in \mathbb{Z}^d}$  of i.i.d. random variables, which is called the *random scenery*, and a random walk  $S = (S_n)_{n \in \mathbb{N}}$  evolving in  $\mathbb{Z}^d$ , independent from the scenery. The RWRS  $Z = (Z_n)_{n \in \mathbb{N}}$  is then defined as the accumulated scenery along the trajectory of the random walk, i.e.,  $Z_n := \sum_{k=1}^n \xi_{S_k}$ . The law of  $Z$  under the joint law of  $\xi$  and  $S$  is called “annealed”, and the conditional law given  $\xi$  is called “quenched”. Recently, central limit theorems under the quenched law were proved for  $Z$  by the first two authors for a class of transient random walks including walks with finite variance in dimension  $d \geq 3$ . In this paper we extend their results to dimension  $d = 2$ .

## 1 Introduction

Let  $d \geq 1$  and  $(\xi(x), x \in \mathbb{Z}^d)$  be a collection of independent and identically distributed (i.i.d.) real random variables, further referred to as *scenery*, and  $(S_n)_{n \geq 0}$  a random walk evolving in  $\mathbb{Z}^d$ , independent from the scenery. The random walk in random scenery (RWRS) is the process obtained by adding up the values of the scenery seen by the random walk along its trajectory, that is

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$Z_n = \xi(S_1) + \dots + \xi(S_n)$ , for all  $n \geq 1$ . This model was introduced independently by Kesten and Spitzer [23] and by Borodin [4, 5].

RWRS appears naturally in a variety of contexts, for instance (i) in the energy function of statistical mechanics models of polymers interacting with a random medium, (ii) in Bouchaud's trap model via the clock process, see [2], (iii) in the study of random walks in randomly oriented lattices, as in [6, 9]. The last example is related to the phenomenon of anomalous diffusion in layered random media, see Le Doussal [18] and Matheron and de Marsily [26] on this matter. Indeed, Kesten and Spitzer's original motivation was to build a new class of self-similar stochastic processes with non-standard normalizations.

Results were first established under the *annealed* (or product) measure, that is when one averages at the same time over the scenery and the random walk. Let us suppose here that  $d = 1$  and that the scenery and the random walk are in the domains of attraction of different stable laws with index  $1 < a \leq 2$  and  $0 < b \leq 2$ , respectively. Kesten and Spitzer [23] proved in that case that the process  $(n^{-\delta} Z_{\lceil nt \rceil})_{t \geq 0}$  converges weakly, as  $n \rightarrow \infty$ , to a continuous  $\delta$ -self-similar process, where  $\delta = 1 - a^{-1} + (ab)^{-1}$ . Later on, Bolthausen [3] proved that for  $a = 1$  and  $b = 2$  a functional central limit theorem is satisfied for  $((n \log n)^{-1/2} Z_{\lceil nt \rceil})_{t \geq 0}$  and his result covers the case of a two-dimensional recurrent random walk as well (see also Borodin [4]). More recently, Castell, Guilloin-Plantard and Pène [8] proved that when  $0 < a < 1$  and for arbitrary  $b$ , or  $d = a \in \{1, 2\}$  and  $0 < b < 2$ , then  $(Z_{\lceil nt \rceil})_{t \geq 0}$  has to be normalized by  $n^{1/b}$ , respectively  $n^{1/b}(\log n)^{1-1/b}$  so that it converges to a limiting process. The case of transient random walk has been dealt with in Spitzer [28] and Kesten and Spitzer [23]: the standard  $\sqrt{n}$ -scaling and Gaussian limit hold. Other results on RWRS include strong approximation results and laws of the iterated logarithm [15, 16, 24], limit theorems for correlated sceneries or walks [14, 22], large and moderate deviations results [1, 7, 10, 20], ergodic and mixing properties [19].

Distributional limit theorems for *quenched* sceneries (that is, conditionally given the scenery) are more recent. The first result in this direction that we are aware of was obtained by Ben Arous and Černý [2], in the case of a heavy-tailed scenery and planar random walk. Recently, the first two authors proved in [21] that a quenched central limit theorem (with the usual  $\sqrt{n}$ -scaling and Gaussian law in the limit) holds for a class of transient random walks. With one of the methods used there, namely convergence of moments, they could prove convergence along a subsequence for sceneries having finite moments of all orders and planar random walks with finite non-singular covariance matrices, after a non-standard scaling by  $\sqrt{n \log n}$ . The question was raised whether the convergence takes place along the full sequence. In this paper we are able to answer this question in the positive when the scenery has a finite moment of order strictly greater than two.

## 2 Notation, assumptions and results

Let us start with a few words about notation. We will denote by  $\mathbb{N} := \{0, 1, 2, \dots\}$  the set of non-negative integers and put  $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ .

For two sequences  $a_n, b_n$  of positive real numbers, we will write  $a_n \sim b_n$  to denote that  $\lim_{n \rightarrow \infty} a_n/b_n = 1$ .

We will write  $K$  and  $C$  to denote generic positive constants that may change from expression to expression; we will consistently use  $K$  in statements of lemmas, etc, and  $C$  inside proofs.

We now proceed to define the model. Let  $S = (S_n)_{n \geq 0}$  be a random walk in  $\mathbb{Z}^2$  starting at 0, i.e.,  $S_0 = 0$  and

$$(S_n - S_{n-1})_{n \geq 1} \text{ is a sequence of i.i.d. } \mathbb{Z}^2\text{-valued random variables.} \quad (2.1)$$

We denote the local times of the random walk by

$$N_n(x) := \sum_{1 \leq k \leq n} \mathbf{1}_{\{S_k=x\}}, \quad \forall x \in \mathbb{Z}^2. \quad (2.2)$$

Let  $\xi = (\xi_x)_{x \in \mathbb{Z}^2}$  be a field of i.i.d. real random variables independent of  $S$ . The field  $\xi$  is called *the random scenery*.

The random walk in random scenery (RWRS)  $Z = (Z_n)_{n \geq 0}$  is defined by setting  $Z_0 := 0$  and, for  $n \in \mathbb{N}^*$ ,

$$Z_n := \sum_{i=1}^n \xi_{S_i} = \sum_{x \in \mathbb{Z}^2} \xi_x N_n(x). \quad (2.3)$$

We will denote by  $\mathbb{P}$  the joint law of  $S$  and  $\xi$ , and by  $P$  the marginal of  $S$ . The law  $\mathbb{P}$  is called the *annealed* law, while the conditional law  $\mathbb{P}(\cdot | \xi)$  is called the *quenched* law.

We will make the following two assumptions on the random walk and on the random scenery:

**(A1)** The random walk increment  $S_1$  has a centered law with a finite and non-singular covariance matrix  $\Sigma$ . We further suppose that the random walk is aperiodic in the sense of Spitzer [28], which amounts to requiring that the walk is not confined to a proper subgroup of  $\mathbb{Z}^2$ .

**(A2)**  $\mathbb{E}[\xi_0^2] = 1$  and there exists a  $\gamma > 2$  such that

$$\mathbb{E}[|\xi_0|^\gamma] < +\infty. \quad (2.4)$$

The aim of this paper is to prove the following quenched central limit theorem.

**Theorem 2.1.** *Under assumptions (A1) and (A2), for  $\mathbb{P}$ -a.e.  $\xi$ , for all  $x \in \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( Z_n \geq x \sqrt{n \log n} \mid \xi \right) = \frac{1}{\sqrt{2\pi\sigma}} \int_x^\infty e^{-u^2/2\sigma^2} du, \quad (2.5)$$

where  $\sigma^2 = (\pi \sqrt{\det \Sigma})^{-1}$ .

**Remark:** The conclusion of this theorem still holds if, alternatively, the assumption (A1) is replaced by the following one:

(A1') The sequence  $S = (S_n)_{n \geq 0}$  is an aperiodic random walk in  $\mathbb{Z}$  starting from 0 such that  $(\frac{S_n}{n})_n$  converges in distribution to a random variable with characteristic function given by  $t \mapsto \exp(-a|t|)$ ,  $a > 0$ . In this case,  $\sigma^2 = 2(\pi a)^{-1}$ .

Indeed, the proof of Theorem 2.1 depends on the random walk  $S$  only through certain properties of its local times (including self and mutual intersection local times), which are known to be the same under assumptions (A1) or (A1'). These properties are listed in Section 4.

### 3 Outline of the proof of Theorem 2.1

In [21], the result for  $d = 2$  is obtained by applying the method of moments under the quenched law. More precisely, it is shown that, under the assumption that the scenery has moments of all orders, the random variables  $\mathbb{E}[Z_n^k | \xi]$ ,  $k \in \mathbb{N}^*$ , when properly normalized converge a.s. to the corresponding moments of a normal random variable along a certain subsequence. This subsequence however grows too fast (super-exponentially) to allow for a comparison with the process along the integers, preventing the conclusion of the quenched CLT for  $Z_n$ .

Here we will use a similar approach, but with a number of differences. First of all, the quenched moments are not all well defined under assumptions (A1)–(A2) alone, so in order to apply the method of moments we will first need to truncate the scenery in an appropriate way. For the truncated version, we are able to prove convergence of quenched moments along the subsequence

$$\tau_n := \lceil \exp n^\alpha \rceil, \quad \alpha \in (2/\gamma, 1), \quad (3.1)$$

which grows slowly enough to allow for a comparison with the original process at the end. The most important new ingredient that enables us to prove convergence along  $\tau_n$  are certain  $L^p$  bounds for the quenched moments, which are derived with the help of moment inequalities for random fields. These bounds give good enough concentration of the quenched moments around their mean, which in turn can be shown to converge to the right limits.

The following two propositions directly imply Theorem 2.1.

**Proposition 3.1.** *Under (A1)–(A2), for  $\mathbb{P}$ -a.e.  $\xi$ , for all  $x \in \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( Z_{\tau_n} \geq x \sqrt{\tau_n \log \tau_n} \mid \xi \right) = \frac{1}{\sqrt{2\pi\sigma}} \int_x^\infty e^{-u^2/2\sigma^2} du. \quad (3.2)$$

**Proposition 3.2.** *Define  $i(n) \in \mathbb{N}$  by  $\tau_{i(n)} \leq n < \tau_{i(n)+1}$ . Then, for  $\mathbb{P}$ -a.e.  $\xi$ , the difference*

$$\frac{Z_n}{\sqrt{n \log n}} - \frac{Z_{\tau_{i(n)}}}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \quad (3.3)$$

*converges in probability to 0 as  $n \rightarrow \infty$  under  $\mathbb{P}(\cdot | \xi)$ .*

The rest of the paper is organized as follows. In Section 4 we collect all results concerning two-dimensional random walks that will be used in the other sections. In Section 5, we introduce an  $L^p$ -norm inequality for certain weighted sums of products of random fields; this inequality plays a key role in the proof of Proposition 3.1 given in Section 6. In Section 7 we prove Proposition 3.2, while the proof of the  $L^p$ -norm inequality itself is delayed to Section 8.

## 4 Two-dimensional random walks

We gather here some useful facts concerning the local times of two-dimensional random walks. In the following we always assume (A1). We note that all the results below hold also under the alternative assumption (A1').

### 4.1 Maximum local times

Let

$$N_n^* := \sup_{x \in \mathbb{Z}^2} N_n(x) \quad (4.1)$$

be the maximum local time of the random walk  $S$  up to time  $n$ .

**Lemma 4.1.**

(i) For all  $k \in \mathbb{N}$  and  $\varepsilon > 0$ , there exists a  $K > 0$  such that

$$\sup_{x \in \mathbb{Z}^2} E [N_n^k(x)] \leq Kn^\varepsilon \quad \forall n \in \mathbb{N}^*. \quad (4.2)$$

(ii) For all  $\varepsilon > 0$ , there exists a  $K > 0$  such that

$$P(N_n^* > n^\varepsilon) \leq Kn^{-3} \quad \forall n \in \mathbb{N}^*. \quad (4.3)$$

*Proof.* The two statements follow from the proof of Lemma 2.5 in [3] (note that  $N_n(x)$  is stochastically dominated by  $N_n(0)$ ).  $\square$

### 4.2 Self and mutual intersection local times

For  $p \geq 1$ , the  $p$ -fold self-intersection local time  $I_n^{[p]}$  of  $S$  up to time  $n$  is defined by

$$I_n^{[p]} := \sum_{x \in \mathbb{Z}^2} N_n^p(x). \quad (4.4)$$

When  $p = 2$  we will omit the superscript and write  $I_n$ . Now let  $\widehat{S}$  be an independent copy of  $S$ . We will denote by  $P^{\otimes 2}$  the joint law of  $S$  and  $\widehat{S}$ . The  $p$ -fold mutual intersection local time  $Q_n^{[p]}$  of  $S$  and  $\widehat{S}$  up to time  $n$  is defined by

$$Q_n^{[p]} := \sum_{x \in \mathbb{Z}^2} N_n^p(x) \widehat{N}_n^p(x), \quad (4.5)$$

where  $\widehat{N}_n(x) = \sum_{i=1}^n \mathbf{1}_{\{\widehat{S}_i=x\}}$  is the local time of  $\widehat{S}$  at  $x$  up to time  $n$ . When  $p = 1$  we will omit the superscript and write  $Q_n$ .

**Lemma 4.2.**

(i) For all  $p \geq 2$  and  $k \in \mathbb{N}$ , there exists a  $K > 0$  such that

$$E \left[ (I_n^{[p]})^k \right] \leq K n^k (\log n)^{k(p-1)} \quad \forall n \geq 2. \quad (4.6)$$

(ii) For all  $k \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \frac{E [I_n^k]}{(n \log n)^k} = \sigma^{2k}. \quad (4.7)$$

(iii) For all  $k \in \mathbb{N}$ , there exists a  $K > 0$  such that

$$E^{\otimes 2} [Q_n^k] \leq K n^k \quad \forall n \in \mathbb{N}^*. \quad (4.8)$$

*Proof.* Statement (i) can be found in [21] (Proposition 2.3) and statement (ii) comes as a combination of statement (i) and Theorem 1 in [11]. The last statement follows from the following facts:

1- (Proof of Lemma 5.2 in [12]) For all  $k \in \mathbb{N}$ , there exists some  $C > 0$  such that

$$E^{\otimes 2} [Q_n^k] \leq C E^{\otimes 2} [Q_n]^k \quad \forall n \in \mathbb{N}^*.$$

2- (Proof of Corollary 3.2 in [25]) There exists some  $C > 0$  such that

$$E^{\otimes 2} [Q_n] \leq C n \quad \forall n \in \mathbb{N}^*.$$

□

## 5 An $L^p$ -norm inequality

In this section we formulate, in a more general context, a statement (Proposition 5.1 below) regarding an  $L^p$ -norm inequality for certain weighted sums of products of random fields. This result will play an important role in the proof of Proposition 3.1 in Section 6.

For  $\theta \in \mathbb{N}^*$ , let  $X_i = (X_i(x))_{x \in \mathbb{Z}^d}$ ,  $i = 1, \dots, \theta$  be random fields on  $\mathbb{Z}^d$  with joint law  $\mathbb{P}$ , satisfying the following assumptions:

1. The random fields are all centered, i.e.,

$$\mathbb{E} [X_i(x)] = 0 \quad \forall x \in \mathbb{Z}^d \text{ and } 1 \leq i \leq \theta; \quad (5.1)$$

2. There exists a  $p \geq 2$  such that

$$\Lambda_p(i) := \sup_{x \in \mathbb{Z}^d} \|X_i(x)\|_p < +\infty \quad \forall 1 \leq i \leq \theta; \quad (5.2)$$

3. For any  $k \in \mathbb{N}^*$  and any choice of  $i_1, \dots, i_k \in \{1, \dots, \theta\}$ , the random variables  $X_{i_1}(x_1), \dots, X_{i_k}(x_k)$  are independent if the points  $x_i \in \mathbb{Z}^d$ ,  $i = 1, \dots, k$  are all distinct.

Consider the lexicographical order in  $\mathbb{Z}^d$ , i.e., if  $x = (x(1), \dots, x(d))$  and  $y = (y(1), \dots, y(d))$  are vectors in  $\mathbb{Z}^d$ , write  $y < x$  to denote that there exists a  $k \in \{1, \dots, d\}$  such that  $y(k) < x(k)$  and  $y(i) \leq x(i) \forall 1 \leq i < k$ . We also write  $y \leq x$  to mean that either  $y < x$  or  $y = x$ .

Let  $a : (\mathbb{Z}^d)^\theta \rightarrow \mathbb{R}$  such that

$$\sum_{x_1, \dots, x_\theta} |a(x_1, \dots, x_\theta)| < \infty, \quad (5.3)$$

i.e.,  $a \in \ell_1((\mathbb{Z}^d)^\theta)$ , and that

$$a(x_1, \dots, x_\theta) = 0 \text{ if } x_{i_2} \leq x_{i_1} \text{ for some } 1 \leq i_1 < i_2 \leq \theta. \quad (5.4)$$

Then, by (5.2)–(5.3),

$$m := \sum_{x_1, \dots, x_\theta} \prod_{i=1}^{\theta} X_i(x_i) a(x_1, \dots, x_\theta) \quad (5.5)$$

is well-defined.

We have the following inequality for the  $L^p$ -norm of  $m$ .

**Proposition 5.1.**

$$\|m\|_p^2 \leq (2p)^\theta \left( \prod_{i=1}^{\theta} \Lambda_p^2(i) \right) \sum_{x_1, \dots, x_\theta} a^2(x_1, \dots, x_\theta). \quad (5.6)$$

The proof of this proposition is given in Section 8.

## 6 Proof of Proposition 3.1

The proof is split into several steps. First, we define in Section 6.1 a truncation of the scenery that will allow us to apply the method of moments. This truncation will be again useful in the proof of Proposition 3.2 below. Next we show in Section 6.2 how to decompose the quenched moments in terms that can be analysed with the help of the  $L^p$ -norm inequality of Section 5. This analysis is carried out in Section 6.3. Finally in Section 6.4 these results are combined to conclude the proof of Proposition 3.1.

### 6.1 Truncation

Fix  $\beta \in (1/\gamma, 1/2)$ . For  $n \in \mathbb{N}^*$ , let  $\xi^{(n)}$  and  $\widehat{\xi}^{(n)} \in \mathbb{R}^{\mathbb{Z}^2}$  be defined by

$$\begin{aligned} \xi_x^{(n)} &:= \xi_x \mathbf{1}_{\{|\xi_x| \leq n^\beta\}} \\ \widehat{\xi}_x^{(n)} &:= \xi_x^{(n)} - \mathbb{E} \left[ \xi_x^{(n)} \right] \end{aligned} \quad \text{for } x \in \mathbb{Z}^2, \quad (6.1)$$



and let  $Z^{(n)}$  and  $\widehat{Z}^{(n)}$  be defined by

$$\begin{aligned} Z_k^{(n)} &:= \sum_{i=1}^k \xi_{S_i}^{(n)} = \sum_{x \in \mathbb{Z}^2} \xi_x^{(n)} N_k(x) \\ \widehat{Z}_k^{(n)} &:= \sum_{i=1}^k \widehat{\xi}_{S_i}^{(n)} = \sum_{x \in \mathbb{Z}^2} \widehat{\xi}_x^{(n)} N_k(x) \end{aligned} \quad \text{for } k \in \mathbb{N}^*. \quad (6.2)$$

The following two propositions show that, in order to prove Proposition 3.1 for  $Z_n$ , it is enough to prove it for  $\widehat{Z}_n^{(n)}$ .

**Proposition 6.1.** (Comparison between  $Z$  and  $Z^{(n)}$ )

There exists  $\mathbb{P}$ -a.s. a random time  $T_0 \in \mathbb{N}^*$  such that, if  $n \geq T_0$ , then  $Z_k^{(n)} = Z_k$  for all  $1 \leq k \leq n$ .

*Proof.* Let

$$R_n := \{x \in \mathbb{Z}^2 : N_n(x) > 0\} \quad (6.3)$$

be the range of the random walk  $S$  up to time  $n$ , and set

$$\mathcal{W}_n := \{x \in R_n : \xi_x^{(n)} \neq \xi_x\}. \quad (6.4)$$

We have

$$\mathcal{W}_n \setminus \mathcal{W}_{n-1} = \begin{cases} \{S_n\} & \text{if } S_n \notin R_{n-1} \text{ and } |\xi_{S_n}| > n^\beta, \\ \emptyset & \text{otherwise.} \end{cases} \quad (6.5)$$

Since

$$\mathbb{P}(|\xi_{S_n}| > n^\beta) = \mathbb{P}(|\xi_0| > n^\beta) \leq \frac{\mathbb{E}[|\xi_0|^\gamma]}{n^{\gamma\beta}} \quad (6.6)$$

which is summable since  $\beta > 1/\gamma$ , by Borel-Cantelli there exists a random index  $N_0 \in \mathbb{N}^*$  such that  $\mathcal{W}_n \subset \mathcal{W}_{N_0}$  for all  $n \geq N_0$ . Therefore if we set

$$T_0 := \inf \left\{ n \geq N_0 : n^\beta > \sup_{x \in \mathcal{W}_{N_0}} |\xi_x| \right\}, \quad (6.7)$$

we have  $\mathcal{W}_n = \emptyset$  for  $n \geq T_0$ .  $\square$

**Proposition 6.2.** (Comparison between  $Z^{(n)}$  and  $\widehat{Z}^{(n)}$ )

$$\lim_{n \rightarrow \infty} \frac{|\widehat{Z}_n^{(n)} - Z_n^{(n)}|}{\sqrt{n \log n}} = 0 \quad \mathbb{P}\text{-a.s.} \quad (6.8)$$

*Proof.* Since  $\xi$  is centered,

$$\begin{aligned} |\mathbb{E}[\xi_0 \mathbf{1}_{\{|\xi_0| \leq n^\beta\}}]| &= |\mathbb{E}[\xi_0 \mathbf{1}_{\{|\xi_0| > n^\beta\}}]| \leq \mathbb{E} \left[ |\xi_0| \left( \frac{|\xi_0|}{n^\beta} \right)^{\gamma-1} \right] \\ &\leq \frac{C}{n^{\beta(\gamma-1)}}. \end{aligned} \quad (6.9)$$

Therefore,

$$\frac{|Z_n^{(n)} - \widehat{Z}_n^{(n)}|}{\sqrt{n \log n}} = \frac{n |\mathbb{E}[\xi_0 \mathbf{1}_{\{|\xi_0| \leq n^\beta\}}]|}{\sqrt{n \log n}} \leq \frac{C}{n^{\beta(\gamma-1) - \frac{1}{2}}} \quad (6.10)$$

which goes to 0 as  $n \rightarrow \infty$  since  $\beta \in (1/\gamma, 1/2)$ .  $\square$

## 6.2 Decomposition of quenched moments

From now on, we will work with the truncated and recentered version  $\widehat{Z}^{(n)}$  of the RWRS. Since the  $\widehat{\xi}_x^{(n)}$  are bounded, the quenched moments

$$\widehat{m}_n^{(k)} := \sum_{x_1, \dots, x_k} \prod_{i=1}^k \widehat{\xi}_{x_i}^{(n)} E \left[ \prod_{i=1}^k N_n(x_i) \right] \quad (6.11)$$

are all well defined and satisfy

$$\widehat{m}_n^{(k)} = \mathbb{E} \left[ (\widehat{Z}_n^{(n)})^k \mid \xi \right] \quad \mathbb{P}\text{-a.s.} \quad (6.12)$$

We aim to prove that the  $\widehat{m}_n^{(k)}$  when properly normalized converge a.s. along  $\tau_n$  to the corresponding moments of a Gaussian random variable. In order to do that, we will first show how they can be decomposed into sums of terms that can be controlled with the help of Proposition 5.1.

The first step is to note that  $\widehat{m}_n^{(k)}$  may be decomposed as follows:

$$\widehat{m}_n^{(k)} = \sum_{j=1}^k \sum_{\substack{l=(l_1, \dots, l_j) \in (\mathbb{N}^*)^j \\ l_1 + \dots + l_j = k}} \binom{k}{l_1, \dots, l_j} \widehat{m}_n^{(k)}(j, l), \quad (6.13)$$

where

$$\binom{k}{l_1, \dots, l_j} = \frac{k!}{l_1! \dots l_j!} \quad (6.14)$$

are multinomial coefficients and

$$\widehat{m}_n^{(k)}(j, l) := \sum_{x_1 < \dots < x_j} \prod_{i=1}^j (\widehat{\xi}_{x_i}^{(n)})^{l_i} E \left[ \prod_{i=1}^j N_n^{l_i}(x_i) \right]. \quad (6.15)$$

The  $\widehat{m}_n^{(k)}(j, l)$  are close to the correct form for Proposition 5.1, except that the random fields  $(\widehat{\xi}_x^{(n)})^{l_i}$  are not necessarily centered if  $l_i > 1$ . Using the identity

$$\prod_{i \in \mathcal{I}} (a_i + b_i) = \sum_{A \subset \mathcal{I}} \prod_{i \in A} a_i \prod_{i \notin A} b_i, \quad (6.16)$$

we may further decompose the  $\widehat{m}_n^{(k)}(j, l)$  as

$$\widehat{m}_n^{(k)}(j, l) = \sum_{A \subset \{1, \dots, j\}} \widehat{m}_n^{(k)}(j, l, A), \quad (6.17)$$

where  $\widehat{m}_n^{(k)}(j, l, A) :=$

$$\sum_{x_1 < \dots < x_j} \prod_{i \in A} \mathbb{E} \left[ (\widehat{\xi}_{x_i}^{(n)})^{l_i} \right] \prod_{i \notin A} \left( (\widehat{\xi}_{x_i}^{(n)})^{l_i} - \mathbb{E} \left[ (\widehat{\xi}_{x_i}^{(n)})^{l_i} \right] \right) E \left[ \prod_{i=1}^j N_n^{l_i}(x_i) \right]. \quad (6.18)$$

For fixed  $j$  and  $l$ , let

$$\mathcal{A} := \{i \in \{1, \dots, j\} : l_i > 1\}. \quad (6.19)$$

Noting that  $\widehat{m}_n^{(k)}(j, l, A) = 0$  if  $A \cap \mathcal{A}^c \neq \emptyset$ , and that  $\widehat{m}_n^{(k)}(j, l, \{1, \dots, j\}) = \mathbb{E} \left[ \widehat{m}_n^{(k)}(j, l) \right]$ , we have

$$\widehat{m}_n^{(k)}(j, l) - \mathbb{E} \left[ \widehat{m}_n^{(k)}(j, l) \right] = \sum_{A \subset \mathcal{A} : A^c \neq \emptyset} \widehat{m}_n^{(k)}(j, l, A). \quad (6.20)$$

Moreover, when  $A^c \neq \emptyset$  we may write

$$\widehat{m}_n^{(k)}(j, l, A) = \sum_{x_i : i \notin A} \prod_{i \notin A} \left\{ (\widehat{\xi}_{x_i}^{(n)})^{l_i} - \mathbb{E} \left[ (\widehat{\xi}_{x_i}^{(n)})^{l_i} \right] \right\} \widehat{a}_n((x_i)_{i \notin A}, l, A), \quad (6.21)$$

where

$$\widehat{a}_n((x_i)_{i \notin A}, l, A) := \sum_{x_i : i \in A} \prod_{i \in A} \mathbb{E} \left[ (\widehat{\xi}_{x_i}^{(n)})^{l_i} \right] E \left[ \prod_{i=1}^j N_n^{l_i}(x_i) \right] \mathbf{1}_{\{x_1 < \dots < x_j\}}, \quad (6.22)$$

to see that  $\widehat{m}_n^{(k)}(j, l, A)$  has the same form as  $m$  in (5.5) with

$$X_i(x) := (\widehat{\xi}_x^{(n)})^{l_i} - \mathbb{E} \left[ (\widehat{\xi}_x^{(n)})^{l_i} \right], \quad i \notin A, \quad x \in \mathbb{Z}^2, \quad (6.23)$$

which are centered and satisfy (5.2) for all  $p \geq 2$ , and  $a := \widehat{a}_n$ , which satisfies (5.3) and (5.4). Therefore, Proposition 5.1 may be used to analyse such terms. This will be done in the following section.

### 6.3 Analysis of the terms

We begin with the terms in which  $A^c = \emptyset$ , i.e., the ones corresponding to  $\mathbb{E}[\widehat{m}_n^{(k)}(j, l)]$ .

**Proposition 6.3.**

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ \widehat{m}_n^{(k)}(j, l) \right]}{(n \log n)^{k/2}} = \begin{cases} \frac{1}{j!} \sigma^{2j} & \text{if } 2j = k \text{ and } l_i = 2 \quad \forall 1 \leq i \leq j, \\ 0 & \text{otherwise.} \end{cases} \quad (6.24)$$

*Proof.* We have

$$\begin{aligned} \mathbb{E} \left[ \widehat{m}_n^{(k)}(j, l) \right] &= \sum_{x_1 < \dots < x_j} \prod_{i=1}^j \mathbb{E} \left[ (\widehat{\xi}_{x_i}^{(n)})^{l_i} \right] E \left[ \prod_{i=1}^j N_n^{l_i}(x_i) \right] \\ &= \prod_{i=1}^j \mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^{l_i} \right] \sum_{x_1 < \dots < x_j} E \left[ \prod_{i=1}^j N_n^{l_i}(x_i) \right]. \end{aligned} \quad (6.25)$$

If  $2j > k$ , then at least one of the  $l_i$ 's is equal to one; therefore, by (6.13),  $\mathbb{E}[\widehat{m}_n^{(k)}(j, l)] = 0$  for all  $n \in \mathbb{N}^*$ .

If  $2j = k$  and all  $l_i$ 's are larger than 1, then  $l_i = 2$  for all  $1 \leq i \leq j$ , and

$$\begin{aligned} \mathbb{E} \left[ \widehat{m}_n^{(k)}(j, l) \right] &= \mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^2 \right]^j \sum_{x_1 < \dots < x_j} E \left[ \prod_{i=1}^j N_n^2(x_i) \right] \\ &= \mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^2 \right]^j \frac{1}{j!} \sum_{\substack{x_1, \dots, x_j \\ \text{all distinct}}} E \left[ \prod_{i=1}^j N_n^2(x_i) \right]. \end{aligned} \quad (6.26)$$

Since  $\lim_{n \rightarrow \infty} \mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^2 \right] = 1$ , the claim will follow from Lemma 4.2(ii) once we show that

$$\sum_{\substack{x_1, \dots, x_j \\ \text{all distinct}}} E \left[ \prod_{i=1}^j N_n^2(x_i) \right] \sim E \left[ I_n^j \right] = \sum_{x_1, \dots, x_j} E \left[ \prod_{i=1}^j N_n^2(x_i) \right]. \quad (6.27)$$

But

$$\begin{aligned} \sum_{\substack{x_1, \dots, x_j \\ \exists i_1 \neq i_2 : x_{i_1} = x_{i_2}}} E \left[ \prod_{i=1}^j N_n^2(x_i) \right] &\leq j^2 \sum_{x_2, \dots, x_j} E \left[ N_n^4(x_2) \prod_{i=3}^j N_n^2(x_i) \right] \\ &= j^2 E \left[ I_n^{[4]} I_n^{j-2} \right] \\ &\leq C n^{j-1} (\log n)^{j+1}, \end{aligned} \quad (6.28)$$

where we used the symmetry of the summands, Hölder's inequality and Lemma 4.2(i). Since (6.28) divided by  $(n \log n)^j$  goes to 0 as  $n \rightarrow \infty$ , (6.27) follows.

Consider now the case  $2j < k$ . We may assume that  $l_i \geq 2$  for all  $1 \leq i \leq j$ . Estimating

$$\mathbb{E} \left[ |\widehat{\xi}_0^{(n)}|^{l_i} \right] \leq 2^{l_i} \mathbb{E} \left[ |\xi_0^{(n)}|^{l_i} \right] = 2^{l_i} \mathbb{E} \left[ |\xi_0^{(n)}|^{l_i-2} |\xi_0^{(n)}|^2 \right] \leq 2^{l_i} n^{\beta(l_i-2)}, \quad (6.29)$$

we obtain

$$\begin{aligned} \mathbb{E} \left[ \widehat{m}_n^{(k)}(j, l) \right] &\leq 2^k n^{\beta(k-2j)} E \left[ \prod_{i=1}^j I_n^{[l_i]} \right] \\ &\leq C n^{\beta(k-2j)+j} (\log n)^{k-j} \end{aligned} \quad (6.30)$$

where we used the multidimensional version of Hölder's inequality and Lemma 4.2(i). Hence

$$\frac{\mathbb{E} \left[ \widehat{m}_n^{(k)}(j, l) \right]}{(n \log n)^{k/2}} \leq C \left( \frac{\log n}{n^{1-2\beta}} \right)^{\frac{k}{2}-j} \quad (6.31)$$

which goes to 0 as  $n \rightarrow \infty$  since  $\beta < 1/2$ .  $\square$

The rest of the analysis consists in showing that all other terms with  $A^c \neq \emptyset$  converge to zero a.s. along  $\tau_n$  when normalized.

**Proposition 6.4.** *For any fixed choice of  $k, j, l$ , if  $A^c \neq \emptyset$  then*

$$\lim_{n \rightarrow \infty} \frac{\widehat{m}_{\tau_n}^{(k)}(j, l, A)}{(\tau_n \log \tau_n)^{k/2}} = 0 \quad \mathbb{P}\text{-a.s.} \quad (6.32)$$

*Proof.* We may suppose that  $A \subset \mathcal{A}$ . As mentioned at the end of Section 6.2, when  $A^c \neq \emptyset$  we may apply Proposition 5.1 to  $\widehat{m}_n^{(k)}(j, l, A)$ . Doing so we obtain

$$\begin{aligned} \left\| \widehat{m}_n^{(k)}(j, l, A) \right\|_p^2 &\leq \\ (2p)^{|A^c|} \prod_{i \notin A} \left\| (\widehat{\xi}_0^{(n)})^{l_i} - \mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^{l_i} \right] \right\|_p^2 &\sum_{x_i: i \notin A} \widehat{a}_n^2((x_i)_{i \notin A}, l, A) \end{aligned} \quad (6.33)$$

for all  $p \geq 2$ , where  $\widehat{a}_n$  is as in (6.22). Ignoring the indicator function there we may estimate

$$\begin{aligned} \left\| \widehat{m}_n^{(k)}(j, l, A) \right\|_p^2 &\leq \\ (2p)^{|A^c|} \prod_{i \notin A} \left\| (\widehat{\xi}_0^{(n)})^{l_i} - \mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^{l_i} \right] \right\|_p^2 &\prod_{i \in A} \mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^{l_i} \right]^2 B_n(l, A) \end{aligned} \quad (6.34)$$

where

$$\begin{aligned} B_n(l, A) &:= \sum_{x_i: i \notin A} \left( \sum_{x_i: i \in A} E \left[ \prod_{i=1}^j N_n^{l_i}(x_i) \right] \right)^2 \\ &= \sum_{x_i: i \notin A} \left( E \left[ \prod_{i \in A} I_n^{[l_i]} \prod_{i \notin A} N_n^{l_i}(x_i) \right] \right)^2. \end{aligned} \quad (6.35)$$

We proceed to bound  $B_n(l, A)$ . Denoting by  $\widehat{N}_n(x)$  and  $\widehat{I}_n^{[p]}$  the analogues of  $N_n(x)$  and  $I_n^{[p]}$  for an independent copy  $\widehat{S}$  of  $S$ , we can rewrite (6.35) as

$$\begin{aligned} &\sum_{x_i: i \notin A} E^{\otimes 2} \left[ \prod_{i \in A} I_n^{[l_i]} \widehat{I}_n^{[l_i]} \prod_{i \notin A} N_n^{l_i}(x_i) \widehat{N}_n^{l_i}(x_i) \right] \\ &= E^{\otimes 2} \left[ \prod_{i \in A} I_n^{[l_i]} \widehat{I}_n^{[l_i]} \prod_{i \notin A} Q_n^{[l_i]} \right] \\ &\leq \prod_{i \in A} E \left[ (I_n^{[l_i]})^j \right]^{2/j} \prod_{i \notin A} E^{\otimes 2} \left[ (Q_n^{[l_i]})^j \right]^{1/j} \\ &\leq C n^{2|A|} (\log n)^{2 \sum_{i \in A} (l_i - 1)} \prod_{i \notin A} E^{\otimes 2} \left[ (Q_n^{[l_i]})^j \right]^{1/j}, \end{aligned} \quad (6.36)$$

where, for the next-to-last inequality, we use the multidimensional version of Hölder's inequality and, for the last one, Lemma 4.2(i).

Since  $A \subset \mathcal{A}$ , we have

$$2|A| + |A^c| \leq \sum_{i \in A} l_i + \sum_{i \notin A} l_i = k. \quad (6.37)$$

There are two cases. If the inequality in (6.37) is strict, we estimate

$$\begin{aligned} E^{\otimes 2} \left[ (Q_n^{[p]})^j \right] &= \sum_{x_1, \dots, x_j} E \left[ \prod_{i=1}^j N_n^p(x_i) \right]^2 \\ &\leq \sum_{x_1, \dots, x_j} E \left[ \prod_{i=1}^j N_n^{2p}(x_i) \right] \\ &= E \left[ (I_n^{[2p]})^j \right] \leq C(n(\log n)^{2p-1})^j, \end{aligned} \quad (6.38)$$

where we used Hölder's inequality and Lemma 4.2(i). From (6.36) we obtain that, in this case,

$$B_n(l, A) \leq Cn^{2|A|+|A^c|}(\log n)^{2k-2|A|-|A^c|}. \quad (6.39)$$

If there is equality in (6.37), then  $l_i = 2$  for all  $i \in A$  and  $l_i = 1$  for all  $i \notin A$ . Thus we may estimate, using (6.36) and Lemma 4.2(iii),

$$B_n(l, A) \leq Cn^{2|A|+|A^c|}(\log n)^{2 \sum_{i \in A} l_i - 2|A|} = Cn^k(\log n)^{k-|A^c|}. \quad (6.40)$$

Summarizing, we have

$$B_n(l, A) \leq \begin{cases} Cn^k(\log n)^{k-|A^c|} & \text{if } 2|A| + |A^c| = k, \\ Cn^{2|A|+|A^c|}(\log n)^{2k-2|A|-|A^c|} & \text{if } 2|A| + |A^c| < k. \end{cases} \quad (6.41)$$

For convenience we define

$$t := k - 2|A| - |A^c| \geq 0. \quad (6.42)$$

We will consider separately the cases  $t \geq 1$  and  $t = 0$ . In both cases we will show that, for every  $\epsilon > 0$ ,

$$\sum_{n=1}^{\infty} \mathbb{P} \left( |\widehat{m}_{\tau_n}^{(k)}(j, l, A)| > \epsilon \sqrt{\tau_n \log \tau_n} \right) < +\infty. \quad (6.43)$$

This will be done via Markov's inequality by bounding  $\|\widehat{m}_{\tau_n}^{(k)}(j, l, A)\|_p$  for some  $p \geq 2$  with the estimates above.

Case  $t \geq 1$ : For  $i \in A$ , we have

$$\mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^{l_i} \right]^2 \leq 2^{2l_i} n^{2\beta(l_i-2)} \quad (6.44)$$

as in (6.29) and, for  $i \notin A$ , we can estimate in a similar fashion

$$\begin{aligned} \left\| (\widehat{\xi}_0^{(n)})^{l_i} - \mathbb{E} \left[ (\widehat{\xi}_0^{(n)})^{l_i} \right] \right\|_2^2 &\leq 2^{2l_i} \mathbb{E} \left[ (\xi_0^{(n)})^{2l_i} \right] \\ &= 2^{2l_i} \mathbb{E} \left[ (\xi_0^{(n)})^{2(l_i-1)} (\xi_0^{(n)})^2 \right] \\ &\leq 2^{2l_i} n^{2\beta(l_i-1)}. \end{aligned} \quad (6.45)$$

Using (6.34) and (6.41) with  $p = 2$  we get

$$\frac{\mathbb{E} \left[ \left| \widehat{m}_n^{(k)}(j, l, A) \right|^2 \right]}{(n \log n)^k} \leq C n^{-t(1-2\beta)} (\log n)^t, \quad (6.46)$$

which is summable along  $\tau_n$  since  $\beta < 1/2$ .

Case  $t = 0$ : As mentioned above, in this case  $l_i = 2$  for  $i \in A$  and  $l_i = 1$  for  $i \notin A$ . Using  $\mathbb{E} [|\widehat{\xi}_0^{(n)}|^p] \leq 2^p \mathbb{E} [|\xi_0|^p]$ , we get from (6.34) and (6.41) for  $p = \gamma$  that

$$\frac{\|\widehat{m}_n^{(k)}(j, l, A)\|_\gamma^\gamma}{(n \log n)^{\gamma k/2}} \leq C (\log n)^{-\gamma/2} \quad (6.47)$$

which is summable along  $\tau_n$  since  $\alpha > 2/\gamma$ . □

## 6.4 Conclusion

From the results of Section 6.3 we obtain the following two propositions. Together with Propositions 6.1 and 6.2, they will allow us to finish the proof of Proposition 3.1.

**Proposition 6.5.** (Convergence of annealed moments)

For every  $k \in \mathbb{N}^*$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ \widehat{m}_n^{(k)} \right]}{(n \log n)^{k/2}} &= \frac{1}{\sqrt{2\pi}\sigma} \int x^k e^{-x^2/2\sigma^2} dx \\ &= \begin{cases} \sigma^k (k-1)!! & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases} \end{aligned} \quad (6.48)$$

*Proof.* Taking expectations in (6.13) and applying Proposition 6.3, we see that all the terms converge to zero if  $k$  is odd, while, if  $k = 2j$ , then the only surviving term is the one with  $l_i = 2$  for all  $1 \leq i \leq j$ . The corresponding combinatorial factor in (6.13) is

$$\binom{k}{2, \dots, 2} = \frac{(2j)!}{2^j} = 1 \cdot 3 \cdots (2j-1)j! = (k-1)!! j!, \quad (6.49)$$

so (6.48) follows from Proposition 6.3. □

**Proposition 6.6.** (Convergence of quenched moments)

For every  $k \in \mathbb{N}^*$ ,

$$\lim_{n \rightarrow \infty} \frac{\widehat{m}_{\tau_n}^{(k)} - \mathbb{E} \left[ \widehat{m}_{\tau_n}^{(k)} \right]}{(\tau_n \log \tau_n)^{k/2}} = 0 \quad \mathbb{P}\text{-a.s.} \quad (6.50)$$

*Proof.* Combining (6.13) and (6.20), we see that  $\widehat{m}_n^{(k)} - \mathbb{E} \left[ \widehat{m}_n^{(k)} \right]$  is a sum of terms  $\widehat{m}_n^{(k)}(j, l, A)$  with  $A \subset \mathcal{A}$ ,  $A^c \neq \emptyset$ , so the result follows from Proposition 6.4.  $\square$

*Proof of Proposition 3.1.* The conclusion is now straightforward: Propositions 6.5–6.6 give us (3.2) with  $\widehat{Z}_n^{(n)}$  in place of  $Z_n$  by the method of moments, and this is passed to  $Z_n$  by Propositions 6.1–6.2.  $\square$

## 7 Proof of Proposition 3.2

Before we start, we note some properties of the subsequence  $\tau_n$  that will be used in the sequel: there exist positive constants  $K_1, K_2$  such that

$$\begin{aligned} \text{(p1)} \quad & \lim_{n \rightarrow \infty} \tau_{n+1}/\tau_n = 1; \\ \text{(p2)} \quad & K_1 \exp(n^\alpha/2) \leq \tau_{n+1} - \tau_n \leq K_2 \tau_n / n^{1-\alpha} \quad \forall n \in \mathbb{N}^*; \\ \text{(p3)} \quad & \tau_n \leq K_2 \exp(n^\alpha) \quad \forall n \in \mathbb{N}^*. \end{aligned} \quad (7.1)$$

*Proof.* For  $b > a \in \mathbb{N}$ , let

$$Z_{a,b} := Z_b - Z_a = \sum_{j=a+1}^b \xi_{S_j} = \sum_{x \in \mathbb{Z}^2} \xi_x N_{a,b}(x), \quad (7.2)$$

where  $N_{a,b}(x) := \sum_{j=a+1}^b \mathbf{1}_{\{S_j=x\}}$ . Once we show that

$$\lim_{n \rightarrow \infty} \sup_{\tau_n \leq k \leq \tau_{n+1}} \frac{|Z_{\tau_n, k}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.}, \quad (7.3)$$

Proposition 3.2 will follow by noting that

$$\begin{aligned} \left| \frac{Z_n}{\sqrt{n \log n}} - \frac{Z_{\tau_{i(n)}}}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \right| & \leq \frac{|Z_n - Z_{\tau_{i(n)}}|}{\sqrt{n \log n}} \\ & + \frac{|Z_{\tau_{i(n)}}|}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \left( 1 - \sqrt{\frac{\tau_{i(n)} \log \tau_{i(n)}}{n \log n}} \right). \end{aligned} \quad (7.4)$$

Then, by (7.3) and since  $\lim_{n \rightarrow \infty} n^{-1} \tau_{i(n)} = 1$ , the first term in the r.h.s. of (7.4) converges a.s. to 0. Moreover, the second term converges for  $\mathbb{P}$ -a.e.  $\xi$  in  $\mathbb{P}(\cdot|\xi)$ -probability to 0 since, by Proposition 3.1,  $Z_{\tau_n}/\sqrt{\tau_n \log \tau_n}$  is a.s. tight



under  $\mathbb{P}(\cdot|\xi)$ . Therefore, we only need to show (7.3). For this end, we will again make use of a truncation argument.

Analogously to (6.2), let

$$\begin{aligned} Z_{a,b}^{(n)} &:= \sum_{i=a+1}^b \xi_{S_i}^{(n)} = \sum_{x \in \mathbb{Z}^2} \xi_x^{(n)} N_{a,b}(x), \\ \widehat{Z}_{a,b}^{(n)} &:= \sum_{i=a+1}^b \widehat{\xi}_{S_i}^{(n)} = \sum_{x \in \mathbb{Z}^2} \widehat{\xi}_x^{(n)} N_{a,b}(x). \end{aligned} \tag{7.5}$$

We will now show that it is enough to prove (7.3) for  $\widehat{Z}_{\tau_n, k}^{(\tau_{n+1})}$  in place of  $Z_{\tau_n, k}$ . First, a direct consequence of Proposition 6.1 is

$$\lim_{n \rightarrow \infty} \sup_{\tau_n \leq k \leq \tau_{n+1}} \frac{|Z_{\tau_n, k} - Z_{\tau_n, k}^{(\tau_{n+1})}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.} \tag{7.6}$$

On the other hand,

$$\left| Z_{a,b}^{(n)} - \widehat{Z}_{a,b}^{(n)} \right| = \left| \mathbb{E} \left[ \xi_0^{(n)} \right] \right| (b-a) \leq \frac{C(b-a)}{n^{\beta(\gamma-1)}} \tag{7.7}$$

as in (6.9); thus for  $k \leq \tau_{n+1}$  we have

$$\left| Z_{\tau_n, k}^{(\tau_{n+1})} - \widehat{Z}_{\tau_n, k}^{(\tau_{n+1})} \right| \leq C \frac{\tau_{n+1} - \tau_n}{\tau_{n+1}^{\beta(\gamma-1)}} \leq C \frac{\tau_n}{n^{1-\alpha} \tau_{n+1}^{\beta(\gamma-1)}} \leq C \frac{\tau_n^{1-\beta(\gamma-1)}}{n^{1-\alpha}} \tag{7.8}$$

by properties of  $\tau_n$ . Since  $\beta(\gamma-1) > 1/2$ , we obtain

$$\lim_{n \rightarrow \infty} \sup_{\tau_n \leq k \leq \tau_{n+1}} \frac{|Z_{\tau_n, k}^{(\tau_{n+1})} - \widehat{Z}_{\tau_n, k}^{(\tau_{n+1})}|}{\sqrt{\tau_n \log \tau_n}} = 0. \tag{7.9}$$

Therefore by (7.6) and (7.9) it is enough to show that

$$\lim_{n \rightarrow \infty} \sup_{\tau_n \leq k \leq \tau_{n+1}} \frac{|\widehat{Z}_{\tau_n, k}^{(\tau_{n+1})}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.} \tag{7.10}$$

To prove (7.10), we will use a maximal inequality for demimartingales proved by Newman and Wright in [27], and Bernstein's inequality. We first discuss the maximal inequality.

For fixed  $a$  and  $n$ , we can see using (7.5) that the process  $(\widehat{Z}_{a,k}^{(n)})_{k>a}$  has positively associated increments under  $\mathbb{P}(\cdot|S)$ , as they are all increasing functions of  $\widehat{\xi}^{(n)}$  which is an i.i.d. random field. By Proposition 2 in [27],  $\widehat{Z}_{a,k}^{(n)}$  is a demimartingale under  $\mathbb{P}(\cdot|S)$ , which implies, by the definition of a demimartingale,

that it is also a demimartingale under  $\mathbb{P}$ . Therefore, by Corollary 6 of [27], for all  $u > 0$ ,

$$\mathbb{P} \left( \sup_{a+1 \leq k \leq b} |\widehat{Z}_{a,k}^{(n)}| > 2u \right) \leq \frac{\sqrt{2\mathbb{E} \left[ |\widehat{Z}_{a,b}^{(n)}|^2 \right]}}{u} \mathbb{P} \left( |\widehat{Z}_{a,b}^{(n)}| > u \right)^{\frac{1}{2}}. \quad (7.11)$$

Note that

$$\mathbb{E} \left[ (\widehat{Z}_{a,b}^{(n)})^2 \middle| S \right] \leq CI_{a,b} \quad (7.12)$$

where  $I_{a,b} := \sum_x N_{a,b}^2(x)$  has the same distribution as  $I_{b-a}$ . Using Lemma 4.2(i), (7.11) and the properties of  $\tau_n$  we see that for all  $\epsilon > 0$ ,

$$\begin{aligned} \mathbb{P} \left( \sup_{\tau_n \leq k \leq \tau_{n+1}} |\widehat{Z}_{\tau_n,k}^{(\tau_{n+1})}| > 2\epsilon\sqrt{\tau_n \log \tau_n} \right) \\ \leq \frac{C}{\epsilon} \mathbb{P} \left( |\widehat{Z}_{\tau_n,\tau_{n+1}}^{(\tau_{n+1})}| > \epsilon\sqrt{\tau_n \log \tau_n} \right)^{\frac{1}{2}}. \end{aligned} \quad (7.13)$$

We proceed to bound the r.h.s. of (7.13). For that we use Bernstein's inequality, which can be found e.g. in [13], Exercise 4.3.14. From (7.5), we can see that  $\widehat{Z}_{a,b}^{(n)}$  is under  $\mathbb{P}(\cdot|S)$  a finite sum of independent and centered random variables, which are uniformly bounded by  $n^\beta N_{a,b}^*$ , where  $N_{a,b}^* := \sup_x N_{a,b}(x)$ , and with variance bounded by (7.12). Using Bernstein's inequality applied under  $\mathbb{P}(\cdot|S)$  and then integrating, we obtain

$$\mathbb{P} \left( |\widehat{Z}_{a,b}^{(n)}| > u \right) \leq E \left[ \exp \left\{ -C \frac{u^2}{I_{a,b} + n^\beta N_{a,b}^* u} \right\} \right]. \quad (7.14)$$

Since the pair  $I_{a,b}, N_{a,b}^*$  has the same law as  $I_{b-a}, N_{b-a}^*$ , we may write

$$\begin{aligned} \mathbb{P} \left( |\widehat{Z}_{\tau_n,\tau_{n+1}}^{(\tau_{n+1})}| > \epsilon\sqrt{\tau_n \log \tau_n} \right) \\ \leq E \left[ \exp \left\{ -C \frac{\tau_n \log \tau_n}{I_{\tau_{n+1}-\tau_n} + \tau_{n+1}^\beta N_{\tau_{n+1}-\tau_n}^* \sqrt{\tau_n \log \tau_n}} \right\} \right]. \end{aligned} \quad (7.15)$$

Fix  $0 < \epsilon < 1/2 - \beta$ . By Lemma 4.1(ii), we have

$$P(N_k^* > k^\epsilon) \leq Ck^{-3} \quad \forall k \in \mathbb{N}^*. \quad (7.16)$$

Now fix  $0 < \delta < \alpha^{-1} - 1$  and an integer  $\theta > 2/(\alpha\delta)$ . By Markov's inequality and Lemma 4.2(i) we have

$$P(I_k > k(\log k)^{1+\delta}) \leq \frac{E[I_k^\theta]}{k^\theta (\log k)^{(1+\delta)\theta}} \leq \frac{C}{(\log k)^{\theta\delta}} \quad \forall k \geq 2. \quad (7.17)$$

By (7.15), (7.16), (7.17), the subadditivity of  $\sqrt{\cdot}$ , and the fact that  $e^{-2x/(y+z)} \leq e^{-x/y} + e^{-x/z}$  for any  $x, y, z > 0$ , we see that (7.13) is at most

$$C_1(\tau_{n+1} - \tau_n)^{-3/2} + C_2(\log(\tau_{n+1} - \tau_n))^{-\frac{\theta\delta}{2}} + e^{-C_3 a_n/b_n} + e^{-C_3 a_n/c_n}, \quad (7.18)$$

where  $C_1, C_2$  and  $C_3$  are positive constants and

$$\begin{aligned} a_n &:= \tau_n \log \tau_n, \\ b_n &:= (\tau_{n+1} - \tau_n) [\log(\tau_{n+1} - \tau_n)]^{1+\delta}, \\ c_n &:= \tau_{n+1}^\beta (\tau_{n+1} - \tau_n)^\varepsilon \sqrt{\tau_n \log \tau_n}. \end{aligned} \quad (7.19)$$

Using the properties of  $\tau_n$ , we see that the first term of (7.18) is summable; by our choice of  $\theta$ , so is the second. Furthermore,

$$a_n/b_n \geq Cn^{1-\alpha}/(\log \tau_n)^\delta \geq Cn^{1-\alpha(1+\delta)}, \quad (7.20)$$

so the third term is summable by our choice of  $\delta$ , while

$$a_n/c_n \geq C\tau_n^\lambda, \quad (7.21)$$

where  $\lambda := 1 - (\beta + \frac{1}{2} + \varepsilon) > 0$  by our choice of  $\varepsilon$ , so the last term is again summable. Thus, by the Borel-Cantelli lemma, (7.10) holds, and the proof is complete.  $\square$

## 8 Proof of Proposition 5.1

Here we assume the same context as in Section 5. As before, endow  $\mathbb{Z}^d$  with the lexicographical order, denoted by  $<$ .

Let  $V = (V_x)_{x \in \mathbb{Z}^d}$  be a random field and, for  $x \in \mathbb{Z}^d$ , let

$$\mathcal{G}_x := \sigma(V_y : y < x). \quad (8.1)$$

We say that  $V$  is a random field of martingale differences (denoted AM for “accroissements d’une martingale”) if, for all  $x \in \mathbb{Z}^d$ ,

$$\mathbb{E}[V_x | \mathcal{G}_x] = 0 \text{ a.s.} \quad (8.2)$$

The following moment inequalities for random fields of type AM can be found in [17], Proposition 1 (see also Remark 6 therein).

**Lemma 8.1.** *Let  $V$  be a random field of type AM. Then, for all  $p \geq 2$ , and any finite subset  $B$  of  $\mathbb{Z}^d$ ,*

$$\left\| \sum_{x \in B} V_x \right\|_p^2 \leq 2p \sum_{x \in B} \|V_x\|_p^2. \quad (8.3)$$

*Proof of Proposition 5.1.* We will show (5.6) by induction in  $\theta$ . Suppose that (5.6) is valid for  $\theta = 1, \dots, n$ , and take  $\theta = n + 1$ . Let  $V_x := X_{n+1}(x)Y_x$  with

$$Y_x := \sum_{x_1, \dots, x_n} \prod_{i=1}^n X_i(x_i) a_x(x_1, \dots, x_n),$$

where  $a_x(x_1, \dots, x_n) := a(x_1, \dots, x_n, x)$ . Then  $m = \sum_x V_x$ . Moreover,  $V_x$  is a field of type AM; this can be easily checked using the assumptions made on the random fields  $X_i$  and on  $a$ , in particular (5.4) and the fact that the  $X_i$  are all centered and independent on distinct points. Therefore, by Lemma 8.1, for any finite  $B \subset \mathbb{Z}^d$ ,

$$\begin{aligned} \left\| \sum_{x \in B} V_x \right\|_p^2 &\leq 2p \sum_{x \in B} \|X_{n+1}(x)Y_x\|_p^2 \\ &\leq 2p \Lambda_p^2(n+1) \sum_{x \in B} \|Y_x\|_p^2, \end{aligned} \quad (8.4)$$

where we used the independence of  $X_{n+1}(x)$  and  $Y_x$  and (5.2). Letting  $B \uparrow \mathbb{Z}^d$ , an application of Fatou's inequality gives

$$\left\| \sum_{x \in \mathbb{Z}^d} V_x \right\|_p^2 \leq 2p \Lambda_p^2(n+1) \sum_{x \in \mathbb{Z}^d} \|Y_x\|_p^2. \quad (8.5)$$

Since  $Y_x$  is for each  $x$  a sum of the same form as  $m$  but with  $\theta = n$ , by the induction hypothesis (8.5) is at most

$$\begin{aligned} &2p \Lambda_p^2(n+1) \sum_{x \in \mathbb{Z}^d} (2p)^n \prod_{i=1}^n \Lambda_p^2(i) \sum_{x_1, \dots, x_n} a_x(x_1, \dots, x_n)^2 \\ &= (2p)^{n+1} \prod_{i=1}^{n+1} \Lambda_p^2(i) \sum_{x_1, \dots, x_{n+1}} a(x_1, \dots, x_{n+1})^2, \end{aligned} \quad (8.6)$$

which proves the induction step. The case  $\theta = 1$  is proved analogously.  $\square$

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