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# POISSON STRUCTURES AND STAR PRODUCTS ON QUASIMODULAR FORMS

FRANÇOIS DUMAS AND EMMANUEL ROYER

**ABSTRACT.** We construct and classify all Poisson structures on quasimodular forms that extend the one coming from the first Rankin-Cohen bracket on the modular forms. We use them to build formal deformations on the algebra of quasimodular forms.

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## 1. INTRODUCTION

In [Coh75, Theorem 7.1], Henri Cohen defined a collection of bi-differential operators on modular forms. Let  $n$  be a positive integer,  $f$  a modular form of weight  $k$  and  $g$  a modular form of weight  $\ell$ . The  $n$ -th Rankin-Cohen bracket of  $f$  and  $g$  is the modular form of weight  $k + \ell + 2n$  defined by

$$\mathrm{RC}_n(f, g) = \sum_{r=0}^n (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} D^r f D^{n-r} g, \quad \left( D = \frac{1}{2\pi i} \frac{d}{dz} \right).$$

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The algebraic structure of these brackets has been studied in Zagier's seminal work [Zag94]. That Rankin-Cohen brackets define a formal deformation of the algebra of modular forms has been widely studied. Important contributions are [UU96, CMZ97, Yao07, BTY07, Pev12, KP13].

In this paper, we construct formal deformations of the algebra  $\mathcal{M}_*^{\leq\infty}$  of quasimodular forms. This algebra is generated over  $\mathbb{C}$  by the three Eisenstein series  $E_2, E_4$  and  $E_6$ . The algebra  $\mathcal{M}_*$  of modular forms is the subalgebra generated by  $E_4$  and  $E_6$ . As a first step, we classify the admissible Poisson structures of  $\mathcal{M}_*^{\leq\infty}$ . A Poisson bracket  $\{ , \}$  on  $\mathcal{M}_*^{\leq\infty}$  is admissible if

- i) the restriction of  $\{ , \}$  to the algebra  $\mathcal{M}_*$  of modular forms is the first Rankin-Cohen bracket  $\text{RC}_1$
- ii) it satisfies  $\{\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}\} \subset \mathcal{M}_{k+\ell+2}^{\leq s+t}$  for any even integers  $k, \ell$  and any integers  $s$  and  $t$  where  $\mathcal{M}_k^{\leq s}$  is the vector space of quasimodular forms of weight  $k$  and depth less than  $s$ . The vector space of parabolic modular forms of weight 12 is one dimensional. We choose  $\Delta = E_4^3 - E_6^2$  a generator.

**Proposition A** (First family of Poisson brackets)– *For any  $\lambda \in \mathbb{C}^*$ , there exists an admissible Poisson bracket  $\{ , \}_\lambda$  on the algebra of quasimodular forms defined by the following values on the generators:*

$$\{E_4, E_6\}_\lambda = -2\Delta, \quad \{E_2, E_4\}_\lambda = -\frac{1}{3}(2E_6E_2 - \lambda E_4^2), \quad \{E_2, E_6\}_\lambda = -\frac{1}{2}(2E_4^2E_2 - \lambda E_4E_6).$$

Moreover,

- i) for any  $\lambda \in \mathbb{C}^*$ , the Poisson bracket  $\{ , \}_\lambda$  is not unimodular.
- ii) The Poisson algebras  $(\mathcal{M}_*^{\leq\infty}, \{ , \}_\lambda)$  and  $(\mathcal{M}_*^{\leq\infty}, \{ , \}_{\lambda'})$  are Poisson modular isomorphic for all  $\lambda$  and  $\lambda'$  in  $\mathbb{C}^*$ .
- iii) For any  $\lambda \in \mathbb{C}^*$ , the Poisson center of  $(\mathcal{M}_*^{\leq\infty}, \{ , \}_\lambda)$  is  $\mathbb{C}$ .

*Remark*– A Poisson isomorphism  $\varphi$  on  $\mathcal{M}_*^{\leq\infty}$  is modular if  $\varphi(\mathcal{M}_*) \subset \mathcal{M}_*$ .

Thanks to ii) in Proposition A, we restrict to the bracket  $\{ , \}_1$ . Following [Zag94, Eq. (38)], we consider the derivation  $w$  on  $\mathcal{M}_*^{\leq\infty}$  defined by

$$w(f) = \frac{\{\Delta, f\}_1}{12\Delta}.$$

A derivation  $\delta$  on  $\mathcal{M}_*^{\leq\infty}$  is complex-like if  $\delta(\mathcal{M}_k^{\leq s}) \subset \mathcal{M}_{k+2}^{\leq s+1}$  for any  $k$  and  $s$ . The set of complex-like derivations  $\delta$  such that  $kf\delta(g) - \ell g\delta(f) = 0$  for any  $f \in \mathcal{M}_k^{\leq s}$  and  $g \in \mathcal{M}_\ell^{\leq t}$ , for any  $k, \ell, s, t$ , is a one dimensional vector space over  $\mathbb{C}$ . Let  $\pi$  be a generator. The following Theorem provides a first family of formal deformations of the algebra  $\mathcal{M}_*^{\leq\infty}$ .

**Theorem A**– *For any  $a \in \mathbb{C}$ , let  $d_a$  be the derivation on  $\mathcal{M}_*^{\leq\infty}$  defined by  $d_a = a\pi + w$ .*

- i) For all quasimodular forms  $f$  and  $g$  of respective weights  $k$  and  $\ell$ , we have

$$\{f, g\}_1 = kf d_a(f) - \ell g d_a(f).$$

ii) More generally, for any  $a \in \mathbb{C}$ , the brackets defined for any integer  $n \geq 0$  by

$$[f, g]_{d_a, n} = \sum_{r=0}^n (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} d_a^r(f) d_a^{n-r}(g), \quad (f \in \mathcal{M}_k^{\leq \infty}, g \in \mathcal{M}_\ell^{\leq \infty})$$

satisfy

$$[\mathcal{M}_k^{\leq \infty}, \mathcal{M}_\ell^{\leq \infty}]_{d_a, n} \subset \mathcal{M}_{k+\ell+2n}^{\leq \infty}$$

and define a formal deformation of  $\mathcal{M}_*^{\leq \infty}$ .

iii) Moreover,  $[\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}]_{d_a, n} \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}$  for all  $n, s, t, k, \ell$  if and only if  $a = 0$ .

*Remark-* A generator is  $\pi$  defined by linear extension of  $\pi(f) = kf E_2$  for  $f$  any quasimodular form of weight  $k$ . For this choice, the derivation  $d_a$  is defined on the generators by

$$d_a E_2 = 2a E_2^2 - \frac{1}{12} E_4, \quad d_a E_4 = 4a E_4 E_2 - \frac{1}{3} E_6, \quad d_a E_6 = 6a E_6 E_2 - \frac{1}{2} E_4^2.$$

We construct a second family of Poisson brackets.

**Proposition B** (Second family of Poisson brackets)– For any  $\alpha \in \mathbb{C}$ , there exists an admissible Poisson bracket  $(, )_\alpha$  on the algebra of quasimodular forms defined by the following values on the generators:

$$(E_4, E_6)_\alpha = -2\Delta, \quad (E_2, E_4)_\alpha = \alpha E_6 E_2, \quad (E_2, E_6)_\alpha = \frac{3}{2} \alpha E_4^2 E_2.$$

Moreover,

- i) for any  $\alpha \in \mathbb{C} \setminus \{4\}$ , the Poisson bracket  $(, )_\alpha$  is not unimodular. For  $\alpha = 4$ , the Poisson bracket  $(, )_4$  is Jacobian (hence unimodular) of potential  $k_0 = -2\Delta E_2$ .
- ii) The Poisson algebras  $(\mathcal{M}_*^{\leq \infty}, (, )_\alpha)$  and  $(\mathcal{M}_*^{\leq \infty}, (, )_{\alpha'})$  are Poisson modular isomorphic if and only if  $\alpha = \alpha'$ .
- iii) For any  $\alpha \in \mathbb{C}$ ,
  - a) if  $\alpha \notin \mathbb{Q}$ , the Poisson center of  $(\mathcal{M}_*^{\leq \infty}, (, )_\alpha)$  is  $\mathbb{C}$
  - b) if  $\alpha = 0$ , the Poisson center of  $(\mathcal{M}_*^{\leq \infty}, (, )_\alpha)$  is  $\mathbb{C}[E_2]$
  - c) if  $\alpha = p/q$  with  $p \in \mathbb{Z}^*$  and  $q \in \mathbb{N}^*$ ,  $p$  and  $q$  coprimes, the Poisson center of  $(\mathcal{M}_*^{\leq \infty}, (, )_\alpha)$  is

$$\begin{cases} \mathbb{C} & \text{if } p < 0 \\ \mathbb{C}[\Delta^p E_2^{4q}] & \text{if } p \geq 1 \text{ is odd} \\ \mathbb{C}[\Delta^u E_2^{2q}] & \text{if } p = 2u \text{ for odd } u \geq 1 \\ \mathbb{C}[\Delta^v E_2^q] & \text{if } p = 4v \text{ with } v \geq 1. \end{cases}$$

*Remark-* The bracket  $(, )_0$  is the trivial bracket.

This second family provides a new set of formal deformations of the algebra of quasimodular forms. Following [Zag94, Eq. (38)], we consider the derivation  $v$  defined

on  $\mathcal{M}_*^{\leq\infty}$  by

$$v(f) = \frac{(\Delta, f)_\alpha}{12\Delta}.$$

Let us define  $\mathcal{K}_\alpha: \mathcal{M}_*^{\leq\infty} \rightarrow \mathbb{C}$  by setting  $\mathcal{K}_\alpha(f) = k - (3\alpha + 2)s$  if  $f$  has weight  $k$  and depth  $s$ . The set of complex-like derivations  $\delta$  such that  $\mathcal{K}_\alpha(f)f\delta(g) - \mathcal{K}_\alpha(g)g\delta(f) = 0$  for any  $f \in \mathcal{M}_k^{\leq s}$  and  $g \in \mathcal{M}_\ell^{\leq t}$ , for any  $k, \ell, s, t$ , is a one dimensional vector space over  $\mathbb{C}$ . Let  $\pi_\alpha$  be a generator. We note

$$\mathcal{M}_k^s = \mathcal{M}_k E_2^s.$$

**Theorem B**– Let  $\alpha \in \mathbb{C}$ . For any  $b \in \mathbb{C}$ , let  $\delta_{\alpha,b}$  be the derivation on  $\mathcal{M}_*^{\leq\infty}$  defined by  $\delta_{\alpha,b} = b\pi_\alpha + v$ .

i) For all  $f \in \mathcal{M}_k^s$  and  $g \in \mathcal{M}_\ell^t$ , we have

$$(f, g)_\alpha = (k - (3\alpha + 2)s)f\delta_{\alpha,b}(g) - (\ell - (3\alpha + 2)t)g\delta_{\alpha,b}(f)$$

for any  $f \in \mathcal{M}_k^s$  and  $g \in \mathcal{M}_\ell^t$ .

ii) Moreover, the brackets defined for any integer  $n \geq 0$  by

$$[f, g]_{\delta_{\alpha,b}, n}^{\mathcal{K}_\alpha} = \sum_{r=0}^n (-1)^r \binom{k - (3\alpha + 2)s + n - 1}{n - r} \binom{\ell - (3\alpha + 2)t + n - 1}{r} \delta_{\alpha,b}^r(f) \delta_{\alpha,b}^{n-r}(g),$$

for any  $f \in \mathcal{M}_k^s$  and  $g \in \mathcal{M}_\ell^t$  define a formal deformation of  $\mathcal{M}_*^{\leq\infty}$  satisfying

$$\left[ \mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t} \right]_{\delta_{\alpha,b}, n}^{\mathcal{K}_\alpha} \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}$$

for all  $k, \ell$  in  $2\mathbb{N}$  and  $s, t$  in  $\mathbb{N}$  if and only if  $b = 0$ .

*Remark*– A generator  $\pi_\alpha$  is defined by linear extension of:

$$\pi_\alpha(f) = [k - (3\alpha + 2)s]f E_2 \quad (f \in \mathcal{M}_k^s).$$

For this choice, the derivation  $\delta_{\alpha,b}$  is defined on the generators by

$$\delta_{\alpha,b} E_2 = -3b\alpha E_2^2, \quad \delta_{\alpha,b} E_4 = 4b E_4 E_2 - \frac{1}{3} E_6, \quad \delta_{\alpha,b} E_6 = 6b E_6 E_2 - \frac{1}{2} E_4^2.$$

To complete the classification of Poisson structures, we introduce a third family of Poisson brackets. We note however that, when  $\mu \neq 0$ , this third family does not lead to a formal deformation of  $\mathcal{M}_*^{\leq\infty}$  with the shape of Rankin-Cohen brackets (see §4.3).

**Proposition C** (Third family of Poisson brackets)– For any  $\mu \in \mathbb{C}$ , there exists an admissible Poisson brackets  $\langle \cdot, \cdot \rangle_\mu$  on the algebra of quasimodular forms defined by the following values on the generators:

$$\langle E_4, E_6 \rangle_\mu = -2\Delta, \quad \langle E_2, E_4 \rangle_\mu = 4E_6 E_2 + \mu E_4^2, \quad \langle E_2, E_6 \rangle_\mu = 6E_4^2 E_2 - 2\mu E_4 E_6.$$

Moreover,

i) this Poisson bracket is Jacobian with potential

$$k_\mu = -2\Delta E_2 + \mu E_4^2 E_6.$$

ii) The Poisson algebras  $(\mathcal{M}_*^{\leq\infty}, \langle \cdot, \cdot \rangle_\mu)$  and  $(\mathcal{M}_*^{\leq\infty}, \langle \cdot, \cdot \rangle_{\mu'})$  are Poisson modular isomorphic for all  $\mu$  and  $\mu'$  in  $\mathbb{C}^*$ .

iii) For any  $\mu \in \mathbb{C}$ , the Poisson center of  $(\mathcal{M}_*^{\leq \infty}, \langle \cdot, \cdot \rangle_\mu)$  is the polynomial algebra  $\mathbb{C}[k_\mu]$ .

*Remark*- We note that  $\langle \cdot, \cdot \rangle_0 = (\cdot, \cdot)_4$ .

Finally, the following result implies that our classification is complete.

**Theorem C**– Up to Poisson modular isomorphism, the only distinct admissible Poisson brackets on the algebra of quasimodular forms are  $\{ \cdot, \cdot \}_1, \langle \cdot, \cdot \rangle_1$  and the family  $(\cdot, \cdot)_\alpha$  for any  $\alpha \in \mathbb{C}$ .

*Remark*- We could endow the algebra of modular forms with another Poisson structure  $\mathfrak{b}$ . If we require  $\mathfrak{b}(\mathcal{M}_k, \mathcal{M}_\ell) \subset \mathcal{M}_{k+\ell+2}$ , then  $\mathfrak{b}$  is necessarily defined by  $\mathfrak{b}(E_4, E_6) = \alpha E_4^3 + \beta E_6^2$  for some complex numbers  $\alpha$  and  $\beta$ . If  $\alpha\beta \neq 0$  then  $(\mathcal{M}_*, \mathfrak{b})$  is Poisson isomorphic to  $(\mathcal{M}_*, \text{RC}_1)$  and is indeed studied by this work. If  $\alpha\beta = 0$ , the Poisson algebras are no more Poisson isomorphic (they do not have the same group of automorphisms). This degenerate case deserves another study.

*Remark*- From an algebraic point of view, classifications of Poisson structures and associated (co)homology for polynomial algebras in two variables appear in [Mon02], [Pic06a] or [RV02] for a Poisson bracket on  $\mathbb{C}[x, y]$  defined by  $\{x, y\} = \varphi(x, y)$  with  $\varphi$  an homogeneous or a square free weight homogeneous polynomial in  $\mathbb{C}[x, y]$ . The algebra of modular forms  $\mathcal{M}_* = \mathbb{C}[E_4, E_6]$  with the Poisson bracket defined by  $\text{RC}_1$  is the case  $A_2$  in the classification theorem 3.8 in [Mon02]. Applying propositions 4.10 and 4.11 of [Pic06a], or theorems 4.6 and 4.11 of [Mon02], we can deduce that the Poisson cohomology spaces  $\text{HP}^1(\mathcal{M}_*)$  and  $\text{HP}^2(\mathcal{M}_*)$  are of respective dimensions 1 and 2. In three variables, the Poisson structures on the algebra  $\mathcal{M}_*^{\leq \infty} = \mathbb{C}[E_2, E_4, E_6]$  of quasimodular forms arising from theorem C above do not fall under the classification of [DH91] since they are not quadratic. The (co)homological study of [Pic06a] and [Pic06b] does not apply to the brackets  $\{ \cdot, \cdot \}_1$  and  $(\cdot, \cdot)_\alpha$  since they are not Jacobian, or to the Jacobian bracket  $\langle \cdot, \cdot \rangle_1$  because its potential  $k_1$  does not admit an isolated singularity at the origin.

## 2. NUMBER THEORETIC AND ALGEBRAIC BACKGROUND

**2.1. Quasimodular forms.** The aim of this section is to provide the necessary background on quasimodular forms. For details, the reader is advised to refer to [Zag08] or [MR05]. On  $\text{SL}(2, \mathbb{Z})$ , a modular form of weight  $k \in 2\mathbb{N}$ ,  $k \neq 2$ , is a holomorphic function on the Poincaré upper half plane  $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}z > 0\}$  satisfying

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = f(z)$$

for any  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$  and having Fourier expansion

$$f(z) = \sum_{n \geq 0} \widehat{f}(n) e^{2\pi i n z}.$$

We denote by  $\mathcal{M}_k$  the finite dimensional space of modular forms of weight  $k$ . The algebra of modular forms is defined as the graded algebra

$$\mathcal{M}_* = \bigoplus_{\substack{k \in 2\mathbb{N} \\ k \neq 2}} \mathcal{M}_k.$$

Let  $k \geq 2$  be even. We define the Eisenstein series of weight  $k$  by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{+\infty} \sigma_{k-1}(n) e^{2\pi i n z}.$$

Here the rational numbers  $B_k$  are defined by their exponential generating series

$$\sum_{n=0}^{+\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}$$

and  $\sigma_{k-1}$  is the divisor function defined by

$$\sigma_{k-1}(n) = \sum_{\substack{d|n \\ d>0}} d^{k-1} \quad (n \in \mathbb{N}^*).$$

If  $k \geq 4$ , the Eisenstein series  $E_k$  is a modular form of weight  $k$  and  $\mathcal{M}_*$  is the polynomial algebra in the two algebraically independent Eisenstein series  $E_4$  and  $E_6$ . In other words,

$$\mathcal{M}_* = \mathbb{C}[E_4, E_6], \quad \mathcal{M}_k = \bigoplus_{\substack{(i,j) \in \mathbb{N}^2 \\ 4i+6j=k}} \mathbb{C} E_4^i E_6^j.$$

However, the Eisenstein series  $E_2$  is not a modular form. It satisfies

$$(cz + d)^{-2} E_2\left(\frac{az + b}{cz + d}\right) = E_2(z) + \frac{6}{\pi i} \frac{c}{cz + d} \quad (z \in \mathcal{H})$$

for any  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ . Moreover, the algebra of modular forms is not stable by the normalised complex derivation

$$D = \frac{1}{2\pi i} \frac{d}{dz}.$$

For example, we have the following Ramanujan differential equations

$$DE_2 = \frac{1}{12} (E_2^2 - E_4)$$

$$DE_4 = \frac{1}{3} (E_4 E_2 - E_6)$$

$$DE_6 = \frac{1}{2} (E_6 E_2 - E_4^2).$$

To account for these observations, and using the fact that  $E_2$ ,  $E_4$  and  $E_6$  are algebraically independent, we introduce the algebra  $\mathcal{M}_*^{\leq \infty}$  of quasimodular forms defined as the polynomial algebra

$$\mathcal{M}_*^{\leq \infty} = \mathbb{C}[E_2, E_4, E_6] = \mathcal{M}_*[E_2].$$

More intrinsically, if for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$  we define

$$X(\gamma) = z \mapsto \frac{c}{cz + d}$$

and

$$f|_k \gamma = z \mapsto (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right)$$

then a quasimodular form of weight  $k \in 2\mathbb{N}$  and depth  $s \in \mathbb{N}$  is a holomorphic function  $f$  on  $\mathcal{H}$  such that there exist holomorphic functions  $f_0, \dots, f_s$  ( $f_s \neq 0$ ) satisfying

$$f|_k \gamma = \sum_{j=0}^s f_j X(\gamma)^j$$

for any  $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ . Moreover, it is required that any  $f_j$  has a Fourier expansion

$$f_j(z) = \sum_{n \geq 0} \widehat{f}_j(n) e^{2\pi i n z} \quad (z \in \mathcal{H}).$$

The zero function is supposed to have arbitrary weight and depth 0. We write  $\mathcal{M}_k^{\leq \infty}$  for the space of quasimodular forms of weight  $k$  and  $\mathcal{M}_k^{\leq s}$  for the space of quasimodular forms of weight  $k$  and depth less than or equal to  $s$ . We have  $\mathcal{M}_k^{\leq 0} = \mathcal{M}_k$  and

$$\mathcal{M}_k^{\leq s} = \bigoplus_{j=0}^s \mathcal{M}_{k-2j} E_2^j, \quad \mathcal{M}_*^{\leq \infty} = \bigoplus_{k \in 2\mathbb{N}} \mathcal{M}_k^{\leq k/2}.$$

Moreover  $D\mathcal{M}_k^{\leq s} \subset \mathcal{M}_{k+2}^{\leq s+1}$ . Since the depth of a quasimodular form is nothing but its degree as a polynomial in  $E_2$  with modular coefficients we note that

$$\mathcal{M}_k^{\leq s} = \bigoplus_{j=0}^s \mathcal{M}_k^t, \quad \mathcal{M}_*^{\leq \infty} = \bigoplus_{k \in 2\mathbb{N}} \bigoplus_{t=0}^{k/2} \mathcal{M}_k^t$$

where

$$\mathcal{M}_k^t = \mathcal{M}_{k-2t} E_2^t = \bigoplus_{\substack{(i,j) \in \mathbb{N}^2 \\ 4i+6j=k-2t}} \mathbb{C} E_4^i E_6^j E_2^t.$$

An important element in our study will be the discriminant function  $\Delta = E_4^3 - E_6^2$ . We note that  $D\Delta = \Delta E_2$ .

Let  $n$  be a non negative integer,  $f$  a modular form of weight  $k$  and  $g$  a modular form of weight  $\ell$ . The  $n$ -th Rankin-Cohen bracket of  $f$  and  $g$  is

$$\mathrm{RC}_n(f, g) = \sum_{r=0}^n (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} D^r f D^{n-r} g.$$

This is a modular form of weight  $k + \ell + 2n$ . If  $f$  and  $g$  are quasimodular forms of respective weights  $k$  and  $\ell$  and respective depths  $s$  and  $t$ , their  $n$ -th Rankin-Cohen



bracket is defined in [MR09] by

$$\text{RC}_n(f, g) = \sum_{r=0}^n (-1)^r \binom{k-s+n-1}{n-r} \binom{\ell-t+n-1}{r} D^r f D^{n-r} g. \quad (1)$$

This is a quasimodular form of weight  $k + \ell + 2n$  and minimal depth (that is  $s + t$ ).

**2.2. Poisson algebra.** The aim of this section is to give a brief account on what is needed about Poisson algebra. For more details, the reader is advised to refer to [LGPV13]. A commutative  $\mathbb{C}$ -algebra  $A$  is a Poisson algebra if there exists a bilinear skew-symmetric map  $b: A \times A \rightarrow A$  satisfying the two conditions:

- Leibniz rule  $b(fg, h) = f b(g, h) + b(f, h)g$
- Jacobi identity  $b(f, b(g, h)) + b(g, b(h, f)) + b(h, b(f, g)) = 0$

for all  $f, g$  and  $h$  in  $A$ . The bilinear map  $b$  is given the name of Poisson bracket. If  $A$  is a finitely generated algebra with generators  $x_1, \dots, x_N$ , a Poisson bracket  $b$  is entirely determined by its values  $b(x_i, x_j)$  for  $i < j$  where  $A$  is generated by  $x_1, \dots, x_N$ . More precisely, we have

$$b(f, g) = \sum_{0 \leq i < j \leq N} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right) b(x_i, x_j) \quad (2)$$

for  $f$  and  $g$  expressed as polynomials in  $x_1, \dots, x_N$ .

If  $A = \mathbb{C}[x, y]$ , any  $p \in A$  determines a Poisson bracket satisfying  $b(x, y) = p$ . However, if  $A = \mathbb{C}[x, y, z]$ , for any  $p, q$  and  $r$  in  $A$ , there exists a Poisson bracket on  $A$  defined by

$$b(x, y) = r, \quad b(y, z) = p, \quad \text{and} \quad b(z, x) = q$$

if and only if

$$\text{curl}(p, q, r) \cdot (p, q, r) = 0 \quad (3)$$

where

$$\text{curl}(p, q, r) = \left( \frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}, \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}, \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right).$$

If condition (3) is satisfied, then  $(p, q, r)$  is called a Poissonian triple. A particular case is obtained if there exists  $k \in \mathbb{C}[x, y, z]$  such that  $\text{curl}(p, q, r) = (p, q, r) \wedge \text{grad} k$ . The bracket  $b$  is said then to be unimodular. Among unimodular brackets are the Jacobian brackets. A bracket  $b$  is Jacobian if  $(p, q, r) = \text{grad} k$  for some polynomial  $k$ . The bracket  $b$  satisfies then

$$b(f, g) = \text{jac}(f, g, k) \quad (f, g \in A)$$

In this case,  $\mathbb{C}[x, y, z]$  is said to have a Jacobian Poisson structure (JPS) of potential (or Casimir function)  $k$ . The Poissonian triple  $(p, q, r)$  is said then to be exact.

The Poisson center (or zeroth Poisson cohomology group) of a Poisson algebra  $A$  is the Poisson subalgebra

$$\text{HP}^0(A) = \{g \in A: b(f, g) = 0, \forall f \in A\}.$$

The Poisson center is contained in the Poisson centraliser of any element in the algebra: let  $f \in A$ , its Poisson centraliser is  $\{g \in A: b(f, g) = 0\}$ . The following lemma computes the Poisson center of polynomial algebras in three variables equipped with a Jacobian Poisson structure. It allows one to recover for example [Pic06b, Proposition 4.2] in the

particular case where the potential is a weight homogeneous polynomial with an isolated singularity. A polynomial  $h \in \mathbb{C}[x, y, z]$  is indecomposable if there is no polynomial  $p \in \mathbb{C}[x]$  with  $\deg p \geq 2$  such that  $h = p \circ \ell$  for some  $\ell \in \mathbb{C}[x, y, z]$ .

**Lemma 1**— *Let  $\mathbb{C}[x, y, z]$  be endowed with a Jacobian Poisson structure of non constant potential  $k$ . Its Poisson center is  $\mathbb{C}[k]$  if and only if  $k$  is indecomposable.*

*Proof.* Assume that  $k$  is not indecomposable:  $k = p \circ \ell$  with  $p \in \mathbb{C}[x]$ ,  $\deg p = 2$ . Then  $\text{jac}(\ell, g, k) = (p' \circ \ell)\text{jac}(\ell, g, \ell)$  hence  $\ell$  is in the Poisson center however not in  $\mathbb{C}[k]$ . Assume conversely that  $k$  is indecomposable. Let  $f$  be in the Poisson center, then the rank of the Jacobian matrix of  $(f, g, k)$  is at most 2 for any  $g$ . If it is 1 for any  $g$  then  $\text{grad } f$  and  $\text{grad } k$  are zero, it contradicts the fact that  $k$  is not constant. Hence, for some  $g$ , the rank is 2. It follows (see e.g. [GS06, Theorem 6]) that there exists  $q \in \mathbb{C}[x, y, z]$ ,  $F \in \mathbb{C}[x]$  and  $K \in \mathbb{C}[x]$  such that  $f = F \circ q$  and  $k = K \circ q$ . Since  $k$  is indecomposable and non constant, we have  $\deg K = 1$  hence  $q$ , and  $f$ , are polynomials in  $k$ .  $\square$

If  $A$  and  $B$  are two Poisson algebras, with respective Poisson brackets  $\mathfrak{b}_A$  and  $\mathfrak{b}_B$ , a map  $\varphi: A \rightarrow B$  is a morphism of Poisson algebras when it is a morphism of algebras that satisfies

$$\varphi(\mathfrak{b}_A(f, g)) = \mathfrak{b}_B(\varphi(f), \varphi(g))$$

for any  $f$  and  $g$  in  $A$ . Two Poisson-isomorphic Poisson algebras have isomorphic Poisson centres.

We detail now a canonical way to extend a Poisson structure from an algebra  $A$  to a polynomial algebra  $A[x]$ . This construction is due to Sei-Qwon Oh [Oh06]. A Poisson derivation of  $A$  is a derivation  $\sigma$  of  $A$  satisfying

$$\sigma(\mathfrak{b}(f, g)) = \mathfrak{b}(\sigma(f), g) + \mathfrak{b}(f, \sigma(g))$$

for all  $f$  and  $g$  in  $A$ . If  $\sigma$  is a Poisson derivation of  $A$ , a Poisson  $\sigma$ -derivation is a derivation  $\delta$  of  $A$  such that

$$\delta(\mathfrak{b}(f, g)) = \mathfrak{b}(\delta(f), g) + \mathfrak{b}(f, \delta(g)) + \sigma(f)\delta(g) - \delta(f)\sigma(g)$$

for all  $f$  and  $g$  in  $A$ .

**Theorem 2** (Oh)— *Let  $(A, \mathfrak{b}_A)$  be a Poisson algebra. Let  $\sigma$  and  $\delta$  be linear maps on  $A$ . The polynomial ring  $A[x]$  becomes a Poisson algebra with Poisson brackets  $\mathfrak{b}$  defined by*

$$\begin{aligned} \mathfrak{b}(f, g) &= \mathfrak{b}_A(f, g) \\ \mathfrak{b}(x, f) &= \sigma(f)x + \delta(f) \end{aligned}$$

*for all  $f$  and  $g$  in  $A$  if and only if  $\sigma$  is a Poisson derivation and  $\delta$  is a Poisson  $\sigma$ -derivation. In this case, the Poisson algebra  $A[x]$  is said to be a Poisson-Ore extension of  $A$ . It is denoted  $A[x]_{\sigma, \delta}$ .*

We describe also a general process to obtain Poisson brackets from a pair of derivations. A pair  $(\delta, d)$  of two derivations of  $A$  is solvable if there exists some scalar  $\alpha$  such that  $\delta \circ d - d \circ \delta = \alpha d$ . In particular, a solvable pair  $(\delta, d)$  is abelian when  $\alpha = 0$ .

**Proposition 3**— *Let  $A$  be a commutative algebra,  $d$  and  $\delta$  two derivations of  $A$ . Let  $\mathfrak{b}: A \times A \rightarrow A$  be defined by*

$$\mathfrak{b}(f, g) = \delta(f)d(g) - d(f)\delta(g) \quad (f, g \in A).$$

Then

- i) the map  $\mathfrak{b}$  is bilinear skew-symmetric and satisfies Leibniz rule.
- ii) If  $(\delta, \mathfrak{d})$  is solvable, then  $\mathfrak{b}$  satisfies Jacobi identity and so becomes a Poisson bracket.
- iii) Under the hypothesis in ii),  $\mathfrak{d}$  is a Poisson derivation for  $\mathfrak{b}$ .

*Proof of Proposition 3.* Point i) is immediate. Point ii) is a consequence of the following computation. If  $(\delta, \mathfrak{d})$  is solvable with  $\delta \mathfrak{d} - \mathfrak{d} \delta = \alpha \mathfrak{d}$  and if  $B: A \otimes A \otimes A \otimes A \rightarrow A$  is defined by  $B(f, g, h) = \mathfrak{b}(f, \mathfrak{b}(g, h))$  then

$$\begin{aligned} B &= \alpha(\mathfrak{d} \otimes \mathfrak{d} \otimes \delta - \mathfrak{d} \otimes \delta \otimes \mathfrak{d}) \\ &\quad + (\delta \otimes (\mathfrak{d} \circ \delta) \otimes \mathfrak{d} - \mathfrak{d} \otimes \delta \otimes (\mathfrak{d} \circ \delta)) \\ &\quad + (\mathfrak{d} \otimes (\mathfrak{d} \circ \delta) \otimes \delta - \delta \otimes \mathfrak{d} \otimes (\mathfrak{d} \circ \delta)) \\ &\quad + (\delta \otimes \delta \otimes \mathfrak{d}^2 - \delta \otimes \mathfrak{d}^2 \otimes \delta) \\ &\quad + (\mathfrak{d} \otimes \mathfrak{d} \otimes \delta^2 - \mathfrak{d} \otimes \delta^2 \otimes \mathfrak{d}). \end{aligned}$$

Point iii) is obtained by direct computation. □

A direct consequence of this Proposition is the following Corollary. If  $A = \bigoplus_{n \geq 0} A_n$  is a commutative graded algebra, a map  $\kappa: A \rightarrow \mathbb{C}$  is graded-additive if, for any  $k$  and  $\ell$ , for any  $f \in A_k$  and any  $g \in A_\ell$ , we have  $\kappa(fg) = \kappa(f) + \kappa(g)$ .

**Corollary 4**— Let  $A = \bigoplus_{n \geq 0} A_n$  be a commutative graded algebra. Let  $\kappa: A \rightarrow \mathbb{C}$  be a graded-additive map. Let  $\mathfrak{d}$  be a homogeneous derivation of  $A$  (there exists  $e \geq 0$  such that  $\mathfrak{d}A_n \subset A_{n+e}$  for any  $e \geq 0$ ). Then, the bracket defined on  $A$  by the bilinear extension of

$$\mathfrak{b}(f, g) = \kappa(f)f \mathfrak{d}(g) - \kappa(g)g \mathfrak{d}(f) \quad (f \in A_k, g \in A_\ell)$$

is a Poisson bracket for which  $\mathfrak{d}$  is a Poisson derivation.

We turn on formal deformations of a commutative  $\mathbb{C}$ -algebra  $A$ . Assume we have a family  $\boldsymbol{\mu} = (\mu_i)_{i \in \mathbb{N}}$  of bilinear maps  $\mu_i: A \times A \rightarrow A$  such that  $\mu_0$  is the product. Let  $A[[\hbar]]$  be the commutative algebra of formal power series in one variable  $\hbar$  with coefficients in  $A$ . The family  $\boldsymbol{\mu}$  is a formal deformation of  $A$  if the non commutative product on  $A[[\hbar]]$  defined by extension of

$$f * g = \sum_{j \geq 0} \mu_j(f, g) \hbar^j \quad (f, g \in A)$$

is associative. This condition is equivalent to

$$\sum_{r=0}^n \mu_{n-r}(\mu_r(f, g), h) = \sum_{r=0}^n \mu_{n-r}(f, \mu_r(g, h)) \quad \text{for all } f, g, h \in A \quad (4)$$

for all  $n \geq 0$ . In this case, the product  $*$  is called a star product. If  $\boldsymbol{\mu}$  is a formal deformation and if moreover  $\mu_1$  is skew-symmetric and  $\mu_2$  is symmetric, then  $(A, \mu_1)$  is a Poisson algebra.

2.3. **Problems at issue.** The first Rankin-Cohen bracket

$$\text{RC}_1(f, g) = kfD(g) - D(f)\ell g \quad (f \in \mathcal{M}_k, g \in \mathcal{M}_\ell)$$

gives  $\mathcal{M}_*$  a structure of Poisson algebra. This is a consequence of Corollary 4. Cohen, Manin & Zagier [CMZ97] and Yao [Yao07] or [RTY11] proved that the family of Rankin-Cohen brackets is a formal deformation of  $\mathcal{M}_*$ . In this case, the star product is called the Eholzer product. This subject has been widely studied then. See for example [OS00, Pev08].

Can we construct formal deformations of  $\mathcal{M}_*^{\leq \infty}$ ? In other words, can we construct suitable families  $(\mu_n)_{n \in \mathbb{N}}$  of bilinear maps on  $\mathcal{M}_*^{\leq \infty}$  that increase the weight by  $2n$ , preserve the depth and define an analogue of Eholzer product? The brackets defined in (1) do not lead to a solution since  $\text{RC}_1$  does not even provide  $\mathcal{M}_*^{\leq \infty}$  with a Poisson structure. Our first step is to obtain admissible Poisson brackets on  $\mathcal{M}_*^{\leq \infty}$  with the following definition.

**Definition 1**– A Poisson bracket  $\mathfrak{b}$  on  $\mathcal{M}_*^{\leq \infty}$  is admissible if

- 1)  $\mathfrak{b}(f, g) = \text{RC}_1(f, g)$  if  $f$  and  $g$  are in  $\mathcal{M}_*$
- 2) it satisfies  $\mathfrak{b}(\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}) \subset \mathcal{M}_{k+\ell+2}^{\leq s+t}$  for all  $k, \ell, s, t$ .

*Remark*– We could have replace condition 2 by the following one: there exists  $e \geq 0$  such that  $\mathfrak{b}(\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}) \subset \mathcal{M}_{k+\ell+e}^{\leq s+t}$  for all  $k, \ell, s, t$ . However, condition 1 implies that necessarily  $e = 2$ .

Equivalently, a Poisson bracket  $\mathfrak{b}$  on  $\mathcal{M}_*^{\leq \infty}$  is admissible if and only if

$$\begin{aligned} \mathfrak{b}(E_4, E_6) &= -2\Delta \\ \mathfrak{b}(E_2, E_4) &\in \mathcal{M}_8^{\leq \infty}, \quad \mathfrak{b}(E_2, E_6) \in \mathcal{M}_{10}^{\leq \infty} \\ \mathfrak{b}(E_2, \mathcal{M}_*) &\subset \mathcal{M}_* E_2 + \mathcal{M}_*. \end{aligned}$$

In order to classify the admissible Poisson brackets, we introduce the notion of Poisson modular isomorphism.

**Definition 2**– A Poisson isomorphism  $\varphi: (\mathcal{M}_*^{\leq \infty}, \mathfrak{b}_1) \rightarrow (\mathcal{M}_*^{\leq \infty}, \mathfrak{b}_2)$  is called a Poisson modular isomorphism if  $\varphi(\mathcal{M}_*) \subset \mathcal{M}_*$ .

Indeed, if  $\varphi$  is a Poisson modular isomorphism then, its restriction to the subalgebra  $\mathcal{M}_*$  is the identity. This is a consequence of the following Proposition.

**Proposition 5**– The group of Poisson automorphisms of the Poisson algebra  $(\mathcal{M}_*, \text{RC}_1)$  is trivial.

*Proof.* Let  $\varphi$  be a Poisson automorphism of  $\mathcal{M}_*$ . There exist two polynomials  $s$  and  $t$  in  $\mathbb{C}[x, y]$  such that  $\varphi(E_4) = s(E_4, E_6)$  and  $\varphi(E_6) = t(E_4, E_6)$ . By (2) we have

$$\text{RC}_1(\varphi(E_4), \varphi(E_6)) = \text{jac}(s, t)(E_4, E_6) \cdot \text{RC}_1(E_4, E_6).$$

Since  $\varphi$  is an automorphism,  $\text{jac}(s, t)$  is a non zero scalar, say  $\lambda$ . We get

$$\varphi(\text{RC}_1(E_4, E_6)) = \lambda \text{RC}_1(E_4, E_6) \quad \text{hence} \quad s^3 - t^2 = \lambda(x^3 - y^2) \quad \text{in } \mathbb{C}[x, y]. \quad (5)$$

We develop  $s$  and  $t$  into homogeneous components with respect to the weight:

$$s = \sum_{\substack{i=0 \\ i \neq 1}}^m s_{2i} \quad \text{and} \quad t = \sum_{\substack{i=0 \\ i \neq 1}}^n t_{2i}$$

where

$$s_{2i} = \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ 2a+3b=i}} \sigma_{a,b} x^a y^b \quad \text{and} \quad t_{2i} = \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ 2a+3b=i}} \tau_{a,b} x^a y^b \quad (\sigma_{a,b}, \tau_{a,b} \in \mathbb{C})$$

for all  $i$  (where  $m = 0$  or  $m \geq 2$  and  $n = 0$  or  $n \geq 2$ ). Equation (5) implies that  $t(E_4, E_6)^2 - s(E_4, E_6)^3$  has weight 12. Then only three cases are possible.

- 1) If  $3m > 2n$  then  $m = 2$  and so  $n \in \{0, 2\}$ . It implies that  $s = \sigma_{00} + \sigma_{10}x$  and  $t = \tau_{00} + \tau_{10}x$ . This contradicts  $\text{jac}(s, t) \neq 0$ .
- 2) If  $3m < 2n$  then  $n = 3$  and  $m = 0$ . This contradicts  $\text{jac}(s, t) \neq 0$ .
- 3) If  $3m = 2n$ . We differentiate (5) with respect to  $x$  and  $y$  and get

$$3s^2 \frac{\partial s}{\partial x} - 2t \frac{\partial t}{\partial x} = 3\lambda x^2, \quad 3s^2 \frac{\partial s}{\partial y} - 2t \frac{\partial t}{\partial y} = -2\lambda y.$$

This implies

$$2t = 3x^2 \frac{\partial s}{\partial y} + 2y \frac{\partial s}{\partial x}, \quad 3s^2 = 3x^2 \frac{\partial t}{\partial y} + 2y \frac{\partial t}{\partial x}. \quad (6)$$

From the first differential equation of (6) we have

$$2t(E_4, E_6) = 3E_4^2 \frac{\partial s}{\partial y}(E_4, E_6) + 2E_6 \frac{\partial s}{\partial x}(E_4, E_6).$$

The highest weight of the right hand side is less than or equal to  $2m + 2$ . This implies  $n \leq m + 1$ . From the second differential equation of (6) we have

$$3s^2(E_4, E_6) = 3E_4^2 \frac{\partial t}{\partial y}(E_4, E_6) + 2E_6 \frac{\partial t}{\partial x}(E_4, E_6)$$

hence  $2m \leq n + 1$ . We deduce  $(m, n) \in \{(0, 0), (2, 3)\}$ . Since  $n = m = 0$  would imply  $\text{jac}(s, t) = 0$  we have  $n = 3$  and  $m = 2$ . Then  $s = \sigma_{00} + \sigma_{10}x$  and  $t = \tau_{00} + \tau_{10}x + \tau_{01}y$ . The first differential equation in (6) implies that  $\tau_{00} = \tau_{10} = 0$  and  $\tau_{01} = \sigma_{10}$  whereas the second one implies that  $\sigma_{00} = 0$  and  $\sigma_{10} = 1$ . Finally  $\varphi(E_4) = s(E_4, E_6) = E_4$  and  $\varphi(E_6) = s(E_4, E_6) = E_6$ .  $\square$

Since  $\text{RC}_1(\Delta, f) = (12D(f) - kfE_2)\Delta$  for any  $f \in \mathcal{M}_k$ , the first Rankin-Cohen bracket defines a derivation on  $\mathcal{M}_*$  called the Serre's derivative by linear extension of:

$$\vartheta f = \frac{\text{RC}_1(\Delta, f)}{12\Delta} = D(f) - \frac{k}{12}fE_2 \quad (f \in \mathcal{M}_k). \quad (7)$$

This derivation is characterised by its values on the generators

$$\vartheta E_4 = -\frac{1}{3}E_6, \quad \vartheta E_6 = -\frac{1}{2}E_4^2.$$

We shall need the following result.

**Proposition 6**– *The kernel of Serre’s derivative is the Poisson centraliser of  $\Delta$  for the first Rankin-Cohen bracket. This is  $\mathbb{C}[\Delta]$ .*

*Proof.* If  $f \in \mathcal{M}_k$  is in  $\ker \vartheta$  then  $kf D(\Delta) = 12\Delta D(f)$ . Solving the differential equation we find that 12 divides  $k$  and that  $f \in \mathbb{C}\Delta^{k/12}$ .  $\square$

We note that, for any  $g \in \mathcal{M}_\ell$ , we have

$$\text{RC}_1(\Delta^m, g) = m\Delta^m (12D(g) - \ell g E_2)$$

and deduce that, for any  $f \in \mathbb{C}[\Delta]$  and  $g \in \mathcal{M}_*$  we have

$$\text{RC}_1(f, g) = 12\xi(f)\vartheta(g) \quad (8)$$

where  $\xi$  is the Eulerian derivative on  $\mathbb{C}[\Delta]$  defined by  $\xi = \Delta \frac{\partial}{\partial \Delta}$ .

### 3. POISSON STRUCTURES ON QUASIMODULAR FORMS

**3.1. First family.** This section is devoted to the proof of Proposition A.

We fix  $\lambda \in \mathbb{C}^*$  and introduce in  $\mathbb{C}[x, y, z]$  the three polynomials

$$\begin{aligned} r(x, y, z) &= \frac{1}{3}(\lambda y^2 - 2xz) \\ p(x, y, z) &= -2(y^3 - z^2) \\ q(x, y, z) &= -\frac{1}{2}(\lambda yz - 2xy^2). \end{aligned}$$

Since  $(p, q, r) \cdot \text{curl}(p, q, r) = 0$ , we define a Poisson bracket on  $\mathcal{M}_*^{\leq \infty}$  if we set

$$\begin{aligned} \{E_4, E_6\}_\lambda &= p(E_2, E_4, E_6) \\ \{E_2, E_4\}_\lambda &= r(E_2, E_4, E_6) \\ \{E_6, E_2\}_\lambda &= q(E_2, E_4, E_6). \end{aligned}$$

Let us prove that  $\{ , \}_\lambda$  is not unimodular. If it were, we would have  $k \in \mathbb{C}[x, y, z]$  such that  $\text{curl}(p, q, r) = (p, q, r) \wedge \text{grad } k$ . Identifying the first components would lead to

$$\frac{7}{6}\lambda y = \frac{1}{2}(-\lambda yz + 2y^2x) \frac{\partial k}{\partial z} - \frac{1}{3}(\lambda y^2 - 2zx) \frac{\partial k}{\partial y}$$

that has no solution in  $\mathbb{C}[x, y, z]$ .

A Poisson modular isomorphism  $\varphi_\lambda$  between  $(\mathcal{M}_*^{\leq \infty}, \{ , \}_\lambda)$  and  $(\mathcal{M}_*^{\leq \infty}, \{ , \}_1)$  is determined by

$$\varphi_\lambda(E_2) = \lambda E_2, \quad \varphi_\lambda(E_4) = E_4, \quad \varphi_\lambda(E_6) = E_6.$$

Finally, we determine the Poisson center of the Poisson algebra  $(\mathcal{M}_*^{\leq \infty}, \{ , \}_1)$ . Let us define a derivation on  $\mathcal{M}_*$  by  $\sigma = 2\vartheta$  (see (7)) and a derivation on  $\mathcal{M}_*$  by linear extension of

$$\delta(f) = \frac{k}{12} f E_4 \quad (f \in \mathcal{M}_k).$$

We note that  $(\mathcal{M}_*^{\leq \infty}, \{ , \}_1)$  is the Poisson-Ore extension  $\mathbb{C}[E_4, E_6][E_2]_{\sigma, \delta}$ . Now, consider any  $f \in \mathcal{M}_*^{\leq \infty}$  written as

$$f = \sum_{i=0}^s f_i E_2^i, \quad f_i \in \mathcal{M}_*.$$

We compute

$$\{E_2, f\}_1 = \delta(f_0) + \sum_{i=1}^s (\sigma(f_{i-1}) + \delta(f_i)) E_2^i + \sigma(f_s) E_2^{s+1}.$$

If  $\{E_2, f\}_1 = 0$  then  $\delta(f_0) = 0$  hence  $f_0 \in \mathbb{C}$  and  $\sigma(f_0) = 0$ . We obtain inductively that  $f_i \in \mathbb{C}$  for all  $0 \leq i \leq s$  so that the Poisson centraliser of  $E_2$  is  $\mathbb{C}[E_2]$ . Suppose that the Poisson center contains a non scalar element. Then it is in the Poisson centraliser of  $E_2$  and can be written

$$f = \sum_{j=0}^p \alpha_j E_2^j, \quad p \geq 1, \alpha_j \in \mathbb{C}, \alpha_p \neq 0.$$

We compute

$$\{E_4, f\}_1 = \sum_{j=0}^p j \alpha_j E_2^{j-1} \cdot \{E_4, E_2\}_1$$

and find that the coefficient of  $E_2^p$  is non zero. It follows that  $f$  is not in the Poisson center.

**3.2. Second family.** This section is devoted to the proof of Proposition B. We fix  $\alpha \in \mathbb{C}$  and introduce in  $\mathbb{C}[x, y, z]$  the three polynomials

$$\begin{aligned} r(x, y, z) &= \alpha x z \\ p(x, y, z) &= -2(y^3 - z^2) \\ q(x, y, z) &= -\frac{3}{2} \alpha x y^2. \end{aligned}$$

Since  $(p, q, r) \cdot \text{curl}(p, q, r) = 0$ , we define a Poisson bracket on  $\mathcal{M}_*^{\leq \infty}$  if we set

$$\begin{aligned} (E_4, E_6)_\alpha &= p(E_2, E_4, E_6) \\ (E_2, E_4)_\alpha &= r(E_2, E_4, E_6) \\ (E_6, E_2)_\alpha &= q(E_2, E_4, E_6). \end{aligned}$$

Assume  $\alpha \neq 4$ . Let us prove that  $( , )_\alpha$  is not unimodular. If it were, we would have  $k \in \mathbb{C}[x, y, z]$  such that  $\text{curl}(p, q, r) = (p, q, r) \wedge \text{grad } k$ . Identifying the second components would lead to

$$(4 - \alpha)z = \alpha x z \frac{\partial k}{\partial x} + 2(y^3 - z^2) \frac{\partial k}{\partial z}$$

that has no solution in  $\mathbb{C}[x, y, z]$ .

If  $\alpha = 4$ , then  $(p, q, r) = \text{grad } k_0$  where  $k_0 = -2(y^3 - z^2)x$ . As a consequence, the bracket  $( , )_4$  provides  $\mathcal{M}_*^{\leq \infty}$  with a Jacobian Poisson structure of potential  $k_0 = -2\Delta E_2 = -2D(\Delta)$ .

If  $\varphi: (\mathcal{M}_*^{\leq \infty}, (\cdot, \cdot)_\alpha) \rightarrow (\mathcal{M}_*^{\leq \infty}, (\cdot, \cdot)_{\alpha'})$  is a Poisson modular isomorphism, let us prove that  $\alpha = \alpha'$ . By Proposition 5, we have  $\varphi(E_4) = E_4$  and  $\varphi(E_6) = E_6$ . By surjectivity, it follows that  $\varphi(E_2) = \eta E_2 + F$  for some  $\eta \in \mathbb{C}^*$  and  $F \in \mathcal{M}_*$ . We compute

$$\varphi((E_2, E_4)_\alpha) = \alpha \eta E_6 E_2 + \alpha E_6 F$$

and

$$(\varphi(E_2), \varphi(E_4))_{\alpha'} = \alpha' \eta E_6 E_2 + (F, E_4)_{\alpha'}.$$

Since  $(F, E_4)_{\alpha'} \in \mathcal{M}_*$  we get  $\alpha' = \alpha$ .

Finally, we determine the Poisson center of the Poisson algebra  $(\mathcal{M}_*^{\leq \infty}, (\cdot, \cdot)_\alpha)$ . We note that  $(\mathcal{M}_*^{\leq \infty}, (\cdot, \cdot)_\alpha)$  is the Poisson-Ore extension  $\mathbb{C}[E_4, E_6][E_2]_{\sigma, \delta}$  where  $\sigma = -3\alpha\partial$  (see (7)) and  $\delta = 0$ . Let

$$f = \sum_{j=0}^s f_j E_2^j, \quad f_j \in \mathcal{M}_*.$$

We have

$$(E_2, f)_\alpha = \sum_{j=0}^s \sigma(f_j) E_2^{j+1}$$

hence  $f$  is in the Poisson centraliser of  $E_2$  if and only if each  $f_j$  is in the Poisson centraliser of  $\Delta$  for  $\text{RC}_1$ . By Proposition 6 we deduce that the centraliser of  $E_2$  is  $\mathbb{C}[\Delta, E_2]$ . Let

$$f = \sum_{j=0}^s f_j(\Delta) E_2^j \in \mathbb{C}[\Delta, E_2].$$

We use (8) to compute

$$(f, E_4)_\alpha = \sum_{j=0}^s (-4\xi(f_j) + j\alpha f_j) E_6 E_2^j$$

and

$$(f, E_6)_\alpha = \frac{3}{2} \sum_{j=0}^s (-4\xi(f_j) + j\alpha f_j) E_4^2 E_2^j.$$

We deduce that  $f$  is in the Poisson center of  $(\cdot, \cdot)_\alpha$  if and only if

$$\xi(f_j) = \frac{j\alpha}{4} f_j$$

for all  $j$ , i.e. if and only if any  $f_j$  is of the form  $f_j = \lambda_j \Delta^{m_j}$  for some  $\lambda_j \in \mathbb{C}$  and  $m_j \in \mathbb{N}$  such that  $j\alpha = 4m_j$ . If  $\alpha \notin \mathbb{Q}$  or if  $\alpha < 0$  then  $j = 0$  and  $m_j = 0$  hence  $f = f_0 \in \mathbb{C}$ . If  $\alpha = p/q$  with  $p \geq 1$ ,  $q \geq 1$  and  $(p, q) = 1$  then  $\lambda \Delta^{m_j} E_2^j$  is in the Poisson center if and only if  $pj = 4qm_j$ . The result follows by obvious arithmetical consideration. Finally, if  $\alpha = 0$ , then  $(\cdot, \cdot)_0$  is the trivial bracket and its Poisson center is  $\mathbb{C}[E_2]$ .



**3.3. Third family.** In this section, we study the third family, i.e. we prove Proposition C.

For any  $\mu \in \mathbb{C}$ , let us introduce

$$k_\mu = -2\Delta E_2 + \mu E_4^2 E_6.$$

Then

$$\begin{aligned} \text{jac}(E_4, E_6, k_\mu) &= \frac{\partial k_\mu}{\partial E_2} = -2E_4^3 + E_6^2 \\ \text{jac}(E_2, E_4, k_\mu) &= \frac{\partial k_\mu}{\partial E_6} = 4E_6 E_2 + \mu E_4^3 \\ \text{jac}(E_2, E_6, k_\mu) &= -\frac{\partial k_\mu}{\partial E_4} = 6E_4^2 E_2 - 2\mu E_4 E_6. \end{aligned}$$

The third family of Poisson bracket is then defined by  $\langle f, g \rangle_\mu = \text{jac}(f, g, k_\mu)$ . With the notation of Proposition B, we have in particular  $\langle f, g \rangle_0 = (f, g)_4$ .

For any  $\mu \in \mathbb{C}^*$ , we define a Poisson modular isomorphism  $\varphi_\mu$  between  $(\mathcal{M}_*^{\leq \infty}, \langle \cdot, \cdot \rangle_\mu)$  and  $(\mathcal{M}_*^{\leq \infty}, \langle \cdot, \cdot \rangle_1)$  by setting  $\varphi_\mu(E_2) = \mu E_2$ ,  $\varphi_\mu(E_4) = E_4$  and  $\varphi_\mu(E_6) = E_6$ .

Since the degree in  $E_2$  of  $k_\mu$  as a polynomial in  $E_2, E_4, E_6$  is 1, Lemma 1 implies that the Poisson center of  $(\mathcal{M}_*^{\leq \infty}, \langle \cdot, \cdot \rangle_\mu)$  is  $\mathbb{C}[k_\mu]$ .

**3.4. Classification.** This section is devoted to the proof of Theorem C.

Let  $\{ \cdot, \cdot \}$  be an admissible bracket on  $\mathcal{M}_*^{\leq \infty}$ . By Definition 1 and Theorem 2, there exist a Poisson derivation  $\sigma$  of  $\mathcal{M}_*$  and a Poisson  $\sigma$ -derivation  $\delta$  of  $\mathcal{M}_*$  such that

$$\{E_2, f\} = \sigma(f)E_2 + \delta(f) \quad (f \in \mathcal{M}_*).$$

By definition,  $\sigma(\mathcal{M}_k) \subset \mathcal{M}_{k+2}$  and  $\delta(\mathcal{M}_k) \subset \mathcal{M}_{k+4}$  for any  $k$ . The admissible bracket  $\{ \cdot, \cdot \}$  is then defined by the four scalars  $\alpha, \beta, \gamma$  and  $\varepsilon$  such that

$$\sigma(E_4) = \alpha E_6, \delta(E_4) = \beta E_4^2, \sigma(E_6) = \gamma E_4^2, \text{ and } \delta(E_6) = \varepsilon E_4 E_6.$$

The condition that  $\sigma$  is a Poisson derivation imposes the condition

$$\{\sigma(E_4), E_6\} + \{E_4, \sigma(E_6)\} = -2\sigma(E_4^3 - E_6^2)$$

or equivalently  $3\alpha = 2\gamma$ . The condition that  $\delta$  is a Poisson  $\sigma$ -derivation imposes

$$\delta(\{E_4, E_6\}) = (2\beta + \varepsilon)E_4\{E_4, E_6\} + \alpha\varepsilon E_4 E_6^2 - \beta\gamma E_4^4$$

or equivalently

$$\begin{cases} 4\beta + (\alpha - 2)\varepsilon = 0 \\ (3\alpha - 4)\beta + 4\varepsilon = 0. \end{cases}$$

Either  $\beta = \varepsilon = 0$  is the only solution, or  $\alpha \in \{-2/3, 4\}$  and  $\varepsilon = \frac{4}{2-\alpha}\beta$ .

- The case  $\beta = \varepsilon = 0$  leads to the second family:  $\{ \cdot, \cdot \} = ( \cdot, \cdot )_\alpha$ .
- The case  $\alpha = -2/3$  and  $\varepsilon = 3\beta/2 \neq 0$  leads to the first family:  $\{ \cdot, \cdot \} = \{ \cdot, \cdot \}_{3\beta}$ .
- The case  $\alpha = 4$  and  $\varepsilon = -2\beta \neq 0$  leads to the third family:  $\{ \cdot, \cdot \} = \langle \cdot, \cdot \rangle_\beta$ .

Using Propositions B and C we conclude that the only admissible Poisson brackets, up to Poisson modular-isomorphisms are  $\{ , \}_1$ ,  $\langle , \rangle_1$  and  $( , )_\alpha$  for any  $\alpha \in \mathbb{C}$ . Looking at the centres, it is clear that the Poisson algebras  $(\mathcal{M}_*^{\leq \infty}, \langle , \rangle_1)$  and  $(\mathcal{M}_*^{\leq \infty}, \{ , \}_1)$  are not Poisson modular isomorphic. Suppose that there exists a Poisson modular isomorphism  $\varphi$  from  $(\mathcal{M}_*^{\leq \infty}, ( , )_\alpha)$  to  $(\mathcal{M}_*^{\leq \infty}, \{ , \}_1)$ . We know (see § 3.2) that

$$\varphi(E_4) = E_4, \quad \varphi(E_6) = E_6, \quad \text{and} \quad \varphi(E_2) = \eta E_2 + F$$

for some  $\eta \in \mathbb{C}^*$  and  $F \in \mathcal{M}_*$ . From  $\varphi((E_2, E_4)_\alpha) = \{\varphi(E_2), \varphi(E_4)\}_1$  we obtain

$$\alpha \eta E_6 E_2 + \alpha F E_6 = -\frac{2}{3} \eta E_6 E_2 + \frac{1}{3} \eta E_4^2 + \{F, E_4\}_1$$

hence

$$\begin{aligned} \alpha = -\frac{2}{3}, \quad \frac{1}{3} \eta E_4^2 &= -\frac{2}{3} F E_6 - \{F, E_4\}_1 \\ &= -\frac{2}{3} F E_6 + 2(E_4^3 - E_6^2) \frac{\partial F}{\partial E_6} \end{aligned}$$

by (2). We get a contradiction. Replacing  $\{ , \}_1$  by  $\langle , \rangle_1$  we get

$$\alpha = 4, \quad \eta E_4^2 = 4F E_6 - 2(E_4^3 - E_6^2) \frac{\partial F}{\partial E_6}$$

and again, we get a contradiction.

#### 4. STAR PRODUCTS ON QUASIMODULAR FORMS

**4.1. Extension of the first family.** This section is devoted to the proof of Theorem A. We will use the following result of Zagier [Zag94, Example 1]. Let  $A = \bigoplus A_k$  be a commutative graded algebra with a derivation  $d$  homogeneous of degree 2 (i.e.  $d(A_k) \subset A_{k+2}$ ). Let us define, for any  $f \in A_k$ ,  $g \in A_\ell$ ,  $r \geq 0$ :

$$[f, g]_{d,r} = \sum_{i=0}^r (-1)^i \binom{k+r-1}{r-i} \binom{\ell+r-1}{i} d^i(f) d^{r-i}(g) \in A_{k+\ell+2r}. \quad (9)$$

Then  $A$  equipped with these brackets is a Rankin-Cohen algebra, that means that all algebraic identities satisfied by the usual Rankin-Cohen brackets on modular forms are also satisfied, in particular those expressing of the associativity of the corresponding star product. We obtain the following result.

**Theorem 7**– *The star product defined by*

$$f \# g = \sum_{n \geq 0} [f, g]_{d,n} \hbar^n$$

*defines a formal deformation on  $A$ .*

In particular, we recover the fact, given by Corollary 4, that  $[ , ]_{d,1}$  is a Poisson bracket. Note also that this Theorem can be obtained from Connes & Moscovici's result cited below (see § 4.2).

Let  $a \in \mathbb{C}$  and  $d_a$  be the homogeneous derivation of degree 2 on  $\mathcal{M}_*^{\leq \infty}$  defined by

$$d_a(E_2) = 2aE_2^2 - \frac{1}{12}E_4, \quad d_a(E_4) = 4aE_4E_2 - \frac{1}{3}E_6, \quad d_a(E_6) = 6aE_6E_2 - \frac{1}{2}E_4^2.$$

A direct computation proves that the two Poisson brackets  $[ \ , \ ]_{d,a,1}$  and  $\{ \ , \ }_1$  coincide on generators hence are equal on  $\mathcal{M}_*^{\leq \infty}$ .

*Remark-* A derivation  $d$  on  $\mathcal{M}_*^{\leq \infty}$  is complex-like if  $d\mathcal{M}_k^{\leq s} \subset \mathcal{M}_{k+2}^{\leq s+1}$  for all  $k$  and  $s$ . Let  $\pi$  be the derivation on  $\mathcal{M}_*^{\leq \infty}$  defined by linear extension of  $\pi(f) = kfE_2$  for all  $f \in \mathcal{M}_k^{\leq \infty}$ . The set of complex-like derivations  $d$  such that  $[ \ , \ ]_{d,1} = 0$  is the vector space of dimension 1 over  $\mathbb{C}$  generated by  $\pi$ . Let us define  $w$  on  $\mathcal{M}_*^{\leq \infty}$  by

$$w(f) = \frac{\{\Delta, f\}_1}{12\Delta}.$$

Then,

$$d_a = w + a\delta.$$

This implies in particular that if a complex-like derivation  $d$  satisfies  $[ \ , \ ]_{d,1} = \{ \ , \ }_1$  then  $d = d_a$  for some  $a \in \mathbb{C}$ .

Point ii) of Theorem A is obtained by a direct application of Theorem 7. We prove now iii). The term of highest degree with respect to  $E_2$  in  $[E_2, E_4]_{d_a,2}$  is  $8a^2 E_4 E_2^3$ . This forces  $a = 0$ . Conversely, if  $a = 0$  then  $d_0 \mathcal{M}_*^{\leq \infty} \subset \mathcal{M}_*$ . For any  $f = f_i E_2^i$  with  $f_i \in \mathcal{M}_*$  we have

$$d_0(f) = d_0(f_i)E_2^i - \frac{1}{12}i f_i E_4 E_2^{i-1}$$

hence  $\deg_{E_2} d_0(f) \leq \deg_{E_2} f$  and  $\deg_{E_2} d_0^j(f) \leq \deg_{E_2} f$  for any  $f \in \mathcal{M}_*^{\leq \infty}$  and  $j \geq 0$ . This implies that

$$[\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}]_{d_0, n} \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}.$$

**4.2. Extension of the second family.** The aim of this section is to prove Theorem B. The proof of i) is similar to the proof of i) in Theorem A. Let  $\mathcal{K}: \mathcal{M}_*^{\leq \infty} \rightarrow \mathbb{C}$  a graded-additive map. For any integer  $n \geq 0$ , we define a bilinear application  $[ \ , \ ]_{d,n}^{\mathcal{K}}$  by bilinear extension of

$$[f, g]_{d,n}^{\mathcal{K}} = \sum_{r=0}^n (-1)^r \binom{\mathcal{K}(f) + n - 1}{n-r} \binom{\mathcal{K}(g) + n - 1}{r} d^r f d^{n-r} g.$$

By Corollary 4, we know that  $[f, g]_{d,1}^{\mathcal{K}}$  is a Poisson bracket.

Let us fix  $\mathcal{K}_\alpha$  to be the linear extension on  $\mathcal{M}_*^{\leq \infty} = \bigoplus_k \bigoplus_s \mathcal{M}_k^s$  of

$$\mathcal{K}_\alpha(f) = (k - (3\alpha + 2)s) \quad (f \in \mathcal{M}_k^s).$$

Let  $\pi_\alpha$  be the derivation on  $\mathcal{M}_*^{\leq \infty}$  defined by  $\pi_\alpha(f) = \mathcal{K}_\alpha(f)fE_2$  for all  $f \in \mathcal{M}_*^{\leq \infty}$ . The set of complex-like derivations such that  $[ \ , \ ]_{d,1}^{\mathcal{K}_\alpha} = 0$  is the vector space of dimension 1 over  $\mathbb{C}$  generated by  $\pi_\alpha$ . Define derivations  $v$  and  $\delta_{\alpha,b}$  on  $\mathcal{M}_*^{\leq \infty}$  by

$$v(f) = \frac{(\Delta, f)_\alpha}{12\Delta}$$

and

$$\delta_{\alpha,b} = v + b\pi_\alpha.$$

Note that  $v$  does not depend on  $\alpha$ . By comparing the values on the generators, it is immediate that  $(\ , \ )_\alpha = [ \ , \ ]_{\delta_{\alpha,b,1}}^{\mathcal{K}_\alpha}$ .

*Remark-* Direct computations show that if  $d$  is a homogeneous derivation of degree 2 and  $\mathcal{K}$  is such that  $(\ , \ )_\alpha = [ \ , \ ]_{d,1}^{\mathcal{K}}$  then we necessarily have  $\mathcal{K} = \mathcal{K}_\alpha$  and  $d = \delta_{\alpha,b}$  for some  $b \in \mathbb{C}$ .

The condition that  $[E_4, E_6]_{\delta_{\alpha,b,2}}^{\mathcal{K}}$  has to be a modular form implies  $b = 0$  or  $\alpha = -1/3$ . For  $\alpha = -1/3$  condition (4) for  $\mu_r = [ \ , \ ]_{\delta_{\alpha,b,r}}^{\mathcal{K}}$  and  $n = 3$  is not satisfied (this can be done by using computer assistance, for example with Sage [S<sup>+</sup>13]). We assume then that  $b = 0$ .

Connes & Moscovici [CM04, Remark 14] (see also [Yao07, § II.2] for a nice presentation of this result) proved that, if  $E$  and  $H$  are two derivations of an algebra  $R$  such that  $HE - EH = E$ , then the applications  $\mu_n: R \times R \rightarrow R$  defined by

$$\mu_n(f, g) = \sum_{r=0}^n \frac{(-1)^r}{r!(n-r)!} [E^r \circ (2H+r)^{\langle n-r \rangle}(f)] \cdot [E^{n-r} \circ (2H+n-r)^{\langle r \rangle}(g)] \quad (10)$$

define a formal deformation on  $R$  with the notation

$$F^{\langle m \rangle} = F \circ (F+1) \circ (F+2) \circ \cdots \circ (F+m-1).$$

Let  $\omega$  be the derivation defined on  $\mathcal{M}_*^{\leq \infty}$  by  $\omega(f) = \mathcal{K}(f)f$ . Then we have

$$\omega \circ \delta_{\alpha,0} - \delta_{\alpha,0} \circ \omega = 2\delta_{\alpha,0}.$$

We use Connes & Moscovici's result with  $E = \delta_{\alpha,0}$  and  $H = \omega/2$  to obtain

$$\mu_n(f, g) = \sum_{r=0}^n (-1)^r \binom{k - (3\alpha + 2)s + n - 1}{n-r} \binom{\ell - (3\alpha + 2)t + n - 1}{r} \delta_{\alpha,0}^r(f) \delta_{\alpha,0}^{n-r}(g).$$

This implies Theorem B.

*Remark-* We could have apply Connes & Moscovici's result to extend the first family. Indeed Zagier's result is a consequence of Connes & Moscovici's one. Let  $d$  be a derivation homogeneous of degree 2 of the commutative graded algebra  $A = \bigoplus A_k$ . It is obvious that the linear map defined on each  $A_k$  by  $H(f) = \frac{k}{2}f$  is a derivation of  $A$ . It is also clear that it satisfies  $H \circ d - d \circ H = d$ . In particular, for any  $f \in A_k$  and  $g \in A_\ell$  we calculate

$$(2H+r)^{\langle n-r \rangle}(f) = \frac{(k+n-1)!}{(k+r-1)!} f$$

and

$$(2H+(n-r))^{\langle r \rangle}(g) = \frac{(\ell+n-1)!}{(\ell+n-r-1)!} g.$$

Hence a direct application of formula (10) gives formula (9).

**4.3. Extension of the third family.** We do not extend the third family since, for  $\mu \neq 0$ , the bracket  $\langle \cdot, \cdot \rangle_\mu$  has not the shape of a Rankin-Cohen bracket. More precisely, if there exists a function  $\kappa: \mathcal{M}_*^{\leq \infty} \rightarrow \mathbb{C}$  and a complex-like derivation  $\delta$  of  $\mathcal{M}_*^{\leq \infty}$  such that

$$\langle f, g \rangle_\mu = \kappa(f)f\delta(g) - \kappa(g)g\delta(f)$$

for all  $f$  and  $g$  in  $\mathcal{M}_*^{\leq \infty}$ , then  $\mu = 0$ . Indeed, assume  $\kappa$  and  $\delta$  exist, then

$$\begin{cases} \delta(E_2) = AE_2^2 + BE_4 \\ \delta(E_4) = CE_4E_2 + DE_6 \\ \delta(E_6) = EE_6E_2 + FE_4^2 \end{cases}$$

for some complex numbers  $A, B, C, D, E$  and  $F$ . Since we know the values of  $\langle \cdot, \cdot \rangle_\mu$  on the generators, we get a system relying  $A, B, C, D, E, F, \kappa(E_2), \kappa(E_4)$  and  $\kappa(E_6)$ . It is not difficult to prove that this system has a solution if and only if  $\mu = 0$ .

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