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POISSON STRUCTURES AND STAR PRODUCTS ON QUASIMODULAR FORMS

FRANÇOIS DUMAS AND EMMANUEL ROYER

Abstract. We construct and classify all Poisson structures on quasimodular forms that extend the one coming from the first Rankin-Cohen bracket on the modular forms. We use them to build formal deformations on the algebra of quasimodular forms.

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1. Introduction

In [Coh75, Theorem 7.1], Henri Cohen defined a collection of bidifferential operators on modular forms. Let n be a positive integer, f a modular form of weight k and g a modular form of weight ℓ. The n-th Rankin-Cohen bracket of f and g is the modular form of weight $k+\ell+2n$ defined by

$$\mathrm{RC}_n(f,g)=\sum_{r=0}^{n}(-1)^r\binom{k+n-1}{n-r}\binom{\ell+n-1}{r}\mathrm{D}^r f\,\mathrm{D}^{n-r}g,\qquad\left(\mathrm{D}=\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}\right).$$

The algebraic structure of these brackets has been studied in Zagier's seminal work [Zag94]. That Rankin-Cohen brackets define a formal deformation of the algebra of

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modular forms has been widely studied. Among other contributions, we cite Cohen, Manin & Zagier [CMZ97] and Yi-Jun Yao [Yao07, Chapitre IV].

In this paper, we construct formal deformations of the algebra $\mathcal{M}_*^{\leq\infty}$ of quasimodular forms. This algebra is generated over $\mathbb{C}$ by the three Eisenstein series E_2, E_4 and E_6. The algebra $\mathcal{M}_*$ of modular forms is the subalgebra generated by E_4 and E_6. As a first step, we classify the admissible Poisson structures of $\mathcal{M}_*^{\leq\infty}$. A Poisson bracket $\{\ ,\ \}$ on $\mathcal{M}_*^{\leq\infty}$ is admissible if

i) the restriction of $\{\ ,\ \}$ to the algebra $\mathcal{M}_*$ of modular forms is the first Rankin-Cohen bracket RC_1
ii) there exists $e \geq 0$ such that $\{\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}\} \subset \mathcal{M}_{k+\ell+e}^{\leq s+t}$ for any even integers k, ℓ and any integers s and t

where $\mathcal{M}_k^{\leq s}$ is the vector space of quasimodular forms of weight k and depth less than s. The vector space of parabolic modular forms of weight 12 is one dimensional. We choose $\Delta = \mathrm{E}_4^3 - \mathrm{E}_6^2$ a generator.

***Proposition* A** (First family of Poisson brackets)– *For any $\lambda \in \mathbb{C}^*$, there exists an admissible Poisson bracket $\{\ ,\ \}_\lambda$ on the algebra of quasimodular forms defined by the following values on the generators:*

$$\{\mathrm{E}_4, \mathrm{E}_6\}_\lambda = -2\Delta, \quad \{\mathrm{E}_2, \mathrm{E}_4\}_\lambda = -\frac{1}{3}\Big(2\,\mathrm{E}_6\,\mathrm{E}_2 - \lambda\,\mathrm{E}_4^2\Big), \quad \{\mathrm{E}_2, \mathrm{E}_6\}_\lambda = -\frac{1}{2}\Big(2\,\mathrm{E}_4^2\,\mathrm{E}_2 - \lambda\,\mathrm{E}_4\,\mathrm{E}_6\Big).$$

Moreover,

i) for any $\lambda \in \mathbb{C}^*$, the Poisson bracket $\{\ ,\ \}_\lambda$ is not unimodular.
ii) The Poisson algebras $\Big(\mathcal{M}_*^{\leq\infty}, \{\ ,\ \}_\lambda\Big)$ and $\Big(\mathcal{M}_*^{\leq\infty}, \{\ ,\ \}_{\lambda'}\Big)$ are Poisson modular isomorphic for all λ and λ' in $\mathbb{C}^*$.
iii) For any $\lambda \in \mathbb{C}^*$, the Poisson center of $\Big(\mathcal{M}_*^{\leq\infty}, \{\ ,\ \}_\lambda\Big)$ is $\mathbb{C}$.

Remark- A Poisson isomorphism φ on $\mathcal{M}_*^{\leq\infty}$ is modular if $\varphi(\mathcal{M}_*) \subset \mathcal{M}_*$.

Thanks to ii) in Proposition A, we restrict to the bracket $\{\ ,\ \}_1$. Following [Zag94, Eq. (38)], we consider the derivation w on $\mathcal{M}_*^{\leq\infty}$ defined by

$$\mathrm{w}(f) = \frac{\{\Delta, f\}_1}{12\Delta}.$$

A derivation δ on $\mathcal{M}_*^{\leq\infty}$ is complex-like if $\delta\Big(\mathcal{M}_k^{\leq s}\Big) \subset \mathcal{M}_{k+2}^{\leq s+1}$ for any k and s. The set of complex-like derivations δ such that $kf\delta(g) - \ell g\delta(f) = 0$ for any $f \in \mathcal{M}_k^{\leq s}$ and $g \in \mathcal{M}_\ell^{\leq t}$, for any k, ℓ, s, t, is a one dimensional vector space over $\mathbb{C}$. Let π be a generator. The following Theorem provides a first family of formal deformations of the algebra $\mathcal{M}_*^{\leq\infty}$.

***Theorem* A**– *For any $a \in \mathbb{C}$, let d_a be the derivation on $\mathcal{M}_*^{\leq\infty}$ defined by $\mathrm{d}_a = a\pi + \mathrm{w}_{\{,\}}$.*

i) For all quasimodular forms f and g of respective weights k and ℓ, we have

$$\{f, g\}_1 = kf\,\mathrm{d}_a(f) - \ell g\,\mathrm{d}_a(f).$$

ii) More generally, for any $a \in \mathbb{C}$, the brackets defined for any integer $n \geq 0$ by

$$[f, g]_{\mathrm{d}_a, n} = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r}\binom{\ell+n-1}{r} \mathrm{d}_a^r(f)\,\mathrm{d}_a^{n-r}(g), \quad (f \in \mathcal{M}_k^{\leq\infty}, g \in \mathcal{M}_\ell^{\leq\infty})$$

satisfy

$$\left[\mathcal{M}_k^{\leq\infty},\mathcal{M}_\ell^{\leq\infty}\right]_{\mathrm{d}_a,n}\subset\mathcal{M}_{k+\ell+2n}^{\leq\infty}$$

and define a formal deformation of $\mathcal{M}_*^{\leq\infty}$.

iii) *Moreover,* $\left[\mathcal{M}_k^{\leq s},\mathcal{M}_\ell^{\leq t}\right]_{\mathrm{d}_a,n}\subset\mathcal{M}_{k+\ell+2n}^{\leq s+t}$ *for all* n,s,t,k,ℓ *if and only if* $a=0$.

Remark- A generator is π defined by linear extension of $\pi(f)=kf\,\mathrm{E}_2$ for f any quasimodular form of weight k. For this choice, the derivation d_a is defined on the generators by

$$\mathrm{d}_a\mathrm{E}_2=2a\mathrm{E}_2^2-\frac{1}{12}\mathrm{E}_4,\qquad \mathrm{d}_a\mathrm{E}_4=4a\mathrm{E}_4\mathrm{E}_2-\frac{1}{3}\mathrm{E}_6,\qquad \mathrm{d}_a\mathrm{E}_6=6a\mathrm{E}_6\mathrm{E}_2-\frac{1}{2}\mathrm{E}_4^2.$$

We construct a second family of Poisson brackets.

***Proposition* B** (Second family of Poisson brackets)– *For any* $\alpha\in\mathbb{C}$, *there exists an admissible Poisson bracket* $(\ ,\ )_\alpha$ *on the algebra of quasimodular forms defined by the following values on the generators:*

$$(\mathrm{E}_4,\mathrm{E}_6)_\alpha=-2\Delta,\qquad (\mathrm{E}_2,\mathrm{E}_4)_\alpha=\alpha\mathrm{E}_6\mathrm{E}_2,\qquad (\mathrm{E}_2,\mathrm{E}_6)_\alpha=\frac{3}{2}\alpha\mathrm{E}_4^2\mathrm{E}_2.$$

Moreover,

i) *for any* $\alpha\in\mathbb{C}\setminus\{4\}$, *the Poisson bracket* $(\ ,\ )_\alpha$ *is not unimodular. For* $\alpha=4$, *the Poisson bracket* $(\ ,\ )_4$ *is Jacobian (hence unimodular) of potential* $k_0=-2\Delta\mathrm{E}_2$.

ii) *The Poisson algebras* $\left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_\alpha\right)$ *and* $\left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_{\alpha'}\right)$ *are Poisson modular isomorphic if and only if* $\alpha=\alpha'$.

iii) *For any* $\alpha\in\mathbb{C}$,

   a) *if* $\alpha\notin\mathbb{Q}$, *the Poisson center of* $\left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_\alpha\right)$ *is* $\mathbb{C}$

   b) *if* $\alpha=0$, *the Poisson center of* $\left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_\alpha\right)$ *is* $\mathbb{C}[\mathrm{E}_2]$

   c) *if* $\alpha=p/q$ *with* $p\in\mathbb{Z}^*$ *and* $q\in\mathbb{N}^*$, p *and* q *coprimes, the Poisson center of* $\left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_\alpha\right)$ *is*

$$\begin{cases}\mathbb{C} & \text{if } p<0\\ \mathbb{C}\left[\Delta^p\mathrm{E}_2^{4q}\right] & \text{if } p\geq 1 \text{ is odd}\\ \mathbb{C}\left[\Delta^u\mathrm{E}_2^{2q}\right] & \text{if } p=2u \text{ for odd } u\geq 1\\ \mathbb{C}\left[\Delta^v\mathrm{E}_2^{q}\right] & \text{if } p=4v \text{ with } v\geq 1.\end{cases}$$

Remark- The bracket $(\ ,\ )_0$ is the trivial bracket.

This second family provides a new set of formal deformations of the algebra of quasimodular forms. Following [Zag94, Eq. (38)], we consider the derivation $\mathrm{w}_{(\alpha)}$ defined on $\mathcal{M}_*^{\leq\infty}$ by

$$\mathrm{w}_{(\alpha)}(f)=\frac{(\Delta,f)_\alpha}{12\Delta}.$$

Let us define $\mathcal{K}\colon\mathcal{M}_*^{\leq\infty}\to\mathbb{C}$ by setting $\mathcal{K}(f)=k-(3\alpha+2)s$ if f has weight k and depth s. The set of complex-like derivations δ such that $\mathcal{K}(f)f\delta(g)-\mathcal{K}(g)g\delta(f)=0$ for any

$f \in \mathcal{M}_k^{\leq s}$ and $g \in \mathcal{M}_\ell^{\leq t}$, for any k, ℓ, s, t, is a one dimensional vector space over $\mathbb{C}$. Let π_α be a generator. We note

$$\mathcal{M}_k^s = \mathcal{M}_k \mathrm{E}_2^s.$$

***Theorem* B–** *Let $\alpha \in \mathbb{C}$. For any $b \in \mathbb{C}$, let $\delta_{\alpha,b}$ be the derivation on $\mathcal{M}_*^{\leq\infty}$ defined by $\delta_{\alpha,b} = b\pi_\alpha + \mathrm{w}_{(\alpha)}$.*

i) For all $f \in \mathcal{M}_k^s$ and $g \in \mathcal{M}_\ell^t$, we have

$$(f,g)_\alpha = (k-(3\alpha+2)s) f \delta_{\alpha,b}(g) - (\ell-(3\alpha+2)t) g \delta_{\alpha,b}(f)$$

for any $f \in \mathcal{M}_k^s$ and $g \in \mathcal{M}_\ell^t$.

ii) Moreover, the brackets defined for any integer $n \geq 0$ by

$$[f,g]^{\mathcal{K}}_{\delta_{\alpha,b},n} = \sum_{r=0}^{n} (-1)^r \binom{k-(3\alpha+2)s+n-1}{n-r} \binom{\ell-(3\alpha+2)t+n-1}{r} \delta_{\alpha,b}^r(f) \delta_{\alpha,b}^{n-r}(g),$$

for any $f \in \mathcal{M}_k^s, g \in \mathcal{M}_\ell^t$ define a formal deformation of $\mathcal{M}_*^{\leq\infty}$ satisfying

$$\left[\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}\right]^{\mathcal{K}}_{\delta_{\alpha,b},n} \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}$$

for all k, ℓ in $2\mathbb{N}$ and s, t in $\mathbb{N}$ if and only if $b = 0$.

Remark- A generator π_α is defined by linear extension of:

$$\pi_\alpha(f) = [k-(3\alpha+2)s] f \mathrm{E}_2 \qquad (f \in \mathcal{M}_k^s).$$

For this choice, the derivation $\delta_{\alpha,b}$ is defined on the generators by

$$\delta_{\alpha,b}\mathrm{E}_2 = -3b\alpha \mathrm{E}_2^2, \qquad \delta_{\alpha,b}\mathrm{E}_4 = 4b\mathrm{E}_4\mathrm{E}_2 - \frac{1}{3}\mathrm{E}_6, \qquad \delta_{\alpha,b}\mathrm{E}_6 = 6b\mathrm{E}_6\mathrm{E}_2 - \frac{1}{2}\mathrm{E}_4^2.$$

To complete the classification of Poisson structures, we introduce a third family of Poisson brackets. We note however that, when $\mu \neq 0$, this third family does not lead to a formal deformation of $\mathcal{M}_*^{\leq\infty}$ with the shape of Rankin-Cohen brackets.

***Proposition* C** (Third family of Poisson brackets)– *For any $\mu \in \mathbb{C}$, there exists an admissible Poisson brackets $\langle\ ,\ \rangle_\mu$ on the algebra of quasimodular forms defined by the following values on the generators:*

$$\langle \mathrm{E}_4, \mathrm{E}_6\rangle_\mu = -2\Delta, \quad \langle \mathrm{E}_2, \mathrm{E}_4\rangle_\mu = 4\mathrm{E}_6\mathrm{E}_2 + \mu\mathrm{E}_4^2, \quad \langle \mathrm{E}_2, \mathrm{E}_6\rangle_\mu = 6\mathrm{E}_4^2\mathrm{E}_2 - 2\mu\mathrm{E}_4\mathrm{E}_6.$$

Moreover,

i) this Poisson bracket is Jacobian with potential

$$k_\mu = -2\Delta\mathrm{E}_2 + \mu\mathrm{E}_4^2\mathrm{E}_6.$$

ii) The Poisson algebras $\left(\mathcal{M}_*^{\leq\infty}, \langle\ ,\ \rangle_\mu\right)$ and $\left(\mathcal{M}_*^{\leq\infty}, \langle\ ,\ \rangle_{\mu'}\right)$ are Poisson modular isomorphic for all μ and μ' in $\mathbb{C}^*$.

iii) For any $\mu \in \mathbb{C}$, the Poisson center of $\left(\mathcal{M}_*^{\leq\infty}, \langle\ ,\ \rangle_\mu\right)$ is the polynomial algebra $\mathbb{C}[k_\mu]$.

Remark- We note that $\langle\ ,\ \rangle_0 = (\ ,\ )_4$.

Finally, the following result implies that our classification is complete.

***Theorem* C–** *Up to Poisson modular isomorphism, the only distinct admissible Poisson brackets on the algebra of quasimodular forms are* $\{\ ,\ \}_1$, $\langle\ ,\ \rangle_1$ *and the family* $(\ ,\ )_\alpha$ *for any* $\alpha \in \mathbb{C}$.

Remark- We could endow the algebra of modular forms with another Poisson structure b. If we require $\mathrm{b}(\mathcal{M}_k, \mathcal{M}_\ell) \subset \mathcal{M}_{k+\ell+2}$, then b is necessarily defined by $\mathrm{b}(\mathrm{E}_4, \mathrm{E}_6) = \alpha \mathrm{E}_4^3 + \beta \mathrm{E}_6^2$ for some complex numbers α and β. If $\alpha\beta \neq 0$ then $(\mathcal{M}_*, \mathrm{b})$ is Poisson isomorphic to $(\mathcal{M}_*, \mathrm{RC}_1)$ and is indeed studied by this work. If $\alpha\beta = 0$, the Poisson algebras are no more Poisson isomorphic (they do not have the same group of automorphisms). This degenerated case deserves another study.

2. Number theoretic and algebraic background

2.1. **Quasimodular forms.** The aim of this section is to provide the necessary background on quasimodular forms. For details, the reader is advised to report to [Zag08] or [MR05]. On $\mathrm{SL}(2,\mathbb{Z})$, a modular form of weight $k \in 2\mathbb{N}$, $k \neq 2$, is a holomorphic function on the Poincaré upper half plane $\mathcal{H} = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ satisfying

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and having Fourier expansion

$$f(z) = \sum_{n \geq 0} \widehat{f}(n) \mathrm{e}^{2\pi \mathrm{i} n z}.$$

We denote by $\mathcal{M}_k$ the finite dimensional space of modular forms of weight k. The algebra of modular forms is defined as the graded algebra

$$\mathcal{M}_* = \bigoplus_{\substack{k \in 2\mathbb{N} \\ k \neq 2}} \mathcal{M}_k.$$

Let $k \geq 2$ be even. We define the Eisenstein series of weight k by

$$\mathrm{E}_k(z) = 1 - \frac{2k}{\mathrm{B}_k} \sum_{n=1}^{+\infty} \sigma_{k-1}(n) \mathrm{e}^{2\pi \mathrm{i} n z}.$$

Here the rational numbers B_k are defined by their exponential generating series

$$\sum_{n=0}^{+\infty} \mathrm{B}_n \frac{t^n}{n!} = \frac{t}{\mathrm{e}^t - 1}$$

and σ_{k-1} is the divisor function defined by

$$\sigma_{k-1}(n) = \sum_{\substack{d \mid n \\ d > 0}} d^{k-1} \qquad (n \in \mathbb{N}^*).$$

If $k \geq 4$, the Eisenstein series E_k is a modular form of weight k and $\mathcal{M}_*$ is the polynomial algebra in the two algebraically independent Eisentsein series E_4 and E_6. In other

words,

$$\mathcal{M}_* = \mathbb{C}[\mathrm{E}_4, \mathrm{E}_6], \quad \mathcal{M}_k = \bigoplus_{\substack{(i,j)\in\mathbb{N}^2 \\ 4i+6j=k}} \mathbb{C}\,\mathrm{E}_4^i\,\mathrm{E}_6^j.$$

However, the Eisenstein series E_2 is not a modular form. It satisfies

$$(cz+d)^{-2}\,\mathrm{E}_2\left(\frac{az+b}{cz+d}\right) = \mathrm{E}_2(z) + \frac{6}{\pi\mathrm{i}}\frac{c}{cz+d} \qquad (z\in\mathcal{H})$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. Moreover, the algebra of modular forms is not stable by the normalised complex derivation

$$\mathrm{D} = \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}.$$

For example, we have the following Ramanujan's differential equations

$$\mathrm{D}\,\mathrm{E}_2 = \frac{1}{12}\left(\mathrm{E}_2^2 - \mathrm{E}_4\right)$$

$$\mathrm{D}\,\mathrm{E}_4 = \frac{1}{3}\left(\mathrm{E}_4\,\mathrm{E}_2 - \mathrm{E}_6\right)$$

$$\mathrm{D}\,\mathrm{E}_6 = \frac{1}{2}\left(\mathrm{E}_6\,\mathrm{E}_2 - \mathrm{E}_4^2\right).$$

To account these observations, and using the fact that E_2, E_4 and E_6 are algebraically independent, we introduce the algebra $\mathcal{M}_*^{\leq\infty}$ of quasimodular forms defined as the polynomial algebra

$$\mathcal{M}_*^{\leq\infty} = \mathbb{C}[\mathrm{E}_2, \mathrm{E}_4, \mathrm{E}_6] = \mathcal{M}_*[\mathrm{E}_2].$$

More intrinsically, if for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ we define

$$\mathrm{X}(\gamma) = z \mapsto \frac{c}{cz+d}$$

and

$$f|_k\gamma = z \mapsto (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

then a quasimodular form of weight $k \in 2\mathbb{N}$ and depth $s \in \mathbb{N}$ is a holomorphic function f on $\mathcal{H}$ such that there exist holomorphic functions $f_0, \ldots, f_s$ ($f_s \neq 0$) satisfying

$$f|_k\gamma = \sum_{j=0}^{s} f_j\,\mathrm{X}(\gamma)^j$$

for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Moreover, it is required that any f_j has a Fourier expansion

$$f_j(z) = \sum_{n\geq 0} \widehat{f_j}(n)\mathrm{e}^{2\pi\mathrm{i}nz} \qquad (z\in\mathcal{H}).$$

The zero function is supposed to have arbitrary weight and depth 0. We write $\mathcal{M}_k^{\leq\infty}$ for the space of quasimodular forms of weight k and $\mathcal{M}_k^{\leq s}$ for the space of quasimodular forms of weight k and depth less than or equal to s. We have $\mathcal{M}_k^{\leq 0} = \mathcal{M}_k$ and

$$\mathcal{M}_k^{\leq s} = \bigoplus_{j=0}^{s} \mathcal{M}_{k-2j}\,\mathrm{E}_2^j, \qquad \mathcal{M}_*^{\leq\infty} = \bigoplus_{k\in 2\mathbb{N}} \mathcal{M}_k^{\leq k/2}.$$

Moreover $\mathrm{D}\,\mathcal{M}_k^{\leq s} \subset \mathcal{M}_{k+2}^{\leq s+1}$. Since the depth of a quasimodular form is nothing but its degree as a polynomial in E_2 with modular coefficients we note that

$$\mathcal{M}_k^{\leq s} = \bigoplus_{j=0}^{s} \mathcal{M}_k^t, \qquad \mathcal{M}_*^{\leq\infty} = \bigoplus_{k\in 2\mathbb{N}} \bigoplus_{t=0}^{k/2} \mathcal{M}_k^t$$

where

$$\mathcal{M}_k^t = \mathcal{M}_{k-2t}\,\mathrm{E}_2^t = \bigoplus_{\substack{(i,j)\in\mathbb{N}^2\\ 4i+6j=k-2t}} \mathbb{C}\,\mathrm{E}_4^i\,\mathrm{E}_6^j\,\mathrm{E}_2^t.$$

An important element in our study will be the discriminant function $\Delta = \mathrm{E}_4^3 - \mathrm{E}_6^2$. We note that $\mathrm{D}\,\Delta = \Delta\,\mathrm{E}_2$.

Let n be a non negative integer, f a modular form of weight k and g a modular form of weight ℓ. The n-th Rankin-Cohen bracket of f and g is

$$\mathrm{RC}_n(f,g) = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r}\binom{\ell+n-1}{r} \mathrm{D}^r f\,\mathrm{D}^{n-r} g.$$

This is a modular form of weight $k+\ell+2n$. If f and g are quasimodular forms of respective weights k and ℓ and respective depths s and t, their n-th Rankin-Cohen bracket is defined in [MR09] by

$$\mathrm{RC}_n(f,g) = \sum_{r=0}^{n} (-1)^r \binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r} \mathrm{D}^r f\,\mathrm{D}^{n-r} g. \tag{1}$$

This is a quasimodular form of weight $k+\ell+2n$ and minimal depth (that is $s+t$).

2.2. **Poisson algebra.** The aim of this section is to give a brief account on what is needed about Poisson algebra. For more details, the reader is advised to refer to [LGPV13]. A commutative $\mathbb{C}$-algebra A is a Poisson algebra if there exists a bilinear skew-symmetric map $\mathrm{b}\colon \mathrm{A}\times\mathrm{A}\to\mathrm{A}$ satisfying the two conditions:

- Leibniz rule $\mathrm{b}(fg,h) = f\,\mathrm{b}(g,h) + \mathrm{b}(f,h)g$
- Jacobi identity $\mathrm{b}(f,\mathrm{b}(g,h)) + \mathrm{b}(g,\mathrm{b}(h,f)) + \mathrm{b}(h,\mathrm{b}(f,g)) = 0$

for all f, g and h in A. The bilinear map b is given the name of Poisson bracket. If A is a finitely generated algebra with generators $x_1,\dots,x_\mathrm{N}$, a Poisson bracket b is entirely determined by its values $\mathrm{b}(x_i,x_j)$ for $i<j$ where A is generated by $x_1,\dots,x_\mathrm{N}$. More precisely, we have

$$\mathrm{b}(f,g) = \sum_{0\leq i<j\leq \mathrm{N}} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\mathrm{b}(x_i,x_j) \tag{2}$$

for f and g expressed as polynomials in $x_1, \dots, x_N$.

If $A = \mathbb{C}[x,y]$, any $p \in A$ determines a Poisson bracket satisfying $b(x,y) = p$. However, if $A = \mathbb{C}[x,y,z]$, for any p, q and r in A, there exists a Poisson bracket on A defined by

$$b(x,y) = r, \quad b(y,z) = p, \quad \text{and} \quad b(z,x) = q$$

if and only if

$$\operatorname{curl}(p,q,r) \cdot (p,q,r) = 0 \tag{3}$$

where

$$\operatorname{curl}(p,q,r) = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}, \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}, \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right).$$

If condition (3) is satisfied, then (p,q,r) is called a Poissonian triple. A particular case is obtained if there exists $k \in \mathbb{C}[x,y,z]$ such that $\operatorname{curl}(p,q,r) = (p,q,r) \wedge \operatorname{grad} k$. The bracket b is said then to be unimodular. Among unimodular brackets are the Jacobian brackets. A bracket b is Jacobian if $(p,q,r) = \operatorname{grad} k$ for some polynomial k. The bracket b satisfies then

$$b(f,g) = \operatorname{jac}(f,g,k) \qquad (f,g \in A)$$

In this case, $\mathbb{C}[x,y,z]$ is said to have a Jacobian Poisson structure (JPS) of potential (or Casimir function) k. The Poissonian triple (p,q,r) is said then to be exact.

The Poisson center (or zeroth Poisson cohomology group) of a Poisson algebra A is the Poisson subalgebra

$$\mathrm{HP}^0(A) = \{g \in A \colon\ b(f,g) = 0, \forall f \in A\}.$$

The Poisson center contains the Poisson centraliser of any element in the algebra: let $f \in A$, its Poisson centraliser is $\{g \in A \colon\ b(f,g) = 0\}$. The following lemma computes the Poisson center of polynomial algebras in three variables equipped with a Jacobian Poisson structure. It allows to recover for example [Pic06, Poposition 4.2] in the particular case where the potential is a weight homogeneous polynomial with an isolated singularity. A polynomial $h \in \mathbb{C}[x,y,z]$ is indecomposable if there is no polynomial $p \in \mathbb{C}[x]$ with $\deg p \geq 2$ such that $h = p \circ \ell$ for some $\ell \in \mathbb{C}[x,y,z]$.

***Lemma* 1–** *Let $\mathbb{C}[x,y,z]$ be endowed with a Jacobian Poisson structure of non constant potential k. Its Poisson center is $\mathbb{C}[k]$ if and only if k is indecomposable.*

Proof. Assume that k is not indecomposable: $k = p \circ \ell$ with $p \in \mathbb{C}[x]$, $\deg p = 2$. Then $\operatorname{jac}(\ell,g,k) = (p' \circ \ell)\operatorname{jac}(\ell,g,\ell)$ hence ℓ is in the Poisson center however not in $\mathbb{C}[k]$. Assume conversely that k is indecomposable. Let f be in the Poisson center, then the rank of the Jacobian matrix of (f,g,k) is at most 2 for any g. If it is 1 for any g then $\operatorname{grad} f$ and $\operatorname{grad} k$ are zero, it contradicts the fact that k is not constant. Hence, for some g, the rank is 2. It follows (see e.g. [GS06, Theorem 6]) that there exists $q \in \mathbb{C}[x,y,z]$, $F \in \mathbb{C}[x]$ and $K \in \mathbb{C}[x]$ such that $f = F \circ q$ and $k = K \circ q$. Since k is indecomposable and non constant, we have $\deg K = 1$ hence q, and f, are polynomials if h. □

If A and B are two Poisson algebras, with respective Poisson brackets b_A and b_B, a map $\varphi \colon A \to B$ is a morphism of Poisson algebras when it is a morphism of algebras that satisfies

$$\varphi(b_A(f,g)) = b_B(\varphi(f), \varphi(g))$$

for any f and g in A. Two Poisson-isomorphic Poisson algebras have isomorphic Poisson centres.

We detail now a canonical way to extend a Poisson structure from an algebra A to a polynomial algebra $A[x]$. This construction is due to Sei-Qwon Oh [Oh06]. A Poisson derivation of A is a derivation σ of A satisfying

$$\sigma(b(f,g)) = b(\sigma(f),g) + b(f,\sigma(g))$$

for all f and g in A. If σ is a Poisson derivation of A, a Poisson σ-derivation is a derivation δ of A such that

$$\delta(b(f,g)) = b(\delta(f),g) + b(f,\delta(g)) + \sigma(f)\delta(g) - \delta(f)\sigma(g)$$

for all f and g in A.

***Theorem* 2** (Oh)– *Let* (A, b_A) *be a Poisson algebra. Let* σ *and* δ *be linear maps on* A. *The polynomial ring* $A[x]$ *becomes a Poisson algebra with Poisson brackets* b *defined by*

$$\begin{aligned} b(f,g) &= b_A(f,g) \\ b(x,f) &= \sigma(f)x + \delta(f) \end{aligned}$$

for all f *and* g *in* A *if and only if* σ *is a Poisson derivation and* δ *is a Poisson* σ*-derivation. In this case, the Poisson algebra* $A[x]$ *is said to be a Poisson-Ore extension of* A. *It is denoted* $A[x]_{\sigma,\delta}$.

We describe also a general process to obtain Poisson brackets from a pair of derivations. A pair (δ, d) of two derivations of A is solvable if there exists some scalar α such that $\delta \circ d - d \circ \delta = \alpha d$. In particular, a solvable pair (δ, d) is abelian when $\alpha = 0$.

***Proposition* 3**– *Let* A *be a commutative algebra,* d *and* δ *two derivations of* A. *Let* $b\colon A \times A \to A$ *be defined by*

$$b(f,g) = \delta(f)d(g) - d(f)\delta(g) \qquad (f,g \in A).$$

Then

i) the map b *is bilinear skew-symmetric and satisfies Leibniz rule.*
ii) If (δ, d) *is solvable, then* b *satisfies Jacobi identity and so becomes a Poisson bracket.*
iii) Under the hypothesis in ii), d *is a Poisson derivation for* b.

Proof of Proposition 3. Point i) is immediate. Point ii) is a consequence of the following computation. If (δ, d) is solvable with $\delta d - d\delta = \alpha d$ and if $B\colon A \otimes A \otimes A \otimes A \to A$ is defined by $B(f,g,h) = b(f, b(g,h))$ then

$$\begin{aligned} B = \alpha&(d \otimes d \otimes \delta - d \otimes \delta \otimes d) \\ &+ (\delta \otimes (d \circ \delta) \otimes d - d \otimes \delta \otimes (d \circ \delta)) \\ &+ (d \otimes (d \circ \delta) \otimes \delta - \delta \otimes d \otimes (d \circ \delta)) \\ &+ (\delta \otimes \delta \otimes d^2 - \delta \otimes d^2 \otimes \delta) \\ &+ (d \otimes d \otimes \delta^2 - d \otimes \delta^2 \otimes d). \end{aligned}$$

Point iii) is obtained by direct computation. □

A direct consequence of this Proposition is the following Corollary. If $\mathrm{A} = \bigoplus_{n\geq 0} \mathrm{A}_n$ is a commutative graded algebra, a map $\kappa\colon \mathrm{A} \to \mathbb{C}$ is graded-additive if, for any k and ℓ, for any $f \in \mathrm{A}_k$ and any $g \in \mathrm{A}_\ell$, we have $\kappa(fg) = \kappa(f) + \kappa(g)$.

***Corollary* 4–** *Let $\mathrm{A} = \bigoplus_{n\geq 0} \mathrm{A}_n$ be a commutative graded algebra. Let $\kappa\colon \mathrm{A} \to \mathbb{C}$ be a graded-additive map. Let d be a homogeneous derivation of A (there exists $e \geq 0$ such that $\mathrm{d}\,\mathrm{A}_n \subset \mathrm{A}_{n+e}$ for any $e \geq 0$). Then, the bracket defined on A by the bilinear extension of*

$$\mathrm{b}(f,g) = \kappa(f) f\,\mathrm{d}(g) - \kappa(g) g\,\mathrm{d}(f) \qquad (f \in \mathrm{A}_k,\, g \in \mathrm{A}_\ell)$$

is a Poisson bracket for which d is a Poisson derivation.

We turn on formal deformations of a commutative $\mathbb{C}$-algebra A. Assume we have a family $\boldsymbol{\mu} = (\mu_i)_{i\in\mathbb{N}}$ of bilinear maps $\mu_i\colon \mathrm{A}\times\mathrm{A} \to \mathrm{A}$ such that μ_0 is the product. Let $\mathrm{A}[[\hbar]]$ be the commutative algebra of formal power series in one variable $\hbar$ with coefficients in A. The family $\boldsymbol{\mu}$ is a formal deformation of A if the non commutative product on $\mathrm{A}[[\hbar]]$ defined by extension of

$$f * g = \sum_{j\geq 0} \mu_j(f,g)\hbar^j \qquad (f,g \in \mathrm{A})$$

is associative. This condition is equivalent to

$$\sum_{r=0}^{n} \mu_{n-r}(\mu_r(f,g),h) = \sum_{r=0}^{n} \mu_{n-r}(f,\mu_r(g,h)) \qquad \text{for all } f,g,h \in \mathrm{A} \tag{4}$$

for all $n \geq 0$. In this case, the product $*$ is called a star product. If $\boldsymbol{\mu}$ is a formal deformation anf if moreover μ_1 is skew-symmetric and μ_2 is symmetric, then (A,μ_1) is a Poisson algebra.

2.3. **Problematic of this work.** The first Rankin-Cohen bracket

$$\mathrm{RC}_1(f,g) = kf\,\mathrm{D}(g) - \mathrm{D}(f)\ell g \qquad (f \in \mathcal{M}_k,\, g \in \mathcal{M}_\ell)$$

gives $\mathcal{M}_*$ a structure of Poisson algebra. This is a consequence of Corollary 4. Cohen, Manin & Zagier [CMZ97] and Yao [Yao07] or [RTY11] proved that the family of Rankin-Cohen brackets is a formal deformation of $\mathcal{M}_*$. In this case, the star product is called the Eholzer product. This subject has been widely studied then. See for example [OS00, Pev08].

Can we construct formal deformations of $\mathcal{M}_*^{\leq\infty}$? In other words, can we construct suitable families $(\mu_n)_{n\in\mathbb{N}}$ of bilinear maps on $\mathcal{M}_*^{\leq\infty}$ that increase the weight by $2n$, preserve the depth and define an analogue of Eholzer product? The brackets defined in (1) do not lead to a solution since RC_1 does not even provide $\mathcal{M}_*^{\leq\infty}$ with a Poisson structure. Our first step is to obtain admissible Poisson brackets on $\mathcal{M}_*^{\leq\infty}$ with the following definition.

***Definition* 1–** *A Poisson bracket b on $\mathcal{M}_*^{\leq\infty}$ is admissible if*

1) $\mathrm{b}(f,g) = \mathrm{RC}_1(f,g)$ if f and g are in $\mathcal{M}_*$
2) there exists $e \geq 0$ such that $\mathrm{b}\left(\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}\right) \subset \mathcal{M}_{k+\ell+e}^{\leq s+t}$ for all k,ℓ,s,t.

Equivalently, a Poisson bracket b on $\mathcal{M}_*^{\leq\infty}$ is admissible if and only if

$$\begin{aligned} &\mathrm{b}(\mathrm{E}_4,\mathrm{E}_6) = -2\Delta \\ &\mathrm{b}(\mathrm{E}_2,\mathrm{E}_4) \in \mathcal{M}_8^{\leq\infty}, \quad \mathrm{b}(\mathrm{E}_2,\mathrm{E}_6) \in \mathcal{M}_{10}^{\leq\infty} \\ &\mathrm{b}(\mathrm{E}_2,\mathcal{M}_*) \subset \mathcal{M}_*\mathrm{E}_2 + \mathcal{M}_*. \end{aligned}$$

In order to classify the admissible Poisson brackets, we introduce the notion of Poisson modular isomorphism.

***Definition* 2–** *A Poisson isomorphism* $\varphi\colon (\mathcal{M}_*^{\leq\infty},\mathrm{b}_1) \to (\mathcal{M}_*^{\leq\infty},\mathrm{b}_2)$ *is called a Poisson modular isomorphism if* $\varphi(\mathcal{M}_*) \subset \mathcal{M}_*$.

Indeed, if φ is a Poisson modular isomorphism then, its restriction to the subalgebra $\mathcal{M}_*$ is the identity. This is a consequence of the following Proposition.

***Proposition* 5–** *The group of Poisson automorphisms of the Poisson algebra* $(\mathcal{M}_*,\mathrm{RC}_1)$ *is trivial.*

Proof. Let φ be a Poisson automorphism of $\mathcal{M}_*$. There exist two polynomials s and t in $\mathbb{C}[x,y]$ such that $\varphi(\mathrm{E}_4) = s(\mathrm{E}_4,\mathrm{E}_6)$ and $\varphi(\mathrm{E}_6) = t(\mathrm{E}_4,\mathrm{E}_6)$. By (2) we have

$$\mathrm{RC}_1(\varphi(\mathrm{E}_4),\varphi(\mathrm{E}_6)) = \mathrm{jac}(s,t)(\mathrm{E}_4,\mathrm{E}_6)\cdot \mathrm{RC}_1(\mathrm{E}_4,\mathrm{E}_6).$$

Since φ is an automorphism, $\mathrm{jac}(s,t)$ is a non zero scalar, say λ. We get

$$\varphi(\mathrm{RC}_1(\mathrm{E}_4,\mathrm{E}_6)) = \lambda\,\mathrm{RC}_1(\mathrm{E}_4,\mathrm{E}_6) \quad \text{hence} \quad s^3 - t^2 = \lambda(x^3 - y^2) \quad \text{in } \mathbb{C}[x,y]. \tag{5}$$

We develop s and t into homogeneous components with respect to the weight:

$$s = \sum_{\substack{i=0\\ i\neq 1}}^{m} s_{2i} \quad \text{and} \quad t = \sum_{\substack{i=0\\ i\neq 1}}^{n} t_{2i}$$

where

$$s_{2i} = \sum_{\substack{(a,b)\in\mathbb{N}^2\\ 2a+3b=i}} \sigma_{a,b}x^a y^b \quad \text{and} \quad t_{2i} = \sum_{\substack{(a,b)\in\mathbb{N}^2\\ 2a+3b=i}} \tau_{a,b}x^a y^b \qquad (\sigma_{a,b},\,\tau_{a,b}\in\mathbb{C})$$

for all i (where $m = 0$ or $m \geq 2$ and $n = 0$ or $n \geq 2$). Equation (5) implies that $t(\mathrm{E}_4,\mathrm{E}_6)^2 - s(\mathrm{E}_4,\mathrm{E}_6)^3$ has weight 12. Then only three cases are possible.

1) If $3m > 2n$ then $m = 2$ and so $n \in \{0,2\}$. It implies that $s = \sigma_{00} + \sigma_{10}x$ and $t = \tau_{00} + \tau_{10}x$. This contradicts $\mathrm{jac}(s,t) \neq 0$.
2) If $3m < 2n$ then $n = 3$ and $m = 0$. This contradicts $\mathrm{jac}(s,t) \neq 0$.
3) If $3m = 2n$. We differentiate (5) with respect to x and y and get

$$3s^2\frac{\partial s}{\partial x} - 2t\frac{\partial t}{\partial x} = 3\lambda x^2, \qquad 3s^2\frac{\partial s}{\partial y} - 2t\frac{\partial t}{\partial y} = -2\lambda y.$$

This implies

$$2t = 3x^2\frac{\partial s}{\partial y} + 2y\frac{\partial s}{\partial x}, \qquad 3s^2 = 3x^2\frac{\partial t}{\partial y} + 2y\frac{\partial t}{\partial x}. \tag{6}$$

From the first differential equation of (6) we have

$$2t(\mathrm{E}_4,\mathrm{E}_6) = 3\mathrm{E}_4^2\frac{\partial s}{\partial y}(\mathrm{E}_4,\mathrm{E}_6) + 2\mathrm{E}_6\frac{\partial s}{\partial x}(\mathrm{E}_4,\mathrm{E}_6).$$

The highest weight of the right hand side is less than or equal to $2m+2$. This implies $n \leq m+1$. From the second differential equation of (6) we have

$$3s^2(\mathrm{E}_4,\mathrm{E}_6) = 3\mathrm{E}_4^2\frac{\partial t}{\partial y}(\mathrm{E}_4,\mathrm{E}_6) + 2\mathrm{E}_6\frac{\partial t}{\partial x}(\mathrm{E}_4,\mathrm{E}_6)$$

hence $2m \leq n+1$. We deduce $(m,n) \in \{(0,0),(2,3)\}$. Since $n = m = 0$ would imply $\mathrm{jac}(s,t) = 0$ we have $n = 3$ and $m = 2$. Then $s = \sigma_{00} + \sigma_{10}x$ and $t = \tau_{00} + \tau_{10}x + \tau_{01}y$. The first differential equation in (6) implies that $\tau_{00} = \tau_{10} = 0$ and $\tau_{01} = \sigma_{10}$ whereas the second one implies that $\sigma_{00} = 0$ and $\sigma_{10} = 1$. Finally $\varphi(\mathrm{E}_4) = s(\mathrm{E}_4,\mathrm{E}_6) = \mathrm{E}_4$ and $\varphi(\mathrm{E}_6) = s(\mathrm{E}_4,\mathrm{E}_6) = \mathrm{E}_6$. □

Since $\mathrm{RC}_1(\Delta,f) = (12\,\mathrm{D}(f) - kf\,\mathrm{E}_2)\Delta$ for any $f \in \mathcal{M}_k$, the first Rankin-Cohen bracket defines a derivation on $\mathcal{M}_*$ called the Serre's derivative by linear extension of:

$$\vartheta f = \frac{\mathrm{RC}_1(\Delta,f)}{12\Delta} = \mathrm{D}(f) - \frac{k}{12}f\,\mathrm{E}_2 \qquad (f \in \mathcal{M}_k). \tag{7}$$

This derivation is characterised by its values on the generators

$$\vartheta\,\mathrm{E}_4 = -\frac{1}{3}\mathrm{E}_6, \qquad \vartheta\,\mathrm{E}_6 = -\frac{1}{2}\mathrm{E}_4^2.$$

We shall need the following result.

Proposition 6– *The kernel of Serre's derivative is the Poisson centraliser of Δ for the first Rankin-Cohen bracket. This is $\mathbb{C}[\Delta]$.*

Proof. If $f \in \mathcal{M}_k$ is in $\ker\vartheta$ then $kf\,\mathrm{D}(\Delta) = 12\Delta\,\mathrm{D}(f)$. Solving the differential equation we find that 12 divides k and that $f \in \mathbb{C}\Delta^{k/12}$. □

We note that, for any $g \in \mathcal{M}_\ell$, we have

$$\mathrm{RC}_1(\Delta^m,g) = m\Delta^m(12\,\mathrm{D}(g) - \ell g\,\mathrm{E}_2)$$

and deduce that, for any $f \in \mathbb{C}[\Delta]$ and $g \in \mathcal{M}_*$ we have

$$\mathrm{RC}_1(f,g) = 12\xi(f)\vartheta(g) \tag{8}$$

where ξ is the Eulerian derivative on $\mathbb{C}[\Delta]$ defined by $\xi = \Delta\dfrac{\partial}{\partial\Delta}$.

3. Poisson structures on quasimodular forms

3.1. **First family.** This section is devoted to the proof of Proposition A.

We fix $\lambda \in \mathbb{C}^*$ and introduce in $\mathbb{C}[x,y,z]$ the three polynomials

$$r(x,y,z) = \frac{1}{3}(\lambda y^2 - 2xz)$$
$$p(x,y,z) = -2(y^3 - z^2)$$
$$q(x,y,z) = -\frac{1}{2}(\lambda yz - 2xy^2).$$

Since $(p,q,r)\cdot \mathrm{curl}(p,q,r)=0$, we define a Poisson bracket on $\mathcal{M}_*^{\leq\infty}$ if we set

$$\begin{aligned}\{\mathrm{E}_4,\mathrm{E}_6\}_\lambda &= p(\mathrm{E}_2,\mathrm{E}_4,\mathrm{E}_6)\\ \{\mathrm{E}_2,\mathrm{E}_4\}_\lambda &= r(\mathrm{E}_2,\mathrm{E}_4,\mathrm{E}_6)\\ \{\mathrm{E}_6,\mathrm{E}_2\}_\lambda &= q(\mathrm{E}_2,\mathrm{E}_4,\mathrm{E}_6).\end{aligned}$$

Let us prove that $\{\ ,\ \}_\lambda$ is not unimodular. If it were, we would have $k\in\mathbb{C}[x,y,z]$ such that $\mathrm{curl}(p,q,r)=(p,q,r)\wedge \mathrm{grad}\, k$. Identifying the first components would lead to

$$\frac{7}{6}\lambda y=\frac{1}{2}(-\lambda yz+2y^2x)\frac{\partial k}{\partial z}-\frac{1}{3}(\lambda y^2-2zx)\frac{\partial k}{\partial y}$$

that has no solution in $\mathbb{C}[x,y,z]$.

A Poisson modular isomorphism φ_λ between $\left(\mathcal{M}_*^{\leq\infty},\{\ ,\ \}_\lambda\right)$ and $\left(\mathcal{M}_*^{\leq\infty},\{\ ,\ \}_1\right)$ is determined by

$$\varphi_\lambda(\mathrm{E}_2)=\lambda\,\mathrm{E}_2,\quad \varphi_\lambda(\mathrm{E}_4)=\mathrm{E}_4,\quad \varphi_\lambda(\mathrm{E}_6)=\mathrm{E}_6\,.$$

Finally, we determine the Poisson center of the Poisson algebra $\left(\mathcal{M}_*^{\leq\infty},\{\ ,\ \}_1\right)$. Let us define a derivation on $\mathcal{M}_*$ by $\sigma=2\vartheta$ (see (7)) an a derivation on $\mathcal{M}_*$ by linear extension of

$$\delta(f)=\frac{k}{12}f\,\mathrm{E}_4\qquad (f\in\mathcal{M}_k).$$

We note that $\left(\mathcal{M}_*^{\leq\infty},\{\ ,\ \}_1\right)$ is the Poisson-Ore extension $\mathbb{C}[\mathrm{E}_4,\mathrm{E}_6][\mathrm{E}_2]_{\sigma,\delta}$ Now, consider any $f\in\mathcal{M}_*^{\leq\infty}$ written as

$$f=\sum_{i=0}^{s}f_i\,\mathrm{E}_2^i,\qquad f_i\in\mathcal{M}_*.$$

We compute

$$\{\mathrm{E}_2,f\}_1=\delta(f_0)+\sum_{i=1}^{s}(\sigma(f_{i-1})+\delta(f_i))\mathrm{E}_2^i+\sigma(f_s)\mathrm{E}_2^{s+1}\,.$$

If $\{\mathrm{E}_2,f\}_1=0$ then $\delta(f_0)=0$ hence $f_0\in\mathbb{C}$ and $\sigma(f_0)=0$. We obtain inductively that $f_i\in\mathbb{C}$ for all $0\leq i\leq s$ so that the Poisson centraliser of E_2 is $\mathbb{C}[\mathrm{E}_2]$. Suppose that the Poisson center contains a non scalar element. Then it is in the Poisson centraliser of E_2 and can be written

$$f=\sum_{j=0}^{p}\alpha_j\,\mathrm{E}_2^j,\qquad p\geq 1,\ \alpha_j\in\mathbb{C},\ \alpha_p\neq 0.$$

We compute

$$\{\mathrm{E}_4,f\}_1=\sum_{j=0}^{p}j\alpha_j\,\mathrm{E}_2^{j-1}\cdot\{\mathrm{E}_4,\mathrm{E}_2\}_1$$

and find that the coefficient of E_2^p is non zero. It follows that f is not in the Poisson center.

3.2. **Second family.** This section is devoted to the proof of Proposition B. We fix $\alpha \in \mathbb{C}$ and introduce in $\mathbb{C}[x,y,z]$ the three polynomials

$$\begin{aligned} r(x,y,z) &= \alpha xz \\ p(x,y,z) &= -2(y^3 - z^2) \\ q(x,y,z) &= -\frac{3}{2}\alpha xy^2. \end{aligned}$$

Since $(p,q,r)\cdot \operatorname{curl}(p,q,r) = 0$, we define a Poisson bracket on $\mathcal{M}_*^{\leq\infty}$ if we set

$$\begin{aligned} (\mathrm{E}_4,\mathrm{E}_6)_\alpha &= p(\mathrm{E}_2,\mathrm{E}_4,\mathrm{E}_6) \\ (\mathrm{E}_2,\mathrm{E}_4)_\alpha &= r(\mathrm{E}_2,\mathrm{E}_4,\mathrm{E}_6) \\ (\mathrm{E}_6,\mathrm{E}_2)_\alpha &= q(\mathrm{E}_2,\mathrm{E}_4,\mathrm{E}_6). \end{aligned}$$

Assume $\alpha \neq 4$. Let us prove that $(\ ,\ )_\alpha$ is not unimodular. If it were, we would have $k \in \mathbb{C}[x,y,z]$ such that $\operatorname{curl}(p,q,r) = (p,q,r) \wedge \operatorname{grad} k$. Identifying the second components would lead to

$$(4-\alpha)z = \alpha xz\frac{\partial k}{\partial x} + 2(y^3 - z^2)\frac{\partial k}{\partial z}$$

that has no solution in $\mathbb{C}[x,y,z]$.

If $\alpha = 4$, then $(p,q,r) = \operatorname{grad} k_0$ where $k_0 = -2(y^3 - z^2)x$. As a consequence, the bracket $(\ ,\ )_4$ provides $\mathcal{M}_*^{\leq\infty}$ with a Jacobian Poisson structure of potential $k_0 = -2\Delta\,\mathrm{E}_2 = -2\,\mathrm{D}(\Delta)$.

If $\varphi\colon \left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_\alpha\right) \to \left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_{\alpha'}\right)$ is a Poisson modular isomorphism, let us prove that $\alpha = \alpha'$. By Proposition 5, we have $\varphi(\mathrm{E}_4) = \mathrm{E}_4$ and $\varphi(\mathrm{E}_6) = \mathrm{E}_6$. By surjectivity, it follows that $\varphi(\mathrm{E}_2) = \eta\,\mathrm{E}_2 + \mathrm{F}$ for some $\eta \in \mathbb{C}^*$ and $\mathrm{F} \in \mathcal{M}_*$. We compute

$$\varphi\left((\mathrm{E}_2,\mathrm{E}_4)_\alpha\right) = \alpha\eta\,\mathrm{E}_6\,\mathrm{E}_2 + \alpha\,\mathrm{E}_6\,\mathrm{F}$$

and

$$(\varphi(\mathrm{E}_2),\varphi(\mathrm{E}_4))_{\alpha'} = \alpha'\eta\,\mathrm{E}_6\,\mathrm{E}_2 + (\mathrm{F},\mathrm{E}_4)_{\alpha'}.$$

Since $(\mathrm{F},\mathrm{E}_4)_{\alpha'} \in \mathcal{M}_*$ we get $\alpha' = \alpha$.

Finally, we determine the Poisson center of the Poisson algebra $\left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_\alpha\right)$. We note that $\left(\mathcal{M}_*^{\leq\infty},(\ ,\ )_\alpha\right)$ is the Poisson-Ore extension $\mathbb{C}[\mathrm{E}_4,\mathrm{E}_6][\mathrm{E}_2]_{\sigma,\delta}$ where $\sigma = -3\alpha\vartheta$ (see (7)) and $\delta = 0$. Let

$$f = \sum_{j=0}^{s} f_j\,\mathrm{E}_2^j, \qquad f_j \in \mathcal{M}_*.$$

We have

$$(\mathrm{E}_2,f)_\alpha = \sum_{j=0}^{s} \sigma(f_j)\,\mathrm{E}_2^{j+1}$$

hence f is in the Poisson centraliser of E_2 if and only each f_j is in the Poisson centraliser of Δ for RC_1. By Proposition 6 we deduce that the centraliser of E_2 is $\mathbb{C}[\Delta,\mathrm{E}_2]$. Let

$$f = \sum_{j=0}^{s} f_j(\Delta)\,\mathrm{E}_2^j \in \mathbb{C}[\Delta,\mathrm{E}_2].$$

We use (8) to compute

$$(f,\mathrm{E}_4)_\alpha = \sum_{j=0}^{s}\Big(-4\xi(f_j)+j\alpha f_j\Big)\mathrm{E}_6\,\mathrm{E}_2^j$$

and

$$(f,\mathrm{E}_6)_\alpha = \frac{3}{2}\sum_{j=0}^{s}\Big(-4\xi(f_j)+j\alpha f_j\Big)\mathrm{E}_4^2\,\mathrm{E}_2^j.$$

We deduce that f is in the Poisson center of $(\ ,\ )_\alpha$ if and only if

$$\xi(f_j) = \frac{j\alpha}{4}f_j$$

for all j, i.e. if and only if any f_j is of the form $f_j = \lambda_j\Delta^{m_j}$ for some $\lambda_j \in \mathbb{C}$ and $m_j \in \mathbb{N}$ such that $j\alpha = 4m_j$. If $\alpha \notin \mathbb{Q}$ or if $\alpha < 0$ then $j = 0$ and $m_j = 0$ hence $f = f_0 \in \mathbb{C}$. If $\alpha = p/q$ with $p \geq 1$, $q \geq 1$ and $(p,q) = 1$ then $\lambda\Delta^{m_j}\mathrm{E}_2^j$ is in the Poisson center if and only if $pj = 4qm_j$. The result follows by obvious arithmetical consideration. Finally, if $\alpha = 0$, then $(\ ,\ )_0$ is the trivial bracket and its Poisson center is $\mathbb{C}[\mathrm{E}_2]$.

3.3. **Third family.** In this section, we study the third family, i.e. we prove Proposition C.

For any $\mu \in \mathbb{C}$, let us introduce

$$k_\mu = -2\Delta\,\mathrm{E}_2 + \mu\,\mathrm{E}_4^2\,\mathrm{E}_6.$$

Then

$$\begin{aligned}
\mathrm{jac}(\mathrm{E}_4,\mathrm{E}_6,k_\mu) &= \frac{\partial\mu_k}{\partial\mathrm{E}_2} = -2\,\mathrm{E}_4^3 + \mathrm{E}_6^2\\
\mathrm{jac}(\mathrm{E}_2,\mathrm{E}_4,k_\mu) &= \frac{\partial\mu_k}{\partial\mathrm{E}_6} = 4\,\mathrm{E}_6\,\mathrm{E}_2 + \mu\,\mathrm{E}_4^3\\
\mathrm{jac}(\mathrm{E}_2,\mathrm{E}_6,k_\mu) &= -\frac{\partial\mu_k}{\partial\mathrm{E}_4} = 6\,\mathrm{E}_4^2\,\mathrm{E}_2 - 2\mu\,\mathrm{E}_4\,\mathrm{E}_6.
\end{aligned}$$

The third family of Poisson bracket is then defined by $\langle f,g\rangle_\mu = \mathrm{jac}(f,g,k_\mu)$. With the notation of Proposition B, we have in particular $\langle f,g\rangle_0 = (f,g)_4$.

For any $\mu \in \mathbb{C}^*$, we define a Poisson modular isomorphism φ_μ between $\left(\mathcal{M}_*^{\leq\infty},\langle\ ,\ \rangle_\mu\right)$ and $\left(\mathcal{M}_*^{\leq\infty},\langle\ ,\ \rangle_1\right)$ by setting $\varphi_\mu(\mathrm{E}_2) = \mu\,\mathrm{E}_2$, $\varphi_\mu(\mathrm{E}_4) = \mathrm{E}_4$ and $\varphi_\mu(\mathrm{E}_6) = \mathrm{E}_6$.

Since the degree in E_2 of k_μ as a polynomial in $\mathrm{E}_2,\mathrm{E}_4,\mathrm{E}_6$ is 1, Lemma 1 implies that the Poisson center of $\left(\mathcal{M}_*^{\leq\infty},\langle\ ,\ \rangle_\mu\right)$ is $\mathbb{C}[k_\mu]$.

3.4. **Classification.** This section is devoted to the proof of Theorem C.

Let $\{\ ,\ \}$ be an admissible bracket on $\mathcal{M}_*^{\leq\infty}$. By Definition 1 and Theorem 2 , there exist a Poisson derivation σ of $\mathcal{M}_*$ and a Poisson σ-derivation δ of $\mathcal{M}_*$ such that

$$\{\mathrm{E}_2,f\} = \sigma(f)\,\mathrm{E}_2 + \delta(f) \qquad (f \in \mathcal{M}_*).$$

By definition, $\sigma(\mathcal{M}_k) \subset \mathcal{M}_{k+2}$ and $\delta(\mathcal{M}_k) \subset \mathcal{M}_{k+4}$ for any k. The admissible bracket $\{\ ,\ \}$ is then defined by the four scalars α, β, γ and ε such that

$$\sigma(E_4) = \alpha E_6,\ \delta(E_4) = \beta E_4^2,\ \sigma(E_6) = \gamma E_4^2, \text{and } \delta(E_6) = \varepsilon E_4 E_6.$$

The condition that σ is a Poisson derivation imposes the condition

$$\{\sigma(E_4), E_6\} + \{E_4, \sigma(E_6)\} = -2\sigma(E_4^3 - E_6^2)$$

or equivalently $3\alpha = 2\gamma$. The condition that δ is a Poisson σ-derivation imposes

$$\delta(\{E_4, E_6\}) = (2\beta + \varepsilon) E_4 \{E_4, E_6\} + \alpha\varepsilon E_4 E_6^2 - \beta\gamma E_4^4$$

or equivalently

$$\begin{cases} 4\beta + (\alpha - 2)\varepsilon = 0 \\ (3\alpha - 4)\beta + 4\varepsilon = 0. \end{cases}$$

Either $\beta = \varepsilon = 0$ is the only solution, or $\alpha \in \{-2/3, 4\}$ and $\varepsilon = \dfrac{4}{2-\alpha}\beta$.

- The case $\beta = \varepsilon = 0$ leads to the second family: $\{\ ,\ \} = (\ ,\ )_\alpha$.
- The case $\alpha = -2/3$ and $\varepsilon = 3\beta/2 \neq 0$ leads to the first family: $\{\ ,\ \} = \{\ ,\ \}_{3\beta}$.
- The case $\alpha = 4$ and $\varepsilon = -2\beta \neq 0$ leads to the third family: $\{\ ,\ \} = \langle\ ,\ \rangle_\beta$.

Using Propositions B and C we conclude that the only admissible Poisson brackets, up to Poisson modular-isomorphisms are $\{\ ,\ \}_1$, $\langle\ ,\ \rangle_1$ and $(\ ,\ )_\alpha$ for any $\alpha \in \mathbb{C}$. Looking at the centres, it is clear that the Poisson algebras $\left(\mathcal{M}_*^{\leq\infty}, \langle\ ,\ \rangle_1\right)$ and $\left(\mathcal{M}_*^{\leq\infty}, \{\ ,\ \}_1\right)$ are not Poisson modular isomorphic. Suppose that there exists a Poisson modular isomorphism φ from $\left(\mathcal{M}_*^{\leq\infty}, (\ ,\ )_\alpha\right)$ to $\left(\mathcal{M}_*^{\leq\infty}, \{\ ,\ \}_1\right)$. We know (see § 3.2) that

$$\varphi(E_4) = E_4, \quad \varphi(E_6) = E_6, \quad \text{and} \quad \varphi(E_2) = \eta E_2 + F$$

for some $\eta \in \mathbb{C}^*$ and $F \in \mathcal{M}_*$. From $\varphi((E_2, E_4)_\alpha) = \{\varphi(E_2), \varphi(E_4)\}_1$ we obtain

$$\alpha\eta E_6 E_2 + \alpha F E_6 = -\frac{2}{3}\eta E_6 E_2 + \frac{1}{3}\eta E_4^2 + \{F, E_4\}_1$$

hence

$$\begin{aligned} \alpha = -\frac{2}{3}, \quad \frac{1}{3}\eta E_4^2 &= -\frac{2}{3} F E_6 - \{F, E_4\}_1 \\ &= -\frac{2}{3} F E_6 + 2(E_4^3 - E_6^2)\frac{\partial F}{\partial E_6} \end{aligned}$$

by (2). We get a contradiction. Replacing $\{\ ,\ \}_1$ by $\langle\ ,\ \rangle_1$ we get

$$\alpha = 4, \quad \eta E_4^2 = 4 F E_6 - 2(E_4^3 - E_6^2)\frac{\partial F}{\partial E_6}$$

and again, we get a contradiction.

4. Star products on quasimodular forms

4.1. **Extension of the first family.** This section is devoted to the proof of Theorem A. We will use the following result of Zagier [Zag94, Example 1]. Let $\mathrm{A} = \bigoplus \mathrm{A}_k$ be a commutative graded algebra with a derivation d homogeneous of degree 2 (i.e. $\mathrm{d}(\mathrm{A}_k) \subset \mathrm{A}_{k+2}$). Let us define, for any $f \in \mathrm{A}_k$, $g \in \mathrm{A}_\ell$, $r \geq 0$:

$$[f,g]_{d,r} = \sum_{i=0}^{r} (-1)^i \binom{k+r-1}{r-i} \binom{\ell+r-1}{i} \mathrm{d}^i(f)\,\mathrm{d}^{r-i}(g) \in \mathrm{A}_{k+\ell+2r}.$$

Then A equipped with this brackets is a Rankin-Cohen algebra, that means that all algebraic identities satisfied by the usual Rankin-Cohen brackets on modular forms are also satisfied, in particular those expressing of the associativity of the corresponding star product. We obtain the following result.

Theorem 7– *The star product defined by*

$$f \# g = \sum_{n \geq 0} [f,g]_{\mathrm{d},n} \hbar^n$$

defines a formal deformation on A.

In particular, we recover the fact, given by Corollary 4, that $[\ ,\ ]_{\mathrm{d},1}$ is a Poisson bracket. Note also that this Theorem can be obtained from Connes & Moscovici's result cited below (see § 4.2).

Let $a \in \mathbb{C}$ and d_a be the homogeneous derivation of degree 2 on $\mathcal{M}_*^{\leq\infty}$ defined by

$$\mathrm{d}_a(\mathrm{E}_2) = 2a\,\mathrm{E}_2^2 - \frac{1}{12}\mathrm{E}_4, \qquad \mathrm{d}_a(\mathrm{E}_4) = 4a\,\mathrm{E}_4\,\mathrm{E}_2 - \frac{1}{3}\mathrm{E}_6, \qquad \mathrm{d}_a(\mathrm{E}_6) = 6e\,\mathrm{E}_6\,\mathrm{E}_2 - \frac{1}{2}\mathrm{E}_4^2.$$

A direct computation proves that the two Poisson brackets $[\ ,\ ]_{\mathrm{d}_a,1}$ and $\{\ ,\ \}_1$ coincide on generators hence are equal on $\mathcal{M}_*^{\leq\infty}$.

Remark 8- A derivation d on $\mathcal{M}_*^{\leq\infty}$ is complex-like if $\mathrm{d}\,\mathcal{M}_k^{\leq s} \subset \mathcal{M}_{k+2}^{\leq s+1}$ for all k and s. Let π be the derivation on $\mathcal{M}_*^{\leq\infty}$ defined by linear extension of $\pi(f) = kf\,\mathrm{E}_2$ for all $f \in \mathcal{M}_k^{\leq\infty}$. The set of complex-like derivations d such that $[\ ,\ ]_{\mathrm{d},1} = 0$ is the vector space of dimension 1 over $\mathbb{C}$ generated by π. Let us define w on $\mathcal{M}_*^{\leq\infty}$ by

$$\mathrm{w}(f) = \frac{\{\Delta, f\}_1}{12\Delta}.$$

Then,

$$\mathrm{d}_a = \mathrm{w} + a\delta.$$

This implies in particular that if a complex-like derivation d satisfies $[\ ,\ ]_{\mathrm{d},1} = \{\ ,\ \}_1$ then $\mathrm{d} = \mathrm{d}_a$ for some $a \in \mathbb{C}$.

Point ii) of Theorem A is obtained by a direct application of Theorem 7. We prove now iii). The term of highest degree with respect to E_2 in $[\mathrm{E}_2, \mathrm{E}_4]_{\mathrm{d}_a,2}$ is $8a^2\,\mathrm{E}_4\,\mathrm{E}_2^3$. This forces $a = 0$. Conversely, if $a = 0$ then $\mathrm{d}_0\,\mathcal{M}_*^{\leq\infty} \subset \mathcal{M}_*$. For any $f = f_i\,\mathrm{E}_2^i$ with $f_i \in \mathcal{M}_*$ we have

$$\mathrm{d}_0(f) = \mathrm{d}_0(f_i)\,\mathrm{E}_2^i - \frac{1}{12} i f_i\,\mathrm{E}_4\,\mathrm{E}_2^{i-1}$$

hence $\deg_{\mathrm{E}_2} \mathrm{d}_0(f) \le \deg_{\mathrm{E}_2} f$ and $\deg_{\mathrm{E}_2} \mathrm{d}_0^j(f) \le \deg_{\mathrm{E}_2} f$ for any $f \in \mathcal{M}_*^{\le\infty}$ and $j \ge 0$. This implies that

$$[\mathcal{M}_k^{\le s}, \mathcal{M}_\ell^{\le t}]_{\mathrm{d}_0,n} \subset \mathcal{M}_{k+\ell+2n}^{\le s+t}.$$

4.2. **Extension of the second family.** The aim of this section is to prove Theorem B. The proof of i) is similar to the proof of i) in Theorem A. Let $\mathcal{K}\colon \mathcal{M}_*^{\le\infty} \to \mathbb{C}$ a graded-additive map. For any integer $n \ge 0$, we define a bilinear application $[\ ,\ ]_{\mathrm{d},n}^{\mathcal{K}}$ by bilinear extension of

$$[f,g]_{\mathrm{d},n}^{\mathcal{K}} = \sum_{r=0}^{n} (-1)^r \binom{\mathcal{K}(f)+n-1}{n-r}\binom{\mathcal{K}(g)+n-1}{r} \mathrm{d}^r f\, \mathrm{d}^{n-r} g.$$

By Corollary 4, we know that $[f,g]_{\mathrm{d},1}^{\mathcal{K}}$ is a Poisson bracket.

Let us fix $\mathcal{K}$ to be the linear extension on $\mathcal{M}_*^{\le\infty} = \bigoplus_k \bigoplus_s \mathcal{M}_k^s$ of

$$\mathcal{K}(f) = (k - (3\alpha+2)s) \qquad (f \in \mathcal{M}_k^s). \tag{9}$$

Let π_α be the derivation on $\mathcal{M}_*^{\le\infty}$ defined by $\pi_\alpha(f) = \mathcal{K}(f) f\, \mathrm{E}_2$ for all $f \in \mathcal{M}_*^{\le\infty}$. The set of complex-like derivations such that $[\ ,\ ]_{\mathrm{d},1}^{\mathcal{K}} = 0$ is the vector space of dimension 1 over $\mathbb{C}$ generated by π_α. Define derivations $\mathrm{w}_{(\alpha)}$ and $\delta_{\alpha,b}$ on $\mathcal{M}_*^{\le\infty}$ by

$$\mathrm{w}_{(\alpha)} f = \frac{(\Delta, f)_\alpha}{12\Delta}$$

and

$$\delta_{\alpha,b} = \mathrm{w}_{(\alpha)} + b\pi_\alpha.$$

By comparing the values on the generators, it is immediate that $(\ ,\ )_\alpha = [\ ,\ ]_{\delta_{\alpha,b},1}^{\mathcal{K}}$.

Remark 9- Direct computations show if d is a homogeneous derivation of degree 2 and $\mathcal{K}$ is such that $(\ ,\ )_\alpha = [\ ,\ ]_{\mathrm{d},1}^{\mathcal{K}}$ then we necessarily have $\mathcal{K}$ defined by (9) and $\mathrm{d} = \delta_{\alpha,b}$ for some $b \in \mathbb{C}$.

The condition that $[\mathrm{E}_4, \mathrm{E}_6]_{\delta_{\alpha,b},2}^{\mathcal{K}}$ has to be a modular form implies $b = 0$ or $\alpha = -1/3$. For $\alpha = -1/3$ condition (4) for $\mu_r = [\ ,\ ]_{\delta_{\alpha,b},r}^{\mathcal{K}}$ and $n = 3$ is not satisfied (this can be done by using computer assistance, for example with Sage [S+13]). We assume then that $b = 0$.

Connes & Moscovici [CM04, Remark 14] (see also [Yao07, § II.2] for a nice presentation of this result) proved that, if E and H are two derivations of an algebra R such that $\mathrm{HE} - \mathrm{EH} = \mathrm{E}$, then the applications $\mu_n\colon \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ defined by

$$\mu_n(f,g) = \sum_{r=0}^{n} \frac{(-1)^r}{r!(n-r)!} [\mathrm{E}^r \circ (2\mathrm{H}+r)^{<n-r>}(f)] \cdot [\mathrm{E}^{n-r} \circ (2\mathrm{H}+n-r)^{<r>}(g)]$$

define a formal deformation on $\mathrm{R}[[\hbar]]$ with the notation

$$\mathrm{F}^{<m>} = \mathrm{F} \circ (\mathrm{F}+1) \circ (\mathrm{F}+2) \circ \cdots \circ (\mathrm{F}+m-1).$$

Let ϖ be the derivation defined on $\mathcal{M}_*^{\le\infty}$ by $\varpi(f) = \mathcal{K}(f) f$. Then we have

$$\varpi \circ \delta_{\alpha,0} - \delta_{\alpha,0} \circ \varpi = 2\delta_{\alpha,0}.$$

We use Connes & Moscovici's result with $E = \delta_{\alpha,0}$ and $H = \varpi/2$ to obtain

$$\mu_n(f,g) = \sum_{r=0}^{n} (-1)^r \binom{k-(3\alpha+2)s+n-1}{n-r} \binom{\ell-(3\alpha+2)t+n-1}{r} \delta_{\alpha,0}^r(f)\delta_{\alpha,0}^{n-r}(g).$$

This implies Theorem B.

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