Task Design in Mathematics Education. Proceedings of ICMI Study 22

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Introduction

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The study aims to produce a state-of-the-art summary of relevant research and to go beyond that summary to develop new insights and new areas of knowledge and study about task design. In particular, we aim to develop more explicit understanding of the difficulties involved in designing and implementing tasks, and of the interfaces between the teaching, researching, and designing roles – recognising that these might be undertaken by the same person, or by completely separate teams.

Background

In her plenary address to the International Group for Psychology of Mathematics Education (PME) Sierpinska (2003) identified task design and use as a core issue in research reports and in mathematics education research more generally. She commented that research reports rarely give sufficient detail about tasks for them to be used by someone else in the same way. Few studies justify task choice or identify what features of a task are essential and what features are irrelevant to the study. In some studies using intervention/treatment comparisons to investigate cognitive development, the intervention tasks are often vague, as if the reader can infer what the learning environment was like from a few brief indications. A similar view had been expressed by Schoenfeld (1980). Yet we learn from applications of variation theory to learning study (e.g., Runesson, 2005), from studies of learning from worked examples (e.g., Renkl, 2005), and from the Adaptive Control of Thought model (ACT-R) (e.g., Anderson & Schunn, 2000) that seemingly minor differences in tasks can have significant effects on learning.

At the same time Burkhardt has drawn attention to the importance of design, with the founding of an international society and a journal, *Educational Designer* (www.educationaldesigner.org). (Schoenfeld, 2009) makes a plea for more communication between designers and researchers, making the point, among others, that many designers are not articulate about their design principles, and may not be informed by research. In 2008, the International Congress on Mathematics Education (ICME) hosted a topic study group (TSG), *Research and development in task design and analysis*, which provided a forum for that kind of interaction (http://tsg.icme11.org/tsg/show/35). Designers had to be explicit about their principles and demonstrate how they used them. Participants were given the opportunity to experience various tasks, and compare and critique design principles. Drawing from a wide international field, an overview of the papers makes it apparent that:

(1) it is necessary to have theories about learners' intellectual engagement to have successful design; and

(2) most design principles included the use of several representations, several kinds of sensory engagement, and several question types.

The TSG increased its membership during the conference, indicating that a serious, organised look at task design was of growing interest. A further TSG is due to take place at ICME 12 in July 2012 in Seoul, Korea. Working groups on task design using digital technologies, and design of digital learning environments, proliferate, but we are not aware of a similar level of activity in other environments.

Mathematics educators have focused to a great extent on the social cultures of classrooms and designed learning environments, on patterns of argumentation, on emotional aspects of engagement, and on measures of learning. A distinct mathematical contribution can be made in understanding whether and how doing tasks, of whatever kind, enables conceptual learning. For example, Lagrange (2002) suggests that applying routine techniques can achieve results, and also provide the basis for conceptual understanding and new theorising; (Watson & Mason, 2006) have shown how a set of procedural exercises, seen as one object, can provide raw material for conceptualisation; Realistic Mathematics Education (RME) from the Netherlands and Mathematics in Context materials (from the United States) show how carefully designed situational sequences can turn a learners' attention to abstract similarities.

Our statement that task design is core to effective teaching is well-illustrated by the success of theoretically-based long term design-research projects resulting in publications such as those from Shell Centre (Swan, 1985), Realistic Mathematics Education (de Lange, 1996) and Connected Mathematics (Lappan & Phillips, 2009). In these, design and research over time have combined to develop materials and approaches that have appealed to teachers. In addition, research related to the QUASAR project (Quantitative Understanding: Amplifying Student Achievement and Reasoning) found that the cognitive demand of designed tasks was often reduced during implementation (Henningsen & Stein, 1997). A research forum at PME in Mexico (Tzur, Sullivan, & Zaslavsky, 2008) offered cogent explanations for the inevitability and even desirability of teachers' alteration of the cognitive demand of tasks. Further, Choppin (2011) suggests how adaptation differs among teachers. Thus, a possible area of investigation is how published tasks are appropriated by teachers for complex purposes. In variation theory, a distinction is made between the intended, enacted, and lived objects of learning. The Documentational Approach of Didactics (Gueudet & Trouche, 2009, 2011) also refers to the practitioner perspective in terms of the resources on which teachers draw. Didactic engineering was the topic of the 15th summer school in mathematics didactics in 2009 (Margolinas, Abboud-Blanchard, Bueno-Ravel, Douek, Fluckiger, Gibel, Vandebrouck, & Wozniak, 2011). The discussion focused not only on various principles of task design (see the contributions of Bessot, Chevallard, Boero, and Schneider) but also on the problem of the influence of task design on the development of actual mathematics teaching (see contributions of Perrin-Glorian, René de Cotret and Robert). The tasks in these references are all complex, multi-stage tasks which address complex purposes, such as those usefully summarised in Kilpatrick, Swafford, & Findell (2011), namely the development of conceptual understanding; procedural fluency; strategic competence; adaptive reasoning; and productive disposition.

We would like to encourage an interest in tasks that have more limited but valid intentions, such as tasks that have a change in conceptual understanding as an aim, or tasks that focus only on fluency and accuracy. Research can investigate how students perceive and conceptualise from the examples they are given, or on which they work. Most mathematics learners world-wide learn procedures and possibly concepts through ‘practice’, regardless of the de-emphasis on procedures held by reform enthusiasts. Thus, the design of sequences of near-similar tasks deserves attention. For reasons of global reality and equity, the study conference shall also focus on textbook design partly because textbooks are often informed by tradition or by an examination syllabus rather than through research and development (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002), but also because in some countries textbooks are the major force for change. Textbooks are not the only medium in which sequences of tasks, designed to afford progressive understanding or shifts to other levels of perception, can be presented, and we expect that study conference participants will look also at the design of online task banks.

Work from Sullivan indicates the need to educate new teachers in the use of complex tasks (Sullivan, 1999) and it is inevitable that teacher education will cross several of our suggested areas. A volume of the *Handbook of Mathematics Teacher Education* was devoted to the tasks and processes of teacher education (Tirosh & Wood, 2009). A particular relationship between teacher education and task design is the design of tasks for teacher education purposes. Mathematics teacher education, as a subfield of mathematics education, has paid significant recent attention to the nature, role and use of tasks with a triple special issue of the *Journal of Mathematics Teacher Education* (volume 10, 4-6) edited by Mason, Watson and Zaslavsky, and a book edited by Zaslavsky & Sullivan (2011)

The meaning of ‘task’

The word ‘task’ is used in different ways. In activity theory (Leont'ev, 1975) task means an operation undertaken within certain constraints and conditions (that is in a determinate situation, see Brousseau (1997)). Some writers (Christiansen & Walter, 1986; Mason & Johnston-Wilder, 2006) express ‘task’ as being what students are asked to do. Then ‘activity’ means the subsequent mathematical (and other) motives that emerge from interaction between student, teacher, resources, environment, and so on around the task. By contrast, in some professional traditions, ‘activity’ means a situation set up by the teacher in which a student has to engage in a certain way. Other traditions (e.g.Chevallard, 1999) distinguish between tasks, techniques, technology and theories, as a way to acknowledge the various aspects of a praxeology. We are also aware that ‘task’ sometimes denotes designed materials or environments which are intended to promote complex mathematical activity (e.g. Becker & Shimada, 1997), sometimes called ‘rich tasks’. In this study, we use ‘task’ to mean a wider range of ‘things to do’ than this, and include repetitive exercises, constructing objects, exemplifying definitions, solving single-stage and multi-stage problems, deciding between two possibilities, or carrying out an experiment or investigation. Indeed, a task is anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something. Task can also be anything that students decide to do for themselves in a particular situation. Tasks, therefore, are the mediating tools for teaching and learning mathematics and the central issues are how tasks relate to learning, and how tasks are used pedagogically.

Task design

The design and use of tasks for pedagogic purposes is at the core of mathematics education (Artigue & Perrin-Glorian, 1991). Tasks generate activity which affords opportunity to encounter mathematical concepts, ideas, strategies, and also to use and develop mathematical thinking and modes of enquiry. Teaching includes the selection, modification, design, sequencing, installation, observation and evaluation of tasks. This work is often undertaken by using a textbook and/or other resources designed by outsiders.

The extent and detail of design varies widely among those who work on task design. For some (e.g., Shell Centre) design includes full necessary materials, task sequences and advice about effective choices, and detailed pedagogic advice about ways of working, verbal interventions, likely misconceptions and possibly extensions. For others (Ainley, Bills, & Wilson, 2004, 2005) there may be provision of a question, or a microworld, or some physical material, with no written object to describe ‘the complete task’, but rather a series of things that the teacher might say, perhaps supported by some written prompts. During the resulting activity, learners may ask questions or make comments to which the teacher needs to respond, and part of the design is trying to anticipate these and have a general picture of the shape of responses which would complement the task design. Another form of design is to refine a question or problem-situation until it is most likely to promote intriguing mathematical reactions (e.g., (ATM, various dates)). Sullivan, Zevenbergen, & Mousley (2006) have identified a need to design whole lesson sequences around certain types of tasks. All of these approaches have implications for implementation, with some relying on teachers’ existing skills, some providing advice to extend teachers’ skills, and others dependent on teachers maintaining or adapting the original task intentions (see, e.g., Kieran, Tanguay, & Solares, 2011).

Tasks also arise spontaneously in educational contexts, with teachers and/or learners raising questions or providing prompts for action by drawing on a repertoire of past experience. We are interested in how these are underpinned with implicit design principles.

Task sequences

This discussion of tasks may lead readers to assume that we are focused only on tasks as single events, but it is important to address also the question of sequences of tasks. There are different aspects embedded in the design of sequences and, while this is an obvious consideration when designing textbooks, it also stretches across the whole field of task design.

To achieve the goal of teaching a whole conceptual field (e.g., rational numbers), we have to describe the different aspects of this knowledge and the way the aspects are linked (for interesting examples see Brousseau, Brousseau, & Warfield, 2004a, 2004b, 2007, 2008, 2009). In Brousseau’s Theory of Didactic Situations (Brousseau, 1997), particular situations (or single tasks) are generated from more general situations. The earlier tasks in a sequence should provide experiences that scaffold the student in the solution of later tasks, allowing them to engage in more sophisticated mathematics than would otherwise have been the case. In some published sequences, the earlier tasks might be technical components to be used and combined later; in others, the earlier tasks might provide images or experiences which enable later tasks to be undertaken with situational understanding.

To understand how tasks are linked in order to support teaching, it is important to understand the nature of the transformation of knowledge from implicit knowledge-in-action (see Vergnaud, 1982) to knowledge which is formulated, formalized, memorized, related to cultural knowledge, and so on.

However, there are different ways to create sequences of tasks, some of them are more commonly known by teachers themselves. One of these types of task sequences is that in which the problem formulation remains constant but the numbers used increase the complexity of the task, say moving from small positive integers (for which answers might be easy to guess) to other ranges of numbers for which a method might be needed. Another type of sequence is one in which the problem is progressively made more complex by the addition of steps or variables, such as in a network task where additional nodes are added. A third type of sequence may be one where the concept itself becomes more complex, such as in a sequence of finding areas or progressively more complex shapes from rectangles, to composite shapes, to irregular shapes. These different types of sequences, and their relation to the teaching unit as a whole, are often the focus of lesson study cycles, such as those reported in for example Corey, Peterson, Lewis, & Bukarau (2010); Huang & Bao (2006) Yoshida, (1999).

The importance of sequencing is explicit in Realistic Mathematics Education. In that tradition, a task sequence starts with situated problems (Gravemeijer, 1999), like dividing large numbers of people into smaller groups (quotative division problems) to evoke informal strategies and representations, and continues by changing the focus to formalizing and generalizing solution procedures, i.e. in this case a general algorithm that can be used for various division problems. In this type of task sequence the idea of 'guidance with didactical models' from informal to formal is important as an alternative strategy for the increasing mathematical complexity of problems students encounter (Van den Heuvel-Panhuizen, 2003). The situated problems are often already rather complex and can be solved before you know 'the' mathematical solution procedure, and therefore can be good starting points for problematizing a concept.

Design communities and methods

Of course, teachers also design tasks explicitly and deliberately. Whereas some authors think it desirable that designing and teaching are separate acts carried out by separate groups of people (e.g. Wittman, 1995), the experience of the authors of this discussion document indicates that the communities involved in task design are naturally overlapping and diverse. Design can involve designers, professional mathematicians, teacher educators, teachers, researchers, learners, authors, publishers and manufacturers, or combinations of these, and individuals acting in several of these roles. In the study, we wish to illuminate the diverse communities and methods that lead to the development and use of tasks. In all methods, the central consideration is the interaction between teachers and learners through the designed artefacts and/or the design process. A major focus in the study will therefore be on learning how design impacts on learners and learning, rather than research which focuses solely on the design process. For example, research which identifies implicit design principles would be of interest if connections are made between these principles and the impact on learning; research about identities of different players in the design process would be of interest if it contrasted ‘teacher-as-task-designer’ and ‘teacher-as-task-user’.

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Theme A
Tools and Representations

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In the mathematics classroom, concrete tools (for example, compasses and ruler, unit blocks, interactive ICT platforms) are usually used as resources to enhance the teaching-learning activity (see for example, Bartolini Bussi & Maschietto, 2008; Maschietto & Trouche, 2010; Radford, 2011). In this context, tools are broadly interpreted as physical or virtual artefacts that have potential to mediate between mathematical experience and mathematical understanding. This theme concerns designing teaching-learning tasks that involve the use of tools in the mathematics classroom and consequently how, under such design, tools can represent mathematical knowledge. A task here is a teacher designed purposeful 'thing to do' using tools for students in order to activate an interactive tool-based environment where teacher, students and resources mutually enhance each other in producing mathematical experiences. On a meta-level, it is about possible tool-driven relationships within the design, teaching and learning triad. In this connection, this type of task design rests heavily on the complex relationship between artefacts and mathematical knowledge.

There are a few theoretical grounds on which to build and expand this discussion. Instrumental genesis explicates how the usage of a tool can be turned into a cognitive instrumentation process for knowledge acquisition. A Vygotskian approach examines how an artefact can be turned into a psychological tool in the context of social and cultural interaction developed through the zone of proximal development and internalization processes. Semiotic mediation can be used as an integrated approach to explore the mathematics classroom under which a tool takes on multiple pedagogical functions (Bartolini Bussi & Mariotti, 2008). Embodiment theory proposes that there are strong relationships among sensory activities and cultural artefacts in the appropriation of mathematical practices, and in particular, their application to inclusive mathematics education (Healy & Fernandes, 2011). The guided re-invention principle of RME (Realistic Mathematics Education) practiced by the Freudenthal school can be used to direct the design of tool-based mathematical tasks. These theoretical orientations, and/or others, may serve to facilitate discussion on tool-based task design and representation in the mathematics classroom.

An important question to address in this theme is: How to design tasks that can bring about situated discourses (hence representations) for the mathematical knowledge mediated by tools in the mathematics classroom and how these discourses relate to mathematics knowledge? This in turn comprises several additional questions.

Possible questions about tools and representation:

- What mathematics epistemological considerations are taken into account when designing tasks using tools?

- How do we create a tool environment for the mathematics classroom to support the design of teaching and learning tasks for specific mathematic topics?
- How do different types of tools afford different mathematical activities/tasks, different representations and/or discourses, and different interactions between representations?
- How do different task designs using tools impact on students' learning and understanding of mathematics?
- How do we design mathematical tasks that can transform an artefact into a pedagogical instrument?
- Are there models (theoretical or pragmatic) of tool-based task design for the teaching and learning of mathematics?

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Using crises, feedback and fading for online task design

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A recent discussion involves the elaboration on possible design principles for sequences of tasks. This paper builds on three principles, as described by Bokhove and Drijvers (2012b). A model with ingredients of crises, feedback and fading of sequences with near-similar tasks can be used to address both procedural fluency and conceptual understanding in an online environment. Apart from theoretical underpinnings, this is demonstrated by analysing a case example from a study conducted in nine schools in the Netherlands. Together with quantitative results of the underlying study, it shows that the model described, could be a fruitful addition to the task design repertoire.

Keywords: task, design, sequence, near-similar, crisis, feedback, fading

Introduction

In recent years more and more attention has been paid to task design. In the call for papers for the 22nd ICMI study on task design the reasons for this are clearly described. One problem is that tasks are often only described vaguely. Furthermore, Schoenfeld (2009) advises on having more communication between designers and researchers. In this way educational research and design can be bridged, as the communities involving task design are naturally overlapping and diverse. This paper was triggered by some of the remarks that were made in the call for papers:

The topic of understanding whether and how doing tasks, of whatever kind, enable conceptual learning. The study reported in this paper supports Lagrange (2002) who suggested that applying routine techniques can achieve results, and also provide the basis for conceptual understanding and new theorizing.

To not only address tasks as single events, but also address the question of sequences of tasks.

It is suggested that the design of sequences of near-similar tasks deserves attention. Therefore, the paper makes a point of defining a ‘task’ to mean a wider range of ‘things to do’ than just one task, and include repetitive exercises.

Several types of task sequences are mentioned. “One of these types of task sequences is that in which the problem formulation remains constant but the numbers used increase the complexity of the task, say moving from small positive integers (for which answers might be easy to guess) to other ranges of numbers for which a method might be needed.”. Building on an earlier article (Bokhove & Drijvers, 2012b) an extra type of task sequences is proposed, whereby the complexity of tasks first increases, and then –with the help of feedback- decreases.

In one sense this can be seen as an adaptation of the ‘variation’ Watson and Mason (2006) coined: “From a modelling perspective the term micromodelling’ may be helpful to describe learners’ response to exercises in which dimensions of variation have been carefully controlled, because the aim is to promote generalization of the dimensions being varied in the exercise, and thence to focus on mathematical relationships between dimensions.” (p.104)

Summarizing these points we would want to formulate design principles for: (i) sequences of tasks; sequences of tasks; (ii) near similar and/or repetitive tasks; and (iii) addressing both conceptual understanding conceptual understanding and procedural fluency. This paper synthesizes, elaborates on and illustrates design illustrates design principles of a study first described by Bokhove and Drijvers (2012a, b). The principles The principles regarding crises, feedback and fading are applied to a sequence of digital tasks. This paper sets tasks. This paper sets out to describe the three principles in an additional cohesive model, and describes one case describes one case example of student work. Bearing the aforementioned goals in mind it should be a model that should be a model that could prove to be fruitful while designing tasks. The model bears elements of both my elements of both my roles as a designer and researcher when doing my PhD as a teacher at a secondary school in secondary school in the Netherlands. The study called ‘Algebra met Inzicht’ [Algebra with Insight] was designed Insight] was designed in the Digital Mathematical Environment (http://www.fi.uu.nl/dwo/en)[1]. The intervention (http://www.fi.uu.nl/dwo/en)[1]. The intervention consists of a pen-and-paper pre-test, four digital modules, a digital modules, a digital diagnostic test, and a final digital test and, finally, a pen-and-paper post-test. It was post-test. It was deployed in fifteen 12th grade classes from nine Dutch secondary schools (N=324), involving (N=324), involving eleven mathematics teachers. The schools were spread across the country and showed a and showed a variation in school size and pedagogical and religious backgrounds. The participating classes participating classes consisted of pre-university level ‘wiskunde B’ students (comparable to grade 12 in Anglo-grade 12 in Anglo-Saxon countries). As this article is about design principles, I refer to different articles for different articles for more details of the set-up of the study and the actual effects of the digital intervention intervention (Bokhove & Drijvers, 2012a, b).

figure 1 shows the proposed model for sequences of (near-similar) tasks.

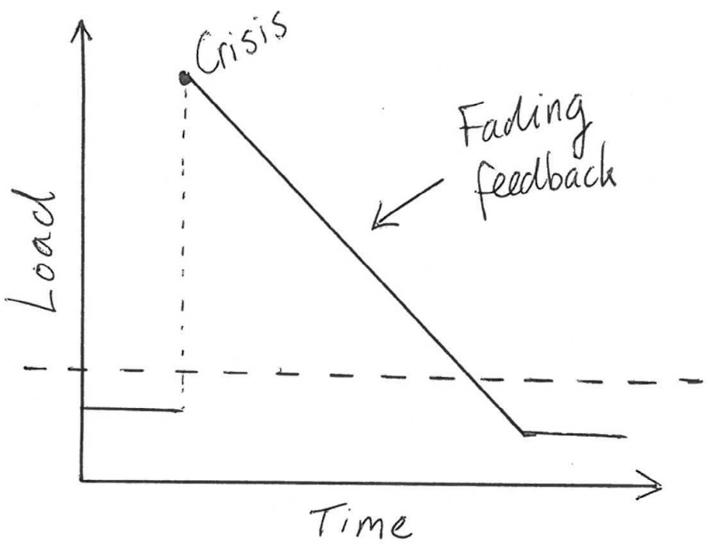

figure 1: proposed model for crises, feedback and fading

[1] An English translation of part of the module can be found at http://www.fi.uu.nl/dwo/soton. Log in as guest, and choose ‘Demo for 22nd ICMI study’. Java is needed.

I contend that it is okay to use near-similar tasks and repetitive exercises, but suggest that the sequence is interspersed by *intentional crises* i.e. tasks that are hard or impossible to solve with skills and knowledge that are available. In other words, the 'load' of the task is too high. I will not go into the word 'load' in detail. There is a vast body of knowledge connected to the term Cognitive Load Theory (Sweller, 1988). There also is, rightly so, criticism (De Jong, 2010). For the purpose of this paper, we will only assume that knowledge that isn't known potentially will bear a larger load than unknown knowledge. Then, let students overcome crises by *providing feedback*. To avoid a dependency on feedback for the summative assessments *fade the feedback* during the course of the sequence of tasks.

A model for sequences of near-similar tasks

I will first elaborate on the three main ingredients of the model: crises, feedback and fading.

With a crisis we refer to a principle that the poet John Keats so eloquently described in the early 19th century 'failure is the highway to success'. This principle corresponds to similar concepts that have been described during the years. Piaget (1964) used the concept of equilibrium and disequilibrium. Essentially, whenever the child's experience/interaction with the environment yielded results that confirmed her mental model, she could easily assimilate the experience. When the experience resulted in something new and unexpected, the result was disequilibrium, and a child may experience this as confusion or frustration. Eventually, the child changes his or her cognitive structures to accommodate the new experience and moves back into equilibrium. Piaget studied the individual case, on a societal level Kuhn (1962) referred to a paradigm shift, arguing that scientific advancement is not evolutionary, but rather a "series of peaceful interludes punctuated by intellectually violent revolutions", and that in those revolutions "one conceptual world view is replaced by another". Tall (1977) refers to cognitive conflicts: "one of the distinguishing factors in catastrophe theory is the existence of discontinuities, or sudden jumps in behaviour when certain paths are taken.". In his 'levels of thinking' van Hiele (1985) discerns structure and insight. There can be a 'crisis of thinking', which has a link to the vygotskian zone of proximal development. The common ground between the two is that there is a need for challenge. More recently, Kapur (2010) uses the term productive failure and cites Clifford (1984): "However, allowing for the concomitant possibility that under certain conditions letting learners persist, struggle, and even fail at tasks that are complex and beyond their skills and abilities may in fact be a productive exercise in failure requiring a paradigm shift". Kapur explains this by stating that it is reasonable to reinterpret their central findings collectively as an argument for a delay of structure in learning and problem-solving situations, be it in the form of, feedback and explanations, coherence in texts, or direct instruction. The difference with my own work (Bokhove & Drijvers, 2012b) seems to be whether crisis are an inherent part of learning when solving open problems, or actually embedding tasks that could intentionally cause a crisis. It is proposed that intentional crisis tasks are added to sequences of near-similar tasks, for example in the way depicted in table 1, which illustrates the way in which crisis items are integrated within the digital tool. The general structure of a sequence is: pre-crisis items, crisis item, post-crisis items. But then the question becomes: how can students address this crisis? Can we add another principle which enables students to use assessment *for* learning. One way would be to make use of *formative* assessment. Black and Wiliam (1998) define assessment as being 'formative' only when feedback from learning activities is actually used to modify teaching to meet the learner's needs. From this it is clear that feedback plays a pivotal role in the process of formative assessment. Hattie and Timperley (2007) conducted a meta-review of the effectiveness of different types of feedback. The feedback effects of hints and corrective feedback are deemed best. However, in my personal experience as a teacher I have seen there can be an over-reliance on feedback that is provided. Assuming that students finally have to pass an exam themselves, it makes sense to address this over-reliance on feedback. In a follow-up paper Kapur (2011) notes that scaffolding implies help to overcome failure (Pea 2004). It turns out that when dealing with previously stored information in the long term memory, these limits tend to disappear.

1.1	Tasks: "Solve the following equation:"	**Pre-crisis items** In the initial items students are confronted with equations they have experience with. Students may choose their own strategy. Many students choose to expand brackets as that is the strategy that they have used often: work towards the form $ax^2 + bx + c = 0$ and use the Quadratic Formula. There is some limited feedback on the task.
1.2	$(4x-3)\cdot(4x-1)=(4x-3)\cdot 2$	
1.3	$\sqrt{3x+2}\cdot(3x+3)=\sqrt{3x+2}\cdot(6x-2)$	
1.4	$(x-4)\cdot(2x-5)=(x-4)\cdot(-3x+3)$	
1.5	**Opgave 1.5** Los de volgende vergelijking op:	meer abc $(5x-13)\cdot(4x-3)-(5x-13)\cdot(-2x+3)=0$
1.6	meer $(x^2+3x-3)\cdot(8x-6)=(x^2+3x-3)\cdot(4x+12)$ $8x^3+18x^2-42x+18=4x^3+24x^2+24x-36$ $4x^3-6x^2-66x=-54$ $4x\left(x^2-1\frac{1}{2}x-16\frac{1}{2}\right)=-54$ ✓	**Crisis item** Students are then confronted with an intentional crisis: if a student uses his/her conventional strategy of expanding the expression. The yellow tick at the bottom of the screen denotes that the equation is algebraically equivalent to the initial one, but that it is not the final answer. This is accompanied by a partial score for an item and some feedback in Dutch: 'You are rewriting correctly'. Although these students showed good rearranging skills, in the end they were not able to continue, as they did not master the skill to solve a third order equation. There is some limited feedback on the task.
1.7	**Opgave 1.7** Los de volgende vergelijking op: voorbeeldfilm	meer tip stap losop abc $(2x^2-3x-2)\cdot(7x-3)=(2x^2-3x-2)\cdot(3x+12)$
1.8	$(x^2-3x-2)\cdot(6x-3)=(x^2-3x-2)\cdot(4x+12)$	**Post-crisis items** After the crisis item students are offered help by providing a 'voorbeeldfilm', an instructional screencast, and buttons to get hints ('tip'), the next step in the solution ('stap') or a worked solution ('losop'). These features have in common that they provide feedforward information at the task level and self-regulation.
1.9	$\sqrt{3x+3}\cdot(2x+4)=\sqrt{3x+3}\cdot(6x-5)$	
1.10	$(4x+4)\cdot\sqrt{-2+3x}=\sqrt{3x-2}\cdot(7x-5)$	
1.11	$(-5+{}^{2}\log(x-2))\cdot(6x-6)=(-5+{}^{2}\log(x-2))\cdot(3x+14)$	
1.12	$(4x-13)\cdot(3x-3)=(4x-13)\cdot(-3x+2)$	
1.13	$(-4x+5)\cdot(8x-5)=(-4x+6)\cdot(3x+14)$	

table 1: Sequence of items illustrating crises and feedback

As Kirschner et al (2006) argued "any instructional theory that ignores the limits of working memory when dealing with novel information or ignores the disappearance of those limits when dealing with familiar information is unlikely to be effective." As a design principle it is therefore proposed that initially a lot of feedback is provided to foster learning, but the amount is decreased towards the end, to facilitate transfer. Using scaffolding this way is based on the concept of *fading* (Renkl, Atkinson, & Große, 2004). Formative scenarios (Bokhove, 2008) are a variation of this concept, starting off with much feedback, and providing a gradually decreasing level of feedback. figure 1 shows how this principle was implemented in the intervention. At the start feedback is provided for all intermediate steps of a solution. The subsequent part of the intervention concerns self-assessment and diagnostics: the student performs the steps without any feedback and chooses when to check his or her solution by clicking a "check" button. Feedback is then given for the whole of the exercise.

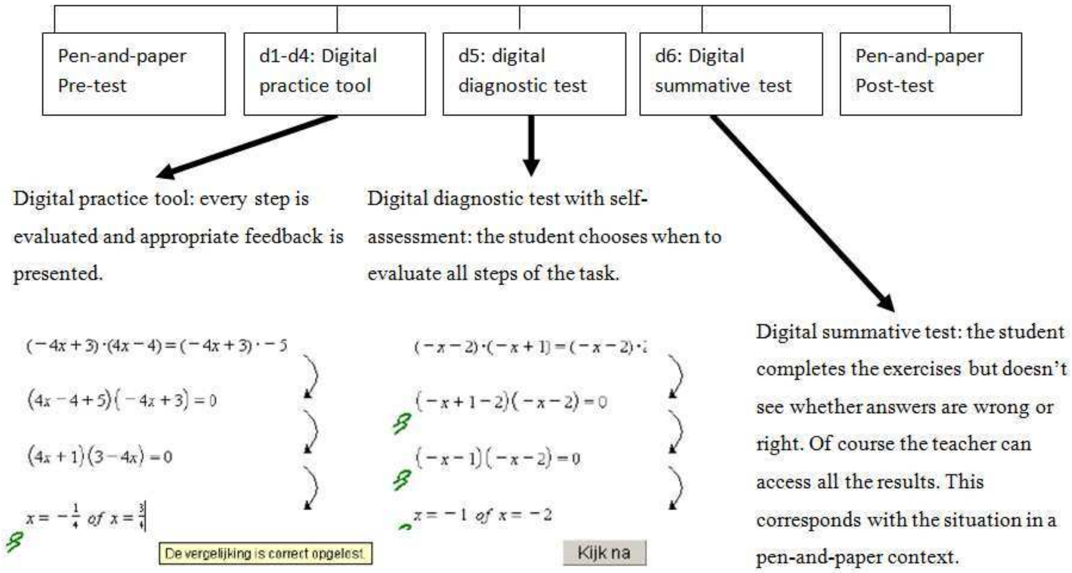

figure 2: Outline of fading feedback in formative scenarios

Finally, students get a final exam with no means to see how they performed. Just as is the case with a paper test, the teacher will be able to check and grade the exam (in this case automatically) and give students feedback on their performance. A student needs to be able to accomplish tasks independently, without the help of a computer. An implicit advantage of implementing feedback in a sequence of tasks is that teachers and designers have to think upfront about possible student responses.

Principles at work: a case example

Let's look at one student named Paula during the course of this module. The student starts off with a pre-test. Apart from the calculation error on the right hand side of the equation, figure 3 clearly shows that Paula's strategy here is to expand the expressions, similar to students in earlier phases of the study (Bokhove & Drijvers, 2010).

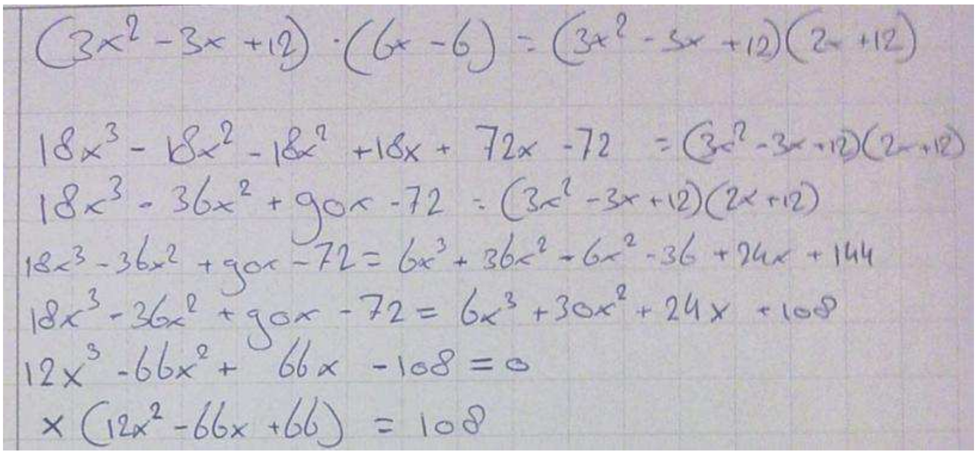

Figure 3: Example of Paula's pre-test pen-and-paper work

Not surprisingly this strategy fails in the case of this equation. Paula only scores 14 out of 100 for the whole pre-test. With regard to symbol sense, Paula scores a -4 for (for details on the calculation I refer to Bokhove & Drijvers, 2012a). Paula then starts with the sequence of digital tasks. In the first task the student has to get acquainted with the digital environment. The pre-crisis items pose no problem for most students, including Paula. On arriving at the crisis item students roughly exhibit three behaviours, roughly corresponding with the ones already observed in the pre-test: (i) students solve the equation correctly, (ii) students recognize the pattern of the equation but subsequently make mistakes (for example by losing solutions in the process), and (iii) students expand the expressions and get stuck with an equation with a third power. Figure shows that our case student Paula again shows the third type of behaviour. At this moment feedback is still restricted to correct/incorrect. In addition, students are allowed to choose their own strategies, even when they aren't efficient or would lead to problems. In the post-crisis items, as well as feedback correct/incorrect, Paula is provided with buttons for hints, and a movie clip demonstrating the solution. From the log-files of the online environment it becomes clear that Paula fails at the crisis-item (0 out of 10), but succeeds at the post-crisis item with feedback (10 out of 10). When looking at attempts made, Paula attempts the crisis-item 73 times, and the post-crisis item, being aided by feedback, only 3 times.

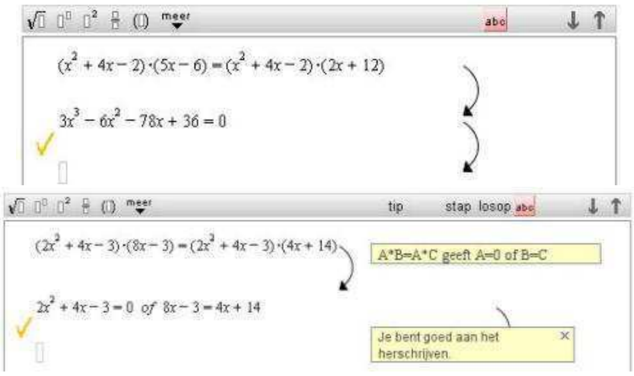

Figure 4: Paula's digital work. Left: crisis item left. Right: post-crisis item

Finally, in the post-test Paula shows a significant increase in the total score (70 out of 100) and symbol sense behaviour (+1, an increase of 5). Even though mistakes are made they were not caused by a lack of symbol sense any more but errors in calculations. Focusing only on similar types of equations it becomes clear that Paula manages to solve these equations correctly. Paula is not a unique case in this school. Overall, students in participating schools improved on their scores and symbol sense behaviour.

Conclusion

Illustrated by the theoretical underpinnings, the overall results in the study, and case example, it is concluded it would be a good idea to study design principles that can be used to design sequences of near-similar tasks in more detail. By combining three principles from an initial study -crises, feedback and fading- in one model for sequences of tasks, three important aspects of this ICMI study are addressed: (i) sequences of tasks; (ii) near similar and/or repetitive tasks; and (iii) addressing both conceptual understanding and procedural fluency. We propose that educators, teachers, designers and researchers alike can adopt these principles when designing and implementing sequences of (near-similar) tasks. It is, however, important to note some points of discussion. The difficulty of every task or sequence of tasks depends on the context. What can be a simple task for one year eight student can prove to be difficult for another student, even when at first sight they seem fairly similar. Also, the way in which a crisis is overcome can differ: some students learn from repeating near-similar tasks, others seem to recognize 'a pattern' immediately and apply this to new tasks. Given this diversity, it is important to field-test and evaluate sequences of tasks, again combining the power of teaching, researching and designing. I think it would be unfair to indefinitely classify certain tasks as 'more creative' and other tasks as 'less creative'. This, too depends on the background and context of the learner: a wonderful, new and creative task can become a repetitive task the second time around. Looking back on this paper I wonder whether the predicate 'near-similar' does not actually apply to *all* tasks, if a student has seen a task before, even the elaborate, creative ones. It is my wish that we look at the total picture, and integrate all these tasks in one clear picture for the learning student. One way would be to not so much study the nature of solitary tasks but place them in sequences and their corresponding contexts. Hopefully, this paper provides general design principles that can be used, and task design can be taken forward.

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A Holistic Approach for Designing Tasks that Capture and Enhance Mathematical Understanding of a Particular Topic: The Case of the Interplay between Examples and Proof

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This study[2] illustrates an approach to task design that aims at revealing and enhancing students' understandings of a particular mathematical topic: the roles of examples in proving and refuting mathematical statements. As a preliminary stage we conducted a content analysis that led to the construction of a comprehensive mathematical framework that describes the various aspects of understanding this topic. Based on this framework, we constructed six prototypes of tasks that complement each other in assessing and enhancing students' understanding of the logical connections between examples and statements. In this paper we describe the design principles that guided the construction of the collection of tasks that can be considered a special kind of tool and provide illustrations of how this tool was used.

Keywords: task design, examples, reasoning and proof

Introduction and Background

With the current curricular emphasis on students' engagement in mathematical investigations and proving, it becomes increasingly important that students develop the understanding of the logical connections between empirical and formal aspects of mathematics. Such understanding manifests itself in understanding of the roles of examples proving and refuting conjectures. Despite the fact that several studies indicate that students encounter difficulties in this area (Fishbein, 1987, Antonini et al., 2011, Bills & Watson, 2008), such connections are not explicitly addressed in the school curriculum and are left to students to develop indirectly mostly on their own. Moreover, there is only scarce work in conceptualizing what constitutes an understanding of the roles of examples in proving (e.g., Zaslavsky & Ron, 1998, Barkai et.al., 2008, Buchbinder & Zaslavsky, 2009, Tabach et. al., 2010).

The goal of our study was to develop a tool that could be used both to assess students' overall understanding of the above topic and to enhance these understandings. Moreover, we were looking for a tool that would help capture the

[2] This study was conducted at the Technion – Israel Institute of Technology with support of Israeli Ministry of Education.

nature of this understanding. The first question we posed was: What does it mean to understand the roles of examples in determining the validity of mathematical statements?

In order to answer this question, we developed a general framework[3] that describes four types of examples (confirming, non-confirming, contradicting, and irrelevant) and their status in proving or refuting the two types of mathematical statements (universal and existential). We describe the framework in the next section.

The framework

Every mathematical statement can be characterised by the domain (D) of objects (x) to which it refers to and a proposition (P(x)) that specifies some property. A *Universal* statement states that a proposition is true for all the objects in the domain: $\forall x \in D, P(x)$. An *Existential* statement asserts that there exists an object in the domain for which the proposition is true: $\exists x \in D, P(x)$. For example, if the domain D is '*all integers for which the sum of their digits is divisible by 6'* and the proposition P is '*multiple of 6'*, we can formulate two types of statements: 1. A Universal statement, "All integers, for which the sum of their digits is divisible by 6 are multiples of 6"; and 2. An Existential statement, "There exists an integer for which the sum of its digits is divisible by 6 that is a multiple of 6".

With respect to a given domain D and a property P, four types of examples can be defined, depending on whether (*a*) the object x is an element of D or not ($x \in D$ or $x \notin D$), and (*b*) the proposition P(x) is true for it or not ($P(x)$ or $\neg P(x)$). The logical status of these examples depends on the type of statement (Figure 1).

The first type of object x, which we term a *confirming example*, is an element of D for which the proposition P(x) is true ($x \in D, P(x)$). In the above case, 24 is a confirming example. While 24 is insufficient for proving the universal statement, it proves the corresponding existential statement. The second type of object, which is termed a *counterexample,* is an element of D that does not satisfy the proposition P(x) ($x \in D, \neg P(x)$). For example, 33 is a *counterexample,* thus, it contradicts the false universal statement; we refer to such an example as a *contradicting* example (of the universal statement). However, with respect to the corresponding existential statement 33 is insufficient for refuting it. It neither contradicts nor confirms it; thus we term such an example a *non-confirming* example (of the existential statement). All of the above types of examples are *relevant* for examining the validity of a given statement. The other two types of examples are objects that do not belong to the domain D: $x \notin D, P(x)$ and $x \notin D, \neg P(x)$. From a logical stand both types are equally *irrelevant* for determining the validity of either type of statement. However, example of type $x \notin D, \neg P(x)$ (e.g., 36) is potentially misleading.

[3] For a detailed description and discussion of the framework see Buchbinder & Zaslavsky, 2009.

Type of Statement	Universal statement $\forall x \in D,\ P(x)$		Existential Statement $\exists x \in D,\ P(x)$	
Goal / Type of Example	**To prove**	**To disprove**	**To prove**	**To disprove**
Confirming $x \in D,\ P(x)$	**Insufficient**	Non applicable	**Sufficient**	Non applicable
Contradicting (the universal statement) **Non-confirming** (the existential statement) $x \in D, \neg P(x)$	Non applicable	**Sufficient**	Non applicable	**Insufficient**
Irrelevant $x \notin D,\ P(x)$	Non applicable	Non applicable	Non applicable	Non applicable
Irrelevant $x \notin D,\ \neg P(x)$	Non applicable	Non applicable	Non applicable	Non applicable

Figure 1: A framework for examining the logical status of examples in determining the validity of mathematical statements

We take "understanding of the logical status of examples in determining the validity of mathematical statements" as becoming fluent with the types of inferences that can and cannot be made based on the four types of examples with respect to the two types of statements. We consider the following four types of cognitive activities as eliciting evidence of such understanding: construction of examples (spontaneously or on demand); recognition of and differentiation between types of statements and types of examples (without necessarily using our terms); making logical inferences based on examples, and correctly justifying then. Such operational approach to conceptualizing understanding is consistent with Borgen & Manu (2002), Schoenfeld (1989) and Zaslavsky (1997) and sets the grounds for task design.

The design principles behind the collection of tasks

Based on our framework, we constructed 6 prototypes of tasks, each of which addresses a particular aspect of understanding. The set of tasks as a whole covers all the aspects of the framework and proved effective as a diagnostic and facilitating tool for 10th grade students in Israel.

The overarching design principle was to create a *balanced* representation of all of the following variables: 1. Type of statement (universal or existential); 2. Type of example (confirming, contradicting, non-confirming, and [two kinds of] irrelevant); 3. The truth-value of the statement (true/false); 4. The content (secondary level algebra or geometry, all of which students are supposed to be sufficiently knowledgeable); 5. Number of relevant examples that exist for each statement (e.g., no example, a single example or a finite number, an infinite number of examples of a particular type).

From a mathematical point of view, the specific number of examples that exist for each statement is irrelevant[4]. A single contradicting example is sufficient for refuting a statement, even if there exist an infinite number of confirming examples. However, the number of examples may affect students' inferences, their perseverance in evaluating the truth-value of the statement, and the methods they use.

[4] If a domain of a universal statement has a finite number of elements, it is possible to prove that the statement is true by showing that the proposition holds for each one of them. However, showing that the domain is indeed finite is an imperative part of the proof.

Thus, the collection of tasks included statements for which all existing examples are supporting ones, statements that have both supporting and contradicting (or non-confirming examples), and statements which have only contradicting or non-confirming examples. The rationale for such design was to create a rich scope of situations for examining students' understanding of the roles of examples in proving.

While we wanted to ensure that students have the necessary knowledge to approach the tasks, we also wanted to confront the students with statements that were *unfamiliar* to them of which truth-values would not be immediately evident. Such statements have the potential to *evoke uncertainty and doubt*, which are widely recognized as powerful diagnostic tools as well as vehicles for creating situations in which the need to prove arises intrinsically (e.g. Zaslavsky, 2005; Buchbinder & Zaslavsky, 2011; Hadas, Hershkowitz & Schwarz, 2000). We expected that the uncertainty evoked by the tasks would trigger students' discussion and attempts to convince each other through argumentation, which would allow for various aspects of their understanding to be revealed.

The design of our tool (i.e., the collection of tasks) was inspired to some extent by Harel's (2007) DNR principles for learning environments. In particular, the *necessity principle* that emphasizes the importance of evoking students' intellectual need to prove; and the *repeated reasoning* principle that states that students need to practice reasoning in various settings. Thus, the design of our tool aimed at providing multiple opportunities for students to practice reasoning, to reflect on their knowledge of the logical connections between examples and proving, and to enhance their understanding of these connections through resolving uncertainty.

The collection of tasks

The collection includes 6 types of tasks: (1) What kind of example is this? (2) True or false? (3) Always, sometimes, or never? (4) Who is right? (5) Is this a coincidence? (6) Does it exist? For space constraints we cannot go into equal length of detail for each task.

Task 1:What kind of example is this?

This task has two parts, each involving different statements: a false universal or a true existential. Each statement was accompanied by 7 examples of various types. For each part, the students were asked to determine the types of the 7 examples; to construct two additional examples (one confirming and one non-confirming); to decide whether the statement is true or false and to justify the decision.

Figure 2 shows one of the *false universal statements* and the analysis of the examples used for the task.

This type of task was the only one that explicitly calls for examining irrelevant examples and determining their logical status, and that explicitly requires construction of examples that satisfy a given criteria. Note, that the students did not receive any explanation of the framework including its vocabulary prior to engaging in the task.

The Given Statement: The product of any two numbers a and b, for which the sum is positive, is also positive.		Design-Analysis of the examples
$a=2$, $b=8$	is a confirming / contradicting / irrelevant example of the statement?	$x\in D, P(x)$ confirming
$a=-3$, $b=10$	is a confirming / contradicting / irrelevant example of the statement?	$x\in D, \neg P(x)$ contradicting
$a=4$, $b=-11$	is a confirming / contradicting / irrelevant example of the statement?	$x\notin D, \neg P(x)$ irrelevant
$a=-6$, $b=-5$	is a confirming / contradicting / irrelevant example of the statement?	$x\notin D, P(x)$ irrelevant
$a=-\frac{1}{2}, b=\frac{3}{4}$	is a confirming / contradicting / irrelevant example of the statement?	$x\in D, \neg P(x)$ contradicting
$a=\frac{1}{3}, b=7$	is a confirming / contradicting / irrelevant example of the statement?	$x\in D, P(x)$ confirming
$a=\frac{2}{5}, b=-\frac{4}{6}$	is a confirming / contradicting / irrelevant example of the statement?	$x\notin D, \neg P(x)$ irrelevant

Figure 2: Examples and design-analysis of the categorization part of Task 1 for the given statement

Task 2: True or False?

Tasks of this structure are rather common. We adapted this familiar model to fit our mathematical framework. In this task a set of 6 statements were given: 3 universal and 3 existential. Within each type of statement, we included one statement for which all relevant examples are confirming, one statement which has both confirming and non-confirming examples, and one statement for which all relevant examples are contradicting or non-confirming.

Figure 3 presents the structure of the algebraic version of this task. Note that constructing a case for each row is a non-trivial task for the designer.

This task addresses most of the aspects of understanding of the roles of examples with respect to the framework. In order to solve the task correctly, students need to distinguish between the different types of statements and the corresponding types of examples; they also need to construct counterexamples to disprove false universal statements (1 & 3), and confirming examples to prove true existential statements (4 & 5). The task calls for demonstrating the understanding that examples could be insufficient to prove or refute statements (2 & 6), and that in such cases a general logical argument is required. Students may arrive at this conclusion either by constructing and checking examples or by applying algebraic procedures.

The Statement	Type of statement: U / E	Truth value: T / F	Possible types of examples for the statement: Only confirming	Possible types of examples for the statement: Confirming and non-confirming / contradicting	Possible types of examples for the statement: Only non-confirming / contradicting	Type of justification required
Every three numbers *a, b, c* satisfy the equation: $\frac{a}{b+c}=\frac{a}{b}+c$.	U	F		✓		Refutation by contradicting example
The (positive) difference between the squares of two consecutive natural numbers is equal to their sum.	U	T	✓			General proof
Every two numbers *n, m* satisfy the equation: $\frac{1}{m}+\frac{1}{n}=\frac{1}{n+m}$.	U	F			✓	Refutation by contradicting example
There exist four numbers *a, b, c, d* that satisfy: $\frac{a+c}{b+d}=\frac{a}{b}+\frac{c}{d}$.	E	T		✓		Proof by a confirming example
There exists a number $a \neq 1$ that satisfies the equation: $a+\left(1+\frac{1}{a-1}\right)=a\cdot\left(1+\frac{1}{a-1}\right)$	E	T	✓			Proof by a confirming example
There exist three distinct positive integers a, b, c that satisfy: $\frac{a+c}{b+c}=\frac{a}{b}$.	E	F			✓	General refutation

Figure 3: Design structure of the algebraic version of Task 2
Note: E stands for Existential, U for Universal, T for True, and F for False

Task 3: Always, Sometimes, Never

This task is related to the "True or False?" task, with a shift of focus. Given the *propositions* from the statements in Task 2, students are required to determine whether the proposition is always, sometimes or never true (Figure 4). From a logical point of view, a statement involving quantifiers is either true or false. Thus, a universal statement cannot be "sometimes" true, even if both confirming and contradicting examples exist. However, a proposition P(x) can be true for some values of x and false for others. For example the proposition "x is greater than 5" is true for x=3, but false for x=13. Thus, a proposition P(x) can be always, sometimes, or never true (Rosen, 2003).

Figure 4 presents one of the items that were included in Task 3. It is a different take on item 1 that appears in Figure 3.

> Choose the correct answer and explain your reasoning.
> [Note that the letters *a, b, c* represent numbers].
> Is the equation: $\frac{a}{b+c}=\frac{a}{b}+c$ always / sometimes / never true?

Figure 4: An example of Task 3

Task 4: Who is right?

This task has two parts involving one false universal and one true existential statement. Each statement is followed by utterances of five hypothetical students stating their opinion on the truth-value of the statement. The task requires, for each utterance, to determine whether it is correct or not and to justify the decision.

Figure 5 presents the design structure of the Task 4.

Hypothetical student	Given: A False Universal Statement.	Given: A True Existential Statement.	Correctness
Student A	Uses multiple confirming examples to “prove” the statement.	Uses multiple non-confirming examples to "refute" the statement.	×
Student B	Refutes the statement using a counterexample.	Proves the statement using a confirming example.	✓
Student C	Requires multiple counterexamples.	Requires multiple confirming examples.	×
Student D	Maintains that the statement is false but does not accept counterexamples as sufficient. Requires a general argument.	Maintains that the statement is true but does not accept confirming examples as sufficient. Requires a general argument.	×
Student E	Maintains that it is impossible to determine whether the statement is true or false since there are both confirming and contradicting examples.	Maintains that it is impossible to determine whether the statement is true or false since there are both confirming and contradicting examples.	×

Figure 5: Design structure of Task 4

Task 5: Is this a coincidence?

This task presents a hypothetical student's actions and his/her observation based on a single example. The question "is this a coincidence?" invites the students to evaluate the generality of the observed phenomenon in order to determine whether it holds for every relevant case or just for some specific cases, one of which the student examined. Successful completion of the task involves either proving that the described phenomenon is a general one or constructing a counterexample.

> A student chose two fractions: $\frac{1}{2}$ and $\frac{3}{4}$. He generated another fraction by adding the two nominators and the two denominators in the following way: $\frac{1+3}{2+4}=\frac{4}{6}=\frac{2}{3}$. The student observed that the resulting fraction $\frac{2}{3}$ is between the two original ones: $\frac{1}{2}<\frac{2}{3}<\frac{3}{4}$. **Is this a coincidence?**

Figure 6: An algebraic example of Task 5

Note, that there is no explicit requirement in the task to prove any claim. The intention was to draw students' attention to the extent of generality of the phenomenon, and to study the ways in which they deal with the task, focusing on their need to form an assertion and justify it by means of proof or refutation.

Task 6: Does it exist?

This task focuses on existential statements and the roles of examples in determining their validity. The task included four statements, two in algebra and two in geometry worded in the form "Does an object with a certain property exist?" (figure 7). In some ways this task resembles the "True or false?" one, however, in the pilot study we found that the wording of the statements as questions appeared to be more appealing to the students.

Does there exist a triangle with two heights that are perpendicular to each other?
Does there exist a triangle with two angle bisectors that are perpendicular to each other?

Does there exist a pair of natural numbers *a*, *c* that satisfy: $\frac{a}{a+c}=\frac{1}{c}$?

Does there exist a pair of natural numbers *a*, *b* that satisfy:
$a-b[a-b(a-b)] = (a-b)(a-b)(a-b)$?

Figure 7: Examples of Task 6

Concluding remarks

In this paper we offer a holistic approach to task design. Figure 8 illustrates how the conceptual framework that we developed guided the design of a whole collection of tasks that formed a diagnostic tool.

The collection of tasks constituted the research instrument for a study of 10th grade students' understanding of the roles of examples in determining the validity of mathematical statements. This tool was also used for task-based interviews with pairs of students who worked aloud on these tasks. Overall, the rich scope of types of tasks, as well as the specific statements and examples that were chosen for each type of task, proved useful in tapping into students' thinking and enhancing their understandings. Space does not permit to include examples of such manifestations in this paper but they will be included in a presentation.

Type of Statement	**Universal statement** 1, 2, 3, 4-U, 5		**Existential Statement** 1, 2, 3, 4-E, 6	
Goal / **Type of Example**	**To prove**	**To disprove**	**To prove**	**To disprove**
Confirming 1, 2, 3, 4, 5, 6	1, 2, 3, 4-U, 5	1, 2, 3, 5	1, 2, 3, 4-E, 6	1, 2, 3, 6
Contradicting (the universal statement) **Non-confirming** (the existential statement) 1, 2, 3, 4, 5, 6	1, 2, 3, 5	1, 2, 3, 4-U, 5	1, 2, 3, 6	1, 2, 3, 4-E, 6
Irrelevant 1, 2, 3, 5, 6	1, 2, 3, 5, 6	1, 2, 3, 5, 6	1, 2, 3, 5, 6	1, 2, 3, 5, 6

Figure 8: Connections between the overall collection of tasks and the conceptual framework. [Note that the numbers refer to the numbers of the task-types]

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Rotational symmetry: semiotic potential of a transparency toolkit

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In this paper, excerpts of lessons on using tool-based tasks to teach the concept of 'rotational symmetry' were analyzed. Theory of semiotic mediation was adopted as theoretical framework. The analysis focused on how the tasks could (or could not) bring out the semiotic potential of the tool used. We hope that this analysis could provide some insights on how the design of tool-based tasks may enhance the bringing about of semiotic potential of the tools.

Keywords: semiotic potential, rotational symmetry, tool-based task

Introduction

It has been a long history to use concrete manipulative objects to teach mathematics (Dienes, 1960, 1971). Designing tool-based learning tasks becomes popular in recent decades because of the new development of virtual manipulative software such as Geometer's Sketchpad or Carbri. By a task we mean a teacher designed 'thing-to-do' using the tool, either concrete or virtual manipulatives, for students to experience potential mathematical meanings carried by this tool. Empirical studies suggest that teachers can promote the evolution of mathematics knowledge through "orchestrating" tool-based learning tasks and post-task mathematics discussions. (See for example, Jones, 2000; Mariotti, 2002; Falcade, Laborde, & Mariotti, 2007.) We follow Mariotti & Maracci (2012) to use "orchestration" as a metaphor for classroom discussions with the aim "of developing shared meanings, having an explicit formulation, de-contextualized from the artifact [tool] use, recognizable and acceptable by the mathematicians' community" (p.60). In this paper, excerpts of lessons on using tool-based tasks to teach the concept of 'rotational symmetry' were analyzed under the framework of semiotic mediation. The analysis focused on how teacher's orchestration of tool-based learning tasks and post-task mathematics discussions could (or could not) bring out the semiotic potential of the tool. It is hoped that this analysis could provide some insights on how the design of tool-based tasks may enhance the bringing out of semiotic potential of tools.

Theoretical Perspective

The framework of "semiotic mediation" (Bartolini Bussi & Mariotti, 2008) which is rooted from Vygotskian perspective on "social construction of knowledge" is adopted as theoretical framework. A tool (artefact) [5]which carries mathematical meanings can become a "tool of semiotic mediation" by which students can experience the development of mathematical concepts. In particular, a tool of semiotic mediation provides a means to express mathematical ideas. Bartolini Bussi & Mariotti (2008) points out that there is "double semiotic link" between a tool, a task and mathematical knowledge when the tool is used to accomplish a specific task. They further point out that:

> "The main point is that of exploiting the system of relationships among artefact, task and mathematical knowledge. On the one hand, an artefact is related to a specific task … that seeks to provide a suitable solution. On the other hand, the same artefact is related to a specific mathematical knowledge." (Bartolini Bussi & Mariotti, 2008, p.753)

This double semiotic relationship is called the "semiotic potential" of the tool (artefact) (Bartolini Bussi & Mariotti, 2008, p. 754). In the mathematics classroom, teacher plays a crucial role in the process of semiotic mediation. As "the voice of mathematics culture", the teacher guides mathematical discussions which aim at bringing out the semiotic potential of the tool: a progression from students' production of mathematical discourse to mathematical knowledge. Building on this perspective, in Azrarello, Bartolini-Bussi, Leung, Mariotti & Stevenson (2012, pp. 107-108), a model was used to describe the process of semiotic mediation which highlights the classroom dynamic relationship among the tool (artefact), the task, students' productions, the teacher, and mathematical knowledge. The dynamism consists of a network of interactions wherein the teacher uses a tool (artefact) as a mediator between mathematical knowledge and tasks performed by the students. The model highlighted two responsibilities of a teacher: 1. choose suitable tasks based on the tool used; and 2. monitor and manage the process of progression from students' production to mathematical knowledge to be taught.

The Context

The excerpts of lesson episodes chosen for discussion in this paper were taken from a research lesson study carried out in Grade 5 classroom at a Hong Kong primary school[6] based on the Japanese's Lesson Study (Fernandez & Yoshida, 2004) and the Learning Study (Lo, Pong & Chik, 2005) models. The research lesson aimed at improving mathematics teaching through tool-based tasks. Five Grade 5 mathematics teachers worked together over period of five months to design lessons for a selected topic. The topic selected for the research lesson was rotational symmetry. One of the teachers implemented the lesson whereas other teachers observed and evaluated the lesson. A modified lesson was agreed by the teachers and was implemented by another teacher (at another class). This cycle was repeated until all the five teachers have taught the lesson to their own classes. A researcher in charge (one of the authors) acted as a participant observer and gave advice from theoretical

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[5] Some theoretical perspectives such as instrumental approach make distinction between artefact and instrument in order to highlight the differences between the potentialities and the actual uses of the tool respectively. In this paper tool and artefact are regarded as synonyms which refer to both the potentialities and the actual uses.

[6] The authors would like to express their gratitude to the team of mathematics teachers from St Edward Catholic Primary who designed and implemented this Lesson Study.

consideration. All the teacher preparation meetings, lessons and the post-lesson discussion were video-taped.

The Tool and the Task

The objective of the lesson was to introduce to students the idea of a rotational symmetric figure; that is, a figure 'overlaps itself' at least two times during one cycle of rotation. A toolkit was specially designed for this lesson. The toolkit, which we will call the 'transparency toolkit', consisted of blu-tack, push pins, overhead transparency, pen and a Styrofoam board (figure 1). It was designed for the purpose to verify whether a given/constructed figure has rotational symmetry. Students were given a figure (or constructed a figure using given plastic shapes) and the figure was copied on the overhead transparency. The copied figure acted as an identical copy of the original figure and was placed on top of the original figure. Students used the push pin to locate and fasten the position of the centre of rotation. While rotating the transparency, the original figure and the rotated copied figure could be seen at the same time. This synchronic simultaneity enables the concept of rotational symmetry to become visible and manipulative, thus forming a semiotic potential for the transparency toolkit.

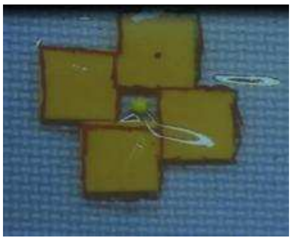

Figure 1: Transparency Toolkit

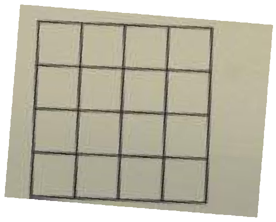

Figure 2: Sheet of square grids

The lesson structure was basically the same among all the five lessons whereas there were slightly differences in details. The teacher started the lesson by giving a brief review on line symmetry through a whole-class discussion and in transition the idea of rotational symmetry was introduced. Then a figure was given to students and they were asked to verify/explore whether the figure has rotational symmetry using the transparency toolkit. The last part was the main part of the lesson. Four identical plastic square pieces were given to the students (see Figure 1). Students (worked in pair) were asked to design rotational symmetric figures using these four plastic squares and used the transparency toolkit to verify their works. Afterwards, a whole-class discussion was led by the teachers along with student presentations.

There were variations among different classes. In one class, a sheet of square grids (figure 2) was used instead of plastic squares. In another class, a 'virtual toolkit' using computer software that was designed to be equivalent to the transparency toolkit was used instead of the plastic squares and the transparency toolkit.

In the following, we will analyze the implementations of the last part (i.e. designing rotational symmetric figures) in two of the lessons (Lesson A and Lesson B) where the transparency toolkit was used. Sheets of square grids were used in Lesson A and plastics squares were used in Lesson B.

Analysis

Part 1: Producing rotational symmetric figures using squares

In Lesson A, square grid sheets were provided for students (in groups) to produce rotational symmetric figures by colouring selected squares. Generally speaking, students found it a difficult task to handle. After guidance from the teacher, some students rotated the square grid sheet and could produce some simple figures with rotational symmetry. In contrast, in Lesson B plastic squares were given to students. The plastics shapes were concrete manipulative which seemed to provide students a more tangible experience to comprehend the meaning of rotational symmetry. It allowed students to rotate individual square piece and consequently could produce more complicated figures. Some students could detect and correct mistakes by rotating the individual square pieces. For instance, a pair of students initially designed an incorrect figure (**Erreur ! Source du renvoi introuvable.**). While rotating (some of) the square pieces, they found that the figure has no rotational symmetry (figure 4). Then they modified the figure into a rotational symmetric one (figure 5). In this self-correction process enabled by the tool, the students (implicitly) experienced the fact that the upper half and the lower half of the figure they produced have the same distance from the centre but in the opposite direction. This may emerge in students' mind an intuitive understanding of the meaning of a 2-fold rotational symmetry.


Figure 3: incorrect figure

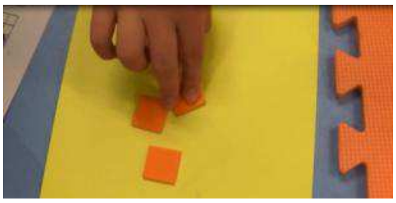
Figure 4: rotate part of the figure

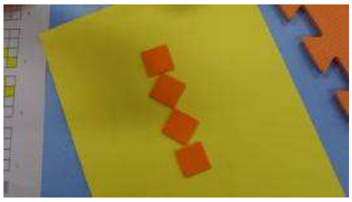
Figure 5: corrected figure

When comparing lessons A and B, it was observed by the teachers and the researcher that the square grid sheet was not as conducive as the plastics square pieces to bring about the (intuitive) meaning of rotational symmetry. What made the difference? One possibility was that square grid sheet could not be separated into different parts. Whenever the whole square grid sheet was rotated, all the individual squares were changed in the same way. This lack of variation in parts in the creation process may result in a limited experience to what are typical in rotational symmetry. In contrast, the plastics square pieces could be manipulated as separate entities and changed with respect to each other in different ways. This opened up wider variation and opportunity to produce more complicated figures. As seen in the 'self-correction' example of Lesson B, students could rotate *parts of* the created figure (the middle two squares). This may lead to discernment of critical features of rotational symmetry and hence an intuitive understanding of the concept may consequently emerge. Thus a shift of attention between the parts and the whole of the object of exploration (the rotational symmetric figure) could occur through manipulation of the square pieces but may not be the case for the square grid sheet.

Part 2: Initiating mathematics discussion through verification with the transparency toolkit

After the above group activity of figure creation, students were asked to verify whether their created figures have rotational symmetry or not by using the transparency toolkit. The teacher then selected a few student groups to report their works in front of the class and then initiated mathematics discussion which aimed to bring out the semiotic potential of the tool. In Lesson A, two groups were asked to report their works by demonstrating the rotational symmetry using the transparency toolkit. The teacher only focused on whether the students' answers were correct or not. It was not clear that the semiotic potential of the tool could be brought out by the teacher led discussion. In contrast, three groups were asked to report their works in Lesson B. The structure of reporting for each group was basically the same. First, the teacher asked the students whether the figure has rotational symmetry and asked the students to locate the centre of rotation (which was the location of the push pin). Then, the students were asked to demonstrate the figure's rotational symmetry using the transparency toolkit. While rotating the transparency, it was observed that the students paid attention to when the original figure (i.e. the figure composed by the square pieces) and the rotated figure (i.e. the figure on the transparency) overlapped. When the two figures overlapped, the whole class counted the number of times of overlapping. After a cycle of rotation, the teacher asked the whole class whether this was a rotational symmetric figure and highlighted the reason; that is, the figures overlapped at least two times in one cycle. It was interesting that the work of the last group initiated further mathematics discussion. The created figure was a 3-fold rotational symmetric figure composed of three square pieces (Figure 6). Although the explanation given by the students was basically correct, the teacher intentionally extended the discussion by pointing out that the original figure and the rotated figure did not overlapped exactly (Figure 7). He proceeded to ask the whole class how to modify the figure in order to make these two figures overlapped exactly. After thinking for a while, a student suggested that the sizes of the angles between each of the adjacent squares should be the same. In order to further elaborate this idea, the teacher compared this figure with a 4-fold rotational symmetric figure which was produced by another group (Figure 8). After a brief discussion, it was concluded that the more number of times of overlapping in one cycle, the smaller the size of the angle between adjacent squares. This property was actually beyond the mathematics knowledge that the teacher intended to teach.

Figure 6: a 3-fold rotational symmetric figure

Figure 7: verify the figure by transparency toolkit

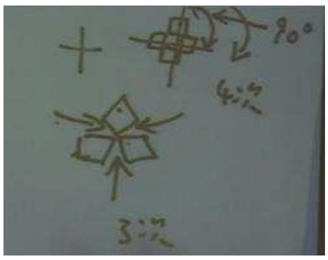

Figure 8: teacher's further elaboration

What could be learnt about tool based task implementation from these lesson episodes? Firstly, as seen in Lesson A, simply focusing on the correctness of students' productions may not bring about the mathematical meaning embedded in the tool and the task. In contrast, as in Lesson B, the teacher highlighted the key concepts associated to rotational symmetry by a well-structured group reporting procedure:

locate the center, rotate the figure on the transparency, count the number of times of overlapping, and lastly, recall prior knowledge, and address the *discrepancy caused by the tool utilization*. In this orchestration process, the semiotic potential of the transparency toolkit as a tool of semiotic mediation is emerged through evaluation of the students' productions by manipulating the tool and speaking out the key concepts explicitly *at the same time.* An interesting double semiotic link occurred when the teacher made use of the tool's 'inaccurate representation' to extend the conceptual understanding of rotational symmetry from merely recognizing a descriptive definition to discerning a critical feature about angle size in rotational symmetry. The occurrence of this discussion could be regarded as an incidental opportunity created by the semiotic potential of the tool. The original task was to create a rotational symmetric figure by using four (identical) square pieces. It happened that one pair of the students did not follow the instruction and used three square pieces instead of four. It can be calculated easily (by us not the students) that the angle between two adjacent square pieces needs to be 30 degrees in order to create a rotational symmetry figure using three (identical) squares. For Grade five students, it is difficult visually to arrange the square pieces so that all three gaps are of 30 degrees. Rotating the figure by the transparency toolkit made this discrepancy explicit which gives rise to an opportunity for the teacher to orchestrate meaningful mathematics discussion with the students. This suggests that a tool-based task which can capitalize the *potential discrepancy embedded in the tool* may bring about a higher level of conceptual understanding. The insight and flexibility of the teacher is a key factor of whether the task can be orchestrated to the emergence of mathematical meaning. In the above teaching episode, if the teacher ignored the discrepancy or he simply told the students that all the four squares should be used, the mathematics discussion would not be as rich and fruitful.

Discussion and Conclusion

In this last section, some lesson episodes on tool-based tasks were analyzed. In the following, we will extend the discussion to propose some suggestions (hypothesis) on the design of tool-based tasks which may enhance the bringing out of semiotic potential of the tools. We will focus on four aspects: 1. the tool, 2. the task, 3. students' productions, and 4. the teacher.

The tool

Semiotic potential of the tool may be enhanced if the tool allows students to freely shift their attentions between the parts and the whole of the object of exploration

This suggestion is consistent to the perspective of discernment and variation (Marton, Runesson & Tsui, 2004). Based on this perspective, Leung (2003) pointed out that "discernment comes about what parts (features) are being focused and temporarily demarcated from the whole (background)" (p.198). For instance, in dynamic geometry environment (DGE), part of the geometric configuration can be varied via dragging while keeping the other parts fixed. The invariant patterns of the configuration could then be "separated-out" (Leung, 2008). The lesson episodes above showed an example of a non-ICT tool. The square pieces (without the blu-tack) allow students to see the object of exploration (the rotational symmetric figure) in parts and hence 'separate' some parts from the whole object. Whereas, the square pieces with blu-tack allow the students to see the object as a whole. This flexibility of the tools may give an opportunity to have a better understanding on the part-whole relationship of the object and discern the essential feature of the object from the incidental one.

The task

A task which provides an opportunity to make use of the discrepancy embedded in the tool may initiate meaningful mathematics discussion which could lead to deeper conceptual understanding.

Making mistakes may not be a bad thing. Meaningful mistakes may initiate mathematics discussion which leads to construction of mathematical knowledge. Some tasks may have higher possibility to make (meaningful) mistakes. For instance, in the lesson episodes above, creating a rotational symmetry figure by using three (identical) square pieces is easier to make ‘mistakes’ than by using four (identical) square pieces. Though the ‘mistake opportunity’ occurred in our lesson episodes was rather incidental, we suggest that the ‘mistake opportunity’ should be planned together with the task design. In order to make these mistakes meaningful to the students, the mathematics implied in the ‘mistakes’ should be considered beforehand. The tool used in a task may influence the chance of making mistakes. For instance, if instead of concrete square pieces, an ICT tool (e.g. DGE) was used for the task described in this lesson episode, it was unlikely that the ‘mistakes’ discussed above would occur because drawing in DGE is usually accurate (at least the discrepancy cannot be discerned visually). However, there may be other discrepancies in DGE that are conducive to rich mathematical discussion. (Some empirical studies found that sometime students deliberately drew a ‘wrong’ picture in DGE in the process of exploration and argumentation. See for example, Leung & Lopez-Real (2002); Mariotti & Antonini (2009); Baccaglini-Frank, Antonini, Leung & Mariotti, (2011).)

Student’ productions

Requiring the students to evaluate their productions using the tool while simultaneously speaking out the key mathematics concepts with respect to the tool may effectively mediate the progression from students’ production to mathematics knowledge.

The progression from students’ production to mathematics knowledge is not automatic. Teacher, as an expert representative of mathematics culture, needs to use students’ production as “a tool of semiotic mediation” (Bartolini Bussi & Mariotti, 2008). For instance, Mariotti and her colleagues conducted a series of long term teaching experiments in DGE to develop students’ mathematical thinking in axiomatic approach (Mariotti, 2002). Students were mediated to experience the development process of mathematical theory by Cabri (a DGE) commands. There is a correspondence between these commands and the elements of mathematical theory (definitions, axioms and theorems). Indeed, the ‘labels’ of the Cabri commands make the correspondence explicit. However, for non-ICT tools, such correspondence may not be explicit because this kind of ‘labels’ may not exist in the tools. It is the role of teachers to make the correspondence explicit. The teachers in the lesson episodes made use of a well-structured orchestration procedure which involves students’ evaluation of their own productions by manipulating the tool and speaking out the key mathematics concepts explicitly at the same time to achieve this ‘labelling effect’. In this orchestration, the teacher deliberately highlighted the mathematics terminologies realized in the semiotic potentials of the tool. For example, the push pin corresponds to the centre of rotation. The correspondence of the students' productions and the mathematics knowledge may be seen through this process of "situated abstraction" (Noss & Hoyles, 1996).

The teacher

Developing teacher's ability to determine whether student production (correct or incorrect) is conducive to conceptual learning is an important aspect in training teachers to use tool-based tasks in the mathematics classroom.

Teachers play a significant role in the process of semiotic mediation. As an expert representative of mathematics culture, the teacher guides the evolution of mathematical meanings related to the tool and its use within the mathematics classroom (Bartolini Bussi & Mariotti, 2008). The lesson episodes above suggested that the quality of the guidance depends on the insight and flexibility of the teacher. In the second part of Lesson B, the teacher made use of the students' incorrect (or not-so-correct) production to extend the mathematics discussion so that deeper conceptual understanding of rotational symmetry was evolved. This was incidental in the sense that it was out of the teacher's original planning. Certainly, not all students' mistakes are equally worthwhile for extending the discussion. Therefore, a teacher's ability to determine whether a student's (mistaken or unexpected) production (or response) is worthwhile for further discussion is one of the criterions of success to bring about the semiotic potential of a tool-based task. We suggest that developing such ability is an important aspect in training teachers to use tool-based mathematics tasks.

In this paper, four suggestions on tool-based tasks are proposed according to our analysis of the design and implementation of a tool-based geometry task at primary school level. More empirical studies will be conducted in order to verify the validity of our suggestions.

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Designing tasks within a multi-representational technological environment: An emerging rubric.

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The advent of digital technological environments that offer linked multiple mathematical representations present a substantial challenge to secondary mathematics teachers as they begin to design tasks for their learners. This paper concludes a design rubric for mathematical tasks as one of the outcomes of a longitudinal doctoral study in which fifteen teachers began to integrate such a tool (TI-Nspire handheld and software) within their classroom practice (Clark-Wilson, 2010a). The research, which was carried out in two phases, provided evidence for the predominant use of the technology to support explorations of variance and invariance. It also gave an insight into the process through which the teachers integrated the technology into their classroom practices. This paper offers a joint perspective of the project from the mathematics education researcher (Clark-Wilson) alongside one of participating teachers (Timotheus).

Keywords: Hiccup, instrumental genesis, mathematics education, mathematical generalisation, multiple representational technology, task design, teacher development, variance and invariance.

Introduction

The research reported within this paper was situated within a pilot evaluation project that introduced a prototype technology 'TI-Nspire' (Texas Instruments, 2007) to a group of English secondary mathematics teachers between July 2007 and November 2009. The project community comprised educational technology developers (Texas Instruments), a mathematics education researcher (the first author), mathematics education consultants and secondary mathematics teachers (including the second author). The first author was solely responsible for the design, implementation and reporting of the project and the second author was one of the participating teachers. In common with other teacher development projects, with and without ICT, there was a clear rationale underpinning the approach taken to establish the ethos and ways of working within the project (Ahmed, 1987; Watson, De Geest, & Prestage, 2003). There was a clear sense of 'researching-with', which resulted from privileging the teachers' classroom stories and by supporting teachers to substantiate their own claims of changed mathematical experiences and outcomes on the part of their students. Within the research, the word 'activity' was used to describe the set of situations that the teacher had designed to initiate the students' mathematical work. These activities would include one or more 'tasks' and the outcomes of these tasks were subsequently described as 'episodes' within the teaching sequence. The broad aims for the study were to try to articulate the nature of the teachers' cognitive and pedagogical learning as they began to use the aforementioned multi-representational technology in their classrooms with students.

Theoretical perspective

The research was framed within an activity-theoretic approach that interprets the Vygotskian notion of activity as a 'unit of analysis that included both the individual and his/her culturally defined environment' (Wertsch, 1981). The elaboration of the instrumental approach within technological environments (Verillon & Rabardel, 1995) was developed further by a body of researchers within the context of technology use in mathematics education, leading to the notions of instrumentation, instrumentalisation, instrumental genesis, instrumental orchestration and documentational genesis (Drijvers, 2011; Drijvers & Trouche, 2008; Gueudet & Trouche, 2009; Guin & Trouche, 1999; Trouche, 2004). These ideas concern the complex and interrelated processes of:

- learning to use a new technology for purposeful mathematical activity;
- designing tasks for students to initiate purposeful mathematical activity;
- collating the various artefacts that comprise the 'document system' for the activity;
- supporting students to learn to use technology for purposeful mathematical activity;
- articulating the teacher's role in supporting the students to navigate their respective routes through the various artefacts that comprise the activity to include interaction with the technology.

It is important to comment that within the research contexts from which these notions emanate, the chosen technologies are best described by Pierce and Stacey (2008) as 'mathematical analysis tools', which include technologies such as computer algebra software (CAS), dynamic geometry software (DGS), graphing software and spreadsheet software. The TI-Nspire technology adopted within this research afforded a range of 'applications' that included: calculator; spreadsheet; dynamic geometry; function graphing; statistical calculation and graphing; built in commands i.e. factor(n); and text editing. In all of these environments, the facility to save numeric outputs as variables supported the linking of variables within and between these different applications.

A second important theoretical construct of significance to the study was that of a multiple representational environment, which was postulated initially by Kaput (Kaput, 1986) as he presented a vision for the way in which technology might support higher–level engagement with mathematics. In the intervening years, the aforementioned genres of technologies have afforded opportunities to engage with mathematics dynamically by observing the simultaneous views of different representations. For example, the representations of a function, its graph and a table of its associated coordinate values. The development of 'dragging' a figural image through the interface of a mouse (or pen or finger) has afforded further forms of mathematical interaction.

Finally, the underlying principle that informed the pedagogical approaches within the study were informed by Mason et al's assertion that 'a lesson without the opportunity for learners to express a generality is not, in fact, a mathematics lesson' (Mason, Graham, & Johnston-Wilder, 2005, p.297). This promoted the use of the technology to provide the students with opportunities to explore variance and invariance within a range of mathematical contexts.

Research methodology

In concluding their research study into teachers' perceptions of the use of technology, which involved secondary mathematics departments in seven English schools, Ruthven and Hennessy (2002) , highlighted the need for,

> naturalistic studies which provide analyses of the perspectives and practices of groups of teachers across different settings within the context of using technology to support teaching and learning of mathematics. (p.50)

Although this study cannot claim to be wholly naturalistic, as the technology had been prescribed, it did aim to gain a deep insight into the perspectives and practices of secondary mathematics teachers as they developed their use of the MRT in the classroom.

The research was carried out in two phases, the first between July 2007 – November 2008 and the second between April 2009 and December 2009. In each of these phases, a group of teachers was selected and a series of methodological tools developed to capture rich evidence of their use of the technology in classrooms to enable the aims of the study to be realised. The methodological approaches for each phase were distinct, and each is described briefly below.

Phase 1

Fifteen teachers were selected (in pairs) from seven English schools where there was some history of use of technology use within mathematics, although some of the individual teachers had limited personal experience. During the first phase of the research (July 2007 to November 2008), the teachers reported 66 lesson activities that had used the TI-Nspire technology, which resonate with Gueudet and Trouche's 'document systems' (2009) comprised a number of artefacts that included some or all of the following:

- an activity plan in the form of a school lesson planning proforma or a hand-written set of personal notes;
- a lesson structure for use in the classroom (for example a Smart NoteBook or PowerPoint file);
- a TI-Nspire software file developed by the teacher for use by the teacher (to introduce the activity or demonstrate an aspect of the activity);
- a TI-Nspire software file developed by the teacher for use by the students, which would normally need to be transferred to the students' handhelds in advance or at the beginning of the lesson;
- a activity or instruction sheet developed by the teacher for students' use;
- students' written work resulting from the activity;
- students' TI-Nspire software files captured during and/or at the end of the activity;
- a compulsory post-activity questionnaire, which prompted the teacher to reflect deeply on the lesson(s). (Clark-Wilson, 2008)

This data was used to analyse each of the lesson activities with respect to the teachers' planned instrument utilisation schemes and, as a result nine diagrammatic 'instrument utilisation schemes' were developed. For more detail, see Clark-Wilson (2010b).

The analysis of the phase 1 data led to 268 thickly coded examples of 'teacher learning' from 100 data sources and a process of constant comparison led to 38 categories that were subsequently grouped into four domains: activity (task) design, expectations of students, instrument utilisation schemes and meta-level ideas. As this paper is concerned with the first theme, the broader description of the categories related to task design included aspects concerning:

- the teacher's initial choice of examples, its appropriateness and potential for non-trivial mathematical exploration and extension;
- the features of a good 'first activity' when introducing the technology or new instrumental techniques to the students;
- the need to balance the construction and exploration elements within the design of activities;
- teachers' reflections on their personal instrument utilisation scheme when introducing the activity to the students and during whole class discussions;
- the relationship between the students' learning of relevant mathematical concepts with technology and the traditional 'by hand' or 'paper and pencil' approaches - and reflections surrounding the teacher's role in supporting students to connect these experiences;
- an appreciation of the need for a tighter focus on or attention to specific mathematical generalisations within activities.

The first phase of the project revealed rich data concerning teachers' perceptions of explorations of regularity and variation, which seemed to substantiate Stacey's claim that, from the Australian research, this was the most common form of mathematical activity with technology (Stacey, 2008). There was less evidence of activities that made use of opportunities to link representations. However, the analysis of the teachers' developing instrument utilisation schemes suggested that this could be a function of both teachers' time and familiarity with the technology.

Phase 2

The second phase sought to probe the nature of teachers' situational learning through a focussed and systematic case study of two of the teachers who had participated in the first phase (Eleanor and Tim). These two teachers had a level of technical competency that showed that they had grasped the skills needed to create activities using a range of applications within the MRT. The second consideration was that the teachers adopted pedagogical approaches that placed the students' mathematical experiences at the centre of the classroom environment. By allowing the students more mathematical choices within the activities they developed, they adopted a more socio-constructivist philosophy in which their students could form their own mathematical meanings with a collaborative, supportive classroom ethos. My rationale for this choice was related to my desire for the study to generate new knowledge about the way that teachers conceive and learn from their own innovations with complex new technologies.

The data collected as part of the teacher's 'document system' during phase one was supplemented by the following:

- classroom observation data (video, audio and field notes);

- additional data relating to the teacher's interactions with students, for example, a software file at the time of the interaction;
- data resulting from interviews with the teachers, which included pre- and post lesson discussions;
- field notes, which incorporated extracts from email exchanges with the teachers.

A further 14 lessons were observed between May 2009 – December 2009 and for each one, the data set was imported into Nvivo 8 software to support the subsequent data coding and analysis processes (QSR International, 2008). The analysis of the phase two data concluded the important notion of the lesson 'hiccup', that is the perturbation experienced by teachers during lessons stimulated by their use of the technology, which illuminated discontinuities within the teachers' knowledge. The cross-case analysis attributed the hiccups to seven considerations: Aspects of the initial activity design; Students' interpretations of the mathematical generality under scrutiny; Unanticipated student responses as a result of using the MRT; Perturbations experienced by students as a result of the representational outputs of the MRT; Instrumentation issues experienced by students when making inputs to the MRT and whilst actively engaging with the MRT; Instrumentation issue experienced by one teacher whilst actively engaging with the MRT; and Unavoidable technical issues. However, as this paper is concerned with the subsidiary research findings concerning the emergent principles for task design within MRT environments and the tool-driven implications for the design, teaching and learning triad the discussion that follows will focus on these aspects.

What did the teachers learn about designing tasks within the MRT?

A central tenet of this thesis was that activities and approaches that privileged an exploration of mathematical variance and invariance constituted a legitimate pedagogical opportunity for the use of technology in secondary mathematics classrooms. The teachers' evolving conceptions of variance and invariance were revealed in several ways during the study. This included how they conceptualised the initial representation of the variant property under exploration and the related representations that added insight to, or progressed, the exploration and the way in which any invariant properties would become explicit as a result of the exploration.

Eleanor and Tim learned that there were several important aspects when making the choice of initial example within their activity design and its subsequent representation within the MRT. Firstly, it needed to be an example of the generalisation being sought. A hiccup that Tim experienced during the activity 'Linear equations' led him to conclude that his choice of functions for the initial example, which was a spontaneous decision, was unhelpful. His selection of y=1x+1 and y=2x did not support the students to begin to generalise about the gradients of the linear functions, partly because there were two digit ones inherent in y=1x+1, but also due to the way that the MRT displayed the measured equations[7] (see Figure 3).

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[7] This hiccup was compounded by the output from the MRT which displayed an unhelpful 'measured' equation. (A measured equation is one which for which the MRT numerically analyses a set of points that define a geometrically constructed line, often revealing unexpected numerical errors.)

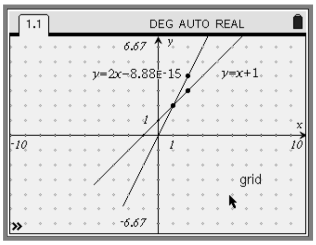

Figure 3 Tim’s handheld screen during the activity ‘Linear graphs’ [STP10(tns-T)].

Eleanor and Tim both seemed to grapple with the notion of the ‘example space’ in ways that were resonant with Goldenberg and Mason’s elaboration of the difficulties encountered when trying to identify appropriate examples for different mathematical generalisations (Goldenberg & Mason, 2008). Tim’s choice of functions in the ‘Linear equations’ activity could be interpreted as him having a ‘general sense’ of the representation of any two straight lines on a dotty grid alongside the functionality of a ‘gradient measure’ tool as a sufficient space in which to be able to generalise about slope. However, a more informed choice of functions, alongside the utilisation of the MRT’s functionality to measure the slope of the line, rather than its equation, all formed aspects of Tim’s redesign of this activity.

Having decided upon the initial input representation, another important consideration for the teachers related to their decisions about the syntactic labelling or notation of variable objects. They learned that this was a necessary element of activities when classroom discourse would be initiated to support students to notice and verbalise their generalisations. As both teachers privileged whole-class discourse focussed on key mathematical generalisations in their classroom practices, they both experienced a number of hiccups that indicated that there was insufficient notation or labelling to enable this discussion to be suitably focussed. This is an aspect that Pierce and Stacey have noted in their recent research into the way that Australian secondary mathematics teachers are developing the use of the TI-Nspire handhelds to privilege linked mathematical representations when teaching about quadratic functions (Pierce & Stacey, 2009). They have highlighted the conflict between the labelling of variables within the ‘ideal’ or pencil and paper mathematics alongside the conventions built into the MRT, as an important element for teachers to consider in the design of tasks.

A final consideration was the development of the teachers’ knowledge concerning the grouping or sequencing of examples, that is, determining the scale and structure of the example space. This aspect of their learning not only related strongly to their personal mathematical knowledge and interpretation of the mathematical progression of ideas within the topic matter, but was also intrinsically linked to the pedagogic skill of considering the most suitable activity structure for their students.

In general, having selected an initial example for each activity, Eleanor and Tim expanded the example spaces to add insight to, or support the analysis of the variant and invariant properties under exploration in one, or a combination of the following approaches:

- Further examples were explored within the same mathematical representation by making new inputs in the same numeric, syntactic or geometric form.
- Further examples were explored within the same mathematical representation by dragging objects dynamically.
- Further examples were explored within a different mathematical representation by making new numeric, syntactic or geometric inputs.

- The same example was explored within a new mathematical representation.

Conclusion

The scrutiny of over eighty classroom activities within this study has provided a substantial data set on which to draw a number of conclusions relating to their experiences of designing tasks within the chosen MRT environment.

Mason et al's premise regarding the role of expressing generality within mathematical activity could be considered to have been an initial constraint for the teachers involved in this study when designing their activities to explore variance and invariance (Mason et al., 2005, p.297). However, an element of the teachers' epistemological development was related to their realisation that expressing generality was a very important aspect of the activities that they went on to design. The teachers' increasing attentions to the way that the MRT environment supported or hindered this, and the design of the associated supporting resources and their role in mediating the associated classroom discourse, was another element of their professional development.

Evidence from the study suggests that the process of designing tasks that utilise the MRT to privilege explorations of variance and invariance is a highly complex process which requires teachers to carefully consider how variance and invariance might manifest itself within any given mathematical topic. The relevance and importance of the initial example space, and how this might be productively expanded to support learners towards the desired generalisation is a crucial aspect of activity design.

The starting point for any classroom activity is its initial design and the following set of questions, generated as a result of this study offer a research-informed approach:

- What is the generalisable property within the mathematics topic under investigation?
- How might this property manifest itself within the multi-representational technological environment – and which of these manifestations is at an accessible level for the students concerned?
- What forms of interaction with the MRT will reveal the desired manifestation?
- What labelling and referencing notations will support the articulation and communication of the generalisation that is being sought?
- What might the 'flow' of mathematical representations (with and without technology) look like as a means to illuminate and make sense of the generalisation?
- What forms of interaction between the students and teacher will support the generalisation to be more widely communicated?
- How might the original example space be expanded to incorporate broader related generalisations?

Responses to these questions uncover a generic 'top level' of thinking which makes little sense in the absence of a clear mathematical context. The next level of thinking is closely related to the mathematical concept under scrutiny for which a number of existing structures and approaches can support teachers to develop further this aspect of their practice.

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Digital design: RME principles for designing online tasks

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The theory of Realistic Mathematics Education provides principles that can be applied to task design. In this paper, we investigate how these principles apply to the design of online tasks. To do so, we present examples of tasks on algebra, calculus and geometry designed in the Digital Mathematics Environment. As a result, we conclude that the principles of guided reinvention, didactical phenomenology, and emergent modeling can inform and guide digital design, but that some aspects work out differently compared to the design of paper-and-pencil tasks.

Keywords: Didactical Phenomenology, Digital Design, Digital Mathematics Environment, Emergent Modeling, Guided Reinvention, Realistic Mathematics Education

Introduction

Task design is widely recognized as an important, but complex and subtle activity. Based on the experience of skilled designers, design guidelines and heuristics have been identified (e.g., see Watson & Mason, 2006). Whereas the design and use of digital content nowadays plays an increasingly important role in mathematics education, most of these design principles are based on and applied to the design of paper-and-pencil tasks. The question, therefore, is how such design principles apply to digital design, and how the rich experience and knowledge in the field of designing paper-and-pencil tasks can be transferred to the case of digital design.

To address this question, we limit ourselves to three principles that emerge from the theory of Realistic Mathematics Education and may inform design: guided reinvention, didactical phenomenology, and emergent modeling. We revisit some tasks designed in the Freudenthal Institute's Digital Mathematics Environment in the field of algebra, calculus and geometry education.

RME Design Principles

Realistic Mathematics Education (RME) is a domain-specific instruction theory for the teaching and learning of mathematics. According to RME, mathematics should be seen as an activity (Freudenthal, 1973), and students, rather than being receivers of ready-made mathematics, should be active participants in the educational process, in which they develop mathematical tools and insights by themselves. This point of departure led to the following RME principles that inform task design: guided reinvention, didactical phenomenology, and emergent modeling (Freudenthal, 1973; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996).

According to the principle of *guided reinvention,* students should be given the opportunity to experience a process similar to that by which a given mathematical topic was invented. Even if this primarily is a teaching principle, it has consequences for task design: tasks – or sets of tasks – should invite students to develop 'their own' mathematics. This process, however, needs guidance from the teacher, to help to further develop sensible directions, to leave 'dead-end streets' and to ascertain convergence towards shared knowledge according to the standards within the mathematical community.

Didactical phenomenology concerns the relation between the thought object −the 'nooumenon'− and the phenomenon –the 'phainomenon'− from the perspective of teaching and learning. In particular, it addresses the question how mathematical 'thought objects' can help in organizing and structuring phenomena in reality. The challenge for the task designer, then, is to find such meaningful phenomena that beg to be organized and structured by the targeted mathematical knowledge.

According to the *emergent modeling* perspective, a model may play different roles during different phases of activity. Initially, a model is context-specific: it refers to a meaningful problem situation that is experientially real for the student, and is a model of that situation. Then, through working with the model, it gradually acquires a more generic character and develops into a model for mathematical reasoning that is possible because of the development of new mathematical objects in a more abstract framework of mathematical relations that the model starts to refer to. This notion is elaborated into a four-level structure that represents levels of mathematical activity (Gravemeijer, 1994). For the task designer, the challenge is to find suitable situations that ask for the development of such models, and allow for a process of progressive abstraction.

While applying these RME principles to the design of paper-and-pencil tasks is not straightforward, their use for the design of digital content is even more challenging. Although early research on the use of graphing calculators identified opportunities for an RME-based teaching approach (Drijvers & Doorman, 1996), later studies describe the tension between RME principles and the integration of computer algebra software (Drijvers, 2000). Apparently, the match between RME and ICT is not self-evident.

The Digital Mathematics Environment

As technology for teaching mathematics the Freudenthal Institute's *Digital Mathematics Environment* (DME) is used. The DME integrates a content management system, a learning management system and an authoring environment. The content consists of online modules in the form of Java applets. The learning management system offers means to distribute content among students and to monitor student progress.

The authoring tool is the DME's design environment. Authors, such as teachers, text book authors, or educators, can use the tool for adapting existing online modules or for designing new ones, based on existing materials and basic tools such as graphing and equation editing facilities. While designing, the author can split up the screens in different windows, add applications and tasks, and design feedback. Knowledge of the underlying Java programming language is not required; rather, an intuitive and mathematical interface makes the digital design accessible to a wide audience (Figure 1).

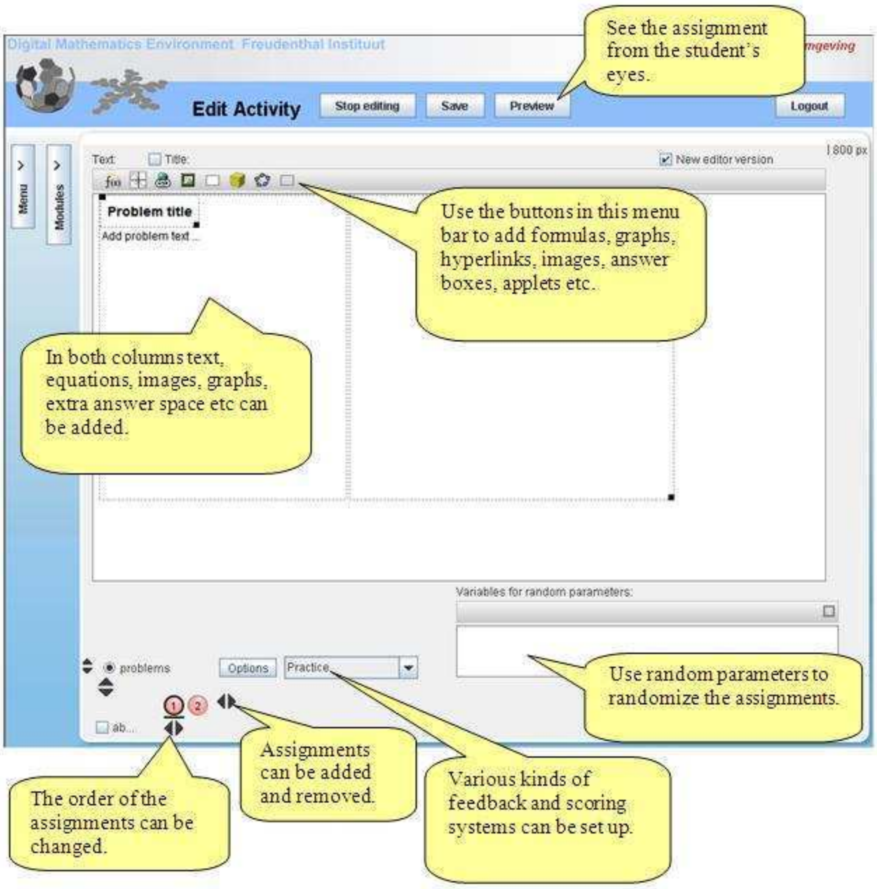


Figure 1 The Digital Mathematics Environment Authoring Tool

Examples of the results of digital design using the DME's authoring tool can be found at www.fi.uu.nl/dwo/demo/en. In this paper, we will briefly discuss some examples from algebra, calculus and geometry from the perspective of the above RME principles.

Digital Tasks for Algebra

As an example of the design of digital tasks for algebra, we consider the work by Bokhove, available at www.fi.uu.nl/dwo/voho/. The digital tasks focus on solving polynomial equations. However, the online units go beyond procedural practice and also focus on the development of symbol sense and strategic skills (Bokhove & Drijvers, 2010, 2012). Crucial factors in the design are the sequencing of the tasks (with sometimes thought-provoking equations) and the design of feedback, and its timing and fading in particular.

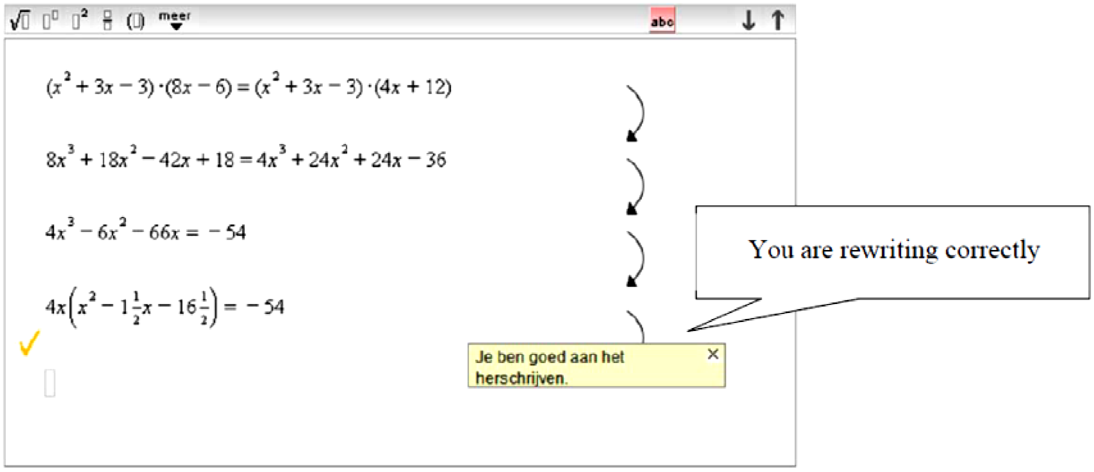


Figure 2 Solving an equation with unexpected difficulties (Bokhove & Drijvers, 2010, 2012)

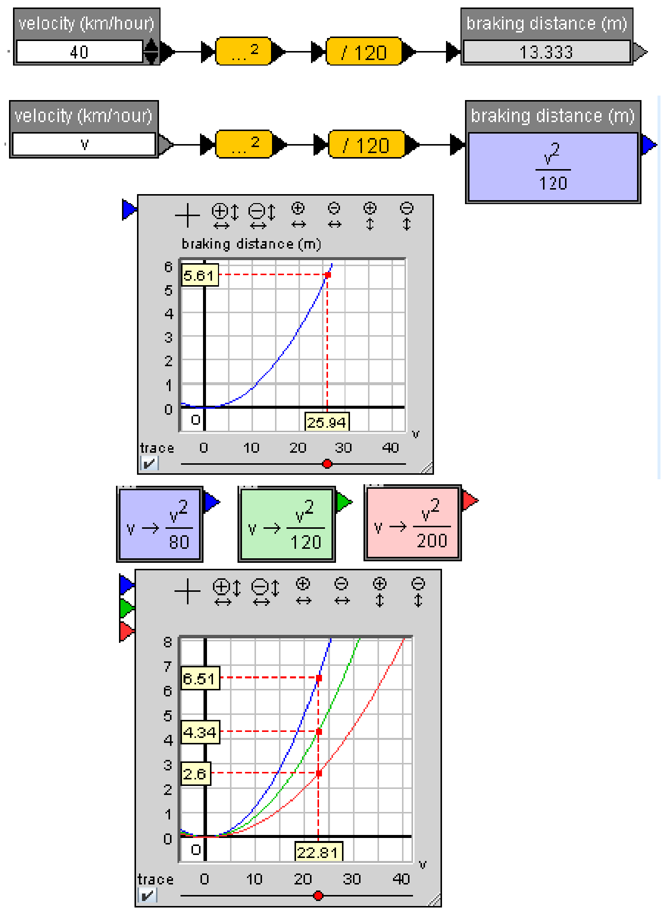
Figure 2 shows an exemplary equation in line 1, and a part of a student's work in lines 2-4. The feedback just refers to the algebraic equivalence of the subsequent equations the student enters; no feedback is provided on the problem solving strategy. This can be seen as a manifestation of the guided reinvention heuristic: this combination of task and tool provides the opportunity for students to reinvent efficient ways of solving equations. Eventually, students can continue entering equivalent equations without coming closer to the solution, but this exploration space is supposed to elicit a wish for efficiency.

From the didactical phenomenology perspective, this task may seem quite poor: what is the phenomenon at stake that would motivate students to engage in mathematics? In defense of this, one can argue that the target group of this online module consists of students in grade 12, who were to do the national examination soon, and who were familiar with the 'world of polynomial equations'. This familiarity makes that these tasks can be appropriate for the mathematization of the field and the development of new problem solving strategies.

Data shows that students work easily with this type of online tasks (Tacoma, Drijvers, & Boon, 2011). Instrumental genesis, the process of developing schemes to use tools to solve the tasks (Artigue, 2002), was an issue to a much lesser extent than in the case of computer algebra (Drijvers, 2002; Drijvers et al., 2012). Apparently, using CAS puts higher demands on instrumental genesis, and this is something to take into account as a designer.

Digital Tasks on Functions and Calculus

As an example of the design of digital tasks for (pre-)calculus we consider the online module Function and Arrow Chain, available at www.fi.uu.nl/dwo/prootool/en. The digital tasks focus on the development of conceptual understanding of the notion of functions, where a function is seen as an input-output assignment, as a dynamic process of co-variation, and as a mathematical object with different representations (Doorman et al., 2012). A crucial factor in the design is sequencing the tasks so these function models emerge in a natural way of increasing complexity and abstraction.



A function as an input-output assignment: braking distance as a function of velocity

Investigation of the co-variation of velocity and braking distance through tracing the graph

Investigation of a family of functions, representing braking distances for three different vehicles

Figure 3 Different views on function in the Function and Arrow Chain module

Figure 3 shows some screens from this module, in which the function gradually develops from a numerical input-output engine, to a process of co-variation and, finally, a mathematical object that is part of a family of functions that can be compared. This sequence of screen shots reflects the emergent modeling heuristic: the context, in this case one of a vehicle's braking distance as a function of its velocity, leads to function models of increasing complexity and abstraction. The digital tool supports this development by means of offering techniques of increasing richness and an increasing repertoire of connected function representations. Of course, this approach requires a context that is suitable for this emergent modeling process. Finding such a context is a question of didactical phenomenology.

In an online calculus course for university freshmen, the co-variation idea is supported by a Geogebra applet for tracing graphs (www.fi.uu.nl/dwo/sk/en/). However, in the context of this intensive and short-period remedial course, the emergent modeling design heuristics was exploited to a lesser extent than was the case for the Function and Arrow Chain course.

Digital Tasks for Geometry

An example of the design of digital tasks for geometry is the module available at www.fi.uu.nl/dwo/dpict/en/. The digital tasks focus on exploring, discovering, and proving properties of bisectors, altitudes and medians in triangles. The latter aspect, the proving, provides a particular design challenge: how to design tasks that offer support and guidance for a proof, but leave room for reinvention?

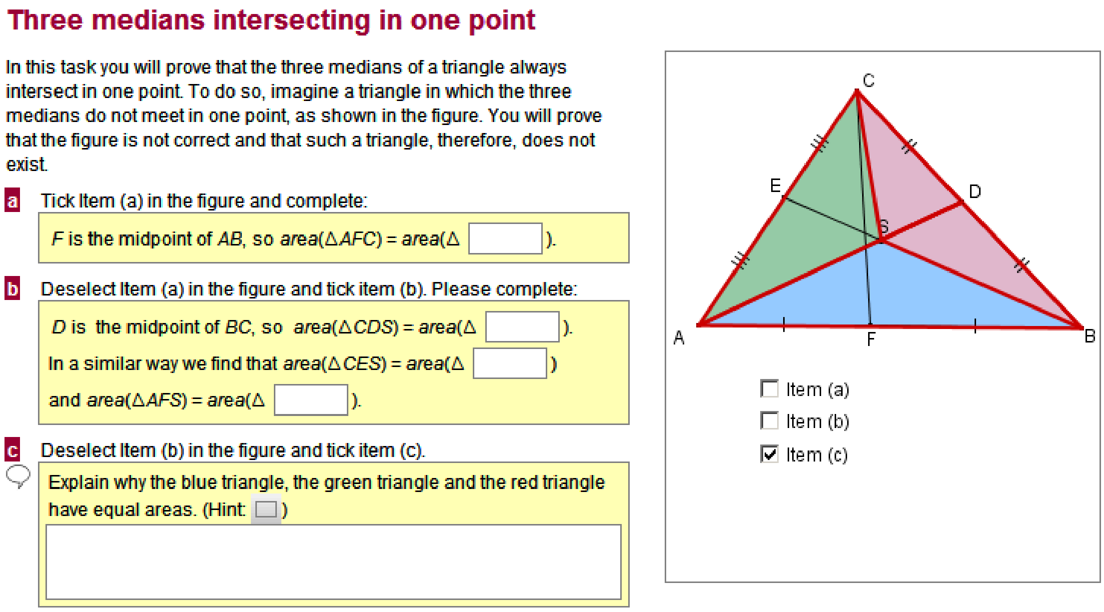


Figure 4 Proving that three medians of a triangle intersect in one point

While reading the proving task shown in Figure 4, it should be noted that Dutch grade 8 students, the target group for this module, have little experience in proving, and, therefore, need strong guidance: the structure of the proof and the corresponding sequence of images is suggested to the students. Still, writing down the final argument in the reasoning, as is requested in task c, is very difficult to them. The design principle of guided reinvention is easier to apply to tasks in which students explore the properties of bisectors, altitudes and medians in triangles: the dragging options of the dynamic geometry system, in this case Geogebra, in collaboration with DME's feedback, provide a strong learning environment in which students can really experience the geometrical situation and discover the targeted properties.

In the case of this online module, classroom observations show that attention needs to be paid to students' and teachers' instrumental genesis: the interplay between Geogebra and the DME in this module is powerful but it may also be demanding and initially complex to novice users.

Conclusion

In this short paper, we set out to investigate the application of the RME principles of guided reinvention, didactical phenomenology, and emergent modeling for the case of the design of digital tasks in the online DME. To do so, we considered three exemplary tasks. The first example, the task on solving a relatively complex equation, shows that guided reinvention in this case concerns the development of new problem solving strategies, invited by tasks that cannot be solved with the strategies available so far. The didactical phenomenology heuristic here does not lead to the use of real life contexts, but rather takes the world of polynomial equations as a point of departure. In the second example, a real life context –in this case the stopping distance situation– does form the starting point. Emergent modeling heuristics are manifest in the gradual abstraction of the students' view on function. The third example on geometrical proof again takes a guided reinvention perspective, in which the designers chose strong guidance. Again, the didactical phenomenology here does not lead to a real life problem situation, but to a problem in the world of geometry, that is expected to be experientially real to the students.

What can we conclude about the three RME principles and their application to digital design? The principle of guided reinvention seems to apply well to digital design. ICT offers opportunities for exploration and investigation, and in this way for reinvention. The design choice to confront students with unexpected examples can be seen as a way to invite reinvention as well. In general, this guided reinvention approach might suffer from constraints of the technology that may limit the students' exploration space, such as requirements for input formats and styles, and pre-designed tools that may incorporate too much guidance.

The didactical phenomenology heuristics is also valuable for digital design, but it seems that the phenomena that play a central role in the task do not necessarily come from real life: ICT already forms a meaningful 'world' on its own for the student, in addition to the world of mathematics. Having a real life context as an entrance to these two worlds may lead to cognitive overload. This is a question to be considered carefully in the design process.

Emergent modeling can be a fruitful design heuristics for digital design. As in the case of paper-and-pencil design, the models need to lend themselves for further development towards increasing mathematical abstraction and complexity. Specific for the case of digital design is that these emerging models need to be supported by the digital tools available, for example by an increasing repertoire of representations and techniques in the digital environment, or by increasing options to dynamically use these representations, connect them, and switch between them (Duval, 2006).

As an overall conclusion, this brief exploration of the issue suggests that the three RME principles are valuable for digital design, even if some appropriation is needed compared to the design of paper-and-pencil tasks, such as taking into account the constraints of the digital tools and the fact that the technological environment forms an additional 'world' to the student. The transfer of skills, developed in the ICT environment, to paper-and-pencil, for example, may need specific attention, as well as the instrumental genesis involved in the learning process.

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Designing tasks for a more inclusive school mathematics

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In this contribution to Theme A (tools and representations), we detail our approach to designing tasks to be incorporated into inclusive mathematics learning scenarios. These scenarios also involve tools created to represent mathematical knowledge in forms appropriable by students with sensory disabilities and which are developed to privilege multimodal experiences of mathematical objects, relationships and properties. We begin by introducing the theoretical influences which underpin the processes of task design and our attempts to take into consideration the complex relationships between artefacts, their mathematical affordances and the embodied practices they engender in the context of task resolution. We go on to outline the collaborative approach we adopt to simultaneously develop both tasks and tools, and how teachers, students and researchers bring different, complementary, expertise to this process. To further illustrate our approach, we consider two examples from our work with blind learners and deaf learners.

Keywords: Collaborative design, Inclusion, Blind learners, Deaf learners, Tool mediation, Embodied cognition.

Introduction

In recent years, Brazil has experienced large changes in the educational paradigm. One of these relates to the growing influence of political and social movements that defend inclusive education, and the organization of schools to attend the diverse needs of all students, without any kind of discrimination or segregation. Inclusive schools are those which see difference as a factor which enriches the educational process. They aim to support all learners in overcoming barriers to learning as they become participants in a more equitable system. The political policies related to the process of including students with special educational needs have resulted in a significant increase in their presence within mainstream schools, with statistical data from the most recent school census showing an increase of 234% between 2003 and 2010. At the same time, these policies of inclusion have been associated with taking the educational community out of the "comfort" zone and, amongst the many uncertainties, insecurities and conflict the actors in these communities are facing, questions related to curriculum demands and pedagogical actions have a central role. In particular, the increasing diversity of students within the same classroom setting raises questions related to task design: what principles and procedures might be adopted in the design of tasks for the inclusive mathematics classroom?

It is within this context that we began work on a research project aiming to (1) investigate forms of accessing and expressing mathematics which respect the diverse needs of all our students, (2) contribute to the development of teaching strategies which recognize this diversity, and (3) explore the relationships between sensory experience and mathematical knowledge[8]. The project involves the development and analysis of inclusive scenarios for mathematics learning, though a collaborative process involving researchers, teachers and students. In this article, we intend to exemplify our approach to task design and present some examples from our work in São Paulo schools.

In this approach, the process of task design is accompanied simultaneously by the development of the material and digital tools that are also incorporated in the learning scenarios. Together, tasks and tools aim to enable the interaction of different students with mathematical objects and relationships. To this end, they are designed to facilitate multiple ways of interacting with these objects and relations and to respect the diverse experiences of the students with whom we work. The approaches we use involve representing mathematical ideas through colour, sound, music, movement and texture, and hence appeal to different sensory canals, and particularly to the skin, the ears and the eyes. The multimodal nature of the mathematical representations reflects our attempts to offer stimuli appropriate to the particularities of each and every student: for those with visual impairments, the tools enable tactile and auditory stimuli, for deaf learners, tactile and visual approaches are privileged and students who can both hear and see have access to all three modes, allowing even those with specific difficulties in learning mathematics to have a variety of ways to think mathematically. Before describing in more detail the process behind the design of task and tools, we begin by delineating our understanding of the term "task" and how this term figures in our view of learning mathematics.

Learning scenarios: Tasks and activity

Our view is that tasks represent one of the elements that compose the learning scenarios we enact within. In a similar way to Laborde (2002), we see learning scenarios as consisting of specific tasks, or sequences of interrelated tasks, the mediating tools available for their resolution, along with the activities of the participating actors (which may include different combinations of students, teachers and researchers). More precisely, we distinguish between task and activity in the same ways as Dejours (1997, p.39). He argues that "a task is that which is to be achieved or that which must be done. Activity is, in the context of the task, that which is actually done by the operator to arrive as close as possible to objectives fixed by the task". That is, tasks are proposed to the collective, and might be realized by differing individuals in different ways. Dejours (1997) was concerned with the work context and how different people might employ different techniques to attain a particular objective, depending on the tools available, of course, but also on the individuals themselves. The same is also the case in educational settings. The task proposed to a

[8] Funded by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, No. 23038.019444/2009-33)

group of mathematics students might be the same, but the interpretation of the task and the activity that results will be shaped by both intrapersonal and interpersonal aspects of the particular students involved. The actions of each of the individuals who engage in a particular given task, individually or collectively, are a function, not only of the task itself, or the means available to interact with it, but also of the meanings that are associated with the activity itself (Leontiev, 1978).

Here, another question is raised, what is the relationship between completing a task and learning mathematics? In our view, tasks are proposed to motivate learners to engage with practices associated with the set of artefacts that have, historically and culturally, come to represent the body of knowledge we call mathematics. In the socio-cultural perspective which informs this view, learning can be defined as participating in, and appropriating, these practices. Using the concept of activity defended by Leontiev (1978), appropriation is a social process in which participants aim to make their own objects already steeped in cultural meaning. The process of appropriation occurs, necessarily in the case of mathematics, on the basis of actions mediated by semiotic systems. Mathematical activity, then, occurs as a dialectical process, in which individuals interact with the environment and with other individuals to attribute sense to aspects of the knowledge and experiences developed in the course of human history. As a result of this activity, objects of the environment, recognised by the senses, acquire the character of objects of reflection (Fernandes, 2008, p.47). This brings us to another aspect central to the theoretical framework that guides our approach to task design: the role of the senses in mathematical cognition. We see links between the socio-cultural perspective developed by Soviet psychologists in the last century with the views of researchers in the area of embodied cognition today, who argue that our mathematical understandings, like all others, are structured by our encounters and interactions with the worlds we experience via our bodies and our brains (Gallese and Lakoff, 2005). Indeed, Radford (2006) argues that the body itself can serve as a semiotic system, through acts of perception, gestures and other movements. Other semiotic systems for Radford include artefacts, language and signs. Jointly, then these represent the tools through which task demands are to be mediated.

With this view of mathematics learning in mind, we return to the question of designing tasks, tools and teaching interventions for inclusive learning scenarios. More specifically, in this article, we concentrate on learning scenarios in which students who lack access to one or other sensory field act; that is, we focus on learners who are deaf and learners who are blind. To enable the participation of students with sensory limitations, it is important to understand how the different channels through which they experience the world mediate the processes by which they appropriate mathematical knowledge, and to recognise that the mathematical practices that are depend on these experiences. Once again, we draw here from the work of the Vygotsky and his colleagues in the former Soviet Union and particular to work in the area of what at the time was called *Defectology*. Vygotsky (1997) proposed an approach to understanding the learning of students with sensory, motor or cognitive disabilities which involves considering how and when the substitution of one (non-functioning) tool by another may engender different forms of activity (Healy & Fernandes, 2011; Healy & Powell, in press). His approach stressed the potential for development of learners with disabilities, rather than positioning them as deficit in relation to some supposed "norm".

> "The positive particularity of a child with a disability is created not by the failure of one or other function observed in a normal child but by the new structures which result from this absence […]. The blind or deaf child can achieve the same

> level of development as the normal child, but through *a different mode, a distinct path, by other means*. And for the pedagogue, it is particularly important to know the uniqueness of the path along with the child should be led" (Vygotsky, 1997; p.17 – emphasis in the original)

In relation to empowering those without access to one or other sensory field to participate in social (cultural) activities, for Vygotsky, the solution lies in seeking ways to substitute the traditional means of interacting with information and knowledge with another. For example, he suggested that the eye and speech are "instruments" to see and to think respectively, and that other instruments might be sought to substitute the function of sensory organs (Vygotsky, 1997). As mathematics educators, we interpret this message from Vygosky as implying we need to pay attention to – and where necessary create – a multitude of (substitute) semiotic systems to mediate mathematics learning.

Our approach to design

The research strategy we adopt is based on the establishing of partnerships between school- and university-based participants – researchers, school teachers and school students – who collaborate on the design of tasks and tools for use in the mathematics classrooms of the teachers and students. In these partnerships, participants work together to conduct a process of co-generative inquiry (Greenwood & Levin, 2000), a kind of participatory action research in which all participants co-generate knowledge through a process of collaborative communication.

The process of designing learning activities is not a simple one and passes through a number of stages. Not all the participants are necessarily involved in all the stages, although usually at least one member from the school and one from the university is present in each. The first stage involves designers in identifying particular challenges associated with the learning of the mathematical topic in question and in developing and testing hypotheses about how best to engender the intended learning. The topics selected are those that are emphasised in the mathematics curriculum that the schools are following and the starting point for the design process is twofold, aiming to combine both pragmatic and theoretical concerns. On the one hand, the teachers and sometimes also the students themselves, brings examples of particular difficulties and problems they have experienced. At the same time, we also consult the existing literature to attempt to determine what previous research tells us about students' conceptions in the chosen topic. On the whole, we have found relatively little research addressing the mathematical learning processes of either blind students or deaf students with respect to the majority of topic areas we have addressed. This means that the first versions of the task are often developed on the basis of what we know about sighted and hearing learners and hence may not be fine-turned to the particular strengths of those who do not see with their eyes or who do not speak with their mouths.

This is one of the reasons that we believe it is crucial to involve the students, as well as teachers and researchers in the design process. Student participation occurs early in the design process, as students are invited, either individually or in small groups, to work on the first prototypes of the tasks and tools under development. For these first tests, we have tended to work exclusively either with blind students or with deaf students. The meetings are videotaped and represent a means for reviewing our developing theoretical models and revising our hypotheses so that they can be operationalised given the particularities of the schools and students involved. Our tendency has been to develop tools and task sequences simultaneously and to modify both as the sequence is applied in practice during these tests. It is only after the scenarios have been tested and analysed, that we consider re-enacting the scenarios in teacher's mathematics classrooms.

Examples of the design process in practice

To illustrate this process in a little more detail, we have selected two examples from the collection of learning scenarios we have investigated: one in which tasks were to be mediated using material objects and a second involving the use of a tactile, digitally-controlled tool designed to permit the exploration of graphs of polynomial functions.

The first example involves the topic Matrices, a topic that is introduced to students in the second year of High School (that is the 11th year of compulsory schooling) in the curriculum currently followed in the state of São Paulo. One motivation for focussing on this topic comes from the comments of deaf students in one of our partnership schools and blind students in another. The deaf students, for example, described Matrices as "*something that has brackets and numbers*", but were not confident in manipulating paper-based representations these objects. Their teacher, fluent in Libras (the Brazilian sign language), suggested that the lack of specific signs for the vocabulary associated with matrices served as a complicating factor in teaching the topic. The blind students, too, spoke of their difficulties in solving tasks with matrices, which they described as "drawings with numbers inside". We found no research studies which addressed interactions of either blind or deaf students with matrix representations. We decided to attempt to construct a form of representing matrices which would permit both deaf students and blind students to construct them and operate upon them (more details of the design process are available in Silva, G. G., 2012). The tool MATRIZMAT is a very simple one, made up of plastic boxes (5cm by 5cm by 3cm) which could be joined to each other by magnets fixed to each of the boxes' four sides (Figures 1 and 2). In the version for deaf students, the numbers written on foam-rubber rectangles could be placed in the cells of the matrices (Figure 1), whereas, for the blind students, we made use of the lids of the boxes, with numbers in Braille stuck onto the top (Figure 2).

Figure 1: MATRIZMAT with written numerals

Figure 2: MATRIZMAT with numbers in Braille

The tasks for both student groups had the aim of introducing the language associated with matrices, their organisation in rows and columns, determination of the order of a given matrix, identification of equal matrices and matrix addition. It is not our intention in this article to present in detail the interactions of the students with these tasks, but perhaps it is worth stressing that the material tools enabled to both student groups to develop efficient ways of expressing matrix structure. Figure 3, for example, shows Maria using the signs developed by the students themselves to indicate position $\mathbf{a}_{12}$ of a 3 by 3 matrix**.** It seemed that the layout of the matrix in a physical, palpable form helped the group to develop ways of talking about its structure – something that they had had difficulty to do when operating with the paper and pencil representation.

Figure 3: Maria signing position $\mathbf{a}_{12}$ of a 3 by 3 matrix

In the case of the blind students, the importance of the tactile tool was most evident when they were comparing or adding two matrices (see, for example, Figure 4). Being able to explore spatially the positions of the elements of the matrix enabled them to experience matrices in ways that correspond to those of their sighted contemporaries. This was rather different to their previous experiences, which had involved Braille representations in which matrices were presented in a form that did not emphasise the spatial layout of the elements and in which they had found it very difficult to locate the elements in different matrices which should be added to each other.

Figure 4: Adding matrices of order 3x2

The second example evolved as we attempted to develop tasks related to polynomial functions that would be accessible to blind students. We knew from the students themselves that tasks involving graphical representations of functions were something that their teachers tended to avoid assigning (Silva, B. J., 2012). From the research literature about (sighted) students' conceptions of functions, we conjectured that a tool in which blind students could experience not only static representations of the locations of particular points on the Cartesian plane or static representations of the graphs of specific functions, but could also feel the graphs of different functions appear as the independent variable changed, might afford more dynamic views of function and help them understand the dependence relationship that exists between the independent and dependent variables. The tool that we designed to permit blind student to engage in such tasks was composed of a digitally controlled board made up of a rectangular matrix of pins, each of which represented a point on the plane. When particular point is requested or a graph of a given function plotted, the relevant pins are raised up (sequentially as the value of the independent variable increases in the case of the graph of a function), allowing the student to feel the image as it is produced. Figure 5 presents the final version of the tool, while Figure 6 shows a blind user, Alice, as she feels the graph of a function as it reveals itself to her.

Figure 5:The current version of the tactile graphing tool and its digital interface

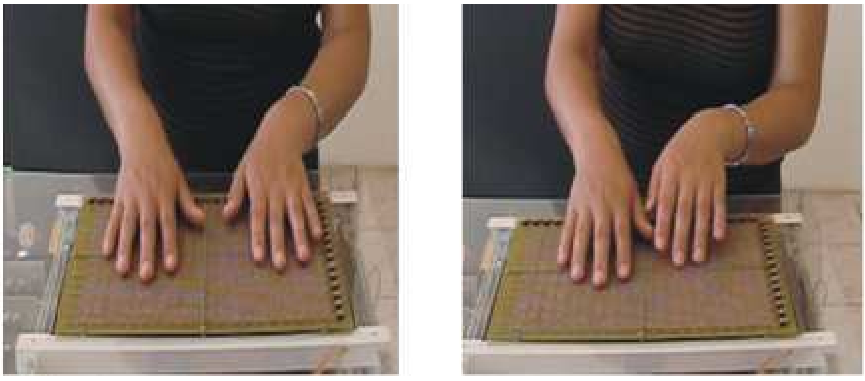

Figure 6: Alice feeling a graph as it is plotted

To date, we have only tested this tool with students who do not see with their eyes. Of course, it could also be used by the sighted, but perhaps represents a rather expensive option to the on screen dynamic graphing tools which already exist. There is a question though: are the experiences of seeing a graph as it emerges in front of one's eyes and feeling it emerge with one's hands cognitively identical? Our conjecture is that they may not be – the act of moving one's hands to find points in ways not guided by one's eyes seems rather different to having a kind of global access to the whole plane upon which the graph appears. The difference in the strategies afforded by these different ways of perceiving and the properties of the graphs emphasized in these two conditions is something we believe merits further research.

Reflections on the relationships between task and tool design

The theoretical influences that inform our approach to design indicate that the processes of creating tasks and the tools by which they are to be mediated are best tackled simultaneously. They also lead us to recognize, as Cole e Wertsch (1996, p.255) have pointed out, that the insertion of tools into situations with instructional intent does not simply serve to facilitate the mental processes that occur within them, it fundamentally forms and transforms these mental processes, conditioning the practices of the learners who operate the tools to the particular practices associated with their use. Moreover, it is not only the learners whose practices are transformed: the introduction of any artefact into a given situation might offer new – and even more efficient means – to resolve a problem, but it also changes the very nature of the task (Béguin & Rabardel, 2000, p.2). In this paper, we have concentrated on the process of design and not on the resulting interactions of those who participate in the learning situations into which the resulting tasks and tools are incorporated, but we believe that the particularities of the students with whom we work help to illustrate the extent to which it is not only the material and semiotic tools we provide that impact upon the practices which emerge in the scenarios. Equally important are the bodily resources through which tool and task are experienced, with different sensory-motor systems potentially affording different modes of acting mathematically and, hence, different paths by which mathematical meanings might be appropriated.

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Using computers in classroom mathematical tasks: revisiting theory to develop recommendations for the design of tasks

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The context for this paper is the ordinary mathematics classroom lesson, in which the tasks set for students involve the use of computer software. The paper highlights the importance of paying attention to the intended mathematical learning of students as they work through the task, adapting their strategies as they negotiate epistemological obstacles. It draws on theoretical notions of student activity, or 'dialectics', within the classroom environment to suggest the sorts of activity that are likely to provoke mathematical learning, particularly highlighting the role of computers within these activities. This provides the background against which approaches to the design of mathematical tasks are recommended. The approaches focus on the intended mathematical learning of the students and the obstacles built intentionally and unintentionally into the tasks.

Keywords: dialectics, computer, validation, feedback, obstacles, schools

Introduction

This theoretical paper concerns classroom mathematical tasks and activity, where, in common with the conventions adopted in much of the literature, (e.g. Ainley & Pratt, 2002; Love & Mason, 1992; Sierpinska, 2004), I use the word 'task' to mean the work the teacher asks the students to do. The teacher has in mind what it is she wants the students to learn ('didactical intention'), but, as Brousseau (1997) explains in detail, she cannot just tell the students and hope that they will learn. Instead the teacher takes the decontextualised and depersonalised knowledge she wishes them to learn and recontextualises and repersonalises it by embedding it in a task. Tasks can therefore be seen as 'the fabric of student learning' (Sierpinska, 2004, p 25) ; through tasks, teachers provide students with opportunities to engage in thinking and sense-making in the classroom, which in turn leads to learning (Stein, Grover, & Henningsen, 1996). Tasks influence learners by directing attention to particular aspects of content and by specifying ways of processing information (Doyle, 1983).

The literature distinguishes three main task types; exercises (or routine problems), problems and investigations (Christiansen & Walther, 1986; Henningsen & Stein, 1997; Orton & Frobisher, 1996). To some extent, these can be distinguished by their goals, which are discussed below. However, there is also an argument that they can be distinguished by the prior learning of the students. As Orton and Frobisher (ibid) explain 'not all children in a class may view what is taught as a problem… what is a mathematics problem for one learner may be an exercise for another' (p 25) and this, they suggest, depends on the prior learning and experience of the individual students. Christiansen and Walther (ibid) make a similar argument, suggesting that what is for one student a task that provides potential for learning can be a routine task for another, depending on their current developmental levels.

For most tasks, the goal for the student is an answer or some answers (Orton & Frobisher, 1996). These goals can be seen as the end point of the task from the students' point of view; this is what the student must do to complete the task. Further, as Sierpinska (2004) and Mason (2000) point out, the assumption is that the students' agenda is to complete the task; and as a result the design of the task should include the mathematical thinking and reasoning that are '*strictly necessary and sufficient to complete the task'* (Sierpinska op cit p12, italics in the original); as she suggests, completing the task may result in problem solving but does not necessarily do so. Hence, while the goals described above are (usually) explicit products, it is the mathematics which the task address which should perhaps be seen as more important.

In my empirical research (Joubert, 2007), I observed authentic mathematics lessons, where teachers chose the tasks they wanted to students to complete, and I analysed student activity in order to provide evidence of their mathematical learning (in vain). The lessons I observed could be described as 'disasters of well taught mathematics courses' (Schoenfeld, 1988); mathematics lessons in which students were engaged in activity which *looked like mathematics* but their activity led to limited learning of mathematics. In all these lessons, computers were used by the students and my research confirms how difficult and complex it is to integrate computers into the teaching and learning of mathematics (see for example, the 17th ICMI study book, (Hoyles & Lagrange, 2009)).

An analysis of the tasks the students were set emphasised the importance of understanding a) the mathematics the teacher intends them to learn and the ways in which the design of the tasks provokes activity through which students learn b) the relationship between student activity and their mathematical learning, and c) the role of the computer, which can perform some or all of the mathematics involved in the task. In this paper, I draw on my research, revisiting some theoretical perspectives underpinning task design, particularly the work of Brousseau (1997), to propose an approach to task design taking into account a), b) and c) above, which might begin to address the challenge of embedding the computer in classroom mathematics.

Intended learning and the design of tasks

The need for a clear idea of the mathematical learning intended to take place as a result of student engagement with mathematical activity is frequently highlighted in the literature about tasks; Henningsen and Stein (1997), for example, suggest that teachers should be attentive to emphasising meaning and explicitly drawing out the mathematics underlying the activities. Similarly, Christiansen and Walther (1986) emphasise the need for teachers to be aware of the intended learning as both process and product and as addressing different 'dimensions of learning'; on one hand the 'object for the activity… the mathematical core of the given task' and on the other hand 'establishing the schemata for learning' (p 262).

It may seem obvious that teachers should be clear about what it is they want their students to learn, but, as Brousseau argues, the detail of the intended learning is frequently not sufficiently well understood by teachers, can be obscured (as, for example, in the early introduction of decimal numbers through a measuring system

Brousseau (1997)) or is unclear (Love & Mason, 1992). This point is also made in the discussion document accompanying the call for papers for the study group:

> There is a tacit assumption that the completion of mathematical tasks chosen or designed by the teacher will result in the student learning the intended mathematics.

Even when the intended learning is thoroughly analysed, however, students will not necessarily learn what the teacher intended (Love & Mason, 1992, Sutherland, 2007). This may particularly be the case when tools, including computer software, are used and the meanings students make from the feedback on the computer screen are sometimes at odds with the intended learning (e.g. see Sutherland (2007) and Hoyles and Noss (2003)). Further, their learning may be influenced by the design of the software which frequently has some mathematical and design assumptions built into it (Sutherland, 2007). The implication is that the design of the task should attend to the mathematics embodied in the tools, and to working out how this might mediate the learning. Further, the fact that software can perform some of the mathematical processes can be confusing; if the computer does the mathematics, what learning is there for the students to do? For example, whereas plotting a quadratic graph is a valid activity in a mathematics classroom as an end in itself, generating the same curve using software is perhaps less valid as an end in itself.

To sum up: although the importance of the intended learning of a task seems to be obvious, to understand what it is that the task is meant to 'teach' requires careful analysis of the mathematics that might be provoked by the task, of the tools used to mediate the activity of the students and – crucially – the prior learning of the students (as argued in the introductory section above).

Obstacles

As discussed above, students will do only what is necessary to complete the task. It follows that the design of the task needs to 'force' the students to identify, adopt and adapt strategies to complete the task in order to reach target knowledge (Brousseau, 1997). Adaptation may take place, Brousseau suggests, if the students encounter and overcome 'obstacles' as they work through the task. He describes an obstacle as:

> 'a previous piece of knowledge which was once interesting and successful but which is now revealed as false or simply unadapted'. (p 82)

Obstacles can take a variety of forms; Brousseau, for example, identifies obstacles of ontogenic, didactical and epistemological origin. The first of these relates to the 'student's limitations' (p 86), including 'developmental' limitations (Swan, 2006) such as the lack of prior learning which is an obstacle because the student is in some way not ready for the mathematics required to complete the task (Love & Mason, 1992). For example, suppose a task requires students to generate graphs using computer software and to 'notice' some feature or other. For students unfamiliar with 'noticing' tasks , this presents an ontogenic obstacle.

Obstacles of didactic origin 'are seen as obstacles for the students because of ill thought out presentation of subject matter, or 'the result of narrow or faulty instruction' (Harel & Sowder, 2005, p 34). Harel and Sowder give the example of the didactical practice of teaching students to look for 'key words' in mathematics problems (such as 'altogether' signals that addition is required).

Both ontogenetic and didactic obstacles should be avoided (Brousseau, 1997, Harel & Sowder, 2005). The third type of obstacle, however, the epistemological obstacle, should **not** be avoided and is key to the design of 'good' tasks.

Epistemological obstacles are perhaps most clearly explained by Balacheff (1990, p 264) who discusses how mathematical concepts are learnt through their use as tools in the process of problem solving with some content; this content is supported by the students' prior knowledge. However, although the old knowledge is a necessary basis for the content, it may cause problems for the students as they work through the problem; in other words it becomes an obstacle which causes the students to stumble. If, in order to overcome the obstacle, the students are required to construct the meaning of the new piece of knowledge, then this obstacle is an epistemological obstacle. Harel and Sowder (2005) suggest that 'MMB' (multiplication makes bigger) can often be seen as an epistemological obstacle and Sierpinska (1987) identifies a complete mapping of epistemological obstacles with respect to limit; scientific knowledge, infinity, function, real number.

As discussed at length in Joubert (2007), it is not always easy to distinguish between didactical and epistemological obstacles and what may be epistemological in one context may be didactical in another. A key consideration is the development and prior learning of the students and the design of any task will need to take these into account when designing the task.

Student activity and mathematical learning

In completing the task, the students interact with the *milieu*, described by Balacheff and Sutherland (1994) as 'the specific part of the environment of the learner which is accessible and relevant to his or her actions' (p 141). However, each student experiences the *milieu* differently, depending on, for example, their prior learning.

The interactions of students can be conceptualised as a 'dialogue' between the student and the feedback from the *milieu.* The feedback from the *milieu* can take many forms; Brousseau includes verbal feedback from other students and the teacher, an outcome of a game and drawings produced by a pantograph as examples. The importance of feedback should not be underestimated;

> 'The pupils' behaviour and the type of control the pupils exert on the solution they produce strongly depends on the feedback given during the situation. If there is no feedback, then the pupils' cognitive activity is different from what it could be in a situation in which the falsity of the solution could have serious consequences' (Balacheff , 1990, p 260).

When computers are used, feedback from the *milieu* will also include feedback from the computer. The general conception of computer feedback seems to be that it is beneficial in teaching and learning mathematics; it can be seen to provide opportunities 'to quickly test ideas, to observe invariants … and, generally, to be bolder about making generalisations' (Hillel, 1992, p 205). However, students sometimes interpret feedback from the computer in unexpected ways, which may have a negative or distracting impact on developing the understandings the teacher intended (Hoyles & Noss, 2003, Sutherland, 2007), and which may lead to the development of strategies which are at odds with the intended learning, such as trial and error, trial and improvement and 'intellectual passivity' (Hillel, 1992, p 217).

A range of different types of interactions between students and feedback from the *milieu* may take place. For example, the TIMSS study (Hiebert et al., 2003) classifies interactions in terms of public and private interactions, and then goes on to categorise the mathematical processes for solving problems as ‘Giving results only’, ‘Using procedures’, ‘Stating concepts’ and ‘Making connections’. Frobisher (1994) provides a more detailed categorisation, including processes such as guessing, pattern-seeking, predicting, hypothesising and proving. However, these do not take into account the **dialectical**[9] nature of the student interactions with the *milieu* which, as explained above, may be important in understanding and explaining the students’ progress through the task.

Modes of production

Brousseau (1997) uses the notion of ‘modes of production’ to describe the different types of dialectic interactions between students and the *milieu*; he suggests that as they work through a mathematical task, they will engage **action** dialectics, in which the students know how to proceed; they implement a procedure or choose it in preference to another or apply a knowing (the object of a student’s construction activity), all of these with or without communication. In the context of my research, an example of an action dialectic was to generate a graph using software and to copy the computer-generated graph onto a worksheet. In action dialectics within, the student does not need to extend her mathematical knowledge or understanding to do so; ‘simple familiarity, even active familiarity … never suffices to provoke a mathematization’ (Brousseau, 1997 p 211). Some action dialectics are *necessary* within any didactical situation; and Brousseau suggests that it is when the action dialectics are motivated by the mathematics rather than the didactics, these actions may lead to a sequence of mathematical dialectical interactions which could include the development of hypotheses, alternative strategies, justifications and proofs.

Dialectics of **formulation** occur when students meet a difficulty or problem as they engage in mathematical activity; Brousseau explains that when a solution to a problem is inappropriate, the situation should feed back to the students in some way, perhaps by providing a new situation. The means that the student may become conscious of her strategies and begin to make suggestions. Brousseau includes in this category ‘classifying orders, questions etc….’ (p 61). He goes on to say that in these communications students do not ‘expect to be contradicted or called upon to verify … information’ (p 61). When the didactical intention is that the student should make some observations, or to ‘notice’, it is likely that the students will begin to formulate choices; noticing requires the student to make conscious choices about what to notice, which aspects to attend to, which to suppress (Mason, 1989) and how to express and articulate these. ‘Noticing’ can therefore be seen as a key formulation activity and in some cases noticing may lead to making suggestions or to conjecturing. When computers are used, an important implication of the computer’s ability to ‘do the mathematics’ is in the opportunities which can be developed for the students to work inductively rather than deductively as is more usual in mathematics classrooms. For example, when computers are used to draw graphs, students may be able to notice

[9] This is the term Brousseau uses to emphasise the notion that the student interactions can be seen as a ‘dialogue’ with the *milieu.* Sierpinska discusses the term in detail, referring to its roots in philosophy, and concluding by remarking that, in *Didactique*, it means the method by which the student manages contradictions between the expectations from the *milieu* and the feedback. Feedback, she explains, is just communication of information; the dialectic turns the information into knowledge. See Sierpinska, A. (2000)

characteristics of families of graphs, develop conjectures based on their observations and test their conjectures on the computer.

Both action and formulation involve manipulating ‘moves in the game’ or mathematical objects; **validation** however involves manipulating ‘statements about the moves’ (Sierpinska, 2000). Validation therefore takes place when an interaction intentionally includes an element of proof, theorem or explanation and is treated thus by the interaction partner (or interlocutor) ‘this means that the interlocutor must be able to provide feedback…’ (Brousseau, 1997 p 16). Dialectics of validation include justification (perhaps of a procedure, a word, a language or a model), organising theoretical notions, ‘axiomization’ (ibid p 216), and a range of proofs. An example of a validation dialectic might be an explanation to justify a prediction that a computer-generated graph will have particular intercepts by noticing a relationship between the intercepts and the equations of graphs generated previously.

Brousseau, while suggesting that all three modes of production are ‘expected from students’, (ibid p 62) argues that it is through situations of validation that genuine mathematical activities take place in the classroom. Romberg and Kaput, (Romberg, Kaput, Fennema, & Romberg, 1999) amongst others (for example, Lakatos, Worrall, & Zahar, 1976) echo Brousseau’s sentiments.

Brousseau suggests that situations of validation do not occur very often and are unlikely to occur spontaneously and it is probable that validation will not take place unless it is explicitly called for. This has important implications for task design.

Working across different ‘fields’

It can be that the mathematical activity of the students relates only to the physical or concrete characteristics of the mathematical objects they work with. On the other hand, students may relate their activity to mathematical notions underpinning the objects. For example, on the one hand students using dynamic geometry software may construct a tangent to a circle by rotating a straight line so that it touches the circle, and on the other hand they may use the geometrical property that the tangent is perpendicular to the radius to construct the tangent (Laborde, 2005).

These different ways of working in different contexts are described in a variety of ways; for Laborde (1998) the distinction is between ‘spatio-graphical’ and ‘theoretical’, for Fischbein (1993) and Mariotti (1995) it is between ‘figural’ and ‘conceptual’, for Tall (for example, 1996) it is between ‘visuo-spatial’ and ‘verbal/logical’. While the authors’ varying theoretical perspectives imply variations on these ideas, the point for all of them is that it is useful to make the distinction and this paper uses the terminology adopted by Noss, Healy, & Hoyles, (1997) pragmatic/empirical’ and ‘mathematical/systematic’ to make the distinction.

In addition to the perceived need to distinguish between the ‘pragmatic/empirical’ and ‘mathematical/systematic’ fields, there seems to be a consensus that a movement between the two fields is required for mathematical learning to take place (Balacheff, 1991; Brousseau, 1997, Dörfler, 2000; Laborde, 2005; Noss et al., 1997). Brousseau, for example, describes different types of proofs, including contingent and experimental proofs and proofs by exhaustion. These, he suggests, relate to implicit models students hold and therefore they are likely to take place in the pragmatic/empirical field and not relate explicitly to theoretical mathematical knowledge. However, for ‘mathemization’ to take place, according to Brousseau, mathematical proofs which relate to the theoretical mathematics involved in the situation, are required. Similarly, Mariotti (op cit) claims that the solution of geometry problems requires continuous moves between the two fields and Laborde (op cit) suggests that there is a need for interactions between images and concepts. The implication, in terms of the relationship between the students' dialectics (particularly of formulation and validation), and student learning is that, where these dialectics remain in the pragmatic/empirical field, mathematical learning will be limited.

When computers are used, very often the task students are given requires them to construct an artefact on the screen, such as, for example, a graph or a geometric transformation and they begin by working in the pragmatic/empirical field. It seems that frequently the task does not require movement between this and mathematical/systematic fields and mathematical learning is limited (Joubert, 2007).

In terms of implications for task design, the theoretical discussion related to student dialectics above suggested that a task should provoke students to engage in dialectics of action, formulation and, crucially, validation. Further, the students should move between the pragmatic/empirical field and the mathematical/systematic field in order to reach the goal of the task.

Concluding comments

This paper has made an argument that task designers should pay attention to the mathematical learning a task addresses, emphasising the need for a task to provoke student activity that does not merely look like mathematics, but actually is learning mathematics. The task should be designed in such a way that it provides situations for action, formulation and validation dialectics, so that students begin by adopting a strategy and, in the course of stumbling over the epistemological obstacles built into the task, they adapt their strategy and eventually reach the goal of the task.

The importance of understanding the role of the computer software in classroom tasks was also made. In particular, the paper has suggested that the designers of tasks need to be clear about what the work of the computer is, and what the work of the students is. As explained, computer software is able to do the mathematics that is frequently, in traditional classrooms, seen as the end point of a task (such as creating a graph), and if computers are used in this role, then teachers and designers of tasks need to be clear about what mathematics they want the students to do. Further, because of the potential power of feedback from the computer, designers need to take into account the meanings students read into this feedback.

The point was also made, several times, that the context and prior learning of the students is a crucial concern. The implication of this is that no designed task can be 'set in stone' as teachers will frequently want to adapt the task to the needs of their students (as discussed, for example, by Kieran, Tanguay, & Solares, 2012). My concluding suggestion is that task designers should therefore take teachers' need to adapt tasks into account in their design of tasks.

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The interaction between task design and technology design in creating tasks with Cabri Elem

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Both the design of tasks and the design of technology have been identified as important factors in the effective use of technology-based tasks in the classroom. By analyzing both the design of a sequence of tasks (based on didactical principles from Brousseau's (1998) theory of situations) and the affordances of Cabri Elem software it will be shown that technology can be designed in such a way as to enhance the implementation of didactical principles.

Keywords: Task design, technology, instrumental genesis, feedback, didactical variable

Design of the technology and design of the task

The potential for the contribution of Cabri II Plus and Cabri 3D to student learning is well known. However, Laborde and Laborde (2011) note that there is a gap between situations used in research and everyday teaching practice and suggest the provision of ready-made resources to help teachers take more advantage of dynamic mathematics environments. Sinclair (2003) studied pre-constructed files in a dynamic geometry environment and concluded that the design of the files was critical to their potential to support or impede student learning. There is an increasing concern about the design of technology itself. "*Design detail counts*" states Jackiw (Butler et al., 2009, p. 431). Research on resources and the problematic of quality indicates the importance of the question (Gueudet *et al.* 2011; Trgalova *et al.* 2011). Mackrell (2011) has shown that particular dynamic geometry technologies may have unclear interfaces, a lack of affordances and poor mathematical representations, all of which will hamper the use of such technology as an environment for the design of resources.

The above concerns, together with the perceived need to create a version of Cabri that was suitable to the needs of primary students has led to the development of the Cabri Elem technology. A key difference with earlier versions of Cabri is that the focus of the technology is on creating an environment for task design, acknowledging the importance of both the design of the technology itself and of the design of tasks using the technology.

Tasks in Cabri Elem are presented in "activity books" which consist of a succession of pages incorporating a sequence of tasks. Once an activity book has been created in the Cabri Elem Creator task design environment, it can be used directly by teachers and students in the more restricted environment of Cabri Elem Player.

In this paper we will analyze a particular Cabri Elem activity book in order to illustrate the interconnections between the affordances of the technology and the ability to implement particular didactic principles.

Part I. Theories of Task and Software Design

The theory of didactical situations (Brousseau, 1998) offers a theoretical framework and a number of conceptual tools to study and also to design tasks. In this theory, knowledge is a property of a system constituted by a subject and a "milieu" in interaction. Learning occurs through this interaction: the subject acts within and receives feedback from the milieu. But there is an added requirement. The signification of the knowledge that can be constructed in the interaction depends on the existence of a space of uncertainty and freedom for the subject about appropriate action and strategy. If the student has no choice, the learning outcome may have no mathematical meaning. Together with criteria for success or failure, the goal, whether teacher or student determined, is made clear, unlike in the common dynamic geometry task "drag this point and observe" where the student has no choice of action and is uncertain about what is relevant to observe. In the case of mathematical learning with technology, the relation between the technology and the milieu is complex. The milieu cannot be reduced to the technology. Technology may be one component of the milieu, (Laborde and Capponi 1994) but only the part of technology relevant to the mathematics concerned (Brousseau 1998). The same technology will also not constitute the same milieu for every subject. The milieu related to a student changes as student knowledge, both technical and mathematical, develops.

The mathematical problem and the task are key elements of a didactical situation. In the context of this contribution, we characterize a "task" by the following requirements:

- a task involves learning objectives; when a teacher proposes a task to a student, he assumes that achieving the task will cause learning;
- it involves the student encountering a mathematical problem;
- it is performed by concrete and conceptual student actions;
- it corresponds to phases of the didactical situation (in the sense of Brousseau) and is related to different values of a set of didactical variables.

Didactical variables are parameters of the situation, with values that affect solution strategies. The effects can be of three kinds: (i) a change in the validity of a strategy, where a strategy that produces a correct answer with a certain value of a didactical variable will produce an incorrect answer with another value, (ii) a change in the cost of the strategy (for example counting elements one by one is efficient for a small number but much more costly for a larger number) (iii) the impossibility of using the strategy. A combination of the different didactical variable values contributes to the task definition. The learning situation is a choice of different tasks that lead the students to construct the appropriate strategy. Thus task design will consist, for a part, in identifying the didactical variables of the situation and then choosing the succession of appropriate combinations of didactical variable values. This will be one of the foci in describing the task design process.

When creating a learning task in a computer environment, the author has to create all the elements the student will deal with: the objects the student will manipulate, the possibilities of actions on these objects and the feedback provided by the environment. The elements chosen will determine the possible milieu and the potential for learning. The feedback given by the milieu will be a second focus in describing the task design process.

Instrumental genesis involves the processes of instrumentation whereby a person builds personal utilization schemes for an artefact and instrumentalization whereby a person adapts an artefact to their own purposes, with the result that an artefact becomes an instrument to be used in the pursuit of a goal (Rabardel and Bourmaud, 2003). Instrumentation is clearly relevant to task design using tools, as it has been found to be problematic (Trouche, 2005). However, instrumental genesis is also highly relevant to the design of the technology itself. In the human computer interaction literature, "design is perhaps the most common issue addressed within the approach" (Kaptelinin and Nardi, 2006, p. 110):

> "artefacts should be designed to enable efficient transformation into instruments. […] the importance of designing flexible, open artefacts that can be modified by users and adjusted for various tasks, including unanticipated tasks and the need for designers to take into account the actual transformation of practices and the real needs of users over the course of appropriating an artefact".

The most important design principle in the development of Cabri software is that of direct manipulation (Laborde and Laborde, 2011), which involves both action and feedback on action. This is a fundamental concept of human computer interaction, developed in the 1980's when the current almost-ubiquitous graphical user interfaces were replacing interfaces requiring text commands (Kaptelinin and Nardi, 2006, p. 82). The general principles below (Shneiderman and Plaisant, 2010, p. 196) are illustrated by examples from dynamic geometry.

1. Continuous representations of the objects and actions of interest with meaningful visual metaphors. Geometric objects and tools.

2. *Physical actions, or presses of buttons, instead of complex syntax*. Dragging to move objects, constructing a custom tool or macro.

3. Rapid, incremental, reversible actions whose effects on the objects of interest are visible immediately. The response of any geometric object when one of the points on which it is dependent is dragged.

Some of the benefits claimed for systems designed using these principles are that novices can learn basic functionality quickly, users can immediately see whether their actions are furthering their goals, users experience less anxiety and gain a sense of confidence and mastery (Shneiderman and Plaisant, 2010, p. 196) This has obvious implications for instrumentation.

However, Rabardel (2002, p. 148) suggests that direct manipulation may not always be the most appropriate means of interaction with an artefact in a learning situation. The aim, rather than making the action easier, may be to construct constraints that lead the subject to use and elaborate cognitive constructions of knowledge (p. 148). An implication is that the task designer in a technological environment must have the facility to vary the type of action required and response received as appropriate.

An important additional principle for mathematics educational software is "epistemic fidelity" in which the representations of mathematical objects must avoid any contradictions with the abstract objects they represent, both in appearance and in behaviour when manipulated (Laborde and Laborde, 2011). Rabardel (2002, p. 160) also warns that while instruments may be designed to favour the construction and manipulation of conceptualizations these impose a "world view" on users. This has consequences for design at many levels, ranging from the choice of perspective (to ensure that a cube does not appear to "pulsate" when moved) in Cabri 3D to the importance of distinguishing geometric labels and variables (Mackrell, 2011). Laborde and Laborde also feel that physical action is preferable to pressing buttons: that by directly acting on the representation of mathematical objects the user eventually perceives the abstract object itself.

Part II. Analysis of a Cabri Elem activity book

In what follows, we will look at an activity book and discuss the task design decisions and the technology affordances, which enable these decisions. The "Target" activity book addresses the French primary school level CE1 (7 year old students) and deals with the representation of numbers using place value notation. The idea arose from comparing counters on a scoreboard, where the value of the counter depends on its position on the board, with the way that the value of a digit depends on its position in a written number. It was designed by a team of ten researchers (including two of the authors of this paper), teacher educators and teachers involved in a French national project[10] whose purpose is to create resources for the teaching of mathematics in kindergarten and primary school.

Cabri Elem has the affordances of earlier Cabri technology for direct manipulation and feedback involving geometrical objects and numbers. It has been extended to more effectively model the real world with new objects (such as 3D models) and new tools (such as a realistic compass). It also enables a 3D view, as shown below.

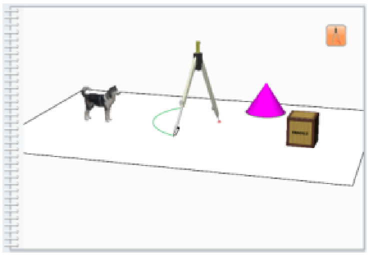

Figure 1. 3D objects and use of the realistic compass tool in Cabri Elem

The affordances for interaction with objects have also been extended, both by enabling restrictions on default behaviours (objects may be locked to prevent changes, or feedback on student actions may be delayed), and by enabling new possibilities for action, such as feedback given at the click of a button.

A key feature of Cabri Elem for task design is that the task author is entirely responsible for the student interface, as the Cabri Elem Player interface is empty at the beginning of the design. Instrumentation and student focus can be better managed by controlling the complexity of the interface, and the author is free to elaborate the milieu by choosing appropriate objects, possible actions and resultant feedback according to the aims of the task. In our example, the objects are essentially the scoreboard with three different regions, the counters, the target number and the score, as shown below.

[10] The « Mallette » project is supported by the French Ministry of Education and conducted in collaboration between the IFE Institut Français de l'Education and the COPIRELEM Commission of IREM http://educmath.ens-lyon.fr/Educmath/recherche/equipes-associees/mallette/.

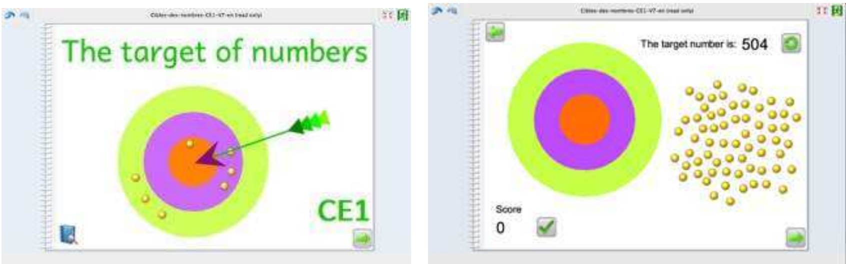

Figure 2. Title page and a task page from the "Target" activity book

The actions on the objects are simple: dragging the counters, clicking on a button to get a new target number and reset position of the counters and clicking on a button to get an evaluation.

Another key feature of Cabri Elem is the ability to include multiple pages. This enables activities to be planned as a sequence of tasks, in which an evolution in student strategies may be provoked through changes in the value of the various didactical variables. The ability to choose the way in which pages are linked also enables the provision of optional help in tool use, differentiated tasks, notes for teachers, etc.

In the task design process with Cabri Elem, didactical variables play an important role. Some are identified a priori, while others emerge during the design process as the author becomes more aware of what aspects of the situation may be changed. Once a potential variable is identified, an analysis of the ways in which this variable may be changed produces a better understanding of the possible tasks and their consequences. It also enables the creation of strategy feedback.

We will also examine three kinds of feedback. Evaluation feedback is related to the achievement of the task or part of the task. Strategy feedback aims to support the student in the course of task resolution, like scaffolding (Wood and al. 1976). It is a response to the strategy used by the student. The author needs to identify (i) configurations that are typical of a strategy and hence enable a diagnosis and (ii) new objects or actions that can be provided to help the student without changing the nature of the task. Such feedback may consist of help messages, or a graphic enlightening of contradictory elements. Another possibility is to modify the values of some didactical variables in order to make the student aware of the current strategy limitations. Direct manipulation feedback is the response of the environment to student action, and may serve the function of either of the previous types of feedback.

Page 2: a page to initiate instrumental genesis

Page 1 is a title page (Figure 2). In page 2, the main objects are presented. The student may interact with these objects, by dragging counters to different positions on the scoreboard and noticing how this affects the score. This is dynamically calculated: one, ten and one hundred for each counter in the green outside region, the purple intermediate region and the orange central region respectively. The aim of the page is to give time for instrumentation to both teachers and students. They can explore interactions with the elements that will constitute the milieu without the constraints of a particular task. It also contains a reset button which, when clicked, replaces counters in their initial positions, and a button which allows students to move on to the next page.

The changing score is direct manipulation feedback that shows students not only the effect of their action, but also that action on one object (moving a counter to a different region of the scoreboard) will affect another object (the score). The values of the different scoreboard regions were chosen by the authors in accordance with learning objectives: in a version of the activity book used with younger students the scoreboard regions only gave values of one or ten. The score is always displayed in some pages, but displayed only after a specific sequence of actions in other pages.

The score is generated by means of the point counter tool a new tool, which counts the number of points within geometric shapes (in this case the three regions of the scoreboard[11]). The idea for this tool was initiated by a primary teacher educator for work with early number: she wanted a means of creating and counting a collection of objects using technology. Other affordances used in this page are the ability to evaluate expressions (the total score being calculated from the number of points in each region), and the ability to reset the position of points.

Page 3: evaluation feedback

On page 3 the student is given a specific task: to reach a score equal to a target number, randomly generated between 1 and 999 (see Figure 2). Clicking on the reset button now in addition generates a new target number. Another new action is that the student may, in addition to comparing whether the score matches the scoreboard, click on a new button for evaluation feedback: a red frowning face if the answer is wrong, and a yellow smiling face if the answer is correct. In case of failure, the student can continue to drag counters and ask for a new evaluation: a new smiley will appear to the right of the previous one. It is important that new feedback is only generated at the student's request: otherwise a trial and error strategy not stemming from mathematical considerations could lead to success.

The key new software affordance used in this page is the Boolean function, which enables a comparison between the student response and the target number. When the feedback button is pressed, the value of the resultant Boolean (TRUE or FALSE) will determine which of the faces is shown.

Pages 4 to 7: suppression of a direct manipulation feedback and evolution of the task

From page 4 to 7 students are no longer given the direct manipulation feedback of seeing the score. They hence need to take into account the value of the counters in the different regions of the scoreboard to determine the score. "Score" was identified a priori as a possible didactical variable, with two values: visible or hidden. All objects may be either visible or invisible in Cabri Elem, enabling the author to control the level of direct manipulation feedback given.

In page 5, the number of counters is reduced so that, if the target number is over 27, a strategy that consists in placing counters only in the green units region will fail. A strategy which takes into account that a single counter can have another value than 1, i.e. using the inside regions of the scoreboard, is necessary. Therefore, another potential didactical variable is identified: the number of available counters, with two values, 3x9=27 and >27. In page 6, the target number is a multiple of ten, between 10 and 990. As there are enough counters to either leave the green region empty or to fill it with multiples of ten counters, a change of strategy is not necessary. In page 7, however, a single counter is fixed in the green region. Therefore, new strategies are required, involving the placement of a multiple of ten counters into the units region of

[11] The outer rings show an example of instrumentalization: as a ring is not a Cabri Elem geometrical shape, the authors have created non-convex polygons with touching sides.

the scoreboard. The “fixed counter” didactical variable is identified, with four values: no fixed counters, or fixed counters in the units, tens, or hundreds region.

Page 8 contains input boxes for the student to enter the values of a counter in each region of the scoreboard. The aim of this task is to summarize the key idea of the activity book, i.e. that the value of a counter depends on the scoreboard region.

Other pages of the activity book not devoted to student tasks

The first page is designed with the aim of attracting teachers and students to the activity book with an iconic representation of some of the main objects.

Pages 9 and 10 contain commentaries for teachers, reporting the main aspect of the task, the evolution from one page to another, possible student strategies (correct or not) and also the solution. The structure of the pages of the activity book was used to organise these notes and the didactical variable analysis helped to determine what information was useful. Trgalova et al. (2011) also points out that teachers find teacher notes of value.

Part III. Feedback from the school

The “Target” activity book was trialled in the spring of 2012 in two primary school classes: CE1 with the version presented here and CP (six year old students) with a version where the target number size was limited to 99. Teachers used the activity book as one resource for learning about place value and instrumentalised the book by printing pages to construct related paper and pencil tasks. They were enthusiastic about student engagement, mathematical reasoning and the evolution of strategies, but raised a number of issues. We present here some findings related to instrumentation by teachers and students, instrumentalization, strategies feedback and didactical variables.

It was expected that the strong metaphor between the task situation and real situations involving a scoreboard would, as well as providing a meaningful context, minimize the need for instrumentation: however, students expected that, as in the real situation, moving a counter would require tossing it in some way and were initially uncertain about how to do this using the software. Teachers also proposed that instrumentation would be enhanced by modifying page 2 to include a target number chosen either by the teacher according to the constraints of the class, or chosen by students in order to challenge each other.

Some students used the target number update not only to get a new number after finding a previous target but also, unexpectedly, to get a number they knew they were able to deal with. They were able to diagnose their level of expertise and this, important ability to the learning process must not be ignored by the environment. It is planned to modify pages to provoke problem resolution, but also to locally enable this usage. This example of students’ instrumentalization of a functionality to adapt it to their level of expertise is a new, generalizable element in activity book design.

The number of available counters was not a didactical variable for most CE1 students, who used each region of the scoreboard and limited the number of counters they needed to drag. Many of them did not notice the reduced number of counters on page 5 and were surprised to apparently have to solve the same task again. However, for a few CE1 students and many of the younger CP students who used only the units region of the scoreboard and placed as many counters as the target number the number of available counters was indeed a didactical variable. The status of page 5 will hence be changed in further developments of the book. Instead of being automatically displayed to CE1 students, it will only be displayed as necessary, i.e. if the unit region is repeatedly filled with many more than 10 counters. The strategy feedback, resulting from our analysis in terms of didactic variables, will consist in reducing the number of counters and choosing a target number over 50.

Conclusion

In this paper the following elements have been considered: task design, construction of milieu in the context of a particular technology, didactical variable, feedback, and instrumental genesis for authors and students. In order to discuss tasks, we tried to show their mutual relationships. We have shown that software affordances and theories related to task design together enable the effective creation of resources for learning and their introduction into the classroom.

However, many elements remain to be controlled and articulated in the analysis of tasks and task design. This paper, drawn from the initial stages of a study, attempts to illustrate some of these and to suggest theoretical means to study the whole process. On the one hand the analysis has shown the potentialities of Cabri Elem, but on the other hand it has shown the complexity of the process: different instrumental geneses are involved, subjects' knowledge comes into play, didactical constraints are present, etc. However, we have also shown that there is not a distinct divide between software affordances, task design and classroom implementation: Cabri Elem affordances are based on theories of software design, but also come from the requests of researchers and educators. In turn, designing a task in the Cabri Elem environment enables a greater awareness of the potential of different didactical variables. Feedback from the classroom suggests ways in which the task may be improved.

The next stage in our study is to explore ways in which activity books could be usefully modified by any teacher. The Cabri Elem task design environment[12] will enable us to consider this new level of instrumental genesis.

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Designing Tasks to Foster Operative Apprehension for Visualization and Reasoning in Dynamic Geometry Environment

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This paper presents how to design tasks in dynamic geometry environment (DGE) to foster operative apprehension for visualization and reasoning in Duval's model of geometrical reasoning. The meaning and significance of operative apprehension in DGE would first be proposed, and the principle of using soft constructions to design tasks to foster operative apprehension would then be discussed, illustrated by some tasks designed in the DGE GeoGebra. Finally a model of task design in DGE to foster operative apprehension for visualization and reasoning is proposed for further discussions in future research of task design in DGE.

Keywords: Task design, Dynamic Geometry Environment, Visualization, Reasoning, Operative Apprehension, GeoGebra

1. Introduction

One of the important aims of the Hong Kong Secondary Mathematics Curriculum is to develop students' ability to conceptualize, inquire, reason and communicate (Curriculum Development Council, 1999 p.4). Hence the terms "explore" and "justify" appear in the learning objectives of many topics. For example, it is expected that students could "explore the formula for the area of circles" (ibid p.20), or could "explore and justify the methods of constructing centres of a triangle such as in-centre, circumcentre, orthocentre, centroids etc" (ibid p.23).

However, it seems that the learning tasks proposed in Hong Kong textbooks could not always fulfil this aim. For instance, when constructing the circumcircle of a triangle, students are told directly to first construct the perpendicular bisectors of the three sides using rulers and compasses, and then use their intersection as a centre to draw a circle passing through one of the vertices, and finally see that this circle also passes through the other two vertices (Figure 1). In this task students are neither provided the opportunity to explore nor asked to justify how the circumcircle could be constructed, but instead just to verify the correctness of the procedures to construct the circumcircle given by the textbooks.

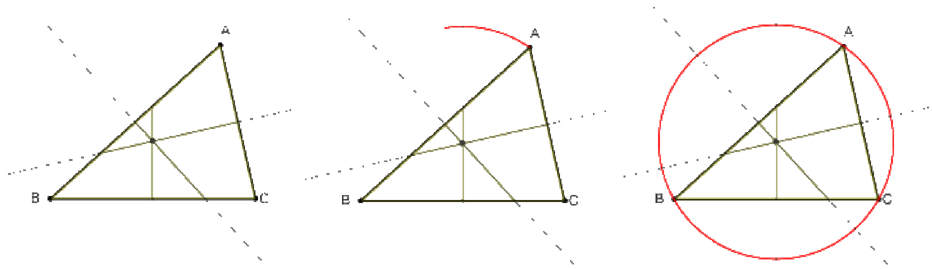


Figure 1: Construction of the circumcircle in textbooks

As a curriculum officer of the Education Bureau in Hong Kong, I have to design tasks to support teachers to implement the curriculum in their classrooms. In this paper, I am going to propose a framework on task design in dynamic geometry environment (DGE) to facilitate visualization and reasoning based on Duval's model of the role of visualization in the development of geometrical reasoning (Duval, 1998). In particular, I suggest that the use of soft constructions (Healy, 2000) is an effective approach for designing tasks to foster operative apprehension for visualization and reasoning in DGE. The discussion would be illustrated by some tasks I designed for Hong Kong teachers.

2. Duval's Model of Geometrical Reasoning

Duval (1998) suggests that geometry involves three kinds of closely connected cognitive processes fulfilling specific epistemological functions, namely, *visualization*, *construction* and *reasoning*. Their epistemological functions and connections are represented by Figure 2 below.

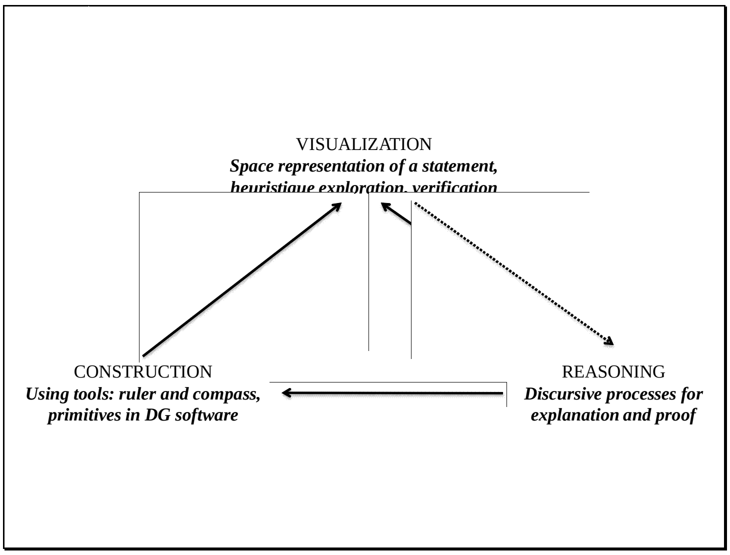


Figure 2: Duval's model of the underlying cognitive interactions involved in geometrical activity

In the figure each arrow represents the way a kind of cognitive process can support another kind in a task. The dotted arrow suggests that visualization does not always help reasoning. For example, visualization can be misleading if our visualized image is a special case. Duval states that these three kinds of cognitive processes are quite different and must be developed separately, and the significance of the teaching of geometry is to develop visual representation and reasoning abilities and to favour the synergy of these processes.

To facilitate visualization as well as reasoning, Duval suggests the necessity of a kind of apprehension of geometric figures called *operative apprehension*, which means operations on the figure or its subfigure, either mentally or physically, that gives insight into the solution of a problem. He emphasizes that operative apprehension is crucial and teachers have to identify factors triggering or inhibiting it so as to make visualization possible and gives rise to various transfers.

With regard to the use of dynamic geometry software (DGS), Duval states that DGS provides enormous possibilities of visualization through the introduction of the aspect of movement, and allows manipulations of geometric objects and hence true explorations of geometrical situations. However, the construction-centered design of DGS does not develop all functions of visualization, in particular the operative apprehension.

Duval's theory emphasizes the importance of operative apprehension to facilitate visualization and also reasoning in the teaching and learning of geometry. In view of his comments on the uses and limitation of DGS in visualization, I shall discuss how to design tasks in DGE to foster operative apprehension for visualization and reasoning. I would first define what operative apprehension means in the DGE, and how the use of soft constructions proposed by Healy (2000) could be an effective approach of designing task to foster operative apprehension.

3 Operative Apprehension in DGE

A task is a set of pre-designed, environmentally situated materials aims to engage learners in activities that could transform the ways they see and do mathematics (Leung, 2011). A task has to be pre-designed in the way that through these pre-designed means, learners are guided to construct insights and the meaning of the mathematics knowledge. A task is also environmentally situated in the sense that the qualities or tools of the environment have been made use to empower learners with extended or amplified abilities to acquire knowledge which could not be acquired in the same ways in other environments (Leung, 2011). In what follows, I shall discuss how to design tasks situated in DGE to foster operative apprehension for visualization and reasoning in Duval's framework. In particular, I will focus on pre-designed DG figures and interpret Duval's operative apprehension as the following:

> Operative apprehension of a mathematical concept or problem in DGE is the insights into the concept or the solution of the problem revealed by operating on a pre-designed figure in the environment through dragging.

Let me illustrate the significance of operative apprehension in DGE using a task I designed in GeoGebra (a free DGS). This task originates from the following problem in a textbook.

A quadrilateral is dissected by a line joining the mid-points of one pair of opposite sides, and the perpendiculars to this line from the mid-points of the other pair of opposite sides. (See Figure 3(a) below.) What shape can you get from this dissection?

A task is designed in GeoGebra to help learners to explore this problem (http://www.geogebratube.org/student/m3459). In this task, a quadrilateral is dissected into four pieces as described in the problem. Each piece can be rotated through dragging the red point at the vertex. In this way we can see that how the four pieces could form a rectangle (Figure 3). Also, the operation gives us the insights to reason why this dissection gives a rectangle, by, for instance, thinking about why the four angles at the vertices give a sum 360° (Figure 3(d)).

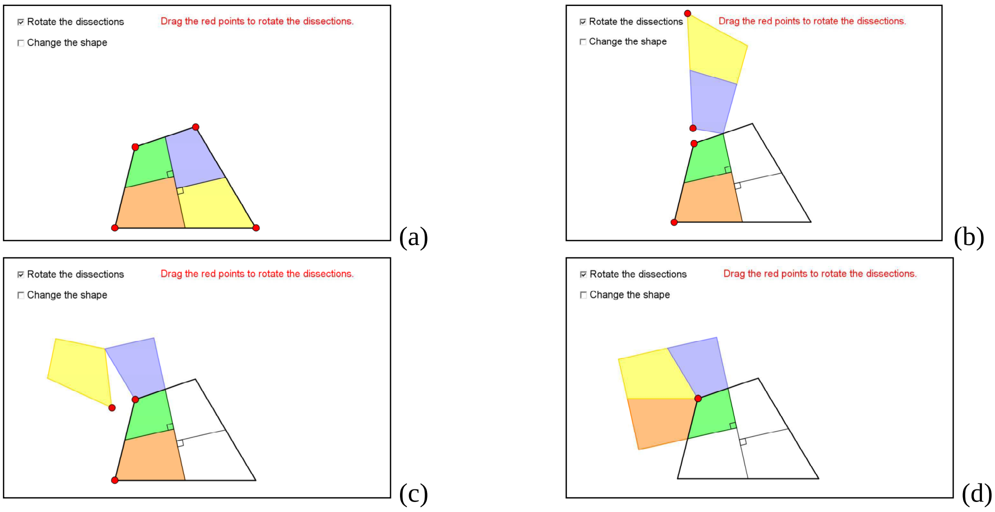


Figure 3: The GeoGebra task of dissecting a quadrilateral into a rectangle

Besides rotating the four pieces, learners can also operate on the shape of the quadrilateral. After checking the "Change the shape" box, four green points appear at the vertices of the quadrilateral and the shape of the quadrilateral could be changed by dragging them. Through dragging the vertices, I see that the dissection would give a square for some shapes of the quadrilateral (Figure 4).

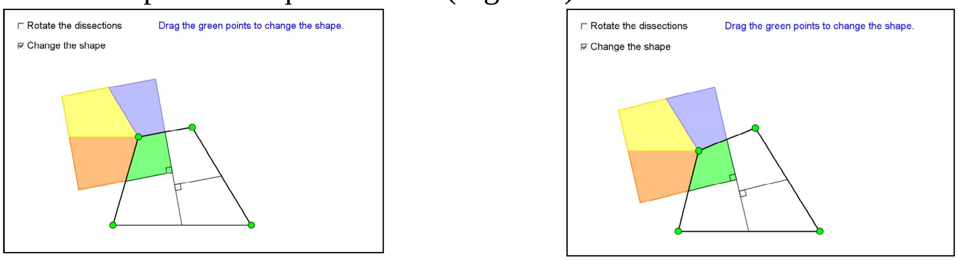


Figure 4: Some shapes of the quadrilateral that give a square after the dissection

This problem reminds me the famous Haberdasher Puzzle composed by English mathematician Dudeney (Dudeney, 1907). This puzzle shows how an equilateral triangle could be dissected into a square (Figure 5). Although I know this puzzle for a long time, I never understand how Dudeney could think of this method of dissection, nor have any idea how to generalize his method to dissect an arbitrary triangle into a square.

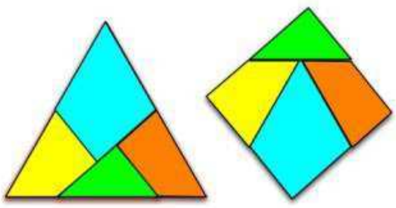
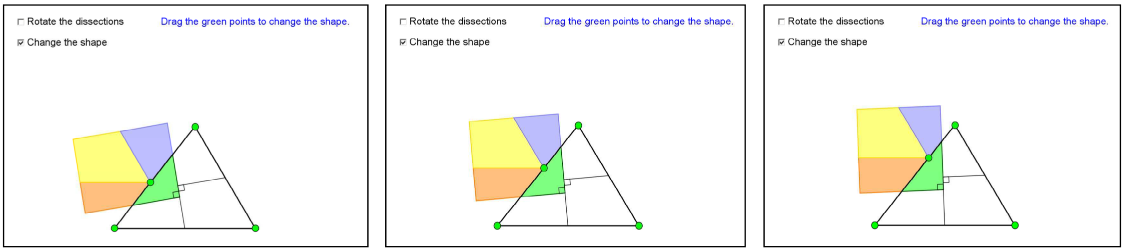


Figure 5: Dudeney's Haberdasher Puzzle of dissecting an equilateral triangle into a square

When I try to compare the quadrilateral problem with Dudeney's puzzle, I suddenly realize that if I drag a vertex, say the upper-left one, to a position at which it is collinear with the other two adjacent vertices (Figure 6(a)), the quadrilateral would be degenerated into a triangle which is dissected into a rectangle. Furthermore, if I drag this vertex along the side of the triangle (Figure 6(b)), since the shape of this triangle is unchanged the area of the rectangle is kept constant while its length and width are decreasing and increasing respectively through dragging. Hence I should get a square somewhere on this side (Figure 6(c)).



(a) (b) (c)

Figure 6: Dissecting a triangle degenerated from the quadrilateral into rectangles or a square

After the above exploration I see how an arbitrary triangle could be dissected into rectangles of various sizes, and there should be a particular dissection that gives a square. Through operating on the shape of the quadrilateral, I get important insights of comprehending how a general triangle could be dissected into a rectangle, and investigating when the dissection would give a square.

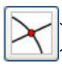
This example illustrates the advantage of fostering operative apprehension in DGE. If we use a paper quadrilateral, although we could cut it to see how it could be dissected into a rectangle, it is impossible for us to operate on its shape. In DGE we can operate on the shape of the quadrilateral so that we can degenerate it to a triangle to get the insights of how a triangle could be dissected into a rectangle.

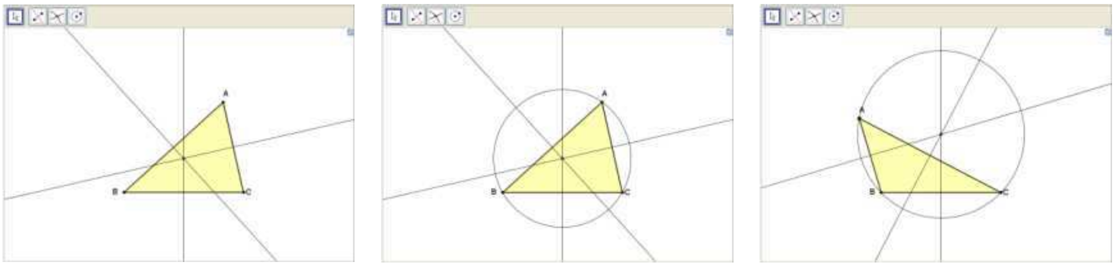
4 Operative Apprehension for Visualization and Reasoning: Soft Construction

At the beginning of research in dynamic geometry, tasks in robust constructions, i.e. constructions preserve relationships upon dragging, were recognized as promoting for the learning of geometry. However, Healy (2000) discovered through observation that, rather than robust constructions, students preferred to investigate constructions "in which one of the chosen properties is purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student". Healy called these constructions *soft constructions*.

Healy differentiates the roles of dragging in robust and soft constructions. In a robust construction, dragging provides a visual verification of the validity of the construction through dragging. In a soft construction, dragging is not verification but part of the construction itself. Through dragging, the general can emerge from the specific by searching empirically for the locus of figures fulfilling the given conditions. Soft constructions offer a transition from an empirical approach to a theoretical approach in solving a geometry problem.

In the lens of Duval's model of geometrical reasoning, tasks in robust and soft constructions can be considered as operative apprehension on figures serving different functions of visualization: a robust construction provides a verification of the construction, while a soft construction provides heuristics or insights through an empirically searched locus which mediates reasoning. I shall illustrate this point with two GeoGebra tasks of drawing the circumcircle of a triangle, one in robust construction and one in soft construction.

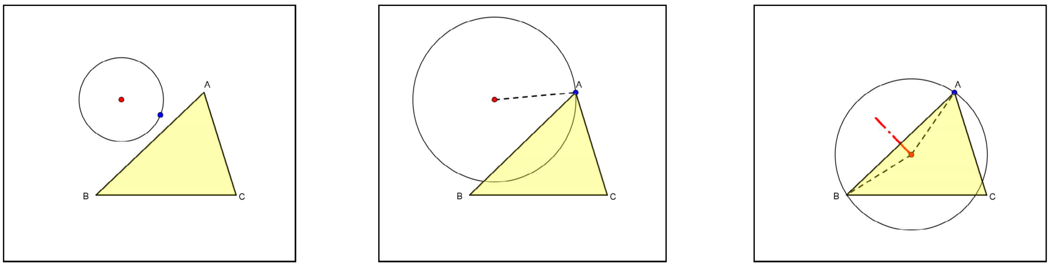
In the robust construction task, perpendicular bisectors of the three sides are first constructed using the "Perpendicular Bisector" tool . A circle centred at their intersection (found by the "Intersection" tool ) and passing through either one vertex (say *A*) is constructed using the "Circle" tool , and it could be seen that this circle also passes through the other two vertices (*B* and *C*). By dragging the vertices of the triangle, learners can check the validity of the construction by seeing that the circle always passes through the vertices. They can also see that the circumcentre lies outside the triangle when the triangle is obtuse (Figure 7).



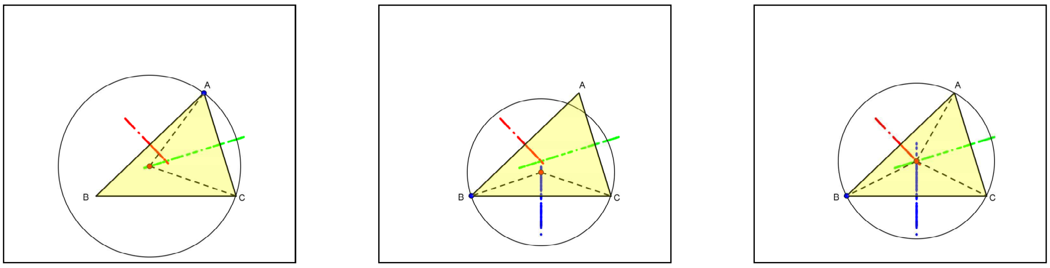
(a) (b) (c)

Figure 7: A robust construction task of finding the circumcircle of a triangle

In the soft construction task (http://www.geogebratube.org/student/m3958), learners are first given the triangle and a circle which can be moved by dragging its centre (in red) and a blue point on its circumference (Figure 8(a)). Learners first drag the blue point to either one vertex, say *A*, and a dotted line joining *A* and the centre would then be shown (Figure 8(b)). Then they drag the red centre to different positions at which the circle also passes through another vertex *B*, and when this happens a dotted line joining *B* and the centre would be shown. These positions of the centre are marked in red, and learners can see that the locus of the centres of circles passing through *A* and *B* is a straight line (Figure 8(c)). Learners can then be asked what this line of locus should be, and the two dotted lines from the centre to *A* and *B* providing hints for them to reason that this line is the perpendicular bisector of *AB* (through looking at two congruent triangles). Once they recognize that the locus of the centres should be the perpendicular bisector, they can find empirically the loci of the centres when the circle passes through *A*, *C* (Figure 8(d)) and *B*, *C* (Figure 8(e)), and finally visualize that the circumcircle should centred at the intersection of the three loci, i.e. the intersection of the perpendicular bisectors of the three sides (Figure 8(f)).



(a) (b) (c)



(d) (e) (f)

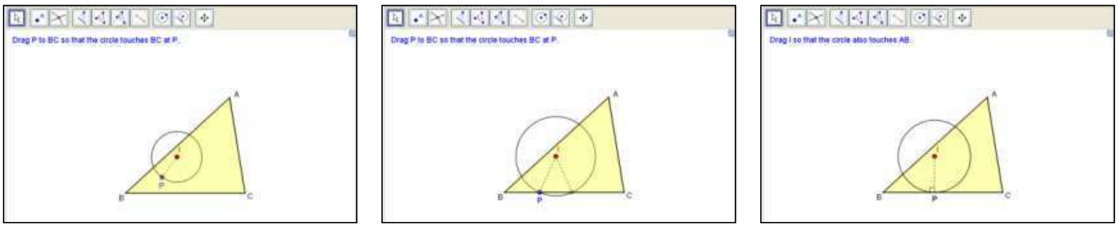
Figure 8: A soft construction task of finding the circumcircle of a triangle

The above example illustrates how a task in soft construction could foster operative apprehension by recording the loci of positions at which the eye construction satisfies the given conditions. These loci of positions provide insights to solve the problem, and also mediate the reasoning of why the problem could be solved in this way. I now propose the following principle of using soft constructions to design task fostering operative apprehension for visualization and reasoning in DGE.

Principle of using soft constructions to foster operative apprehension

Learners are provided opportunities to perform soft (eye) construction by dragging. The loci of the dragging satisfying the given conditions, together with the other elements supporting their visualization and reasoning, would be shown to the learners so that theoretical elements could emerge from the empirical evidences.

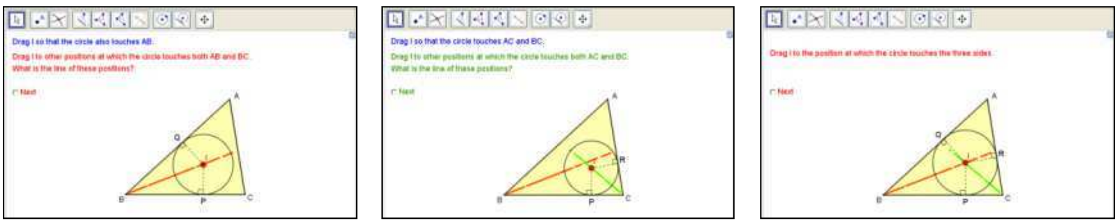
I further elaborate the above principle using a more sophisticated task of finding the incircle of a triangle (http://www.geogebratube.org/student/m4363). In this task the triangle and a circle of centre *I* and passing through *P* are given, and the radius *IP* is also shown. Learners are first asked to drag *P* to the side *BC*, then another dotted line would be shown to indicate that there are two intersections (Figure 9(b)). By dragging *P* towards the other intersection learners would visualize that for the circle to touch *BC*, the two dotted radii should overlap to form one radius *IP* perpendicular to *BC* (Figure 9(c)). I also anticipate that this process of dragging, together with the overlapping of the two radii, would help learners to reason why the tangent of a circle should be perpendicular to the radius.



(a) (b) (c)

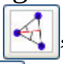
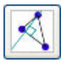
Figure 9: A task of finding the incircle of a triangle – touching one side

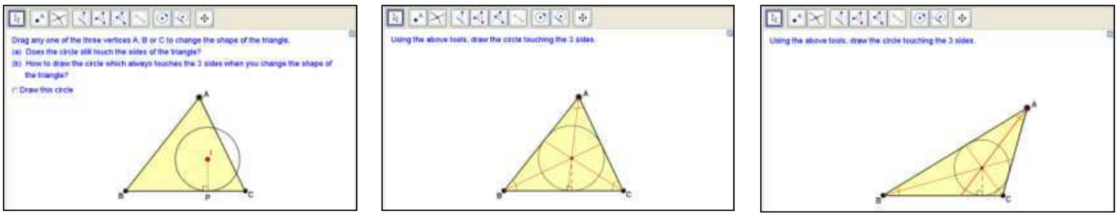
Once the circle touches *BC*, *P* can no longer be dragged and learners are asked to drag the centre *I* to different positions so that the circle would also touch *AB*, and the locus of *I* is marked in red (Figure 10(a)). Learners are prompted to identify this line of locus as the angle bisector at *B*, and could explain this by looking at the congruent triangles *IBP* and *IBQ*. Similarly learners identify the locus of *I* at which the circle touches *BC* and *AC* as another angle bisector at *C* (Figure 10(b)), and see that the circle would touch the three sides when *I* is at the intersection of the angle bisectors (Figure 10(c)).



(a) (b) (c)

Figure 10: A task of finding the incircle of a triangle – finding the incentre

Finally, the three vertices of the triangle are made draggable to the learners and they are asked to drag the vertex *A* to change the shape of the triangle, and see that the original circle no longer touches the three sides (Figure 11(a)). They are then asked to perform robust construct of the incircle by constructing the suitable lines in a triangle using the given tools (median , angle bisector , altitude  and perpendicular bisector ) and the touching circle tool  (Figure 11(b)). They can then check the validity of their construction by dragging the vertices (Figure 11(c)).



(a) (b) (c)

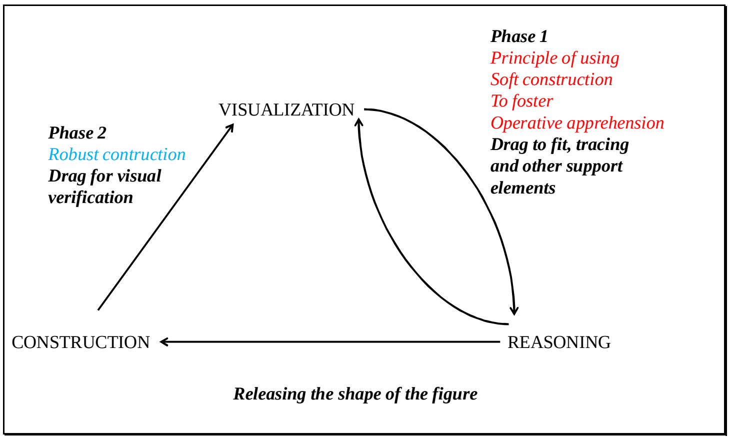
Figure 11: A task of finding the incircle of a triangle – from soft to robust construction

5 Discussions and Implications

Based on the above illustrations, I propose a model of task design in DGE to foster operative apprehension for visualization and reasoning by modifying Duval's model of geometrical reasoning as follows:

Task Design Model in DGE to foster Operative Apprehension for

Visualization and Reasoning through Dragging



Task design in this model consists of two phases. In Phase 1, the *Principle of using Soft Construction to foster Operative Apprehension* is applied so as to foster students' operative apprehension through *soft construction*, i.e. to use the *drag to fit* strategy to find solutions satisfying the given conditions. In the process of soft construction, the *trace* of the *locus of validity* (Leung and Lopez-Real, 2002) and other support elements that mediate reasoning would be shown. Use my in-centre task as an example (p.7), the dotted radii and their overlapping through dragging (Figure 9) are the support elements which are shown to students to mediate the insight and reasoning of perpendicularity of the radius and the side when the circle touches it. Similarly, the traces and the radii shown by the software in Figure 10 support the reasoning that the traces are the angle bisectors of the triangle and in-centre lies on their intersection.

In this phase dragging and tracing are the *cognitive tools* (Leung, 2011) to start a recursive cycle between visualization and reasoning until a solution and its explanation is reached. In the design of the in-circle task, students are guided to first visualize through dragging that the radius has to be perpendicular to the side when the circle touches it. Then with this property students are further guided to visualize through dragging and tracing that centre of the circle must lie on a certain line when the circle touches two sides of the triangle. They are then guided to reason, using the trace and the dotted radii, that this line is in fact the angle bisector. Finally they further visualize that when the centre lies on the intersection of the two angle bisectors, the circle would touch all the three sides and at this stage they should be able to explain why this happens.

When the solution and its explanation are reached in Phase 1, the task is then transited to Phase two in which students are required to use the construction tools given by the software to do a *robust* construction to verify the solution and explanations they obtained in Phase 1. This is done by releasing the shape of the figure in the problem so that students observe that the soft construction in Phase 1 no longer works when the shape of the figure is changed (Figure 11(a)). Students are then asked to use the tools of the software to construct a robust in-circle that always touch the three sides (Figure 11(b)(c)). In this phase dragging is a tool for visual verification of the construction.

This model shows how the different roles of robust and soft constructions could foster operative apprehension, through which the synergy of visualization, reasoning and construction can be facilitated. If we agree with Duval that developing visualization and reasoning abilities to favour the synergy of the three cognitive processes is of crucial importance for the teaching of geometry, designing tasks to fostering operative apprehension for visualization and reasoning in DGE effectively would then be very promising to promote the teaching of geometry. This is also a great challenge to all teachers, educators and researchers. It is hoped that the principle and the model of task design in DGE proposed in this paper could provide a useful initiation for further discussions, challenges and refinement in future task design research.

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Dynamic representations for algebraic objects available in AlNuSet: how develop meanings of the notions involved in the equation solution.

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This article presents a research work aimed to study the impact of new representations for algebraic objects handled by AlNuSet, an artefact of dynamic algebra recently developed. In particular this article focus on equations and identities. Traditionally, conceptual construction of algebraic equality is pursued through solving equations using techniques of symbolic manipulation. Several researches have highlighted that this approach does not favor the construction of an appropriate sense for the notion of algebraic equality, identity or for that of solution of equation. To this aim, teachers need new tools that provide effective representations for algebraic objects. Thus, this article presents new semiotic representations available in AlNuSet that support both students in the conceptualization of algebraic objects and teachers in the design of innovative educational sequences of tasks allowing to tackle algebra not only in syntactic aspects but also in the semantic ones.

Keywords : dynamic algebra, representations, dynamic representations, equation, identity, existential quantifier, universal quantifier

Introduction

With a significant percentage of students, the current teaching of algebra seems not to be sufficient to effective develop skills and knowledge to master this domain of knowledge (Sfard & Linchevski, 1994; Kieran, 2006).

To clarify students' difficulties in algebra I refer to Chevallard's (1985) *para-mathematical* end *proto-mathematical* notions.

On the base of these definitions, the algebraic concepts taught at school are mathematical notions. However, Chevallard highlights that there are some notions, called para-mathematical notions (i.e. variable, parameter, unknown, universal or existential quantifier, algebraic expressions, …) that have a name and that can be defined in class, but often no educational activity is performed about them. Moreover, there are proto-mathematical notions (for example, recognize a simple factorization in an expression of the second degree or the equivalence between expressions), that are not explicitly defined in class, but they live in educational practice and students are not aware of their existence. The development of para-mathematical and proto-mathematical notions, seem to be very important to give meanings to the algebraic objects and to better control the algebraic manipulation activities. Students' difficulties in algebra to develop para-mathematical and proto-mathematical notions, seem to be due to the fact that these notions can be grasp by the control on their algebraic meaning (Arzarello, Bazzini, Chiappini, 1995). To make explicit proto-mathematical notions and to ensure that students grasp meaning of para-mathematica notions, teachers need new artifacts, which make available new algebraic objects' representations. Our research hypothesis is that AlNuSet (ALgebra on the NUmerical SETs), can be effectively used to mediate conceptual development necessary to master the notion of algebraic equality.

AlNuSet was developed in the context of ReMath (IST - 4 - 26751) EC project and is designed for students of lower and upper secondary school (yrs 12-13 to 16/17). It was developed by the research group of ITD (Istituto per le Tecnologie Didattiche)-CNR (Centro Nazionale di Ricerca) of Genoa (Italy) that I belong. Alnuset allows, through visual-spatial approach and dynamic representations, to built meanings of para-matematical and proto-mathematica notions.

In this article I show how this kind of representations implemented in AlNuSet can support the teaching and learning algebra making explicit para-matematical and proto-matematical notions involving in equality and equation process solution. The sequence of tasks was designed in collaborations with teachers involved in ReMath project and it was experimented with them. This article focus on a sequence of two tasks, aimed to presents a new educational approach to equation and identity, which is part of a more completed Teaching Guide for teachers that I have designed with my group of research (authors of AlNuSet) and experimented in ReMath project context.

Short description of AlNuSet

Alnuset is constituted of three strictly integrated components: Algebraic Line component, Symbolic Manipulator component and Functions component. Since this paper concern tasks where only the first two components are used, I will describe in the following only the Algebraic Line component and the Symbolic Manipulator component (for more details see www.alnuset.com).

The main educational characteristics of these components are:

making available, through a visuo-spatial approach, dynamic representations of expressions and algebraic propositions (in Algebraic Line component);

making available, through an operative and deductive approach, axioms and rules to transform and manipulate algebraic expressions/propositions (in Manipulator component).

The main characteristic of Algebraic Line component is the possibility to represent an algebraic variable as a mobile point on the line, namely, a point that can be dragged with the mouse along the line. Dragging the mobile point along the line, the letter associated to that point assumes the values of numerical set instantiated. This new visuo-spatial approach which exploits dynamic representations, allows to make explicit the notion of *variable* as mobile point on the line that assume, in numerical set instantiated, all its possible values. When dragging the mobile point on the line, all algebraic expressions containing such a variable move accordingly. This feature has transformed the number line into an algebraic line where it is possible to operate with algebraic expressions and propositions through techniques of quantitative and dynamic nature. This visuo-spatial approach to algebra allows student to handle dynamic representations as new semiotic representations of algebraic objects in Algebraic Line. This makes a dynamic algebra possible and supports students in the conceptualization of algebraic objects. The most important new semiotic representations available in Algebraic Line of AlNuSet that are involved in the tasks presented in this paper are:

The post-it (see Fig 1b). The belonging of two expressions to the same post-it can be connected to the notion of equality and equivalence between expressions.

Color of the dot associated to a proposition and/or to the numerical set constructed by the user (see Fig 1a and 1b). The color accordance between the dot associated to a proposition and that associated to the numerical set constructed by the user, can be connected to the notion of truth set of the proposition and can be used to validate the constructed numerical set as the truth set of the proposition.

The main characteristic of Symbolic Manipulator component is the possibility to transform algebraic expressions and propositions through a set of particular commands. These commands correspond to basic properties of operations, properties of equality and inequality, logic operations among propositions, operations among sets. Another characteristic is the possibility to create a new transformation rule once it has been proved. These characteristics support the development of skills regarding the algebraic transformation and they contribute to assign a meaning of proof to it. Note that not all the commands are available in the same time: only the commands that can be applied on the selected part of expression are available. The user can easily control the whole process of algebraic transformation exploiting feedback given by the system. Moreover, the user can verify the preservation of the equivalence in the transformation representing the transformed forms on the Algebraic Line.

Mediation provides by AlNuSet is profoundly different from that proposed by the software used for the traditional teaching of algebra: both new dynamic representations, based on visuo-spatial approach, which allow to reify semiotic representations and its strictly integrated components, allow students to link semantic and symbolic nature of algebraic objects. Moreover, AlNuSet's components allow teachers to make explicit these links and, in particular, to make explicit para-matematical notions (i.e. variable, unknown, universal or existential quantifier, algebraic expressions,…) and proto-matematical notions (for example, choose the appropriate rule to transform an expression to obtain another; reconizing a factorization in an expressions…) involved in equality and equation notions.

Several researches (Chiappini, Pedemonte, & Robotti, 2008; Chiappini, Robotti, Trgalova, 2010;) highlighted the educational potentialities of this software showing how the approaches described above are effective in order to understand the basic mathematical concepts (fractions, expressions, equations,…)

Some particular tasks of the Teaching Guide

Note that, the standard teaching approach to algebra is to find, within the algebraic formalism, the meanings of algebraic notions (for instance, the manipulation of an equation allows to find values that, replaced in the initial equality, make it true; thus, the meaning of solutions of equation is found in the algebraic manipulation itself). The new approach to algebra offered by AlNuSet enables teachers to change this sequence: at first construct the meanings of algebraic notions (for instance, what does mean solving an equation) and, only then, deal with the formalism, finding in it the meanings previously built. This new approach is possible using dynamic representations that are available in AlNuSet, as I will show in the following sessions concerning the sequence of two tasks.

The tasks aim to exploit operative and representative possibilities of the Algebraic Line component to design explorative activities that address the construction of meanings for the notions involved in the solution of equations and identities. Then, tasks aim to consider formal and syntactical aspect of algebraic manipulation involved in the solution of equations and identities. These algebraic manipulations are performed in Manipulator component by the application of rules and axioms available on its interface. The equivalence of the expressions obtained by these transformations can be verified on the Algebraic Line. In this way, the links between semantics aspects and syntactic aspects can be made perceptively evident.

The sequence of tasks presented in this paper is composed of different activities that focus on the solution of a first-degree equation and the comparison between equation and identity. For each activity, I will present the task assigned to the students and I will discuss how we have designed task taking into account the dynamic representations available in Algebraic Line of AlNuSet and the rules and axioms available in Manipulator component of AlNuSet. Finally, I will describe the mediation role of AlNuSet in teaching-learning process of the algebraic notions taken into account.

I underline that Teaching Guide aims to offer indications to better exploit the potentialities of AlNuSet in order to improve teaching and learning algebra (Chiappini G., Pedemonte B., Robotti E., 2010; for more details see Educational Activities in www.alnuset.com).

Linear algebraic equation

In order to master algebraic equality, the conceptual development of notions of equation, identity, truth-value and truth set is necessary. Moreover, to express the way in which a letter can condition the truth-value of an equality, you must be able to consciously use universal and existential quantifiers, even though in implicit way (that is, as para-mathematical notion). For this reason, the sequence of tasks presented in this paper aims to answer the following questions concerning equations, their process resolution and identities:

What does solving an equation mean? What could we intend for truth-value of an equality? What is an algebraic identity and what differentiates it from a conditioned equality?

In order to promote the construction of meanings for the notions of equation, identity, truth-value and truth set involved in answering these questions, the sequence of tasks promotes the para-mathematical notions of unknown, algebraic expressions, variable, universal and existential quantifier, equality. Similarly, it makes explicit the proto-mathematical notion concerning equivalent expressions.

To this aims, I show how the use of Algebraic Line’s functionnalities and Manipulator’s functionnalities allow students to approach the notions of equation and identity in different ways: perceptive way (by a visuo-spatial approach mediated by dynamic representations) in Algebraic line component, and syntactic way (by operative and deductive approach) in Manipulator component.

To make explicit the connection between AlNuSet, as tool for teaching and learning algebra, the representations and the task design, I highlight that, the design of the following tasks is based on integration of these two different approaches (perceptive and syntactic), which are possible only in the AlNuSet’s components, in order to promote the development of semantic competences and operational competences engaged in the solution of equations and identities.

Task 1

Consider the following two polynomials: x+2; 2*x+3.

Explain what does it mean putting the equal sign between them, or, in other words, explain how you interpret the following writing x+2=2*x+3.

Now, represent on the Algebraic Line of AlNuSet the two polynomials: x+2 and 2*x+3 and drag the mobile point x to verify your hypothesis. Insert in Manipulator the equality x+2=2*x+3 and solve it using the appropriate commands. What does it obtains?

The first part of the task is aimed to make explicit students' conceptions on the notion of algebraic equality. As matter of fact, on the semantic plan, equality denotes a truth-value (true/false) related to the statement of a comparison.

When the expressions composing the equality are strictly numerical, it is easy verifying their truth-value through some simple calculations (e.g., 2*4+2=10 is true while 2*3+1=10 is false). Experiences with numerical equality contribute to structure a sense of computational result for the "=" sign. This sense can be an obstacle in the conceptualization of algebraic equality as relation between two terms, as highlighted by several researches (Kieran 1989, Filloy et al. 2000). Coherently with this meaning, we expect that the equal sign between two expressions could suggest that the computation related to the two terms of the equality has to produce the same result whatever the values of the variable x is.

The second part of the task aims to discuss this misconception and to construct the idea that the equality between two members (polynomials) is conditioned by the value of x. As matter of fact, when the expressions composing the equality are literal, the equality can present different senses because the value assumed by the letter can condition differently its truth-value. In this cases the "=" sign should suggest to verify numerical conditions of the variable for which two terms are equal. In others words, the equality is conditioned by the values of x.

To make explicit this condition, usually teachers have recourse to substitution of truth-values in the literal equality in order to obtain a true equality. This kind of approach does not seem very effective to grasp the sense of conditioned equality. For this reason, we designed the second part of the task asking students to use Algebraic Line of AlNuSet: by exploiting dynamic representations of the Algebraic Line it is possible to built the meaning of "=" sign between the expressions as conditioned equality in perceptive way. Thus, these representations allow students to grasp the meaning of conditioned equality by means of a visuo-spatial approach rather than a computational approach.

Solution in Algebraic Line component

On the Algebraic Line, students insert the mobile point x, that represent a *variable*, and the expressions x+2 and 2*x+3. Dragging the point x along the line, students can observe that there is only a value of x for which the points of the two expressions assume the same value. This dynamic representation contributes to build the meaning for the equal sign between the expressions as *conditioned equality*. As matter of fact, when student try to verify the equality between expressions, the drag of x is made with a specific aim: to ensure that the two expressions take equal values, that is, they are associated to the same point on the line and they belong to same post-it (yellow square). If dragging is realized with this aim, then the variable can assume the meaning of *unknown*: search for value to be assigned to x so that the two expressions have equal value and equality is true.

In Figure 1, two moments corresponding to the drag of x along the line are represented (Fig. 1a, Fig.1b). Only when x assume value -1, the expressions x+2 and 2*x+3 are on the same point corresponding to the value 1 and they belong to the same post-it. The expressions are equals so that the equality is true.

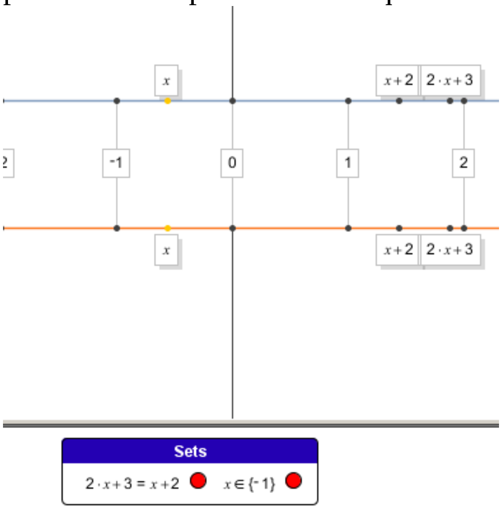

Fig 1a

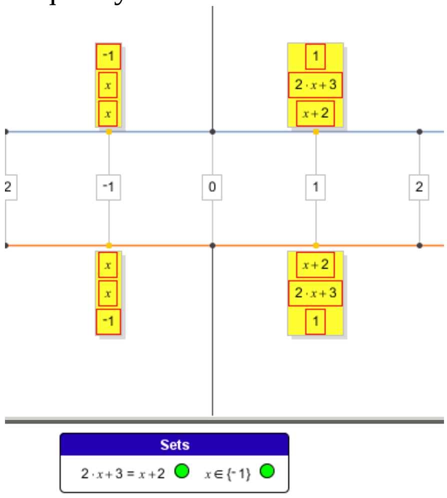

Fig. 1b

Note that software automatically associates a colored dot to the equality: this color is red whatever value of x different to -1 (Fig. 1a: equality is false); instead, the color is green for the value -1 (Fig. 1b: equality is true, so that the value -1 is a *truth value* for the equality). A specific command allows student to construct and visualize the truth set associated to the equality (Fig. 1). AlNuSet associates a colored dot to that set. The color red (Fig 1a) /green (Fig. 1b) of the dot means that the current value of x is an element/not is an element of the constructed set. So that the concordance of color between dot associated to the equality and that associated to the corresponding truth set during the drag of x along the line, is a visual representation allowing students to construct meaning for *truth set for equality*, as set of the only values making true the equality.

The functionality “Tracking” (Fig 2) allows a more complete exploration of the line: students can perceptively experiment that x = -1 is the only value that condition the truth of equality.

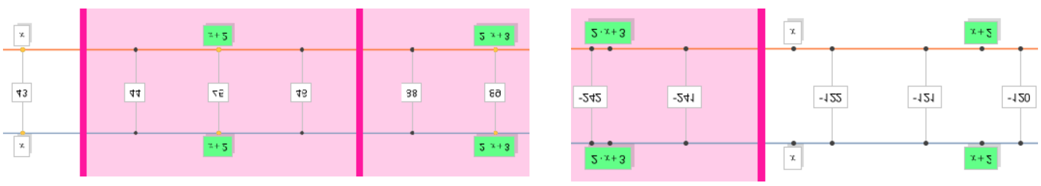

Fig. 2

The following paragraph discusses the syntactic solution of the equation developed in Manipulator component where the development of operational competences necessaries to solve equation is promoted.

Solution in Manipulator component

Manipulator Component allows student to solve equation by different approaches: automatic solution, simplified solution and solution by sequence of steps.

The equation can be automatically solved selecting the command “Simplify Domain” providing the following result (Fig 3). Teacher can use this command to achieve heuristic or pragmatic aims. This command can be used to solve any polynomial equation.

$x+2=2\cdot x+3$
$x\in\{-1\}$

Figure 3

A simplified solution can be performed using the “Simplify” command to perform calculus considered standard (Fig. 4).

$x+2=2\cdot x+3$
$x+2-(2\cdot x+3)=0$
$-1\cdot x-1=0$
$-1\cdot x=1$
$x=-1\cdot 1$
$x=-1$
$x\in\{-1\}$

Figure 4

Moreover, to solve equation, students can use the commands of the interface that refer to basic properties of operations and propositions (equalities and inequalities). The solution appears as a sequence of steps (Fig. 5):

$x+2=2\cdot x+3$
$x+2-(2\cdot x+3)=0$
$x+2+-(2\cdot x+3)=0$
$x+2+-1\cdot(2\cdot x+3)=0$
$x+2+-1\cdot 2\cdot x+-1\cdot 3=0$
$x+-1\cdot 2\cdot x+2+-1\cdot 3=0$
$x\cdot 1+-1\cdot 2\cdot x+2+-1\cdot 3=0$
$1\cdot x+-1\cdot 2\cdot x+2+-1\cdot 3=0$
$(1+-1\cdot 2)\cdot x+2+-1\cdot 3=0$
$(-1)\cdot x+2+-1\cdot 3=0$
$(-1)\cdot x+-1=0$
$(-1)\cdot x-1=0$
$-1\cdot x-1=0$
$-1\cdot x=1$
$x=-1\cdot 1$
$x=-1$
$x\in\{-1\}$

Fig 5

Transformation of the equation by the rules available on the interface of Manipulator component.

The solution is provided in the same form of the Set window presents in the Algebraic Line component.

The solution performed in Manipulator component allows students to focus on the meaning of the formal transformation: obtaining the result by means of chain of deductive steps. The expressions obtained in each step transform the form of the expression maintaining the equivalence. As matter of fact, each step is obtained applying a rule available on the interface. The equivalence between expressions can be verified on the Algebraic line: for instance, the value of x for which the expression x+2 is equal to 2*x+3, is the same value for which the expression x+2-(2*x+3) is equal to zero (expressions referring to first and second line in Fig. 5). Moreover, note that Manipulator, making available rules and axioms on its interface, supports student's memory: he shall not recall to memory the possible rules or axioms applicable to the part of the expression selected, but he should consider which of the rules and axioms provided by Manipulator, are the most convenient and effective to reach the educational aim. Again, note that the calculus performed in application of the rule is completely charged by software. So that student can concentrate his/her attention on the meaning of the procedure (deductive sequence) rather than in calculus. This fact, as well as the support on students' memory in recalling rules and axioms, is particularly interesting for students having difficulties in calculus or dyscalculia.

The solution obtained in Manipulator component and that obtained in Algebraic Line component have the same form (the last line of Fig 5 and the window Sets in Fig. 1b). Thus, the solution obtained by means the formal approach is charged of the meaning of truth set (truth value) for the equation. The integration between visual-spatial approach, developed in Algebraic Line, and syntactical and formal approach, developed in Manipulator, is realized.

In this way, the teaching and learning of the notions concerning solution of linear algebraic equation is radically changed: the formal manipulation, performed in Manipulator component, is arrival point and not starting point in order to develop meanings related to the notions of equation, truth-value and truth set, unknown, algebraic expressions, variable.

For this reason, it seems possible to say that our hypothesis, concerning the possibility to perform innovative teaching and learning algebra, is realized.

Identity

As presented above, when the expressions composing the equality are literal the equality can present different senses related to the value assumed by the letter. To interpret the equality on the semantic plane, it is necessary to distinguish if it has to be considered as equation or as identity. The "=" sign assigns to the equality the sense of equation when its two members are equal only for specific values of the letter as presented in taks1. Instead, the "=" sign gives to the equality the sense of identity when its two members are equal whatever the numerical value of the letter is, as in the following task 2

Task 2

Consider the following equalities: 2x + 3 = 5x and 2x + 3x = 5x

Represent on the Algebraic Line of AlNuSet the expressions composing the two equalities. Drag x along the line and find the values of x for which the expressions are true.

The following figures (Fig. 6a and Fig. 6b) show the expressions composing equalities on the line in two moments corresponding to different values of x.

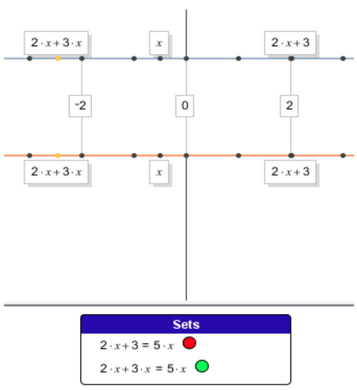

Fig 6a

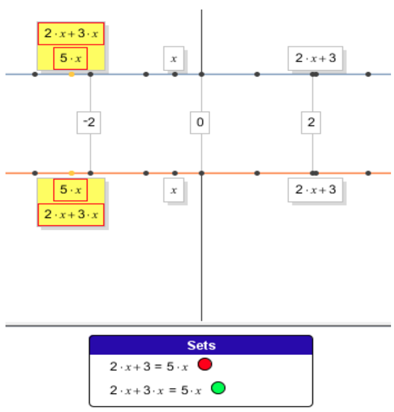

Fig 6b

Dragging x along the line it is possible to verify that the expressions 2x+3x and 5x refer, for all values of x, to the same point and they belong to the same post-it. The equality is verified for all values of x.

Note that the colour of the dot associated to the equation 2x+3x=5x is always green so that the equality is always true and it is an *identity*. Instead, the expressions 2x+3 and 5x refer to the same point and they belong to the same post-it only for x=1. This is a conditioned equality that is true for the only value 1 of x. This kind of task and the support of the Algebraic Line, allow teachers to explicitly speak about universal and existential quantifiers.

The dynamic representations available on the Algebraic Line allow teacher to promote meaning of universal and existential quantifiers exploiting the perceptive and visuo-spatial approach.

Conclusion

The sequence of tasks proposed in this article shows the impact of new representations performed in Algebraic Line and Manipulator components of AlNuSet in the solution process of equations and identities. The efficacy of visuo-spatial and dynamic representations available in Algebraic Line, to built algebraic meanings of notions involved in solution of equations and identities, is presented. Moreover, in this paper I described how the formal procedure of equation' solution, performed in Manipulator component, can be linked effectively to the meanings developed by by perceptive approach in Algebraic Line component. The algebraic transformations performed in Manipulator allow students to focus on the educational aim concerning the equivalence between algebraic propositions. This equivalence is performed on the base of rules and axioms that have to be chosen among that presented on the interface. Moreover, the proof of the equivalence is not charged of calculus. The sequence of tasks in which the use of Algebraic Line of AlNuSet is required, shows how it is possible to promote the para-mathematical notions of unknown, algebraic expressions, variable, universal and existential quantifier, equality. Moreover, the tasks show how the use of Manipulator can support proto-mathematical notion concerning equivalent expressions. Finally, the sequence of tasks performed by means of AlNuSet' functionalities shows how teachers were able to innovate the teaching learning algebra starting with the development of the meanings of algebraic notions before moving to the formal aspects related to syntactic manipulations. The discussion of tasks shows that this is possible by means of new dynamic representations available in AlNuSet.

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Designing Tasks for Use With Digital Technology

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In this paper we use a research-oriented approach to describe some of the features of technology-integrated tasks with epistemic value. Some examples of the construction of these tasks are given along with teacher factors required to employ them. A brief case study of the collaboration of the authors, a researcher and a teacher, in the development and implementation of tasks addressing Riemann integration is presented.

Keywords: Tasks, Technology, Epistemic value, Pedagogy, Teachers

Background

Many educators promote digital technology as having a role to play in helping students to develop mathematical thinking. However, some of the key affordances arising from technology use emanate from the tasks we use with it. In turn the implementation of these tasks depends in large measure on the knowledge and orientations (Schoenfeld, 2010) of the teacher. Thus while engagement with digital technologies may change or challenge traditional student learning trajectories (Anthony & Walshaw, 2007), it can, and will, also challenge teaching trajectories. Hence, one of the central issues in the use of technology[13] is the design and implementation of tasks that will encourage the learning and understanding of mathematics, and in particular mathematical thinking (*ibid*). To design these tasks we have to consider how this mathematical thinking is mediated by the tasks that are employed with the technology. In this paper we address the design aspects of such tasks from a theoretical standpoint, present a brief case study and then discuss some implications for teachers.

We take a relatively broad view of the idea of a task and take it to mean an activity that students are asked to attempt (or choose to) in a mathematics lesson (and possibly complete at another time). It may be short, as in solving a single linear algebraic equation, or long, such as a structured or semi-structured investigation of the graphical properties of polynomial functions of degree one and two. Solving the tasks they are given will require certain techniques from students, which depend on the task and, in the words of Artigue (2002), may go beyond routine actions to include a complex combination of reasoning and routine work. To generate a wide range of

[13] The term technology should be read as meaning digital technology throughout this paper.

such techniques tasks need to take students beyond the 'routine', placing the task at centre stage. These classroom tasks will require students to exhibit techniques that have either *epistemic or pragmatic value* (Artigue, 2002). In the former the focus is on the production of knowledge of the mathematical object under study, whereas in the latter the techniques are perceived and evaluated in terms of their productive potential. Techniques with pragmatic value alone include what are often termed procedural tasks. In a technological classroom setting Thomas (2009a, p. 152) suggests that procedural tasks such as:

1. Solve $x^2 - 3x + 2 = 0$
2. Draw the graph of $y = (x - 1)(x + 2)$
3. Differentiate $y = 3x^3 - 9x$
4. Find $\int_1^3 (3x - 2)$

have little or no epistemic value, since solving these with 'black box' technology use does not assist students to focus on, or understand, the constructs of mathematics.

Other tasks that do have well known by-hand techniques can quickly assume the character of procedural tasks when technology is used. One example of this is when a well-known by-hand scheme is transferred to a technological tool, with a subsequent change of technique. Using the scheme for finding the inverse of a function given as $y = f(x)$ that makes x the subject of the formula and then replaces x with y can be partly executed on a CAS instrument using the technique of solving the equation for x. Figure 1 shows two examples of how students given the task of finding the inverse function by hand have integrated the use of CAS into this scheme, employing a direct command to perform the first step of making x the subject of the equation. The use of the CAS is indicated by the ♦ symbol, as requested. We note that in each case the student writes down precisely the form given by the CAS, and the first student (top) has also written down the details of each step of the scheme. Neither student has used the technique of entering the function and simply using the CAS to provide the inverse function, but instead have integrated it into their by-hand method.

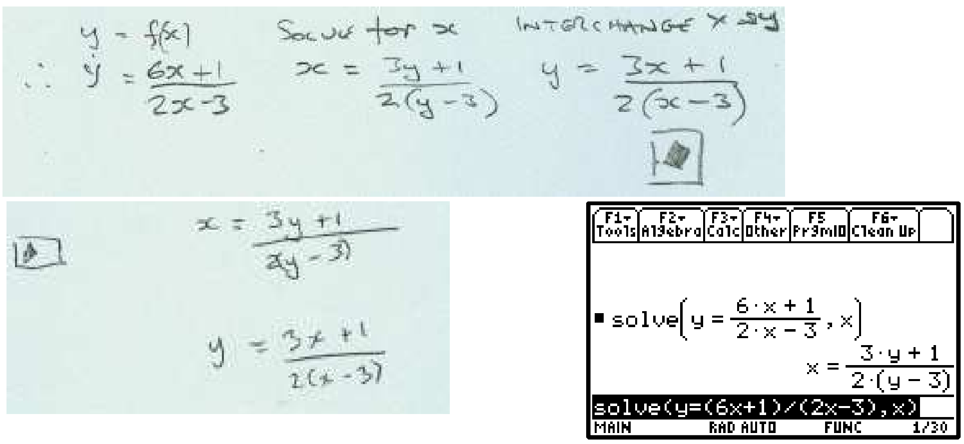


Figure 1. Integration of CAS in the task:

Given $f(x) = \dfrac{6x+1}{2x-3}$ is invertible ($x \neq 1.5$), find $f^{-1}(x)$.

What should be considered when constructing tasks employing technology that have epistemic value? Based on a consideration of the benefits of a Task-Technique-Theory (TTT) approach (derived from the anthropological theory of didactics of Chevallard), Kieran and Drijvers (2006) suggest a number of features of a task with epistemic value. Based on his research, these have been added to by Thomas (2009b), and include addressing a mathematical concept or idea, examining the role of language and asking students to write about how they interpret their work, considering dynamic multi-representations, involving treatments and, especially, conversions between representations (Duval, 2006) and versatile interactions with representations (Thomas, 2008), integrating technological and by-hand techniques, aiming for generalisation, and getting students to think about proof and development of mathematical theory. We consider how the following task, based on one in Thomas (2009a), seeks to follow these research-based design principles.

> Which of the following equations have the same solutions? Explain how you worked out your answers and write down reasons for your answers. Use a graphic calculator or computer to help you work out and support your answers with an explanation. In general, what equations of this type would have the same solutions, and why?
>
> (a) $x^2 + x + 1 = 2x^2 - x - 3$ (b) $x^2 + x + 5 = 2x^2 - x + 1$
> (c) $x^2 - x + 1 = 2x^2 - 3x - 3$ (d) $x^2 + 2x + 1 = 2x^2 - 2x - 3$
> (e) $2x^2 + 3x - 1 = 3x^2 + x - 5$

First, the students are encouraged to think about language and to write about their work giving reasons. The mathematical theory focussed on in this task includes the notion of the nature of a solution, the transformations under which solutions are conserved, and the difference between *legitimate transformations* of an equation, ones that are mathematically correct, and *productive transformations*, ones that move more rapidly towards the solution (Hong, Thomas & Kwon, 2000). While this task is approachable using a graphic calculator, a computer program such as GeoGebra (see Figure 2) can enhance the dynamic nature of the links to a graphical representation, allowing one to interact with the representations to ask questions that involve treatments—Why does the x solution remain on the same vertical line?—and conversions—What is the effect on the graphs of adding ax^2 or bx to both sides of an equation, and why? Thus the aim is that the graphs support the learner in allowing versatile interactions with the graphs and providing an important algebra-graph linking of representations (an algebra view is available in GeoGebra). To complete the task requires the integration of technological and by-hand techniques, students are asked to generalise from the given equations to any equations of the same form, and the inclusion of the word 'why' may motivate them to think about giving reasons, leading to argumentation and possibly the notion of proving a conjecture.

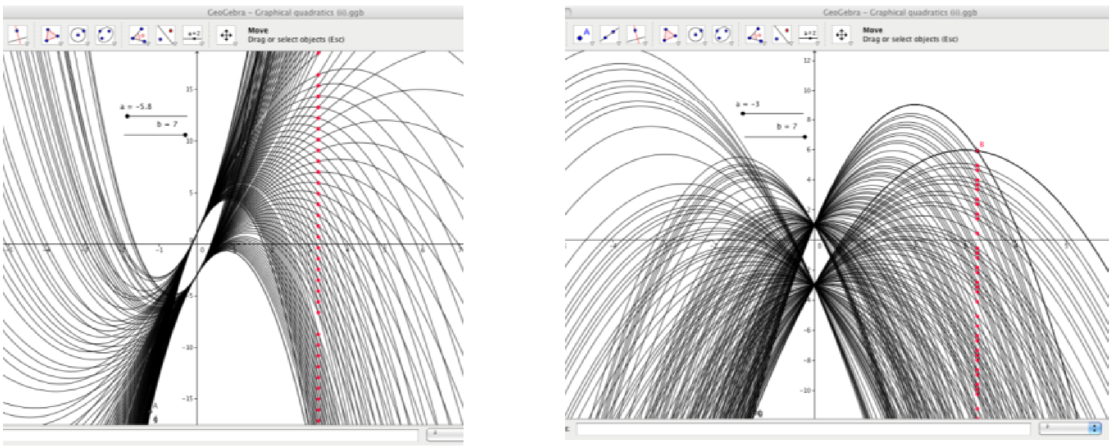


Figure 2. Using GeoGebra to investigate adding ax^2 (left) and bx (right) to both sides of an equation.

Another example of a task that exemplifies using technology to engage students with a mathematical concept in the manner described above is presented in Heid, Thomas and Zbiek (2012, in press). This requires a consideration of a linear transformation, asking for the algebraic form a function f would take when its graph is reflected in the line $y = k$, for some real k. Assuming that students do not know a procedure for writing this down, and may not even have a mental scheme for a suitable technique, technology, such as a CAS calculator or computer program, can help them solve the general problem inductively through specific examples. This may start with reflecting the graph of, say, $y = x^2 + 3x$ in the line $y = 2$. The technology can be used to draw the graphs, possibly linking the algebraic and graphical representations in a dynamic way. In Figure 3 we see an example of how a slider is used to vary the function (left) and then the value of k (right) and the program is set up to reflect the function. The first step shows that the points of intersection of the functions and $y = 2$ are invariant under reflection, and hence all lie on the line $y = 2$ and the second that all the points of intersection lie on the graph of the function as k varies. Likewise the x-value of the vertex is invariant under reflection in both cases. The algebra view of GeoGebra also gives access to the algebraic form of the reflected function. It is hoped that students may develop a technique for the reflection involving translating the graph vertically by $-k$, then reflecting in the x-axis ($g(x)=-f(x)$), and then translating vertically by $+k$. This means linking translations and reflections across representations, relating graphs to algebraic forms $f(x) + k$ and $-f(x)$.

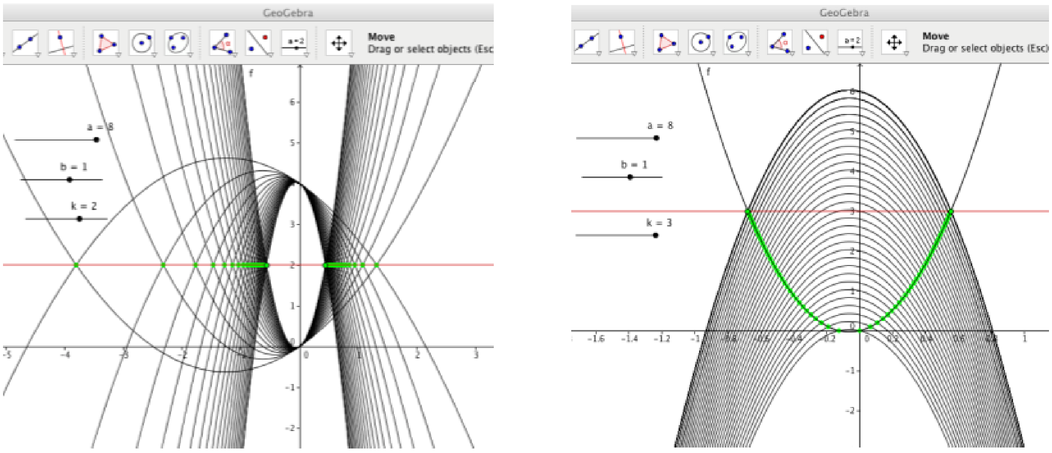


Figure 3. Dynamic reflection of graphs in the line $x = k$ using GeoGebra.

Of course a key feature of the task is the encouragement to generalise algebraically for any function and the line $y = k$. Since $g(x)$ is a reflection of $f(x)$ in $y = k$ *every* point of the graph (and the plane) is similarly reflected. Thus taking a general point $(x, f(x))$, distance n above the line $y = k$, $n = f(x) - k$, and so for our reflected function, $g(x) = k - n = k - (f(x) - k) = 2k - f(x)$. Hence, the result of reflecting the graph of the function $f(x)$ in the line $y = k$ is to obtain a function $g(x) = 2k - f(x)$. For example, reflecting of the graph of $f(x) = 3x^4 - 2x + 1$ in the line $y = -2$ gives the graph of the function $g(x) = -4 - (3x^4 - 2x + 1) = 2x - 3x^4 - 5$. Involving students in inductive reasoning from examples might thus enable them to abstract this general result, describe it, and seek to prove it, thus engaging deeply with the mathematics.

Teacher Knowledge

The ability of a teacher to construct and use tasks with epistemic value, such as those described above, requires what Thomas (Thomas & Hong, 2005; Hong & Thomas, 2006) calls *pedagogical technology knowledge* (PTK—see Figure 4). This emerging framework comprises positive orientations (Schoenfeld, 2010—includes beliefs and attitudes), goals and affect regarding technology use, strong *mathematical knowledge for teaching* (Hill & Ball, 2004) and good *instrumentalisation* and *instrumentation* of the technological tool. Thus it comprises a teacher's perspective on the technology, their familiarity with it as a teaching tool, and their understanding of mathematics and how to teach it. A teacher with good PTK can understand the principles and techniques required to build didactical situations incorporating technology, with the milieu containing tasks that enable mathematical learning to emerge, mediated by the technology.

A range of factors was identified by Goos (2005) as influencing teacher technology use. This includes: skill and previous experience in using technology; time and opportunities to learn (pre-service education, guidance during practicum and beginning teaching, professional development); availability of appropriate teaching materials; support from colleagues; knowledge of how to integrate technology into mathematics teaching; and beliefs about mathematics and how it is learned. Thus to assist teachers with development of PTK some way of targeting these has to be devised. One way is through partnerships with researchers. Some of the results of research involving such a partnership relating to task design are recorded below.

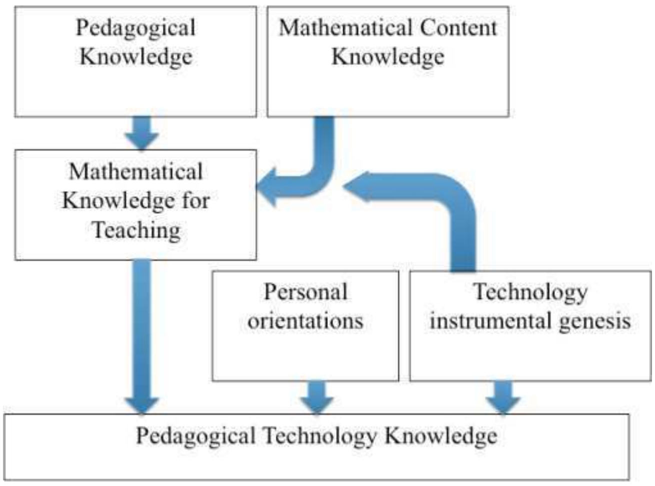


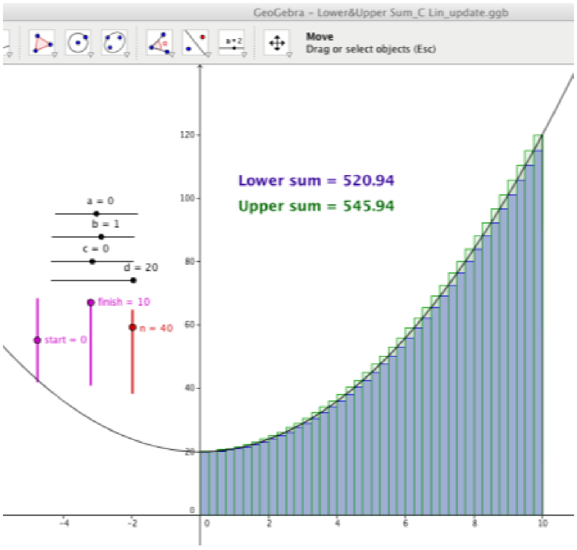
Figure 4. An outline of the construction of Pedagogical Technology Knowledge (PTK).

In this research the two authors of this article worked together using the principles above to develop suitable tasks incorporating technology use that would assist students to construct a fuller understanding of Riemann integration (see Lin & Thomas, 2011). This included helping students develop a rich and comprehensive concept image of definite integral and the ability to coordinate the visual schema of Riemann sums and the analytical schema of the limit of the numerical sequences (Czarnocha, Loch, Prabhu, & Vidakovic, 2001). To do this we developed an embodied cognition approach (Tall, 2008) to the notion of infinity and limit through carefully constructed tasks that would extend students' previous experience and broaden their concept image of definite integral (Rasslan & Tall, 2002). A joint decision was made to use the GeoGebra software and the first-named author (the experienced researcher) passed on to the teacher his ideas on the benefits of the software and how it might be used in this case to construct suitable tasks. The teacher was enthusiastic about using the software, could see the potential pedagogical benefits and was committed to trying it out. The decision was made to use algebraic, graphical and numerical (spreadsheet) features of the GeoGebra program. The task construction then proceeded through cycles of suggestions by the researcher, experimental construction by the teacher, and then refinement through discussion with the researcher, until both were happy with the outcomes. Clear goals for the role and implementation of the technology were agreed.

One of the tasks constructed, relating to the mathematical concept of Riemann sums is presented in more detail here. The teacher established the following teaching trajectory, integrating the technology, for this concept.

> Introductory questions and two thought experiments on finding out distance covered over a period of time given the rate of change of the distance.
> Obtain a range (min&max) of estimates of the area by summing up the rectangles.
> Define the minimum (under-estimate) of the range as the "lower sum" and maximum (over-estimate) as the "upper sum".
> Explore what happens to the estimates when increasing the number of rectangles.
> Explore what happen when we making the time interval really small by comparing the results with actual area using *GeoGebra* software.
> Establish the idea that as the number of rectangles increases, the lower sums are almost equal to the upper sums.

The task used[14] varying numbers of left-hand and right-hand rectangles (from 5 to 500) to approximate the area under the curve from $t = 0$ to $t = 10$ hours (see Figure 5). The students recorded the approximate value of the area on each occasion, as the number of subinterval, or rectangles, was changed dynamically with the slider, in the correct cells in the table provided, shown in Figure 5 (right).



Number of subintervals (rectangles)	Actual Area = 533.33			Other method for better approximation?
	Area Approximation			
	Left Rectangle (Left-hand sum)	Right Rectangle (Right-hand sum)	Accuracy Difference	Using Trapezium
5	440	640	200	540
10	485	585	100	535
20	508.75	558.75	50	533.75
30	516.85	550.19	33.33	533.52
40	520.94	545.94	25	533.44
50	523.4	543.4	20	533.4
60	525.05	541.71	16.67	533.38
80	527.11	539.61	12.5	533.36
100	528.35	538.35	10	533.35
500	532.33	534.33	2	533.33

Filling in the table provided

Figure 5. Using the GeoGebra program to investigate Riemann sums.

In line with the task design principles, the students were then encouraged to write about what they noticed from the program and the results in the table, using the questions: What did you notice about the difference between the left-hand sum and

[14] Note the program was used in a form allowing general investigation of polynomial functions up to degree 4. The spreadsheet view showing numerical sums was also visible to the right of the graph.

right-hand sum (Accuracy Difference) as the number of rectangles increases?; As we increase the number of rectangles drawn between $t = 0$ to $t = 10$ hours, what did you notice about the relationship between the rectangles and the area under the curve? How does this feature relate to the accuracy of the area approximation?; and Can you think about other methods to approximate the area to produce a better estimate? A method using trapeziums followed. This prepared students for presentation of an algebraic generalisation of the limit of Riemann sums.

Overall the mean improvement of the class based on tests before and after the teaching was significantly better than that of a second group of students who covered the same ground but did not do the computer tasks ($t = –2.9095$, $p < 0.01$), and the students had an improved understanding of Riemann sums and numerical methods graphically, although they were less able to carry out the method numerically. The teacher talked to six of the 23 students in the class and when asked how the program had influenced their understanding. All six commented that they liked the dynamic graphics view of GeoGebra and that the visual aspect of the software really helped them learn and understand the integration topic better. Two of their comments were:

> Student 1: I think the visual display and the dynamic image of the program help my understanding of seeing where the formula is derived from. By looking at the different shapes…I understand better how we found the area using different shapes and how it is related to the definite integral.
>
> Student 2: I really like it [GeoGebra] when we were learning about the area approximation using rectangle and also trapezium. I can see it on the graphics view instantly that the approximation is better when you started changing the number of rectangles or trapeziums...it is easier for me to understand because it is not just graphical, it is also instant.

It appears from their comments that the task using GeoGebra particularly helped these students understand better how the concept of integration can be seen to begin with area approximations that improve if we increase the numbers of rectangles or trapeziums, matching more closely the actual area under the graph. This idea can be difficult to illustrate and explain without the aid of such technology. With regard to the way their overall understanding from the module of work improved they said:

> Student 2: I think my knowledge of the relationship between integration and area has changed. It [GeoGebra] probably enhances my knowledge of area and improves it. I mean before I see integration just as a technique or a formula opposite to differentiation. But now…I understand more why we use integration for area.
>
> Student 3: Before I think integration just a formula that is opposite to differentiation. But now I know more why you do integration, like it is for finding out the area. Basically I see more connection between differentiation and integration now.
>
> Student 4: At Year 12, I just think integration as a simple algebra algorithm. Now after you taught us the topic with the computer program, I think I have learned more about integration such as how it relates to the area. I understand it better graphically now.

The teacher enjoyed using the approach developed in her lessons, was confident in doing so, and has continued to use it since. Her growing PTK is partly evidenced in her comment that: “Teachers need to use GeoGebra…for investigative activities in order to increase the interactions between students and the software. More importantly, teachers need to pose interesting and meaningful problems [tasks] so that students can be actively engaged, and try to answer questions and re-formulate concepts in their own terms…an ideal way of using GeoGebra is where students investigate mathematical ideas with the software by making conjectures and testing them out.” She also comments that although the experience of writing the tasks with GeoGebra was initially not easy, it improved her understanding of Riemann integral since she needed to understand the concept thoroughly before she could use all the program functions available in GeoGebra to achieve what we wanted to show the students. Overall she felt “It was a very good experience for me collaborating with [the researcher] trying to write tasks that can improve students' understanding.”

In this paper we have taken a theoretical stance to making suggestions on the kind of features that tasks incorporating technology use might need in order to have epistemic value. These have been exemplified with several tasks that have their basis in the research of the first author. We suggest that the development of tasks like these is not a trivial matter for classroom mathematics teachers and have described how a teacher-researcher collaborative effort is one possible way to assist teachers to build the PTK they need to become confident task designers.

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Designing mathematical modelling tasks in a technology rich secondary school context

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The potential of digital tools to enhance student learning is well researched, however, the potential of technology to promote students' engagement with mathematical modelling tasks has received limited consideration. This paper draws on a research study that aimed to investigate the possibilities that exist for student learning when teachers from six secondary schools designed tasks that anticipated for the use of digital tools within mathematical modelling tasks. The paper describes and analyses the collaboration which took place in identifying principles of design for such tasks.

Keywords: Mathematical modelling and Applications; Digital tools; Technology

Introduction

While there are strong research traditions in the fields of mathematical modelling and applications and the use of digital tools in mathematics classrooms, few studies have explored the potential of the nexus which exists between these two powerful approaches to thinking (Geiger, Faragher and Goos, 2010). Mathematical modelling is often described as a process involving the formulation of a mathematical representation of a real world situation and then using mathematics to derive results, interpret the results in terms of the given situation and if necessary, revising the model. The purpose of models is to interpret real world situations and/or make predictions about the future or past states of modelled systems (English, Fox, & Watters, 2005).

There is now a large corpus of literature devoted to the way in which digital tools can enhance teaching and learning opportunities in mathematics classrooms. Studies, however, have tended to report on advantages to instruction in mathematical thinking and learning within content specific domains such as number (e.g., Kieran & Guzmán, 2005), geometry (e.g., Laborde, Kynigos, Hollebrands & Straesser, 2006), algebra and calculus (e.g., Ferrara, Pratt & Robutta, 2006) or social aspects of classroom practice such as collaborative investigative practice (e.g., Beatty & Geiger, 2010). Thus, there is little research on how digital tools can be used in tandem with mathematical knowledge to work on problems that exist in the real world, as Zevenbergen (2004) observes:

> While such innovations [ICTs] have been useful in enhancing understandings of school mathematics, less is known about the transfer of such knowledge, skills and dispositions to the world beyond schools. Given the high tech world that students will enter once they leave schools, there needs to be recognition of the new demands of these changed workplaces. (p. 99)

Given this identified need for students to be provided opportunity to use digital tools when working on real world problems consideration needs to be given to the nature of the learning experiences, and the tasks at the centre of these experiences, students should encounter within school mathematics classes. The aim of this paper is to explore an approach to the design and implementation of tasks which focus on the a mathematical modelling approach to teaching and learning that is supported by digital tools. In doing so, the paper will address Theme A, Tools and Representations, through the following research question.

What are the principles of design for technology rich modelling and applications tasks that result in effective learning experiences for students?

Artefacts as mediators of mathematical learning

In developing principles of design for technology integrated modelling and applications tasks the role of artefacts, in this case the task and the digital tool(s), must be examined. Verillon and Rabardel's (1995) iconic work on the distinction between an artefact and an instrument provides insight into the role of artefacts in mediating learning by distinguishing between an artefact, which includes both physical and sign tools that have no intrinsic meaning of their own, and an instrument in which an artefact is used in a meaningful way to work on a specific task. Different tasks make different demands on the user and their relationship with the artefact. The development of this relationship, and thus how the artefact is used, is known as instrumental genesis. Instrumental genesis is a complex process in which, firstly, the potentialities of the artefact for performing a specific task are recognised which transforms the artefact into an instrument (instrumentalisation), and, secondly, there is a process that takes place within the user in order to use the instrument for a particular task (instrumentation) (Artigue, 2002). Instrumentation generates schemas of instrumented action that are either original creations by individuals or pre-existing entities that are appropriated from others. An instrument, therefore, consists of the artefact and the user's associated schemas of instrumented action. The process of instrumental genesis is also dynamic between the instrument and the user as the constraints and affordances of the artefact shape the user's conceptual development while at the same time the user's perception of the possibilities of the artefact during instrumentation can lead to the use of the artefact in ways that were not originally intended by the designers of a tool (Drijvers & Gravemeijer, 2005).

Instrumental genesis has been used to explain how digital tools are transformed into instruments for learning through interaction with teachers and students (e.g., Artigue, 2002). A teacher's activity in promoting a student's instrumental genesis is known as instrumental orchestration (Trouche, 2005). This process recognises the social aspects of learning as it allows for the sharing of schemas as of instrumented action that individuals have developed within a small group or whole class. A teacher can facilitate the appropriation of these schemas by other students by making the nature of these schemas explicit by orchestration classroom interaction around the schemas through careful and selective questioning

More recently, others have attempted to extend our understanding of an instrumental approach to the role of artefacts in mediating learning by recognising that the genesis of an artefact into an instrument takes place within highly interactive environments, such as school staff rooms or mathematics classrooms, where a number of artefacts are used simultaneously. Gueudet and Trouche (2009) extend the definition of artefact by introducing the term *resources* to encompass any artefact with the potential to promote semiotic mediation in the process of learning. *Resources* include entities such as computer applications, student worksheets or discussions with a colleague. A *resource* is appropriated and reshaped by a teacher, in a way that reflects their professional experience in relation to the use of resources, to form a schema of utilisation – a process parallel to the creation of a schema of instrumented action within instrumental genesis. The combination of the resource and the schema of utilisation is called a *document*. The process of documental genesis is an ongoing one as utilisation schemas will be reshaped as a teacher gains more experience through the use of a resource.

A modelling task oriented research project

Six teachers were recruited from six secondary schools; three from each of two different Australian states. Schools were drawn from across different schools systems (government and non-government) and were representative of a range of socio-economic characteristics. Teachers were invited into the project because of their reputations as highly effective teachers with particular skills in the use of digital tools in mathematics learning and their commitment to improving the learning outcomes of their students. The project was managed by two university based researchers – one in each state. The researchers were primarily responsible for the: conceptual development of the project; classroom data collection including lesson observations, teacher and student interviews, and collection of student samples. Teachers were primarily responsible for the development and implementation of technology demanding mathematical modelling tasks. Researchers played a vital role in providing feedback about the effectiveness of tasks trialled in teachers' classrooms. Together teachers and researchers developed principles of design for effective tasks based on their shared experiences while trialling tasks in mathematics classrooms.

This paper reports, specifically, on the work of one teacher and on his students in a Year 11 (15-16 years of age) mathematics class. The curriculum context in which he taught mandated the teaching, learning and assessment of mathematical modelling as a key objective of a state-wide syllabus (educational authorities are state based in Australia). The use of technology in mathematics teaching and learning was also prescribed in the Mathematics B program (incorporating the study of functions, calculus and statistics) in which his students were enrolled. Students had almost unrestricted access to digital technologies including: powerful handheld digital devices with mathematical facilities such as data and function plotters and Computer Algebra Systems; computers with mathematically enabled applications; the internet; and electronic white boards.

The research design consisted of three components: (1) two whole day teacher professional learning meetings which took place at the beginning and middle of the project; (2) three classroom observations for each teacher; and (3) a focus group interview near the end of the project that involved all teachers. The detail and purpose of each of these activities is outlined in Table 1. Further detail on the research methodology can be found in (Geiger, Faragher and Goos, 2010).

Time	Activity
Sept-Dec Year 1	Teacher workshops in each state: research team outline the aims of the project; offer prototype tasks; discussion of principles which underlie prototype tasks.
Jan-April Year 2	Lesson observations ; teacher and student interviews; collection of student work samples; feedback on effectiveness of trialed tasks in relation to modeling and the use of digital tools.
April-June Year 2	Lesson observations; teacher and student interviews; collection of student work samples; feedback on effectiveness of trialed tasks in relation to modeling and the use of digital tools.
July Year 2	Teacher workshops in each state: teachers share exemplars of digital tool and modelling tasks; discussion on principles which underlie teacher developed tasks; research team offer accounts of practice from classroom observations.
Aug-Sept Year 2	Lesson observations; teacher and student interviews; collection of student work samples; feedback on effectiveness of trialed tasks in relation to modeling and the use of digital tools.
Oct-Dec Year 2	Final project meeting and focus group interview in each state; teachers share exemplars of modelling and digital tool tasks; further discussion on principles which underlie teacher developed tasks.

Table 1: Research design

Principles of task design in technology demanding modelling tasks

The teacher (the co-author of this paper) who is the focus of this paper, proved to be an effective designer of technology demanding modelling tasks while, at the same time, demonstrated keen insight into his own design processes and how these developed through the duration of the project. This teacher, in particular, contributed to the development of principles for designing modelling tasks. These principles and their descriptions are presented in Table 2. While these are useful insights they confirm rather than extend what is widely accepted as approaches to designing effective modelling tasks or general advice on good teaching practice.

Principles	Description
Syllabus compliance	The task must meet the requirements of the syllabus for content knowledge and the dimensions related to applications and technology.
Authenticity and relevance	Tasks must be set in an authentic or life-related context. The task must be of interest to the teacher and be of potential interest to the student.
Open-endedness	The mathematics necessary to solve the problem set up in the task should not be immediately apparent. The task must be open-ended in nature providing for opportunity for multiple solution pathways.
Connectivity	Ideally the task must make links to different content areas within the syllabus.
Accessibility	The task must provide opportunity for students to link to their previous learning. There should be provision for multiple entry and exist points. The task should allow for the introduction of scaffolding prompts or hints.
Development	The task must provide challenge and so encourage students to go beyond what they presently know and can do through the modelling process. Students' engagement with the task should provide feedback to the teacher about the development of their understanding.

Table 2: Characteristics of effective modelling tasks

The teacher also provided valuable input into the role technology played in the design of modelling tasks, and indicated that digital tools served as an enabler of each of the identified principles. He provided comment on the role of digital tools in relation to each principle of design.

The use of digital tools is a mandatory element of the state-wide senior secondary mathematics *syllabuses*. Genuinely *authentic* problems are mathematically complex. The representational capabilities of digital tools allow students to accommodate this complexity and thus provide *access* to *authentic* problems that otherwise might be considered beyond the scope of their capabilities.

> If we didn't have the CAS calculators we couldn't do half the stuff that we do. From my perspective it is the integration of the whole lot together. We have a set of data and we try and build a model from that. We do a scatter plot and we make decisions about the model. We build a model and make some sorts of predictions.

Digital tools also provide the means for students with gaps in their content knowledge to *access* challenging problem scenarios.

> Lower achievers may be struggling with differentiation or integration at that particular point in time…but they can still have access to the problem. My lower achieving kids can still engage in the problem and still make some meaningful contributions. If they don't get caught up in all that manipulation they can still be thoughtful about it.

The nature of *authentic open-ended* problems means there is no clear solution pathway and students need to evaluate options as they progress toward a solution. The teacher argued that digital tools offer facilities that are essential for exploring possible solution pathways. Technology also provides the means for *connecting* different types of mathematical knowledge, for example, data representations and functional relationships that modelled patterns in the data.

> Selecting authentic, open tasks to model generally implies the students will need to make use of technology. Even if the teacher has scaffolded the task to facilitate access to the context, there is a requirement that the task be sufficiently open for there to be multi-representations of the solution and perhaps different solutions.

The *authenticity* and *open-endedness* of a problem is enhanced if students are required to collect data relevant to a problem from an original source; a capacity provided by digital tools in his classroom.

> There is often a need to collect data and then to determine whether a relationship exists within that data. Students may need to collect primary data, through the use of probes, or from a video that is then analysed using the technology or use secondary data collected from a newspaper, magazine, web site or some other source.

Used effectively, digital tools provide immediate feedback to students about their initial attempts to build models and solve problems thus progressing students' understanding of the underlying mathematics at the core of the task and hence their mathematical *development*.

> Technology has a significant role to play in the provision of feedback to the student in the first instance, about the models they have built and how well they fit the context being investigated. In mathematical modelling it is important to look for consensus between the mathematics and the context, hence, it is necessary to consider the validity of the conclusions in terms of the context.

Exemplar task and commentary

The principles for design of technology demanding modelling tasks are evident in the following description of a task developed and then implemented by the teacher in his Year 11 mathematics classroom – the *Algal Bloom Problem* outlined in the Figure 1. In developing this task, the teacher had expected his students to build a mathematical model for these data by first creating a scatterplot using their CAS active calculator. A plot of this data suggests a piecewise function (one part linear and one part power function) would be appropriate. The teacher anticipated that students would then use the plot to determine the general form of the functions that would best fit the data and, in due course, develop an equation that would best fit the data. Students were then expected to use the model they had created to respond to the question at the end of the task and also to list any assumptions they made in developing their model and also comment on any limitations they believed were inherent in the repose they provided.

In observing the lesson in which this task was used, the researcher noticed that while every student was able to produce a plot of the data using their handhelds, few had drawn the conclusion that a piecewise function was necessary to model the data. Most students attempted to model the data using a single function, generally by trying to generate a model for the data using the digital handhelds regression model facility. When their single functions were plotted on their screens with the original data points it was obvious that their various functions were a poor fit. When students asked the teacher for assistance he simply encouraged them to have a closer look at their data and explore a wider range of possibilities for fitting a model to the data. After a period of time, two students, working together near the researcher, attempted to fit a piecewise function to the data, and after performing fine adjustments to each part of their function were happy with the result. Their success prompted a subdued celebration by the two students which attracted the teacher's attention. After discussing their conjectured model with the teacher students went on to complete the task. A short period of time after his discussion with these students, the teacher called for the attention of the class and asked them about their progress. The two students near the researcher volunteered and were asked to outline their attempt at the task. When they announced they had decided to make use of a piecewise function, sections of the class responded in different ways. A small number of students indicated agreement with the approach the pair of students were proposing even though the details of the functions other students had used differed. Most students, however, expressed exasperation that they had not noticed what was now an obvious feature of the plotted data. These students then returned to the task and were able to develop a piecewise function that fitted the data for themselves. A small minority of students needed more direct help from the teacher but were also able to develop a model based on a piecewise function by the end of the lesson. The lesson concluded when the teacher asked the students to work further on their assumptions and limitations for homework.

> The CSIRO has been monitoring the rate at which Carbon Dioxide is produced in a section of the Darling River. Over a 20 day period they recorded the rate of CO_2 production in the river. The averages of these measurements appear in the table below.
>
> The CO_2 concentration $[CO_2]$ of the water is of concern because an excessive difference between the $[CO_2]$ at night and the $[CO_2]$ used during the day through photosynthesis can result in algal blooms which then results in oxygen deprivation and death of the resulting animal population and sunlight deprivation leading to death of the plant life and the subsequent death of that section of the river.
>
> From experience it is known that a difference of greater than 5% between the $[CO_2]$ of a water sample at night and the $[CO_2]$ during the day can signal an algal bloom is imminent.
>
> Rate of CO_2 Production versus time

Time in Hours	0	1	2	3	4	5	6	7	8	9
Rate of CO_2 Production	0	-0.042	-0.044	-0.041	-0.039	-0.038	-0.035	-0.03	-0.026	-0.023

Time in Hours	10	11	12	13	14	15	16	17	18	19
Rate of CO_2 Production	-0.02	-0.008	0	0.054	0.045	0.04	0.035	0.03	0.027	0.023

Time in Hours	20	21	22	23	24					
Rate of CO_2 Production	0.02	0.015	0.012	0.005	0					

Is there cause for concern by the CSIRO researchers?
Identify any assumptions and the limitations of your mathematical model.

Figure 4 : Algal Bloom Problem

Discussion and conclusion

This task satisfies each of the principles developed during the project for mathematical modelling tasks and for the use of digital tools within tasks. The use of modelling tasks and digital tools are consistent with mandatory requirements of the relevant state *syllabus*. As national scientific bodies monitor the blue-green algae in the various river systems because of the effect on aquatic wildlife this represents a task set in a near *authentic* life-related context. The task is *open-ended* in that a variety of mathematical models are plausible and the use of different models will lead to different, but still valid, responses to the problem. The available digital tools provided the facility to trial a range of functions to fit a complex underlying pattern and offered immediate feedback on the appropriateness of a conjectured function allowing students to develop specific solutions from a wide range of possibilities. Different types of mathematics were necessary to explore the data (data representation, different forms of function) and so, students were expected to make *connections* to different types of mathematical knowledge. The available technology provided the option of viewing different types of mathematical representations (e.g., scatterplots and function graphs) on a screen at the same time, so enhancing the connecting between these types of mathematical knowledge. Students found the task to be *accessible* as it linked to mathematical knowledge they had studied in previous classes and the teacher made use of progress made by other students to provide a prompt when many were experiencing difficulty. The opportunity to trial a function against the data and receive immediate feedback provided an entry point to most students and so made the problem *accessible*. As the task required students to make use of mathematical knowledge they had already studied in previous lessons within an unfamiliar context it provided opportunity for students' *development* in mathematical knowledge and their capacity to apply this knowledge in real world contexts. Here, digital tools acted as a catalyst for this development by providing feedback which indicated students' first single function conjectures were not consistent with the data.

As outlined above, there is an inseparable interplay between the task and digital tools. The teacher has created the task by drawing on principles for developing effective technology active modelling tasks. These principles are based on the potentialities of both types of resource – the task and the digital tool. In implementing the task, the teacher anticipated how students would interpret the potentials of the task for learning and of the digital tool to act as a resource. The relationship between student, teacher, task and digital tool represents a documental genesis as each element within this genesis transforms the other in some way. The task is transformed, from the perspective of the students when they realise they need to make use of a piecewise rather than a single function in order to model the data presented in the problem. This transformation occurs as a result of an attempt by the students to use a single function and receiving feedback via the digital device that this was an inappropriate model. The use of the digital tool changes from that of a device that provided a specific solution for students once they had made a decision on the general form of the function to model the data into a tool used to explore the data and eventually find a model that fitted the data to their level of satisfaction. Students' learning is also transformed during this same process as they realise the purpose of the task and the digital tool is not to algorithmically implement prior learning but to apply their knowledge and understanding in an original way. The teacher had to transform his approach to the lesson when students took a path he had not anticipated – attempting to fit a single function to the data. He changed his approach by orchestrating the resources at his disposal, in this case the two students who had eventually solved the problem, to provide an insight into the problem other students were yet to see.

At the same time, nearly all of the teacher's principles of design, the characteristics of effective modelling tasks, acted as enablers of the process of instrumental genesis of both digital tools and of the task. The principle of *authenticity and relevance* requires students to recognise the potential of the available digital tools to assist them in exploring and solving the problem described in the task from within both purely mathematical and real world contexts. There was a necessary duality about the schemas of instrumented action required to accommodate the purely mathematical and contextual demands of the task. Students needed to recognise that the real world context demanded the development of a piecewise rather than single function to model the inhalation and exhalation of CO_2. This required a specific use of the digital tool that was different from the development of a single function to model the provided data. Having decided that two functions were needed to model the data, a specific instrumentation of the digital tool was needed to find the most appropriate functions for each section of the piecewise function. This second process takes place within a purely mathematical context.

The *open-endedness* of the task placed students in a position where they were challenged to make choices among multiple potential solution pathways. Thus, students were required to make choices among existing schemas of instrumented action or to generate new schemas. To generate new schemas students must firstly recognising the potential of the digital tool for meeting the challenge defined by the task and then, secondly, develop processes for the use of their digital tool that are specific to the set task.

The principle of *connectivity* designed into this task required students to generate schemas of instrumented action that were inclusive of different types of mathematical content. The CAS active calculator students used while working with the task included the capacity to link statistical plots with the graphs of specific functions, and these functions could be developed using the regression facility of the calculator. With these facilities available, students needed to find ways of taking advantage of the capabilities of their digital tool in engaging with the demands of the task and pursuing a solution. This is a type of instrumental genesis in which the potential of an artefact is only realised through its instrumented action.

The task was designed to link the demands of the activity to students' previous learning as the separate functions required to build an appropriate piecewise function had been studied and applied to real world contexts in earlier classes. Thus, the task was created to be *accessible* to students but, at the same time, required students to apply this previous learning in a more complex context – one in which multiple functions were needed to model a phenomena rather than a single function. This meant that students' existing schemas of instrumented action required adaptation in order to accommodate a more complex scenario. The CAS enabled calculator was the tool the teacher believed would mediate this adaptation through the provision of a medium that provided for the representation of multiple functions against complex data.

The *development* aspect of the design is most apparent in the way the way the teacher invited the pair of students who had found that an appropriate solution required a piecewise function to offer their solution to the whole class and the subsequent realisation by most of the class that this was an insight they had missed. This revelation changed both the ways in which these students used the available digital tools and also the way they viewed the task. In this circumstance the teacher *orchestrated* changes in students' schemas of instrumented action related to both the digital tool and also the task

The episode included in this paper demonstrates it is possible to design for effective technology demanding mathematical modelling tasks, and so the approach offers direction for curriculum designers, teachers and teacher educators. While the teacher had designed an engaging task based on principles developed during the project, students took an approach that was not anticipated by their teacher. The teacher, however, was able to take advantage of students' original but inappropriate approaches, generating a dynamic learning environment where students' knowledge of using mathematics within real world contexts was transformed. This raises a challenge for teachers in how such triggers can be deliberately embedded in planned learning experiences in a way that provides space for the type of documental genesis described in this paper. This also indicates that further research is necessary to investigate how to take advantage of unanticipated events in a well planned lesson and in turn for how teacher educators provide advice about task design and implementation in pre-service and in-service programs.

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Optimizing through geometric reasoning supported by 3-D models: Visual representations of change

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The representations, tools and task discussed were designed as a response to a pedagogical challenge: How can rate of change be investigated meaningfully at stages and ages below calculus? The task involves 3-D models of volume and surface area, to represent and explore differences in volume between pairs of open-topped boxes, as well as an exploration of a 2-D representation with dynamic geometry software. Reasoning during the task, with the affordances offered by the tools, learners as young as 14 were able to decide when and why an optimum volume was reached. Through the lens of conceptual blending, we discuss what mathematical insight, activity, and understanding is available to learners via engagement with our task. We further suggest that the representation of change developed through this task extends as an accessible spatial reasoning technique applicable in a range of other problems in 2-D and 3-D.

Key words: task sequence design, optimization, visual spatial reasoning, 3-D model exploration, dynamic geometry software

Background

In this paper, we report on research about the design of a tool and task intended to develop concepts of change and rate of change with secondary students, pre-service and in-service teachers. The mathematical focus of the task is on geometric reasoning – developing the concept of rate of change through visual spatial reasoning, without reliance on calculation or computation. We will illustrate how different types of tools can afford different mathematical activities, representations, and interactions between representations, as well as how specific tools – in our case 3-D models supported by dynamic geometry software (Geometer's Sketchpad) – can impact student learning and understanding of mathematics. Our understanding of 'task' in this case is in line with the definition offered by Theme A: a teacher designed purposeful 'thing to do' using tools for students in order to activate an interactive tool-based environment to produce mathematical experiences. This task, developed by Whiteley and researched by Mamolo and Whiteley, offers a means to address the following problem:

The Popcorn Box Problem

> Given a square sheet of material, cut equal squares from the corners and fold up the sides to make an open-top box (see Appendix A). How large should the square cut-outs be to make the box contain maximum volume?

A version of this optimization problem was initially developed by the Ontario Association for Mathematics Education (OAME, 2005) for grade 9 and 12 students (ages 14 and 17), as an introduction to optimization prior to calculus, primarily with numerical calculations to compare overall volumes of boxes. Our task affords an avenue of richer investigation of this problem through visual and kinesthetic reasoning prior to the development of algebraic/symbolic reasoning of calculus. It includes the use of 3-D models (pictured in Appendix B) made from clear plastic sheeting and pieces of coloured foam, as well as a dynamic geometry exploration via Geometer's Sketchpad (sample screens of which are provided in Appendix C and D). The original tool, task, and an associated novel representation presented here were designed as a 'proof of concept' that deep, effective reasoning with change, and rate of change, could be enabled for students prior to symbolic manipulation and the algebraic techniques of calculus. This was part of the response by one of the authors (Whiteley) to a challenge during recent curriculum writing in Ontario: Can we being reasoning about 'change' down to earlier years, with minimal algebraic load and a focus on big ideas? In addition, do these tasks support unpacking of concepts of change and optimization by pre-service and in-service teachers?

In what follows, we outline the design principles and development of the task, highlighting the mathematical epistemological goals and principles to the design, as well as the pedagogical considerations and modifications which resulted from implementing the task with diverse sets of learners. These considerations speak to how different types of tools may afford different learning possibilities for learners – providing them with different types of experiences and activity, as well as different ways to represent ideas and concepts. We then go on to address how experience with this task may impact learners' understanding of optimization and, more generally, problem solving. We use as a lens of analysis the framework of conceptual blending developed by Fauconnier and Turner (2002), and offer suggestions of different possible blends afforded to different participant groups from secondary school pupils to their teachers.

Design Principles and Task Development

The initial task was to visually/spatially reason about the optimum shape of an open-topped box (see Whiteley & Mamolo, 2012 for details of the task), enriching a simple investigation geared for pre-calculus students. This investigation required developing a new tool of 'paired boxes with physically represented changes' (see Appendix B). Working with this tool, with a variety of participants from in-service teachers and curriculum writers, through pre-service teachers, senior high school students and students just completing elementary school (14 year olds) pushed the development of the task and associated tools and representations in multiple directions. In this section we describe specifics of the task, how participants engaged with the tools, and the emergent principles which informed the task design.

The 3-D models included manipulative tools of paired boxes, with inserts of foam for volume lost and gained (see Appendix B). The tools were designed, and re-designed after testing, to encourage a visual spatial way of reasoning about rate of change that could emerge naturally via their use. The task was designed to scaffold from exploring changes in volume with physical models, into reasoning supported by visual representations through the dynamic geometry software GSP (see Appendix C), and finally to the important idea of how geometric features can identify non-optimal objects (leaving the optimal objects as the only remaining choice!).

Testing the design

Given that there were analogous examples in both 2-D and 3-D, and that the physical models were more difficult to make for the 3-D situation, in the initial pilot, with a group of classroom teachers, we first presented the 2-D problem in a GSP sketch. The problem involved optimizing the area of a rectangle along a fixed barrier, given a fixed perimeter (the fence on the river) and participants worked in pairs on the computer to solve this. Then a demonstration with the physical model (Appendix B) was presented, during which attention was directed to the materials filling in 'between' the two boxes as physical representations of the change in volume (and not the size of the particular individual volumes). The participants took turns examining the models, taking the 'change in volume' pieces out and comparing them by overlaying the pieces of foam, and concluding which volume was larger. The response of these adults was that the 3-D model was more effective in directing their attention to the change in volume, than the analogous 2-D model had been. As a result, further models were constructed so that every pair of learners could explore a pair of physical boxes for themselves.

All subsequent testing and teaching with the task has started with the 3-D activity, and occurred with high school students (14 and 17 years old), pre-service teachers, in-service teachers. The novelty of this setting invites more attention to the physical representation of the change in volume, and less attention to efforts to convert the problem to symbolic or numerical formulas. The focus on spatial reasoning provoked discussions of important curricular ideas and the accessibility of mathematical concepts when lifted from their symbolic representations. In addition, the unusual appearance of the pairs of models focussed the participants' attention on key built-in 'errors' to the first-approximation of the change in volume (such as the visibly missing 'corner volumes', see Appendix B) and away from incidental defects in the cutting and model construction. This particular feature of the tools (the missing volume in the corners) triggered a discussion of the roles and implications of estimation and refinement in a meaningful and concrete way. This discussion led to the suggestion that the size of the difference in cuts could be reduced by using different materials – e.g. using Bristol board in addition to foam allows exploration of thin differences in cut-size (where the Bristol board illustrates instantaneous rates of change, while the foam illustrates average rates of change).

A bonus of starting with the 3-D tools was that participants made an initial guess about the optimal shape (the echo of folklore that "the cube has the maximum volume for a given surface…"). This was quickly found to be incorrect, just through participants' own viewing of a pair of models. We found during these testing stages that participants approached the task with the sense that there was a puzzle to be sorted out that they 'owned', and were now motivated to understand why and what the maximum might be. In addition, we found that younger students who are less channelled into 2-D situations, brought stronger memories and naïve intuitions about 3-D settings, were less distracted by one-variable algebra (or any algebra), and posed novel questions for exploration. As a result they were more willing to engage in the ‘play’ that got them started along a trajectory reasoning with the affordances built into the models.

Mathematical ideas represented in the task

Via engagement with the task and models, several mathematical ideas and observations emerge in a manner accessible to learners of various ages and mathematical sophistication. Here we identify a few key ones:

i. Volume and surface area can be physically represented.

ii. Change in volume is equal to surface area for the gains minus the surface area for the loss, and these changes in volume (loss and gain) between pairs can be compared qualitatively by naïve overlay strategies (see Appendix B).

iii. Reasoning about ‘change of volume’ shifts to comparing surface areas and recording the loss and the gain – with a visual focus, not with numerical calculation.

iv. If the surface area of the loss from one box does not equal the surface area of the gain of the other box – i.e. the pieces of foam do not overlap completely – then the volume of the smaller box is clearly not the optimum (i.e. we have a non-zero average rate of change).

v. Equivalently, the optimum occurs at the shape when the Bristol board representing ‘volume lost’ completely covers the Bristol board representing ‘volume gained’ (i.e. we have a zero instantaneous rate of change) – which occurs by creating a box with a ‘corner cut’ of 1/6 of the way along an edge (or 1/3 of the way to the centre).

vi. Errors are reduced, and then vanish, as the size of the changes in the cut corners is reduced (a hands-on way of passing to the limit and representing this limiting value).

vii. The invariance of the optimal shape under scaling (proportional reasoning) for the optimum is immediate to people working with this tool.

viii. Geometric optimization of other shapes can be explored by the ‘geometric loss-gain’ variational paradigm.

ix. For students taking calculus, these steps in the task can then be connected to algebraic reasoning, including a model-based introduction to the basic conceptual layers of differential calculus (e.g. average change or secants (thick foam); small differences as limits (Bristol board); optimal volume as rate of change equal to zero).

x. For pre- or in-service teachers, this exploration supports unpacking of the process steps and the big ideas of optimization with differential calculus.

Extending the Task

To the designer, and to the teacher, other questions for investigation arise. Here are some that we have explored with at least some of the groups of participants:

a. Is there a 2-D analogue? Yes: a 2-D variant starts with a fixed perimeter seeking maximum area, e.g. the ‘fence on the river’ problem where three sides of a rectangle must be fenced to produce a maximum area, or variants;

b. A 3-D qualitative variation: Does the volume get bigger if we tilt the sides of the box out (as is done in theatres)? Yes: initially the volume increases – explored by a cross-sectional comparison, supported by dynamic geometry software (see Appendix D);

c. What if we cut corners from a triangle, or a hexagon or…? (There is a delightful general principle: if the edges of the paper are all tangent to an in-circle – cut 1/3 of the way to the centre!);

d. If prompted, a deeper puzzle arises within the more common 2-D variant with all four sides included: Why would the plane shape giving maximum area when constrained by a fixed perimeter (a square) also be the shape with a minimum perimeter when constrained to a fixed area? Interestingly, people anticipate this duality will work ("it sounds right!") – but the supportive reasoning, linked to a fundamental duality principle in operations research, requires further shift in representation to points on an area/perimeter graph, accessible only to the most adept problem solvers.

We are reminded that problems become 'similar' in the mind of the problem solver when the representation, or more generally, the reasoning, is transferred. One key to transferring this geometric 'change in pairs' or 'geometric loss-gain' reasoning is to provide other problems where it can be applied either quantitatively, or qualitatively. The above list provides some starting points for exploring this transfer.

In the remainder of the paper, we turn our attention to how engagement with our task can impact student learning and understanding of mathematics. For our analysis, we use the framework of conceptual blending (Fauconnier & Turner, 2002), and offer different possible blends that emerged from participants' engagement with the task. The blends afforded differ based on learners' mathematical background and sophistication, along with their own learning goals. We highlight some of the important blends available for a variety of learners.

Conceptual Blending: A Framework for Analysis

Conceptual blending (Fauconnier & Turner, 1998, 2002) is a theory which describes how new inferences can arise when two representations and associated ways of reasoning (or 'input spaces') are brought together in a 'blended concept'. The 'blend' can be thought of as a mapping which combines certain features of the two input spaces and projects them onto a third (newly formed) mental space. (In the blend, other features are not mapped, shifting the focus and reducing the cognitive load for further reasoning.) Blending processes are used to conceptualize actual things such as computer viruses, fictional things such as talking animals, and impossible things. Although blends may sometimes be bizarre, "the inferences generated inside them are often useful and [can] lead to productive changes in the conceptualizer's knowledge base" (Couson & Oakley, 2005, p.1513). Blending is not a metaphorical or analogical map, but rather it is a specific way to combine and infer from and about information from two or more mental spaces (Fauconnier & Turner, 2002). The partial representations from an individual's perceptions and concepts that are contained in the prior mental spaces blend by "the establishment and exploitation of mappings, the activation of background knowledge, and frequently involve the use of mental imagery and mental simulation" (Couson & Oakley, 2005, p.1513).

The emergent blended space arises in three ways: "through *composition* of projections from the inputs, through *completion* based on independently recruited frames and scenarios, and through *elaboration*" (Fauconnier & Turner, 2002, p. 48, emphasis in original). Specifically, composition creates new relations not previously existent in the separate input spaces, while completion allows the composite structure in the blended space to be thought of as part of a larger structure in the blend, and elaboration, or 'running the blend' consists of cognitive work performed within the blend to exploit and elaborate upon the composite structure (Fauconnier, 1997, p.150-1). The blend continues to offer the individual ways to access each of the original representations, in a flexible manner. For instance, in our case, the three input spaces, are (i) the word problem of maximum measured volume, (ii) the classical approach to optimization through calculus and rate of change (often held as a symbolic procedure with formulas), and (iii) the spatial reasoning (i.e. the variational exploration of change in volume represented physically with pieces of foam, and then Bristol board see Appendix B). Such a blend would allow an individual to:

i. compose - to explore change physically by starting with a pair of representative examples (e.g. boxes) and focus primarily on the change in volumes (ΔV for the secant in symbols);
ii. compose - to conceptualize change in volume physically by considering surface area for loss and gain (ΔV in symbols, pieces of foam in the model);
iii. complete - to focus on the sign of the change, with a simple physical comparison via foam inserts, to determine which changes will make the volume larger;
iv. complete - to consider sources of error – and minimize them (an informal invitation to a limit process, which is natural in the physical model);
v. elaborate - to consider what boxes cannot be the optimum and why (rate of change is not zero in both the models and the symbols);
vi. elaborate and combine – to eliminate the boxes with non-zero rates of changes and determine the single box shape which remains as the optimum (in both the model and the symbols).

We note that a blend is both an internal cognitive process and a cultural artefact. Once achieved by someone and shared – this new blend becomes a possible cognitive approach requiring less cognitive load for others who have the appropriate parts to develop their own internal blend. Our task provides support for such transmission because the tools are external and the task focuses attention on key features listed above. The use of an external representation further supports shared conversations about the blended concept, the process and the reasoning.

The original background to such model-based visual reasoning was scattered reflections by Whiteley which evolved over several decades of classroom teaching practice, observing patterns of deep, but often hidden, connections between: perimeter and area; and between surface area and change in volume, trying to anticipate why at answer would be what the symbolic computations produced. The goal was to have both a symbolic answer and a visually sensible answer that matched! Implicitly – the key blend was being developed and repeatedly explored in pedagogical contexts. However, that form of reasoning was not part of shared experiences or curricular practices, and therefore was not accessible, or even communicable, without the external support provided by our task. The physical models as tools and representations provided an essential support for communicating, for refining the representation, for reasoning, and then for embedding this one clear example into a widening mix of problems and solutions that fit with the new representation, and the larger conceptual blend. Whiteley (2011) provides a more extensive discussion of how the framework of blending gives insight into this modelling and model building process, illustrated with the task presented here. We turn our attention now toward different blends created by prospective and practicing teachers, university students, and secondary school students.

Emergent blends from engagement with the popcorn box task

The representation of change and optimization in the model(s) is 'sensible' to a wide variety of problem solvers. The key steps with the model are similar for a very wide variety of users. The conceptual blends and the context, however, are very different:

- **a.** For practicing teachers, we interpret both a completion and composition of a blend. Specifically, the task allows an opportunity to un-pack the basic processes of calculus and optimization in an unusual context, supporting careful reflection on the steps (completion), and an alternative visual way to reason through the problems (composition). The teachers create a new blend between a well-grasped symbolic sense of the processes of calculus, and a novel spatial sense of optimizing in geometric problems. The 'aha' moment when the 'loss-gain' representation is also found in the symbolic derivative set to zero is exciting to witness.
- **b.** For prospective teachers, a similar situation arises. The task is also an opportunity to un-pack the concepts of calculus, and invites reflection to develop a more flexible and 'thicker' conceptual basis for the study of change and optimization (completing a blend). Individuals are creating (composing) a new blend between a sometimes fragile symbolic sense of the processes of calculus, and a novel spatial sense of optimizing and checking optima in geometric problems, which will strengthen both (elaboration).
- **c.** Alternatively, for practicing and prospective teachers, the geometric paradigm available through our task can be blended with the prior formula-based procedural calculus knowledge, to re-infuse it with sense and visual estimation that grounds the algebraic solutions to geometric optimization (elaboration and completion).
- **d.** For senior secondary students or university students starting calculus, or doing an initial review, the task affords a blend where key processes are experienced twice, in two representations, supporting an initial development of both procedural steps and conceptual reasoning. Students are creating an 'immediate' balanced blend which supports flexible approaches to solving problems (composition and completion).
- **e.** For younger secondary students, even down to age 14, this task provides an accessible tool to solve problems they had heard about but had not been able to approach (e.g. why does a square have the maximum volume for a fixed perimeter?). The task affords a reasoning power, and a representation, which solves optimization problems and lays an initial foundation for a later mature blend with the second symbolic / algebraic approach, with sufficient parallels in the reasoning to make the calculus 'make sense' (composition).

Concluding Remarks

The task and task design are intimately linked to a novel visual representation of loss-gain in geometric optimization, and to reasoning processes that provide an alternative to symbolic manipulations – an alternative accessible to young students. The task and activity emergent from the task can be analysed via the lens of conceptual blending, which can inform the teacher/researcher about both the conceptual development of his/her students as well as the possibilities or deficiencies of the tools and models. The task also offers a context in which the actions and expressed reflections of students/research participants can give insight into spatial reasoning, and the possibility of new rich blends, given initial procedural symbolic ways of solving problems. The task was carried out in pairs (and then debriefed in larger groups), which encouraged development of shared blends, and allowed the researchers to observe more of the student activity and reasoning. There are many questions that arise when observing how participants act (and fail to act) with the materials, which can lead to an enriched understanding of what mathematical experiences are necessary for the learner.

The tool representing loss and gain, the associated visual representation of volume changes and the associated task provided a way “to change what we see” (Hoffman 1998) and ‘how’ we see. With the directed ‘seeing’ flowing from the affordances in the task, such as the ‘missing corners’, the colour coded pieces of foam, and the clear plastic sheeting. This change in ‘seeing’ / change in ‘thinking’ would otherwise not have been accessible to, or at least noticed by, most participants. With practice, the problem solver internalizes the representations, and can reason ‘spatially’ in the mind – imaging and imagining situations, and changes, which carry on the reasoning internally. In the language of Fauconnier and Turner (1998, 2002), the task affords an opportunity for composing a blend from a physical input space – the models – and a mental input space – the procedural knowledge of calculus – to create a newly formed understanding of optimization. This may allow completion or elaboration of a blend for individuals well-versed in calculus, such as prospective and practicing teachers. In the blended space, each ‘visual step’ in the task has an algebraic representation, and each key step in algebraic reasoning has visual support to confirm that the ‘solution makes sense’. In our view, variants of this intended blend support a flexible problem solving cycle.

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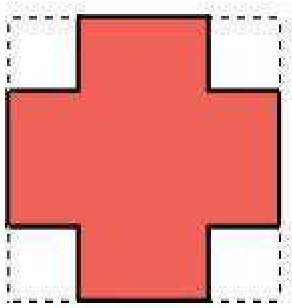
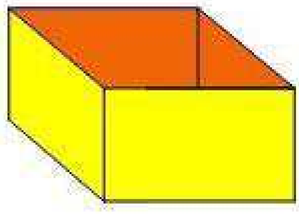
Appendix A: The popcorn box problem:

(A) The problem being explored is a classical optimization / calculus problem:

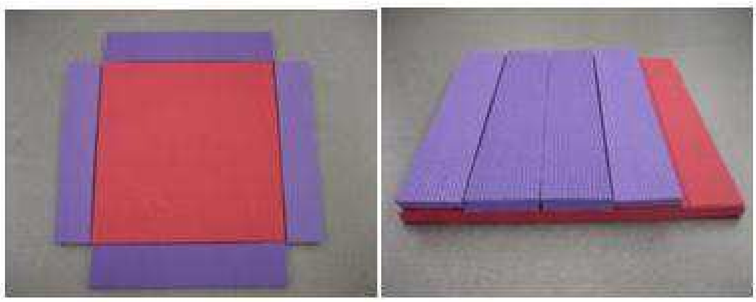
Given a square sheet of material, equal squares are cut out of the corners, and the sides are folded up to make an open topped box. Which size of corners should be cut out in order to make the maximum volume? Why?

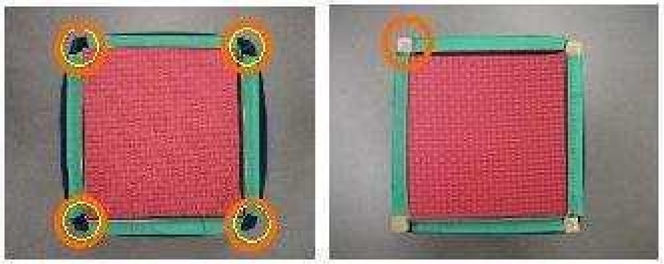

Appendix B: The 3-D Models

Pairs of clear plastic 'popcorn boxes' and physical representations of loss and gain in volume (blue/purple and red pieces of foam, respectively):

(i) Two pairs of popcorn boxes with foam inserts representing changes in volume

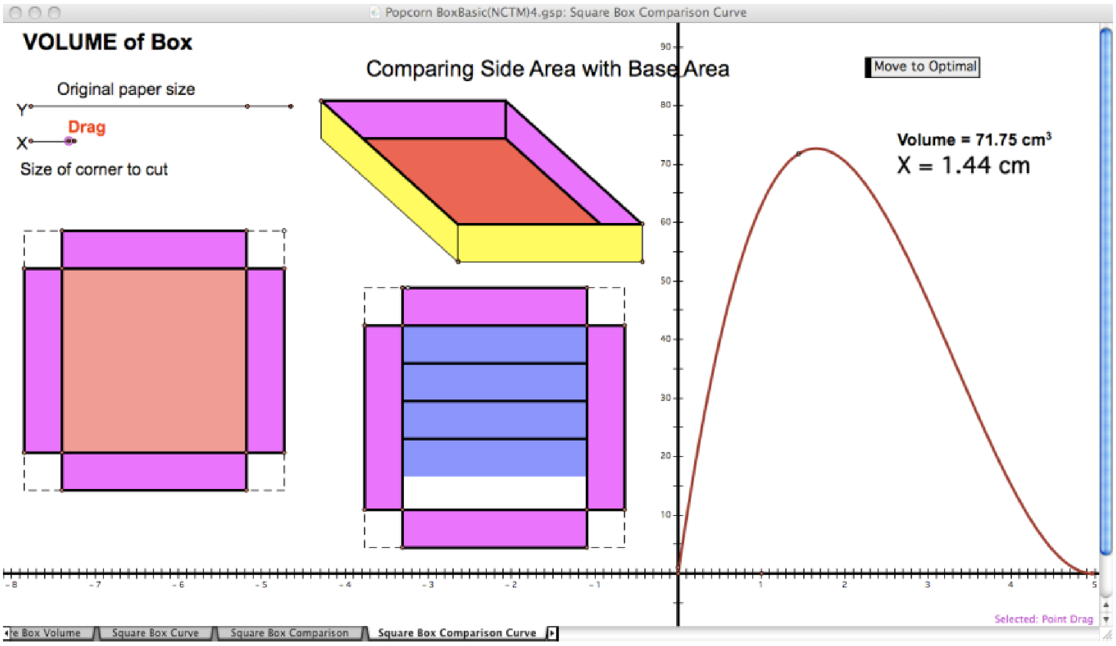


(ii) Foam inserts removed and compared via overlaying

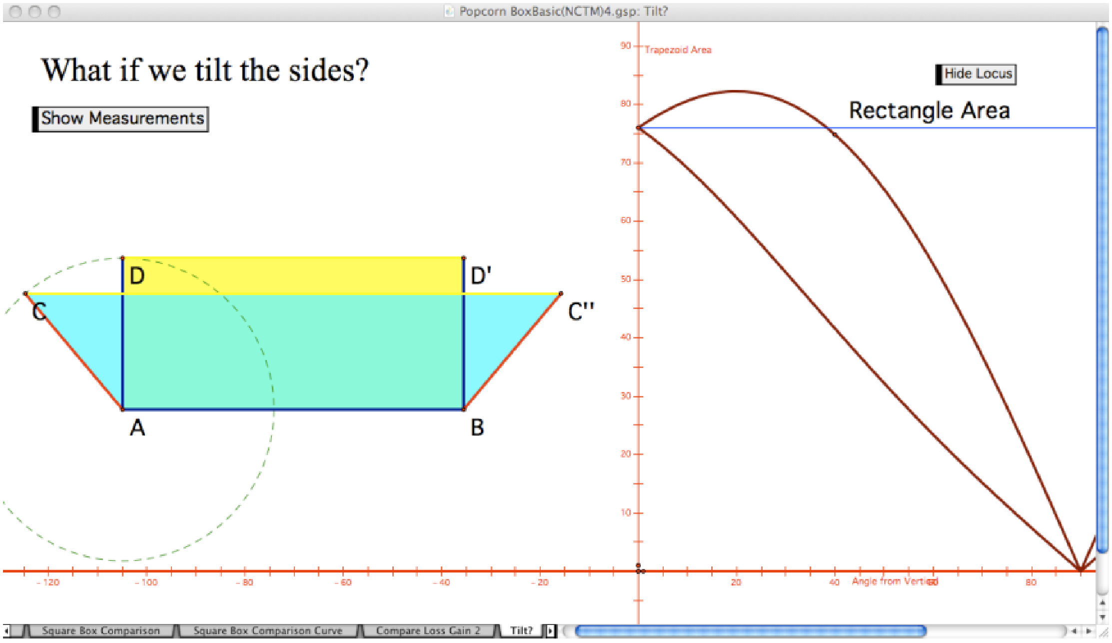


(iii) 'Missing corners' source of error – volume missing (left) and then filled in (right)

Appendix C: GSP Screen of solution method – supplemented by graph



Appendix D: Extending the exploration to make predictions about boxes with 'tilted' sides



Instrumental value and semiotic value of ostensives and task design

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This text presents, by means of various examples, how the notions of instrumental value and semiotic value of an ostensive, developed in Anthropological Theory of the Didactic, make possible to design or analyse mathematical activity , in particular tool-based task design.

Keywords: Instrumental value, semiotic value, ostensive, non-ostensive Anthropological Theory of the Didactic.

Introduction

This text presents a theoretical point of view about conception and analysis of task design. Our framework is the Anthropological Theory of the Didactic (hereafter ATD).We thus consider the mathematical activity in a large sense: the task, but also the techniques to perform it and all the discourses about it. In the research paradigm of this theoretical framework, the didactic facts and the mathematics facts are the two sides of the same coin. Thus, we consider that didactical knowledge serves mathematical knowledge, and in the same way mathematical knowledge impacts on situations (see Margolinas and Wozniak, 2012). Therefore we look what does happen and what could happen in the classroom from these different angles. In this text, we focus on a particular point about the observed activities: the objects that are used to do mathematics are considered as elements of constraints and conditions of the mathematical activity.

In a "classe de cinquième de college" in France – children between twelve and thirteen years old –, a teacher asks pupils to build a perpendicular bisector of a given line segment [AB]. He obtains two kinds of constructions (see figure 1).

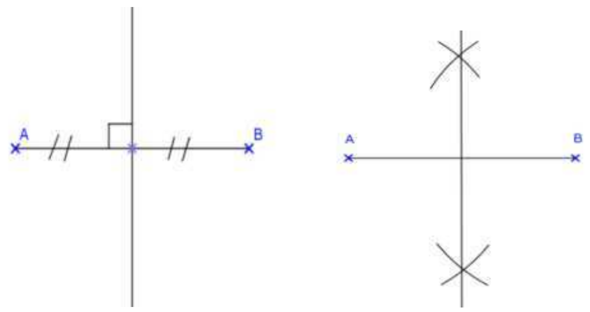

Figure 1- Constructions of the perpendicular bisector of the given line segment [AB]

Coming with the required line, several features appear: small parallel lines on two parts of the line segment [AB], a small square and a cross with the intersection of the perpendicular bisector and the line segment for the first drawing or arcs for the second one. These graphic ostensives evoke or even "say", the geometrical properties used to build the perpendicular bisector of the segment. In the left-hand drawing, the pupil has traced the perpendicular with the segment passing by his midpoint, whereas in the right-hand drawing, the second pupil has built the whole points equidistant from the ends of the segment.

These graphic ostensives do not evoke only the geometrical properties that they incarnate but also others ostensives, materials now: graduated ruler and set square in one case, ruler (not graduated) and compass in the other case. Resorting to the ostensives is thus controlled by objects that have no materiality, no perceptibility, but that are only evoked by the ostensives. In return, depending on the characterization of the perpendicular bisector used by the pupil, he uses different geometrical tools and draws lines and signs characteristic of the properties used. The ostensives embody and reveal the property of the perpendicular bisector underlying the geometrical construction. Thus, the concept of ostensive is linked in a consubstantial way to the concept of non-ostensive – notion, concept, idea – which cannot be handled but only evoked through the associated ostensives. As Chevallard (1994) emphasized, every implementation of a technique assumes the resort to two kinds of objects: ostensives and non-ostensives.

We can see through this small example how the tools used to perform a task, but also the way they are used, lead to a different mathematical activity. Therefore, the didactical knowledge on the relationship between ostensive and non-ostensive objects is essential for the teacher when he designs a mathematical activity. In this example, if a teacher wants the pupils to use a geometrical property instead of another, he privileges some tools rather than others. Thus, the first stage for understanding and analysing the task-design, teaching, learning triad is to reveal ostensives and non-ostensives in each technique used to perform a task. The notion of ostensive was introduced in Anthropological Theory of the Didactic, (Chevallard 1994, Bosch 1995, Bosch & Chevallard 1999) to describe and analyse the mathematical activity. A very broad definition is: "One calls ostensives, the objects which have for us a material form, sensitive, notwithstanding unspecified" (Chevallard, 1994, p.4, our translation). Therefore, any object which can be handled concretely by the body, the voice, the vision is an ostensive: material object, gesture, language, diagram, drawing, graphics, formalism, etc. Then non-ostensives are the objects which "cannot, strictly speaking, be handled: they can only be evoked, through the handling of ostensives associated." (Chevallard, 1994, p. 5, our translation). Thus for example, the concept of quantity related to a thumb, an index and a major raised is a non-ostensive that is not accessible by perception apart from its representation by an ostensive like the fingers which were raised, the figure "3", the word "three", or a drawing like ***. Consequently any work of conceptualization seems the fruit of a work on ostensives which gives access to non-ostensives. The analysis and the design of a task to teach and learn mathematics go through the identification of everything that makes signals and "speaks" in the situations, i.e. the whole ostensives and associated non-ostensives.

The aim of this text is to present how in ATD the artefacts are analysed in their two dimensions of tool to act and tool to think within task design for the teaching and learning of mathematics. In the first section we come back on the concepts of ostensive and non-ostensive as element for describing the mathematical activity from ATD point of view. In the second section, we clarify the concepts of instrumental value and semiotic value of ostensive starting from an example; these are essential concepts for the design and the analysis of mathematical activity. In the third section we show how an ostensive can stop being a tool for taking an action because the non-ostensive that it evokes is not operational and does not make sense any more. Then, in the fourth section we show how the choice of an ostensive for designing a task is based on a dialectical play between the instrumental and semiotic values. Lastly, we will show how the notion of praxeology, developed in ATD, can be a useful tool to understand the relationship between ostensives and non-ostensives evoked in a mathematical activity.

1. Ostensives, institution and praxeologies

Let us consider now the following problem[15]: if 8 lollipops cost 10 €, how much are 3 lollipops? Until the middle of the 20th century in France, the expected answer was based on the theory of the ratios and proportions: "8 is to 10 as 3 is to *x*". It results symbolically in writing the proportion "8: 10:: 3: *x*" and is calculated by using the fact that the product of the extremes terms is equal to the product of the middle terms: $8 \times x = 10 \times 3$ thus $x = 10 \times 3/8 = 3.75$. In the 1970's, the reform "of modern mathematics" has modified the curriculum. Then, the modelling of the situations of proportionality by a linear function is expected: if f(8) = 10, then f(3) = f(3/8 × 8) = 3/8 × f(8) = 3/8 ×10 = 3.75. In the 1990's, such problems are solved with "the cross-product" starting with writing the values in a "table of proportionality" (see figure 2): $x = (10 \times 3)/8 = 30/8 = 3.75$.

Number of lollipops	8	3
Price of lollipops (€)	10	x

Figure 2 – Table of proportionality

Nowadays, at the primary school the pupil must use the "rule of three" which uses the return to the unity: if 8 lollipops cost 10 €, then 1 lollipop costs 10 €/8 = 1.25 €. The price of 3 lollipops thus equals 3 × 1.25 € = 3.75 €.

Thus, the techniques used to solve a task and the discourses that justify them depend on the institutions within which they are used. However the institutions generate practices and discourses on the practices that are specific because, as Radford (2002) underlines, the artefacts/ostensives contain within themselves the culture of the institution that produces them but also the culture of the institution that resorts to them.These are the two founder points of the Anthropological Theory of the Didactic. The model of the mathematical activity can be built in terms of praxeologies within institutions, like any other human activity. The personal relationship with an ostensive is regulated by the institutional relationship with this ostensive, inside the institution where it has been activated. In ATD, a praxeology is structured into two components. The praxis component contains techniques to achieve a kind of tasks. The logos component includes the theoretical discourses, called technology, that describe, explain, justify or develop the techniques used, and the justification of technology, called theory. Thus, the mathematical activity is carried out through the handling of ostensives whose instrumental value is perceived through the praxis component whereas the semiotic value nourishes the logos component of praxeologies.

[15] We borrow the idea of this example from Chevallard (1994).

2. Instrumental value and semiotic value of an ostensive

In a “classe de CP” of a primary school – children between six and seven years old – pupils try to calculate the result of the addition “12 + 6”. A pupil counts 12, 13, 14… 17, 18 and raises an additional finger each time. The fingers that raise during the counting constitute a gestural ostensive. Raising a finger at the same time a number name is enunciated allows controlling how much numbers names were enunciated. Thus the gesture makes the achievement of the task possible: it is an element of the implementation of the technique. But the fingers that raise gradually say what the pupil does: he continues the counting from 12. The same ostensive, the gesture of the fingers raising gradually, has two values: an instrumental value that expresses what has been carried out by the ostensive and a semiotic value that allows seeing what is made by the ostensive and reveals the evoked non-ostensive. In the same way, the number name enunciated when the fingers raise is a linguistic ostensive which lets us see (or rather understand) the technique used by the pupil: the counting. But the declamation of number names has also an instrumental value: the last number name enunciated is the result of the addition. The work made by the pupil on the words is an element of the technique of addition. Thus the language does not have only a communication function for describing, for instance, a technique. It is also an ostensive like another: it is simultaneously a tool which makes possible to achieve a task and makes apparent the meaning of what is made.

Instrumental and semiotic values of an ostensive “appear, within a given technique, linked like the recto and back of a sheet of paper” (Chevallard 1994, p.6, our translation). The mathematical activity is carried out through the handled ostensives and of the non-ostensives thus evoked. Then, a didactic analysis, in ATD, takes into account what the ostensives make possible to do, their instrumental value, and what they let us see of the work done, being done or to be made, their semiotic value. In this example, we have seen how the efficiency of the process gives sense to what has been done: the instrumental value of an ostensive nourishes its semiotic value. With the next example, we observe the case where the loss of sense makes an ostensive loses its instrumentality.

3. The loss of instrumentality of an ostensive

In a “classe de CM2” of a primary school – pupils between ten and eleven years old – pupils solve a proportionality problem (Wozniak, 2012). It is asked to determine the height of a giant in an amusement park from a photograph[16] of 16.1 cm horizontally by 12 cm vertically (see figure 3).

[16] Copyright Richard Phillips (2001-2009): www.problempictures.co.uk

This photo was taken in an amusement park in England. One sees there part of the leg of a giant. Which is about the size of this giant?

Figure 3 – Proportionality problem

To help pupils, the teacher distributes a document (our translation):

1. Observe the photograph. According to you, what is roughly the size of an adult man in cm?
2. Observe the photograph. What is roughly the size of the giant foot in cm?
3. According to you, what is roughly the measure of an adult man foot in cm?
4. Observe this table. Give a headline to each line and fill in them with the information you have found.

Calculate the size of the giant in cm, then converted this measurement into the suitable unit. Sentence-answer:

After discussions with the pupils, the teacher adopts the size of an adult man as being 180 cm, the foot of the giant 200 cm and the foot of a man 30 cm. He declares: “Look at this table, it could help you. Only with this table we could find an accurate calculation”. After a collective working time two answers are produced (our translation):

	entire	foot
Size for man	180 cm	30 cm
Size for giant		200 cm

	man	giant
Size for feet	30 cm	200 cm
Whole size	180 cm	1200 cm

Figure 4 – Tables made by pupils

Then pupils discover the real challenge of this collective work when the teacher declares (our translation): “It was the first step, and from this table we will continue this work on Monday and we will try to see how we can use this table to calculate the size of the giant. What does this kind of table make you think?” The pupils do not allude to a proportional table, thus the teacher insists:

Teacher: Perhaps you remember it. Some time ago, a few weeks ago, we worked several times with small problems with tables which looked roughly like this one.
Pupil: Yes
Pupil: Yeah! On the sheet of mathematics. It is like the sheet we have made there with… where there had written Diams and all… tricks like that.
Pupil: No. About strawberries.
Teacher: There you mix up. It was a work about durations. You are right, Ouiam we have done a… we have made several small problems where it was necessary to calculate the price of one kilo of strawberries and then to find the price of 5 kg, 6 kg. It is what is named a proportionality table.

Thus it is the evocation of an ostensive and its use – resorting to a table of proportionality – which reveals to pupils the kind of problems they solve. At the next session, the teacher becomes even more precise to give some meaning to the numbers arranged in the (proportionality) tables that pupils are obviously not able to make use of: “How many times we need the size of an human foot to obtain the size of an adult man? ” (our translation). And to make sure of the understanding of the question, the teacher makes a drawing. Finally, a solution is collectively found and the teacher concludes: “To calculate the size of the giant, one takes the size of his foot and multiplies it by six” (our translation). Then pupils must redo individually the work which was made collectively by using only the photograph given at the beginning. The analysis of these last writings allows noting that the pupils did not understand what have occurred: one third answers nothing on the sheet and only another third gives the good answer (with a badly filled out table for two of them).

If we report this episode, it is in order to illustrate how much an ostensive – here, the table of proportionality – has lost of its instrumentality when it has lost its semioticity. A single table of proportionality is not enough to solve the problem given. However it seems that, for this teacher, the presence of a table of proportionality alone must be enough to make understandable for the pupils that they have to solve a problem of proportionality and how to proceed. Thus the teacher has got rid of the genuine challenge of the learning: recognizing that a situation has to get modelling by the proportionality. The table of proportionality can become an instrument only if the pupil has recognized the situation of proportionality: the instrumentality of ostensive also depends on its semioticity.

With these two last examples – add 12 + 6 or determine the height of a giant – we observe that instrumental value and semiotic value are inseparable and how they nourish one another. An ostensive is efficient only if the non-ostensive evoked makes sense for the mathematical activity. But in the same way, the efficiency of the technique that uses an ostensive makes sense about what has been done. Therefore, tool and representation are embedded and the ATD modelling of mathematical activity as a praxeology allows clarifying the relationships by means of the praxis component and the logos one.

4. Choose an ostensive for a design task: study the values according to the learning objective

Now, we illustrate with an example how the concepts of instrumental and semiotic values make possible to answer two essential questions to design a mathematical activity: « What mathematics epistemological considerations are taken into account when designing tasks using tools? How do different types of tools afford different mathematical activities/tasks, different representations and/or discourses, and different interactions between representations? »

Let us consider a teacher of a “classe de CP” of a primary school who wishes to introduce an ostensive representing the integers as a reference tool. This teacher hesitates between a linear or tabular presentation (see figures 5 and 6).

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Figure 4- Linear presentation of integers from 0 to 20

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49

Figure 5- Tabular presentation of integers from 0 to 49

Which ostensive must be chosen? The linear presentation, or number line, is present in pre-primary school – children between three and six years old – and is found in various board games where it is necessary to push or to move back a pawn according to a dice throw. It is a familiar representation whereas the table of numbers is a tool which belongs to the school culture. The number line enforces the continuity of the series of numbers and makes possible the association between cardinal number and ordinal number. It also allows determining for each number what is the previous and the next number, bounding it between two whole tens, adding or subtracting two numbers while moving forth or back by analogy with games. They are there the ingredients of its instrumental value. The same techniques can be used with a table of numbers. However the tabular structure arranges the numbers according to their figures of tens and units, and breaks somehow the continuity of the number line. Then it is less easy to perceive than 19 is between 18 and 20 or 10 and 20 with a table of numbers than with a number line.

If the aim of the teacher is to obtain the result by counting, he can consider that the instrumental value is more important for the number line. Of course, the same technique of counting can be used with both ostensives, but the table of numbers makes easier the use of the decomposition of tens and units because of its structure. For example, to add 5 + 12, the pupil can advance of 12 boxes on the number file from 5 (see figure 5). With the table of numbers, it is possible to find the result by moving down from a line from 5 and then by moving forth by two boxes (see figure 6). This last technique is based on the decomposition of 12 into 10 + 2. Obviously, a moving on the number file by a jump of ten boxes and then by two jumps of one box is possible. Nevertheless, this technique is easier to use with the table of numbers than with the number file. If the goal of the teacher is to develop a work on the numeration and the decomposition of the numbers rather than counting, he can consider that the instrumental value is more important for the table of numbers. Therefore, the choice of the representation will be determined by the mathematical praxeologies that the teacher wishes to make alive in the classroom: the greatest instrumentality and the greatest semioticity are then estimated according to the aim of the study. It is finally noteworthy that the semiotic value of ostensive is not “already there”. It is a work around the structure of the table which reveals 26 as being at the crossing of the line of 20 and column of 6 and led to the decomposition 26 = 20 + 6. It is because the table of numbers is transformed into a tool making easier the use of the decomposition of the numbers that its semiotic value, connected with the numeration, develops. The instrumental genesis (Rabardel, 1995) does not facilitate only the use, and thus the instrumentality, of an ostensive. It develops also its semiotic value: an increase of the instrumental value allows also an increase of the semiotic value.

We have discussed above that the instrumental value of the table of numbers, from a didactic point of view, is to make easier the transition from techniques relying on counting to techniques based on numeration and additive decomposition. The teacher will determine his choice between the number file and the table of numbers according to the non-ostensives that it evokes and thus techniques that it makes possible to implement. The instrumental value of an ostensive from the point of view of the teacher's activity to design a teaching situation is thus based on the semiotic value of this ostensive from the point of view of the pupil's mathematical activity. But not only, because the semiotic value of an ostensive is also built starting from its instrumentalization. Thus, the choice of a particular artefact for a design task is determined according to the analysis of its semiotic *and* instrumental value to develop particular mathematical praxeologies. The notion of praxeology is thus central in ATD to design a mathematical activity. The teacher must consider the techniques he wants to make alive in the classroom to perform a specific task; he must consider the technological discourses he wants the pupils to develop for describing, explaining, justifying and developing the techniques used. In the above example, the choice of tools is determined by the questions on the kinds of praxeologies that the teacher intends to make alive in the classroom.

Conclusion

Any ostensive is an instrument to do something and the representative of a non-ostensive that allows thinking what is made or has to be made. It is this bivalence that is recognized through the concepts of instrumental value and semiotic value of an ostensive which are inseparable and nourish each other within praxeologies that activate the ostensive. In fact, an ostensive can acquire a higher instrumentality thanks to a technological or theoretical work giving a higher intelligibility to new technical uses. Reciprocally an ostensive can acquire a greater semioticity thanks to a work on instrumentation.The identification of the instrumental and semiotic values seems an essential didactic gesture for the teacher during the design of a teaching situation. It is based on an epistemological analysis of mathematical praxeologies that he wishes to make alive in his classroom depending on the learning challenge. The example in the fourth section illustrates this point of view.

What happens when the instrumental and semiotic values of ostensives are insufficiently taken into account during the design of a mathematical activity? In this case, the milieu of the situation (according to Brousseau, 1997) is insufficient to create a dynamics that allows the construction of the knowledge. In Bulf, Mathé, Mithalal, Wozniak (in press) we show how a teacher in this case uses a specific ostensive, the language. In fact, he resorts to maieutic as a driver of the mathematical activity which cannot start from the others ostensives and non-ostensives of the situation. The teacher intervenes directly with pupils in order to guide them in their activity. Thus, through the questions he asks, he finally ends by suggesting the expected answer. This kind of situation is named by Brousseau (1986) the Topaze's effect.

In fact, the instrumentality of the notions of instrumental and semiotic values of an ostensive to design a mathematical activity is born from the anthropological point of view which we adopted: identification, interpretation and description of the instrumental and semiotic values of an ostensive depend on the studied practices and the institutions in which they are activated. ATD is an epistemological approach and its main theoretical tool is the notion of mathematical praxeology as model of mathematical knowledge. The notion of praxeology allows analysing the tool-driven relationships within the design, teaching and learning triad. What type of techniques does the teacher want the pupils use? What kind of technological discourses does the teacher want the pupils develop? These are fundamental questions to design a mathematical activity and analyse its teaching and learning. Any mathematical activity activates ostensives and evokes associated non-ostensives. Then, the analysis of the instrumental value and the semiotic value of the ostensives allows to choice the tools for performing the task in function of techniques or technological discourses the teacher wants to be used in the classroom. In conclusion, it seems that the notions of praxeologies, instrumental value, and semiotic value of an ostensive, are useful theoretical tools to study task design.

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Theme B
Accounting for student perspectives in task design

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It is obvious that tasks or sequences of tasks are designed to embody mathematical knowledge in ways that are accessible to students, and to improve students' mathematics thinking. However, if we look beyond the intentions of those who design and select tasks, the actual impact on students' mathematical learning raises important questions. One of the aims of this thematic group is to gain insights into students' perspectives about the meanings and purposes of mathematical tasks, and to better understand how appropriate task design might help to minimise the gap between teacher intentions and student mathematical activity.

There is a tacit assumption that the completion of mathematical tasks chosen or designed by the teacher will result in the student learning the intended mathematics. This view is persistent despite research that suggests that this is not a direct relationship (Margolinas, 2004, 2005). This can result in completion of the task (rather than mathematical learning) becoming the priority for students and even sometimes for teachers. This can be particularly true for younger and lower achieving students, who are 'helped' by the teacher to complete the task in order to 'keep up' with their peers. Teachers are encouraged to differentiate tasks for different students in order to facilitate learning. However, changes that make it easier for the student to complete the task may have the effect of undermining the designers' intentions, and reinforcing students' attention of completion as the priority.

Research about learners' perceptions of the use of contexts in mathematical tasks has suggested that these can differ considerably from intentions of designers (Cooper & Dunne, 2000). Whilst designers may choose contexts to offer real world models to think with or to illustrate the usefulness of mathematical concepts in real life, pedagogic practice may lead students to adopt 'tricks' to bypass the contextual elements (e.g. Gerofsky, 1996), Verschaffel, Greer, & Torbeyns, 2006)), or fail to appreciate the extent to which everyday knowledge should be utilised in the mathematical task (Cooper & Dunne, 2000). Tasks or sequences which draw on real world contexts, but which do not reflect the purposes for which mathematics is used in the real world, may be perceived by students as evidence of the gap between school mathematics and relevance to their everyday lives (Ainley, Pratt, & Hansen, 2006).

Another issue is a methodological one. One possibility for measuring the impact of tasks or sequences on students' learning is the use of pre- and post-tests. However, since it is highly likely that any teaching may result in some outcome on posttests, it is not so obvious what should be considered as a significant posttest outcome. For instance, if we consider only the mean value of an entire cohort of students, we may not understand whether the low achieving students (as determined by the pretest) have really benefited from the task or sequence. Moreover, the goal of the task or sequence may not be easily (or even possibly) assessed in a written test. Often, it is only by observing the evolution of students' strategies that we can understand the effect of a task or sequence (Brousseau, 2008). Task design is generally initially implemented in favourable contexts: the teachers are members of the research team or closely linked to the designers. In this context, the impact on students is not only linked to the tasks but also to the impact on teacher or students of a collaborative way of dealing with teaching (Arsac, Balacheff, & Mante, 1992). These methodological issues are only examples of those that can be addressed in our group. An aim of this thematic group is therefore to reflect on methodological issues related to studying task impact on students.

Possible questions might be:

- How is it possible to assess the impact of task or sequence on students' mathematical learning?
- What is the intended and actual impact of a task or sequence on low achieving students?
- What do students actually do and attend to when confronted with tasks?
- How do students understand the purposes of tasks they are given in the classroom?
- How do students' reactions influence teachers' adaptation of the task?
- Might what appears to be 'only' a change in presentation convey a different meaning to the student, and result in different mathematical activity?

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Emergent tasks—spontaneous design supporting in-depth learning

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In this paper the concept of emergent tasks is presented. It has been developed through empirical investigations of interest-dense situations comparing them with situations where learning opportunities were not taken up and enables teachers to shape learning opportunities in everyday lessons in an effective way. Three forms of emergent tasks, their conditions of success and possible uses are discussed.

Keywords: Interest-dense situations, learning opportunity, social interaction, epistemic process

Introduction

Interest-dense situations provide epistemic and interest-supporting learning opportunities to the students by the adaptive behavior of the teacher (Bikner-Ahsbahs, 2003; 2005). So far, we have understood how these situations can be built through social interaction but we did not exactly know in what way the teacher aligns his behavior to the students' needs. Investigating this problem brought to the fore a new task format which is not designed beforehand and hence accompanied by the problems of given tasks described in the call of ICMI Study 22 (ICMI, 2012) but is constructed instantaneously deepening mathematical thinking. In the long run, this investigation has the potential to answer some notable problems of task design, for example "to better understand how appropriate task design might help to minimize the gap between teacher intentions and student mathematical activity" (ICMI, 2012, 15) or how tasks developed in favorable contexts can be made more accessible in ordinary classrooms.

Regularly in mathematics classrooms, tasks designed beforehand are implemented to initiate learning activities. In this process, time is needed to understand the task affordances, especially when they are complex and the students are unfamiliar with their forms. The time on task and with it the possible outcomes are reduced when a considerable amount of time is needed for understanding its requirements (Caroll, 1989). However, time on task alone does not induce successful learning, but must be supported by sophisticated teacher instruction (Caroll, 1989, 30; Christiansen & Walther, 1986, 261). For instance, Prediger and Scherres (2012) show in a case study that with so called self-differentiating tasks optimal learning takes place only if it is supported by a teacher intervention that transforms the mathematical learning potential into a fruitful epistemic situation. The two problematic aspects discussed so far—the reduction of learning time needed to understand the affordances of tasks and the necessity to take learning opportunities in the classroom—are addressed effectively by the concept of *emergent tasks* presented in this paper. More aligned to the specific situation than in ad hoc tasks (Christiansen & Walther, 1986, 296), in emergent tasks the teacher conceives the mathematical potential of a learning opportunity and translates it into a task, so that

1. the students' interest present in the situation is taken up and
2. acute mathematical problems and questions are addressed adaptively.

Investigating emergent tasks and their supporting conditions may elucidate how the gap between the students' experiences and the given purposes of tasks can be bridged through the adaptive character of these tasks. In this paper the concept of emergent tasks will be worked out based on qualitative data. Forms of emergent tasks and necessary conditions are presented. Furthermore, based on an empirical example, influences of emergent tasks on further learning processes are illustrated.

Theoretical background and research questions

Given the immediate perceivability of emergent tasks they can contribute to a more effective use of time in class. With their adaptive character, they are embedded in an understanding of mathematics learning that places it in the social togetherness of learners and teacher (Jungwirth, 2003). Both align to the mathematical and situational requirements not only of the teachers' goals, but especially of the students' learning processes. This very condition characterizes the social interaction in interest-dense situations (Bikner-Ahsbahs, 2005). Hence, we assume emergent tasks to be a means to initiate and sustain interest-dense situations and with it deepened mathematical learning.

Interest-dense situations can be described by three characteristics: the learners are deeply involved in working on mathematical questions, they construct successively deep mathematical meanings, and they explicitly or implicitly see the mathematical object as important in the situation (Bikner-Ahsbahs, 2003; 2005). Common to all interest-dense situations is the scheme of students' epistemic actions: Starting from the *gathering* of smallest bits of knowledge via the *connecting* of mathematical knowledge students get to *structure-seeing* (GCSt model). Structure-seeing can refer to known mathematical structures that are seen in new contexts, but also includes the development of new structures. Hence, interest-dense situations can be considered mathematically rich learning processes. The teacher puts aside fixed answer expectations in favor of the developing situation. The students do not try to re-produce the teacher's expected answers but concentrate on their own learning processes. They try to grasp the mathematical content in the social interaction with their peers.

The content-related driving force of such situations is the students' general epistemic need (GEN) (Kidron et al., 2010; 2011). It is often only implicitly visible in epistemic actions. Together with a mathematical gap experienced by the learners it can lead to more specific needs, for example to formalize. In this case, the need can be satisfied by specific actions. These then lead to epistemic steps, hence to the experience of competence and a stabilization of the situational interest (Krapp, 2005), which in turn reproduces the GEN. Thus, a situation-apt emergent task reacting to the needs of a student should be suitable to bring forward epistemic processes.

However, it remains unclear what characteristics emergent tasks account for and how exactly they come about. This study tries to clarify the following questions:

1. Under which conditions do emergent task come about and are there specific forms to be identified? More precisely, we look for conditions of classroom situations and the requirements the teacher must meet to design useful emergent tasks.
2. What is the use of emergent tasks?

Methodological considerations

The research questions are investigated focusing on processes by the use of qualitative methods. As emergent tasks seem to be an attendant phenomenon to interest-dense situations, data from a project dedicated to this subject is used. We identify scenes where teachers translate learning opportunities into tasks and examine how these tasks guide further epistemic processes. We consider these processes successful when students get to mathematical insight regarding the underlying problem. The GCSt model helps in this process: Success in this sense can be stated whenever students see structures (which is the case in all interest-dense situations) *and* this can be ascribed to the emergent task.

We assume with Brousseau (2008, as quoted by ICMI, 2012, 16) that "it is only by observing the evolution of students' strategies that we can understand the effect of a task or sequence", and aim at answering the question what students "actually do and attend to when confronted with task" (ICMI, 2012, 16) - even more so as emergent tasks by definition can only occur in everyday classroom interaction and not in artificial laboratory setups. Thus we ground our ongoing study on the video transcripts from 16 interest-dense situations of 8 lessons. 5 of the lessons are still to be analyzed. They are taken from a total of 89 lessons dedicated to fractions that were conducted for half a year in a 6th grade class. The transcripts are analyzed focusing the question how the teacher deals with learning opportunities marked by the students' GEN. By comparison with scenes where a similar learning potential is not used in an emergent task, fostering and hindering conditions of emergent tasks are gradually worked out. These may vary depending on the form of emergent tasks and thus express a teacher-dependent typing. We reconstruct this typing by first analyzing one rich situation containing an emergent task. In each scene subsequently analyzed a contrasting permanent comparison to the previous scenes is conducted (Jungwirth, 2003). This leads to a typing of emergent tasks regarding the situational conditions.

To describe the tasks designed by the teacher in the situation we use the definition of tasks by Bruder (2000). It describes tasks as calls to action with an initial state I consisting of given information, conditions, etc., a final state F meaning the solution, conclusion, etc., and a transformation T between the initial and the final state. Based on this definition we can describe in each situation what is given in the students' utterances, what the students struggle with, and what is translated by the teacher.

The method of analysis follows the rules of the interpretative approach (Jungwirth, 2003). It assumes that subjects act towards things (here: mathematical content, learning environments, other persons, etc.) according to the meanings these things have for them, and furthermore, that these meanings are built in social interactions through interpretation (Blumer cited by Wagner, 1999, 32). Analyzing then means re-interpreting the interpretations of the subjects. As we work on everyday classes, we can adopt the reflexivity assumption from ethnomethodology: the reasons for acting are visible in acting itself (Lamnek, 1995, 51 ff). Regarding the tasks this means that the persons involved make clear through their actions (i) how the learning opportunity comes about, (ii) if, why, and how the teacher formulates an emergent task, and (iii) how this task influences the epistemic process.

Exemplary data analysis

In the following exemplary display of results we will first describe a situation where the teacher does not see the potential of the learning opportunity and present a first hypothesis regarding possible reasons. Then one successful emergent task is presented and its conditions are worked out. Finally these considerations are completed by a look at the following task sequence which leads from the initial problem to a learning result for many students.

A missed learning opportunity

The students are asked to find one third of a slip of paper. Rosa proposes to tentatively fold the slip in three layers. Some students cannot follow this idea as they have problems implementing it. The teacher, Mr. Kramer, is not content and says:

167 T: ah is there (.) is there yet another possibility (.) some took (..) the view ,one could also fold in the middle
168 /S: no
169 /T: and does that work' in the middle
170 S: no
171 S: doesn't work.

Transcription key:

w-e-l-l	speaking slowly
exact.	dropping the voice
exact'	raising the voice
,exact	with a new onset
EXACT	with a loud voice
(.),(..),...	1, 2, ... sec pause
/S:	interrupts the previous speaker

172 Mira: Yeah ,it works (.) yes
173 Mira: yes but that would only ,work well ,if one would fold it here
174 T: yes'
175 S: uh
176 Mira: and then one would have to fold once more then there would be four pieces ,and ,the one piece one would have had ,to throw that away then that would work
177 S: ooh
178 T: ah like that yes.
179 S: ooh
180 L: yes so it works yes (...) there one would have given away one piece right (..) well okay
181 Eric: but one can also divide the last piece by three (.) if one then divides by four and one divides one piece by three then again
182 /L: yes but then one has the same problem again one piece divided by three
183 /S: yes exactly
184 /L: right

The open question of the teacher stimulates the students. They discuss the question whether it "works" to divide the strip in the middle (168-172). Mira proposes to bisect twice and to "throw away" one of the four pieces (173-176). According to the teacher one would have "given away" one piece in this case. He makes clear that this is a good idea but that not using the whole paper slip is unfortunate—he thus clarifies the *initial state* and the *final state*. In fragile speech, Eric proposes to divide the surplus piece by three (181). With this proposal Eric starts an approximation process. But the teacher does not take up this idea. Once more he marks the situation as unfortunate, as one would have the same problem again. “Yes exactly” is the supporting reaction in the class which in turn gets appreciation by the teacher (182-184). Eric sees a *different final state* than the rest of the class. He does not try to get exactly one third in just one step, but looks for a successive approximation. And his *initial state is different*, too: While Mira and even more explicitly the teacher mark the surplus piece as unfortunate, this very piece constitutes the starting point of Eric’s idea.

Eric’s view constitutes a learning opportunity: If one repeats the process of dividing one of the four pieces by 4 and puts three of the resulting smaller pieces next to the three remaining larger ones, one is again left with one small piece that can again be divided. The result is an approximation algorithm for the division by 3.

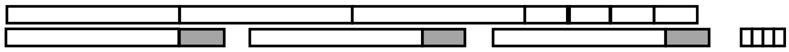

Fig. 1: The first two steps of the approximation algorithm for the division of the paper slip by 3

However, the potential of the situation remains unused. Why does the teacher not adapt to Eric’s view and hence does not use this learning opportunity? He is familiar with approximation algorithms, for example with the bisection method, because he also teaches calculus. In fact, in another lesson he initiates a similar method in the very same class to show the density of rational numbers, albeit by dividing fractions by 2 and not as an approximation. In this case there are some possible reasons to not follow Eric’s idea: Either the teacher really *does not see the mathematical potential of the situation*, or he *does not expect such a complex algorithm* at the beginning of grade 6. Anyway, he does not show interest in Eric’s idea, possibly also because he *has different plans for the class*.

An example of an emergent task

Previous to the scene presented here the students had worked on the question how to divide a round licorice stick evenly among three persons. Among other techniques the students discussed the possibility to cut the stick in length, so that the cross sectional area would be divided in three sectors of 120° each. Because angles had not been introduced before the meaning of 360° was presented using two set squares. The division of a circle by 3 was illustrated with a round trash can where the middle was marked by the cast. This is the point where Anji asks her question.

139 Anji: Mister Kramer I have a question here. they have divided it from the top but how do they know what 120 DEGREES are.

140 T: ah okay' you want to go back to the set square once more.

141 /Anji: no (.) well but when they ,that is such a stick and how do they know that because ,they can’t do that with the set square.

142 /S: yes that’s round now right

143 /Ernst: that is round right

144 /S: it’s round right

145 T: Tom yes. that is kind of a practical problem right' how do I do that when it is such a very small one and not such a BIG circle eh'

146 Tom: one has to put the zero in the central point and then it works nevertheless I think (.) then one only has to imagine the lines from the 120 until one can draw.

147 T: ah yeah I ,there must additionally ,we will exercise that

148 /S: yes but how

149 /Rahel: yes Mister Kramer once more a stupid question ,how does one GET the central point how did they GET that. because that is so small.

150 S: that is just

151 /T: that's another problem right. that's a practical problem (..) oh no ,how does one even find the central point in such a small circle right' (.) exactly. those are questions'

152 Anji: a very small compass right'

153 T: yes one can find out with the compass ,only when one is just drawing the circle' one has the central point. but when one has the circle already right'

154 Rahel: yes

155 L: that's exactly what geometry works with.

156 /Rahel: I know that

157 L: there are possibilities to find out a-n-d you can puzzle at home maybe someone finds a possibility'

158 /S: ah I know-

159 /S: one just puts the thing in there

160 /S: one stings in there

161 T: one puts it in

162 S: what'

163 S: oho

164 T: yes but one can act as if one wouldn't have it. how can one find it then. right maybe there's a possibility right maybe you find something. right and then you just try try to divide in thirds. (..) so first draw a circle' then act as if one wouldn't have the central point' one can't find it anymore and then how can one find it again. and when one has it how can one divide in thirds.

The initial question by Anji shows her GEN. The teacher tries to understand and thus shows interest in her learning process (140). The reaction of the teacher seems imprecise as Anji's does not agree and specifies her question (141). Her question is how the division of the circle in three equal parts can be accomplished in practice as the diameter is very small (ca. 1cm). Some students do not see the problem, i.e. the *initial state*, which Anji and the teacher have agreed upon. They only see the form of the profile as relevant, which is "round" (142-144). So the teacher once more points out Anji's problem by comparing the profile to the larger circle they had worked with before (145). Tom offers a solution. He describes how to measure angles even on very small objects using the set square (146). At this point the teacher probably notices already that the missing central point poses an additional problem and wants to postpone it to an exercise (147). But the students insist on an immediate clarification by asking "but how" (can we exercise that) (148). Rahel reacts by naming the difficulty in dividing the circle without knowing the central point (149). She grasps the epistemic gap and thus *sees* the mathematical *structure*. Now the teacher summarizes the two problems: How does one even find the central point in such a small circle? Commenting "those are questions" he documents wonder about the deep involvement of the students that he tries to take up (151). Anji proposes to use a "very small compass" (152). Mr. Kramer tries to reformulate the problem: If one draws the circle the central point is known. But how can the central point be found if the circle is already given (153)? The teacher probably does not know yet how he can work with this situation. He situates the students' questions as problems of geometry. Again, the students react driven by their epistemic need: "I know that" (this is what geometry works with) is a comment that makes clear that the students do not care about the general context, but about their specific problem (154-156). Now the teacher starts drafting a question that picks up the problem and the epistemic need of the class (157). But for some students the *initial state* is still unclear: they assume the central point to be known, as one can just put the compass there (158-163). The teacher feels prompted to specify his thoughts and poses the following task as a homework assignment: Draw a circle, pretend the central point were unknown, find out how it can be rediscovered (164).

In this situation we see more than just an adaptation of an existing task as a consequence of the students' reactions (ICMI, 2012, 16). Rather, a task *emerges* as a reaction of the teacher to a central problem of the students. The teacher is the designer taking up the students' epistemic need (Kidron et al., 2010; 2011) that he transforms into an action program. The social interaction is fed by the different views the subjects have of the situation. Initially, there are questions, but a *clear initial state for the task is missing*. This changes when Rahel asks for the central point. A brief *clarification of I and F leads to an emergent task* that may seem "to be 'only' a change in presentation" (ICMI, 2012, 17) but proves to be particularly fruitful: At the beginning of the next lesson the teacher asks who has worked alone on the task and gotten to a solution. According to the log of the lesson, 20 of 26 students had tried to solve the task on their own, and 16 students got a solution. Some of the students' solutions are presented. Five different methods can be identified which represent *structure-seeing* at different levels of abstraction:

1. Two maximal, perpendicular chords are found tentatively, they intersect at the central point.
2. To find the maximal chord, a sampling system with nested measurements is presented.
3. The maximal chord is found using the set square. It is placed in the circle so that the circle line passes through the same numbers, which are then maximized. Finally, the central point is marked by the zero.
4. By twofold, perpendicular use of a ruled transparent the central point is found.
5. Two parallel tangents are applied to the circle. Half of the distance is taken as the radius of two circles with their central points on the circle. Their intersection marks the central point of the initial circle.

The last solution, which is *presented* by Andy, is closest to the one that is part of the curriculum in grade 7. However, it seems too complex for many of his peers. Hence, the teacher encourages the class to *ask* Andy about his procedure. The crucial question is why he gets the diameter and the central point this way. The teacher then disseminates prepared paper circles without marked central points and asks the students to *use and check* Andy's method. The students now focus on the question how one can know where on the circle the compass must be placed. They find out that it is irrelevant as all points have the same distance to the central point (*structure-seeing*). Only two points on exact opposite sides would be inconvenient. In the end, the teacher gives a homework assignment as an emergent task, which demands an *expanded use and application* of Andy's method and induces an *institutionalization*: Andy's and two other methods are to be put down in writing "so we don't forget that". Andy's method is not only used to find the central points of the circles, but also incomplete circles are to be reconstructed. The institutionalization contains both, repetition and consolidation of the method. From our analysis we derive five possible design principles that allow for a sequencing based on an emergent task about learning a method:

Principle	**Description of the step**
Emergent task	reacting to a student problem
Presenting and questioning	students' solutions to the task are presented, questions are asked
Using and checking	an interesting method is used and checked by the students
Expanded use and application	the potential in use is evaluated by an expanded task
Institutionalization	(individual) textualization

Tab.1: Five principles for a task sequence about lerning a method

What was the difference between this situation and the one presented before? The teacher tried to grasp the problem from the perspective of the students. Posing questions he makes clear what he has yet understood and thus provokes the students to specify and to make the problem (i.e. the GEN) more visible. This does not necessarily bring clarification. But as soon as the teacher has understood the mathematical problem of the students he can translate it into a task: "how does one find the the central point of a circle"? The task is adapted to the students' situation: they have already shown a GEN, and the question is already clear in the situation. The teacher takes the students' situation and their need seriously and makes it the basis of a longer learning process through the emergent task.

Summary of results and concluding remarks

Emergent tasks can be designed whenever the students' epistemic interest can be translated into a task. In most cases both the initial state and the final state are not clear for all participants. It is the teacher's role to translate and thus clarify.

Based on the analyses of four interest-dense situations we were able to identify three different forms of emergent tasks. When the *students' epistemic need is explicated* the teacher can react to it immediately. We presented a case where an emergent task initiated the learning of a method through a task sequence. When the *students' need is implicit* their problem may be visible to the teacher or not. In the first case emergent tasks may consist of smaller prompts reacting to a student's problem. In the latter case the teacher may help making the problem visible by asking questions like: "show us what you mean", "take the chalk and write (or draw) what you mean". In our data, the students' managed to explicate their problems in this way and especially communicate to the teacher what they were struggling with. On the other hand, we showed how such an opportunity was missed and Eric's idea remained unused. Besides the translations of explicit and implicit problems we also found *emergent tasks that unveiled an epistemic gap* that initially remained unnoticed by the students.

On the part of the teacher our studies point at three necessary conditions that enable her or him to see situations, where an emergent task is suitable, and to then perform the appropriate translations. The teacher must

- have mathematical knowledge that extends the content of the lesson,
- show interest in the students' learning processes,
- and be open for unusual ways on the part of the students. She or he must be willing to abstain from the planned course.

So far, we have only investigated emergent tasks in one class and the study is still ongoing. Further research investigating interest-dense situations and emergent tasks in other classes and other areas is needed. In addition, it remains an open question whether this task format can also be found outside interest-dense situations, and if yes, how this happens and how it influences learning processes. Emergent tasks can be a powerful tool for the teacher to adapt his behavior to the students' actions and interactions in a given mathematical problem situation. However, the implementation of this tool is yet to be investigated.

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Mathematical Investigations: The Impact of Students' Enacted Activity on Design, Development, Evaluation and Implementation

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This paper is based on a year-long action research study that I conducted in a Form Four mathematics class in a girls' state secondary school in Malta. The focus is on presenting the framework that was used to plan, develop, design, classify, implement and evaluate investigative tasks. This framework hence provides principles and guidelines on examining the differentiation between the tasks as designed and foreseen by the teacher and the actual instructional activities as undertaken by the students. For the teacher-researcher, students' perspectives become crucial within the developmental cyclic process of designing, modifying, implementing and evaluating tasks. The study reported in this paper shows that when students' contributions (students' classroom experiences, responses and understandings) are valued, the teacher-researcher gains more informed knowledge and improved understanding about the design and implementation of investigative tasks.

Investigations; active learning; reflective practice; students' perspectives; task resources

Introduction

This paper draws upon an action research study which I conducted with a Form Four (i.e., 14-year-olds) mathematics class during the scholastic year 2008-2009 that explored the integration of a set of 18 investigations within the mathematics syllabus (see Calleja, 2011). Since then, I have moved to a different state school and have also assumed the responsibilities of Head of Mathematics Department. The focus of this paper is to present a view of didactical goals, guidelines and frameworks that may be useful for collaborative design research projects that are specifically aimed at integrating tasks that promote active learning practices (see Anthony, 1996). The embedded understanding is enlightening my current collaborative project, which commenced in September 2012, on promoting the design and implementation of inquiry-based mathematical tasks with all the Form One students (i.e., 11-year-olds) in my new secondary school. As head of mathematics department, I am coordinating this year-long pilot project and acting as a teacher-mentor within a team consisting of the four other mathematics teachers. At present, we are busy selecting, planning, designing, implementing and evaluating a variety of inquiry tasks, including a range of resource materials (e.g., worksheets, measuring instruments, etc.). Working within our community of practice, we follow a teacher-researcher perspective of '*learning-*

to-develop-learning' (Jaworski, 2006) while implementing our task-driven pedagogy (Walls, 2005).

The Research Area and Methodology

At the time of the action research study in my former school, I undertook the role of a reflective practitioner (see Schön, 1983) in the belief that "good research is not about good methods as much as it is about good thinking" (Stake, 1995, p. 19). This involved taking a reflective stance towards planning, designing, implementing, modifying and evaluating a set of mathematical investigations that I integrated within my scheme of work. Data collection included multiple sources. Apart from writing field notes of my classroom observations, I occasionally discussed classroom situations with a critical friend and kept a reflective journal to give my own interpretations as the events unfolded. However, I also valued students' reflections, thoughts and suggestions. Thus, I encouraged students to keep a learning log book through which they could write about their learning experiences, feelings and learning environment. Moreover, midway through the study, I conducted semi-structured qualitative interviews with the students. Data analysis was then an on-going process of searching for themes as these emerged and resurfaced. This sense-making journey was a search for 'some' truth through qualitative triangulation, namely, an account that presented the classroom situations from three different perspectives: those of the teacher-researcher, the students and a critical friend (McKernan, 1996). My qualitative account emerged as an informed story highlighting the salient features of the classroom community while engaged with a range of investigative tasks as an approach to learning mathematics. At the same time, this study offered me the possibility to better understand and eventually improve my practice.

Conceptual Framework

Reform-oriented approaches claim that doing mathematics should involve sense-making activities by using tasks that provide students with a variety of challenging experiences through which they can actively construct their mathematical meanings (Bishop, 1991). Within this 'active learning' approach – which is associated with experiential, collaborative and inquiry-based learning (see Anthony, 1996) – students gain autonomy and take control over the direction of their learning. Referring to the pedagogical implications, Ernest (1991, p. 288) proposes that the role of the teacher

> is understood in ways that support this pedagogy, as manager of the learning environment and learning resources, and facilitator of learning.

Thus, it is within the teacher's remit to include cognitively demanding tasks for their students, encouraging them to work together and to justify their solutions. Learners are thought of in turn as subjects who are responsible for learning, for making decisions and also for their behaviour (Teong, 2002). Put differently, whereas students are trusted and respected as responsible learners of mathematics and as constructors of their own meaningful knowledge, the teacher assumes the role of a facilitator who assists students' learning by observing, listening, questioning and challenging their inquiries.

Learning through active participation also carries an important social dimension. For as Fosnot (2005) claims, learners can negotiate meanings when they engage in cooperative social learning activities. This social constructivist dimension is based on the understanding that students should be active participants in their own learning by communicating and exchanging ideas with the teacher and other students. Learning is believed to occur by being part of and interacting within a social environment. Discussion with peers can assist learning as students develop conceptual understanding when they articulate their thoughts and points of view, when they learn to listen to others and when they ask questions (Orton & Frobisher, 1996). It follows that learning opportunities arise when student contributions are encouraged and when discussions invite students to challenge, argue and offer explanations.

Defining Tasks as Investigations

Mathematical tasks are an important vehicle through which classroom instruction can enhance students' learning. The tasks teachers present students with and the way in which students negotiate mathematical meaning by working on the set tasks largely determine students' classroom experiences and their learning of mathematics (Hiebert, et al., 1997; Shimizu et al., 2010). As Doyle (1983, p. 161) argues, tasks "influence learners by directing their attention to particular aspects of content and by specifying ways of processing information" and are "defined by the answers students are required to produce and the routes that can be used to obtain these answers". Hence, a key decision for teachers lies in their choice of tasks (Sullivan, 2011). In my action research study, I had chosen investigative tasks. The selected investigations required students to participate actively in their learning as a way of constructing mathematical knowledge within a social setting.

In this paper, an investigative task or investigation is defined as an inquiry into a mathematical situation presented by the teacher, but which can also initiate from a statement or a question posed by a student (Greenes, 1996). The topic of the investigation could arise from real-life or from a mathematically designed problem (e.g., *Investigate the sum of angles in polygons*). Again, an investigation can be a very open exploration or it can take the form of a more structured task that guides the learner into discovering mathematics (Yeo & Yeap, 2010).

Investigations offer opportunities for students to be more active in their learning. When students engage with such work they are involved in processes of exploration and explanation (Skovsmose, 2001). Students are expected to engage with finding ways to unravel the task assigned and be able to justify their work by presenting their method/s to the whole class. In the process, students engage in thinking critically, learning to ask, sharing ideas and communicating mathematically.

A Framework for Reflection on Design and Implementation

Integrating investigations takes into consideration how different mathematical processes and strategies can be embedded within the core topics which make up the content of a mathematics curriculum (Frobisher, 1994). Consequently, in planning and designing my 18 investigative tasks, I set out to achieve two aims. The tasks had to: (i) be related to the mathematics content prescribed in the Form Four syllabus; and
(ii) engage students in mathematical inquiry, that is, thinking about, developing, using and making sense of mathematics (Breen & O'Shea, 2010). While a number of the selected investigative tasks integrated mathematical topics in order to allow students to experience different areas of mathematics, other tasks were intended mainly to help consolidate mathematical concepts and skills. Furthermore, in the belief that tasks should provoke curiosity in and be meaningful to students, the selection and creation

of tasks and resource materials took into account the students' interests, mathematical ability and needs. However, when I came to setting future goals I took into consideration the responses/feedback that the students had provided during the study.

Within this conceptual framework, investigative tasks were classified according to the mathematics embedded within the activity, the degree of structure/guidance provided to students and the time devoted for students' activity. The main reason behind the decision to use investigations of varying levels was to smooth the transition for students from working on traditional exercises to engaging in more challenging tasks (Orton & Frobisher, 1996). Moreover, the spread of tasks along this classification helped to gradually introduce students to the cognitive processes of making and doing mathematics. In other words, the different levels offered graded entry points for students to familiarise themselves with the social experiences of mathematical inquiry, discussion and communication.

At the basic level, the investigations were *structured* tasks that lead students to mathematical discoveries. The given instructions guided students, who worked individually or in pairs, to use particular pre-determined mathematical concepts and apply them to arrive at a solution. At the next level, the investigations were *semi-structured*. This meant that they were either less structured or students were initially given some guidance in their work but were then free to explore and engage with the task using their own conceptual mathematical understanding and reasoning. Believing that learners benefit from discussing ideas and solutions when working on these more challenging tasks, the students were instructed to work in small groups of two or three. At the third and higher level, the students encountered *unstructured* investigations that were more process-oriented activities. These required students to investigate the problem posed or the situation presented in as many different ways as they wished and through different methods. These investigations placed greater demands on students to think through a solution, to make inferences and to test their own conjectures. As this type of investigation required students to challenge, argue about and justify their reasoning, the *unstructured* investigations were set as a group activity involving between three to four students.

Other than the level of structure, the investigations were also classified along the three 'reality levels' identified by Skovsmose (2001). Skovsmose sees mathematical investigations as a landscape that ranges across three levels of real-life contexts. These are: (i) *pure mathematics* which simply involves working with numbers or geometric figures; (ii) *semi-reality* which refers to an everyday-life problem that is rendered artificial as it is tackled in a classroom situation where variables can be controlled; and (iii) *real-life* situations where students are directly involved in carrying out the exercise in the actual setting.

Combining these two classifications, I came up with a rubric consisting of nine different types of investigations. Initially, the main purpose was to produce a template along which I could select and position the investigations (see Table 1). The matrix was eventually also useful in exploring how tasks were enacted in class and whether the students' actual engagement shifted the nature of tasks along the rubric.

Investigation	structured	semi-structured	unstructured
pure mathematics	Type 1	Type 4	Type 7
semi-reality	Type 2	Type 5	Type 8
real-life	Type 3	Type 6	Type 9

Table 2: The nine types of investigations

Investigations of types 1, 2 and 3 were *structured* tasks that varied according to the level of reality involved. While 'type 1' tasks resembled typical traditional exercises that are similar to those found in mathematics textbooks, 'type 2' and 'type 3' tasks were situated in a context of more practical mathematical experiences. Investigations of types 4, 5 and 6 were *semi-structured* tasks that again varied from a purely mathematical context to a real-life situation. The unstructured nature of investigations of types 7, 8 and 9, which again differed by context, placed the greatest cognitive demands on students as they were presented as more 'open-investigations'.

Selecting and Planning Investigations within Hypothetical Learning Trajectories

My quest to select tasks with goals in mind (Hiebert et al., 1997) involved thinking about how investigations would provoke inquiry and stimulate learning. As Simon and Tzur (2004, p. 93) argue:

> The tasks are selected based on hypotheses about the learning process; the hypothesis of the learning process is based on the tasks involved.

Along these lines, the day-to-day classroom experiences were vital in informing future planning that sought to integrate the learning goals with the trajectory of students' mathematical thinking and learning. This reflective process included the notion of 'hypothetical learning trajectory' (HLT) through which the teacher considers the learning goals, the instructional activities, and the thinking and learning in which the students might engage (Simon, 1995). A key aspect of this learning trajectory concerns a prediction of how students' thinking will evolve as they participate in the instructional activities. Actually, in classifying the tasks, I always started off with a HLT based on my expectations about students' explorations in learning – indicating the hypothetical learning trajectories from the syllabus (see Table 2). However, the actual learning trajectory cannot be known in advance as it depends on how the teacher and the students enact the tasks throughout the implementation process (see Stein et al., 2000). This eventuality of having students' task-inquiry leading to different learning trajectories than those anticipated by the teacher is indicated in Table 2, which shows a segment of my *task-delineated* scheme. More precisely, in my study, this occurrence rested upon how students actually went about working on the task and, in particular, their decisions regarding the use of the resources provided. This realization is crucial for the success of my current collaborative project. For I have come to appreciate that designing and implementing tasks requires teacher-decisions not only the number of resources, but also on the type and range of resources to be made available to students. The possibility of differentiating the resources that accompany tasks presents an opportunity for teachers to assign the same tasks to students of different abilities (discussed further in the following section).

Week	Investigation Title	Task Resources	Hypothetical Learning Trajectory	Students' Enacted Learning Trajectory
6	The Netball Court	Measuring tape Squared paper 1 cm grid	Measuring lengths Ratio notation Scale drawing Areas of rectangles, circles & semi-circles	Measuring lengths Mean, mode, median Rounding numbers Ratio notation Scale drawing
7	Web Patterns	Handout Graph paper Calculator	Plotting coordinates Finding areas of right-angled triangles Sequences	Plotting coordinates Areas of triangles Pythagoras' theorem Sequences & nth term

Table 2: A section showing how the scheme of work evolved

The scheme of work thus becomes a crucial reflective document for the teacher. By being responsive to students' inquiries, the teacher can occasionally fine-tune it as he or she re-plans and re-designs tasks and instructional practices. It is worth mentioning here that tasks might shift horizontally along the rubric presented in Table 1. In my case, a shift from structured to semi-structured to unstructured (or vice-versa) may results from the way students engage with the task.

Incorporating Students' Perspective in Design and Implementation

Reported and used extensively in research literature is the Mathematical Tasks Framework developed from the QUASAR (Quantitative Understandings: Amplifying Student Achievement and Reasoning) project team (Stein et al., 2000). This framework defines mathematical tasks as they unfold from design into implementation. In the framework outlined, mathematical tasks pass through three distinct yet related phases: as written by curriculum developers, as presented by the teacher in class, and as negotiated by students during classroom instruction. The framework is also useful in studying changes in task features and cognitive demands as instruction passes between any two successive phases.

As one might understand, depending on their beliefs, attitudes and experience, mathematics teachers are likely to implement the same task in different ways. For example, during the task presentation phase, different teachers may provide different kinds of instruction (information, guidelines and hints) to students about the task. I would argue that providing instruction towards the process rather than the product of students' inquiry is less likely to influence or modify the cognitive demands within the task. During this phase, teachers also tend to attribute titles to the tasks they assign. Reflecting on an incident from my research, I have come to understand that the task title might convey meaning to students about the kind of mathematical content involved in their investigation. Perhaps this focus indirectly provides unwarranted closure to the activity, thus directing students' attention to specific lines of inquiry and possibly shifting the intended 'open' nature of the task. I believe that presenting 'open' titles, such as, 'Investigate Right-Angled Triangles' rather than 'Investigate Pythagoras' Theorem' may offer more 'open' learning opportunities for students. In this case, a few students who attended private lessons already knew the theorem whereas others seemed puzzled by the title. These students, hence, offered resistance and lacked engagement. Apparently, the kind of title presented altered students' learning dispositions since they had 'good' reasons not to investigate.

During the presentation phase, teachers usually also provide students with resource materials to support task inquiry. Teachers might adapt tasks for students of lower ability by opting to provide more appropriate resource materials possibly without unduly reducing the cognitive demands within the task. As in my current collaborative project referred to earlier, teachers are investigating the possibility of designing and implementing inquiry-based learning tasks, on a weekly basis, along a whole scholastic year. These tasks are accompanied by a range of resource materials (including worksheets and instruments) with the intention to cater for students in our ability classes. For example, for the task 'Classifying Triangles' students in a high ability class are provided with a worksheet, scissors and glue, while students in a lower ability class are provided with cut-out triangles, glue, protractor and a ruler. Although students in the two classes are expected to classify different triangles, the ones in the lower ability class are provided with additional material to support their mathematical inquiry. Within this design principle, the 'process help' provided by a range of resources makes it possible for teachers to adapt tasks for all students. This prospect also offers students multiple entry points in engaging with the task. I therefore contend that, within a research task design project, incorporating a range of resources may be crucial in minimizing the gap between the intended and the enacted activity and in engaging all students in cognitively demanding mathematical inquiry.

Students may nevertheless interpret and negotiate tasks in ways that may be different from those intended by the teacher – the process of inquiry may either be undermined or sustained/improved during instruction. Occurrence of the latter trait is more likely to manifest itself in environments where students become truly responsible autonomous learners working within social norms of collaborative learning. When students become independent self-regulated learners, their activity might generate different mathematical trajectories to those intended by the teacher. The 'young mathematician' might be confident enough to choose what resources to use and to consider different lines of inquiry. Alternatively, as a number of studies show, teachers have a propensity in reducing the demand level of the task related to classroom norms, task conditions, and teachers' and students' dispositions (Henningsen & Stein, 1997). For example, Desforges and Cockburn (1987) report this tendency occurring when students are struggling or when they give up. Likewise, during my action research, I faced situations where a particular student continuously asked for help when she fell behind compared to the others. During our interview, this student reported that I usually made it easier for her to finish tasks. My task enactment evidently avoided the student's frustration and speeded up task completion. Bound by time constraints, I occasionally reverted to forms of telling to move on the class to the next phase, namely, the whole-class presentation and discussion. Such teacher interventions direct students' attention towards priority in task completion – at times at the expense of more meaningful and deeper understanding. With hindsight I argue that help directed towards the outcome of the activity hinders the process of learning by investigation and inhibits students' cognitive development.

Concluding Remarks

Implementing investigative tasks essentially involves four-phases: tasks as planned and designed by the teacher; tasks as presented to the students; tasks as negotiated by students; and tasks as concluded by the students and the teacher (Ponte, Segurado & Oliveira, 2003). The study of these inter-related phases has informed practitioners, researchers and academics about design issues originating from students' engagement with a range of tasks (see Stein et al., 2000). Of foremost importance here is the formulation of principles, guidelines and reflective frameworks for effectively exploring how students negotiate and engage with tasks. As I see it, this awareness is based on a reflective framework that includes: (i) a clear understanding of the purpose of tasks – one that resonates well with active learning; (ii) a classification within which different types of content-related tasks could be fitted – this alignment would be useful in examining how different tasks are designed and presented by the teacher and eventually enacted by students in class; (iii) a *task-delineated* scheme that defines the teacher's hypothetical learning trajectories and accounts for students' learning trajectories – this scheme hence also defines the didactical approach undertaken by teachers in class; and (iv) an account of the range and type of resources provided to students in order to render tasks more accessible to all students – this also provides a basis for investigating how students make use of the resource materials supplied and consequently on the ensuing activity.

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Writing the Student into the Task: Agency and Voice

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In this paper, we characterize tasks with respect to intention, action and interpretation to generate insight into student agency and voice. It is our contention that a written mathematical problem cannot be seriously discussed as a mathematical task without specification of the intended purpose, participants, and product. In addition, differences between social, cultural and curricular settings, together with differences between participating classroom communities, shape the performative realization of a mathematical task. This challenges reductionist attempts to characterize instructional tasks independent of these considerations. Given this contextual dependence, any commonalities across context carry significant weight. Central to an understanding of "mathematical task" as enacted in classroom settings is the social distribution of responsibility, agency and voice. Our research shows that competent teachers in several countries utilize mathematical tasks in a way that maximizes student agency and voice. We propose that the prioritisation of student agency and voice should be one of the principles of task design.

Keywords: Task Design, Agency; Voice; Didactical Triangle

Introduction

The classroom performance of a task is ultimately a unique synthesis of task, teacher, students and situation. The activity that arises as a consequence of a student's undertaking a task is itself a constituent element of the learning process and the artefacts (both conceptual and physical) employed in the completion of the task serve simultaneous purposes as scaffolds for cognition, repositories of distributed cognition and cognitive products. Task selection by teachers initiates an instructional process that includes task enactment (collaboratively by teacher and student) and the interpretation of the consequences of this enactment (again, by teacher and student). In the context of Theme B, this paper addresses the question: How do students' reactions influence teacher adaptation of the task? We argue that this question does not adequately portray the fundamental reflexivity by which the actions of teacher and student are mutually informing during the performance of a task. There is no doubt that student response influences the teacher's incremental and iterative adjustment of the task as performed in the classroom, but, equally, these progressive adjustments by the teacher iteratively influence the nature of student responses to the task, while continually re-constructing the task itself in the act of its undertaking. Something of this dynamic reciprocality may be captured by the inclusion of other Theme B questions as contributing considerations: "What do students actually do and attend to when confronted with tasks?" and "How do students understand the purposes of tasks they are given in the classroom." In combination, these questions provide our focus.

Marx and Walsh (1988) identified three essential elements to any consideration of the role of 'academic tasks': the conditions under which the tasks are set; the cognitive plans students use to accomplish tasks; and the products that students create as a result of their task-related efforts. This conception either ignores the role of teacher intentionality and mediation, or it relegates this to just another element in the social context in which the task is undertaken. There is a further danger that student agency and authorship are identified solely with the products of the task and their role in the performative realisation of the task itself goes unrecognised.

Our conception of the teacher/student/task triad is highly interconnected and accords significant agency to each in the determination of the actions and outcomes that find their nexus in the social situation for which the task is the pretext. It is our contention that a written mathematical problem such as "Find the average of 13, 15, 17, 19, and 21" cannot be seriously discussed as a mathematical task without specification of the intended purpose, participants, and product. Even with this specification of intention, the performative realisation of the task is a collective mathematical performance by the teacher and students in a particular context for a particular purpose, generating particular products. "Find the average of 13, 15, 17, 19, and 21" could be referred to as the task statement or the task stimuli, but it is only one component of the task as enacted by particular people at a particular time and place for a particular purpose. Even the imprecision of "average" cannot be either praised as a well-intentioned attempt to elicit information about student understandings of mean, median and mode, or criticised as mathematical sloppiness, until we know the teacher's intentions for the task.

We have examined the function of mathematical tasks in classrooms in five countries. A three-camera method of video data generation (see Clarke, 2006), was supplemented by post-lesson video-stimulated reconstructive interviews with teacher and students, and by teacher questionnaires and copies of student work. Our analysis characterized the tasks employed in each classroom with respect to intention, action and interpretation and related the instructional purpose that guided the teacher's task selection and use to student interpretation and action, and, ultimately, to the learning that post-lesson interviews encouraged us to associate with each task. In this paper, we draw on some of the findings from that analysis.

Our analysis employed 'function' as the combination of intention, action and interpretation to examine the functionality of mathematical tasks in classroom settings. Of particular interest were differences in the function of mathematically similar tasks when employed by different teachers, in different classrooms, for different instructional purposes, with different students. The significance of differences between social, cultural and curricular settings, together with differences between participating classroom communities, challenges any reductionist attempts to characterize instructional tasks independent of these considerations. Of equal interest were differences in learning outcomes arising from the use of fundamentally different mathematical tasks, such as highly decontextualised or abstract tasks (see the Chinese examples below) in comparison with contextualized or so-called 'real world' tasks.

In relation to the related advocacies of relevant and authentic mathematics, Kirschner, Sweller and Clark (2006) make the insightful observation that

> It may be an error to assume that the pedagogic content of the learning experience is [should be] identical to the methods and processes (i.e., the epistemology) of the discipline being studied (p. 84).

In particular, their assertion that “The practice of a profession is not the same as learning to practice the profession” (p. 83) highlights a critical issue in the design of instruction in mathematics. How is classroom mathematical activity related to the activity of the mathematician? While we may classify the tasks of the mathematics classroom in a variety of ways, we should not confuse those tasks with the tasks of the mathematician: they are fundamentally different in purpose.

Mathematical tasks employed in educational settings have been variously categorised under designations such as ‘authentic,’ ‘rich’ and ‘complex.’ The classification ‘authentic’ has particularly emotive overtones – suggesting that some mathematical tasks might be classified as ‘inauthentic.’ The most common usage of the term ‘authentic’ in this regard seems to refer to an assumed correspondence between the nature of the task and other mathematical activities that might be undertaken outside the classroom for purposes other than the learning of mathematics. The value attached to ‘authentic mathematical tasks’ seems to appeal to a theory of learning that measures mathematical understanding by the capacity to employ mathematical knowledge obtained in the classroom in non-classroom (‘real-world’) settings and which constructs the process of mathematical learning as ‘legitimate peripheral participation’ (Lave & Wenger, 1991) in the mathematical activities of a community larger than a mathematics class. Such apprenticeship models deny or ignore the significance of the students’ role as agents in their own learning, actively shaping classroom activity through their participation and progressively becoming more skilled, not as mathematicians, but as mathematics students.

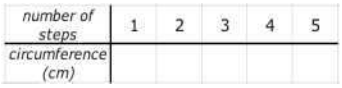
The eighth-grade mathematics classrooms that provided the sites for our analysis were drawn from the data set generated by the Learner’s Perspective Study (LPS) (Clarke, Keitel, & Shimizu, 2006). Our initial goal in the analysis of mathematical tasks undertaken in these classrooms was the selection of tasks that could legitimately be described as distinctive because of the character of the mathematical activity or because of the teachers’ didactical moves in utilising the tasks to facilitate student learning. In this paper, we will use a small selection of very different examples to make two points that we suggest should be central to task design in mathematics education:

(i) The competent teachers that we studied prioritised student agency in the classroom performance of mathematical tasks;
(ii) In promoting high quality mathematical activity, the teacher’s capacity to choreograph sophisticated classroom discourse was as important as the mathematical sophistication of the task statement or question.

Sample Task One: Japan School 1 – Lesson 1 (the Stairs Task)

[For reasons of brevity, 15 minutes of data record has been omitted from the middle of the table]

Educational Context of the Task This was the first lesson in a sequence of lessons concerned with functions, relations and patterns, where particular emphasis was placed on the special terms used in mathematics. The teacher identified her global aims for the entire lesson sequence of about sixteen lessons, as: i) identifying functions and their relationships to everyday life; and ii) understanding how to solve equations using a table, a graph or formal algebraic techniques. This particular lesson was designed by the teacher to focus on: i) different variables and their relationships with one another; ii) understanding the form of a linear equation; and iii) understanding that the investigation of the nature and function of equations is of utmost importance.		
Social Performance of the Task		
Teacher	Mathematical Task as presented	Student
The teacher states her goal to the class, "I'd like to think about change using these figures." -------------------------------------- The teacher asks students to work on drawing the next two figures in the sequence.	The Stairs Task 1) The first three figures have been drawn for you. Draw the next two figures by stacking one cm sided squares on top of each other	(Students attentive) -------------------------------------- Nou is invited to the board to draw the next two figures.
The teacher invites students to present ideas on identifying what aspect of the figures is changing. The teacher invites students to work in small teams to identify as many aspects as they can. (The teacher roams the classroom and speaks with individual students) The teacher addresses the class and invites further suggestions on "what changes?"	2)What changes when the number of steps changes?	Jitsu immediately suggests, "the number of steps." Nobo and Nou add "size" and "area" respectively. Taka adds "height." Students working in small groups) Nii adds "number of sides" and "number of squares." Mika adds "circumference." Taka says, "shape." Nou responds with "the length of the base." Jitsu adds "the time it takes to draw the figures." Nobo adds "sum of the interior angle" and "the number of vertices."
The teacher invites students to suggest how they might go about examining the relationship between the number of steps and the circumference. The teacher reiterates Mawa's mention of the use of a table and reminds students that mathematical expressions are also useful in examining relationships. She proceeds to draw up a table:	3) Examine the relationship between the number of steps and the circumference.	Some students suggest, "graphs." Mawa mentions "table."

 number of steps | 1 | 2 | 3 | 4 | 5 circumference (cm) She asks students to complete the table and to identify a mathematical relationship		
(The teacher roams around the room assisting students)	(Students go to the board and complete the table or write an equation) (~15 minutes)	(Students work on the assigned task at their desks, while some of their classmates go to the board)
The teacher asks for other interpretations – other methods for getting the answer. The teacher highlights this method graphically with Nobo's help: 		Nii responds that you can add four to the previous answer, so that to get the circumference for six steps, one adds four to 20.
The teacher assigns the homework task:	4) Show that 'multiplying by four' works. 5) Examine the relationship between the number of steps and another feature of the diagrams.	(students attentive)

The Stairs Task is well-known and used by mathematics teachers in many countries. This teacher offered students several opportunities to exercise agency while undertaking the task. For example, the students were able to identify a large number of differing variables. The discussion then converged on the particular connection between the number of steps and the circumference and began to model a mathematical approach to defining this relationship. The teacher's focus on the students and the partnership she had formed with them in their learning was further evidenced by students' contributions at the board: Nou drew steps four and five; Yama traced one of the figures with his finger to illustrate the concept of circumference; Ume completed the table, while Taka wrote the relationship in algebraic terms; and Nobo shared his various conjectures of relevant variables.

Once the relationship ($y = 4x$) was proposed, the teacher's attention turned to how to demonstrate the correctness of this equation, "Think about how to show that multiplying by four works." Her recurrent emphasis was on the collaborative exploration of the problem in the interest of promoting student reflection and learning.

Task Two: China School 2 – Lesson 8 (the Numbers task)

The class was learning about the system of linear equations. The students have spent previous lessons working with inequalities, the relationship between the concept of linear equations in two unknowns and their solution, the transformation of equations and solution methods involving substitution and elimination. This was the eighth lesson in this topic and one of the teacher's goals for this lesson was for students to learn to solve some special linear equations in three unknowns. The task statement consisted of:

> There is a three digit number. The sum of the three digits is 12. The sum of the hundreds digit and the tens digit is greater than the ones digit by 2. Three times the hundreds digit equals the sum of the tens and the ones digits. Let the hundreds digit be *x*, and the tens digit be *y* and the ones digit be *z*. Set up the equations according to the question.

The teacher's purpose in using this task was to introduce students to the underlying structure of algebraic representations and to assist them to develop appropriate mathematical language. The task was presented to the class and three students were able to correctly provide the individual equations to form the entire system. At this point, the teacher spent some time identifying the exact nature and definitive characteristics of a system of linear equations in three unknowns. The students were not encouraged to actually solve the equations, algebraically or by other means. The task consisted entirely of setting up the system of equations. Students were then presented with the following system to solve:

$$\begin{cases} x + y = 11 \\ y + z = 17 \\ x + z = 10 \end{cases}$$

Our interest in this task lies in the significant devolution of responsibility to the students to generate solutions and the prioritisation of the development of student facility with the technical language of mathematics.

Task 3. Japan School 3 – Lesson 1 (the Long Task)

In this task, the seemingly simple pair of simultaneous equations $5x + 2y = 9$ and $-5x + 3y = 1$ engaged the class for a fifty-minute lesson (and indeed was the discussion point for the first fifteen minutes of the following lesson). A feature of the performance of this task was the extent to which student suggestions, responses and the articulation of their thinking were regarded as instruments for developing understanding.

The teacher, when asked about his aim for this lesson responded, "It's not that I just wanted them to just solve the problems but also, um, I wanted to teach them that there is a need to think about it a little – what solving equations is all about." The emphasis on prompting student reflection, enacted in this class through social interaction, rather than merely on the solution of a fairly trite mathematical problem distinguished this task performance and this teacher's practice.

This third task provides a stark illustration of the crucial nature of the teacher's role. It also problematises the characterisation of a task as a "good task" independent of the social setting and actions through which the task is undertaken. Our analysis prioritises consideration of the social interactive aspects of task performance and, in particular, interactions that constitute the progressive restructuring or reconception of the task itself.

Theoretical Alternatives in Analysing Classroom Task Performance

There are many different theories currently being employed in mathematics education. Activity Theory, for example, is an obvious contender in considering how the classroom use of mathematical tasks might be situated theoretically. Recent developments in the conceptualisation of Activity Theory (eg Engestrom, 2001) have increased the breadth of phenomena and contexts able to be addressed using Activity

Theory. In particular, mathematical tasks can be situated naturally within the tools available for use in pedagogic activity systems.

Gellert (2008) usefully contrasts 'interactionist' and 'structuralist' perspectives on mathematics classroom practice. In the consideration of the classroom use of mathematical tasks, the interactionist perspective offers insight into the negotiative processes that interact with individuals' use in classroom settings for the socially-mediated constitution of learning. The structuralist perspective potentially offers very different insights into the deployment and function of mathematical tasks in classrooms. Focusing attention on differentiated participation, a structuralist analysis aspires to explain such differentiation in terms of hierarchies and power relationships. In the case of mathematical tasks, these hierarchies reflect the enactment of an entrenched social order and the privileging of particular forms of knowledge. Within the structuralist perspective, particular pedagogies can be seen as embodying systems of social and academic privilege (Bernstein, 1996) and in the mathematics classroom it is primarily through the performance of mathematical tasks that these pedagogies are enacted.

The choice of the theoretical lens focuses analytical attention on some aspects of the role of mathematical tasks and ignores others. This is inevitable. Another entry point employs the three related issues of Abstraction, Context and Transfer. In some discussions, abstract mathematics seems to be treated as simply decontextualised mathematics. Clarke and Helme have argued that there is no such thing as decontextualised mathematics (Clarke & Helme, 1998), since all mathematical activity is undertaken in a context of some sort. If abstraction in mathematics is to have any legitimacy or relevance, then it must reside in some form of generalisability of the mathematical matter under consideration, in the sense that the principle, concept or procedure can be thought of as transcending any particular context or instance. But, to argue that an exercise in Euclidean geometry or in pure number is an abstract task is to deny the social situatedness that has become accepted even from the most cognitivist of perspectives (Lave & Wenger, 1991).

In relation to mathematical tasks, Clarke and Helme distinguished the social context in which the task is undertaken from any 'figurative context' that might be an element of the way the task is posed. In this sense, the task:

> Siu Ming's family intends to travel to Beijing by train during the national holiday, so they have booked three adult tickets and one student ticket, totalling $560. After hearing this, Siu Ming's classmate Siu Wong would like to go to Beijing with them. As a result they buy three adult tickets and two student tickets for a total of $640. Can you calculate the cost of each adult and student ticket?
> (Shanghai School 3, Lesson 7, Train task)

has a figurative context that integrates elements such as the family's need to travel by train and the familiar difference in cost between an adult and a student ticket. The social context, however, could take a wide variety of forms, including: an exploratory instructional activity undertaken in small collaborative groups; the focus of a whole class discussion, orchestrated by the teacher to draw out existing student understandings; or, an assessment task to be undertaken individually. In each case, the manner in which the task will be performed is likely to be quite different, even though we can conceive of the same student as participant in each setting.

The Three Tasks

The three tasks were selected for their disparity across the key attributes: mathematics invoked (both content category and level of sophistication); figurative context (real-world or decontextualised); resources utilised in task completion (diagrams and other representations); and the nature of the role of the task participants. Students were given a significant "voice" in the completion of each task, but the nature of their participation reflected differences in the extent and character of the distribution of responsibility for knowledge constructed in the course of task completion. This distribution of responsibility (or enhanced agency) is a consequence of each teacher's strategic decision, moment by moment, of how best to orchestrate student work on the task. In seeking to understand task performance as the iterative culmination in the joint construction, not only of the task solution, but of the mathematical principles of which the task is model and purveyor.

If we take 'transfer' not as a description of a particular cognitive process, but as a metaphor for a skill developed in one context being used in a different context, then it is reasonable to ask, "Under what conditions (and through the instructional use of what tasks) will the likelihood of transfer be maximised?" A cognitivist might direct attention to the selective variation of task attributes with the intention of successively focusing student attention on salient aspects of the mathematical concept or procedure to be learned. Variation Theory (Marton and Tsui, 2004) identifies learning with an increasing capacity to discern relevant attributes in the object of learning. From such a perspective, particular tasks and particular sequences of tasks can be critiqued as more or less conducive to directing student attention appropriately and thereby to the optimal promotion of the discernment that is identified with learning.

Distributed Cognition (Hutchins, 1995) and other theories with a material semiotic character accord significance to artefacts as participating in cognition. Rezart and Straesser (2012) have expanded our conception of socio-didactical situations in mathematics classes, and include artefacts in addition to the teacher, the students, and the mathematics. Once representational forms are included in the broad class of artefacts, then mathematical tasks cease to be either the objects to which we apply our cognitive tools nor merely the social catalysts for their deployment. Rather, mathematical tasks become the embodiment of performed cognition, integrating, as they do, representational forms, socio-cultural imperatives and mathematical entities. We find it useful to portray mathematical tasks performatively in order to examine the role each task plays in affording or constraining agency and voice in the social settings in which the tasks are communally performed. Our conception of the teacher/student/task triad accords significant agency to *each* in the determination of the actions and outcomes that find their nexus in the social situation for which the task is the pretext.

In each of the three tasks shown, the teacher prioritised student agency in task completion, but in very different ways: The Japanese teacher opened Task One by stimulating student discussion of what had noticeably changed during the recent holiday break. This discussion, drawing on student personal experience, situated the mathematical activity in relation to out-of-class contexts with which the students would be familiar. In Task Two, the teacher's purpose was to introduce students to the underlying structure of algebraic representations, and the task as posed was stripped of any elements that might invoke a figurative context. However, even lacking any sense of familiar context, other than the classroom setting itself, student active participation in the task performance was still prioritised. Task Three was chosen because of the fascinating use by the teacher of a relatively pedestrian pair of simultaneous equations to scaffold students' developing understanding of the general attributes and properties of systems of linear equations. In the performance of this task, student suggestions, responses and the elicited articulation of their thinking became instruments for developing understanding.

Conclusions

Tasks have long been recognized as crucial mediators between mathematical content and the mathematics learner. Of particular interest in our analysis were differences in the function of mathematically similar tasks, dealing with similar mathematical content (those relating to systems of linear equations), when employed by different teachers, in different classrooms, for different instructional purposes, with different students. The "entry point" for our analysis was a tabulation of the details related to the social performance of the task (as shown for Sample Task One). Using these tables, our analysis drew on the video-stimulated, post-lesson interview data to identify intention and interpretation and relate both to social performance of the task.

The conception that the community-at-large holds of the mathematics classroom is intrinsically bound up with the type of tasks that characterise such settings. And this conception is not in error. Mathematical tasks are the embodiment of the curricular pretext that brings each particular set of individuals together in every mathematics classroom. In other contexts, individuals come together to engage in musical performances or dramatic performances. The performances of the mathematics classroom are largely the performance of mathematical tasks and if we are to understand and facilitate the learning that is the ostensible purpose of such settings then we must understand the nature of the performances that we find there.

The thread that we pursued through the examples discussed has been that of the social distribution of responsibility, agency and voice. We commenced our analysis disposed from other studies to believe that these issues were important. Our exploration of responsibility, agency and voice in the context of the classroom performance of mathematical tasks suggests to us that competent teachers of mathematics (within the constraints of culture and curriculum) share a belief in the importance of these elements. The valuing of agency and voice is evident in the task performances in the classrooms of these teachers, rather than in any explicit articulation by them in classroom video data or in interview.

If we are to find pattern and structure in the profound diversity of "well-taught" mathematics classrooms around the world, then the attention given by competent teachers to student voice and student agency, and the mathematical tasks that they employ to catalyse that voice and agency, support our belief that the maximization of student agency and voice in the performative enactment of a mathematical task should be recognized as a key principle of task design and delivery.

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Making distinctions in task design and student activity

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In this article, we articulate design principles that have developed, during the time of the authors' collaboration, over a period of fifteen years. The principles are drawn both from the enactivist theory of cognition and learning (Varela, Thompson and Rosch, 1991) and the pedagogic ideas of Gattegno (1987). We exemplify how task design centres around activities that provoke differences in student response, allowing the opportunity for students to make distinctions and for teachers to introduce new skills. We illustrate how the principles operate to inform teacher planning, teaching actions in the classroom and students' mathematical activity. The fact the principles operate on these three levels means that student activity closely matches teacher intentions.

Keywords: task design principles, mathematics teaching, student activity, enactivism, Gattegno, distinctions

Introduction

In this article we draw out implications of task design principles for our understanding of how teacher intentions and student mathematical activity may begin to converge. We address the questions: what do students actually do and attend to when confronted with tasks? how do students' reactions influence teachers' adaptation of the task?

Our design principles have developed over the fifteen-year period of our collaboration; they arise from our epistemological commitment to enactivism (Varela, Thompson and Rosch, 1991) and the pedagogical influence of the work of Gattegno (1987). Following this introduction, we offer a brief sketch of these perspectives, we then set out and exemplify our design principles and present evidence from the practice of two teachers to analyse the students' mathematical activity.

The particular community that is the focus of this article centred around one school (School S) in the Bristol area of the UK, that was in 'partnership' with the University of Bristol, meaning that the mathematics department took student-teachers from the PGCE course on an annual basis. Alf Coles joined this department in 1996, having already begun a research collaboration with Laurinda Brown. Alf became a school-based mentor for the PGCE course and, in 2001, head of the mathematics department. Between 1996 and 2003, Laurinda made this school her main research site and visited, where possible, weekly. During these visits (often for a day) she would work in Alf's classroom, observing, sometimes co-teaching and always reflecting on events afterwards. This collaboration led to publications (e.g., Brown and Coles, 1997, 2010) and research projects (e.g., Brown et al, 2001).

We take, as our definition of task, the one suggested in the discussion document of this study group, 'a task is anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something. Task can also be anything that students decide to do for themselves in a particular situation' (Watson and Ohtani, 2012, p.4). In any task, as well as learning some mathematics, students are learning about *what learning mathematics is like* in this classroom; for us, the choice to use of any task cannot be dissociated from a choice about *ways* of working.

There is a significant problem, identified in the literature, around the student experience of tasks, compared to the intentions of the designer or teacher. Mason, Graham and Johnston-Wilder (2005, p.131) raise the issue of how an expert's awarenesses get translated into instructions for the learner that do not lead to those same awarenesses. Mason et al (2005) connect this issue to Chevallard's didactic transposition (1988), the problem of moving from the knowledge *used* in a sphere, such as mathematics, to the knowledge to be *taught*. The contexts in which mathematical knowledge is used can never be faithfully replicated in a classroom. We see a similar issue being highlighted by Tahta (1980) when he distinguishes 'outer' and 'inner' aspects of tasks. The outer task is what is made explicit by the teacher, the inner task is the relationship or awareness the teacher hopes students will gain. The problem for teachers is that the more the desired behaviours in students are specified, the less these behaviours are likely to emanate from students' own awareness. We see essentially the issue again being described by Watson and Mason (2007):

> Engaging learners in activity is important, but in order to learn from that activity they need to experience some kind of shift or transformation in what they are sensitised to notice and attend to mathematically. (p.209)

We also agree with Watson and Mason's (2007) emphasis on:

> the importance of developing ways of working, a classroom rubric in which the learners are drawn into patterns of thinking, in which some transforming action takes place (p.210)

In other words, one solution to the problem of how to connect teacher intentions and student activity, is to focus on developing particular ways of working, or patterns of thinking with students. In this article, through analysing examples of what students do when faced with tasks and how teachers respond, we suggest that the issue can also be addressed through a focus on the making of distinctions in task design. However, before getting to this argument, in the next sections we set out our epistemological and mathematical perspectives and the design principles that operated in School S.

Enactivism

Enactivism entails a commitment to the inseparability of thought and action (Varela, Thomson and Rosch, 1991) and a rejection of the view that we operate in the world via making use of inner representations of objects, with thoughts about these representations leading to actions. Rather, we are closed to the world, from an informational point of view. All the world can do is 'trigger'; how we respond is determined by our structure, that is, by the particular relations and physical components that make up ourselves. All living beings are, in this sense, 'structure determined' (Maturana 1988, p.12). Through a recursive process of acting in the world and the world acting back we learn to co-ordinate our actions with others, as we 'co-evolve' (Varela, Thomson and Rosch, 1991, p.201) in a process labeled 'structural coupling' (Maturana & Varela, 1987, p.75). In each interaction my structure changes, as does the world. It is our history of structural coupling (both as a species and in each individual lifetime) that accounts for the smooth functioning of so much of our life. Varela writes, on the enactive view, 'perception is perceptually guided action' (1999, p.12), i.e., perception is not a passive receipt of information (which we represent) but part of an active (on-going and recursive) process of structural coupling with the world. Over time, how we act can change how we perceive (an example being the wine taster, who is able to make distinctions literally imperceptible to the untrained palate).

Within enactivist epistemology, practical knowledge linked to action is fore grounded, with more propositional knowledge arising from awareness of action. Maturana and Varela (1987), key enactivist writers, state, 'cognition is effective action**,** an action that will enable a living being to continue its existence in a definite environment' (1987, p. 29). So an action is effective if it allows me to continue operating in a specific context. Knowledge, which requires cognition, is therefore also equated with effective action. Knowing cannot be separated from the knower nor the context in which the knower acts, hence, '[a]ll doing is knowing and all knowing is doing' (Maturana and Varela, 1987, p.27).

The palate of the wine taster offers a metaphor for the enactive view of learning, in the making of ever-finer distinctions in a sphere of action. If learning is seen as the development of knowing, then learning is equated with being able to act differently (in new ways), which is also the same as saying that learning is equated with being able to perceive differently.

Enactivism also carries implications for *how* learning can occur, which arise from the notion of structural determinism. In a classroom, the most significant features of the 'environment' for any individual are the other individuals in the room. Any interaction between teacher and student or between student and student must alter the structure of both, in the process. Enactivism is a profoundly social theory, we are quite literally changed through interaction with others, or more precisely, we change ourselves through interaction with others who likewise change themselves. We cannot *not* change, however minimally, in every encounter. Equally, we cannot specify any change we want to provoke in others. It is in this sense that we understand Stewart (2010), in a book on enactivism, when he writes, '*instruction*, in the strict sense of the word, is radically impossible' (p.9). All we can do, as teachers, is provoke, stimulate, trigger; how students respond will be a function of their own structures and histories and cannot be determined by us.

Mathematical thinking

Our view of mathematics has been influenced by Gattegno (1987). Gattegno saw mathematics as the awareness of relationships (1965). Awareness was a technical term for Gattegno, whose purpose is, 'to illuminate our fields of action' (1987, p.25). He turned 'awareness' into a countable noun – we can enumerate awarenesses. We are being algebraic whenever we step back from a procedure, or a dynamic, to become aware of it. Writing something as simple as $x+y$, entails a stepping back from the process of addition, to represent that dynamic. In fact, for Gattegno, all mathematics was algebraic, since awareness of relationships and awareness of dynamics are intimately linked, 'all is algebra in mathematics, because to say "algebra" is to say the awareness of the mind at work on whatever content' (Gattegno, 1965, p.22). Gattegno privileged the visible and the tangible, in the learning of mathematics. He worked with imagery and another quotation of his that we have both lived with, is that mathematics 'is shot through with infinity' (1984, p.20). The connection we make here is that, once we become aware of a relationship (i.e., once we are mathematizing, or thinking mathematically) then it becomes possible to imagine that relationship iterated and hence to infinity. Gattegno also distinguished 'powers of the mind' (1971) that he believed all humans possess, which mean all humans are able to think mathematically. These powers (also referred to in Brown and Coles, 2011, p.866) are:

> 1) extraction, finding 'what is common among so large a range of variations'; 2) making transformations, based on the early use of language 'This is my pen' to 'That is your pen'; 3) handling abstractions, evidenced by learning the meanings attached to words; and 4) stressing and ignoring, without which 'we can not see anything' (paraphrased from Gattegno, 1971, pp. 9-11).

Making distinctions play a role in (1) and (4), linking to our enactivist stance.

Principles of task design

Arising directly from the epistemological and mathematical perspectives above, the design principles that operated in School S included the following list, which we will unpack and exemplify. This list is not presented as a finished product and was not made explicit within the department. However, over the course of the last fifteen years, within the community around School S, various of these principles were spoken about and worked on as a group. Two mathematics teachers from School S have moved to be Heads of Mathematics in different schools, and have established their own curricula based on similar principles for activities. The principles are:

1. starting with a closed activity (which may involve teaching a new skill).
2. considering at least two contrasting examples (where possible, images) and collecting responses on a 'common board'.
3. asking students to comment on what is the same or different about contrasting examples and/or to pose questions.
4. having a challenge prepared in case no questions are forthcoming.
5. introducing language and notation arising from student distinctions.
6. opportunities for students to spot patterns, make conjectures and work on proving them (hence involving generalising and algebra).
7. opportunities for the teacher to teach further new skills and for students to practice skills in different contexts.

Principles 2, 3, 5, 7 come from the enactive view of knowing and learning, which is linked to the making of distinctions. By working with at least two examples, we support students in sharing distinctions and, through this sharing, making new distinctions, which is tantamount to learning from the enactive perspective. Language and notation is introduced to label distinctions students make, to support new ways of seeing. By 'examples' (principle 2), we mean to capture a wide range of possibilities, including images, animations and procedures. Principle 6 comes from our view of mathematics as being essentially about the activities of conjecturing, no matter what content is being covered. Principles 1, 4 and 7 derive from Gattegno. The closed activity will, where possible, involved something visible or tangible and which all students can do. The challenge and opportunity to teach skills in different contexts are linked to the power humans have of extraction. When we do something in different contexts, it is more likely we can extract the skill are retain it for use another time.

In the next section we exemplify these principles, first with a task in the scheme of work of School S and then two tasks that arose in lessons.

Principles exemplified

Equable shapes

One exemplification of these design principles, is a task (we call it 'Equable Shapes') that begins with the shapes in Figure 2 drawn on the board. The origins of this task are obscure. The version of the task presented below was developed at School S, from an idea Laurinda had used in her teaching. Often, in School S, tasks would be borrowed from elsewhere but worked on, by Laurinda and Alf in the earlier years of our collaboration, and as a department when Alf became head of mathematics. There was a scheme of work in the school which specified certain 'common tasks' (such as 'Equable Shapes') had to be done in a certain time slot with a certain year group. Discussion amongst teachers often led to changes in task beginnings, which were written up for the following year. What follows is a condensed version of the kind of write up that would be in the scheme of work, for teachers. We have analysed the write up against the seven design principles above.

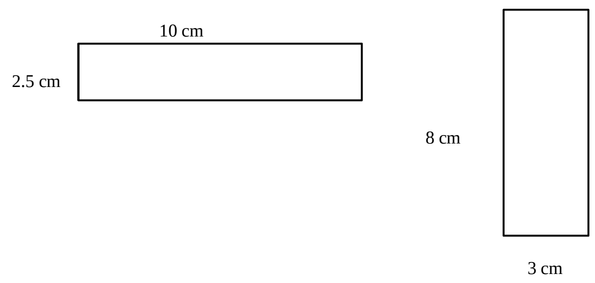

Figure 2: Two contrasting examples

These shapes are 'two contrasting examples' (principle 2). With this image on the board, the teacher asks students, 'what is the same and what is different' (principle 3). Generally, in the UK, students will at some point begin to comment about the different sizes of the rectangles. Students may or may not need support in remembering that mathematicians use the ideas of 'area' and 'perimeter' to judge the size of rectangles. This awareness leads the teacher to invite the students to find the area and perimeter of both shapes (a closed task, principle 1). The results are:

10 by 2.5 cm rectangle: Area = 25cm^2, Perimeter = 25cm

7 by 3 cm rectangle: Area = 21cm^2, Perimeter = 20cm

Again, the offer from the teacher is for students to comment on anything they notice that is the same or different. A student will usually notice that for the first rectangle, the value of the area and perimeter is the same. At this point, the teacher introduces the label 'equable' (principle 5) as a name for the 10 by 2.5 rectangle. The teacher can then ask students what questions they could pose (principle 3) and gather ideas on the board. The teacher-prepared challenge is: what other equable rectangles can we find? are there equable shapes that are not rectangles? (principle 4). Just the work on rectangles, offers opportunities for pattern spotting, generalising, algebra and conjecture (principle 6). In working on generalising what they notice, students are using skills of distinguishing area and perimeter and, depending on the direction in which the activity goes, skills may need to be taught to support students in solving linear equations, using Pythagoras' theorem, using trigonometry, while focused on the idea of finding equable shapes (principle 7).

In keeping with our own design principles, we now offer contrasting examples of tasks that arose more or less spontaneously in the practice of two teachers. These examples allow us to address the questions, when using the design principles above: what do students actually do and attend to when confronted with tasks? and, how do students' reactions influence teachers' adaptation of the task?

Transcript 1: Fractions of quantities

The transcript below is from a video recording of a lesson, taken as part of one of the research projects (Brown et al, 2001) that involved teachers at School S (among other schools). Laurinda directed the project and Alf was a teacher and researcher. The transcript shows the emergence of a task spontaneously within a lesson of Teacher A, who worked in School S. This kind of sequence in a lesson often occurred in School S, evidenced in video recordings of lessons, discussions amongst teachers and visitor observations. The original problem for students had been to find rectangles with area 12. Three examples were on the board, when Student 2 was given the pen. The students were in year 7 (aged 11-12) and the examples on the board were a 3 by 4 rectangle, a 2 by 6 and a 12 by 1 (Teacher A refers to these examples in line 42).

[Student 2 takes the board pen and board ruler and draws a half square by 24 square rectangle, giving it the label $^1/_2$ by 24.]
39 Students: Half a square?
40 Student 2: Half a square.
41 Student 8: Half of 44, half of 48, sorry.
42 Teacher A: Excellent. Oh, lovely. Well done. [Students applaud] So, 3 times 4 is 12, 2 times 6 is twelve, 1 times 12 is twelve and a half times 24 is also 12.
43 Student: And do we do that as well.
44 Teacher A: Pardon.
45 Student 8: And a quarter times 48 is twelve.
46 Teacher A: And a quarter times 48 ...
47 Student 8: And an eighth times ...
48 Student: Three quarters.
49 Teacher A: And an eighth times ...
50 Student: I'm not saying.
51 Student: You can actually go on.

Following this exchange, the teacher then wrote ¼ x 48 and got two students to confirm the answer. Teacher A then wrote on the board (see boxes) as the conversation developed.

57 Teacher A: What about a third?
58 Student: What?
59 Student 9: 36.
60 Student: No, you can't really draw a third.
61 Teacher A: Why, how do you work it out for those that are struggling a bit. [Directed at Student 9.] How do you know? He's right, it is 36. How do you know it's 36?

$\frac{1}{3}$ x
$\frac{1}{3}$ x 36

Two students responded to this question, and then Student 10 said, '3 twelves are 36'. Teacher A picked up this idea and asked what 4 twelves are, pointing to the ¼ x 48 written on the board. Teacher A asked (line 69), what $^1/_6{}^{th}$ would be, a student responded ‘72’. Another student asked (line 72), ‘What about 100? How could you draw it though?’ and three turns later, a student asked (line 76), ‘Sir, what would just a straight line be?’.

In this short episode, that was not planned, we see the design principles in action. The questions the class were initially working on are closed (principle 1), for example, line 49, ‘And an eighth times’. The teacher wrote up ¼ x 48 and $^{1}/_{3}$ x 36, so there were two examples for students to see (principle 2). Students were patently the ones posing the questions and sometimes answering them, for example in lines 45, 47, 72 and 76 (principle 3). The teacher contributed to posing challenges in lines 57, 61, 69 (principle 4). It is the students who first introduced the language of fractions, Teacher A provided the notation (e.g., ¼ x 48) to describe the rectangles they were considering (principle 5). There is clearly scope for spotting patterns in this task, as many students did; generalizing came with the awareness that you could take any unit fraction as the height and still make a rectangle of area 12, taken to its limit by a student in line 76, ‘what would just a straight line be?’ (principle 6). Part of what students were doing, in this short episode, was practicing a relatively mundane skill of finding fractions of quantities, but they were doing it in a novel context and with their attention on the area of the resulting rectangle (principle 7).

There are sections of the transcript (lines 45-51) where, if names were removed, it would be impossible to tell if students or teacher were speaking. This exchange was prompted by a student drawing a shape with a fractional side length. One connection that is made by a student (Student 10) is that to answer, $^{1}/_{3} \times ? = 12$, you can work out 3×12. We interpret this statement from Student 10 as an example of the kind of shift in attention that Watson and Mason (2007, p.209) describe as being central to learning mathematics. Student 10 articulated an awareness of a relationship that then supported other students in extending the implied pattern ($^{1}/_{6} \times ? = 12$, means $6 \times 12 = ?$, etc), this awareness of relationship is mathematical thinking.

What is significant for us, in this lesson, is the way the design principles of the department appear to have influenced how Teacher A adapts in an episode that arose spontaneously from an unplanned student response to an activity. Teacher A was creating the task as the lesson unfolded. Perhaps even more striking is the way the students played a role in creating the task, with Teacher A allowing discussion of student ideas to run and focusing the whole class on certain questions (e.g., lines 49 and 57). We know, from interviews carried out as part of the ESRC project, that ‘going with’ student ideas in the way we see in this transcript was something Teacher A was, at that time, just beginning to experiment with.

Transcript 2: Both Ways

The transcript below is from a video taken in 2008 (as part of an ESRC Studentship) of a teacher (W) who had been in the department for several years and who had worked with this group of (aged 12-13) students for a year and a half. The task we called “Both Ways”, W had drawn the image in Figure 3 and invited students to suggest a number to go in the top left circle.

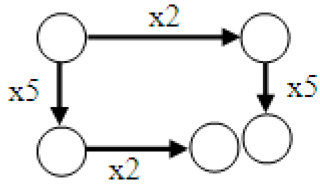

Figure 3. Both ways

A student had suggested 74 as the starting number, which the class then worked through, getting answers of 740 in both circles in the bottom right. Teacher W looked at these answers on the board as she said the first line of the transcript.

Transcript notation: (.) indicates a small pause, (2) a 2 second pause, ( ) undecipherable speech

1 Teacher W: oh right (.) okay
2 Student: yeah but why (.) why does it come to that
3 Student: is that meant to happen miss
4 Student: yeah it is
5 Student: no
6 Student: oh yeah miss
7 Student: because if you do seventy four times ( )
8 Teacher W: um (4) any comments (.) any comments (.) yeah
9 Student: cos it can't be seven hundred and forty which is the bottom one (.) because seventy four times five is three hundred and seventy and then (1) no (1) no ignore me
10 Student: it (.) the two answers in the little circles no matter what you start up there will always be the same (.) because if you start there no matter what (.) if it's times five times two it's like times ten

A little later in the lesson, Teacher W drew a second "Both Ways" image, it was the same as Figure 3 except the "x5" arrows were replaced by "+2".

33 Teacher W: okay I've changed the number machine (.) you may not have noticed (.) okay can we have any thoughts at the moment (1) about what's going to happen in these two circles (1) any thoughts (.) Student 1 yeah go on
34 Student 1: might be (.) still going to be the same
35 Teacher W: same (1) anything else (1) any other comments
36 Student: I think they're going to be different because they're different order um (4) what's it called when
37 Teacher W: order of operations
38 Student: ( )
39 Teacher W: so you think it's different (.) timesing by two and adding two is different to adding two and timesing by two (1)
40 Student: yeah
41 Teacher W: okay so because they're different operations
42 Student: I think it's going to be the same as well (.) because both numbers are times five and added by two (.) so it's kind of the same thing you're doing
43 Student: I reckon they're going to be different (.) because if you start with one again (.) if you times by five and plus two it's going to be seven (.) and if you add two first it'll be three and then you times five which is fifteen (.) so plus-ing on two first will make it a bigger number
44 Teacher W: okay everybody can you draw that back of your book (.) don't worry if it's a mess (.) just draw circles squares and two more circles (.) put those in and I want you to choose your own starting number

This activity meets the design principles presented earlier, however, we offer this transcript to focus on what students do. In line 2 and 3 we see students asking questions having made the, perhaps surprising, distinction that there is no distinction to be made between the final circles of Figure 3. In lines 7, 9 and 10 all the student comments include the word "because" and in line 10 a statement of proof is offered as to why the answers at the end are the same. When W sets up the second problem she asks students what they think will happen. All the responses that follow use the language of similarity and difference, commenting about reasons the answers at the end might be the same or different. As in transcript 1, we see evidence here that students in this department responded to tasks by making distinctions that lead to them asking questions, noticing pattern and generalising. In the classroom of Teacher W, some students' generalisation has taken the form of mathematical proof.

Discussion

We have demonstrated how our design principles are linked to both enactivist and Gattegno's ideas about learning. Both perspectives see the making of distinctions as one of the basic mental functions and a key to learning. The design principles (2) starting with contrasting examples, (3) students comparing/contrasting and (5) naming the distinctions that students make, are all linked directly to the making of distinctions and hence, to learning. We see the principles operating in (at least) three ways. Firstly, the principles inform teacher planning, for example in the activity 'Equable Shapes'. In this task, the examples of the two rectangles inevitably focus students on the distinction between perimeter and area, from which questions and challenges can be generated that provoke further work with that distinction. Secondly, transcript 1 provides evidence that, over time, these principles can also inform teacher actions in the classroom, in adapting tasks in the light of student responses. We are not suggesting there was necessarily any conscious decision making on the part of Teacher A, linked to the principles; we see evidence that the principles have become part of his practice of teaching, as analysed above. Thirdly, there is evidence in transcript 1 and 2 that the principles can inform (again, implicitly) student actions in the mathematics classroom. In answering what, given the use of our design principles, do students do and attend to in response to tasks, the evidence from these transcripts is that through making distinctions, students notice and extend patterns, they ask questions and generalise.

In Teacher A's lesson (transcript 1) we see a snapshot of a way of working in which the design principles of School S have become part of what students see themselves doing in mathematics lessons. Student 2 offers ½ x 24, and another student immediately offers ¼ x 48. In this context, the activities that Teacher A chooses to offer students become less important, as students can make distinctions and generate questions without prompting. We interpret the students in this short excerpt as exhibiting 'inquiry as a form of engagement' (Watson and Mason, 2007, p.213) rather than inquiry being structured into the task. We suggest one reason the design principles are able to operate on three levels (to influence teacher planning, teacher actions in the classrooms and student activity) is that they are based on a theory of learning. The design principles embody how, as a department, we viewed mathematical thinking; our stated aim, as a department, was to develop students' mathematical thinking. These design principles helped close the gap between our intentions for students to be thinking mathematically and what students did in classrooms. By making distinctions about mathematical objects, which was an inevitable part of tasks in this department, students were thinking mathematically.

In transcript 2, there is evidence of students posing questions spontaneously however we see a difference compared to the spontaneity of transcript 1. It was clear from subsequent discussion with Teacher A that the task students ended up engaging with in transcript 1 was not planned. In contrast, Teacher W had crafted her two starting examples on the 'Both Ways' task and there was no surprise in how the students responded and that the distinctions they made lead to a motivation to explore different starting numbers and then later different operations on the 'arms'.

We want to suggest that the making of distinctions within mathematics can become a habit and a normal way of engaging in tasks for students. Creating opportunities for students to make distinctions within mathematics can also become a habit for teachers and a normal way of both planning activity and informing decisions in the classroom. When this happens, there is a convergence of planned and actual activity. With a focus on distinctions, there is a potential route out of the problems highlighted by Mason et al (2005) around the divergence of teacher intention and student activity. With a focus on distinctions, the expert (teacher) can plan, initially via the choice of examples, to support students in making the same distinctions as a mathematician, leading to the same awarenesses. It is of course no easy skill to be able to 'run' a discussion in the manner we see Teacher A or W doing in the transcripts and we are not suggesting that task design is the end of the story; but we do see evidence that task design based on making distinctions supports teachers in working to support and develop the responses (distinctions) of their students.

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Applying the Phenomenographic Approach to Students' Conceptions of Tasks

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Tasks serve a communicative purpose between teacher and student. The student's perception of the task's purpose is communicated in the approaches used and work produced. By applying the descriptions of an outcomes space from a phenomenographic inquiry to student work samples, an assessment of what students tended to focus on and the meaning they assigned to the task were analyzed to determine the depth of student learning. The findings show that the meaning and purpose a student assigns to a task are aligned with the student's meaning of learning, approaches to learning, and capabilities sought as a result of learning.

Keywords: Focal awareness, learning, phenomenography, student conceptions, tasks

Introduction

Tasks are an important component of any unit plan for learning. They serve a communicative purpose between teacher and student, by conveying the teacher's intent for learning and the student's conception of that intent (Crabbe, 2007). Often, responses or work produced from a task reveal a disconnect between the teacher's learning expectation and the true depth of knowledge attained by the student. Attributing to the gap between expected versus actual learning outcomes is that people experience learning differently, and therefore will conceptualize the object of learning differently.

Phenomenography is a research methodology with its own theoretical framework that accounts for the qualitatively different ways people experience learning. From this theoretical stance, the impact a task has on learning may be analyzed using the outcome space of student conceptions about the task. The purpose of this application of phenomenographic inquiry is to present an approach to gauge a task's impact on learning, solely from the student's perspective. By analyzing a student's conception of, and approach to learning, the relationship between focal awareness and task performance is further documented. The analysis is guided by the following questions: a) what do students attend to when assigned a task, b) how do students understand the purpose of an assigned task, and c) how is it possible to assess the impact of the task on students' learning.

Theoretical Framework

Learning

Phenomenography is an empirical approach to studying the various ways people experience a particular object of learning. The learner's perspective is the unit of analysis. Learning achievement is based on the quality of the learner's description of the object of learning, which has often been categorized as ranging from surface learning to deep learning (Marton & Booth, 1997). Learning is defined as perceiving, conceptualizing, or understanding something in a new way by discerning it from and relating it to a context (Marton & Booth, 1997; Pramling, 1996). Furthermore, learning involves two aspects: i) what is to be learned, and ii) how one goes about learning (Marton & Booth, 1997). The learner's perspective of what is to be learned is derived from the student's definition of the direct object of learning. How the learner assigns meaning to the learning object is determined by the learning strategies the student recommends for meeting personal learning goals. Since these aspects of learning will vary, a hierarchical non-linear structure of the qualitative differences is formed, and is called the outcome space (Marton, 1986).

The outcome space in Table #1 used in the study characterizes the learners' perspectives of statistics, and it originates from research by Gardner (2007; 2010). The outcome space has been iteratively refined based on further accumulation of student responses to the phenomenographic inquiry in various levels of statistics courses. Phenomenographic studies on the conceptions of statistics conducted by Gordon (2004) and Petocz & Reid (2003) have yielded outcome spaces with similar conceptions.

Conception 1: Statistics as facts or algorithms	
Definition:	Statistics is a class in which one states terms, evaluates expressions and formulas, solves equations, and makes and describes graphs.
Approach:	Write and study examples or facts the teacher presents, memorize formulas and procedures, manipulate a calculator, solve problems the way they are done in class
Capabilities:	Do well on a statistics test, remember formulas and facts after a long period of time.
Conception 2: Concepts about and procedures for handling data.	
Definition:	Statistics is the study of contextualized techniques for collecting, representing, and analyzing data.
Approach:	Write or state a contextual interpretation of graphs and numerical summaries, execute procedures with and without technology, relate personal experience and knowledge to statistical concepts, determine the appropriate statistical method for a given scenario.
Capabilities:	Explain or teach statistics to another person, read and understand statistics in media, use technology, know when it is appropriate to use a particular procedure or method.
Conception 3: Summarize, estimate, infer and predict	
Definition:	Statistics is the study of processes used to estimate population attributes and to generalize or predict trends.
Approach:	Use multiple approaches, utilize technology to differentiate or discover trends, recognize when data need to be collected, explain assumptions, procedures and results to others, assess the reliability of results, provide support for conclusions drawn or estimates made.
Capabilities:	Write or present a detailed analysis of an inference, estimate, or prediction that includes an assessment of assumptions, interpret statistical output from software, appreciate the practicality of statistics.

Conception 4: Adapting, restructuring, changing viewpoint	
Definition:	Statistics is a way to acquire knowledge about a population and illuminate trends to improve the quality of life, inform decisions, and change one's outlook of the world. It also comes with the responsibility to use and monitor ethical practices.
Approach:	Adapt to the variable nature of statistics, question the ethical treatment of subjects in studies involving humans or animals, employ the highest ethical standards and design principles, disseminates results of studies to illuminate attributes and inform decisions
Capabilities:	Devise a plan of action to change policies or perceptions based on reliable study results, redefine one's understanding of statistics as new processes are learned, formulate theories, re-structure one's view of the world.

Table 1. Outcome space for conceptions of statistics

Task

Traditionally, a task is defined simply as the work given to the student by the teacher to direct the student towards a specified learning goal (Doyle, 1988). To remain consistent with the phenomenographic definition of learning, a task is further characterized by its relationship to the structural and referential aspects of the learning experience, as defined by Marton and Booth (1997). A task is a situation requiring the learner to experience the object of learning in such a way that the learner must discern components of the situation and how they are related (structural aspect), then assign a meaning to the situation (referential aspect).

Prior knowledge, understanding, skills, and connections reside in the structural domain, and they surface to the foreground as a result of a perceived situation to address, or equivalently complete an assigned task. An analysis of the work produced gives some indication of how knowledge and skills were choreographed to complete the task. The application of multiple approaches or applying and honing skills to tasks with increasing degrees of complexity require students to hold more aspects of the learning object in their focal awareness, therefore they discern more properties of the object of learning and how they are connected (Kirshner, F., Paas, & Kirschner, P., 2011; Runesson, 2006). Therefore, students' work is a depiction of their approach to learning, which reveals structural aspects of their focal awareness.

The referential domain is the learning outcome as perceived by the student, which may be communicated through various forms of assessment. A student's conception of learning informs what is attended to when undertaking an assignment. Therefore, the meaning or purpose of the task assigned by the student is directly related to the student's definition of learning. A student's personal learning goals are made evident by the capabilities they seek to acquire as a result of learning. These capabilities are demonstrated in the product of work. Thus the student's meaning or purpose assigned to the task provides evidence of the depth of learning. The collective structural and referential aspects of students' focal awareness are descriptive evaluations of group performance and learning.

Student Conceptions

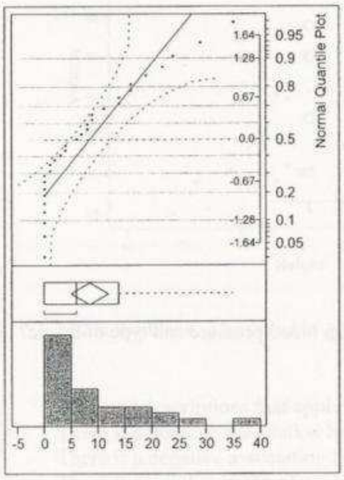
Since the student's conception is the unit of analysis, an explanation of what a student is attentive to when engaged in a task is warranted. The basic components of awareness are appresentation, discernment, and simultaneity (Marton & Booth, 1997; Uljens, 1996). *Appresentation* refers to being conscious of a perceptual or sensual experience in the presence of concrete or abstract entities; *discernment* involves recognizing a foreground-background structure of a situation; *simultaneity* means knowing how the discerned parts are related to the whole structure. The structure of a student's focal awareness directly informs the way the student understands content, which leads the student to perceive that something has been learned. The capacity of focal awareness for an individual is limited, which accounts for the different ways people experience the object of learning (Marton & Booth, 1997).

Methodology

Task Description

Data were collected from one section of a graduate course in data analysis and probability for pre-service and in-service teachers. The task in Figure 1 is an item from the course mid-semester examination. The item assessed the student's performance level on analyzing and reporting summarized data.

27. Class data were collected on scores students made when playing the 1996 Bop It game. The statistical summaries are below. Write a short report summarizing the results.



Distributions: Score96

Quantiles

100.0%	maximum	35
99.5%		35
97.5%		35
90.0%		23.6
75.0%	quartile	14
50.0%	median	6
25.0%	quartile	0
10.0%		0
2.5%		0
0.5%		0
0.0%	minimum	0

Moments

Mean	8.4516129
Std Dev	9.5213399
Std Err Mean	1.7100831
Upper 95% Mean	11.944069
Lower 95% Mean	4.9591572
N	31

Figure 1. Assessment task on descriptive statistics

Four weeks prior to the assessment, students engaged in a five day lesson introducing the data investigation process and data analysis techniques. One of the unit objectives was to recognize the need to generate and analyze data to gain insight to a specific problem. To motivate the lesson, students identified data specific problems, formulated questions, and then generated data from various statistical experiments. Recording the students' scores from the Bop-It game was one such activity, and the dataset was used only for the assessment task. Other experiences students had during the unit lesson were technology labs on producing graphs and numerical summaries of univariate data, and small work groups in which datasets with various features were analyzed and then reported to the class. The lesson activities were informative in determining and expanding the precision and depth of the students' analyses. The Bop-It activity provided a contextual understanding of the assessment task. All student item responses were sorted based on conceptions in the outcome space. One item from each conception was randomly selected for the study, to demonstrate applications of phenomenography.

Approach

To begin the analysis, characteristics of a student's written responses were noted and matched to items in the approach list. The response was categorized by the conception that had the most advanced approaches noted. A grading rubric was applied to rate the performance level (not met, beginning, advancing, accomplished, exceeds) of the response. The conception and performance level were then used to describe the student's focal awareness while engaged in the task, and to explain what the response revealed about the student's understanding of the purpose of the task.

Results and Discussion

Anna

Anna's response to the descriptive statistics task is provided in Figure 2. The performance level was classified as "Beginning" and the approaches used by Anna are aligned with Conception 1.

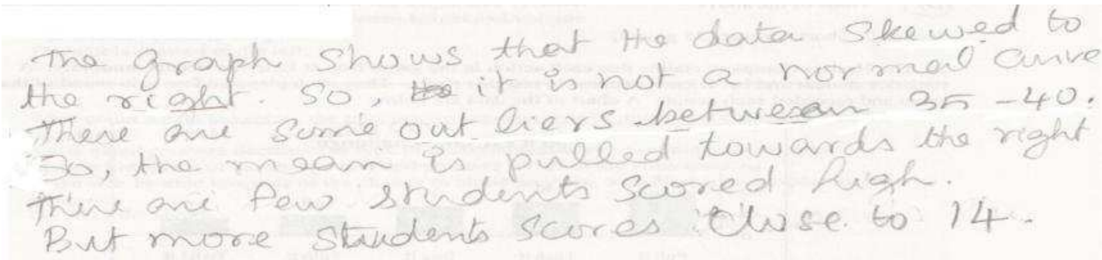

The graph shows that the data skewed to
the right. So, it is not a normal curve
There are some out liers between 35 -40.
So, the mean is pulled towards the right
There are few students scored high.
But more students scores close to 14.

Figure 2. Anna's task response

Anna's response indicates a focus on surface level capabilities. Her approach to the task is a partial recitation of the written distribution summaries presented in the introduction of the learning unit by the teacher and in the textbook. Based on her use of intervals, "…between 35 – 40.", and approximations, "close to 14", it is evident that Anna fixated on providing visual interpretations of the histogram. She did not state features of the histogram used to conclude the distribution is skewed right, although this response is correct. She also was aware of the effect of a high outlier on the mean, but she did not apply the formula for determining specific outliers. Finally, she did not refer to the table that gave the exact mean of the distributions, and only focused on where she perceived the scores clustered, presumably based on frequencies. She did not provide an overall assessment of variability, nor a summative statement about the contextual meaning of the distribution. The capabilities illustrated by Anna suggest she discerned the purpose of the task to be an assessment of her ability to describe features of a histogram. Anna was able to recall facts about histograms that helped her to properly identify the shape of the distribution, sense the presence of outliers, and approximate the center. As demonstrated by her response, to Anna the purpose of the task is to determine whether she can recall facts about statistical summaries.

Byron

Figure 3 is Byron's response to the task. The response was rated as "Advancing" and the most advanced approach resides in Conception 2. Although Byron approached the task similarly to those presented in the class, he used more than one summary in his response, providing some indication that he could determine appropriate methods. By referring to the data as scores, he was also able to relate the features to the task context.

The results come out to show there is not a normal distribution. The box plot shows the uneven spread of the data and the histogram indicates it is skewed right. There are also outliers on the high end of the bop it scores (anything over 27). The median(6) is smaller than the mean(8.45).

Figure 3. Byron's task response

His attention is focused towards two graphs and the table of moments provided in the task. Coming to the forefront is an undisclosed feature of the box plot that lead him to conclude the distribution had "uneven" spread, which to him is not a characteristic of a normal distribution. But the meaning of this is not further clarified. Similarly, an unidentified feature of the histogram informed Byron's description of the distribution shape. His comparison of the mean and median indicates he knew facts about the effect of high outliers on resistant and non-resistant measures of center. He demonstrated correct application of the outlier formula and correctly interpreted its results. Byron's learning is still considered surface level, but advancing towards a higher conception. His conception of the purpose of the task was a means for him to communicate his understanding of concepts about data.

Charles

Charles' task was rated as "Accomplished" because there were a few points deducted for minor misconceptions revealed in his explanations. Based on his approach in Figure 4, Charles attended to all of the summaries provided in the task, thus incorporating multiple approaches to summarize the distribution. He also annotated each graph, thereby utilizing technological output to differentiate or discover trends and to provide support for conclusions drawn. These approaches are in Conception 3.

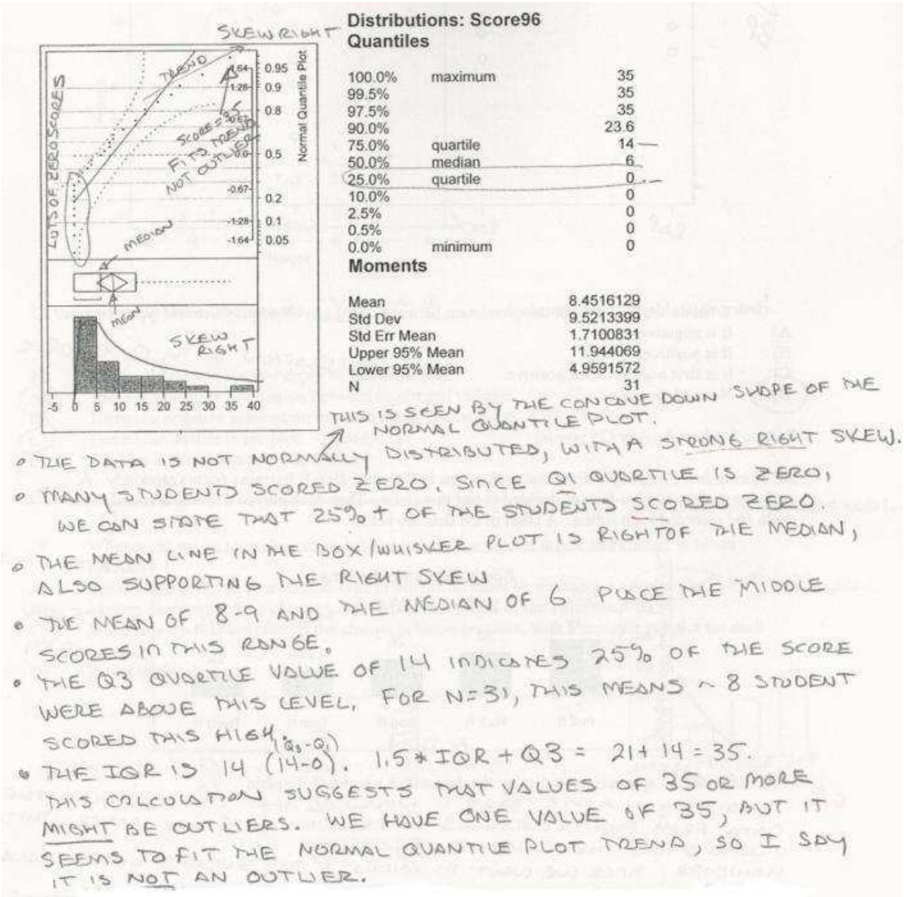


Figure 4. Charles' task response

Charles' response is a strong indication of his deep understanding of descriptive statistics as demonstrated in his capabilities to write a detailed summary of a distribution and interpret statistical output. Charles' conception of the task is best described as a means for him to demonstrate his ability to summarize data and support the conclusions drawn.

Conclusion and Implications

This research demonstrates a method for determining what students attend to when assigned a task, how they understand the purpose of a task, and what their responses reveal about learning, through the applications of phenomenography. The individual student responses contribute to the collective of varying levels in students' understanding. An analysis of their work through the lens of the outcome space communicates the depth and quality of collective learning, thereby informing the degree to which the task or task sequence impacted learning.

The task used in this research assessed student understanding of descriptive statistics and data analysis. Collectively, the various levels of performance in the class are summarized by the first three conceptions of statistics in the outcome space. The presence of student responses in the three conceptions allow for reflection upon and evaluation of the selection and sequencing of tasks throughout the unit lessons. For example, the lesson task sequence was identifying data problems, formulating questions, generating data, using technology to produce summaries, writing an analysis, and reporting the analysis. A lesson requiring students to first design a statistical activity, and then teach it to the class could be contrasted with the results of the lesson for this study to determine whether more students attained higher conceptions. Furthermore, since none of the students responded in ways described by Conception 4, supplementing the lesson sequence with complex tasks requiring the compilation of numerous approaches to learning and demanding deeper cognitive focus may enthusiastically move students towards this level. A response indicative of Conception 4 would have included an illuminating effect or call to action based on the analysis. For example, a reflective statement on the Bop It activity explaining the high frequency of 0 scores, or a statement extending recommendations to the manufacturer based on the findings would have been evidence to categorize a response at this level. It is noted that in the researcher's other statistics courses, some students reach perspectives of Conception 4 in reports where they had to design and conduct comparative studies.

This research also shows that the meaning and purpose a student assigns to a task are aligned with the student's meaning of learning, approaches to learning, and capabilities sought as a result of learning. Lau, Liem, & Nie (2008) discuss similar findings in which the value students assigned to task was shown to be associated with the goal of learning. Students may be asked at integral points in a unit to define the object of learning, so that when or where changes occur from task to task can be noted. To encourage students to aim for conceptions that indicate deep learning, the teaching model, Learning Study (Runesson, 2006) provides methods immersed in the phenomenographic perspective.

The Learning Study model begins with planning a lesson aimed at depicting the critical points of departure in the various ways students understand the object of learning; the lesson is then designed to exploit the patterns of variation to give students multiple ways of experiencing the object of learning (Runesson, 2006). Variation and repeated practice are viewed as effective teaching methods to encourage students to practice varying their perspective. Repeated practice in this context does not mean mundane, rote repetition. Instead, it means to create, invent, adapt, and progress in the light of previous practice where students get numerous opportunities to challenge their perspectives, vary their approaches, and extend or hone the skills they employ (Fazey & Marton, 2002; Runesson, 2006). Designing lessons from the Learning Study perspective inherently leads to developing tasks that provide authentic experiences and require simultaneous, structural focus on multiple details of the learning object.

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On what epistemological thinking brings (or does not bring) to the analysis of tasks in terms of potentialities for mathematical learning

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As recalled by the proposed framework text, the relationship between the completion of tasks by students and the effectively achieved mathematical learning must be questioned. The present contribution will develop and illustrate that this relationship may be adequately analysed by adopting a praxeological perspective (Chevallard, 1999), supported by an epistemological study. Indeed, such a view allows to highlight the fundamental character of the task with respect to the targeted knowledge, and to identify the praxeological level adapted to the level of teaching (Schneider, 2011). Therefore, it allows a certain understanding of the behaviour of students, allowing to assess the impact of the tasks on the learning processes, beyond a possible methodology based on pre/post tests.

Keywords: epistemological obstacle, fundamental task, praxeology, praxeological level, pre/post test

Introduction

Teachers, researchers and the mathematical community in general have an interest in designing tasks to help students, pupils … acquire mathematical knowledge. This goal in mind, a first methodological problem arises regarding their efficiency. A common practice to evaluate the impact of a task is the use of pre/post tests, mimicking procedures found in (hard) sciences like biology and chemistry. The work of Brousseau (Brousseau, 1998) issues a strong warning towards such practices. Although they may be sound in other contexts, in the setting of mathematical education they are subject to great concerns because of sociological considerations pertaining to human behaviour in learning institutions. Brousseau have shown how a teacher and his pupils may engage in some sort of role playing where pupils decode, from the behavior of the teacher, or other hints not drawn from the knowledge they are supposed to acquire, how to answer the given task and how the teacher, consciously or not, gives credit to his pupils doing so, giving an overall illusion of knowledge acquisition. Those initials analysis have been shown to carry over different levels of education (Calmant, 2004, Job, 2011, Rouy, 2007).

Though it puts into perspective the pre/post tests methodology, the sociological viewpoint used by Brousseau might led some people to believe that mathematical understanding ultimately only relies on considerations where mathematical knowledge doesn't need to be analysed e.g. an epistemological study of the targeted knowledge isn't required. Far away from this conclusion, Brousseau, again, shows, thanks to the concept of *epistemological obstacle*, how much the inner peculiarities of mathematical knowledge are at the very heart of mathematical difficulties encountered by students and pupils, and, moreover, that the "toughness" of these difficulties leads, to some extent, to the aforementioned role playing between a teacher and his pupils, as a way of trying to do some mathematics despite the encountered difficulties. As a conclusion the work of Brousseau shows beyond any reasonable doubt that an epistemological analysis of the knowledge of concern cannot be bypassed in the design of mathematical task. But what kind of tool do we have to convey such an inquiry? How can we give credit to a task if the pre/post tests methodology is unsound? Those questions can be addressed, at least to some extent, using the praxeology concept.

Praxeologies, a model of (mathematical) knowledge

The work of Chevallard (Chevallard, 1999) extending some parts of Brousseau's work endow us with the concept of *praxeology*, a model of what (mathematical) knowledge is, whose great strength is to address in a single coherent framework both anthropological and epistemological concerns mentioned above.

According to that theory, any activity, including mathematical ones, can be conceptualized as a *task*, something to do, a *technique* used to solve it, and a justification of the technique used to solve the task that can be split into a *technology* and a *theory*, a theory being a more abstract level of justification than the technology.

We will show in the sequel how praxeologies can be put to good use and among other things shed some more light onto the distinction between technology and theory. But before that we should emphasize that one breakthrough permitted by this approach is to allow mathematical knowledge to gain a form of relativism. This doesn't mean that Pythagoras' theorem will sometimes be true and other times false, but that different institutions may have different views on the same knowledge e.g. a given knowledge may be a simple technique in one institution, like the limit concept in many Belgian secondary schools, or a theory developed to give analysis a Euclidean architecture as has been done by Cauchy who in many ways can be considered the father of analysis and the creator of the modern concept of limit (Job, 2011).

Two different praxeological levels

This *institutional relativity* of mathematical knowledge has led Schneider (Schneider, 2009) to distinguish between two different kinds of praxeology, *modelling* ones and *deductive* ones, allowing us, as we shall explain, to understand the dynamic behind some hard to teach concepts like the limit one (Job, 2011), but also to design tasks that transcend the pre/post tests pitfalls.

In the first kind of praxeology, modelling ones, the fundamental task is to compute features of objects like areas of surfaces, volumes of solids whose existence doesn't rely yet on a formal definition. Those objects exist as mental constructions shared, or believed to be, by some institutions. Justifications given in those praxeologies to techniques developed to address the fundamental task often rely on pragmatic arguments. A technique is validated if the results obtained are in accordance with results derived using other valid techniques that may even belong to other fields of sciences. For instance, early infinitesimals techniques where used and recognized based on the accordance with results obtained using physical arguments of cinematic nature.

In the second kind of praxeology, deductive ones, the fundamental task consist in defining those mental constructions, to make explicit what was left in the shadow in the first kind of praxeology and built a deductive theory. Often the techniques used in modelling praxeology are used in deductive ones as definitions. The definition of integral given by Cauchy is a good example of such a procedure: an approximation procedure is turned into a definition that, in turn, is used to prove theorems about integrals (existence, uniqueness ...).

The two kinds of praxeology are distinct but closely related, the second kind often taking place after the first one. With these two concepts we don't aim, nor claim to encompass the whole mathematics, but important parts of its growth like the birth of analysis from calculus. Indeed, it can be showed that calculus can be roughly speaking represented as a modelling praxeology and analysis a deductive one (Job, 2011). These two praxeologies take into account the institutional relativity of various concepts like the limit one and the derivative that may otherwise be seen as concepts that somehow where born at some place in time almost as they appear nowadays whereas they evolved under the guidance of very different viewpoints. The calculus period was mostly guided by the will to be able to compute areas… and analysis was created out of the will to purge calculus from geometry and physics, forge a new area of mathematics whose rigor would equal that of the ancient Greeks.

Epistemological thickness and fundamental character of a task

The praxeological levels introduced above allow us to get back to our initial asking. How do we assess a task? A partial answer given by Brousseau (Brousseau, 1998) is to consider a task *fundamental* with respect to a given knowledge if the knowledge takes places in a praxeology as a technique where the task cannot be solved without that knowledge. The knowledge is thus seen as a kind of optimal answer to the proposed task. This requirement is legitimated by what we have said earlier in this article about the role playing pupils, students and teachers are prone to engage. It shouldn't be possible to solve the task used to teach a certain knowledge only using hints external to the knowledge like the teachers eyebrows indicating if the students are running along the required lines.

This understanding of the fundamental nature of a task has been shown to be effective to introduce concepts likes the rational numbers (Brousseau, 1998) but doesn't seem to translate well to concepts like the limit one. Indeed, it is one of the twentieth century achievements to have shown with the work of Abraham Robinson that a sound basis could be given to infinitesimal concepts so far rejected as a sound basis for calculus. The limit concept is thus by no means necessary to cast the calculus into a deductive mould.

Anyway, the very heart of Brousseau's idea can be adapted is the following manner, taking into account the institutional relativity of knowledge introduced above. A task is said to be *fundamental (in a broad sense)* with respect to a given knowledge and a given institution if that institution takes for granted the knowledge is optimal to solve the task. In this new definition there is no more necessity in a "mathematical" way but an anthropological necessity that an institution gives to itself.

At this point, we are now able to understand the leading role played by praxeological levels. The structure of a fundamental task and even the fundamental character of that task with respect to, for instance, the limit concept depend on what kind of praxeology we place ourselves in. A fundamental task for the limit concept in a deductive praxeology won't be the same as a fundamental task in a modelling one. Before we dive into some characteristics of these tasks, let us first give an example of the consequences of not being able to clearly state whether a task belong to one praxeological level or the other.

The consequences of blurred praxeological levels in secondary school

In (Job, 2011) we study the teaching of the limit concept in secondary school and are able to support the following views. Secondary school tries to teach the limit concept but fails to do so, unable to identify the praxeological level where it should belong.

Secondary school tries to teach this concept giving students elements that belongs to the deductive praxeology of analysis mathematicians use nowadays in order to place itself under the supervision of that institution from which it draws its legitimacy. This deductive praxeology being out of reach to students of that age, the school praxeology mainly consists of elements acting as blazons, that is, parts of the original praxeology that are able to support the illusion of a real teaching of the limit concept from an outside perspective.

Among these blazons, the definition of a limit plays a key role. Secondary school tries to teach this definition using various tricks to make believe students this definition is a somewhat complicated (mathematical) way of saying something very natural. For instance, it gives students tables with values of x and $f(x)$ for a given function, waiting for the students to recognized some sort of behaviour that should be put into sentences like "as x tends to … $f(x)$ approaches …". Starting from such sentences, teachers gradually turn these into the required forms "$f(x)$ can be made as close as one wishes to …" using arguments that belong more to rhetoric than mathematics.

Such an approach is misleading in nature for the definition of the limit concept was designed by Cauchy *to conduct proofs* and define other key concepts of analysis like the derivative. But except for a few trivial ones, proofs in secondary school are left aside. So the very use of the limit concept in the deductive praxeology where it belongs is left aside. The school praxeology thus bears no fundamental character whatsoever.

Such a fool's game isn't the consequence of any malicious thoughts on the side of secondary school but the resultant of antagonist constraints. On the one hand, it has to teach the limit concept in a way mathematicians would recognize as valid, which is a daunting task. On the other hand it must succeed in that task. The only way secondary school has to its disposal to conciliate the two is to take the deductive praxeology, strip it from most of its content and wrap it in a discourse that can be accepted by students even if the cost is to propose tasks that have no fundamental character. This wrapping is partly a consequence of its unawareness of the existence of another praxeology (a modelling one) where the limit concept is legitimate.

So secondary school's praxeology with respect to the limit concept lies in a no man's land, not being in a deductive or in a modelling praxeology. Similar conclusions are drawn in (Rouy, 2007) regarding the derivative also based on praxeological considerations. This analysis sheds a new light on the pre/post-tests methodology. How could we give credit to a task succeeding a sequence of pre and post tests if that task isn't epistemologically consistent?

What praxeological level for secondary school?

The section above asks a crucial question. Is there a place left for the limit concept in school that would be mathematically legitimate? The answer might be positive if we place ourselves in a modelling praxeology. Although Schneider is critical towards some of the tasks they designed (Schneider, 2001), AHA (AHA, 1999) has proposed a fundamental task for the limit concept in a modelling praxeology, which is declined at the various levels of application of the concept in sub-tasks (areas, speeds, tangents).

On the other hand, Job has studied the teaching of the limit concept in a deductive praxeology (Job, 2011). Its results show how much a deductive approach to the limit concept is a very demanding task. In a few words, the students were asked to propose definitions of a certain behaviour of sequences of real numbers and then to proof properties related to this behaviour. The students were mostly unable to make their definitions evolve. They stayed stuck with definitions that are "descriptions" of what they see of the studied behaviour. They couldn't possibly envision their definitions as something to be chosen to allow proofs despite the many contradictions pointed out by the teacher. This inability is related to epistemological obstacles. Students see definition as a description of some mental concept they believe everyone of them share. They therefore don't understand the rules of the game they are asked to play, feeling they are asked something unnecessary complicated because "everyone agree with the found properties", "nothing has to be proved". This situation seems like a dead end because the teaching school has given them tends to reinforce their vision of mathematics, depriving them from the need to cast theories into a deductive mould.

Different understandings of the task concept

Let us give a second example of the use of praxeologies that will put the task concept itself into question, showing it should sometimes be understood at a different level than is usually done, thus clarifying the concept of a fundamental task understood in a broad sense.

We shall illustrate our views through a task used by our team (Job & Schneider, to be published) to teach negative numbers and specifically the multiplication rule to 12 years old pupils. Being as concise as possible, pupils are asked to devise a single formula that allows them to encompass the motion of two vehicles, being flashed by a radar, driving different roads, but at the same constant speed of 2km/min. A first formula $p=2t$ emerges for positive times where p denotes a location and t a time[17]. They are then asked to elaborate a formula that would also be valid for negative times e.g. times before the two cars are flashed. This requirement of a single formula brings pupils face to face with expressions like -6=2 x (-3) and therefore to an extension of the multiplication rule for positive numbers to negative ones. Pupils are then asked to deal with cars driving in the direction opposite to the one the first two where driving. This introduces "negative" speeds, the minus sign telling which direction the car is driving. The same requirement of a single formula leads in turn to expressions like 6=(-2) x (-3) a completes the multiplication rule for negative numbers.

[17] Aside the multiplication rule, the task allow us to make pupils distinguish between distance and position among many other things we have no space to elaborate on.

Such an introduction of negative numbers and their multiplication meets pupils' global assent but what we are trying to emphasize lies somewhere else. The peculiarity of our task doesn't rely so much on the pupils' assent, but on a characteristic where they are not involved in the first place. This task tries to expose pupils to a choice made by mathematicians/physicists to allow them to model with a single formula the various incarnations of the same motion, in terms pupils should be able to understand. Pushing the structure of our task to the extreme, it doesn't matter so much if the pupils agree with the decision made by mathematicians/physicists as long as they understand there is a choice to be made and its consequences, because it is not their assent we are seeking. We simply try to make as explicit as possible choices made by some institution they have no impact on. Learning mathematics and physics also means learning the conventions of those institutions whether we agree or not with them. It is not to say that pupils have nothing to understand. On the opposite, there is something to understand which is located at a level that is subtle to explain, not to pupils who are living the task, but to the mathematical learning community: if you want to learn mathematics you have to accept its conventions whether or not you agree with them as long as you understand why those conventions have been adopted.

Conclusion

We have argued that a pre/post-tests methodology is unsound to assess the efficiency of a task and that the distinction between two kinds of praxeologies (modelling and deductive ones) plays a key role in designing tasks and understanding the dynamic of ordinary lessons. A task should clearly identify whether it belongs to one praxeology or the other in order to be meaningful. A task that doesn't belong to any of those two levels should be handled with great care, its fundamental character being dubious. Being able to state to what kind of praxeology we belong allow us to interpret students' work in the light of a solid epistemological background, therefore giving us tools to avoid misinterpretations that pre/post-tests a prone to commit due to their very structure: a post test result better than a pre-test one isn't obviously a sign of better understanding but may only be the result of an accommodation from the students that have understood how to answer the tasks without using the targeted mathematical knowledge. Taking advantage of the distinction made between modelling and deductive praxeologies and the relativity of knowledge, we have put into question the very concept of a task showing how much its understanding can be and should broadened as soon as we are dealing with the teaching of concepts like the limit one or the negative numbers. Those tasks should be understood in a broader sense than usual, as a way to highlight choices made by an institution and the reasons underlying these choices.

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Designing Covariation Tasks to Support Students' Reasoning about Quantities involved in Rate of Change

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This paper articulates theoretical and practical considerations for designing a sequence of covariation tasks to support students' reasoning about quantities involved in rate of change. Adapting the well-known bottle problem during two iterations of implementation and analysis, this researcher-developed task sequence incorporates dynamically linked geometric and graphical representations of covarying quantities and prompts fostering students' coordination of quantities that are changing together. Taking into account students' perspective of quantities involved, this task sequence is designed to support students' progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity.

Keywords: Task design, Reasoning, Quantity, Covariation, Rate of Change

Researchers using mathematical tasks involving dynamic representations of covarying quantities have supported middle and high school students' consideration of relationships between quantities involved in rate of change (e.g., Johnson, 2012b; Monk & Nemirovsky, 1994; Saldanha & Thompson, 1998; Stroup, 2002). In this paper I articulate theoretical and practical considerations for designing a sequence of covariation tasks to support middle school students' reasoning about quantities involved in rate of change. The design of this task sequence accounted for students' perspective of quantities involved to support their progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity.

Background

Drawing on Sierpinska's (2004) characterization, by *mathematical task* I mean a purposefully designed problem intended for a particular audience. By problem, I mean a situated problem (Gravemeijer, 1994) involving a particular context. I take an interpretive stance on context, drawing on Van Oers' articulation of context: "What counts as context depends on how a situation is interpreted in terms of activity to be carried out" (1998, p. 481). Acknowledging that individuals for whom a task is intended can interpret the task in myriad ways, I assume that an individual's perspective on the nature of the problem to be solved can influence an individual's reasoning about mathematics he or she perceives to be involved in the task.

I consider mathematical reasoning to be an individual's purposeful mental activity situated within a particular context. The purposeful activity includes making sense of how a mathematical situation holds together (Simon, 1996), making relationships between objects involved in a situation (Thompson, 1996), and engaging in operation that involves carrying out actions both mentally and physically (Piaget, 1970). When characterizing reasoning as quantitative, I consider a quantity to be an individual's conception of the measurability of an attribute of an object (Thompson, 1994). Because individuals do not need to determine actual measurements to reason quantitatively, quantitative mental operations are nonnumerical (Thompson, 1994). By articulating that the object of the reasoning is quantities involved in rate of change, I do not assume that individuals will reason about rate of change as single entity. Focusing on covariation (Carlson et al., 2002), I attend to how individuals make sense of and make relationships between quantities that are changing together.

Adapting the well-known bottle problem to design covariation tasks

I designed covaration tasks by adapting the well-known bottle problem developed by the Nottingham University's Shell Centre (Swan & the Shell Centre Team, 1999). Given the context of a bottle filling with liquid being dispensed into the bottle at a constant rate and a picture of a bottle, the bottle problem requires students to sketch a graph of the changing height of the liquid as a function of the changing volume. Researchers have used the bottle problem to investigate the reasoning of undergraduate and graduate mathematics students (Carlson et al., 2002) and prospective elementary (Carlson, Larsen, & Lesh, 2003) and secondary (Heid, Lunt, Portnoy, & Zembat, 2006) mathematics teachers. My adaptations to the bottle problem have had two iterations of implementation and analysis. The task sequence reported in this paper is from the second iteration.

In the first iteration I developed a covariation task by adapting the bottle problem in two ways: (1) Providing students with a graph and asking students to sketch a bottle that the graph could represent, and (2) Using a graph that represented the changing volume of the liquid as a function of the changing height. Prompts included in this covaration task were: (1) How is the volume of the liquid in the bottle changing as the height of the liquid in the bottle increases? (2) Sketch a bottle that the graph could represent. Intending to implement the task with high school students who had not yet taken a calculus course, I provided students with a graph because previous research (Carlson et al., 2002; Heid et al., 2006) found that even students with extensive mathematics background have difficulty creating graphs. I chose to represent volume as a function of height in part because preservice elementary teachers working on the bottle problem operated with the independent variable, volume, as if it were time (Carlson, Larsen, & Lesh, 2003). I hypothesized that representing volume as a function of height might reduce the likelihood of students treating the independent variable as if it were time.

In the second iteration I adapted Thompson, Byerly, and Hatfield's (in press) version of the bottle problem that dynamically links a pictorial representation of a filling bottle with a graph relating the volume of liquid in the bottle to the height of liquid in the bottle. Key to Thompson et al.'s adaptation was the use of a dynamic environment linking pictorial and graphical representations. I hypothesized that such an environment could foster students' reasoning about covarying quantities by explicitly linking a change in one representation with a change in another representation. Hence, I adapted Thompson et al.'s bottle problem (implemented with beginning calculus students) for use with 7th grade pre-algebra students. Anticipating that 7th grade students might have limited conceptions of volume, I altered the context of the task from a filling bottle to a two-dimensional shape being filled with area. In making this change, the two dimensional pictorial representation would represent a two dimensional rather than a three dimensional quantity.

The adaptation for 7th grade pre-algebra students resulted in a sequence of four tasks. Accompanying each task was a dynamic sketch I developed using Geometer's Sketchpad software (Jackiw, 2009). The filling rectangle sketch (see Fig. 1) linked a rectangle with a graph that related the amount of shaded area to the height of the shaded area. Students could vary the height of the rectangle by animating or dragging point H. Students could drag point F to vary the width of the rectangle, then predict and create corresponding graphs representing the amount of shaded area as a function of its height. The filling triangle sketch (see Fig. 2) linked a right triangle with a graph that related the amount of shaded area to the height of the shaded area. Students could vary the height of the triangle by animating or dragging point D, then predict and create a corresponding graph representing the amount of shaded area as a function of its height. By affording students' manipulation of dynamically linked representations, the dynamic sketches seem to foster students' consideration of relationships between covarying quantities.

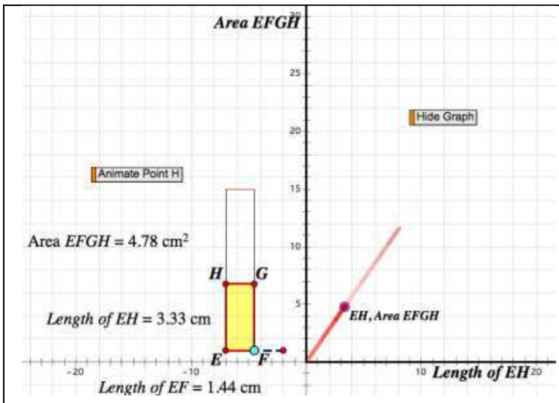

Figure 1. Filling Rectangle Sketch

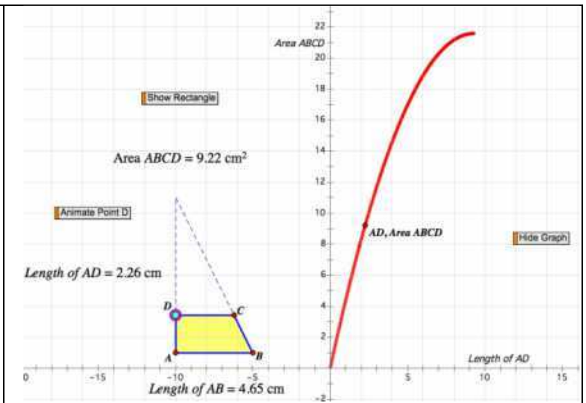

Figure 2. Filling Triangle Sketch

Task sequence

The task sequence is designed to support students' progression in using nonnumerical

quantitative reasoning to coordinate covarying quantities. The description of the task sequence includes: (1) Statement describing the context for the task sequence (2) Identification of dynamic sketch used with each task (filling rectangle or triangle), (3) Quantitative reasoning (QR) objective for each task *(italics)*, and (4) Prompts fostering students' coordination of quantities that are changing together. The context for the task sequence describes the situation on which the tasks are based. A QR objective is distinct from a learning objective because it indicates purposeful activity intended to support a way of reasoning rather than an intended mathematical understanding. These QR objectives indicate purposeful ways of making sense of and making relationships between covarying quantities. Prompts refer to questions and directives designed to foster students' making of relationships between quantities and predictions about characteristics of linked representations.

Context for task sequence

Imagine that a shape (rectangle/triangle) is being filled with area that is increasing at a constant rate.

Task #1: Filling Rectangle Sketch

Create and use non well-ordered tables of values indicating measurements of related quantities to predict characteristics of graphs relating those quantities.

Press *Animate Point* to run the animation of the filling rectangle. What changes and what stays the same?

Given a non-well ordered table of heights for a rectangle with a given base, determine the different amounts of area. For example:

Imagine the side length of EF was **4 cm**. Complete the table:

Length of Side EH	1 cm	3 cm	5 cm	7 cm	10 cm
Area of Rectangle EFGH					

Imagine you created a graph relating the side length of EH and the filling area of EFGH. Would the graph be linear? Explain why or why not.

Repeat b&c for bases of different lengths. How would the graphs be similar/different?

Task #2: Filling Rectangle Sketch

Use an amount of change in one quantity to predict an amount of change in a related quantity.

Given a non-well ordered table of heights for a rectangle with a given base (e.g., the table in 1b), determine the different amounts of area.

Determine at least two different ways to complete this statement: When the height increases by ______, the area increases by ______. How many ways can this statement be completed?

When given the two different heights for the same base, determine the amount of increase in area. For example,

Imagine the side length of EF was **3 cm**. Complete the table:

Length of side EH	14.5 cm	16.5 cm
Amounts of increase in area of rectangle EFGH		

Task #3: Filling Rectangle Sketch

(1) Use a graph relating two quantities to predict a measure for a third, related quantity (not represented explicitly by the graph). (2) Use a relationship between two quantities to predict characteristics of a graph relating one of those quantities to a third quantity.

Given graphs representing the amount of filling area as a function of the length of side EH, predict the length of the base of the rectangle.

Given a relationship between changing area and changing height (e.g., *As the length of EH increases by 2 cm, the area of rectangle EFGH increases by 3 cm^2)*, predict what a graph relating area and the side length of EH would be like.

What would a graph be like for a rectangle with (1) a very short base, (2) a very long base? Why?

Task #4: Filling Triangle Sketch

Use a dynamic geometric representation of covarying quantities to predict characteristics of a graph relating the quantities.

Press *Animate Point* to run the animation of the filling triangle. What changes and what stays the same?

How does area change as the height increases?

Imagine you created a graph relating the side length of AD and the filling area ABCD. What would the graph be like?

Press *Animate Point* to sketch the graph. Was it what you expected? Why do you think it looks that way?

Design principles

Three principles guided my design of the task sequence. Central to each of these principles was my consideration of how students might perceive the nature of the quantities involved and how students' perspectives might influence their consideration of relationship between those quantities.

(1) Anticipate students' perspectives on relationships between changing quantities

Drawing on results of analysis of students' work on the first adaptation of the bottle problem, I anticipated two distinct perspectives students might have on relationships between change in the covarying quantities involved in the task sequence: (a) Quantities can change simultaneously with change in one quantity being independent of change in a related quantity (Johnson, 2012a) and (b) Change in one quantity depends on change in a related quantity (Johnson, 2012b).

(a) By relating covarying quantities as if each quantity were changing independently of the other quantity with respect to time, students can make comparisons between amounts of change in each quantity (Johnson, 2012a). For a filling rectangle, if 7th grade students could determine amounts of area for given amounts of height (The height being the length of EH – see Fig. 1), they could make comparisons between amounts of change in area and amounts of change in height. By sequencing filling rectangle tasks that afforded calculation of amounts of area prior to a filling triangle task that problematized calculation of amounts of area, I intended to support students' gradual move away from making numerical calculations.

(b) Reasoning about change in one quantity as being dependent on change in a related quantity can support students' attention to variation in the intensity of a change (Johnson, 2012b). If 7th grade students were able to envision the area of the filling triangle as varying in relationship to the height (The height being the length of AD – see Fig. 2), they could make claims about the area of the triangle increasing more slowly as the height increased. By requiring students to predict how area would change as height increased when manipulating dynamically linked geometric and graphical representations, I intended to support students' reasoning about variation in area as being dependent on variation in height.

(2) Incorporate key aspects to support students' attention to covarying quantities

The task sequence incorporated three key aspects: (a) Tables that were not well ordered, (b) Questions supporting students' attention to covarying quantities, and (c) Questions supporting students' making of predictions about change.

(a) In the filling rectangle tasks I incorporated tables that were not well ordered. By indicating that a table is not well ordered, I mean that the independent variable contained in the table (in the case of the filling rectangle tasks, the length of EH) does not increase by a uniform amount. When working only with well ordered tables, secondary students did not necessarily attend to change in both independent and dependent variables (Lobato, Ellis, & Munoz, 2003) and tended to pay attention to numerical patterns (Ellis, 2007). I anticipated that including tables that were not well ordered could promote students' attention to and making of relationships between covarying quantities.

(b) Both the filling rectangle and the filling triangle tasks incorporated prompts supporting students' attention to and making of relationships between the changing quantities of area and height. The prompt "What changes and what stays the same?" provided an entry point into the task and fostered students' attention to different quantities involved in the task. Subsequent prompts supported students' making relationships between quantities that were changing together. For example, in task 2b, completing the statement "When the height increases by ______, the area increases by ______" in multiple ways supports students' consideration of multiple relationships between amounts of increase in height and area for a rectangle with a given base. Once students could begin to focus on relationships between quantities, it seems reasonable that students could then draw on those relationships to engage in related activity.

(c) Each task supported students' activity of making predictions about (1) characteristics of a linked representation (tasks 1, 3, 4) or (2) a related amount of change (task 2). Making predictions can foster students' use of nonnumerical reasoning by supporting students' envisioning of running through calculations without actually making the calculations to make relationships between the changing area and the changing height. For example, task 2c required students to predict amounts of change in area given two different heights without actually determining amounts of area. If students determined amounts of area, subtracted those amounts, then arrived at amounts of change, it could indicate that students were not yet relating changes in height with changes in area. By requiring students to predict characteristics of graphs representing rectangles with differently sized bases, task 3c supported students' use of relationships between area and height. Such predictions were intended to support students' attention to relationships rather than to results of calculations.

(3) Sequence tasks to support students' progression from numerically based reasoning to nonnumerically based reasoning

I sequenced the filling rectangle tasks prior to the filling triangle task based on two research hypotheses related to students' perspectives on quantities involved: (a) Students can use numerical calculations (with or without actually engaging in calculations) to make comparisons between amounts of change in covarying quantities, and (b) By focusing on covariation, students can attend to situations involving constant rate of change in ways that can support their attention to variation in the intensity of change in situations involving varying rates of change.

(a) While numerical calculations can be an entry point into the filling rectangle tasks, the design of each filling rectangle task is not intended to support students generalizing from numerical calculations. In contrast, intent is to support students' engagement in the nonnumerical operation of linking a changing area with a changing height. Students working from numerical calculations could begin to imagine running through calculations (without actually completing the calculations) to relate the changing area to the changing height.

(b) The filling rectangle tasks incorporate constant rates of change, and the filling triangle task incorporates a rate of change that increases at a decreasing rate. Although it may seem obvious to position tasks involving constant rate of change task prior to a task involving varying rate of change, research has questioned whether situations involving constant rate of change are sufficiently complex to engender students' reasoning related to varying rate of change (Stroup, 2002). Filling rectangle tasks incorporating constant rate of change supported students' focus on the covarying quantities of volume and height. I anticipated that such tasks would contain sufficient complexity to support students' consideration of situations involving varying rate of change because they could foster students' coordination of covarying quantities.

Task implementation and analysis

During May 2012, I implemented the filling rectangle tasks with 4 sections of 7th grade students at an urban middle school in a large Midwestern U.S. city. The district has identified the school as high performing based on students' academic performance, with approximately 45% of students identified as English Language Learners and over 90% of students receiving free or reduced lunch. I implemented tasks 1-3 during three consecutive days of whole class instruction. Following the lessons, I conducted 40-minute task-based interviews with 7 pairs of students, selecting at least 1 pair of students from each of the 4 sections. I purposefully chose student pairs based on the students' participation in classroom instruction and on evidence of reasoning about quantities involved in rate of change. During the task-based interviews I followed up with task 3 and presented task 4. For this paper I report results of analysis of students' work on task 4.

Analysis of students' work during task-based interviews revealed two main findings: (a) Students may depend on numerical calculations to make claims about how quantities are changing together, and (b) Students may create graphs relating covarying quantities (not including time) as if one quantity were elapsing time.

(a) Students who depended on numerical calculations had difficulty making predictions about how the area of the triangle would change as the height increased. The responses of two students, Navarro and Myra (who participated in different interview pairs), provide insight into the kind of difficulty students might have. When Navarro and Myra were presented with the filling triangle task, both of them attempted to determine amounts of area. Even after prompting to not worry about making calculations, Navarro's persistence in trying to calculate amounts of area made it seem as if he depended on calculating amounts of areas to make such predictions. Unlike Navarro, after my prompt to not worry about how to calculate the area, Myra smiled and exclaimed "Oh, I get you now!" When I asked her to explain, she said "the area is getting bigger, but how much it increases is getting smaller." By no longer attempting to determine amounts of area, Myra was able to describe

variation in how the area was increasing. Future iterations of implementation and analysis could provide further explanation as to how students' nonnumerical reasoning develops when making relationships between quantities.

(b) Although the filling triangle sketch related area with the length of AD, when predicting features of a graph some students seemed to operate as if AD represented time. I designed the animation (See Fig. 2) so that the graph would begin to be sketched from a point when the triangle was partially filled. When predicting the shape of a graph relating area and side length for the filling triangle, Tomas sketched a graph in the air, indicating the graph would curve and then begin to increase again (which is not consistent with the graph shown in Fig. 2). When asked to explain why, Tomas said that area would fill more slowly and then start filling more quickly again. Even though Tomas attended to both area and side length when using the animation, he seemed to sketch the graph as if it were relating increasing area with elapsing time. Future tasks dynamically linking multiple graphs relating changing quantities to a single geometric representation might support students' consideration of the independent variable as something other than time.

Implications for task design in research investigating students' reasoning

Designing a sequence of tasks to support students' reasoning involved theoretical considerations including how students might make sense of and coordinate covarying quantities and practical considerations including how students might manipulate quantities represented with dynamic geometry sketches. The task sequence supports students' progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity. Researchers could draw on and expand design principles underlying this task sequence to develop other task sequences focusing on quantity and covariation. Future research involving implementation of this and other task sequences could support development and expansion of frameworks articulating progressions in quantitative and covariational reasoning.

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Designing tasks that enhance mathematics learning through creative reasoning

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This paper presents a research framework for a design research approach to constructing creative learning opportunities and some related empirical studies.

Keywords: Learning difficulties, rote learning, creative reasoning, task design.

Introduction

A central problem in mathematics education is that we want students to understand mathematics and to become efficient problem solvers, but even after 30 years of research and reform many students still do inefficient rote thinking (Hiebert, 2003; Lithner 2008). This can be seen as one of the main reasons behind learning difficulties in mathematics. The reason that the rote learning problem is (largely) unsolved in many countries is a combination of several factors related to the immense complexity of mathematics learning (Niss, 1999) and to the lack of research insights concerning the effectiveness of different teaching designs (Niss, 2007).

This design research addresses some of the most central mathematics learning goals, which we (teachers, researchers and other actors in the educational systems) largely fail to help students to reach: problem solving ability, reasoning ability and conceptual understanding. Problem solving is defined as "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 51) and includes identifying, posing, and specifying different kinds of problems and solving them, if appropriate, in different ways (Niss 2003). Reasoning is a fundamental aspect of mathematics (NCTM, 2000). It goes beyond constructing reasoning, and includes abilities like following and assessing chains of arguments, knowing what a proof is and how it differs from other kinds of reasoning, uncovering the basic ideas in a given line of argument, and devising formal and informal arguments (Niss 2003). The notion of understanding is very complex (Sierpinska 1996), and will not be pursued here beyond noting that several of the theoretical constructs denoted understanding concern relations between rote learning and deeper understanding, e.g. Skemp (1978) and Hiebert & Lefevre (1986). The notion will here be used in a relatively intuitive way, referring to insights in the origin, motivation, meaning and use (Brousseau 1997) of a mathematical fact, method, concept or other idea.

Rote learning

Rote learning is "the process of learning something by repeating it until you remember it rather than by understanding the meaning of it" (Oxford Advanced Learner's Dictionary). The characteristics, causes, and consequences of rote learning in mathematics can to a large extent be connected to an unwarranted and far-reaching reduction of complexity in terms of an algorithmic focus (Skemp, 1978; Hiebert & Carpenter, 1992; Tall, 1996; Vinner, 1997; Hiebert, 2003; Lithner, 2008). Referring to "massive amounts of converging data" in studies from USA, Hiebert (2003) suggests that the baseline conclusion is that students are learning best the kinds of mathematics that they are having the most opportunities to learn, which is simple calculation procedures, terms and definitions through memorization. Similar opportunities to learn were found in a Swedish large-scale study including observations of 200 mathematics classrooms (Boesen et al., 2012).

Memorising facts and procedures, sometimes without understanding, is a central aspect of mathematics learning. The problem is when rote learning becomes dominating since it is not possible to develop other central competencies like problem solving ability and conceptual understanding by rote learning alone. For example, it is well known (e.g. Schoenfeld, 1985) that there is no transfer from rote learning of basic facts and procedures to the ability to solve non-routine mathematical problems. From literature reviews (e.g. Hiebert, 2003) and from the empirical studies exemplified below, it is reasonable to draw the conclusion that rote learning is one of the main causes behind the difficulties to learn mathematics that large groups of students of all age levels encounter.

Creative reasoning

This and the next section consist of a summary of selected parts of a research framework (Lithner, 2008) that is based on the outcomes of a series of empirical studies on the relationship between reasoning and learning difficulties in mathematics.

Reasoning is defined in this paper as the line of thought that is adopted to produce assertions and reach conclusions when solving tasks. Reasoning is not necessarily based on formal logic and is therefore not restricted to proof; it may even be incorrect as long as there are some sensible (to the reasoner) arguments supporting it. A basic assumption is that the design of the students' tasks (and of the classroom context) affect the reasoning the students will activate which in turn affects their opportunities to learn. A 'task' includes most of the work requested from students in classrooms, such as exercises, tests, group work, etc. Arguments can be anchored in either surface or intrinsic properties, and the relevance of a mathematical property can depend on context. In deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) is a surface property that is insufficient to resolve the problem (a conclusion based on this property alone is that 9/15>2/3 since 9 and 15 are larger than 2 and 3), while the quotient captures the intrinsic property. The intrinsic/surface distinction was introduced because one of the reasons behind students' difficulties was found to be the anchoring of arguments in surface properties (Lithner, 2003). The aspect of creativity that is emphasized in this framework is not ‘genius’ or ‘exceptional novelty,’ but the creation of mathematical task solutions that can be modest but that are original to the individual who creates them (Silver, 1997).

The discussion above leads to a definition of Creative Mathematically Founded Reasoning (CMR) that fulfils all of the following criteria. i) *Creativity*; a new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created, in a way that is sufficiently fluent and flexible to avoid restraining fixations. ii) *Plausibility*; there are arguments supporting the strategy choice and/or strategy implementation explaining why the conclusions are true or plausible. iii) *Anchoring*; the arguments are anchored in the intrinsic mathematical properties of the components that are involved in the reasoning.

Imitative reasoning

The empirical studies that form the basis of this framework have identified three main types of mathematically superficial imitative Algorithmic Reasoning (AR), which may lead to rote learning. The term ‘algorithm’ includes all pre-specified procedures (not only calculations), such as finding the zeros of a function by zooming in on its intersections with the x-axis with a graphing calculator. “An algorithm is a finite sequence of executable instructions which allows one to find a definite result for a given class of problems” (Brousseau, 1997, p. 129). The importance of an algorithm is that it can be determined in advance. The n:th transition does not depend on any circumstance that was unforeseen in the (n-1)st transition - not on finding new information, any new decision, any interpretation, or thus on any meaning that one could attribute to the transitions. Therefore, the execution of an algorithm has high reliability and speed (Brousseau, 1997), which is the strength of using an algorithm when the purpose is only to produce a task solution.

However, if the purpose is to learn something from solving the task, the fact that an algorithm is independent of new decisions, interpretations or meaning implies that all of the conceptually difficult parts are taken care of by the algorithm, and thus only the easy parts are left to the student. This may lead to rote learning. In particular, the resultant argumentation is normally superficial and very limited, as seen in the main AR types that are found in studies: Familiar AR includes a strategy choice that can be characterized by attempts to identify a task as being of a familiar type with a corresponding known solution algorithm or a complete answer. In Delimiting AR, the algorithm is chosen from a set of algorithms that are available to the reasoner, and the set is delimited by the reasoner through the included algorithms’ surface property relationships with the task. For example, if the task contains a second-degree polynomial p(x), the reasoner can choose to solve the corresponding equation as p(x)=0 even if the task asks for the maximum of the polynomial. In Guided AR, the reasoning is mainly guided by two types of sources that are external to the task. In person-guided AR, a teacher or a peer pilots the student’s solution. In text-guided AR, the strategy choice is founded on identifying, in the task to be solved, similar surface properties to those in a text source (e.g., a textbook). Argumentation may be present, but it is not necessary because the authority of the guide ensures that the strategy choice and the implementation are correct.

In students' attempts to resolve problematic task solving situations, the CMR criteria i-iii were found to capture the main differences seen in reasoning characteristics between AR (where i-iii are absent) and constructive CMR (Lithner, 2008). Students often use superficial imitative reasoning of the types presented above in laboratory tests and when working with tasks (e.g. textbooks or assessment) in regular classroom contexts, which is a major obstacle both when it comes to learn and to use mathematics (e.g. Lithner, 2000; 2003: 2008; 2011, Bergqvist, Lithner & Sumpter, 2008; Boesen, Lithner & Palm, 2010). In addition, teaching, textbooks and assessment mainly promote rote learning in the sense that Guided AR is provided by teachers and textbooks, and that most practice and test tasks can be solved by AR (e.g. Bergqvist 2007; Palm, Boesen & Lithner, 2011; Bergqvist & Lithner, 2012; Boesen et al., 2012). Judging from the quote by Hiebert in the introduction this may be the case also outside Sweden, for example as found in common American calculus textbooks (Lithner, 2004).

Design research and the theory of didactical situations

The ongoing research described below can be characterised as design research which in this paper refers to the use of scientific methods to develop theories, frameworks and principles of innovative educational designs. Although the meaning of design experiments have not been settled in the literature (Schoenfeld, 2007) Plomp (2009) argue that authors may vary in the details of how they picture design research, but they all agree that design research comprises of a number of stages or phases: preliminary phase (development of framework), prototyping phase and assessment phase. A key characteristic of design research is thus that it is strongly aligned with effective models linking research and practice (Cobb et al. 2003), which, according to Burkhardt and Schoenfeld (2003), "the traditions of educational research are not". According to Gravemeijer & Cobb (2006) "the purpose of the design experiment is both to test and improve the conjectured local instruction theory that was developed in the preliminary phase, and to develop an understanding of how it works."

The underlying characterisations of students' reasoning in this paper emanate from a cognitive psychology perspective, but extend into sociocultural considerations when addressing potential causes and consequences. The theoretical foundation for the attempts to design better learning opportunities is Brousseau's Theory of Didactical Situations (1997), which is a theory of how mathematics can be learnt through non-routine problem solving. It emphasises "the social and cultural activities which condition the creation, the practice and the communication of knowledge" (p. 23). One central construct is the devolution of problem. The student has to take responsibility for a part of the problem solving process, but she cannot in general learn in isolation. The teacher's task is to arrange a suitable didactic situation in the form of a problem. Between when the student accepts the problem as her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. This part of the didactic situation is called an adidactical situation. The student must construct the piece of new knowledge and the teacher must therefore arrange not the communication of knowledge, but the devolution of a good problem.

The teacher may (e.g. to reduce complexity) try to overcome learning obstacles and force learning by devolving less of the problem to the student. Brousseau exemplifies this by the Topaze effect (p. 25) when the teacher lets the teaching act collapse by taking responsibility for the student's work and letting the target knowledge disappear (as in Guided AR). Telling the student that an automatic method exists relieves her of the responsibility for her intellectual work, thus blocking the devolution of a problem. If this is the normal didactic situation the student meets then the didactical contract is formed accordingly, which may not be the teacher's intention. The teacher expects the student to learn problem solving reasoning, while the student expects that an algorithm should be provided that relieves her of the responsibility of engaging in the adidactical situation.

The key issue with respect to this paper is to find a suitable devolution of problem, with the aim of providing learning opportunities through CMR instead of AR. It is in general easy to design AR tasks, since the structure of the task is based on repeating the algorithmic procedure and follows therefore directly from the procedure. For example, after the procedure to solve linear equations (ax+b=cx+d) is described then a large number of AR tasks are obtained trivially by just formulating different equations. If the purpose is just to design any mathematical non-routine problem suitable for a particular student group, then the situation is a bit trickier but the literature and the Internet is full of good mathematics problems. However, if the purpose is to design a problem that can help the student to construct a particular target knowledge then the design becomes much more complicated. In addition, the central target knowledge within mathematics curricula is often such that a set of tasks (and adidactical situations) rather than a singular task is required. For example, if the goal is that the student shall by herself construct a general method for solving linear equations it is unrealistic that this can be done in a single adidactical situation.

A design experiment

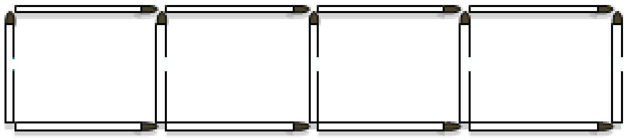
This design experiment is a part of a larger project that studies teaching designs that give students different opportunities to learn with respect to imitation or construction of knowledge. In this experiment two ways of teaching are compared: I) An algorithmic method for solving a type of tasks is presented, and students apply this method on a set of practice tasks. The structure is founded in the framework for AR and the tasks have the same structure as common textbook tasks. II) Guiding the individual into by herself constructing a solution method. This structure is founded in the devolution of problem and in the framework for CMR. See below for examples.

In order to be able to compare these two ways of teaching, it is prioritised a) that similar target knowledge can be reached by both ways and b) that the target knowledge may be learnt both by rote and by CMR. A suitable form of target knowledge is task solving methods that can be economised as mathematical procedures. This is a central aspect of mathematical knowledge (Kilpatrick, Swafford & Findell, 2001). The teaching of such procedures seems to constitute some 50-100% of mathematics teaching (Lithner, 2008; Boesen et al., 2012), at least in Sweden but maybe also in other countries (Hiebert, 2003). Thus the overall background question posed is: "how to best learn mathematical task solving procedures"? Is it to practice standard algorithms by large amounts of drill exercises, or by the students' own construction of the procedures? Concerning this issue the discrepancies between research and practice, and between different research perspectives seem large (Arbaugh et al., 2010). In addition, there seems to be little empirical evidence backing the rather few theoretical claims made.

The teaching mode I is hypothesised to lead the subject into rote learning of algorithms by AR without understanding the foundations of the algorithm. In the mode II the subject is not given a method that can be directly applied to solve the tasks. Instead, a sequence of exploratory tasks is given. This devolution of problem is intended to make pure rote learning impossible and the subject has to understand the method in order to solve the task. One argument behind the hypothesis that tasks that require CMR will lead to a constructive adidactical situation with a real devolution of problem is related to the three defining criteria of CMR: i) Novelty, that the task cannot be solved by familiar imitative reasoning, ensures the devolution of some kind of reasoning that the student has to be responsible for. ii) The presence of arguments, supporting the plausibility of the conclusions, is necessary to guide and verify the construction of new insights. iii) The necessity to anchor the reasoning ensures that the mathematical obstacles are addressed and that the resolutions are based on properties of relevant mathematical facts and concepts.

The research question of this experiment is: What are the characteristics of an adidactical situation that leads to a devolution of problem where learning through CMR is more efficient than learning through AR in the format common in school? The present pre-clinical (Schoenfeld, 2007) experiment is in order to reduce complexity carried out without peer-peer or peer-teacher interaction, and serves to clarify basic phenomena as a preparation to pose the same question in a real classroom context. Several iterations and revisions of task designs have been carried out. In one of the designs two groups of students (n=99) are matched by grades and basic cognitive tests. An example of an AR practice task is given in Figure 1. The corresponding CMR practice task consists of the same introduction and same question at the end, but the sentence with the formula ("If *x* is…) and the solved example is removed. One week after the practice session all students from both groups take the same post-test.

When squares are put in a row it looks like the figure to the right. 13 matches are need for four squares:



If *x* is the number of squares then the number of matches *y* can be calculated by the function $y=3x+1$
Example: If 4 squares are put in a row then
$y=3x+1=3.4+1=13$ matches are needed.
How many matches are needed to get 6 squares in a row?

Figure 1, example of an AR practice task.

One may note that compared the AR group could have an advantage since they are provided with more information. However, the empirical studies mentioned above show that if students are given an algorithmic solution method to a task, they will mainly apply AR to solve the task without considering the underlying meaning of the concepts, representations or connections. Thus they will probably not even try to understand meaning of the algebraic formula, which in this example is the relation between the figure of matches and the formula y=3x+1. Then the Theory of Didactical Situations implies that the CMR group may learn better.

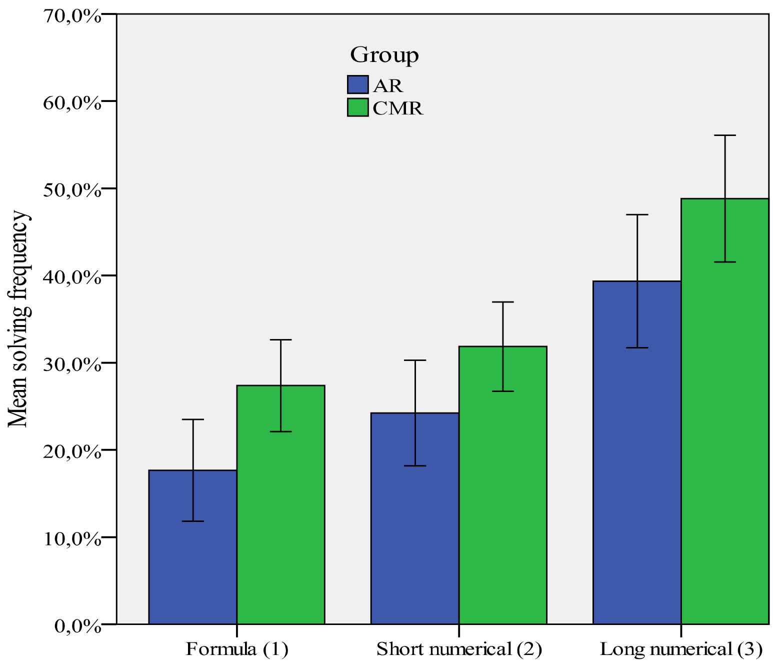


Figure 2

The results show that this is the case, in the sense that the CMR group on average has significantly higher test results and shorter response times (Figure 1). The test tasks “Formula” ask for recalling the formula, “Short numerical” for recalling and applying a solution method and “Long numerical” for (re)constructing a solution method.

In addition, the common belief that only the best pupils can benefit from learning through their own construction of solution methods is not supported by this experiment. On the contrary, in the 30% with lowest cognitive index (a composite grade and pre-test score) the difference to the advantage of the CMR group is even larger than for the whole sample (Figure 2).

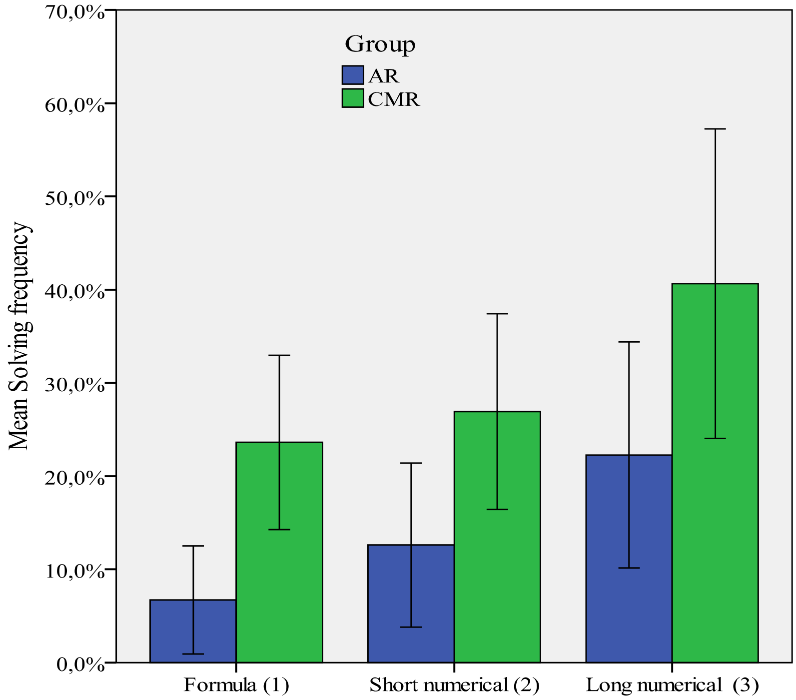


Figure 3

Parallel to the experiment above, other complementary studies are carried out within the research project. One example is an ongoing study using functional Magnetic Resonance Imaging (fMRI) to compare brain activity for students (n=40) from AR and CMR training groups. This study is exploratory with the aim to analyse non-behavioural information about students' thinking processes. One question asked is if students from the two groups activate different neural networks, and how this relates to earlier research findings about the brain and the learning of mathematics. Another question is if students from one group show higher brain activity in specific regions, and what the causes may be. For example, brain activity in the CMR group could be higher if they have created some kind of richer neural networks or lower if they have developed more rational solution methods and/or more efficient understanding. Another example of ongoing research uses eye-tracking methods to compare the strategies used by students learning through AR and CMR designs.

The work reported in this paper has been presented in research journals (see the references), in a large number of Nordic teacher conferences, in international research conferences (e.g. Kristiansand 2011, London 2012, Seoul 2012) and has attracted some interest from national and local media (TV, radio, newspapers and popular scientific journals). During the spring 2012 discussions with leading local school administrators about collaboration in design research and educational development has been initiated, and financial resources are allocated for this purpose.

The research is externally and internally financed and is carried out in collaboration with researchers, teacher educators and teachers from different areas (mathematics education, psychology, neuroscience) in Umeå, Falun and Karlstad. The roles of the authors can be summarised as follows. Johan Lithner (Professor in Mathematics education): Project leader, educational theoretical and conceptual background, design principles, construction of initial task versions. Bert Jonsson (Senior university lecturer in Psychology), psychological theoretical and conceptual background, cognitive measures. Carina Granberg (Senior university lecturer in Educational work) and Jan Olsson (School teacher, Licentiate research student), designing ICT interactive tasks for the classroom. Yvonne Liljekvist and Mathias Norqvist (School teachers, teacher educators, PhD students in Mathematics education), construction of testable tasks. All are engaged in the cyclic design-revision-testing-evaluation-redesign of the tasks.

The ongoing research presented above reside in the pre-clinical stage (Schoenfeld, 2007) and concerns the design of mathematical tasks that are suitable for devolution of problems where students may solve the tasks by CMR. One aim is to form a basis for clinical (classroom) studies. However, it is not just to take tasks designed and evaluated in the pre-clinical phase into the classroom. Stein, Engle, Smith & Huges (2008) argue that teachers who attempt to use inquiry-based, student-centred instructional tasks face challenges that go beyond identifying well-designed tasks and setting them up appropriately in the classroom. Thus one major challenge for the further research is how the task and teaching designs can incorporate class interactions. A second challenge is to design tasks that are more open to the students' own initiatives, and a third to design adidactical situations that encompass wider and deeper target knowledge than the algebraic formulas in the design experiment above.

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Shortcomings in the *milieu* for algebraic generalisation arising from task design and vagueness in mathematical discourse

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This paper presents how the milieu for students' engagement with an algebraic generalisation task is constrained by two factors: first, by the task design; second, by the students' unawareness of the nature of a mathematical statement, combined with the teacher's use of a generic example without the students' awareness of it.

Keywords: Milieu, adidactical situation, algebraic generalisation, mathematical statement, generic example.

Introduction

The formulation of a task, as well as its mathematical, social, psychological, and didactic contexts, are important factors for students' responses on the task (Sierpinska, 2004). This paper presents an analysis of three student teachers' collaborative engagement with a task on algebraic generalisation of a shape pattern. The task is designed by their mathematics teacher educator.[18] The author has had influence neither on the design nor on the implementation of the given task. 'Task' is here understood as an assignment given to students to which they are expected to produce a solution. The paper deals with the question of how students understand the purpose of the task they are given in a *regular* teaching situation (i.e., it is not the result of didactical engineering, Artigue & Perrin-Glorian, 1991). I show how the formulation of the task and the interaction between the teacher and the students about the task constitute a gap between the teacher's intention with the task and the students' mathematical activity.

Inspired by the writing of Whitehead (1947), Devlin (1994), and others, I view mathematics as the science of patterns. A shape pattern in school mathematics is usually instantiated by some consecutive geometric configurations in an alignment imagined as continuing until infinity. Radford (2006) provides a useful characterisation of *algebraic generalisation* of patterns when he proposes that

> generalizing a pattern algebraically rests on the capability of grasping a commonality noticed on some elements of a sequence *S*, being aware that this commonality applies to all the terms of *S* and being able to use it to provide a direct expression of whatsoever term of *S*. (Radford, 2006, p. 5)

[18] In the rest of the paper, "students" is used to refer to student teachers, and "teacher" is used to refer to a teacher educator.

Theoretical framework

In Brousseau's (1997) theory of didactical situations in mathematics, an *adidactical situation* is a situation in which the student takes a mathematical problem as his own and solves it on the basis of its internal logic without the teacher's guidance and without trying to interpret the teacher's intention with the problem. The *devolution* of an adidactical learning situation is the act by which the teacher encourages the student to accept the responsibility for an adidactical learning situation or for a problem, and the teacher accepts the consequences of the transfer of this responsibility (Brousseau, 1997). The student cannot engage in any adidactical situation; the teacher attempts to arrange an adidactical situation that the student can handle.

In the devolution process, which is part of the broader (didactical) situation, the teacher is faced with a system, itself built up from a pair of systems; the student and a *milieu* that lacks any didactical intentions with regard to the student (Brousseau, 1997). The milieu is a subset of the students' environment with only those features that are relevant with respect to the knowledge aimed at by the teacher in the didactical situation. The concept of milieu models the elements of the material or intellectual reality on which the students act and which may be an obstacle to their actions and reasoning (Laborde & Perrin-Glorian, 2005). That is, the milieu of a didactical situation is the part of the environment that can bring feedback to students' actions to accomplish a task.

An adidactical situation is part of the didactical situation that is the broader situation with the system of interaction of the students with the milieu arranged with the purpose of the students' appropriation of the target knowledge without the teacher's intervention (Brousseau, 1997). The teacher can act on the milieu by providing new information or new equipment, for example by asking a question or directing students' attention to certain factors in the classroom situation. When the teacher acts on the milieu, she changes the knowledge needed to solve the problem (Perrin-Glorian, Deblois & Robert, 2008). Whether the student can handle an adidactical situation depends upon two conditions: first, that the student has prior knowledge that enables him to engage with the situation; second, that the milieu created by the teacher provides the student with knowings that enable him to develop the knowledge aimed at (by the teacher).

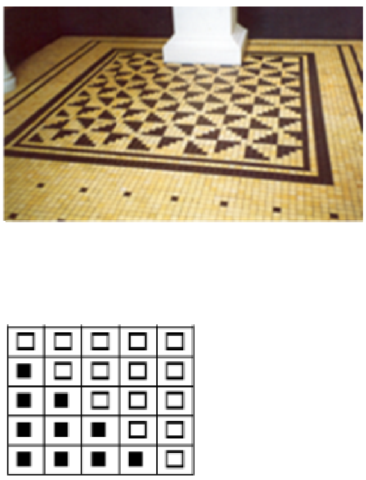
Succeeding the devolution phase, the didactical situation consists of four situations (or phases) in which the role of the teacher and the status of knowledge change (Brousseau, 1997): Situations of action, formulation, and validation are intentionally adidactical situations, whereas the situation of institutionalisation is not adidactical. In the following paragraph, these situations are briefly described (for a more elaborated explanation, see Brousseau, 1997, or Måsøval, 2011, Chapter 2).

The situation of *action* is where the students engage with the presented problem on the basis of its internal logic without the teacher's intervention. The students construct a representation of the situation which serves as a "model" that guides them in their decisions. In the situation of *formulation* the students exchange and compare observations between themselves, where the main purpose is to develop language to formulate their observations and agree on some common meanings. Here, the teacher re-enters the scene to chair the exchanges and make sure that all formulations are made "visible" in the classroom. In the situation of *validation* the students try to explain some phenomenon or verify a conjecture. Here, the teacher acts as a chair in a scientific debate and intervenes only to structure the debate and encourage the students to use more precise mathematical concepts. The situation of *institutionalisation* is where the teacher informs the students about conventional terminology and highlights definitions and theorems considered important for the contextualised knowledge (developed by the students through the preceding situations) to gain the status of cultural knowledge so that it can be used in settings other than in the original one set up by the teacher.

The adopted epistemological perspective in the paper is rooted in the theory of didactical situations: First, teaching involves the devolution to the student of an adidactical, appropriate situation; learning is the student's adaptation to this situation. Second, teaching involves the transformation of the student's responses into a piece of knowledge which can be used beyond the situation in which it is produced.

Methodology

The reported research is derived from the author's PhD project (a case study) reported in (Måsøval, 2011), where the addressed research question was: *What factors constrain students' appropriation of algebraic generality in shape patterns?* The task with which the paper deals (Task 4, presented in Figure 1) was the last of four tasks (during eight lessons of 45 minutes) on algebraic generalisation of shape patterns. Task 4 is designed for students' collaborative self-engagement. It is divided into three subtasks which are part of the milieu for situations of action, formulation, and validation, respectively: Inventing a continuation of the shape pattern represents finding a "model" that guides the students towards the intended theorem (Task 4a: action); identifying what figurate numbers that are part of the invariant structure of the pattern represents developing language to formulate the students' observations (Task 4b: formulation); and, expressing what the pattern tells them represents verifying the intended theorem (Task 4c: validation).



Thorvaldsen's museum in Copenhagen contains several floor mosaics with mathematical content. We looked at one mosaic last Friday, and here we shall look at another one. This pattern can be thought of as built up by equal, squared areas containing bright and dark mosaic tiles.

On the shape below the pattern is reproduced schematically with ■ for each of the dark squared tiles and ☐ for each of the bright ones.

a) If this shape were part of a sequence of shapes, what would the next one look like?

b) What kinds of figurate numbers do you find in the bright and the dark areas, and in the shape as a whole?

c) Express what the shape tells you about these numbers in terms of a mathematical statement.

Figure 1. Task 4 given to the class for work in small groups

It is relevant to notice that this categorisation of the task is based on the author's conceptualisation. At the time the data were collected, the teacher who designed the task was not acquainted with Brousseau's theory. According to the same teacher, the mathematical knowledge aimed at in Task 4 was the formulation of a mathematical statement (a theorem) represented in algebraic notation (e.g.,

$T_{n-1} + T_n = n^2$, where $T_n = 1 + 2 + 3 + \cdots + n$ denotes the *n*-th triangular number). This intention was not communicated to the students.

The data are Task 4 and a video-recorded observation of three students' collaborative engagement with Task 4 (with teacher intervention). The students are Anne, Helen, and Paul (pseudonyms), who were in their first academic year on a teacher education programme for primary and lower secondary school in Norway. The observed teacher (the one who had designed the task) is my colleague, an experienced, male teacher of mathematics. My role during data collection was to be a silent observer while video-recording the classroom interaction analysed in the paper. The teacher interacted with the students during the observed episode on the basis of the students' difficulty in understanding two of the concepts used in the task. The video-recorded episode has been transcribed and analysed through a process of open coding (using an adapted grounded theory approach, Strauss & Corbin, 1998) where concepts from the theory of didactical situations (Brousseau, 1997) have been used to make sense of what factors constrain the students' algebraic generalisation of the actual shape pattern (in this way addressing the research question).

Analysis of students' engagement with a task on algebraic generalisation

Anne, Helen, and Paul have drawn the first three elements of a shape pattern (Figure 2) which is a continuation of the element given in Task 4. They have found that for the first element of this shape pattern (the 5x5 square given in the task), the number of black components (represented by black x-es) is equal to the sum of the first four natural numbers, and the number of white components (represented by turquoise x-es) is equal to the sum of the first five natural numbers. The students have observed that this is a regularity that applies also for the next two elements of the shape pattern. That is, they have verified by inspection that for the second element (a 6x6 square), the number of black components is equal to the sum of the first five natural numbers, and the number of turquoise components is equal to the sum of the first six natural numbers, and likewise for the third element. They have, however, not identified the sums of consecutive natural numbers to be triangular numbers.

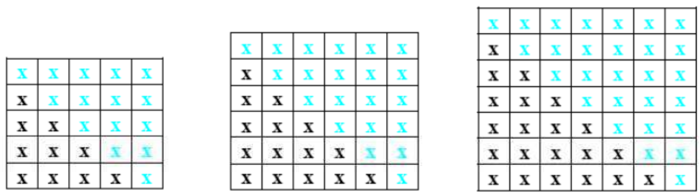


Figure 2. Continuation of the shape pattern invented by Anne, Helen, and Paul (Task 4a)

When they come to Tasks 4b and 4c, they wonder what is meant by "figurate number" and "mathematical statement", and get the teacher to help them. The teacher explains that the question about figurate numbers is about being down on the "bedrock" looking for standard numbers that it is common to have in one's "toolbox". The teacher thereafter asks what the students recognise if they look at the first element (given in the task) as a whole. The following exchange takes place:

591 Paul: Well, that it is five squared.
592 Teacher: Right.
593 Anne: Yes, it is indeed squares, and then [Pause 1-3 s]
594 Teacher: Yes, it is indeed squares, square numbers.
595 Paul: And then you have nine and sixteen as the numbers of [Pause 1-3 s] no, ten perhaps?

596 Teacher: And then there are the black and white ones? Do you recognise them?

597 Anne: Fifteen, twenty one [Pause 1-3 s] ehm

In turn 591 Paul focuses on the first element of the shape pattern (the 5x5 square). I interpret his words here to suggest that he continues to look at this element in turn 595 and refers to its number of black and white components. The numerical values he suggests at first are wrong, but he then makes a new suggestion which is right for the number of black components of the first element. The teacher does not directly respond to Paul's answer, and when the teacher asks if they recognise the black and white components (turn 596), Anne responds by giving the number of black and white components of the second element. This seems to make Paul insecure about what the teacher asks for; he wonders whether it is only the first element or it is the sequence of elements they are supposed to consider:

598 Paul: If we are supposed to see the connection, it is only *this* very shape we shall look at now? [Draws a curve with his pencil around the element given in the task] It is not the next shapes we have made [points at the succeeding elements drawn in his notebook when he says "next"]?

599 Teacher: You may well look at it as it stands there [Pause 1-3 s] uh [Pause 1-3 s] [indecipherable]

600 Paul: Not further, ok.

The teacher's response in turn 599 I interpret as confirming that it is satisfactory that the students look at the element given in the task (a 5x5 square) as a basis for finding answers to Tasks 4b and 4c. It is plausible that the teacher takes this stance as a consequence of seeing the 5x5 square as a generic example. This element of the shape pattern is an example which illustrates that the sum of the fourth and the fifth triangular numbers is equal to the fifth square number. It is generic (Rowland, 2000) in the sense that it is a representative of a class of elements which have the property that they are squares which (by the two colours) illustrate that the n-th square number is the sum of the $(n-1)$-th and the n-th triangular numbers.

These general properties are however not addressed in the classroom situation. The teacher does not express to the students that he uses the 5x5 square in the sense of a generic example, nor does he use the term "generic". What I interpret as the teacher's implicit utilisation of a generic example contributes to vagueness in the discourse: The stance taken by the teacher about the sufficiency of looking at one element of the shape pattern (genericity of the 5x5 square) is consistent with the formulation in Tasks 4b and 4c (reproduced below), a correspondence which may be expected since the task is designed by the same teacher. Application of *singular* number in the noun "the shape" indicates that the shape presented in the task is seen as generic:

> What kinds of figurate numbers do you find in the bright and the dark areas, and in *the shape* as a whole? [Task 4b, emphasis added]
> Express what *the shape* tells you about these numbers in terms of a mathematical statement. [Task 4c, emphasis added]

After having observed that the 5x5 square contains ten black components and fifteen white components, the students describe the structure of the next elements of the shape pattern. They observe that the elements develop by adding to the white components an extra row (at the top) with one more component, and that the number of black components of a successive element is the same as the number of white components of the present element. The teacher reminds the students that they have earlier written ten as a sum of the first four natural numbers, and further, tells them that numbers with this structure are referred to as triangular numbers. He refers to what I interpret as (for him) a generic example when he continues:

640 Teacher E:	So this is actually the clue here. That this element, I think I'll just tell you, that this shape represents a kind of connection between triangular numbers and square numbers.

This is succeeded by a comment by Anne that she had been insecure what was meant by the concept of "figurate numbers". After some exchanges between the teacher and her, she (re)turns attention to the concept of "mathematical statement" which so far has not been addressed explicitly by the teacher:

651 Anne:	Express what the shape tells about these numbers in terms of a mathematical statement [recitation from the task]. Are we supposed to write it as a formula or shall we formulate it?

Based on the students' conclusion on Task 4c (an explanation in natural language of the structure of the first element of the shape pattern), it is plausible that Anne in turn 651 is trying to figure out whether the teacher wants them to present the solution to Task 4c as a formula (potentially in mathematical notation) or as a formulation (potentially in natural language). The teacher responds by reinforcing attention towards the first element of the shape pattern, which I suggest he continues to use as a generic example:

652 Teacher:	Well, then you can think of that one [points at the 5x5 square presented in the task]. If you look at it as a whole, what square number is it that it [Pause 1-3 s] shows us? What position?
653 Helen:	Five or?
654 Anne:	What number in the series or?
655 Teacher:	What number in the series of square numbers, right.
656 Anne:	Well, I can imagine it is [Pause 1-3 s] the fifth then.
657 Teacher:	The fifth, right.
658 Anne:	Because that would have been good for us [smiles]
659 Teacher:	Yes. [Students laugh] Well, but here we don't have much choice, really. It is the fifth, it is twenty five, it is square number five. (Anne: uh huh). And if we think of it as composed by triangular numbers (Anne: yes) then you can think of [Pause 1-3 s] what position in the series of triangular numbers is that which these black and white [components] represent?

I interpret the teacher's utterances in turns 652, 655, 657, and 659 as an incidence of the Topaze effect (Brousseau, 1997): The answer that the students must give is determined in advance (the theorem asserting that the *n*-th square number is equal to the sum of the $(n-1)$-th and the *n*-th triangular numbers); the teacher chooses questions to which the answer can be given (turn 652). The knowledge necessary to produce these answers changes, so does its meaning. Faced with the student's continued difficulty in giving the answer, the teacher poses easier and easier questions: It is possible to answer the teacher's question in turn 659 without having to formulate the intended theorem. Hence the target knowledge has disappeared, a phenomenon referred to as the Topaze effect.

There are features in the milieu which I interpret as giving rise to the Topaze effect: First, the nature of the concept of a mathematical statement is not known to the students and is neither explained to them in plain text. Second, the teacher's use of a particular example in a generic sense, apparently without the students' consciousness about it, contributes to the students' comprehension of the particular example as representing a mathematical statement in its own right. The teacher leaves the students after turn 659, and the students collaborate to find the positions of the triangular numbers from which the fifth square number is constructed (the new task). The outcome of their engagement with Task 4c is the expression in natural language of the property of one particular shape: the fifth square number is constructed from the fourth and the fifth triangular numbers. They make no attempt to generalise this characteristic to apply to an arbitrary element of the shape pattern, neither in natural language, nor in algebraic notation.

682 Paul: Well, a person who could figure out a formula for this, he would be good [laughs].
683 Anne: No, but it is not written (Paul: no) that we shall have a formula (Paul: right). We are supposed to express it as a mathematical statement. We have done that now. It is not very good, but we have emphasised what is relevant, I think.

Recall that Anne asked the teacher if they were supposed to write the mathematical statement as a formula or just formulate it (turn 651). Paul's and Anne's utterances (turns 682 and 683) indicate that they have interpreted the teacher's response (turn 652) to Anne's question (turn 651) to mean that a mathematical statement is a formulation (in natural language) about the numbers in the shape given in the task. Further, Anne seems to conclude that a formula is different from a mathematical statement in the way she claims that they are not asked to find a formula (turn 683). It is likely that she by "formula" understands an expression in mathematical notation.

Discussion

The students remain unaware that the aim of Task 4 is to establish a theorem about a general relationship between numbers (or sequences of numbers). The generalisation process is obscured by two interrelated factors, which are interpreted as weaknesses in the milieu.

The first factor is about the design of the task: It is problematic that the task presents only one element of an imagined pattern, combined with the use of *singular* number ("the shape") when referring to the pattern. Further, there is a problem with the design of the task because the students produce appropriate solutions to the first two subtasks (with input from the teacher on the concept of figurate numbers); this, however, does not afford them with knowings that enable them to formulate the intended theorem. In the context of algebraic generalisation of shape patterns, the knowledge at stake is algebraic generalisation of arithmetic relations mapped from the elements of the pattern. For epistemological reasons, the focus in tasks on algebraic generalisation of shape patterns therefore should be on those arithmetic relations; e.g. in Task 4 (here efficiently represented in mathematical notation): that $T_4 + T_5 = 5^2$, $T_5 + T_6 = 6^2$ and so on, to subsequently encourage generalisation by algebraic thinking.

The second factor that constitutes a weakness in the milieu is about the students' unfamiliarity with the concept of mathematical statement. The students had identified the structure of the fifth, sixth and seventh elements, even if they had not been explicit about their rank (that is, they had not made the point that the fifth element is the sum of the first four natural numbers and the first five natural numbers, and likewise for the next two). They had got the teacher to come to them because they did not know what was meant by the concepts of figurate number and mathematical statement. Anne's recitation of Task 4c and her subsequent question (turn 651) indicates that the adidactical situation devolved to the students is not appropriate because it depends on knowledge they do not have (the concept of mathematical statement). The teacher, however, instead of explaining the nature of a mathematical statement, directs attention to the 5x5 square. It is relevant here that the teacher believes that the students know the concept of mathematical statement (articulated in conversation with the teacher after the lesson). It is therefore plausible that the teacher interprets Anne's question in turn 651 to signify a problem with seeing the invariant structure of the elements of the shape pattern, and not a problem with the concept of mathematical statement *per se*. For that reason, when he acts on the milieu, he tries to help them discover the structure of the elements (by utilising a generic example) so they can develop the knowledge aimed at: an equivalence relation between square numbers and the sum of two triangular numbers. But, as described above, the students' interpretation of the teacher's (generic) example as complete in itself, without attention to general properties, terminates the generalisation process. The milieu is changed, so is the knowledge needed to solve the (new) task. A gap has been created between the teacher's intention with the (original) task and the students' actions on the milieu.

Måsøval (2011) has identified that tasks on algebraic generalisation of shape patterns are of two different types, based on the mathematical object they aim at: The first type (arbitrary shape patterns) aims at a formula for the numerical value of the *n*-th member of the sequence mapped from the shape pattern; the second (conjectural shape patterns) aims at a theorem which asserts equality between two different algebraic expressions for the *n*-th member of the sequence mapped from the shape pattern. It is therefore important that those who design (or choose) tasks analyse the target the tasks aim at, whether it is a formula for the numerical value of the *n*-th element of the actual pattern (a functional relationship), or it is a general numerical statement (a theorem) decontextualised from the actual pattern. This is in order for the milieu to be designed such that the desired relationship can be explored and explicated by the students (e.g, through decomposition of elements according to the invariant structure of the pattern).

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Order of tasks in sequences of early algebra

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This paper is about the selection, construction and application of a sequence of tasks involving patterns, in order to promote algebraic thinking, in two classes, in the period starting in the last trimester of the 3rd year and the first trimester of the 4th year of schooling. Each sequence was composed by twelve tasks (six structural and six sequential). One class got the sequential group of tasks first, and then the structural, while the other group got the reverse order. One of the goals of the study was to ascertain the cognitive suitability of designed sequences of tasks as a situated process.

Keywords: Algebraic thinking; sequence of tasks; patterns

1. About task design and algebra study

We consider task design as a crucial element of the learning environment, and we propose to explore the role that it plays for learners without forgetting the potential role of the teacher in encouraging whole class discussion around tasks. Our perspective relates to one aspect of Realistic Mathematics Education, in which the designer conducts anticipatory thought experiments by envisioning both how proposed instructional activities might be realized in the classroom, and what students might learn as they engage in them. Central instructional design strategies were task analysis and the construction of learning hierarchies in order to identify reification development. Our starting conjecture in terms of designing didactical sequences of tasks, is that we need epistemic and cognitive analysis not only to criticize each task itself, but to adapt its connections as best as possible to the students abilities, found by analyzing teaching experiments as cumulative and with anticipatory purposes. It's important for our task analysis to identify difficulty factors providing frameworks for hypothesizing instructional designs inspired by developmental cognition.

We decided to use an early algebra task as an explanation for a situated study supporting the perspective in which algebraic reasoning could be highly promoted (Kaput, 1995; Malara & Navarra, 2002; NCTM, 2000) as a tool intertwined with arithmetic (Lins & Gimenez 1997) building on their interconnection (Mason, 2008) to promote success by developing together both arithmetic and algebra, one implicated on the other. The study supporting this paper has been done with 8-9 years old students.

About the teaching experiment

The basis to build our sequence of tasks and test analysis, is to promote algebraic thinking by overcoming relational apprehension (Smith, 2011) and the use of patterns (Orton, 1999; Carraher, Martinez and Schliemann, 2008), used here in connection with a search for order or structure and therefore regularity, repetition and symmetry are frequently present (Frobisher et al., 1999), because of the relevance to the development of abstraction, generalization and the establishment of relations (Lins & Gimenez 1997; Mulligan et al., 2006).

The first part of the study consisted of the construction and validation of a questionnaire-task that later was used as pre- and post-test control strictly validated as it's usual in empirical studies (Vieira, Palhares & Gimenez 2012). Next step concerns the experimental process based upon a refined sequence of tasks. The principles for our task design are: (1) assuming the possibility of using arithmetic number sense related to algebraic reasoning; (2) assuming suitability criteria for analyzing mathematical activities; (3) mathematically inspired by using relations and diversity of representations but not letters for the unknowns; (4) hearing the voice of the students for analyzing and promoting mathematisation and retention.

A set of nineteen short tasks was designed, from which twelve tasks were chosen, six of the structural type and six of the sequential type. These separation is closely related to strands 1 and 2 within Core aspect A of algebraic reasoning (Kaput, 2008). In each type, half was figural and the other was numeric. In our sequence of tasks, the main aim is to reflect about the use of some knowledge to build and/or transfer to other mathematical knowledge. Our presentation leaves the problems from everyday life for a subsequent time. Therefore, we don't explain other algebraic activities more related to promote modeling processes. We are facing the problem of complexity and connection of tasks in designing sequences.

The tasks were meant to be diverse, some leading to an exploratory and investigative open activity to improve meaningful construction (Thompson, Carlson, & Silverman, 2007), and also as a problem solving approach (Arcavi & Friedlander 2011), some others involving structured generalization rules (Stacey, 1989). In the experiment, the teacher introduces always the problem by focusing heavily on exploring the situation, leaving the student a maximum degree of freedom for discussion.

About mathematic/epistemic suitability

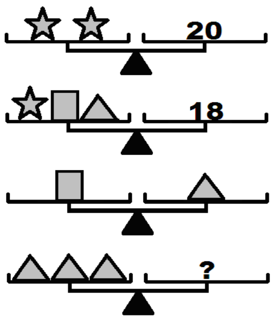
The activities integrate various mathematical aspects. The tasks of the sequential type were characterized by the existence of the first terms of a sequence, asking for the next term or a more advanced as a kind of generalization. The tasks of the structural type include different processes and properties: equality and order properties, finding rules from examples, identify equivalences, assume relational identities, acting by assumption, particularization,… among others.

Cognitive suitability

The implementation sequence was such that numeric and figural were alternated, with an increasing mathematisation in the case of the sequential type. Figural aspects are interpreted as supported by images or pictures in the sense of Pallascio (1992). The patterns and questions are not usual in textbooks, but involve addition and multiplication and addition rules. Only in one case one pattern can be considered as a division. We now give one example for each subtype:

Task type 1 (structural and figural)

In such a task, the aim is to observe how the arguments reveal an algorithmic process (based on Femiano, 2003).



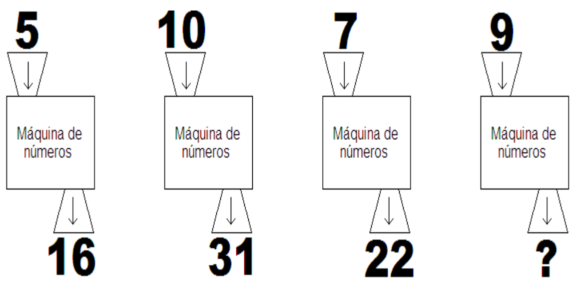
Observe carefully the four dish scales that are in equilibrium. Substitute the question mark by the number that allows the scale to remain in equilibrium. All equal figures have the same value.

Explain how you have found the value to substitute the question mark. You can use words or calculations to describe what you have done.

It's expected that students find the value of a star, and after finding values, identify that the square and triangle sum is 8. Then they can use trial and error, or change the order and assume that two other variables have the same value. Four equal properties are used in this case. We must notice that in such figural tasks, the role of the teacher is to pay attention about the consideration of the same value for the same figure. The teacher asks suddenly the students to explain their ideas in order to focus on mathematical argumentation used.

Task type 2 (structural and numeric)

In such a task, it was planned that the most difficult aspect is the interpretation itself because the sequence of starting numbers is not in order and for the fact that it's not a usual class task.



Observe carefully the four 'number machine'. Substitute the question mark by a number that follows the rule of the other three machines. Explain how you have found the number to substitute the question mark. You may use words or calculations.

We already knew that the use of additive comparison tended to be somewhat challenging to students of this age.

Task type 3 (sequential and figural)

This task was proposed to initiate table representations to establish combined rules made by two operations. The idea is to identify the ways in reaching the rule, and the way to explain. This is similar to those explained by Radford for these ages (Radford 2010) to provoke the use of arguments based on numbers. In this case, the figure can help to find the answer for the first questions.

Observe carefully the sequence of figures below.

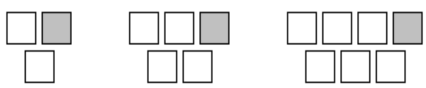


Fig. 1 Fig. 2 Fig. 3 Fig. 4 Fig. 5

Construct the two following figures 4 and 5. Without drawing or constructing it, say how many squares of each colour will have the 8th figure. Complete the table.

Figure number	1	2	3	4	5	…	10	20
Number of white squares	2		6			…		
Number of gray squares	1		1			…		

Explain how you have got to the number of squares of each colour in the 8th figure, using the table. Is it possible to find in this sequence a figure with exactly 50 white squares and one gray square? Explain what your thoughts to answer the question were. Did you find any regularity in the sequence of white squares? And in the gray squares? Write a small text on the conclusions you have reached. Can you establish some relation between the number of the figure and the number of white squares? Justify your answer. What is the rule for this pattern?

Task type 4 (sequential numeric)

It's a typical task in which the students are required to identify the pattern

Observe carefully the sequence of numbers below:

6, 10, 14, 18, 22, ___, ___, ___, …

Place numbers on the lines. What will be the 10th term of the sequence? And what will be the 20th term of the sequence? Explain how you have found the 20th term of the sequence. Will the number 63 be a part of this sequence of numbers? Justify your answer.

The tasks have been answered in a group work setting. After the groups have solved, they had to present their solution with the justification to the whole class and a discussion followed, contributing to provide a sense of purpose and ownership (Ainley and Pratt, 2011). In these classes, one group was chosen from and followed from the start, with their dialogues being recorded. The second author participated as a second teacher, after obtaining permission from the school board and from parents.

These tasks have been introduced in two classes after the pre-test. In one of the classes, the students were confronted with the sequence of sequential tasks, and in the other class students were confronted with the sequence of structural tasks. This has happened near the end of the schooling year and it was followed by the application of the post-test, which happened in the end of May. After vacation time, in September, being the students already on the 4th year of schooling, the test was applied again. The intention was to check for retention. Then students were confronted with the sequence of tasks they had not solved yet. We had then one class solving six sequential tasks and then six structural tasks and another class solving six structural tasks and then six sequential tasks. All students in the two classes solved the same set of twelve problems. Finally, the test was applied one last time, in November.

2. Results and implication for redesigning

The main intention is to introduce a cycle of redesigning interpreting the students' answers both as qualitative and quantitative data. A systematic analysis of student work is not thoroughly described because of space limitation. Contrary to our expectations what these results show is that it is not indifferent to start with sequential or with structural tasks. Sequential tasks are better for starters and apparently provide a solid foundation for the work with structural tasks. As to the two groups that have been followed in more detail, the results have been these:

Structural/sequential group

	April 2011	June 2011	September 2011	November 2011
Daniel	146	156	156	156
Bert	114	144	120	152
Arthur	54	104	53	134
Cindy	42	72	57	69

Sequential/structural group

	April 2011	June 2011	September 2011	November 2011
Gisela	72	102	144	120
Helen	74	93	117	132
Earnest	42	128	117	156
Frank	114	141	156	156

We have taped and later transcribed the dialogues in one group for each class. We will focus on the first two tasks of each type and compare their resolution in the two classes. In terms of the designed tasks, both pairs reached, at the initial stage of predictions, generalizations expected by the designers. In particular, we found some inductive generalization based on the collection and analysis of data (as followed by the sequence of tasks in this activity), but nobody did any deductive reasoning. Let's explain some observations about the tasks above exemplified.

Concerning task 1 described above, which for the structural>sequential class (StSe) was the first task and for the sequential>structural class (SeSt) was the seventh both groups initially fell for attempting trial and error. Both groups easily found the value for the star. Group StSe had some difficulties in concluding that the triangle and the rectangle had to have the same value. The main reason apparently was that being the shapes different, they assumed the value had to be different. They made several attempts with different values. The researcher asked them if they had observed all the scales and from this prompt they returned to the observation of the scales and concluded immediately that the triangle and the rectangle had to have the same value and solved the rest easily. Group SeSt tried at first to guess the final scale value. But abandoned this strategy, found the value for the star and through the analysis of the third scale they found the values of the triangle and rectangle and solved the problem.

Concerning task 2 described above, it was the second task for the StSe group and the eighth for the SeSt group. Group StSe has approached this problem with a strategy based on the difference between the entry number and the exit number. As one of the machines presented the number 10, they started conjecturing with it, as calculations were easier. Eventually they perceived the strategy was not working. Next they tried to establish the difference between the exit number from one machine and the entry number of the following. They gave it up and tried two things: one was comparing the sum of entry and exit numbers of one machine and the entry number of the following, the other was to compare the sum of the three entry numbers of three machine and the exit number of the third. They continued attempting adding and subtracting numbers until they moved to a multiplicative strategy, but this move resulted from an ill constructed comment from the teacher, giving too much away. After this they revealed no difficulty in finding the relation or the rule. Group SeSt rapidly found the value that should be placed in the question mark. Therefore we could not conclude if that was due to the establishment of relations between the numbers or to luck on the first attempt. In the explanation that followed they just describe the calculations performed:

Helen – it's 5 times 3.
Frank – 5 times 3?
Helen – plus 1.
Earnest – it's 5 times 3.
Helen – 7 and 7 is 14 plus 7, 21; 21 plus 1, 22… it's 28 (pointing with the finger to the question mark). 9 and 9 is 18, plus 9, 27.
Earnest – It's 28 (pointing to the question mark).
Helen – 28 (while pressuring Frank's arm, who is responsible for the writing).
Frank – calm down!
Helen – we have to write it down.
Earnest – now we have to explain why.

Concerning task 3 described above, it was the second task for the group SeSt and the eighth for the group StSe. In the group SeSt, one of the elements refers that it is always 'plus two' (speaking to the group partners) while the researcher was still presenting the task, adding that it was always one above and one below:

Earnest – Ah! Figure six has to have on the top 7. Figure…
Frank – eight…
Helen – it has to be 6 down here and the figure eight? Figure seven how much does it have?
Earnest – figure seven, figure seven, figure seven… ah, figure six has 7 here and 6 here and figure seven…
Frank – 8, it has 7 here and here 8.
Helen – 9 and 8
Earnest – what? Figure eight? Then write it.

They work line by line, interiorizing that the top line has one more square than the line below. They do not reveal any difficulty filling the table, establishing the relation 'double' between the figure number and the number of white squares, identifying the invariant (gray square). In the question about wherever there was a figure with exactly 50 white squares and one gray square, one of the group elements refers immediately that it is possible, since they're even (white squares) plus one gray square. In the question about the relation between the figure number and the number of squares they find the relation and manage to establish a distant generalization in the oral conversation, referring that it is the double plus one, however they fail to say so in the written answer.

The group StSe identifies immediately the number of white and gray squares in figure 8, and they reveal no difficulty in filling the table. As to the question about the existence of a figure with 50 whites and one gray, they rapidly answer that it is figure 25. In the question about the relation between the figure number and the number of squares, they find a near generalization mentioning that 'the number of whiter squares doubles is two more than the previous one and the gray stays'. This group has no difficulty finding the relation between the figure number and the number of squares, but does not offer a justification.

Concerning task 4 described above, it was the fourth for the group SeSt and the tenth for the group StSe. Group SeSt have initiated solving this task by searching for a regularity between consecutive terms (plus 4 than the previous term) and later they found that to find a distant term they could multiply the term number by 4 and add 2. They justified that 63 was not a part of the sequence because it was an odd number. In the last question about the regularity found they wrote: "the order number times 4 plus 2 gives the number for the term. They are all even numbers". Group StSe does not reveal difficulties in finding that the difference between consecutive terms is 4. Meanwhile they discover they can multiply the term number by 4 and then add 2, and have written 20x4+2=82 to find the 20^{th} term. However in the written explanation they show some confusion and get into contradictions. As to 63 being part of the sequence, they claim that it is not multiple of 4 (omitting the ulterior adding of 2 units). They say: “all multiples of 4 are even”. Last, in relation to the regularity, they write: “the quadruple of the number for a term plus 2 is equal to the number of the sequence of the term”.

Conclusions

First of all, we know from many previous experiences that structured investigative activities indicate that they provide opportunities for meaningful learning of mathematical concepts. We also know that differences between a designer's planned actions and student work should be expected. One can argue that, these students were unfamiliar with structural algebraic tasks, and they answer them as open questions, making their own interpretation. Cognitive analysis seems not to be enough to decide about ordering. The study is considered as a first step for reconsidering the tasks for redesign in which a new cycle of testing could lead to small or big changes. Apparently, it seems that only because of epistemic values, we consider structural tasks after. It is however clear students that started with the sequential tasks seem to be capable of establishing distant generalization when the other group couldn’t. And certainly the group that started with sequential tasks appears to retain their performance more robustly stable across time. This type of task design proved to be a rich starting point for significant classroom discussions on early algebraic situations.

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A Task Design for Conjecturing in Primary Classroom Contexts

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The purpose of the study was to design tasks for conjecturing to support students engaging in the activities of proving in primary classroom contexts. A task involving perimeter and area of figures as an example was designed on the basis of nine principles of designing for conjecturing and proving suggested in F. L. Lin's team work (Lin, et al., 2012). Twenty seven third grade students' written work was collected for the main data of the study. It was found that the task was characterized as four features in accordance with students' justification, which were analyzed by three components of proofs suggested by Stylianides and Ball (2008) .

Key Words: mathematical conjecturing, proving, task design, primary,

Introduction

Mathematical tasks are recognized as the tool for teachers to shape a teaching design and for students to develop, utilize and understand a certain concept (Stein, Grover, Henningsen, 1996). Thus, mathematical instruction is generally organized and delivered through students' activities on mathematical tasks. Nevertheless, students respond to mathematical tasks very differently, depending on the structure and demands shaped by tasks enacted by teachers. This indicates that to reach high quality of mathematics instruction, mathematical tasks play a crucial role. Thus, selecting and designing appropriate tasks is essential to the success of teaching mathematics (Doyle, 1988; Stein & Lane, 1996).

Doyle (1983) defined academic tasks as (a) the products that students are to formulate, such as the answers to a set of questions; (b) the operations that are to be used to generate the product, such as classifying examples of a concept; and (c) the "givens" or resources available to students while they are generating a product. Doyle (1988) further argues that tasks with different cognitive demands are likely to induce different kinds of learning. According to this definition, tasks can vary not only with respect to mathematics content but also with respect to the cognitive processes involved in working on them. Only worthwhile tasks offer students the opportunity to extend what they know and stimulate their learning. Tasks that require students to solve complex problems can be considered to be cognitively demanding tasks. In contrast, cognitively undemanding tasks are those that give less opportunity for the students to engage in high-level cognitive processes. The conjecturing tasks require high cognitive demands because they are involved in three components: a set of true statements, valid modes of argumentation, and appropriate representation of modes of argumentation (Stylianides, 2007). Therefore, it is worthwhile to develop teachers' knowledge of conjecturing tasks design for enhancing students' proofs.

Moreover, proof is a vehicle to enhance students' understanding of mathematics concepts and promote mathematical proficiency and reasoning (Hanna, 2000). Proving is an important means of exploring in mathematics. Research shows that engagement in proving can support students to explore why things work in mathematics and explain their disagreements in meaningful ways, thus providing them with a solid basis for conceptual understanding (Stylianides, 2007). The previous studies suggest that students should have early and appropriate opportunities to incorporate proof into their mathematical learning (Ko, 2010; Kilpatrick, Swafford & Findell, 2001;Stylianides, 2007).

Conceptual Framework for Designing Conjecturing Tasks

Conjecturing and proving can promote mathematical thinking and launch mathematical inquiry; therefore, tasks of conjecturing and proving should be designed to be embedded into any grade level of classrooms (Kilpatrick, et al., 2001). The tasks of both conjecturing and proving are involved in students' conjecturing, but F. L. Lin and his colleagues distinguished the distinction of the principles for conjecturing tasks design from those for proving task design (Lin, et al., 2012). The four principles for designing conjecturing tasks and the five principles for designing proving tasks are considered as the conceptual framework of the study, since they are achieved to two fundamental functions: relating to the learners' roles or hypothetical learning trajectories and the practical function of easily evaluating.

In the tasks for conjecturing, students are asked not only to generate conjectures according to the given information which could be ill-defined, but also search for proofs to verify or justify whether the conjectures they made are true or not. They further suggest that the designing efficient tasks for conjecturing should consider the provision of opportunities to: (1) observation, (2) construct, (3) transform, and (4) reflect. The observation-based conjecturing refers to activities that involve purposeful or systematic focus on specific cases in order to make a generalization about the cases. The construction is a principle that encourages students to construct new knowledge based on prior knowledge which may lead to conjectures. The transformation design principle means that the task gives students the opportunities to generate conjectures by transforming given algorithms or formula. The conjectures by transformation may lead students to incorrect or meaningless statements. Thus, the reflection principle is essential to design the tasks for conjecturing.

The tasks for proving can promote proofs and proving. The type of tasks for proving is characterized as to ask students to justify whether the given statement in the existing conjectures is true. The false or true statement is given by the instructor's. Lin and his colleagues suggest that the tasks for proving have different difficulties, depending on the sources of conjectures are either from instructor's or students' themselves (Lin, et al., 2012). In the case of conjecture given by the instructor, students may feel it is not necessary to prove it, because they accept the truth based on either epistemic values (Duval, 1998) or instructor's academic authority (Lin &Tsai, 2012a; Reid, 1995). In the case of conjecture given by students themselves, they may have unclear distinction between conjecturing and proving. For instance, they may consider empirical arguments as deductive proof (Stylianides and Stylianides, 2009).

The five principles for designing proving tasks suggested by Lin's team have to do with modes of argumentation and its representations. They include: classifying mathematics statements, expressing arguments in several modes, changing roles in a task, defining efficient and necessary proof, and creating and sharing proof. The first principle is to provide the opportunity for students understanding six possible types of statement between universal and existential statements. It could be always true, sometimes true, and never true for true or false statement, respectively. The second principle is to provide the opportunity for students to understand various modes of argumentation to be used appropriately for different types of statement, such as counter examples, supportive examples, deductive and inductive proof via various representations. The third designing principle promotes the opportunities of switching the role of students as an instructor or an evaluator of the proposed justification from traditionally being seen as the learner. The fourth principle provides students the opportunities to be aware of the sufficiency and necessity for a legitimate proof. The final principle is to provide students the opportunity of creating a proof and then present it to the whole class in public for verifying its validity and truth.

A statement, existing in the tasks for proving, proposed by the instructor can be logically true or false. The existing empirical studies suggest that students at primary or secondary level do not accept counterexamples as refutation, rather, they offer more than one counterexamples for refuting a false statement (Reid & Knipping, 2010). Student even at high school level or undergraduates still have difficulty with mathematical proof in school mathematics (Lin &Tsai, 2012a; Ko, 2010). Some of them accept the truth of empirical induction from finite number of discrete cases for verifying a true statement. Besides, students or future teachers also perform better on false statement than true statement (Lin &Tsai, 2012b). The different difficulties may result from the nature of proving, because the verification of a false statement is easier than a true statement in that one counterexample is enough to refute a false statement. A true statement needs to be verified by inductive or deductive reasoning. The processes of proving a true statement demand rigor and complex argumentation.

On the basis of literature review, it could be a good start for the beginners at primary level for the study to learn proofs with a false statement instead of a true statement for designing proving tasks. However, it is new experience and knowledge that the task design for conjecturing are conducted by the primary teachers involving in the study, because their students have little experience with what conjecturing and proving looks like. Thus, the purpose of the study is intended to support teachers to design various conjecturing tasks in line with the mathematics contents to be taught in primary classroom contexts for student exploring the activities of proving. The paper describes not only the development of the task design but also the effect of conjecturing task on enhancing students' engaging in the activity of proving in the context of the relationship between perimeter and area in two figures. The task referred to in the study was the activities or artifacts such as teaching aids exploring in classroom contexts.

Method

Participants and Context

The task involved in the study was designed by one of the teachers who participated in the first year of a three-year project that was designed to help teachers to create conjecturing tasks for engaging students in the activity of proving. Hence, the tasks design for students engaging in valid proofs is a new experience and novice learning for the teachers, but they were mutually supported in the professional team consisting of six teachers and two researchers, the authors of the paper.

Twenty-seven third grade students in the class have separately learned the concepts of perimeter and area of a figure before engaging the task. They engaged in several activities of conjecturing in the first semester. The task was conducted in the second semester, so that some of the students had slight experience of finding out a counter example to refute a false statement.

They were grouped heterogeneously in groups of 4 or 5. After given the task, the students first worked independently and jotted down their judgement and verification on B4 paper; then they came together in groups to compare their solutions, and finally they shared their arguments to the whole class. The lesson was videotaped throughout the entire class. Each student’s written work was collected throughout the whole year.

Designing Conjecturing Tasks for Students Engaging in the Activities of Proving

The task designed by the teacher was to ask students to make a conjecture and verify whether it is true. The statement is that “*In any two figures, if the area of one figure is bigger than the other, then its perimeter of the figure is greater than the other, too. Do you agree? Why? Show your work on the grid paper.* ” The task for conjecturing is initiated from a false statement. The task design was on the basis of the four principles of the task for conjecturing and five principles of the task for proving, suggested by F. L. Lin’s team work (Lin, et al., 2012), since it is potential to launch the following activities for students engaging in conjecturing and proving: (1) The task provided students an opportunity to engage in *observation* through finding out a pair of two figures and making a generalisation about the cases; (2) The task provided students an opportunity to engage in *construction*. For instance, to solve the task, students needed to create two figures with different areas but same perimeter; (3) The task encouraged students an opportunity to *transform* prior knowledge of the perimeter and area of an irregular figure by counting the number of small squares on the grid paper; and (4) The task provided students an opportunity for *reflection*. For instance, “*Show your work*” as part of the task is to ask students to explain why they believe their conjecture is true for the given condition.

In addition, the task was also characterised as the following features: (1) The task promoted students classifying various statements, such as, the statement of same area in two figures result in same perimeter; (2) The task had the potential to require students to express same argument in several representations of modes of argument. The counter examples for refuting the false statement can be expressed by either “*bigger area but smaller/same perimeter*” or “*same area but greater perimeter*”; (3) The task engaging in the classroom provided the students as the role of an evaluator for the justification proposed by them; (4) After the false statement was proved, the students were asked to modify it and rephrase it as a true statement; and (5) The task engaged in the classroom context provided students an opportunity with creating and sharing their own proofs.

Data Collection and Analysis

The result session was aligned to the Stylianides and Ball's (2008) three components of proof by using students' written work. Taking the consideration of students as mathematical learners, proof suggested by them can be defined as a set of *accepted* statements, *known* modes of argumentation, and *accessible* modes of argument representation to a classroom community. The three components of proofs as the framework of analyzing the data collected for the study, students' written solutions were first split into two piles and then a pile was assigned to each group of the two groups consisting of six school teachers studying in a master program. Afterwards, they took turns to review the other pile for increasing the validity and reliability of analysis.

Results

*The Set of Statements Accepted by the **T**hird **G**raders*

After the task was explored, 8 (30%) students made incorrect judgment by accepting the teacher's statement, while 19 (70%) students judged correctly and verified successfully by finding out a pair of figures to reject the conjecture given by the teacher.

The set of statements referred to the statements accepted by the classroom community. Once they figured out a pair of figures, various accepted set of statements as part of their arguments were generated by 13 students, such as "*bigger area but smaller perimeter*", "*bigger area but same perimeter*", "*smaller area but same perimeter*", or "*same area but greater/smaller perimeter*". For instance, Ming, Jenny, Ron, and Huei, as the examples, successfully showed their accepted statements by their classmates, as shown in Table 1. Thus, possessing various types of statements for students' classification is the first feature of the task.

In addition, the statement "*The bigger area in one figure, it is not necessary to be greater in perimeter*." that 6 students used was another type of statement. Starting from the condition "*bigger area in one figure than the other*" given in the instructor's statement was the most common statements accepted by those who were in favor of the conjecture (in total, 68%, 13 out of 19).

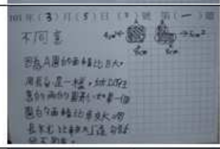
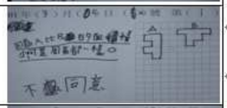
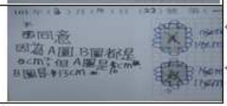
Students Frequen.	Various accepted statements	Examples of the set of statements proposed by students	
		English version	Chinese version
Ming 7	Bigger area but smaller perimeter	Disagree. Because the area of figure A is bigger than B, while the perimeter of figure A is smaller than B.	
Jenny 2	Bigger area but same perimeter	Disagree. Because the area of figure A is bigger than B, while their perimeters are same. Thus, I do not agree with the statement "*In any two figures, the bigger area of a figure has greater perimeter*".	
Ron 1	Smaller area but same perimeter	Disagree. Because the area of figure A is bigger than B, but their perimeters are same in length.	
Huei 2	Same area but greater/smaller perimeter	Disagree. Because the area of figure A and B are 5 cm^2, but the perimeter of figure A and B are 10 cm and 13 cm, respectively.	

Table 1: The Set of Statements Accepted by the Third Graders.

Modes of Argument Known by the Third Graders

Modes of argument are the ways of verifying or justifying a statement. Overall, the mode of argument for the task used mostly by the third graders was the use of counterexample. They seemly knew that one counterexample is sufficient to refute a false statement. Their modes of argument for this task were not like other tasks that students are used to offer more than one counterexamples for refuting a false statement (Lin & Tsai, 2012a). Thus, the task provided the best opportunity for promoting students' understanding on proving that a single counterexample is sufficient to refute a false statement. This is the second feature of the task.

For verifying or refuting the statement given by the teacher, the students needed to find out a pair of figures such that one area in one figure is bigger/smaller than the other. It is followed by observing the relation of their perimeters. To fulfil the work, students needed clear distinction between the two concepts: area and perimeter. Sometimes, they needed to attempt several times, as shown in Figure 1. The Figure 1 displayed that the task was not only providing students an opportunity for exploring the activity of conjecturing and proving but also for clarifying students' confusion of perimeter with area. This is the third feature of the task.

Figure 1: Student' Tries for Finding Out a Pair of Two Figures

Representations of Modes of Argument Accessible by the Third Graders

The representations of modes of argument are the forms of expression for communicating with the classroom community. This conjecture task involving the relationship of perimeter and area, word expressions with figures was the most popular form of argument accessible by the third graders. As shown in Table 2. Eighteen (67%) students verified the statement by drawing at least an irregular figure in the pair of figures on the grid paper. It seems that third graders readily draw an irregular shape on the grid paper. As a consequence, it made students successful in conjecturing and justifying the statement. This is the fourth feature of the task.

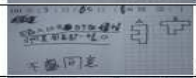
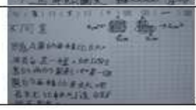
Shapes of a pair of figures	Frequencies	Examples
Two rectangles	9	
Two irregular shapes	8	
Two circles	1	
A rectangle and an irregular shape	7	
A square and an irregular shape	2	

Figure 2: Student' Representation of Figures

Conclusions and Discussions

The quality of task design was evaluated by students' justification on the basis of the three components of mathematical arguments, suggested by Stylianides and Ball (2008). It is said that the task maintained high cognitive demands while it was implemented into a third grade classroom, since the task was characterized as the following four features.

(1) The task provided students new experience that there were various types of statement instead of a single type for a counterexample to refute a false statement. It has to do with the first component of proofs: a set of statements. The students have developed knowledge of proofs for refuting a false statement by unique type of statement for a counterexample from prior tasks. This feature meets the first principle of task designing for proving suggested by F. L. Lin's team (Lin, et al., 2012).

(2) The second feature was that the task provided the best opportunity for promoting students' understanding of a single case as a counterexample to be sufficient to refute a false statement. This feature is documented from the second component of proofs suggested by Stylianides and Ball (2008). The task was beneficial for developing students' knowledge of the way of refuting a false statement. This was different from previous studies in which most of the students at primary or secondary level were used to utilize more than one counterexamples to refute a false statement (Lin &Tsai, 2012a; Reid & Knipping, 2010).

(3) The task, which provided students an opportunity not only engaging the activity of conjecturing and proving but also clarifying students' confusion of perimeter with area, was characterized as the third feature. The feature of clarifying students' misconception or confusion between concepts is not on the list of task design principles suggested in F. L. Lin et al.'s work. The result indicated that the task could be a powerful instructional approach via conjecturing for clear understanding on perimeter and area. However, the effect of conjecturing on learning the relationship of perimeter and area needs further study in the future.

(4) The final feature was that the provision of grid paper as part of the task has potential to make students' successful in conjecturing and proving. The pictorial representation in several modes of argument was matched with the second principle of task design for conjecturing suggested in F. L. Lin et al.'s work (Lin, et al., 2012).

The study suggested that knowledge of students' mathematics concepts and teachers' knowledge of proofs embedded in the conjecturing tasks were two essential factors affecting the quality of proving exploring in the conjecturing activity. Without solid mathematical concepts underpinning the arguments, it is impossible for students to produce logical proofs. The study also suggested that the provision of false statements instead of true statements made it more possible for students to learn successfully on acquiring the knowledge of conjecturing and proving.

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Using Student Solutions to Design Follow-up Tasks to Model-Eliciting Activities

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Model-Eliciting Activities (MEA) are open ended mathematical tasks in which students develop a mathematical model to solve a real world problem. Their open-ended nature often results in students developing and articulating a great diversity of mathematical ideas. What tasks can help students extend these ideas after the MEA has been completed? We describe an innovative approach for designing Follow-up tasks to MEAs, which requires students to analyse other students' solutions to the MEA they have just completed. As students critique and analyse the mathematical strengths and shortcomings in other students' solutions, they simultaneously challenge and extend their own mathematical thinking.

Keywords: Mathematical modelling; task design; model eliciting activities

Introduction

Peter's story: As a teacher, I've often come across intriguing contexts and wondered if I could turn them into mathematical tasks for the classroom. An advertisement in the situations vacant column, asking for a chicken sexer: "*An extremely high level (98%) of accuracy is imperative. Speed is vital, sexing approximately 1000 chicks per hour*." Or the following description of a YouTube video: "*Alexander Overwijk draws a perfect freehand circle 1m in diameter in less than a second*." Experience has taught me that turning these contexts into effective mathematical tasks is not an easy thing. Often, the context outshines the mathematics. Other times, the mathematical content is too fleeting, trivial, broad or complex. I have folders filled with this kind of raw material but for some time, I lacked the design tools to craft them into mathematical tasks. I was intrigued when I heard Caroline describe some task design principles she was using in her research, while I was on a teaching sabbatical at a local university.

Caroline's story: In that talk, I described a sequence of four calculus tasks that I had developed together with Tommy Dreyfus and Mike Thomas. Two of the calculus tasks (described in Yoon, Dreyfus & Thomas, 2010) were created using well-established principles for designing special modelling tasks called Model-Eliciting Activities (MEAs) (Lesh, Hoover, Hole, Kelly & Post, 2000). But the other two calculus tasks were created without any specific guiding design principles. I had just written a grant proposal to develop design principles for the latter kinds of tasks—tasks that followed MEAs and extended students' thinking further. As fate would have it, I was looking for teacher-designers to participate in the project at the same time that Peter was looking for task design principles.

A team of designers: The proposal was funded, and we spent the next two years (with two other teacher-designers, Anne Patel and Nikki Sullivan) designing, testing and revising tasks and their underlying design principles. We applied Lesh et al.'s (2000) design principles to the context of the perfect freehand circle YouTube video to create an MEA we called the Giotto MEA (described below). When we tested the Giotto MEA in classrooms, students typically developed and articulated a diversity of mathematical ideas, but their written solutions often had numerous shortcomings. Peter proposed that these imperfect student-generated solutions were rich fodder for designing Follow-up tasks to MEAs. He proposed that students could extend their own mathematical thinking by analysing the gaps and strengths in *other* students' solutions to the MEA. We will describe how we used this approach to design Follow-up tasks to MEAs, as illustrated in one Follow-up task to the Giotto MEA. We identify with Theme B working group, which aims to "understand how appropriate task design might help minimise the gap between teacher intentions and student mathematical activity". Our Follow-up tasks address this aim as they are designed around students' perspectives (indeed their very solutions) to the initial MEA.

The Giotto MEA

Model-Eliciting Activities (MEAs) are a class of mathematical modelling tasks where students develop a mathematical model in response to a real world problem. MEAs do not stipulate what that model should look like, but only what the model should be able to do. This open-endedness typically leads to MEAs eliciting a diversity of mathematical approaches from students. Consequently, MEAs have been used to identify creatively gifted mathematics students (Chamberlin & Moon, 2005) and to investigate equity issues in undergraduate engineering courses (Diefes-Dux, Hjalmarson, Zawojewski & Bowman, 2006).

We created the "The Giotto MEA" (see website http://icmi22radonich-yoon-paper.wikispaces.com/home) according to Lesh et al.'s (2000) principles for designing MEAs, which are summarised in Table 1. The problem begins with a comic that tells how the Renaissance artist, Giotto, gained the pope's attention by drawing a perfect freehand circle (see Figure 1). After reading the comic, students are asked to draw their own freehand circles and choose the best among them. Next, they watch a short YouTube video of a mathematics teacher who professes to be the world's freehand circle drawing champion and appears to draw a perfect freehand circle.

Figure 1 Students read this comic about the artist Giotto drawing a perfect freehand circle

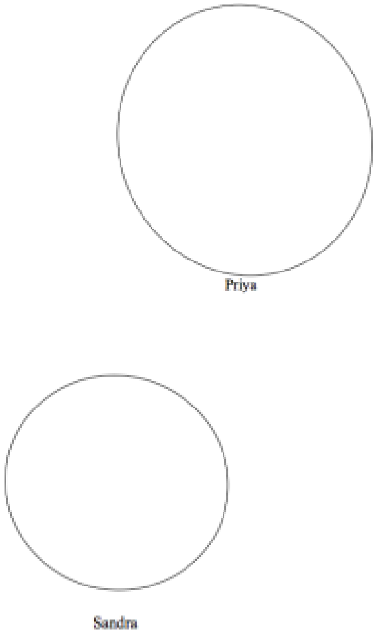


Figure 2 Students are asked to demonstrate their method on some circle attempts.

Students then meet the problem statement, which introduces them to a client, Bonnie, who is holding a circle drawing competition at the local Pancake House. The students are asked to work in teams of three to develop a method for ranking circle attempts from most circular to the least circular, which Bonnie can use to judge the circle drawing attempts on the night of the competition. Students are asked to test their method on some examples of circle attempts (e.g. Figure 2), but their method must also work for any circle attempt that could be drawn on the night of the competition. The student teams write their final method in the form of a letter to Bonnie.

Reality principle The activity should present a problem situation that students can interpret using their own personal knowledge and experiences.	*Model-documentation principle* The activity should require students to describe their mathematical models to others.
Model-construction principle The activity should create a convincing need for a mathematical model.	*Model-generalisability principle* The activity should require a reusable and sharable mathematical model.
Self-assessment principle The activity should articulate the criteria that are used to judge the students' final models.	*Effective prototype principle* The activity should be as simple as possible, while still creating the need for a mathematical model.

Table 1: A summary of Lesh et al.'s (2000) six principles for designing MEAs

Mathematical and epistemological perspectives for analysing student solutions

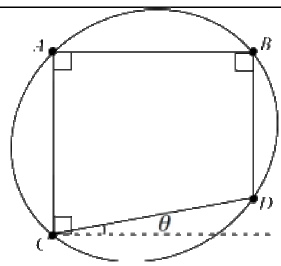
When analysing the mathematical approaches in students' solutions to MEAs, we draw on theoretical perspectives that focus on the *mathematical structures* that students attend to, such as modelling perspectives (Lesh & Doerr, 2003; Niss, Blum and Galbraith, 2007) and Mason's theory of the structure of attention (2004). These perspectives characterise mathematical structure as consisting of mathematical objects or elements (such as counts, measures, sets), attributes of those elements (e.g. few, large, open), operations (e.g. combine, enlarge, invert), and relationships (e.g. greater than, equivalent to). Accordingly, when analysing students' solutions, we concentrate on identifying features the mathematical objects, attributes, operations and relationships that students perceive and work with.

Additionally, we acknowledge that a person's perceived mathematical structures can only be identified through the observable signs the person produces and works with. According to Arzarello, Paola, Robutti and Sabena (2009), these signs or semiotic resources may involve spoken language, gestures, written text, symbols, diagrams, graphs and physical artefacts. Consequently, we analyse the semiotic resources that students produce in order to identify the mathematical structures the students appear to be perceiving, constructing, recalling and manipulating.

Student solutions to the Giotto MEA

In order to demonstrate the diversity of mathematical approaches that are often elicited by the Giotto MEA, we describe some student solutions from one classroom implementation. This class consisted of 29 year-11 (15 year old) students from a New Zealand suburban school with a cross section of socioeconomic groups. Prior to the implementation, the teacher had taught properties of circle geometry to the class. Students worked in groups of three (or two) on the Giotto MEA during one 50-minute class period. A researcher collected the ten group letters as well as written notes students made while working on the activity.

The ten group letters described ten different methods, involving more than 60 different mathematical elements, three different ways of constructing a "centre", four different constructions of a new shape to be used to measure the circle attempt's circularity, and six different measures of the circle attempt. This diversity of mathematical thinking is too large to present fully in this short paper, so we highlight some of the different mathematical elements, relationships and operations in solutions from three groups in Table 2 below.

Penny and Carol: Construct a line segment connecting any two points A and B on the perimeter of the circle attempt. Construct two perpendicular line segments from points A and B to the where they intersect the perimeter (C and D). Construct a line segment between points C and D. Construct a perpendicular line from point C. Measure the angle θ between line $\overline{CD}$ and the perpendicular line. Repeat this process 2 or 3 times, and find the average angle. The circle attempt with the smallest average angle wins.	 Author-generated diagram of Penny and Carol' solution

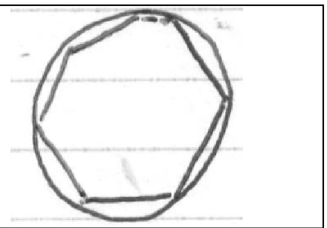
Mike, Mark and Justin: Mark 6 points on the perimeter at 60°, and join adjacent points with line segments. Measure the length of the last line segment, and compare with the lengths of the remaining line segments. The circle attempt whose last line segment is closest to the other line segments wins.	
Nate, Ward and Jim: Construct an exscribed square around the circle attempt and use the intersection of the square's diagonals as the centre. From this centre, find the longest "radius" of the circle attempt and use this to construct a perfect circle from the centre previously found. Measure the area of the perfect circle that is not covered by the circle attempt – this is called the margin of error. The circle attempt with the smallest margin of error wins.	

Table 2: Our summaries of three groups' solutions to the Giotto problem

Penny and Carol's method focuses on relationships between interior angles of an inscribed quadrilateral. They compare the size of the final constructed interior angle of the inscribed quadrilateral to the ideal value of 90°. Mike, Mark and Justin also set about trying to construct an inscribed polygon (this time a hexagon), which should have certain symmetrical properties in a perfect circle. Whereas Penny and Carol focus on the symmetry of the interior angles, however, Mike, Mark and Justin compare the length of the final constructed hexagon side to the lengths of the other sides. Thus, the two groups' measures of circularity focus on different mathematical elements (interior angles and lengths of sides).

Nate, Ward and Jim's measure of circularity focuses on the difference in area between the circle attempt and a perfect circle, which is constructed using the longest "radius" in the circle attempt, and centred on the circle attempt's "centre". Their measure does not, however, take into account differing sizes of circle attempts, and consequently favours small circles (as does Mike, Mark and Justin's). Other solutions in the class attempt to mitigate this by using ratios rather than differences to compare actual and ideal measures. Our Follow-up activities capitalise on this rich mathematical diversity expressed in students' solutions.

Designing and testing Follow-up activities

We used a *Design Experiment* methodology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to develop the Follow-up activities and their underlying design principles. This methodology aligned well to our dual focus on theory and practice:

> Design experiments have both a pragmatic bent – "engineering" particular forms of learning – and a theoretical orientation – developing domain-specific theories by systematically studying those forms of learning and the means of supporting them (Cobb et al., 2003, p. 9).

We sought to "engineer" principles for designing Follow-up tasks that could be implemented immediately after a given MEA, and that built on the mathematics that students encountered in the MEA.

A core team of four designers with diverse expertise (teachers, researchers, and teacher educators) met every 3 weeks over two years to design, test and revise these tasks and their underlying principles. We tested tasks with intermediate, secondary, and tertiary-level students, as well as with pre-service and in-service primary and secondary school teachers. We collected participants' written work and video and audio recordings, and analysed these data to assess the effectiveness of the tasks and the underlying design principles. The results of these implementations led to over 40 Follow-up tasks and at least seven significant revisions of the underlying design principles (Yoon & Radonich, 2011), the latest of which are described in Table 3. Of these 40 tasks, about 25 were based on student-generated solutions to MEAs, and we describe one such task in the next section. We regard the practice of using student solutions a special design *feature or strategy* of some Follow-up tasks, rather than a design *principle* for all Follow-ups.

The Focus principle The Follow-up should focus on a particular aspect of mathematical communication, modelling, or conceptual understanding that is encountered in the initial MEA	The Consolidation principle The Follow-up should ask students to consolidate what they learn in the form of a sharable rule or guideline
The Challenge principle The Follow-up should challenge students to improve on areas of weakness or misconception that are commonly found in students' responses to the initial MEA	The Flow principle The Follow-up should flow on from the context in the initial MEA so that students are likely to relate their new knowledge to that which they developed in the MEA

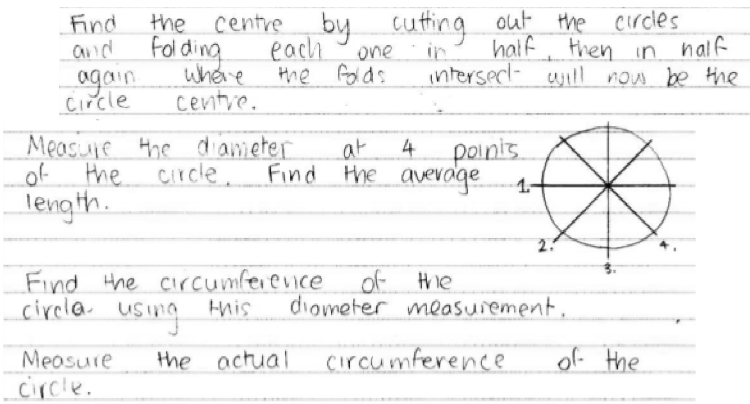
Table 3: Our principles for designing Follow-up activities to MEAs, from Yoon & Radonich (2011)

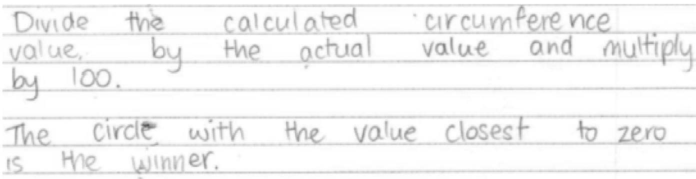
A Follow-up task to the Giotto MEA based on Sonya, Jun and Hayley's solution

We created a Follow-up task that focuses on the solution to the Giotto MEA by Sonya, Jun and Hayley (SJH), a group of students from the classroom implementation that was described previously (for the full task, see website http://icmi22radonich-yoon-paper.wikispaces.com/home). SJH's method (see Figure 3) begins with paper-folding to find a "centre" of a given freehand (imperfect) circle attempt. This "centre" is then used to obtain four "diameter" measurements of the circle attempt. Next, SJH instruct readers to calculate the circumference of a (perfect) circle whose diameter is the average of these four "diameter" measurements. A score of circularity is obtained:

$$\text{Score of circularity for circle attempt} = \frac{\text{calculated circumference}}{\text{actual circumference}} \times 100$$

The (imperfect) circle attempt that yields the score closest to zero is declared the winner of the circle-drawing competition.




Divide the calculated circumference value by the actual value and multiply by 100.

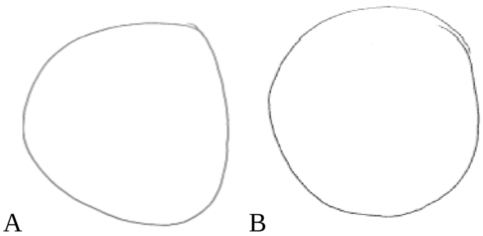
The circle with the value closest to zero is the winner.

Figure 3: Excerpts from Sonya, Jun and Hayley's method.

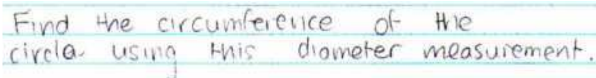
Although most aspects of this method are mathematically correct, one shortcoming is the final ranking, which assigns the attempt that scores closest to "0" as the winner, when in fact it should be the one closest to "100". This error formed the basis of the Follow-up task that we now describe.

After reading SJH's written letter, students are instructed to determine (merely by looking) which of two hand drawn circle attempts appear more circular (see Figure 4). These circle attempts were intentionally chosen so that one attempt (B) is more circular than the other (attempt A). Then, students are asked to determine which circle would be ranked as being more circular under SJH's method (see Figure 4).

Which circle attempt looks more circular?


A B

When Sonya, Jun and Hayley wrote:


Find the circumference of the circle using this diameter measurement.

They should have written:

"Find the circumference of a circle whose diameter is the average length of the four diameters in the circle attempt."

Apply this to the circle attempts *A* and *B*. You can use the formula: $C = \pi d$

Figure 4: Two questions in the Follow-up activity

At this point the students are led to notice that SJH's scoring system ranks circle attempt A as more circular, which contradicts their earlier, common-sense assessment that attempt B is more circular. Students are invited to examine the components of SJH's scoring formula to identify its mathematical flaws. As part of this process, students are instructed to apply the scoring system to a perfect circle, which should score 100. The final instruction in the task is for students to correct SJH's scoring system so that it works effectively, and to rewrite SJH's method.

Discussion and conclusion

The open-endedness of tasks like MEAs is both a blessing and a burden: such tasks elicit a multiplicity of ideas from students, but teachers may struggle to develop these ideas further (Stein, Engle, Smith & Hughes, 2008). We have presented a design approach that utilises this diversity by incorporating real student solutions to MEAs into Follow-up tasks for the same MEAs. One advantage of this approach is that it requires designers to analyse the thinking behind the students' solutions in great detail. This detailed analysis can yield fresh insight into what the students understand (or misunderstand), which can inform the design of subsequent learning tasks. Student misconceptions and mistakes can be particularly rich material for designers as they expose new mathematical avenues to pursue that the designers (who would not typically make those same mistakes) might not have considered themselves. For example, we found that the mistake in SJH's scoring system provided an opportunity to explore ways of comparing actual measures of circularity to ideal measures.

Another advantage of using students' written work in Follow-up tasks is that it shines the spotlight on *student* knowledge. Some of the Follow-up tasks we have designed are more typical of *teacher*-led approaches, which reinforce circle properties such as cyclic quadrilaterals and angles at the centre and circumference. Such tasks run the danger of emphasising the already prevalent perception that teachers are the source of mathematically correct solutions (English & Doerr, 2004), and students could interpret such tasks as subtle hints that these properties of circles should have featured in their solutions to the Giotto MEA. The use of student solutions to design Follow-up tasks can help avert the danger of students thinking the teacher-presented ideas are necessarily more correct than their own.

When other students read Sonya, Jun and Hayley's letter with its original handwriting and diagrams, they may make what is essentially a cultural connection. The Follow-up task is situated in the context of Sonya, Jun and Hayley: three mathematics students who have struggled with the Giotto MEA. Students working on the Follow-up based on SJH's solution may relate to them, as they too are mathematics students who have just worked in groups on the same problem. This connection may lead students to be more engaged and curious about exploring the mathematical concepts that other students described, which in turn, could help students reflect more deeply on their own method (Boaler 1993).

As part of our research, we obtained permission from students to use their solutions in developing future tasks, and we used pseudonyms to protect their identities. However, a teacher may wish to use their own students' work as a resource for creating Follow-up tasks, possibly even with the intention of using those tasks in classes where the students' identities may be revealed to their peers. In such cases, teachers need to be sensitive to students' fears about making their work public. Effort will be required to create a culture where discussing student work is a natural and safe part of the teaching and learning process. Our experience is that students are often less concerned about sharing group solutions with the class, and are more inclined to share individual solutions if names aren't revealed. Even if students give permission to share their solutions, however, teachers may not have enough time to prepare Follow-up tasks for the next lesson. In this case, we advocate using Follow-up tasks based on student solutions from previous years or from other schools or even countries.

This paper has emphasised the diversity of mathematical ideas that are evident in student solutions to MEAs. However, in our research we have found that student solutions also reveal a range of competencies in other related areas, such as mathematical modelling and communication. We have successfully used the same approach of using student solutions to create Follow-up tasks whereby students evaluate the strengths and shortcomings of other students' mathematical communication and modelling as a way of improving their own. A valuable feature of our design approach is that it enables us to create tasks that extend and develop a wide range of mathematical competencies in students.

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Tasks to Promote Holistic Flexible Reasoning about Simple Additive Structures

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In this paper, we focus on an approach that promotes holistic flexible reasoning with simple additive structures in arithmetic problem solving in elementary school. We propose to distinguish two paradigms in which additive problem solving tasks can be seen. We also propose an ethno-mathematical model to analyze the tasks and their implementations. In our experimentation, a new form of problem solving task was designed and tested. The results of our experimentation show that the proposed approach promotes the holistic additive reasoning in students.

Key words: elementary mathematics; problem solving; additive structures; holistic reasoning

Introduction

The current math curriculum for elementary school in Quebec pays a special attention to the development of students' problem solving skills. Some problems that involve using one addition or one subtraction operation can be difficult for some students until the age of 12-14 (Vergnaud, 2009). Many researchers (Carpenter, Fennema, Franke, Linda, & Empson, 1999; Carpenter, Moser, & Bebout, 1988; De Corte & Verschaffel, 1980; Gerofsky, 2004; Julo, 2002; Nesher, Greeno, & Riley, 1982; Riley, Greeno, & Heller, 1984; Vergnaud, 1982a) (to name few) studied the subject from different perspectives and stated that students have difficulty in acquiring a full and flexible understanding of addition and subtraction. Thus, it seems relevant to look at how to improve students' learning by helping teachers to develop new teaching practices.

Our team is conducting a 3-year research project funded by the Quebec Ministry of Education on additive problem solving in early grades of elementary school. The goals of the project are: 1) to develop a pedagogical approach that would promote holistic and flexible reasoning about simple additive structures; 2) to design and test a set of tasks and didactical scenarios that implements the new approach; 3) to propose a related teacher professional development program. Our research team consists of two researchers (Savard and Freiman), a designer (Polotskaia), and a school board consultant responsible for the teachers' professional development (Gervais).

Answering to the request of theme B of the study group, we would like to clarify the following question. How can be the task of additive problem solving transformed the way to minimise the gap between teacher intentions and students activity? First, we will discuss the nature of additive problems as mathematical tasks to better understand their possible educative purposes. Then, we will provide a theoretical framework to analyze students' and teachers' behaviours regarding this kind of tasks in school settings. It will help us to see why students' actual problem-solving activity can differ from teacher's intentions. A short overview of alternatives found in the literature will provide us with specific pedagogical recommendations for the new ways of designing tasks related to word problems having additive structure. As support to our principles for such design we briefly discuss some preliminary findings from the first year of experimentations with elementary school teachers.

Two paradigms in additive problem solving

Nowadays, there are two paradigms in which additive problem solving can be seen. The first one, which we will call the *Operational Paradigm,* puts the focus on addition and subtraction as arithmetic operations. From this position, additive word problems can be seen as exercises where the knowledge about arithmetic operations can be applied or further developed. It corresponds to an order of teaching mathematics where students first learn about operations (how to add and subtract) and then try to solve different word problems to practice this knowledge and to get the *conceptual understanding of these operations*. This paradigm is clearly formulated by Brissiaud (2010) who argues that, "to have a conceptual knowledge of subtraction" means "to have **different senses** of the subtraction operation" like: finding the difference, finding the complement and take over. However, knowing what it means to add or to subtract two quantities, or how to add or subtract two numbers, is not enough to solve additive word problems (Vergnaud, 1982b). As follows from the research (Carpenter, Fennema, Franke, Linda, & Empson, 1999; Gerofsky, 2004; Nesher, Greeno, & Riley, 1982; Riley, Greeno, & Heller, 1984; Vergnaud, 1982) , the senses of subtraction-as-finding-the-difference or subtraction-as-finding-the-complement, are not easy to construct directly from the first and quite intuitive sense - subtraction-as-a-take over.

Several studies conducted in 1980-90s aimed to understand students' difficulties with word problems by proposing classifications of word problems according to their semantic and mathematical structures (Carpenter et al., 1999; Nesher et al., 1982; Riley et al., 1984; Vergnaud, 1982a). Contemporary research (Barrouillet & Camos, 2002; Nunes, Bryant, Evans, Bell, & Barros, 2011; Pape, 2003; Thevenot, 2010) shows that some problems are particularly difficult because they require a flexible and holistic analysis of their mathematical structure while easy problems do not require such analysis. Yet in the Operational Paradigm, additive structures are not seen as a primary mathematical knowledge, but only as different senses of two arithmetic operations.

A different paradigm, which we will name the *Relational Paradigm*, appears in the works by Davydov (1982) and more recent studies (Iannece, Mellone, & Tortora, 2009; Xin, Wiles, & Lin, 2008). According to Davydov (1982), the concept of additive relationship is, "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229). Davydov (1982) advanced that an adequate understanding of the additive relationship is the basis for the learning of addition and subtraction and should be taught prior to counting. According to this view, arithmetic operations are not the mean to understand a situation but serve as tools to modify the situation, ones understood, in a desired way.

In the Relational Paradigm, a problem solving task should be an occasion to analyse the additive relationships present in the situation. This yields the following task design principals.

1. The task should be based on a situation involving a simple additive relationship between three quantities.
2. The task should involve students into the mathematical analysis of the described relationship as a whole. It should help students to discover different properties of the relationship, and to see how different arithmetic operations can be used in the described situation for different purposes.

Existing formats of presenting tasks with additive word problem usually limits student's activity to the goal of finding a solution as final step of problem-solving process, and thus does not directly imply any in-depth analysis of the problem's structure which would help to develop more holistic view. Furthermore, problem solving as a school task has some psychological and social characteristics that, themselves and together with the classroom norms (Bosch, Chevallard, & Gascón, 2005), can potentially contribute to either widening or reducing the gap between pedagogical intentions and students' activity. The socio-cultural analysis is therefore needed to explicit teacher – student behaviour patterns during the activity and related didactic tensions.

Modeling teaching and learning situations for developing mathematical and citizenship competencies

A well-designed curriculum should help students develop profound mathematical knowledge. Furthermore, students should develop competencies applicable in an outside classroom context. To respond to these fundamental requirements, a good mathematical task should be anchored in a socio-cultural context, that reflects the culture and the society including the student's experiences inside and outside the classroom (Savard, 2008).

In order to analyze learning situations, and related tasks as a socio-cultural phenomena, we used an ethno-mathematic model - the Math and Citizenship competencies learning model - created by Mukhopadhyay et Greer (2001) and further developed by Savard (2008). According to this model, implementing problem-solving tasks should start with an analysis of the given "real world" situation involving its socio-cultural context. Then, the focus should move towards the mathematical context. The model compels students to take into consideration the relationships between the quantities involved. Mathematical operations are used to find a result. Then, the result should be evaluated and interpreted within the initial "real world" context (corresponds to the cycle described by (Novotná, 1998)) and eventually in a larger socio-cultural and political context to develop students' critical thinking (Lipman, 2003). Completing the learning cycle *real world–mathematics–real world* can potentially promote meaningful learning in students. Does the traditionally designed problem-solving task help the teacher to organize students' activity in this cyclic way?

The explicit main goal of a problem solving tasks is finding the answer. As described by Gerofsky (2004), very often the answer has no real value in the socio-cultural context of the child. Therefore, students don't see problem solving as an occasion for a meaningful discussion about real-world situations. They often see it as an exercise of addition or subtraction operations. This students' perception of the task prevents the learning cycle to be completed. The task is often finished at the moment when the numerical answer is found and validated by the teacher. A special effort is required from the teacher to organise the class work in a way where the learning cycle can be completed.

Moving too quickly to finding an answer can provoke a specific didactic contract (Brousseau, 1988) in students. For example, students might think that the teacher's expectation is to find the numerical answer the easiest way possible. Under the pressure of this contract, students can develop different strategies that help them rapidly translate the text of the problem to a mathematical expression. According to Hegarty et al. (1995), some students can recognize numerical data and key-words, such as *more*, *less*, *increase*, *take over*, and construct a mathematical expression in a straightforward way, translating these words to arithmetic operations. However, by using this "direct translation" strategy (Hegarty et al., 1995), students move too quickly to the arithmetic operation and do not pay attention to the underlying mathematical concepts and relationships. This problem solving strategy is very efficient for many problems but not for all of them. Yet, the success of using this strategy can prevent students from acquiring a more profound understanding of the mathematical structure of the problem.

The proposed above socio-cultural model (Savard, 2008) helps recognize the difficulties teachers may have while implementing problem solving tasks and obtaining the desired educational goal - a profound and flexible understanding of the additive structures. Furthermore, based on this model, some more task design principals can be deduced.

3. The task should use a socio-cultural context in which students can identify themselves as active agents.
4. The task should not contain any explicit and immediate questions that could be answered by finding one particular number. This criterion is to prevent students from immediately calculating the answer. However, the task should include an intriguing element, which would support students' natural interest and commitment.
5. The goal of the task, which is learning to analyze the situation, should be explicitly communicated to students.

The formulated task design principles do not correspond at hundred percent to the traditional word problem task organisation. What other organizations of such tasks can we find in the literature?

Task organizations overview

As we mentioned earlier, in order to solve difficult or complex problems, students should be able to see the mathematical structure of the problem in a flexible holistic way. In the literature, we can find many approaches and methods potentially supporting this ability development in students. Some of these studies (Bartolini Bussi, Canalini, & Ferri, 2011; Ducharme & Polotskaia, 2008; Gamo, Sander, &

Richard, 2009; Julo, 2002; Nguala, 2005) concern the use of different representations for the problems. Gamo et al. (2009) demonstrate that the comparison of problems and the use of different representations can help students develop efficient problem-solving strategies. Any graphical or schematic representation potentially gives students a rapid visual access to the entire system of quantitative relationships described in the problem. Therefore, using diagrams should promote the holistic vision of the problem in students. Comparison of problems (Bartolini Bussi et al., 2011) and multiple wording of the same problem (Julo, 2002; Nguala, 2005) are also shown to be beneficial for elder students. However, these tasks can be linguistically difficult for very young students (Ducharme & Polotskaia, 2008).

Some other studies (DeBlois, 2006; Neef, Nelles, Iwata, & Page, 2003) propose particular didactic management and class work organisation. Neef and her colleagues (2003) have shown that learning about the roles of each data element in the problem greatly improves the success in problem solving among students with developmental disabilities. DeBlois (2006) suggests that a request for feedback on the solved problem may provoke coordination between representations and procedures and may lead students to reorganize their thoughts. Both approaches clearly reflect the effort to reorganise students' reasoning about the situation in a holistic flexible way.

All mentioned above teaching approaches, explicitly or implicitly, promote in students the ability to see the problem as a whole and to better coordinate relationships between quantities involved. To reinforce this important aspect of the problem solving activity in young students the following principles should be followed.

6. The text of the task should be very short and should contain simple words and expressions that the students are familiar with.
7. The mathematical discussion of the situation should integrate appropriate graphical representations as a method of analysis.

Keeping in mind all formulated above principles, what concrete task organization do we propose?

Task description

We provide here one example of the task that we named **360° situation** to highlight the main goal – holistic analysis of the mathematical structure of the situation. This is an example of a text proposed to students.

> Peter, Gabriel and Daniel are playing marbles. Peter says, "I have 5 marbles." Gabriel says, "I have 8 marbles." Daniel says, "Peter has 4 marbles less than Gabriel".

We introduce this text as a strange situation or as a situation where one of the persons made a mistake. Students are invited to explain why the text is unrealistic and how it can be corrected considering different quantities involved.

The objective of the first is to make explicit the fact that all three quantities are related to each other and that the choice of two values implies one (and only one) third value. At the next step, we invite students to construct a graphical representation, which can support discovering of the appropriate arithmetic operations. Each quantity should be evaluated to figure out a correct numeric value in the condition where the other two quantities are fixed. At this step, the formal use of arithmetic operations can be discussed. Finally, the numbers in the text can be replaced with different ones to further generalise the initially discussed quantitative relations. This will complete the 360° tour around the situation.

Preliminary observation in the classroom

The first year of our project was implemented in rural public schools in Quebec. We worked with two experimental groups of grade 2 elementary students (32 students, age 7-8 years). One other group of students of the same grade (14 students) was observed as a control group. In the experimental group, teachers worked with new and traditional tasks of additive problem solving. The teachers from the experimental group received 6 follow-up sessions during the school year. In the control group, only traditional problem solving tasks design was used and the teacher did not participate in follow-up sessions.

Experiments with the 360° situation tasks started with manipulative activities where we discussed with students some methods and sense of comparison of lengths of two physical objects. We used coloured ropes and paper strips. The 360° tasks were used then to organize the analyse-and-representation activities for students to discuss different comparison situations. One of the central part of these activities was the construction of an “Arrange-all” diagram (Ng & Lee, 2009; Polotskaia, 2010) – a representation of the mathematical structure of the problem similar to the ropes we have just manipulated with. These diagrams were used as an analytical tool to find appropriate arithmetic operations for the given situation. In total, three 360° situation tasks were worked with students prior to solving traditional problems. During our sessions with teachers we recommended the use of similar discussions about all additive problems till the end of the school year.

Although a detail presentation and analysis of collected data lies beyond the scope of this paper, we would like to share some observations made in the experimental classrooms. Below is the partial script of the first lesson when the described above task was proposed to students.

> Teacher writes the story of Peter, Gabriel, and Daniel on the blackboard and asks students to read it aloud. She explains that one of the three friends said something wrong or made a mistake.
> Teacher, showing the three affirmations on the blackboard: Who said something wrong?
> A: Gabriel
> Teacher: You say Gabriel. Why?
> A: There are 8. It’s a lot. There are 5 and 4.
> Teacher: Ok, Who agree with A that Gabriel is wrong? [Some students raised their hands]
> Teacher: Does anybody think differently? Is it another friend who made a mistake? [Many students raised their hands]
> Teacher: You B, what do you think?
> B: I think it is Daniel
> Teacher: You think it is Daniel who is wrong. Why?
> B: Because… Because, let us take the 5 marbles of Peter. Daniel says that Peter has 4 marbles less than Gabriel. Because, 4 plus 4 is 8, but 5 plus 4 will be 9.
> …
> Teacher: Ok. Now we will check each of your propositions.

The intention of the teacher for this part of the lesson was to engage students in the discussion about additive relationship between three quantities. The semantic meaning of the situation described in the task was in contradiction with the numerical values indicated in the text. The numbers were small, so it was easy for students to see this contradiction. We can see from the script that students, while giving their arguments, tried to analyse the three numbers altogether. Therefore, the form of the task helped minimize the gap between the teacher’s intentions and the students’ activity.

We should mention here that at the end of the year, students from the experimental group have generally succeeded to demonstrate a progress in solving problems in which the holistic and flexible mathematical analysis was really necessary. At the same time, students from the control group have progressed more in solving problems that could be solved without such analysis.

Regarding the implementation of the 360° situation tasks, we observed that teachers have a tendency to return to the traditional teaching behaviours as soon as they start to work with traditional problems. For example, once the numerical answer was found for the problem, the discussion of the problem often ended abruptly. Thus, the focus of the activity was often shifted towards the use of the correct (recently discussed) representation or the calculation of the numerical answer.

Conclusion

As response to the call for contribution to the Theme B of the study, we presented our theoretical perspectives on the new task design for problem solving. By collaborating with teachers on the use of different types of tasks (we called them 360° situation), we expected to minimize the gap between teachers' intentions and children's mathematical activity. We are convinced that the Relational Paradigm can be a powerful theoretical tool in didactic engineering and task designs. It helps to construct tasks in which relational analysis is explicitly targeted. The tasks, having this important property and implemented throughout the Math and Citizenship competencies learning model, can play a major role in the reorientation of the classroom work with word problems from operations-oriented toward relations-oriented. It can also help teachers and students to reconsider their immediate teaching/learning targets in mathematical activities. While our preliminary data on students' results and teachers' practices show some progresses, we agree with De Corte (2012), that teachers need a closer follow-up in the new approaches for a considerably long time in order to profoundly change their practice.

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Theme C: Design and use of text-based resources

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This theme focuses on the design of textbooks, downloadable materials, and other forms of text-based communication designed to generate mathematical learning. We recognise that most teachers use textbooks and/or online packages of materials as their total or main source of tasks. Hence the design and use of tasks presented in textbooks is central to many school students' experience of mathematics. The scholarly study of task design should include consideration of theoretically-based textbook development, and can take place at different grain-sizes from individual tasks, through sequences of tasks, to a whole textbook series (Usiskin, 2003).

Some analyses of textbooks draw attention to differences in the use of language, illustrations, cultural and social allusions and some focus more on the mathematical and epistemological content (Askew, Hodgen, Hossain, & Bretscher, 2010; Haggarty & Pepin, 2001; Sutherland, 2002; Thompson, Senk, & Johnson, in press). Significant differences have been found in the conceptual coherence, mathematical challenge, consistency of images, and ordering of tasks between, for example, UK and Singapore textbooks. For example, in some textbooks a new concept is introduced through some everyday questions which are gradually refined to focus on a formal presentation; in others, practice of a technique precedes application through word problems (Ainley, 2010). The design of the order, development, representation and presentation of content is therefore a suitable topic for this ICMI study.

Another way to look at textual presentation is to analyse the content of individual questions or sequences of questions, and variation theory has been used as a tool both for design and analysis at this fine-grained level (Watson & Mason, 2006). For example, control of variation among examples can be used to direct learners towards inductive generalisations about concepts; example sequencing with controlled variation can lead learners towards some cognitive conflict. Textual presentation could be informed by research about how features of page and screen layout affect learners' attention (Ainsworth, 2009; Poole & Ball, 2006).

A third way to look at textbook tasks is to view them as the shapers of the curriculum rather than merely presenting a given curriculum (Senk & Thompson, 2003). The underlying commitments about the nature of mathematics, mathematical activity, and how mathematics is learnt, vary between textbook series and between countries. How these are promoted in the design and content of the tasks in the textbook is an important area of study because a textbook series might have more influence on learners and learning than a national curriculum. Different designers may interpret national standards or recommendations in different ways so that understanding the principles on which they instantiate these recommendations is an important area of study (Hirsch, 2007). Various components of mathematics will be prioritised or marginalised differently through different kinds of tasks and there will be legitimate debate about how students come into contact with mathematical absolutes (if there are any) (e.g. Harel & Wilson, 2011).

Authors' intentions can be different from how tasks or sequences of tasks are used in classrooms, and in this theme we could also look at pedagogic suggestions, particularly for innovative or unusual tasks, and information about conceptual intentions (Thompson & Senk, 2010). Many textbooks now refer users to online resources and tasks, and there is a professional development element to their use. There may be a difference between the adventurousness of students and the conservatism of teachers in their use and vice versa. (See chapters in Reys, Reys, & Rubenstein (2010) for issues related to curriculum and tasks in terms of intentions and enactments.)

Throughout the following set of questions, we consider a textbook and/or online resource to be a collection of tasks, generally sequenced in a given way, and often surrounded by related narrative and/or questions:

- How do curriculum expectations influence authors' design principles?
- How does an intention to promote change influence design?
- How do designers' expectations of teacher knowledge inform the design of dual purpose tasks: to teach students and to facilitate teacher learning?
- How can authors and teachers learn from alignments and misalignments of teachers' adaptations and authors' intent, and the implications for students' learning?
- How can or should new digital formats influence textbook design: e.g. use of podcasts, twitter, and other social media; implications for design and coherence of materials (either original digital design or transfer from print) if teachers are able to select tasks in varied orders?
- How do cultural considerations about instruction and pedagogy influence design: for example, whether teachers are seen as 'facilitators' or 'givers' of knowledge?
- How can designers take account of the language of instruction not being students' home language?
- What research about design of textbooks and other materials should be undertaken to inform the next generation of designers? In particular, how might design experiments (e.g., Clements (2007) or teaching experiments (such as Japanese lesson study)) influence task design in curriculum materials?
- How can design principles from software design, advertising, graphical art and eye-gaze research be used to improve text-based materials?

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Designing Tasks for Engaging Students in Active Knowledge Organization

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Mathematical tasks aim at supporting students to engage in a range of mathematical activities with specific didactical goals. Task design has to take into account the specificity of these different didactical goals (e.g., exploration, concept formation, practising skills). In this study, we focus on tasks intended for the didactical aim of mathematical knowledge organization ("organizing tasks"). In our learning pathways, phases of knowledge organization usually follow a phase of open exploration, of constructing individual concepts; they aim at regularizing and systematizing the students' singular ideas and results. Because such organizing processes conducted in whole-classroom discussion often fall short of engaging every single student, our design research study sets out to develop task formats that promote adequate cognitive activities and formats for this organizing phase. The article describes the efforts in constructing and evaluating organizing tasks and presents – as a result of our study – a conceptual framework for the delicate balance between individual engagement and convergence.

Keywords: organizing tasks, organizing knowledge, regularizing and systematizing

The challenge of organizing knowledge

The construction of mathematical knowledge is a multistep process of "organizing fields of experience" as Freudenthal pointed out (1973, p. 123). In subsequent work of the realistic math education approach (de Lange, 1996), examples of organizing processes are abundant, comprising "horizontal" and "vertical" mathematization (Treffers, 1987, p. 247). Although these ideas lie at the heart of most contemporary approaches for mathematics education, it is undeniable that in mathematics classrooms such organizing processes can rarely be found explicitly (Hiebert et al., 2003). However, there is a need for development of activities which promote systematizing, regularizing, and preserving the results of exploration with several goals:

- structuring the singular and divergent results and connecting them to other facets of knowledge (*systematizing*),
- transforming results into regular and consolidated mathematics (Brousseau, 1997, calls this phase the "institutionalization", we call it *regularizing*);
- writing down in a form that is accessible later (*preserving*).

When and how do such procedures of organizing knowledge occur? When observing German mathematics classrooms, we often find phases of discovery and individual problem solving. For the subsequent phases of regularizing, we actually encounter two different types of classroom procedures.

In the first type of procedure, the teacher conducts a whole-classroom discussion in a Socratic dialogue, collects and evaluates students' contributions, and leads the class to an organized and structured knowledge. This procedure requires whole classroom conversation techniques and proves precarious with respect to the cognitive activation of every single student. In the second type of procedure, the teacher may avoid whole-classroom discussion and instead refer students to the information boxes in the textbook where the correct mathematical concept or result is stated. This ensures a common basis, but there is a danger that the individually constructed knowledge cannot be integrated in this "ready-made-mathematics".

What are alternative options for supporting students in organizing their knowledge, in mastering the step from the singular and individual knowledge to the regular and commonly accepted mathematical knowledge while preserving students' engagement? In our design research study, we developed approaches and tasks on the basis of three premises: (1) Teachers predominantly work with textbooks, so the organizing tasks should be embedded into a comprehensive textbook curriculum; (2) Students must be actively involved in the learning processes; (3) Teachers are not supposed to give up their role in moderating the process of organizing with the whole class. We consider the communication processes within the class as extremely important for attaining a high level of mathematical insight.

Hence, our goals are to construct tasks that support students in actively organizing their knowledge and simultaneously support teachers in guiding this process in an effective manner. We call this task type "organizing task" or, when we need to avoid the misunderstanding of organizing as a purely external, administrative activity, as "knowledge organizing tasks". The guiding questions for our study are:

Q1. Specification of learning goals:

What elements of knowledge have to be organized and preserved?

Q2. Types of Task:s

Which types of tasks can support students' active knowledge organization?

Q3. Principles for the Task Design:

Which principles guide the construction of organizing tasks?

The framework: design research for a middle school curriculum

The study on organizing tasks is embedded in the long term design research project KOSIMA (2006-2016, cf. Hußmann et al., 2011). It is briefly presented here with its methodological framework and the conceptual framework for the design products.

Methodological framework of the long-term design research project

The project KOSIMA (Hußmann et al., 2011) follows the methodology of Didactical Design Research (Gravemeijer & Cobb, 2006; McKenney & Reeves, 2012) with its dual aim of *designing teaching-learning-arrangements* for a complete middle school curriculum (grades 5 to 10 of German Realschule, Gesamtschule, Sekundarschule) and *empirically researching* the teaching-learning-processes and their conditions. The developed curriculum is published as the textbook *Mathewerk-statt* from 2012 to 2017 (Barzel et al., 2012 ff.) and a comprehensive teachers' manual.

All teaching-learning-arrangements of the textbook-to-be are developed in iterative cycles of design, evaluation (by expert discussions and classroom experiments), and redesign. Whereas the design and evaluation steps of the project refer to the entire implementation of the textbook, the deeper research is organized in several smaller design research studies that necessarily have to address more narrow research questions. These studies use different concrete research methods and designs (e.g., intervention studies in quasi-experimental designs, design experiments in laboratory settings with up to four cycles, e.g., Leuders & Philipp, 2012; Prediger & Schnell, 2013). An overall evaluation of summative effectiveness commenced in August, 2012. Results of the quasi-experimental intervention with pre-post-test over two years can be expected in 2014.

The community involved in these processes is a large group of people from different backgrounds who collaborate fruitfully:

- researchers (the four authors of this paper, being the editors of the textbook and leaders of the design research project, supported by many PhD students and student researchers)
- authors of the teaching-learning-arrangements (about 20 experienced reflective practitioners, together with the editors)
- the publisher (with 2-4 copy editors who finalize the design products)
- project teachers (about 10 teachers with their classes, who teach with the curriculum and the textbook material continually in their regular classes).

Conceptual framework for the design product

The design of the middle school curriculum is guided by certain design principles. We only state those which are relevant for the focus of this paper, that is for designing organizing tasks (cf. Hußmann et al., 2011; Prediger et al., 2011). Following socio-constructivist theories of learning, we emphasize the importance of students' active engagement and sense-making by starting from meaningful context problems, and developing conceptual understanding (Leuders et al., 2012, following Wagenschein, 1977 and Freudenthal, 1973). For realizing these principles, every teaching-learning-arrangement (each for 2-6 sessions) is structured into four main phases: activation, exploration, organization of knowledge, and practice.

Activation of students' experiences. For including students' pre-instructional experiences into the learning pathways, every arrangement is situated in an everyday context that allows problems for ready-made mathematics to be reinvented and students to construct meanings for the intended mathematical topics.

The following is an example referred to throughout this article. "Constructing Packages" provides a context for students to think about solid figures and their characteristics, such as parallel and perpendicular lines. To solve problems related to this topic, students put themselves in the role of a package designer who has to create new packaging for a toy. In considering a good design for a package, a designer has to think about what criteria are relevant to create a good package-box. Apart from price, other possible criteria are look, stackability, and ease of construction. Stackability specifically leads to the necessity of having packages with parallel and perpendicular lines.

Exploration. In this extensive phase, the problems are operationalized into open and rich exploration tasks (Flewelling & William, 2001; Freudenthal, 1973). They allow students to actively and collaboratively re-invent ideas, concepts, procedures and relations in the sense of horizontal mathematization. Due to the openness and student-centricity of this phase, it often results in a large diversity of individual ideas, strategies, solutions, findings, pre-concepts, etc. The concrete design principles for exploration tasks that were developed or refined during the design research project are not reported here. Looking at our example in this phase: Students assume the role of the package-designer and they actually construct a package-box for a specific item, e.g. a toy. During this process of construction, many students realize that the "box must be straight and precise, not awry or askew". These experiences prepare the systemization of the concept of *perpendicular and parallel* in the next phase.

Organization of knowledge. The goal of the subsequent phase of organization (in German "ordnen" which also means "ordering", "arranging") is to establish a shared understanding of the core concepts, theorems and procedures and to preserve and document this understanding in the self-written "knowledge-storage" (in German "Wissensspeicher").

The packaging example highlights the kinds of activities required in this phase. Exploration often leads to a great diversity – in our example a lot of different boxes, ideas, images, and students' thoughts. All of these products and ideas have to be shared and compared in order to systematize the new knowledge. To stimulate students´ mental processes, we created focused cognitive activities (called *acquisition activities*) according to the new knowledge.

The type of acquisition activity depends on the types of knowledge. For the concept of perpendicular and parallel, one can differentiate between the following aspects, which have to be learned:

- Students must learn the technical terms "perpendicular" and "parallel". These words are names and just conventions, which cannot be discovered or explored. Students have to be informed about these new words. This has to be done in a language, which can be easily understood by students and with words which link to the experience of the exploration phase (see the first lines of the task in Fig 2).
- Another convention in our example is the marking of a right angle with a dot (a convention that differs throughout several countries). This is information that students must learn (see picture on the left side in Fig. 2).
- Students have to learn to recognize perpendicular and parallel lines. Task 2b (in Fig. 2) involves concretisation and distinction of the new concept. Pupils have to recognize which pairs of lines show parallel or perpendicular ones.
- The extent to which individual students can verbalise the new concept varies and can be described by the extrema: On the one side students

formulate a definition by themselves, a very ambitious task; on the other side students have to copy a given definition (see Fig 3). Choosing a correct definition from several given ones is an activity in between these extrema, and this is demonstrated in our example. (See (2b) in Fig 2.)

When inspecting regular lessons and textbooks, we have only rarely encountered tasks that were constructed for supporting such organizing processes (e.g., Swan, 2005). That is why we had to conduct a design research study for specifying principles for a systematic construction and composition of organizing tasks. The results are reported in the section on Findings.

Practice. In the fourth phase of practice (in German “vertiefen” = deepen, intensify), the students are supposed to render their knowledge and skills more stable and flexible by repeated practice and transfer. Our design principles for these tasks refer to didactical approaches of productive exercises, structured tasks and reflective practising (Büchter & Leuders, 2005; Watson & Mason, 1998; Winter, 1984; Wittmann & Müller, 1990).

It is important to note that the construction of “organizing tasks” as a “carrier of the organizing” phase has only become indispensable within our didactical approach that distinguishes these phases explicitly to give students space for more individual mathematical activities. Within a more integrated approach, concept exploring and organizing activities could be combined or integrated more flexibly.

Methods of the design research study on organizing tasks

The construction of “organizing tasks” and the development of an overarching didactical approach for the principled construction of the tasks are embedded in the larger design research project depicted previously. For the concrete study, the cycles of constructing and evaluating organizing tasks were passed through topic by topic, each in four cycles (mostly including further microcycles).

For each topic, the *first and second cycle* of design, evaluation, and redesign is conducted by expert discussions. After specifying the goals of the learning arrangements, the writing team (authors and one editor) suggests a first draft for the formulation of target knowledge that students are supposed to save in their “knowledge storage”. After discussing the selection and priorities by the editor team with respect to didactical-conceptual considerations, the (exploration and) organizing tasks are formulated, discussed and further developed. The details of formulation are edited by experienced copy editors of the publisher who optimize readability and coherence. The *third cycle of evaluation* is conducted in classroom experiments with 3-10 teachers in their regular classrooms. The data base for the investigation consists of teachers’ written and oral feedback, scans of students’ written texts for tasks, knowledge storages and classroom assessments as well as some videos of classroom interaction and design experiments in laboratory settings on selected tasks. For the current study, the qualitative data analysis is conducted with respect to connections between forms of tasks and a) students’ engagement within the processes, b) the convergence of the processes and c) the results of organized knowledge as stored and especially as performed in the assessments. In the *fourth cycle of redesign*, with theoretical feedback authors, editors and copy editors again are involved in finalizing versions for the textbook and the teachers’ manual. The final version is ready for widespread implementation and effective everyday use.

During these four-step processes for many different mathematical topics, we iteratively accumulated the reflections and experiences, generalized from the specific topics, and developed a conceptual framework for organizing task design. Whereas the answers for research question Q1 on the relevant kinds of knowledge were mostly generated in the first and second steps of *theoretical and conceptual evaluation*, answers for Q2 and Q3 on effects of forms of tasks mostly rely on the *empirical investigation,* involving deeper insights into the mechanisms of the teaching-learning processes as well as practical experiences on robustness for different classroom conditions. Out of the whole conceptual framework for designing organizing task, we present two major aspects in the next section.

Findings

Specification of learning goals: modes and facets of knowledge

The didactical base of a systematic task design is the exact specification of the intended learning goal, here concretely the mode of knowledge that is supposed to be systematized, regularized, and preserved. For concretizing this specification process for each mathematical topic, we developed a conceptual framework of different knowledge elements as printed in Figure 1 (Prediger et al., 2011).

The horizontal dimension follows the classical distinction (Hiebert & Carpenter, 1992) among knowledge about facts, concepts (e.g., numbers, operations, relations), and connections as codified in theorems (*conceptual knowledge*) on the one hand, and the knowledge about mathematical and technical procedures (*procedural knowledge*) on the other hand. We added *metacognitive knowledge* which includes problem solving strategies or steps in modelling processes.

	Which part? (Facet of Knowledge)				
What? (Modes of Knowledge)		**Explicit Verbalisation**	**Concretisation & Distinction**	**Meaning & Connections**	**Conventions**
	Conceptual Knowledge				
	Concepts	definitions	examples/ counterexamples	mental models / representations	technical terms
	Connections	theorems	examples/ counterexamples	(visualized) explanation/ proof	names of theorems, conventional rules
	Procedural Knowledge				
	Mathematical procedures, algorithms	instructions	conditions for applicability, special cases, knowledge about errors	mental models / reasoning as link to conceptual meaning	non-justifiable specifications
	Technical procedures	instructions	realization, conditions		non-justifiable determinations
	Metacognitive Knowledge				

Fig. 1. Conceptual framework for specifying learning goals (Prediger et al., 2011)

During the design research process, we realized the importance of specifying a second (here vertical) dimension that we call *facets of knowledge*: Whereas a piece of knowledge is often only represented by its *explicit verbalisations* (in definitions, theorems or instructions for procedures) or underlying *conventions* (like the technical terms), didactical research has often shown that knowledge acquisition must also comprise *concretization and distinction* (like examples and counterexamples for concepts, cf. Winter, 1984 and Fig. 2 Task 2a) or knowledge of possible errors in procedures (cf. Vollrath, 2010)) and *meanings and connections* to other elements of knowledge, given by visual representations and mental models, explanations and pre-formal proofs.

As mathematical understanding is conceptualized as individual construction of relations, dependencies, or connections between mathematical ideas, procedures and concepts (Hiebert & Carpenter, 1992), we promote that the organizing tasks and the knowledge storage must always include various facets of knowledge. It is the first step of the construction process of an organizing task to specify which cells of Fig. 1 shall be addressed in the task. These modes and facets of knowledge are then subject for initiating the processes of *systematizing* (= connecting different facets of knowledge systematically), *regularizing* (= transforming individual constructions from the exploration phase into regular mathematical concepts, connections and procedures), and *preserving* (= documenting facets in the knowledge storage so that they can be recalled some months later).

Types of tasks: initiating acquisition activities in a balance between students' engagement and convergence

As made explicit previously, students must be actively involved in the processes of systematizing, regularizing, and preserving knowledge. The *necessity of active engagement* is explainable within the socio-constructivist framework and reconstructable in the empirical investigations, because simple inputs (e.g., teachers dictating the information) were not suitable for activating students' mental processes.

That is why for each piece of knowledge (cells in Fig. 1 selected for a specific topic) that is to be systematized and preserved, a focused cognitive activity (*acquisition activity*) must be initiated that supports students' active acquisition of this piece of knowledge.

Fig. 2 shows some formats for how such activities could be initiated. First the technical terms (here parallel and perpendicular) are given - in the frame and context of the experiences and discoveries which have been done before (here the context of exactly folding straight boxes). In 2a), examples and counterexamples are to be identified. In 2b), students shall choose between possible definitions. Independent personal definitions would produce again very divergent solutions, but finding a correct and fitting one among some examples allows an active engagement with higher convergence.

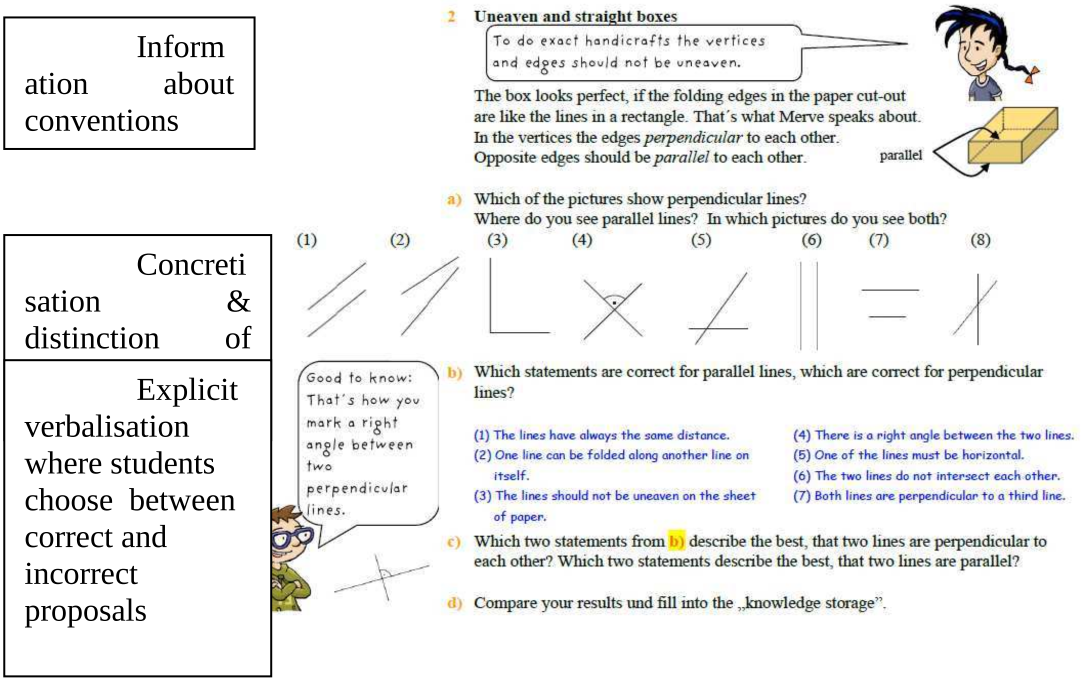

Fig. 2. Examples and counterexamples in 2a), Finding a correct definition in 2b) (Barzel et al., 2012)

The classroom experiments showed clearly that the balance between students' engagement and convergence of the process is delicate. If the openness of the activity is high, students can intensively engage with the content; like in the exploration tasks, they develop very divergent ideas and entries for the knowledge storage. These divergences produce either the need for the teacher to moderate the processes in funnel-like patterns (Bauersfeld, 1988), or to give individual feedback to each individual attempt to write a knowledge storage, which is too much work for each task. If tasks are optimized only with respect to convergence, it might risk students not being engaged enough. The adequate balance between convergence and engagement depends on the concrete topic and the concrete piece of knowledge.

The result of the generalization process was the specification of a range of acquisition activities for each piece of knowledge, as illustrated for two exemplary pieces of knowledge in Fig. 3.

Convergence	**Typical acquisition activities for concretization of concepts**	**Engagement**
low	1. **Realize** the concepts: Find examples and counterexamples and explain why they (don't) fit to the concept.	high
high	2. **Identify** the concept: Which examples fit the concept, which examples don't? Why?	low
	3. **Give reasons** why the examples (don't) fit: Is it a counterexample? Why?	

Convergence	Typical acquisition activities for formulating mathematical definitions or sentences	Engagement
low	1. **Formulate** the definition / sentence by yourself	h
	2. **Rectify** the formulation: Which formulation is wrong? Find the mistake and rectify it.	
high	3. **Understand** the formulation, explain, why it is adequate and find fitting examples.	low

Fig 3. Range of acquisition activities in the balance between convergence and engagement

Discussion

Due to the fact that our design research in the KOSIMA project results in a new textbook, the design part of the work itself has important impact on *several communities*. First of all, it affects the *students*, who get the chance to develop and reinvent mathematical concepts by relating relevant contexts with individual, sustainable conceptions. The developed learning arrangement offers structured tasks so that the teacher can moderate the learning processes and can support students in organizing their knowledge. It has to be mentioned that the *Mathewerkstatt* is one of only a few textbooks in Germany that are put to the test and fully revised and reviewed by sample classes for guaranteeing usability by *teachers*.

The research that was conducted in the iterative interplay with design, evaluation, analysis and revision of the learning arrangements showed the potential of the structure of activation – exploration – organization – practice. The insights gained into the deeper structures of the initiated learning processes allow us to contribute also to *didactical theory*. A further evaluation on the generated learning progress was started in August 2012 for a two year study.

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Developing Young Students' Geometric Insight Based on Multiple Informal Classifications as a Central Principle in the Task Design

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The geometrical (and more generally, mathematical) insight of students, as defined in our previous research, is characterized by four main parameters. Such insight should be, in our opinion, the focus of educators at any level of mathematics. This implies the need for teaching/learning materials designed to serve this purpose. In this paper we illustrate how task design principles aimed at students' mathematical insight are implemented in a series of geometrical textbooks for grades 2-6 of primary school.

Keywords: mathematical insight, teaching solids at primary school, teaching geometry at primary school, informal classifications

Introduction: our model of mathematical insight as a conceptual basis for the task design

In our past and present research, we follow previous studies (see e. g., Barabash & Guberman, 2008; Griffiths, 1971; Sternberg & Davidson, 1999) in the search for characteristics and ways of development of mathematical insight as a feature reflecting the depth of a person's mathematical thinking and understanding. In doing so, we infer that these characteristics should not be dependent upon the person's formal mathematical knowledge (FMK); we found in our research support for this inference.

The concept of mathematical insight is closely related to mathematics understanding and comprehension as a result of learning. Our concept of insight is closely associated with learning theories dealing with procedural vs. conceptual knowledge and includes features of both types in accordance with recent studies indicating that they actually should be intertwined in a fruitful process of learning mathematics (see e.g., Schneider & Stern, 2005).

This may be a good point to indicate our personal professional profiles, since they have a direct implication on our tasks and on their further implementations with various populations. One of us has a PhD in mathematics, and the other one has a PhD in mathematics education. Both of us are teacher educators with more than 20-years of experience. One of us has been a primary school teacher; we both are active in in-service educational programs for mathematics teachers, and have written a number of textbooks and learning materials for primary and secondary school, for mathematics-teachers-to-be, and for teachers educators.

According to our research (Barabash & Guberman, 2008), the characteristics of mathematical insight (***MI***) are:

Implementation: the ability to implement, to employ the material being currently learned in its close "neighborhood", and the measure of closeness of this implementation to the form in which it has been taught in the class ("the farther the better"). This indicates the student's ability to grasp the idea beyond its immediate presentation in a lesson or a handbook.

Skills: the variety of mathematical skills needed or demonstrated by a student in relation to the current issue, and the level of mastering these skills.

Extension / generalization: the ability to extend the acquired knowledge and/or to generalize it, or to incorporate the issue beyond the obvious mathematical context in which it has been originally studied.

Mathematical language: usage of newly acquired terminology; reasoning using relevant mathematical argumentation (appropriate to the formal mathematical (FM) development of the student); usage of formal-mathematical language and ability to appropriately relate it to verbal-non-mathematical expressions and vice versa, etc.

Principal considerations in the basis of our task design

Mathematics learning is known to develop in a spiral-like process, in which each coil is based upon previous ones and unfold out of them. The teaching and learning tasks should support this process in their didactical design and mathematical vision. Tasks can vary not only with respect to mathematics content but also with respect to the cognitive processes involved in working on them (Shimizu, Kaur, Huang, & Clarke, 2010).

In view of these and other similar reference points and in consistency with our own professional and academic experience, we define a task as a series of questions and assignments united by a big mathematical idea. We regard textbooks for primary school as a self-consistent series of such tasks on topics outlined by the curriculum, each task comprising a structured well-planned succession of problems, situations, explanations and exercises aimed at the on-going spiral-like learning of mathematics. By "learning of mathematics", we mean two intertwined processes: the development of students' FM knowledge (i.e., the corpus of notions, facts, techniques, ways of reasoning); and understanding of this corpus, which we characterize as mathematical insight. Accordingly, we claim that the teaching procedures and materials should be aimed at the current state of the students' insight and designed so that they permanently advance it, regarding each of its characteristics as an ongoing objective. It is in view of this intention that we refer to the 5-year perspective of teaching solids in primary school as it is worked out in our series of textbooks for 2nd through 6th grade (excluding 3rd grade).

In our design of learning tasks, we keep in mind mathematical, epistemological and curriculum-related (MEC), cognitive-developmental (CD), and pedagogical and didactical (PD) objectives of the curriculum issue for which the tasks are being designed, all of them having direct implications on the insight development (Fig.1). We refer here briefly to each of these principles in relation to our textbooks. The abridged notation will serve for further reference.

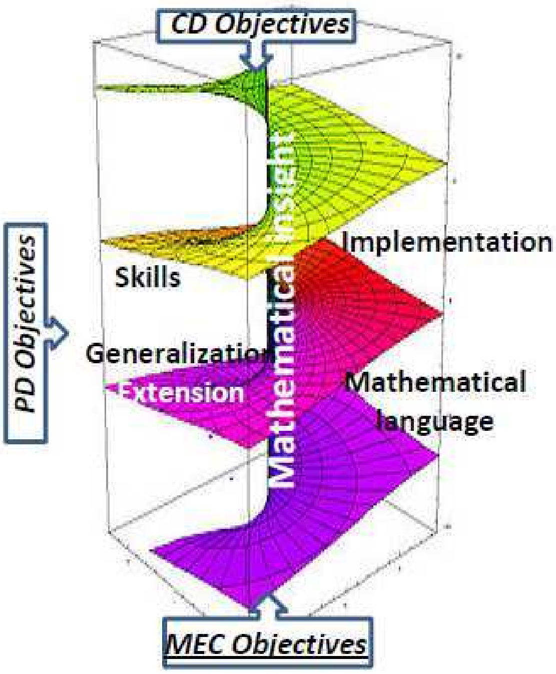

Fig.1. Schematic representation of task design principles aimed at the development of students' mathematical insight at any level of FMK

MEC: When we plan a teaching task, for whichever level of FM learning it is aimed, we first consider it from the perspective of its mathematical content and meaning. This is done in view of curriculum indications, on the one hand, and of our mathematical knowledge and of our pedagogical content knowledge (PCK), on the other hand. Obviously, the mathematical and epistemological considerations concerning the primary school geometry textbooks are curriculum-related. They are:

MEC1: Mathematical correctness. We decline any possibility to present a mathematical notion in a way, however seemingly clear, which contradicts the mathematics of the issue, and hence, will render the notion to be sooner or later learned anew in a different way not consistent with the present one. Mathematical correctness is not a synonym to mathematical precision, rigor, and formality. Any mathematical notion, idea or fact intended for the primary school can be learned in a mathematically correct way at an informal, pre-formal or more-or-less formal mathematical level.

MEC2: The mathematical perspective of past and future development of the issue at school and (possibly) in future higher-education studies (longitudinal perspective), in compliance with the "big mathematical idea" being studied.

MEC3: The links of the new topic to other mathematical or outer-mathematical topics being currently learned (cross-curricular perspective).

MEC4: Intuitive-inductive acquaintance with figures and solids and their properties, with implicit or explicit suggestions for generalization. Where possible and appropriate, we do our best to reveal to the teachers (in the teachers' guide) and/or to the students (in the textbook itself) the motivations for definitions, ways of reasoning, ways of computing, etc. These motivations are conceived so as to pave the future way to more formal mathematical approaches (in compliance with ***MEC2***).

CD: An ability to discern various characteristics of mathematical objects, or to find similarities and differences between them is the basis of the classification ability, which is one of the central thinking abilities. Piaget regarded cognitive developmental ability to be one of the most important cognitive abilities. Following Piaget (1962), class inclusion is an understanding of the fact that a set of objects is simultaneously a subset of another, bigger set. The development of this ability starts at the preoperational stage (ages 2-7). At this stage a child may classify objects by one property. At the concrete operational stage (ages 7-11), the child is able to perform classification by a number of properties, and is also able to rank the objects by one of the properties, for example, by size. At this stage, one of the aims may be to develop the Decentration ability (Piaget, 1962), i.e. an ability to shift between the classifications. The classifications may be formal-mathematical or not.

PD: The results and analysis of TIMSS (see e.g., Valverde et al., 2002) indicate that school mathematical textbooks are the main basis for the lesson planning by teachers and have a powerful influence on what is learned and how it is learned (Ball & Cohen, 1996; Yang, Reys, & Wu, 2010). Textbooks more and more tend to form the actual classroom curriculum and guide teaching practices.

According to the National Curriculum, the Israeli students start getting familiar with three-dimensional geometric solids during second grade. The Curriculum specifies the topics to be learned, but suggests no leading principle as to how this is to be done. Hence, textbook authors are more or less free to implement their own didactic principles.

We discern two major possible approaches to teaching solids:

- Allocating a lesson to each solid, e.g. a lesson to the parallelepiped, another lesson to the cube. The comparison between them may be the topic of the concluding lesson.
- Handling together all the solids included in the curriculum, studying their common properties, and the properties special for some of them. This approach leads to classifications by various geometrical properties.

Our decision was to choose the second approach, i.e. non-formal classifications as the basis for our teaching solids. To make things clearer, we outline here the issue of classifications from the mathematical point of view prior to presenting the examples.

Formal classification of mathematical objects is based on rigorous definitions which reflect the hierarchy of mathematical objects by pure inclusion in a set-theoretical sense. The primary school handbooks are not based on rigorous definitions; at this stage of learning formal definitions are, generally speaking, implausible.

Unlike formal classifications based on rigorous definitions, informal classifications are based on properties not necessarily defining purely inclusive sets, because they do not necessarily either imply or mutually exclude each other. Though informal classifications may not lead immediately to rigorous mathematical hierarchy, we find them important from the didactical point of view, at least at early stages of mathematics learning ***(PD),*** because they enable the concept images to be gradually (***MEC2***, ***MEC4***) and mathematically consistently (***MEC1***) constructed starting from the initial stages of geometry learning.

The following examples from our textbooks clarify how the task design principles serve to enhance the geometrical insight of students. We intentionally chose an example from the second grade and an example from the sixth grade in order to demonstrate the self-consistency of our approach throughout primary school geometry teaching (*MEC2*).

Example 1: 2nd grade

The kit: This is the students' first encounter in their school studies with six solids included in the curriculum for the 2nd Grade: the cone, the cylinder, the ball, the square pyramid, the parallelepiped, the cube. A kit of these solids is attached to every student's copy of the textbook. For the sake of the first encounter, all the revolution surfaces are of the same radius; the square bases of the polyhedra are congruent. The heights of all the solids are equal.

The story: A group of intruders (who are actually the six solids) has broken into the math classroom and left it in a great mess. Fortunately, the intruders have left traces: stains on walls and floor, signs of different forms, etc. The famous Mathy the Detective uses these signs to discover who the intruders were. To succeed, he also watches them at night because they keep breaking into the classroom night after night, so he sees their shadows in the night. This "story" is a rich basis for the variety of classifications by a number of features, among which are the following:

The sign left by a solid on a plane. By a *sign* we mean the result of a momentary contact of a solid with the plane, as if the solid were a stamp. Having experimented with the solid, the students discover that all of them can leave a sign in the form of a point, whereas the segment-like sign may be left by some of them (e.g., a cone) but not by others (e.g., a ball). Actually, the point is the only sign a ball may leave.

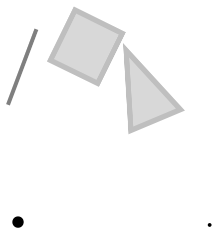

Fig.2a. "Signs" as a result of a contact

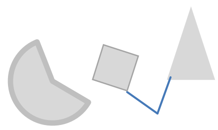

Fig.2b. "Stains" as a result of " rolling"

Stains left by a shape on the plain, when one "rolls" the shape on the plane (without sliding). See Fig 2a, b for some examples of *signs* and *stains*.

The outline of the solid: which solids have "shadows" of similar form, and which do not? By *shadows* we actually mean the outline of orthogonal projections. For example, the pyramid and the cone may have the same isosceles triangle-shaped *shadow*, while the parallelepiped and the cylinder may have congruent rectangular *shadows*.

These and other similar classifications are used to identify the intruders. At the first step, the students must fill in under each shadow the names of all possible "candidates" among the solids (Figs. 3a, b). Later, they must use all their recently and previously acquired knowledge to recognize the solids. One of the main ways for the students to help the detective is to solve logical problems like: "Who is it who has a rectangular shadow and who has left a circular sign?", or "Can we recognize for sure the intruders whose shadows appear in the following two sets?" (Figs. 3a, b). In order to recognize the intruders by their shadows as they appear in Fig. 3a, the students have to reason more or less as follows: there are three circular shadows, hence in this setting the cylinder, the cone and the circle are certainly involved, though we cannot pinpoint any of them. Hence, the rectangular shadow must belong to the parallelepiped. As to the square shadows, they are of the cube and of the pyramid, but we can't know which is which. Thus, in the first setting one recognizes for certain the shadow of the parallelepiped. To compare, in Fig. 3b there are two triangular shadows; hence, these are of the cone and of the pyramid, but we cannot know which is which. The square shadow then belongs to the cube, and the circular shadows are of the ball and of the cylinder, though again we cannot know which is which. Hence, we have pinpointed the cube. To be sure, the wording of second graders is different, but grosso modo this is the reasoning they must use, including the necessarily correct naming of plane figures and of solids, which by itself may be regarded as a success.

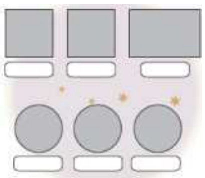

Fig.3a

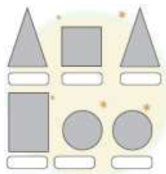

Fig.3b

Fig 3. Using shadows to identify potential solids

Linguistic and thinking skills needed for the solution of these problems are clearly aimed at the students' insight. We will refer later to the generalization possibilities of this task.

Example 2: 6th grade

According to the sixth grade curriculum, the solids are to be studied from several points of view: volume and surface computations; closer acquaintance with solids of revolution and with various types of polyhedra; unfolding solids into plane nets. In addition, we also include a number of assignments whose purpose is to form initial understandings related to the three-dimensional version of isoperimetric problems (***MEC2***). For some of these mathematical purposes (volume and surface computations and related issues, including the isoperimetric issues; nets folding and unfolding), it is worth regarding prisms as generalized cylinders, i.e. as surfaces formed by parallel sliding of a straight line along a plane directrix. A circular directrix leads to a circular cylinder, a polygonal directrix leads to a prism; other directrices may lead to other generalized cylindrical surfaces. Similarly, pyramids are generalized cones: both surfaces are formed by a straight line whose one point is fixed and another is moving along a directrix (circular, polygonal or other) (***MEC1***). This clarifies the identical methods of the volume and surface computations for cones and pyramids, as well as for prisms and cylinders, and is actually an example of a "big mathematical idea" on which the whole series of textbooks is focused concerning 3-d measurements. Naturally, such grouping differs from the classification of solids of revolution vs. polyhedra which serves different purposes. Nets unfolding, in addition to the previous grouping, suggests yet another approach to classifying the solid surfaces: those who may be unfolded into a net and those who may not.

One of the widely used primary-school approaches to the volume computations of cones (and the one recommended by the Israeli school curriculum) is to fill a conic vessel with water or sand to show that three such vessels fill the cylindrical vessel of the same base and height. The same is to be done with a pyramid and a prism, when the volume of a pyramid is studied. This demonstration may be persuasive enough to show that the volume of the cone or of the pyramid is three times less than that of a cylinder or of a prism, but does not give even a hint as to why it is so. This is not compatible with our concept of insight as a result of meaningful learning. Ignoring the similarity of volume computation formulas for cones and pyramids and for cylinders and prisms seems to be an aggrieving omission of the possibility to teach a mathematically meaningful approach to the volume computations in a way that accounts for the future perspective of this topic. The approach which we undertook in our textbooks is actually based on an intuitive mathematical principle known as

Cavalieri's principle, which is an example of a big mathematical idea actually going back to Archimedes. It states: Given two solids of equal heights whose bases have equal areas, if the plane sections of these solids at any height have equal areas, then these solids are of equal volumes. We do not propose to explicitly formulate this principle in the class, but this understanding is our mathematical and epistemological source for the overall approach to volume computations in primary school. The grouping of cones with pyramids and cylinders with prisms serves the initial perception of the intuitive 3-d version of the isoperimetric problem, as well as the consistent understanding of how surfaces of similar structure may unfold into plane nets. One has obviously to discern the revolution solids from the polyhedra in order to study intrinsic properties of each type of solid, which renders a completely different grouping.

The following excerpt from the 6th grade volume-and-surface-computation assignment includes also the intuitive minimal-surface-maximal-volume issue. It originates from designing a tent (MEC3). How do we design a tent (Fig. 4a)? We want a tent to be as light as possible (one has to carry it), to provide as much sleeping place as possible on its floor, and to hold as much air as possible (one expects to breathe while sleeping). What do we need to compute in order to fulfill each of these demands? Which tent will we prefer amongst the following three which are of the same height and of equal bases areas (Fig. 4b)?

Fig.4a How do we design a tent?

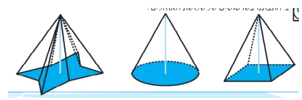

Fig.4b Which tent will we prefer?

The problem thus posed is a situation leading to motivated computation-based decisions: when one needs to compute the basis area, when the surface area is relevant, and when the volume computation is the answer to the question (*MEC4*). Moreover, it is obvious from the context what the net unfolding is for and why it is essentially the same for all the tents. Here, all the tents are of a conic structure, whatever their base forms are. Alternatively, further in the book the pottery is brought as an example in the context of solids, thus distinguishing solids of revolution from polyhedra.

Since the textbooks have been published and approved by the Ministry of Education, about 2000 school teachers have been using them in their primary school geometry lessons. Interviews with the teachers imply that their students understand that the properties of solids may be presented in different ways through two- and three-dimensional notions. The students are able to combine verbal and visual representations, the words they use being similar or identical to mathematical terms. Here is an excerpt from the feedback of one of the teachers:

> "… a student is expected to be familiar with solids already at the primary school. That is, he (or she) must understand them from a number of viewpoints, to be flexible with them…. Flexibility, I would say, means – to feel them, to be able to see them from various directions, to see what they enable us, to see their meanings, when it is convenient to use them… your approach, from all the books I've used, and I've used some textbooks, enables this flexibility for the students, gives them confidence and freedom to "play" with the solids…"

As a result of encouraging reactions of practicing teachers, we included our textbooks as the learning material in our geometry course for students who are being educated as future primary school mathematics teachers or kindergarten teachers.

Analysis of the examples of the tasks from the textbooks

We will use now the examples to illustrate how the previously formulated objectives serve to advance the students' insight by each of its parameters.

Implementation. The situation in which a second-grader must follow and apply the classification immediately after its presentation in the class by the teacher or by one of the classmates is an important opportunity to enhance his or her aptitude for implementation. Different behaviors may indicate different levels of insight. For example, examining the *stains* left by the “rolling” solids, a child may follow closely the “rolling” manner of the teacher, or may decide to roll a solid (e.g., a parallelepiped over its longest side), thus providing a stain in the form of a long and narrow rectangle instead of a shorter and wider one. Performing these activities, the student has to master his knowledge of plane figures acquired previously (during the 1st grade) and discern various signs and stains left by the same solid (***MEC2; CD***).

Skills. The approach which we propose develops the aptitude of comparing and classifying solids by their properties and adequately using these properties for various purposes, which requires much more than mere recognition. We regard these skills as essentially more meaningful for geometry learning, and for mathematics learning in general **(*MEC*)**. Examples of such skills are: an ability to discern a special vertex of a pyramid analogous to the vertex of a cone (which is indispensable for the volume computations); discerning specific parts of solids, including those not directly accessible, like the diameter of a sphere; folding and unfolding nets and recognizing the solids by their nets. These skills comprise the integral learning **(*CD*)**.

Generalization. The approach based on analysis of properties common to a group of objects (such as signs and shadows in 2nd grade, common structure for cones and pyramids vs. structure common to cones and balls in the example of the 6th grade) has direct implication for generalization on this basis. We attempted to demonstrate that this approach is both mathematically (***MEC***) and didactically (***PD***) justified.

Language. We regard the mathematical language to be one of the central factors in the development of one's mathematical insight. The linguistic abilities of the student develop when he or she has to explore, formulate hypotheses, explain, motivate, and relate mathematical ideas, as in the activities described previously. These abilities comprise another component in a student’s cognitive development (***CD***). The example of reasoning needed to recognize solids by their shadows speaks for itself, to our opinion, as far as all these aspects are concerned.

Discussion

We hope to have demonstrated that the mathematical and didactical features of our texts, based on the principal considerations formulated above (***MEC, CD, PD***), serve the on-going purpose of fostering the mathematical insight of young students with that insight as characterized by its four parameters. Moreover, they help to achieve specific objectives which comply with our general line as reflected in these considerations:

Advancing the young students "half-way" to future formal definitions, ideas, notions and formulas in geometry; in particular, to the general understanding of volume and surface area computations and to intuitive perception of the interrelation between the form of a solid and its minimal-surface-maximal-volume properties (***MEC1, MEC2***).

Well-motivated outer-mathematical foundations for correct perception of central notions in geometry of solids which sometimes cause a great deal of ambiguity for students, such as surface area vs. volume (***MEC1, MEC3, CD***).

Flexibility in the students' perception of solids in ways corresponding to specific mathematical and not-necessarily mathematical contexts (***CD***), and more.

The Pedagogical-Didactical (***PD***) objective which is less referenced in the course of the presentation and discussion is obviously central to the series.

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How Do Students Learn Mathematical Proof? A Comparison of Geometry Designs in German and Taiwanese Textbooks

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The learning of mathematical proof varies significantly among topics, tasks, curricular materials, and teaching. In this research, we focus on comparing the content of mathematical proof provided by curricular materials, especially school textbooks, between Germany and Taiwan. We first discuss how mathematics textbooks are designed and then report on the differences of geometric content structure between them (focused on grades 7–9) for unveiling a further comparison of the design relative to two common statements, *the Pythagorean theorem* and *the sum of interior angles of a triangle*. We compared six different textbook series, three from each country, using three different principles, *continuity*, *accessibility*, and *contextualization*, to inspect the content of these two statements. The results reveal that German textbooks employ a more *generic* approach to lead pupils to a theoretical position in mathematical proof and offer tasks in divergent contexts whereas Taiwanese textbooks prefer a *visual-algorithmic* approach to guide pupils to the transition from the pragmatic to the theoretical position by providing physical experiments or algorithmic tasks in the same context.

Keywords: mathematical proof, geometry, design, textbook

Introduction

The design process and usage of curricular materials, e.g., textbooks, are similar in both Germany and Taiwan. However, students' performances on international assessments are discrepant. To study the differences, it might be worthwhile to look deeper into the intended design of mathematics instruction in addition to comparing teaching and learning in classrooms. The importance of textbooks in school cannot be ignored. Although there is an increasing number of comparison studies in mathematics textbooks, most focus on comparing the differences of semantic features or textual presentations; only some discuss the details of the design for specific topics (e.g., Charalambous, Delaney, Hsu, & Mesa, 2010; Stylianides, 2005, 2009; Thompson, Senk, & Johnson, 2012).

Balacheff (2010) indicated the importance of acquiring mathematical knowledge as a process and not only receiving it as a fact.

> …mathematical ideas do not exist as plain facts but as statements which are accepted only once they have been proved explicitly; before that, they cannot be instrumental either within mathematics or for any application (p. 9).

As Balacheff (2010) said, getting involved in mathematics means for pupils (learners) to change their intellectual position and to become a theoretician. He provided two different types of shift, say from practical geometry (the geometry of drawings and shapes) to theoretical geometry (the deductive or axiomatic geometry) and from symbolic arithmetic (computation of quantities using letters) to algebra. Mathematical proof is viewed commonly as a core activity in mathematics (Heinze & Reiss, 2007). The basic features of the curriculum, being content, organization, and sequencing, have an impact on pupils' conception of proof (Chazan, 1993; Healy & Hoyles, 2000; Hoyles, 1997; Stylianides, 2007), and the activities of problem solving might provide an easy way for pupils to experience the process of proof (Balacheff, 1988; Pólya, 1981; Schoenfeld, 1992). Tall, Yevdokimov, Koichu, Whiteley, Kondratieva, and Cheng (2012) collected eight distinct methods of English, German and Taiwanese pupils' proof of the statement that the sum of the interior angles of a triangle is 180°. However, the question remains how the curricular sequences in the three countries differ from each other. Though there are various methods to prove this statement, e.g., empirical argumentation (Lin, 2000), or heuristic worked example (Reiss & Renkl, 2002), it does not mean that all pupils receive the same opportunity to do proof. As we said above, mathematical knowledge (ideas) influences students' learning; once they ascertain the validity of mathematical knowledge (prerequisites) they can apply it to a new statement. In addition, the alignment between designed tasks and how teachers select and use them influences the learners' engagement (Watson & Chick, 2011). Therefore, inspecting the designed tasks is important.

We will argue in the remainder of the paper that the different sequences of acquiring knowledge influence the design of tasks and have an implicit impact on students' opportunities to learn mathematics. We initiate an analysis on comparing the designs of geometry content in different textbooks between Germany and Taiwan and discuss the details of textual presentations as a precursor.

A snapshot of mathematics curricula in Germany and Taiwan

The *school system* (before collegiate education) in the Federal Republic of Germany differs according to each federal state. However, in general there are three different tracks in secondary school with differently designed curricula: Gymnasium (grades 5–12), Realschule (grades 5–10), and Hauptschule/Mittelschule (grades 5–9). For example, geometry proofs are primarily taught in the Gymnasium track. In contrast to Germany, mathematical proof is supposed to be introduced to all pupils in the unitary school track, junior high school (grades 7–9), in Taiwan.

Role of textbook and design process

Regarding the educational ideas and actual materials used in classrooms, it is necessary to allocate and define the position of related elements of them. In Figure 1, we present three layers with their respective trajectory of involved elements from ideal situation to reality. The dashed lines link the implicit relationships between three layers. The middle layer of three different roles of curriculum is the bridge between ideas and actual materials. Textbooks in German or Taiwanese schools can be viewed as intended curriculum or implemented curriculum depending on how teachers use them in classrooms. They are designed based on national standards and teachers use them as a tool to write their own lesson plans or directly include them in their teaching.

The processes of designing mathematics textbooks are similar in Germany and Taiwan. The contents of textbooks are designed by a group which might encompass researchers, mathematicians, mathematics educators, and school teachers, and finally edited by the responsible editor(s). All written textbooks should be designed based on the national standards (and state syllabus in Germany) and approved by the ministry of education, and then can be published. Each individual textbook is designed based on the textbook editors' stated intentions.

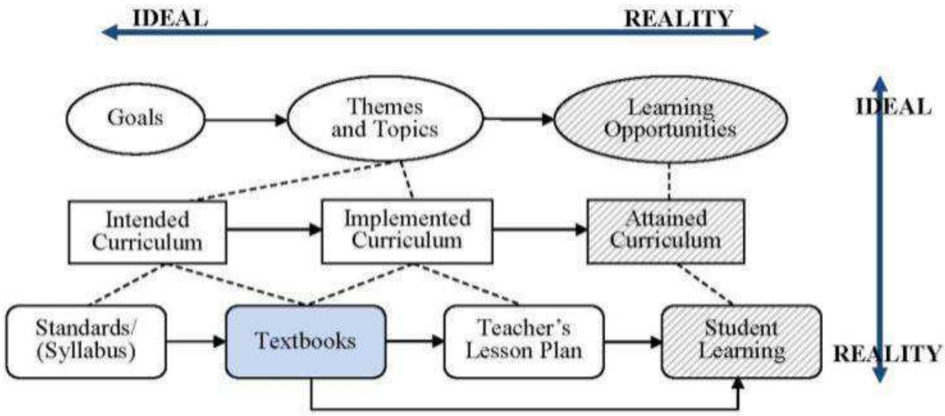


Figure1. The role of German and Taiwanese school textbooks

Distribution and structures of geometric contents

Geometry content is not in the same sequence in German and Taiwanese textbooks. In Germany, most geometry content is arranged in 7th grade, and part in 8th and 9th grades. In Taiwan, geometry is not taught in 7th grade or the second semester of 9th grade, but takes a large amount of classroom work in the second semester of 8th grade and the first semester of 9th grade. A comparison of both countries in different topics is shown in Table 1.

Content	Germany			Taiwan					
	7	8	9	7a	7b	8a	8b	9a	9b
2-D (Plane) Geometry	X	X	X			X	X	X	
3-D (Solid) Geometry			X				X		
[Circles] (tangent, intersection, angles)	X							X	
Perimeter, Area & Volume			X				X		
Constructions (ruler & Compass)	X						X		
Symmetry	X						X		
Parallel Postulate	X						X		
Congruence	X						X		
Similarity		X						X	
Pythagorean Theorem			X			X			
Slope& Trigonometry			X						

Table 1. Distributions of geometric contents between Germany and Taiwan

Comparing the content structures in all the textbook series adopted in this study, we developed a representative structure for each as listed in Table 2.

Germany			
Construction 1	Symmetry		Perpendicularity
			(Angle) Bisector
Angles	Parallel postulate		
Construction 2	Congruence postulate		
Special triangles	Isosceles triangle		
	Right-angled triangle		
	Thales theorem		
Construction 3	Special lines of triangles		
	(Centers of triangles)		
Similarity	Projection		
	Intercept theorem		
Pythagorean theorem	Hypotenuse-Leg theorem (Kathetensatz)		
	Leg-Leg theorem (Höhensatz)		
Trigonometry	**sine, cosine, tangent**		
Solid Geometry	Regular **Oblique**	Prism Cylinder **Pyramid Cone**	Angles
			Surface area
			Volumes [Cavalieri's principle]
			Length of perimeter

Taiwan		
Pythagorean theorem		
Construction 1	Perpendicularity	
	(Angle) Bisection	
(Line) Symmetry		
Solid Geometry	Surface area	
	Volumes [prism, cylinder]	
	Length of perimeter	
Interior and exterior angles		
Construction 2	Congruent postulates	
Side-angle relation of triangles		
Parallelism: Parallelogram		
Similarity	Proportional segment	
	(Intercept theorem)	
Circles		
Geometry Proof		

Table 2. Content structures of German and Taiwanese textbooks

Method

Textbooks are the main materials we used in this study. They are developed and designed with different editors' intents. Therefore, the presentations vary between different textbook series. Nevertheless, they are based on conjoint curricular goals across different settings for instruction. To examine different textbooks from different countries, it is necessary to set principles for comparing them in the same phase, specifically whether they are grounded on the same conceptual knowledge, learning opportunities, and context. Below, we list the materials used in this study and give our exposition of the principles we implemented for the comparison.

Analytical materials

We selected six representative textbook series used in school years 2009-2010, three from the state of Bavaria in Germany and three from Taiwan. In this study, we focus on grades 7, 8, and 9 of the Gymnasium track in Germany and of junior high school in Taiwan. We chose two common statements: the sum of interior angles of a triangle, which is introduced in grade 7 in Germany and the second semester of grade 8 in Taiwan; and the Pythagorean theorem, which is introduced in grade 9 in Germany and in the first semester of grade 8 in Taiwan. The texts from the textbooks can be generally separated into three parts: the *corpus* texts, which provide various activities and tasks to introduce mathematical knowledge; the *summary* part, which summarizes the point of one section or one chapter; and the *exercises* pool, which provides numerous exercises for pupils to practice. We focus only on the corpus texts introducing both statements, and exclude the summary and final exercises pool in order to avoid repetition (summary) and subjective selection without considering the continuity of former mathematical ideas (exercises).

Three principles of content comparison

We set three principles in inspecting the task design to be able to discuss the similarity and dissimilarity of German and Taiwanese tasks on the same scale. Principle one examines the *continuity* of the mathematical knowledge (ideas) involved, the flow of concepts. This principle contemplates the related concepts of a peculiar task from dissimilar societies. Principle two scrutinizes the *accessibility*, which means the learning opportunities provided by the designed tasks. It addresses whether the presentation allows the students access in a reachable way; in other words, it addresses clarity of the final resultant (goal) and the transparency of figural or textual representations. Principle three reviews the *contextualization* of tasks. The contextualization refers to the *sociocultural perspective*, which is regarded as stable physical and discursive elements of a setting in which a learning activity takes place. It refers also to the *constructivist perspective*, meaning that the personal, cognitive context shaped by the learner's personal interpretations of an activity is taken into account (Nilsson & Ryve, 2010).

Results

Two different approaches to the common statements

The geometric content of German and Taiwanese textbooks varies in topics and content structures as mentioned above. In this part, we show the common statements which relate to different mathematical concepts and are designed in different trajectories in both countries. In general, German textbooks design a new statement validly following the origin/hierarchy of related concepts. We call this a *generic* approach. In our view, the generic approach can be linked to the idea of Balacheff's (1988) *generic example*, "involving making explicit the reasons for the truth of an assertion by means of operation or transformation on an object that is not there in its own right, but as a characteristic representative of its class" (p. 219). In Taiwan, the *visual-algorithmic* approach relies highly on the figures and algorithm and is commonly used for geometry content.

Though some German textbooks simultaneously introduce the same content with a visual-algorithmic approach, it is not the mainstream of knowledge arrangement but an alternative method for teachers/pupils to teach/learn mathematics (and reflects probably the influence of the present international discussion). In this study, we compare only the main approach, the intersection of practice, among textbook series in one country. Below, we present the designs regarding both statements between German and Taiwanese textbooks.

Approaches to the Pythagorean theorem and the design

In both countries, instruction of the Pythagorean theorem should lead to the algebraic formula, $a^2 + b^2 = c^2$, where a, b and c are three sides of any right-angled triangle, and then the application of this formula in different types of problems. The relationship between the areas of three similar figures, usually squares as Euclid's *Elements* presents, generated from three sides of a right-angled triangle, is an important method to prove the theorem.

However, the methods of proving the Pythagorean theorem differ in German and Taiwanese textbooks (see Tables 3 and 4). The German curriculum arranges the Pythagorean theorem as part of a series of theorems including hypotenuse-leg theorem (Kathetensatz) and leg-leg theorem (Höhensatz). These are highly connected to the conditions of similarity of triangles, therefore learning the Pythagorean theorem presupposes the topic of similarity. The Taiwanese curriculum arranges this statement after learning the expansion of perfect squares: $(a + b)^2$ and $(a - b)^2$, with various permutations of geometrical figures. In the Taiwanese design, the visual figures and the algorithm coexist to influence the reasoning. Concerning the conceptual *continuity*, the routes to this statement are obviously different in both countries.

Some German textbooks provide additional tasks to construct[19] the squares generated from each side of the right-angled triangle and then encourage students to discuss the relationship between the areas.

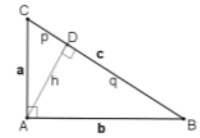
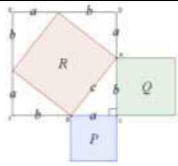
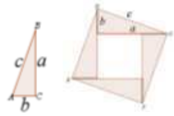
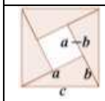
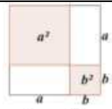
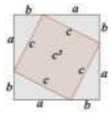
<table>
<tr><th colspan="2">German Main Approach</th><th colspan="3">Taiwanese Main Approach</th></tr>
<tr><td colspan="2"></td><td rowspan="4"></td><td rowspan="4">
</td><td rowspan="4">
</td></tr>
<tr><td colspan="2">Generic</td></tr>
<tr><td>Statements</td><td>Former ideas</td></tr>
<tr><td>$a^2 = p \times c$
$b^2 = q \times c$</td><td>Similarity:
$\triangle ACD \sim \triangle BCA$
$\triangle BAD \sim \triangle BCA$</td></tr>
<tr><td>$h^2 = p \times q$</td><td>Similarity:
$\triangle ACD \sim \triangle BAD$</td><td>Visual-algorithmic</td><td>Visual-algorithmic</td><td>Visual-algorithmic</td></tr>
<tr><td>$a^2 + b^2 = c^2$</td><td>1. Deduced from above formulae
2. Relations between area</td><td>$c^2 = (a+b)^2 - 4 \times \left(\frac{a \times b}{2}\right) = a^2 + b^2$</td><td>$c^2 = 4 \times \left(\frac{a \times b}{2}\right) + (a-b)^2 = a^2 + b^2$</td><td>$(a+b)^2$
$= a^2 + 2ab + b^2$……..(1)
$(a+b)^2 = c^2 + 2ab$…(2)
$a^2 + b^2 = c^2$</td></tr>
</table>

Table 3. Stereotypical approaches to the introduction of *the Pythagorean theorem*

	Germany	**Taiwan**
Continuity	Main idea: • Conditions of similarity • Additional ideas: • Figural construction • Area formulae	Main idea: • Area formulae of triangle and square • (Skills: Perfect squares)
Accessibility	• High accessibility to the stereotypical introduction	• High accessibility to the stereotypical introduction
Contextualization	• Various tasks (mainly, deductive reasoning) • Divergent contexts (e.g., figural construction; algorithm on figural areas)	• Various tasks of algorithm on figural areas, but stable context

Table 4. Three principles in analyzing *the Pythagorean theorem*

Approaches to the sum of interior angles of a triangle and the design

The topic connecting angles and geometric shapes is first introduced in 5th grade in both German and Taiwanese schools. Textbooks in both countries provide a physical experiment for students to experience the quantity of angles by measuring them with a protractor. The difference is that only the Taiwanese curriculum gives the sum of interior angles of a triangle, 180°, in the textbook in this school year. Then the formal reasoning on this topic is set in 7th grade in Germany after the statement of parallel postulates and in second semester of 8th grade in Taiwan.

In Tables 5 and 6, we show the related concepts introducing and applying this statement, and the arrangements of the design. In German textbooks, it is connected to the formerly learned concept that alternate interior angles of a pair of parallel lines with a transversal being equal (parallel postulates) and the concept of straight line (angle). In Taiwanese textbooks, it is not the main work to prove the truth of this statement (the sum of interior angles of a triangle equals 180°) but to apply this "authorized" knowledge in different situations. The introduction of this statement is

[19] There are ample opportunities of experiencing geometric construction (based on ruler and compasses) in German textbooks.

connected again to the students' experiences (physical experiment in 5^{th} grade) to "prove" visually/intuitionally with the concept of straight angle.

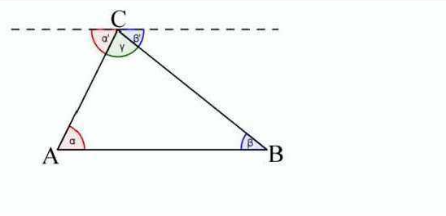
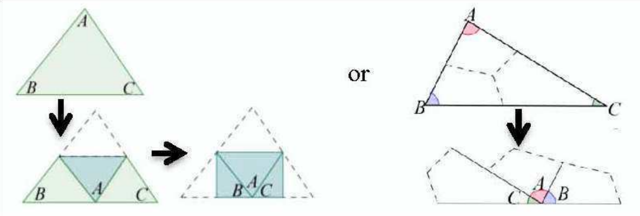
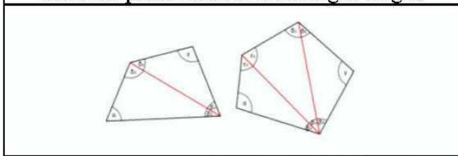
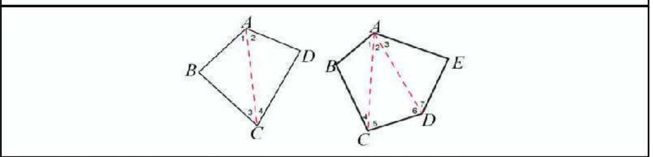
German Main Approach	Taiwanese Main Approach
	
Former ideas: Parallel postulates and Straight angle	Physical/figural experiment
	
(Algorithmic) Generalization of the statement to polygons: *180°·(n-2)*	

Table 5. Stereotypical approaches to the introduction of *the sum of interior angles of a triangle*

	Germany	**Taiwan**
Continuity	Main ideas: • Parallel postulate • Straight angle • Further: • The sum of interior angles of a polygon	Main idea: • Straight angle • Further: • The sum of interior angles of a polygon • The sum of exterior angles of a polygon
Accessibility	• High accessibility to the stereotypical introduction	• High accessibility to the stereotypical introduction
Contextualization	• Various tasks (mainly, deductive reasoning) • Divergent contexts (ex., to experience the invariance of the sum with transformation; physical operation)	• Various tasks with physical operation, but stable context

Table 6. Three principles in analyzing the sum of interior angles of a triangle

One point that needs to be mentioned is that the German design ends with the generalization[20] of this statement to the sum of interior angles of other polygons by the construction of auxiliary line(s). The follow-up of the Taiwanese design continues to apply the three straight angles and this statement: $(180° \times 3) - 180° = 360°$, to introduce the sum of exterior angles of a triangle by calculation, and then to generalize to all polygons by doing algebraic proof, that is, $(180° \times n) - [180° \times (n-2)] = 360°$.

Discussion

There are many factors which influence the design of curricular materials in different countries, such as design intentions for curriculum, content structures, educational goals, policy issues, etc. In this study, we restricted ourselves to a single aspect, the presentation of geometry content in textbooks. It turned out that the approaches to introducing mathematical proof through geometry content significantly differed in Germany and Taiwan. This was especially true for the details of the exemplifications of textbooks above. By examining the designed tasks for the same

[20] The Taiwanese design also presents the similar generalization as the German design does.

statements, the Pythagorean theorem and the sum of interior angles of a triangle, we could identify important differences.

We found that the opportunities to learn mathematical proof differ from Germany to Taiwan. In Germany, *deductive reasoning with hierarchical concepts* in geometry seemed to be pivotal in receiving new mathematical knowledge. In addition, the German textbooks *validated* two statements by deductively reasoning with generic mathematical ideas (theoretical position) and provided various operational tasks, like construction or physical experiments (pragmatic position). These operational tasks in German textbooks expected students to experience the concrete knowledge. Yet, this design of tasks was not always consistent with the central mathematical ideas (divergent contexts). In Taiwan, *authorized knowledge accompanying various applications* dominated the learning. . The Taiwanese textbooks used a physical experiment or reasoning with algorithms accompanied by figures to access both statements, and then arranged various tasks to apply them by using the algorithm in different situations. The truth of the statements was only shown with some empirical examples to "prove" (argue) the conjecture and it seemed the pupils would accept them. Though the differentiation between a pragmatic position and a theoretical position in mathematical proof was not clear in either statement, the consistency between concepts and activities was strong (stable context). Moreover, the application of algebraic proof, which followed these two statements (authorized knowledge), was emphasized more in Taiwanese textbooks. In summary, these differences might correspond to the research which indicates that German students lack strategies in proving while Taiwanese students lack principles to explain why a property is true (e.g. Heinze, 2004; Lin *et al.*, 2003; Reiss *et al.*, 2002).

We do not intend to over-generalize our judgement to the quality of tasks. The usage of tasks might differ with teachers' intentions in using them in class, and hence differ from teacher to teacher. Moreover, this work should not be seen as a general cross-cultural comparison, but it provides some evidence for different teaching styles. We provided an overview of how the same statement can be introduced in different arrangements and in different cultures; we see this as a first step for a more profound and more general comparison of textbooks.

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Designing the *Practical Worksheet* for Problem Solving Tasks

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Problem solving task design is not only the design of a non-routine problem to be solved by students. We conjecture that, in addition, task design also requires a supporting document that would act as a cognitive scaffold for students in the initial stages of the problem solving process before they can internalize the metacognitive strategies and automate the use of these strategies when faced with a new problem. In this paper we describe the design and use of such a document, which we have called *practical worksheet*, within our ongoing problem solving research project.

Keywords: problem solving, cognitive scaffold, practical worksheet

A fundamental aspect of doing mathematics is to solve mathematics problems. Mathematical problems are inherent in the structure of the subject itself and are the raw materials for problem solving, which is a highly valued process in mathematics education. There is now a mounting body of literature pointing to the fact that problem solving is still not implemented in mathematics classrooms, or if implemented, then certain routine approaches to heuristics are being adopted (see English, Lesh, & Fennewald, 2008; Lesh & Zawojeski, 2007; Schoenfeld, 2007; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). Stacey (2005) has suggested that:

> To get closer to the goal requires research directed to understanding the problem solving process for mathematics (in all its aspects), developing effective classroom processes, and designing excellent tasks. Moreover, the research needs to be closely intertwined with curriculum development and teacher development projects so that it can make an impact on practice. (p. 341)

Our research team has identified several issues in the literature in the international context as well as those in the Singapore local context that need to be addressed to facilitate the implementation of problem solving in schools. Our approach still values the problem solving model of Pólya (1945) and the insights from Schoenfeld (1985). To aid in the implementation of problem solving in schools, we have come to the realization that the design of specific problems or problem solving tasks cannot be the only focus of problem solving; rather, cognitive scaffolds (see Holton & Clarke, 2007) that allow students to solve a wider range of problems should also be an important focus. Accordingly, in this paper, we turn our attention to the design of the *practical worksheet* that can be used as a cognitive scaffold in problem solving tasks (see Figure 1). We document the development of the practical worksheet based on our design principles and feedback from teachers.

The Problem Solving Task Design Principles

Based on the literature, our research team drafted the following design principles for the problem solving tasks:

1. Each task is challenging, focused around important mathematical concepts, and interesting to the students.
2. Each task would offer the students an opportunity to extend and/or generalize.
3. The task, although fundamental to the design, is only the means to the loftier goal of transfer of problem solving skills to other situations.
4. A task is most meaningful to a student when presented as an assessment task for which the student earns some credit.
5. The process of solving the problem is as important, if not more important, than the final solution of the problem. As such, assessment of problem solving should consider carefully the problem solving process. The students should see that they earn enough credit for specific steps and/or intermediate strategies that they use.
6. Pólya's problem solving model and Schoenfeld's ideas about problem solving guide the development of the problem solving tasks.

Scrutinising the above principles, we came to the conclusion that, no matter how well problems (as tasks for students) themselves are designed, they can at best address the first two design principles. Good selection of problems alone would have little effect in fulfilling Design Principles 3-6, which are about the need for students to attend to the process of problem solving and the motivation for problem solving. As such, in considering task design for problem solving, we need to include elements of design that attend directly to Design Principles 3-6. To address these, we conjectured that the task for students should include a supporting document that would act as a cognitive scaffold for the students in the initial stages of the problem solving process before they could internalize the metacognitive strategies and automatically use these strategies when faced with a new problem. This is in line with the view that any scaffold should be gradually withdrawn as the learner becomes more competent (Rittle-Johnson & Koedinger, 2005; Yelland & Masters, 2007). We have followed Holton and Clarke's (2007) ideas:

> …cognitive scaffolding allows learners to reach places that they would otherwise be unable to reach. With the right word or question or other device a teacher may put in place the scaffolding that will allow new knowledge to be constructed, incomplete or wrong concepts to be challenged or corrected, or forgotten knowledge to be recalled. (p. 129)

Thus, our team's effort was directed towards the development of this important document that we have since called the *practical worksheet*. The practical worksheet is developed along the same lines as a worksheet that science students would generally use in a science class to help them carry out experiments, record observations, and make inferences. We have also incorporated ideas from Yelland and Masters (2007), who have also used the term cognitive scaffolding in the context of technology use.

> We have used the term *cognitive scaffolding* to denote those activities which pertain to the development of conceptual and procedural understandings which involve either techniques or devices to assist the learner. These include the use of questions, modelling, assisting with making plans, drawing diagrams and encouraging the children to collaborate with their partner. (p. 367)

The way a task is imagined and intended by the teacher may be quite different from the way it is construed and carried out by the students (see Mason & Johnston-Wilder, 2006). The intended learning by students may not happen if the tasks are misconstrued by them. Also, if teachers give too many directions, then the solution process may become too trivial for the students and the solution process may be reduced to a sequence of steps. If the teachers give too few directions, then the students may focus on different things and the implied learning may not happen. As such, the implementation of problem solving in the classroom ultimately hinges on the classroom teacher; in our designing process, we paid careful attention to teacher preparation for problem solving, including the use of the practical worksheet.

Teachers were taught three levels of scaffolding — Pólya stages, specific heuristics, and problem specific hints — to be used when advising students who are doing problem solving (Toh, Quek, Leong, Dindyal, & Tay, 2011). The levels are hierarchical and the next level of scaffolding should be given only after an earlier level has failed. Consider the Lockers Problem given below.

The Lockers Problem

A new school has exactly 343 lockers numbered 1 to 343, and exactly 343 students. On the first day of school, the students meet outside the building and agree on the following plan. The first student will enter the school and open all the lockers. The second student will then enter the school and close every locker with an even number. The third student will then 'reverse' every third locker; i.e. if the locker is closed, he will open it, and if the locker is open, he will close it. The fourth student will reverse every fourth locker, and so on until all 343 students in turn have entered the building and reversed the relevant lockers. Which lockers will finally remain open?

In the crucial level, which we call Level 0, we emphasise the student learning and reinforcing of the Pólya model (see Table 1). We may ask the student if he or she knows what Pólya stage he is in, and what would one normally do in such a stage. We help by asking these *Control* questions. In Level 1, we suggest specific heuristics to get the work moving. Level 2 is to be avoided as much as possible and is included only for the important aspect of ensuring that the self-esteem of the student is not seriously damaged by his or her perceived failure and helplessness on the problem. Here, we give problem specific hints, which essentially is a throwback to the 'usual' help afforded by mathematics teachers.

Level	Feature	Examples based on the Lockers Problem
0	Emphasis on Pólya stages and control	What Pólya stage are you in now? Do you understand the problem? What exactly are you doing? Why are you doing that?
1	Specific heuristics	Why don't you try with fewer lockers (*use smaller numbers*)? Try *looking for a pattern*.
2	Problem specific hints	Think in terms of the locker rather than the student – which student numbers get to touch the locker?

Table 1. Level of scaffolding

The objective of the practical worksheet is for students to internalize Level 0, and ask for Level 1, and to a much lesser extent Level 2, hints only when pressed for time. Assessment of problem solving is certainly another issue that guides students in problem solving tasks. To this end, an accompanying assessment rubric was developed to focus students on what is valued in the problem solving process. At the same time, the rubric gives them feedback on their strengths and weaknesses.

The Practical Worksheet

Holton and Clarke (2007) have claimed that scaffolding is an act of teaching that supports the immediate construction of knowledge by the learner and as well provides the basis for the future independent learning of the individual. These authors have added that scaffolding does not necessarily require the teacher and the student to be actually physically present together. Furthermore, it is essential for an individual student to be able to scaffold for himself or herself when solving a new problem, termed *self-scaffolding* by Holton and Clarke (2007). Accordingly, our aim was to find a way of developing the learner's autonomy in taking charge of his or her own learning when faced with an unfamiliar mathematical problem, whether the teacher was present or not. The artifact that we have developed is called the *practical worksheet*. We relate the evolution of this artifact from 2005 to the present, to scaffold mathematical problem solving behavior within the story of our efforts to teach mathematical problem solving in the schools.

The Development of the Practical Worksheet

In 2005, we developed a problem solving module within the mathematics curriculum of a high ability school in Singapore (see Appendix B). This initial design consisted of a 1.5 hour lecture on problem solving covering Pólya's problem solving model (Pólya, 1945) and Schoenfeld's framework (Schoenfeld, 1985). We had a second attempt in the same school in 2006. In 2009, the research team was invited by another high ability school to develop a module on problem solving. We adopted a design experiment approach (Brown, 1992; Collins, 1999; Gorard, 2004; Wood & Berry, 2003) to produce a workable "design" (an initiative, artefact or intervention, for instance) that could be adapted to other schools. In Gorard's (2004) words, "The emphasis [in design experiments], therefore, is on a general solution that can be 'transported' to any working environment where others might determine the final product within their particular context" (p. 101). We have argued the following theoretical justification for a design experiment for our research on problem solving (Quek, Dindyal, Toh, Leong, &Tay, 2011):

1. Obstacle - Current instruction on problem solving consists mostly of the teaching of heuristics only.

2. The theoretical basis of Pólya and Schoenfeld remain sound.

3. Mathematical problem solving must include the *Looking Back* stage of Pólya's model.

4. Mathematics problem solving is valuable enough to be adequately assessed and must be adequately assessed to be valued.

5. Mathematics problem solving is for every student of mathematics.

We then worked on the curriculum package by referencing the classical view of design where the parameters of the product are to be specified a priori (Ullman, 1992). We thus stated the following parameters, underpinned by the earlier theoretical justification, for the design of a package for teaching problem solving:

1. Place in the curriculum: The problem-solving module must be part of the mainstream mathematics curriculum.

2. Model of mathematical problem solving:

i. Pólya's model – all four stages

ii. Schoenfeld's framework – teach Heuristics and emphasise Control

3. Teacher autonomy: Teachers in school will ultimately teach the module themselves. Build teachers' capacity in problem solving and to teach it.

4. Infusion into regular mathematics content: Problem solving skills and habits learnt in the module must be infused into other mathematics modules to prevent atrophy.

5. Assessment: A valued component in school assessment

The following features were then built into the first prototype of the package to satisfy the demands of the stated parameters. From our experience with the first school, we realised that we must first look for a way out of the perennial quandary of the undervaluation of assessment of problem solving. As before, our package promoted mathematical problem solving as learning the processes of mathematics akin to the established science practice as a way to learn science processes. We designed a mathematics practical worksheet (Figure 1 shows a typical response by a student on a test which used the practical worksheet) and added a rubric for assessment (further details are available in the next section). Parameters 1 and 5 were addressed. A series of lessons to teach students how to use Pólya's model as a scaffold, to teach various heuristics and choice of heuristics, and to make students aware of the need for control in problem solving was developed. Parameter 2 was addressed.

To us, it appeared necessary that the teachers make the proposed instructional approach a routine sufficiently familiar to them so that the approach becomes classroom practice. To reach this stage, it seemed essential for the teachers to adapt the researchers' ideas to make them their own, in the sense that their beliefs of mathematics and of problem solving in mathematics are transformed. Such a process would pass through a stage where the teachers negotiate and change the problem solving lesson. Finally, a community of practice would develop among the teachers to support the change process by providing opportunities to learn to engage the proposed ways – thinking, talking and reflecting on the new teaching experiences and ways of doing mathematics (Shulman & Shulman, 2004). The entire process of transforming an externally proposed instructional approach and curricular change into classroom and school practice appeared to be cyclical, incremental, and emergent in nature. This was our approach to Parameter 3.

To address Parameter 4, problems that were from the regular mathematics curriculum and were rich enough for extended work would be crafted and infused into the regular schedule. The justification was the reference to science practicals going in tandem with science theory lessons, and the motivation was that these difficult problems could be used as assessment *for* learning.

Assessment Using the Scoring Rubric

Traditionally, the assessment of problem solving in the classroom has focused on assessing the products rather than the processes of problem solving. Our efforts to meet the challenge of teaching mathematical problem solving to students call for a curriculum that emphasizes the processes (while not neglecting the products) of problem solving. The assessment strategy must match it so as to drive the mode of teaching and learning of mathematics.

It is common knowledge that most students study mainly those curricular components which are to be assessed. Accordingly, there needs to be a corresponding assessment strategy that drives the teaching and learning of problem solving as described in the preceding paragraphs. Effective assessment practice begins with and enacts a vision of the kinds of learning we most value for students and strives to help them achieve. To assess the students' problem-solving processes (which we value), we developed a scoring rubric based on Pólya's model and Schoenfeld's framework.

The scoring rubric focuses on the problem-solving processes highlighted in the practical worksheet. There are four main components to the rubric, each of which would draw the students' (and teachers') attention to the crucial aspects of an attempt to solve a mathematical problem. In establishing the criteria for each of these components of problem solving, we ask the question: What must students do or show to suggest that (a) they have used Pólya's approach to solve the given mathematics problems, (b) they have made use of heuristics, (c) they have exhibited "control" over the problem-solving process, and (d) they have checked the solution and extended the problem solved (learnt from it)?

The rubric is outlined below. The complete rubric is in Appendix A.

- *Pólya's Stages* [0-10 marks]. This criterion looks for evidence of the use of cycles of Pólya's stages (Understand the Problem, Devise a Plan, Carry out the Plan), and correct solutions.
- *Heuristics* [0-4 marks]. This criterion looks for evidence of the application of heuristics to understand the problem, and to devise/carry out plans.
- *Checking and Extending* [0-6 marks]. This criterion is further divided into three sub-criteria:
  - Evidence of checking of correctness of solution [1 mark];
  - Providing for alternative solutions [2 marks];
  - Extending and generalizing the problem [3 marks]. Full marks for this part is awarded for one who is able to provide (a) two or more problems with solutions or suggestions to solution, or (b) one significant related problem with comments on its solvability.

The rubric was designed to encourage students to go through Pólya stages when they are faced with a problem, and to use heuristics to explore the problem and devise a plan. They would return to one of the first three stages (see practical worksheet) upon failure to realize a plan of solution. Students who show control (Schoenfeld's framework) over the problem-solving process gain marks. For example, a student who did not manage to obtain a completely correct solution would be able to score up to eight and three marks each for *Pólya's Stages* and for *Heuristics*, making a total of eleven, if they show evidence of cycling through the stages, use of heuristics, and exercise of control.

The rubric allows the students to score as much as 70% of the total 20 marks for a correct solution. However, this falls short of obtaining the top marks for the problem. The rest would come from the marks in *Checking and Extending*. Our intention is to push students to check and extend the problem (Stage 4 of Pólya's stages), an area of instruction in problem solving that has not been largely successful so far (see for example, Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005).

M-ProSE

Problem

There are two timers: one for 5 minutes and one for 9 minutes. We want to heat a beaker of water for exactly 11 minutes. How can we do this using only these timers?

Instructions

- You may proceed to complete the worksheet doing stages I – IV.
- If you wish, you have 15 minutes to solve the problem without explicitly using Polya's model. Do your work in the space for Stage III.
  - ➢ If you are stuck after 15 minutes, use Polya's model and complete all the stages I – IV.
  - ➢ If you can solve the problem, you must proceed to do stage IV – Check and Extend.

I Understand the problem

(You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

(a) Write down your feelings about the problem. Does it bore you? scare you? challenge you?

(b) Write down the parts you do not understand or that you misunderstood.

(c) Write down the heuristics you used to understand the problem.

Attempt 1

a) I feel challenged, because I've done a similar question once, although it is simpler.

b) None

c) I noticed that 5, 9, 11 have no common factors, and I tried to do it mentally for a while to get the hang of it.

II Devise a plan

(You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.)

(a) Write down the key concepts that might be involved in solving the question.

(b) Do you think you have the required resources to implement the plan?

(c) Write out each plan concisely and clearly.

Plan 1

1. Define variables (x and y)
2. set up equations
3. Solve the equation
4. Relate back to problem
5. offer solution

III Carry out the plan

(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.)

(i) Write down in the *Control* column, the key points where you make a decision or observation, for eg., go back to check, try something else, look for resources, or totally abandon the plan.

(ii) Write out each implementation in detail under the *Detailed Mathematical Steps* column.

Detailed Mathematical Steps	*Control*
Attempt 1	
Let the number of times the 5 minute timer is used by x, and the 9 minute timer be y.	Define variables
Therefore, $5x+9y=11$	Setting up equation
By the Eucledian Algorithm, $9=5+4$, $5=4+1$	Solving the equation
$1=5-4$ $=5-(9-5)$ $=2(5)-9$	
$11=22(5)-11(9)$ $=4(5)-1(9)+18(5)-10(9)=4(5)-1(9)$	
Hence we have the result: $4(5)-1(9)=11$	
This means that the 5 minute timer is used 5 times, while the 9 minute timer is used only once.	Relating back to problem
*Refer to attached paper	

IV Check and Extend

(a) Write down how you checked your solution.

(b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of.

(c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

Checking

If the water starts to be heated after the 9 minute timer, the 5 minute timer will have 1 minute left. Then the 5 minute timer is played 2 more times, totalling to $5+5+1=11$ minutes.

Alternative Solution 1

In order to form 11, we have the following ways:

$2+9$

$5+6$

$2(5)+1$

$3+8$, $4+7$ } Very low chance of being a solution, since it does not contain 5 nor 9.

Hence our aim is to get 2, 6, or 1.

However, 5 and 9 are coprime, and through the Euclidian algorithm,

$2(5)-9=1$

Now that we have 1, we just need to add 5 twice, and get 11.

$4(5)-9=11$

By the same reasoning as part 3, the solution is to start both timers together, and retiming the 5 minute timer when it stops. When the 9 min. timer stops, ~~play~~ start heating the water. The first time the 5 minute timer times out after start of heating, the water would have been heated 1 minute. Timing 5 minutes twice, we get 11 minutes.

Figure 1. Student's solution using the practical worksheet

Conclusion

The practical worksheet holds promise for teachers who want to elevate problem solving to a prominent position in the mathematics classroom. They can now not only encourage problem solving in their classes, but they can also make transparent to students the criteria for assessment and the processes that are valued. Taken altogether, our task design, which includes the practical worksheet, assessment rubric, a set of problem solving lessons, and teacher scaffolding, has shown great potential in developing student self-scaffolding in problem solving.

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Appendix A

RUBRIC FOR ASSESSING PROBLEM SOLVING

Name: ______________________________

Pólya's Stages		
	Descriptors/Criteria (evidence suggested/indicated on practical sheet or observed by teacher)	Marks Awarded
Correct Solution		
Level 3	Evidence of complete use of Pólya's stages – UP + DP + CP; and when necessary, appropriate loops. [10 marks]	
Level 2	Evidence of trying to understand the problem and having a clear plan – UP + DP + CP. [9 marks]	
Level 1	No evidence of attempt to use Pólya's stages. [8 marks]	
Partially Correct Solution (solve significant part of the problem or lacking rigour)		
Level 3	Evidence of complete use of Pólya's stages – UP + DP + CP; and when necessary, appropriate loops. [8 marks]	
Level 2	Evidence of trying to understand the problem and having a clear plan – UP + DP + CP. [7 marks]	
Level 1	No evidence of attempt to use Pólya's stages. [6 marks]	
Incorrect Solution		
Level 3	Evidence of complete use of Pólya's stages – UP + DP + CP; and when necessary, appropriate loops. [6 marks]	
Level 2	Evidence of trying to understand the problem and having a clear plan – UP + DP + CP. [5 marks]	
Level 1	No evidence of attempt to use Pólya's stages. [0 marks]	
Heuristics		
	Descriptors/Criteria (evidence suggested/indicated on practical sheet or observed by teacher)	Marks Awarded
Correct Solution		
Level 2	Evidence of appropriate use of heuristics. [4 marks]	
Level 1	No evidence of heuristics used. [3 marks]	
Partially Correct Solution (solve significant part of the problem or lacking rigour)		
Level 2	Evidence of appropriate use of heuristics. [3 marks]	
Level 1	No evidence of heuristics used. [2 marks]	
Incorrect Solution		
Level 2	Evidence of appropriate use of heuristics. [2 marks]	
Level 1	No evidence of heuristics used. [0 marks]	

Checking and Expanding		
	Descriptors/Criteria (evidence suggested/indicated on practical sheet or observed by teacher)	Marks Awarded
Checking		
Level 2	Checking done – mistakes identified and correction attempted by cycling back to UP, DP, or CP, until solution is reached. [1 mark]	
Level 1	No checking, or solution contains errors. [0 marks]	
Alternative Solutions		
Level 3	Two or more correct alternative solutions. [2 marks]	
Level 2	One correct alternative solution. [1 mark]	
Level 1	No alternative solution. [0 marks]	
Extending, Adapting & Generalizing		
Level 4	More than one related problem with suggestions of correct solution methods/strategies; or one **significant** related problem, with suggestion of correct solution method/strategy; or one **significant** related problem, with explanation why method of solution for original problem cannot be used. [3 marks]	
Level 3	One related problem with suggestion of correct solution method/strategy. [2 marks]	
Level 2	One related problem given but without suggestion of correct solution method/strategy. [1 mark]	
Level 1	None provided [0 marks]	

Hints given:

Marks deducted: __________________________

Total marks: __________________________

Appendix B

Table 1: Development of the Practical Worksheet

Year	Research Emphasis	Objective	Notes/Issues
2005	Development of a problem solving module in school XX (a 1.5-hour lecture followed by eight 1-hour lessons)	To help students to: understand a problem before rushing in to solve it; devise a plan to attack the problem; use heuristics to understand the problem and as a plan of attack; check a solution attempt; pose new problems with reference to a successful solution of the original problem	Moderate success. Students were reluctant to apply the stages of the Pólya's model. Even higher achieving students did not check and extend the problem
2006	Second attempt in same school. An approach to treat the mathematics lesson as a 'mathematics practical' lesson. Development of a worksheet (first version of the practical worksheet)	Accommodate the different problem solving strategies of students. Not to force all students to go through each of the four Pólya's stages	Some issues still persisted. Some students were unfamiliar with the problem solving heuristics. Some students could not quite differentiate among some of the Pólya's stages.
2007	Teachers in the same school were left on their own to implement the problem solving module.	Teachers to develop autonomy and work independently of the researchers. Develop such a problem solving module sustainable in a mainstream school.	Some students were still not filling the worksheet as desired. The lead teacher decided to put tick boxes for the students to tick rather than write out certain parts.
2009	In another school for more able students	Adopt a design experiment approach in the research and produce a workable design not only of the problem solving module but as well of the practical worksheet. Have a greater focus on teacher development for use of the worksheet in their lessons.	There was better success in this school. A scoring rubric focusing on the processes of problem solving had to be developed to be used in tandem with the worksheet.
2010-2012	Dissemination of the problem solving module to more schools	Adapt the problem solving module using the practical worksheet for implementation in a wider range of schools.	Positive feedback so far on the use of the practical worksheet

Textbooks' Design and Digital Resources

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In this paper we report on the comparison of the design and conceptualization of two very different French lower secondary mathematics textbooks: one which was developed, as it is 'traditionally' done, by 'experts' (teacher educators and researchers); and one which was developed, innovatively, by teachers using a digital platform. These different designs and conceptualizations had repercussions on the content, structure, potential and intended use of the books, which we investigated using specially designed questionnaires given to the two groups of textbook authors. Our results point to re-conceptualisations of the notions of 'quality' and 'coherence' of resources, such as textbooks, taking into consideration teachers' documentation work, in particular their often collective work with digital resources.

Keywords: digital teaching resources, mathematics teaching resources, quality process, textbooks authors' intentions; teacher communities; textbooks' design; teachers' documentation work; textbook use

Introduction: digital means and evolutions of the textbooks' design

As stated in the ICMI study 22 discussion document, "most teachers use textbooks and/or online packages of materials as their total or main source of tasks." Moreover, the textbook content and structure influence teachers' choices (Haggarty & Pepin, 2002; Pepin, 2009), for example, in terms of their choice of tasks or sequencing of the topic area; both content and structure depend on the textbook's design. We investigate textbook design and its developments brought about by digital means. Textbooks are now often complemented by digital materials: files to be projected during the lesson by the teacher; animated figures and exercises using different software and providing feedback to students, available on a CD or on a website. We focus on the following question:

> "How can or should new digital formats influence textbook design: e.g. use of podcasts, twitter, and other social media; implications for design and coherence of materials (either original digital design or transfer from print) if teachers are able to select tasks in varied orders?" (ICMI study 22, discussion document, p.19).

We consider that this question is a complex one. It encompasses indeed several aspects of the evolutions resulting from digital means, in particular the following: digital means provide new opportunities for the structuring of textbooks for their use by teachers. They also open up new possibilities for design and further evolution. In previous works, we provided evidence that digital means foster collective teacher work (Gueudet & Trouche, 2012): teachers discuss by e-mail; or online teacher associations create resources to be shared by all, not only members. Like many other kinds of teaching resources, now the textbook can also be designed by groups of teachers: “bottom-up” designs (in contrast to a traditional “top-down” design) are now developing, in particular in countries where national policies allow it. This constitutes a major evolution, which we retain as a central focus of our study.

Considering these developments, what does the concept of “coherence of a textbook” mean? The discussion document mentions (p. 17), “conceptual coherence.” This can be understood as the correctness of the mathematical content; an alignment or ‘consistency’ with the official curriculum; a sequencing of the notions and properties introduced to avoid gaps in the mathematical progression; a correct articulation between the text of the course and the associated exercises and problems, or an emphasis of ‘abstraction’ as an ‘umbrella concept’ for the coherence of the textbook. In the case of a collective design, coherence can also mean that the individual mathematical intentions of the individual authors, their epistemological stances, are well coordinated; and that the whole textbook corresponds to the same “mode of address” (defined as the positioning of the user induced by the material (Remillard, 2012)). Moreover, in terms of implications for learning, the most important issue linked to coherence is certainly the ‘coherence’ of what the teacher produces when drawing on the textbook. Similar to Shield and Dole (to appear), we consider that analysing textbooks can only inform about their potential to assist in teaching and learning, because teachers interact with textbooks in various ways.

Investigating the potential of a resource leads to an exploration of the concept of quality. We consider that the intrinsic quality of a resource has to be distinguished from its adequacy with respect to institutional and users’ expectations (Trouche, Drijvers, Gueudet, & Sacristan, 2013). The intrinsic quality encompasses mathematical, didactical, and ergonomic (ease of use) aspects. It also depends on the mathematical topic considered. For example, Trgalová, Soury-Lavergne, & Jahn (2011) assessed the quality of dynamic geometry resources, and they differentiated between nine dimensions of quality: mathematical content; technical aspects; instrumental aspects; added-value of dynamic geometry; didactical implementations; pedagogical implementations; the potential of the resource integration into a teaching process/sequence; ergonomic aspects (e.g., presentation and adaptability); and metadata (e.g., accuracy facilitating searchability).

Traditional textbook analysis (e.g., in TIMSS (Valverde et al., 2002)) proposes three aspects: content (e.g., number, measurement, geometry); performance expectation (knowing, using routine procedures, problem solving, mathematical reasoning, and communicating); and perspective (attitudes, careers, participation, interest, and habits of mind). Most textbook analyses focus on tasks (e.g., exercises and working tasks), and many studies have analysed problem-types (Zhu & Fan, 2006), problem solving procedures (Fan & Zhu, 2007), procedural complexity (Vincent & Stacey, 2008), cognitive demand (Jones & Tarr, 2007) or concept treatment (Cai, Lo, & Watanabe, 2002).

However, how textbooks deal with depth of understanding, for example in terms of mathematical abstraction, is largely left untouched. Without going into deep philosophical discussions, textbooks are without doubt didactical materials, and as such can be seen as providing ‘tools and products of abstraction’ (e.g., in their tasks, representations, contexts). Textbooks use different registers of representation (Duval, 2006), e.g. usual language symbols; figures; representations of technological tools. This variety, and the need for conversion between different representations, can be associated with 'depth of understanding'. Another way of analysing depth of understanding might be by conceptualising 'understanding mathematics' in terms of 'making connections' (Pepin, 2008): e.g., connections to what the pupils already know; to authentic situations; across mathematical topics; across other subjects. This is in line with the literature on 'learning mathematics with understanding' (e.g., Hiebert et al., 1997). They contend that students build mathematical understanding by 'reflecting and communicating', and tasks should allow and encourage these processes. This means that such tasks should have the following features:

> "First, the tasks must allow the students to treat the situations as problematic, as something they need to think about rather than as a prescription they need to follow. Second, what is problematic about the task should be the mathematics rather than other aspects of the situation. Finally, in order for students to work seriously on the task, it must offer students the chance to use skills and knowledge they already possess. Tasks that fit these criteria are tasks that can leave behind something of mathematical value for students." (p.18)

In summary, and considering the above, the notion of 'quality' is complex: it involves the notion of coherence and depth of understanding in textbook analysis, and this in the light of the evolutions brought on by digital means. We discuss this further in what follows, drawing on the comparison of two differently designed textbooks. The main research questions are: What are the differences between a textbook designed by a team of experts (researchers, teacher trainers, etc.); and a textbook designed by a group of teachers using a digital platform? What are the 'consequences' of these different designs, in particular in terms of coherence and quality of the textbooks produced?

In the next section, we present the study and subsequently the findings and results. The findings are presented under three headings: (1) the two contrasting teams of authors and their different conceptualisations of their respective textbooks; (2) authors' choices concerning content and structure; and (3) authors' intentions concerning the use of their textbooks.

The study

Our study took place in France, where no institutional control of textbooks exists. France has a National Curriculum, which is presented as a text detailing mathematical objectives and accompanied by detailed comments for teachers, in addition to a booklet giving a structured list of pupil competences. For our study we selected two very different textbooks on the basis of contrasting cases. Both textbooks were grade 6 books (first year of lower secondary school, *collège*, in France). Because no official statistical figures were available on the most commonly bought mathematics textbooks for this grade, we cannot claim that these textbooks were the most used by teachers. However, they were the most commonly used by the teachers we worked with, and most mathematics teachers in the region knew of them.

The differences between the two books not only concerned their content or the material they offered; the differences were also linked to the authors' teams and the design processes. Therefore, we designed a questionnaire for each of the authors' teams. The questionnaires were designed by drawing on our knowledge of previous studies on textbook analysis (e.g., Pepin & Haggarty, 2001) and on our knowledge of the books, including what teachers told us about their use. The questionnaires included questions on the textbook's design mode; the authors' perspectives on mathematics and their teaching; and on the design choices, on a general level and on specific aspects of the textbook studied. The results we present here draw on a cursory analysis of the textbooks, but more importantly on the analysis of the responses to the questionnaires. The analysis involved the identification of similarities and differences, category generation and saturation based on *constant comparison* using a procedure similar to that described by Woods (1996). In particular, we have chosen to focus on the differences in this paper.

Different authors, different conceptualizations

In terms of textbook design, Helice 6[e] (Chesné, Le Yaouanq, Coulange, & Grapin, 2009) has been developed by a team of four "experts": three teacher educators, two of them with a masters' degree in mathematics education; and a researcher in mathematics education. A grade 6 teacher (not considered an author) 'tested' some of the tasks in his class. Asked about the way they evaluated the relevance of the content, the authors declared that they trusted "research results, or [their] training experience" – we call it 'expert evaluation'. These experts were clearly aware of their expert position.

Sesamath 6[e] (Sesamath, 2009) has been developed by a large group of authors: approximately 57 lower secondary school mathematics teachers (being involved in producing both the paper version and the digital complements), their work being coordinated by members of the Sesamath association[21]: teachers who are involved in the design of online resources. The Sesamath association (Gueudet & Trouche, 2012, Sabar & Trouche, 2013) designed many other resources: online exercises; adigital geometry software; and a complete virtual environment, LaboMep[22]. LaboMep allows the co-ordination of various kinds of resources, from Sesamath or from other sources on the web, and their preparation for student use. The "Sesamath 6[e]" textbook was published under a free license; it can be downloaded from a website[23], or used online (Figure 1). A paper textbook exists, corresponding to the text files (which are available in .pdf and .odt). Other complementary files (e.g., dynamic geometry files, spreadsheets, slides, online exercises) can only be accessed using the website.

In terms of evaluation of the content relevance, the teacher authors and other members of the association used the developed textbook in their classrooms and observed and evaluated it, in particular in terms of their students' involvement with particular features of the book. Referring to a distinction introduced in the field of computer supported collaborative learning (Dillenbourg, Baker, Blaye, &O'Malley, 1996), we consider that the collective work of the authors in the case of Helice was *collaborative* (the authors sharing each part and step of the work). In the case of Sesamath, we consider the authors' collective work as *co-operative:* different tasks had been assigned to different authors.

Another important difference which we develop in the discussion was that the content of the digital textbook continuously evolved, according to the experiments

21 The name of the association itself, Sesamath, is interesting and linked to « Open sesame », the famous phrase from the Arabian Nights. The motto of the association is "mathematics for everybody".

22 LaboMep- Laboratory for Mathematics in the Pocket

23 http://manuel.sesamath.net/

and contributions of the teacher users. In fact, Sesamath proposed a website, ‘Sesaprof’, which was open for teachers and comprised of a ‘discussion forum’ section (discussing the textbook). These online discussions led to modifications of the textbook’s content: for example, when a large number of users asked for the solutions of the textbook’s exercises, these were added in the digital textbook.

Beyond these differences, both teams claimed that they were constrained by commercial publishing and user expectations. For example, the Helice team explained that they had wanted to write more guiding comments for the teacher, but had been asked by the publisher to limit these comments. For the Sesamath team, the grade 6 book was the last in a series; they had started with and accomplished textbooks for grades 7, 8 and 9 (in this order) due to curriculum changes. Thus, the structure of these earlier textbooks gradually became a model familiar to users, and the authors were obliged to keep a similar structure for the grade 6 textbook.

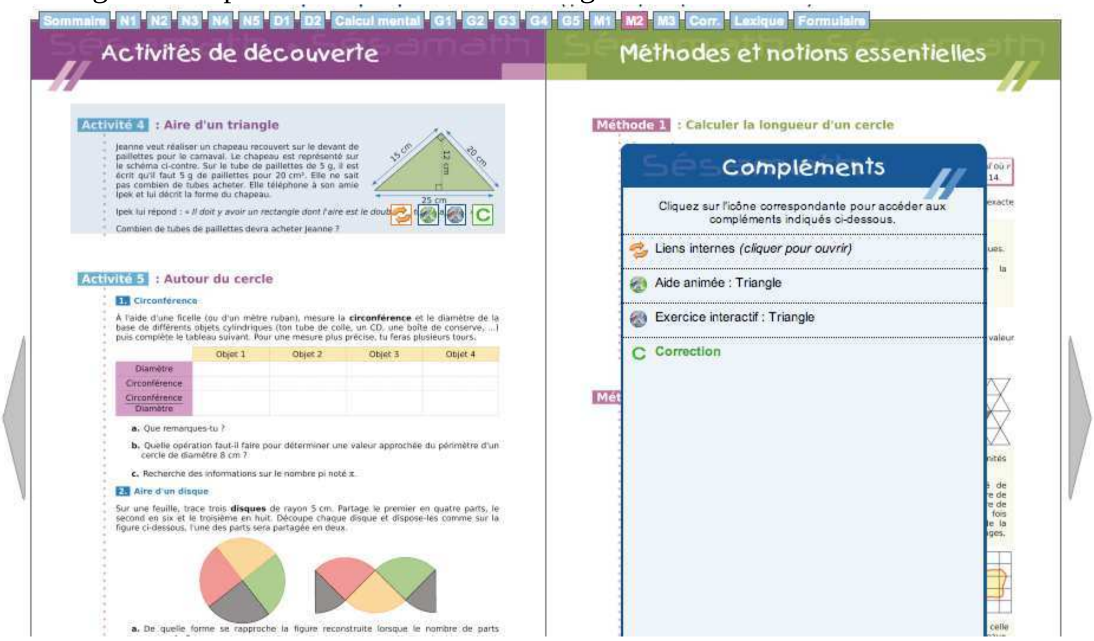

Figure 1. *Sesamath online textbook for grade 6*. The text on the screen background corresponds to the content of the .pdf, or paper version. The “complements” window opens when the mouse is placed on a selected activity.

Authors’ choices concerning content and structure

The analysis of the textbooks and the authors’ answers to the questionnaire showed many differences concerning the textbook content. We illustrate these statements with examples from the mathematical topic of ‘area’.

Organisation of the content

Concerning the textbook’s global structure and the organisation of the mathematical content, Sesamath provided a “classical” organisation: chapters were organised according to the headings of the official National Curriculum. As an example, the topic of ‘area’ was in the section entitled “Area and Perimeter”, which was the second section of the chapter entitled “Quantities and Measures”.

Helice had a very special, *spiral* organisation, hence the title “Helice” (meaning “Helix”). The book was not structured in chapters, but in “units”, working with different ‘thematical’ lessons, and a ‘unit’ finished with problems and exercises linking the different notions learnt. The authors of Helice presented this (spiral organisation) as their central and original structure. Indeed, in France Helice was the only textbook retaining this composition. The authors specified that the intention had been that the learner revisited and deepened the same notion at different stages — a spiral curriculum. At every stage the notions were associated with different representations; links between different chapters were frequently made; and differentiation in terms of pupil learning (e.g., pace) was taken into account. The topic 'area' appeared twice in the table of contents: in the unit entitled "Distance and areas" (Unit 4; lesson called "Area: comparison and sharing") and in the unit entitled "Division and computation of area" (Unit 5; lesson called "Areas: measure and computation"). Moreover, the area of a circle was in fact presented in Unit 7, as an activity demonstrating that the area of a circle and the square of its radius are proportional.

We claim that this difference in terms of structure was influenced by the different design modes and author teams. Following a complex and coherent structure, such as the one retained by Helice, was only possible for a 'steady' author team, and likely to be very difficult for a large and 'variable' author group, such as the Sesamath group. A large 'collective' process, like Sesamath, required splitting the content into different parts, which then were designed by different authors who did not necessarily communicate. This splitting of tasks, we argue, has also influenced the *coherence* of the textbook. Bringing together the work of more than fifty authors and achieving a coherent didactical structure would require an enormous coordination effort, more than the one organised by the Sesamath team.

Different methods, or a single expert method

Helice authors stated that it was important for them to propose different representations and a rich vocabulary in their book. They also favoured exercises with different possible solutions.

In Sesamath, our analysis showed that some additional research activities suggested and fostered the search for several solutions. However, we contend that most of the exercises led to a single and final solution. In particular, the worked examples were all one-solution exercises, the authors called them 'expert solution' in the questionnaire (Figure 2).

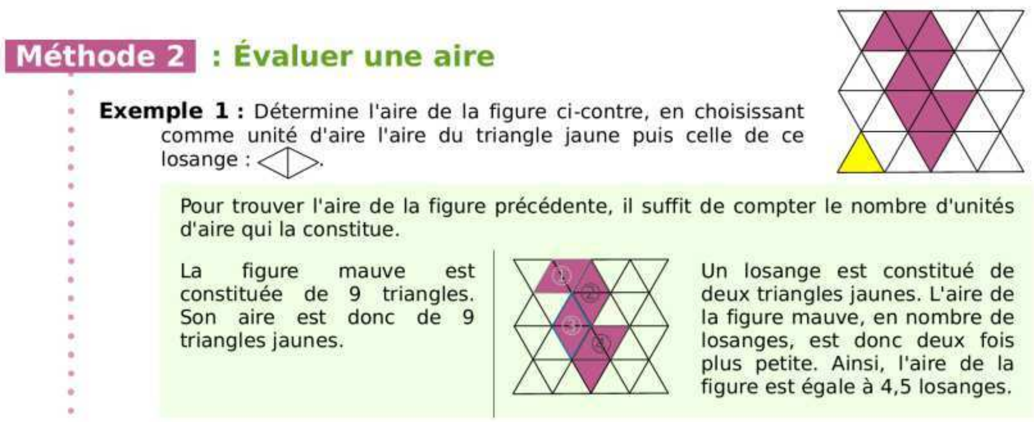

Figure 2. *Extract of Sesamath* 6^e. Method 2: evaluating an area.

Example 1. *Determine the area of this figure by choosing the yellow triangle as the unit, then calculating the area of this rhombus.* To find the area of the figure, you have to count the number of area units of the figure. The purple figure is made of 9 triangles. Its area is thus the area of 9 yellow triangles. The rhombus is made of two yellow triangles. The area of the purple figure, in number of rhombi, is thus twice as smaller. Hence the area of the figure equals 4.5 rhombi.

Primary-secondary link

Another important aspect, given that the textbooks were for grade 6, was whether the primary-secondary link was explicitly addressed. Some of the Helice authors were teacher educators, both for secondary and primary school level. Thus, they had considerable knowledge about the content taught at primary and secondary school. They identified several crucial cognitive and didactical changes in terms of the transition from primary to secondary school mathematics. Concerning 'area', one important choice, according to them, was to start with activities on the comparison of areas of two figures, without using any kind of measures. The activity in Figure 3 illustrates this choice.

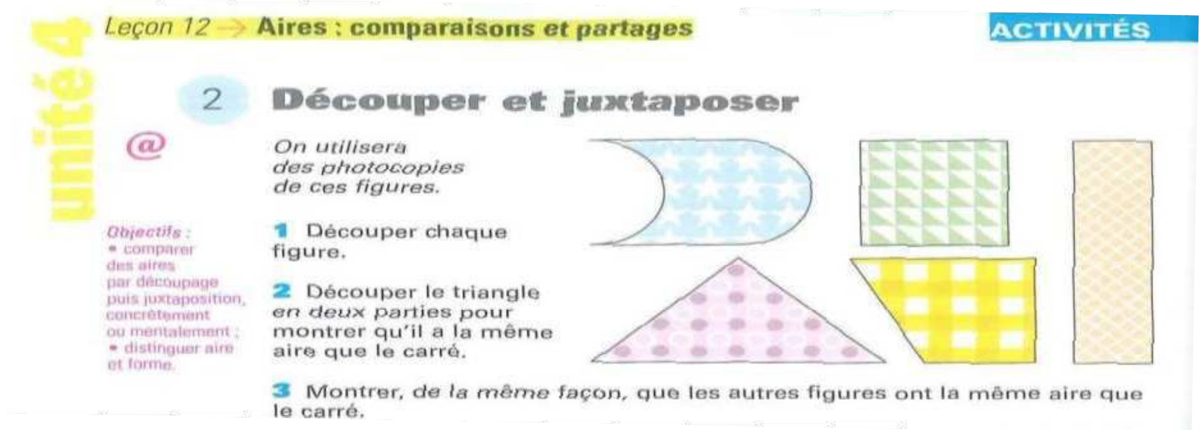

Figure 3. *Extract of Helice 6^{e}*. Lesson 12. Areas: comparing and sharing.

Activity 2: Cutting and arranging. The objective of the activity is to demonstrate that all these figures have the same area as the square, by cutting and adjusting to obtain a superposition with the square.

The official curriculum mentioned "geometrical comparison of areas" as a teaching objective. This aspect was particularly developed in Helice; and this was a deliberate choice of the authors, in order to provide a better link to primary school notions where comparing areas without using measure was a common task. Sesamath authors declared that they had only limited knowledge of primary school mathematics curricular aspects when they started writing the grade 6 book. Concerning 'area', the textbook focused on area computations, whereas the official curriculum also stipulated "geometric comparison of areas". This geometric comparison, without measures, was not evident in the Sesamath textbook. With a better knowledge of primary school, where such tasks were frequent, the authors may have inserted it. However, the Sesamath authors paid attention to one 'classical' difficulty: the potential problem to distinguish between area and perimeter of a figure. This difficulty was known to be important for primary school, and also for grade 6 students, and as teachers of grade 6 students the textbook's authors knew about it.

Writing the book raised the authors' awareness of transition questions (e.g., the importance of mental arithmetic for grade 6.). The paper version of Sesamath did not reflect this, but in the associated online resources, in particular online exercises, the authors attended to this aspect of primary-secondary transition. Whilst the paper book has remained the same since 2009, the authors claimed that they had attended to particular 'shortcomings' and that the online complements had considerably developed since that time (e.g., in LaboMep many online exercises of mental arithmetic for grade 6 had been added).

Authors' intentions concerning the use of their textbooks

In the questionnaire, the Helice designers adopted a general stance about teachers' *adaptations* of the book to their specific contexts. They said that "it [was] impossible to anticipate all possible adaptations" (in terms of contexts) and that they did not regard it as their "responsibility" to attend to these. This view was reflected in several statements throughout the questionnaire, e.g. the following: "the gap between what is planned and what happens in class is large – the gap between the authors' intentions and the teachers' use is even larger". They also declared that they would anticipate that teachers would combine Helice with the use of other textbooks. However, the spiral progression made such a practice difficult and the complex spiral structure was clearly an obstacle for the adaptation, or combination of several textbooks.

For Sesamath, the possibility of adapting the content was an important issue. The authors conceptualised the book, from its inception, as a digital textbook, each chapter file in .odt format. This offered opportunities for teachers to modify the texts of the exercises, and of the lessons. The book also offered a large amount of exercises, with the intention of leaving their choice to teachers. According to the Sesamath authors, the digital textbook was a "collection of bricks". Moreover, they stated that it should be thought of as belonging to a more general set of different kinds of resources available in the virtual environment LaboMep. Sesamath authors considered that it was the teachers' responsibility to 'build coherent lessons' and a 'coherent progression'. In their view this was made possible by providing a *large range of resources* to choose from, the textbook being only one of these. In addition, they wanted to provide 'efficient tools', such as LaboMep, for teachers to build their own teaching from these 'initial bricks'.

Discussion

Going back to our initial questions ("What are the differences between a textbook designed by a team of experts (researchers, teacher trainers, etc.) and a textbook designed by a group of teachers using a digital platform? What are the 'consequences' of these different designs, in particular in terms of coherence and quality of the textbooks produced?), we answer these by comparing the two contrasting textbook cases (Helice and Sesamath).

The investigation of the two textbooks showed that there are **differences** concerning essentially six levels (at least):

- the modes of design: more *collaborative* in the first case, more *co-operative* in the second;
- the nature of the structure: the first book is a *single whole*, with an organised structure (organised by the team of experts); the second is an *atomistic system* that can be arranged differently by different users;
- the organisation of the content: more *didactically original*, linked to the didactical choices of a 'homogeneous' team in the first case; more *aligned with* the institutional instructions in the second;
- the content: more *open* to a variety of ways for solving a given problem in the first case; more *driven* by an expert solution in the second;
- the integration into the whole grades 1-9 mathematics curriculum; links with primary school more taken into account in the first case than in the second; and

- the links to the users; the textbook provided as a *final product* given to the teachers in the first case; and as a *proposal* to be enriched by teachers' contributions in the second.

For Helice, the **coherence** is insured by the authors' didactical expertise, i.e. the mastering of the concepts at stake and of the potential difficulties and misconceptions for learning. It could be said that it is a *transcending* coherence. For Sesamath, the coherence is insured by the link with the curriculum and the institutional prescriptions; by teacher evaluation in class; and by the discussions among authors faced with the different contributions. Sabra and Trouche (2011) describe, for example, the discussion in another Sesamath author team (for a grade 10 book) in terms of reaching a coherence and consistency between the introduction of equations (in one chapter), and the introduction of function (in another chapter). We argue that this is a *collective* and *institutional* coherence.

Helice was, we claim, of high didactical **quality**. It offered many rich tasks, organized according to a carefully considered and complex structure. It took into account central aspects of the primary-secondary transition. The Sesamath textbook appeared to be, in its initial version, of a lower intrinsic quality: it offered fewer problems and fewer rich tasks. In terms of structure it simply followed the structure of the official National Curriculum.

However, the 'digital additions' and possibilities of Sesamath prompted us to re-consider the notions of 'quality' and 'coherence'. Helice remained the same, in both its paper and pdf versions, we call this *static quality*. In contrast, the online version of Sesamath had already been modified several times, to take account of 'user comments', i.e. users' experiences and needs. The digital means offered possibilities for modifications, and these were integrated by Sesamath in the process of re-design. The association perceived this as a necessity for meeting users'/teachers' needs in order to insure the quality of the textbook, we call this *dynamic quality*. Both Helice and Sesamath authors recognized that teachers would select and adapt elements of the textbook for their teaching. However, only Sesamath supported these adaptations and drew on user contributions.

We argue that this was a major development linked to digital means: the involvement of a large group of teachers in the design of resources, which in turn continuously evolved. This also deepens our knowledge of already established phenomena: even teachers who were not involved in the design of textbooks could be considered as 'designers' of their own teaching materials, as teachers selected resources, combined them, and set them up in class – a process that Gueudet & Trouche (2009) called teacher *documentation work*. This leads us to consider that the question raised in the discussion document, "*[Which] implications for design and coherence of materials […] if teachers are able to select tasks in varied orders?*", does not sufficiently recognize the complex link between 'design' and 'coherence' and the evolving role of teachers. Exploiting the potential of digital means for the design of teaching resources, including textbooks, requires the acknowledgement of this new role.

Drawing on the results presented here, we argue that textbooks can be considered as *lived resource* (Gueudet, Pepin, & Trouche, 2012), or even *living resources*, as they get continuously enriched and renewed by teachers' experience. This new conceptualization of the textbook is likely to be associated with new forms of design, for example, in terms of reflection on meta-design (Fischer & Ostwald, 2005), resources which support the design by teachers. It can also lead to favour teams of textbooks designers who combine and involve different 'experts' and expertise (e.g., teachers and other 'experts' such as researchers). From the literature (e.g., Kieran, Tanguay, & Solares, 2012), it is clear that teacher documentation work and professional knowledge are intrinsically intertwined, one leading to the evolution of the other (e.g., Gueudet & Trouche, 2012; Pepin, 2012). This has implications, also in terms of policy, for teacher professional development (of 'users' and of 'designers'); mathematical task design and digital means/possibilities; and the re-conceptualization of the *quality of resources* in mathematics education.

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Pedagogical Content Analysis of Mathematics as a Framework for Task Design

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This paper describes a *pedagogical content analysis* framework for task design in mathematics. This framework has been developed and applied through the author's ongoing work to develop textbooks, classroom materials, professional development materials and programs, and standards, particularly in the area of discrete mathematics.

Keywords: mathematics, discrete mathematics, tasks, teaching, learning, textbooks, curriculum

Introduction

Mathematics instructional tasks are at the heart of teaching and learning mathematics. Effective tasks are the core of good textbooks, classroom lessons, and professional development programs. Designing worthwhile tasks is challenging, but doing so has immense educational payoff. Student learning is developed, deepened, and extended through engagement in rich mathematical tasks; teacher skill and knowledge are strengthened through teaching with and designing tasks; textbooks become teaching-and-learning books rather than just reference books and sources for homework problems when they are developed around worthwhile instructional tasks.

It is difficult to design mathematics tasks that are effective for teaching and learning. This paper describes a useful design framework developed and applied by the author to design sequences of tasks in a variety of settings: for textbooks, through the Core-Plus Mathematics Project and the Transition to College Mathematics Project; through individual tasks for classroom lessons and teacher professional development; through projects for the U.S. state of Iowa (Every Student Counts and Iowa Core Mathematics); through projects for the National Council of Teachers of Mathematics (Illuminations website project); and through a state-funded lesson study project (Important Mathematics and Powerful Pedagogy). Most of the tasks are in the area of discrete mathematics, designed to be used at the high school level.

There are many types and sizes of mathematical tasks. In this paper, the discussion ranges from sequences of tasks to individual tasks, including tasks that generate or clarify a definition and those that develop major concepts or procedural skills. *Task* in this paper means textbook material or classroom lesson material that a teacher uses to help students gain a deeper understanding of mathematics. Several of the examples are what can be called "problem-based instructional tasks," as defined in Iowa Core Mathematics (2010, p. 3):

Problem-Based Instructional Tasks:

- Help students develop a deep understanding of important mathematics

- Emphasize connections, including to the real world
- Are accessible yet challenging to all
- Can be solved in several ways
- Encourage student engagement and communication
- Encourage the use of connected multiple representations
- Encourage appropriate use of intellectual, physical, and technological tools.

Effective mathematical instructional tasks must take into account the nature of mathematics and knowledge, as well as issues of teaching and learning. A key idea in all these domains is connections. Mathematics is a connected and coherent subject, it should be taught that way so students can learn it that way. As Schoenfeld (2009) states, "I truly believe that (a) mathematics is special because of the way it coheres–it really does make sense!–and (b) mathematics can be taught so that students come to see it so" (p. 168). Similarly, "mathematics is an integrated field of study.… When students can connect mathematical ideas, their understanding is deeper and more lasting" (National Council of Teachers of Mathematics, 2000, p. 64).

Understanding is of course our goal. A key perspective on understanding is the construction of knowledge. "Understanding involves the construction of knowledge by individuals through their own activity, so that they develop a personal investment in building knowledge" (Carpenter & Lehrer, 1999, p.23). As stated by Gardner, "When you've encountered an idea in your own way and brought your own thinking to bear, the idea becomes more a part of you … it's a part of your own experience" (1993, p. 6). Teaching for understanding is not just a platitude, there is a large body of research that helps identify what it means, why it is effective, and how one does it (for example, see Hart, 2010). Teaching for understanding can be operationally defined as in the Every Student Counts Project (ESC, 2008, p. 3):

Teaching for Understanding means

- Developing deep conceptual and procedural knowledge of mathematics
- Posing problem-based instructional tasks
- Engaging students in the tasks and providing guidance and support as they develop their own representations and solution strategies
- Promoting discourse among students to share their solution strategies and justify their reasoning
- Summarizing the mathematics and highlighting effective representations and strategies
- Extending students' thinking by challenging them to apply their knowledge in new situations, including in real world situations
- Listening to students and basing instructional decisions on their understanding.

The design framework described in this paper takes the above perspectives into account. The resulting tasks are used in textbooks (Fey et al., 2010; Hirsch et al., in press), in teacher resource books and journals (e.g., Hart, Kenney, DeBellis, & Rosenstein, 2008), in classroom lessons (e.g., *Resources for the Iowa Additions of Iowa Core Mathematics*, 2012), and as objects of reflection for professional development (e.g., in the *Every Student Counts* and *Important Mathematics and Powerful Pedagogy* projects). They are used by teachers to teach, students to learn, educators to provide professional development and teacher training, and scholars to study (e.g., Ziebarth et al., 2009). The framework will now be described, with examples.

Pedagogical content analysis framework for task design

This framework is based on analyzing the target mathematics (i.e., the specific mathematics to be learned in the task or sequence of tasks) from the perspective of mathematics, learning, teaching, curriculum, tools, models, and applications.

Outline of the framework

The framework can be outlined as follows. First, for the targeted mathematical content, *analyze the mathematics* by answering these questions:

- What is it? [deeply, simply, in essence]
- How do you do it? [compute it, compute with it, find it, operate on it, operate with it]
- What is it connected to? [interconnected web of mathematical ideas, concepts, methods, representations]
- What is it good for? [applications, contexts, models]

Next, *analyze the mathematics pedagogically*. That is, understand, identify, and develop strategies for addressing:

- Common misconceptions
- Common student content difficulties
- Sequencing
- Questioning
- Scaffolding
- Tools
- Pedagogical mathematical language.

These analyses are carried out with more or less formality. If time permits, a lesson study approach is very effective. It can be helpful to engage in an explicit process of unpacking and then repacking. The analysis may also take place informally in real time during teaching. In all cases, ultimately the perspective and process become a habit of mind *and* habit of practice.

Example: multiplicative inverses in modular arithmetic (Z_n)

In this example, we see instances of analyzing the mathematics, and addressing the issues of misconceptions, common difficulties, sequencing, scaffolding, questioning, and tools. A sequence of tasks is designed through a process of systematic analysis, development, and revision based on results from classroom tryouts. The tasks are designed for use with high school students in a fourth-year (final year) math course that is an alternative to a pre-calculus course, but with similar prerequisites.

As background to this sequence of tasks, previous analysis and design includes establishing an engaging context (cryptography, in particular the RSA public-key cryptosystem), and developing a prior sequence of tasks for the ideas of equivalence mod *n*, reducing mod *n*, addition and multiplication mod *n*, and the integers mod *n* with the notation Z_n. Also, although the target is multiplicative inverses, because this is what is needed for RSA, additive inverses are also developed to provide an accessible lead-in to multiplicative inverses and to more fully develop the key mathematics.

To begin, a mathematical analysis includes considering the four questions outlined above: What is a multiplicative inverse in Z_n? How do you find a multiplicative inverse? What are connections to other and previous ideas in mathematics? What are multiplicative inverses in Z_n good for? Of course, a huge amount of mathematics emerges from the answers to these questions. This can be thought of as "unpacking." Then we must "repack" to identify the target mathematics for this particular sequence of tasks as it is situated in a particular curriculum and classroom. In this case, the target mathematical ideas are analyzed to be (a) multiplicative inverses in Z_n are similar to and different from multiplicative inverses for real numbers with "regular" multiplication, (b) not every number in Z_n, for all *n*, has a multiplicative inverse, and (c) there are important properties of numbers that determine if a multiplicative inverse exists.

The most fundamental and universal design decision is that *all tasks are structured sequences of questions*. Key decisions pertain to the issues of which questions to ask, in what sequence, and with how much structure.

The first design decision is easy — additive inverses are easier and will help lead into the more difficult idea of multiplicative inverses. Even here, careful decisions are made. For example, how much mathematical formality should be developed, in particular, where and how should the idea of additive identity fit in? How much scaffolding should be provided? *Scaffolding* refers to *guidance* in terms of open-endedness of questions and size of steps in the step-by-step sequence of questions.

The task is initially designed with no explicit mention of additive identity, just use of 0, and little scaffolding. The initial attempt is based on trying out the hypothesis that students will find additive inverses in Z_n to be straightforward analogs of additive inverses with real numbers. Thus, after a question about informally finding additive inverses with real numbers, students are asked to find additive inverses in Z_n, for a few different values of *n*:

a. *What is the additive inverse of 6 in Z_{10}?*
b. *What is the additive inverse of 3 in Z_8?*

Classroom testing showed that students struggled mightily, they could not make the jump from their previous work with real numbers to inverses in Z_n. So the questions were redesigned to provide more scaffolding initially, and then gradually relax the guidance:

2a. *Find a number in Z_{10} that you can add to 6 to get 0 mod 10. Such a number is the additive inverse of 6 in Z_{10}.*

2b. *Find a number in Z_{10} that you can add to 2 to get 0 mod 10. Such a number is the additive inverse of 2 in Z_{10}.*

2c. *What is the additive inverse of 3 in Z_{10}? Explain.*

2d. *What is the additive inverse of 3 in Z_8? Explain why this answer is different than the answer you got for the additive inverse of 3 in Part c.*

Students seemed to have no difficulty with the idea of an additive identity (probably because it is 0 for real numbers and Z_n). For the target high school students, formal understanding of mathematical structures, such as rings and fields, is not deemed appropriate. Nevertheless, for the integrity of the mathematics, additive identities are also more formally defined in the first question about real numbers:

1. *Think about additive inverses. The additive inverse of a number is the number that you add to it to get 0. For example, with real numbers, the additive inverse of 3 is –3, since 3 + –3 = 0. The number 0 is called the additive identity. Find the additive inverse of each of these real numbers: 5, $\frac{3}{4}$, and –1.5.*

After the previous two tasks, students are getting comfortable with inverses in modular systems. They see some differences compared to additive inverses with real numbers, but it is still true that every number has an additive inverse.

Now to the harder idea of multiplicative inverses. Through an iterative process of pedagogical content analysis and classroom testing, the following design issues and decisions emerged.

Start with a task similar to the first additive inverse task (#1 above), to anchor the investigation in students' past experience with real numbers. For Z_n, begin by investigating Z_7, a prime modulus, in which all non-zero numbers have a multiplicative inverse, so that students get introduced to the idea in the simplest setting. They only have to deal with one new idea – a modular system, without yet needing to face the issue that some non-zero numbers in some systems do not have a multiplicative inverse. This task ends with a question to find the multiplicative inverse of 6 in Z_7. Because 6 is its own inverse, students are confronted with their next carefully-planned disequilibration, because no real number (other than 1 and –1) is its own multiplicative inverse.

Which modular system would be best to investigate next? The first try is Z_6. This choice was quite confusing for students. Why might this be so? Part of the reason might be because there are too few numbers with inverses, only 1 and 5, and both are their own inverse. This is a bad pedagogical step because the example is too special; it only illustrates numbers that are their own inverses, and thus raises the danger of students' over-generalizing this pattern.

The next try is Z_8. This choice is better because more numbers have inverses. However, the numbers without inverses are all the even numbers, which could again lead to over-generalizing a pattern. Once again, every number with an inverse is its own inverse. Also, the modulus is a power of 2, that is, the only prime factor is 2, which will not be helpful when looking for properties of numbers that do and do not have inverses. Thus Z_8 also is pedagogically problematic because it has too many special characteristics.

The analysis thus far points to the need for a modular system in which the modulus has more than one prime factor, not all numbers with inverses are their own inverse, not all numbers without inverses are even, and the modulus is not too large so that computing is not a barrier. This leads to the decision to use Z_{10} next.

Now the next big step is to find a pattern for which numbers have an inverse and which do not. Consider Z_{10}. The following design decisions are made:

- For the first time thus far, give students an empty multiplication table, ask them to complete it, and see which numbers have a multiplicative inverse and which do not. It is an important decision when to give support, such as this multiplication table. In previous problems, the intent is for students to think it through on their own in their own way. They may decide to use a multiplication table or not. Such decisions are important for students to consider before the decision is made for them.
- Ask first about patterns for numbers that do *not* have multiplicative inverses. [2, 4, 5, 6, 8]
- Then, after students work for a while, if they need more guidance, ask progressively more focused probing questions:

  How do such numbers relate to 10?

  Do you see any factors of 10? Are they all factors of 10?

  Do you see any connections between the factors of 10 and the factors of these numbers?

• Ask for patterns for numbers that *do* have a multiplicative inverse.

At this point, referring back to the three target mathematical ideas identified at the beginning of this discussion, students are expected to have made substantial progress learning that (a) multiplicative inverses in Z_n are similar to and different from multiplicative inverses for real numbers (in ways that students can describe), (b) not every number in Z_n, for all *n*, has a multiplicative inverse, and (c) there are important properties of numbers that determine if a multiplicative inverse exists.

Teachers may decide that complete attainment of (c) is not necessary, or that it is not attainable in a timely manner, for their students. To help provide differing support for this goal, for some or all students, the design decision is to first ask for a pattern in Z_n in an open-ended question. Then provide guidance to help get the general result, as follows:

You discovered in previous problems that not all numbers in a given modular system have a multiplicative inverse. Think about when multiplicative inverses exist in Z_n.

a. Make some conjectures about which numbers have multiplicative inverses in Z_n, either for a general n or for particular values of n. For each conjecture, try to prove it or disprove it. (You can disprove it by finding a counterexample.)

After trying some of your own conjectures, complete and prove the following three statements.

b. When n is _____________, then every nonzero integer in Z_n has a multiplicative inverse.

c. If n and m have a particular relationship to each other, then m does not have a multiplicative inverse in Z_n. What is that relationship?

d. m has a multiplicative inverse in Z_n if and only if ___________________.

This concludes this example of designing a sequence of tasks using the framework of pedagogical content analysis. We have seen an application of the mathematical analysis outlined in the description of the framework, along with consideration of many of the issues in the pedagogical analysis. These issues and examples from this sequence of tasks include common misconceptions (over-generalizing, for example, that all numbers have a multiplicative inverse), common student content difficulties (making the jump from inverses for real numbers to inverses in Z_n), sequencing (doing additive inverses before multiplicative inverses, or asking about patterns for numbers that do not have inverses before patterns for those that do), scaffolding (numerous examples of strategically providing more or less guidance), tools (when to introduce a multiplication table), and questioning (examples of trying to find the right question, at the right time, with the right open-endedness).

Brief examples illustrating aspects of the pedagogical content analysis framework

The one issue not yet addressed in this task design framework is so-called mathematical pedagogical language. This refers to the use of language that is mathematically accurate, though not conventional, and it is pedagogically powerful. This is language that helps students understand a mathematical idea through the name of the idea or the notation used for the idea.

For example, consider graph theory. The word *graph* is used in many ways, and for students it usually means a graph in the coordinate plane, or perhaps a data plot. But these are not at all what is meant by *graph* in graph theory, where a graph is a collection of vertices and edges. So why not modify the language, maintaining accuracy, to make the name more sensibly reflect the object. Thus, the term *vertex-edge graph* is used when designing tasks for high school students in graph theory.

Another example is recursion. This idea, before it is mathematically formalized, is intuitive for most students. They find it quite natural, for example, to think of the next counting number in terms of one more than the number you have now. So why not make use of the language of "next" and "now." Consider the following incomplete function table:

x	y
0	3
1	7
2	11
3	15
4	19

Based on many classroom trials, when students are asked to find patterns in this table, they are likely to say things like "it goes up by 4" or "add 4 each time." That is, they are seeing the recursive pattern in the y-values. When asked to write an equation that describes that pattern using the words NOW and NEXT, students have little trouble writing NEXT = NOW + 4. In contrast, students have great difficulty with the conventional subscript notation for recursion. Of course, students should also be able to find the x-y pattern: $y = 4x + 3$. To see one strength of the NEXT-NOW language, notice how concretely and meaningfully the slope is shown in the recursive equation, as opposed to the more abstract representation of slope in the $y = 4x + 3$ equation.

This example illustrates some of the characteristics of pedagogical mathematical language, and why it is an important feature of task design. The language must be mathematically accurate and also pedagogically powerful. In the case of NOW/NEXT, this language captures the essence of recursion, as used to describe sequential change. The language helps make an otherwise notoriously difficult idea accessible to all students. It also has substantial payoff in terms of deepening understanding of linear and exponential functions, constant rate per unit interval versus constant percent rate per unit interval, and making concrete the connection to arithmetic and geometric sequences. It promotes "semantic learning" as opposed to only "syntactic learning" (the latter is a danger when moving too fast to subscript notation). It is a "bridging language" in that the formal notation will be used at an appropriate later time. Students will definitely need understanding and facility with subscript notation to utilize the full modelling power of recursion. Finally, note that such pedagogical mathematical language may have limitations that must be carefully monitored. In the case of NOW/NEXT, this notation is not so useful for modelling change related to quadratic functions, and one must be careful to consider the incremental change in the x-values as well when looking for a NOW/NEXT relationship in the y-values.

For one last brief example, consider a lesson on the slopes of perpendicular lines developed as part of a lesson study project. The first iteration of the lesson used transparencies to rotate several lines 90°, gather and record the data on slopes of pairs of perpendicular lines, look for a pattern, and conclude that the slopes are negative reciprocals. After classroom tryouts, several features in the pedagogical content analysis design framework became particularly evident.

First, the use of the transparencies (an instance of the tool issue) proved to be confusing to students and needed to be refined. For example, if no axes are drawn on the transparency, which is on top of the paper with a line drawn, then when the paper is rotated students get confused about how to interpret the new position of the axes. If the transparency is rotated, then students are confused because it seems to them like it is the "same" line so it must have the same slope. The best solution proved to be putting axes on the transparency, and then rotate the paper underneath. This way, two lines appeared on the transparency with the axes fixed. Such small adjustments can often make a big difference in a lesson. Second, it turns out there had been a lack of mathematical analysis, specifically with regards to the first question outlined in the description of the framework, namely, with respect to the target mathematics, What is it? In this case, the answer to the question, "What is the slope of a perpendicular line?," is not that it is a pattern in the data. Rather, the slope is a consequence of the definition of slope and the nature of a 90° rotation, which is exhibited as a pattern in the data. This was an important learning episode for both teachers and students – that finding patterns in data is important, but you must try to understand those patterns in terms of the underlying mathematical concepts and relationships that caused the pattern.

Conclusion

This paper presents a framework for task design in mathematics that has been gradually developed and applied over the last two decades by the author in work on several research and development projects for textbooks, classroom lesson development, and teacher professional development programs. The framework is called a *pedagogical content analysis* to highlight that it involves both analyzing the mathematics and analyzing the mathematics pedagogically. Through iterations of task design and classroom tryouts, specific features that operationally define this framework have been identified, refined, and applied.

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Task Modification and Knowledge Utilization by Korean Prospective Mathematics Teachers

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It has been asserted that mathematical tasks play a critical role in the teaching and learning of mathematics. Modification of tasks included in intended curriculum materials, such as textbooks, can be an effective activity for prospective teachers to understand the role of mathematical tasks in the teaching and learning of mathematics; designing of new tasks requires more knowledge and experience. This study aims to identify the patterns that Korean prospective mathematics teachers seem to follow when they modify the mathematical tasks in textbooks. Knowledge utilized by prospective teachers while they modify textbook tasks is identified and characterized to understand the possible factors that have impact on Korean prospective mathematics teachers' modification of tasks.

Keywords: task modification, prospective teachers, teacher knowledge

Introduction

It has been asserted that mathematical tasks play a critical role in the teaching and learning of mathematics (Crespo, 2003; Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996; Watson & Mason, 2007). Henningsen & Stein (1997), for example, pointed out that the nature of tasks may provide different opportunities to develop students' mathematical thinking and reasoning skills. Furthermore, they argued that teachers' goals, mathematical knowledge, and knowledge of students can influence how the teachers set up the tasks included in intended curriculum materials, such as textbooks. This indicates that students' mathematics learning can be positively influenced by tasks that teachers modify or develop through their content and pedagogical knowledge. Therefore, the issue of teachers' competency to modify and pose meaningful tasks needs to be addressed more actively in teacher education. However, according to Watson & Mason (2007), who reviewed the submitted proposals for the special issue of the *Journal of Mathematics Teacher Education*, only a few papers dealt with issues in teachers' critical analyses such as task analysis, task evaluation, or posing tasks. This tendency is in line with the comment by Prestage and Perks (2007), which is that the issue of how teachers pose and organize tasks to educate students in classrooms has been ignored.

Teachers' ability to analyze and modify tasks to be used in the classroom or pose worthwhile tasks may be a critical factor affecting students' mathematics learning. However, such skills cannot be developed in just a short time, so teachers need continuing and systematic support to develop the skills (Crespo, 2003; Prestage & Perks, 2007).

In the case of Korea, especially, most teachers, in fact about 90% of secondary mathematics teachers, tend to use textbooks as a major resource in classrooms (Mullis, Martin, Gonzalez, & Chrostowski, 2004). Therefore, it seems very important for prospective teachers to have opportunities for performing critical analyses of tasks and the modification of them. The purpose of this study is to identify the patterns that Korean prospective mathematics teachers seem to follow when they modify the mathematical tasks in textbooks. Knowledge utilized by prospective teachers while they modify textbook tasks is identified and characterized to understand the possible factors that have impact on Korean prospective mathematics teachers' modification of tasks.

Theoretical Background

Tasks can be designed in different forms according to the objective and orientation, and every single task form provides a different learning opportunity for students. Prestage & Perks (2007) suggested that transforming only the conditions of closed problems in school textbooks by adding or deleting conditions can foster students' mathematical thinking. When conditions are deleted, students can decide for themselves. When conditions are added, students can extend the knowledge from the national curriculum. Thompson (2012) attempted the transformation of textbook exercises according to the following strategies to reflect the new curriculum's point of view, which emphasizes both reasoning and communication: a) reframe a basic problem by including one or more conditions; b) use relationships to find patterns or predict other results; c) generate conjectures for students to investigate; d) encourage students to solve a problem in multiple ways; e) evaluate student solutions; f) write a question appropriate for a given answer; and g) connect procedural and conceptual knowledge. The different learning opportunities from each task can be identified when tasks in textbooks are modified to require high cognitive demand (Kaur & Yeap, 2009) or to develop critical thinking and creativity (Krulik & Rudnick, 1999).

Competency in task design has been taken into consideration in mathematics teacher education. Zaslavsky (2008) showed changes in the teaching and learning of mathematics after task modification. Analyzing different versions of tasks promotes teachers' development of adaptability; fosters their awareness of similarities and differences; helps them learn how to cope with conflicts, dilemmas and problem situations; encourages them to learn from the study of practice; teaches them to select and use (appropriate) tools and resources for teaching; helps them to identify and overcome barriers to students' learning; and allows them to share and reveal self, peer, and student dispositions. Teachers can foster their professionalism after analyzing the tasks. Voica & Pelezer (2009) investigated the difference between prospective teachers' and inservice teachers' task design of problem posing. Inservice teachers considered the students, reflected the curriculum, and applied their pedagogical knowledge and mathematical knowledge, whereas prospective teachers mostly focused on the context without considering the students or the level of difficulty. Differences between inservice and prospective teachers are typically caused by differences in teacher knowledge.

Ball and her colleagues suggested Mathematical Knowledge for Teaching (MKT) based on Shulman's categories of teacher knowledge (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). According to them, MKT is categorized into two main domains: Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). SMK is subdivided into three categories: Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Horizon Content Knowledge (HCK). PCK is also subdivided into three categories: Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). Each type of teacher knowledge constituting MKT is explained briefly as follows (see Ball et al., (2008) and Hill et al., (2008) for more details). CCK, which is included in SMK, is defined as mathematical knowledge regardless of teaching, whereas SCK is mathematical knowledge and skill for teaching. To distinguish mathematical knowledge for teaching as a mathematics teacher from mathematical knowledge, SMK was conceptualized by Ball and her colleagues, who emphasized the importance of SMK. KCS, which is included in PCK, is the combination of knowledge about students and knowledge about mathematics; for example, teachers with KCS can identify and predict the common errors that students are more likely to commit as well as their misconceptions. Finally, KCT is defined as the combination of knowledge about teaching and knowledge about mathematics. Teachers who possess KCT can identify appropriate examples they should first use to help students understand mathematical concepts or appropriate teaching sequences and methods for the design of instruction.

Method

Participants were 38 prospective secondary mathematics teachers who enrolled in the course, “A Study of Teaching Materials and Teaching Methods,” taught by the first author of this paper. In the course, participants learnt about the Korean mathematics curriculum and teaching materials. At the end of this course, they were asked to analyze the tasks presented in the introduction of each unit of the secondary textbooks and modify them. The selection of tasks and the direction of modification were left up to the prospective teachers. Hence, they chose tasks from different units, such as function, geometry, algebra, or numbers; in particular, the function and geometry units were most frequently selected.

Most of the prospective teachers analyzed and modified more than 2 tasks, so all of the tasks that were done by each participant were analyzed in this study. In addition, the prospective teachers tended to use different types of teacher knowledge simultaneously in the processes of task analysis and modification, so all of those types of teacher knowledge were counted in the data analysis.

In this study, the task modification of prospective teachers was classified into three types: *context modification, condition modification,* and *question modification.* Context modification refers to modification by changing the context of tasks, making them student-friendly or diverse. Condition modification refers to modification by adding, deleting, or transforming the conditions in tasks (Prestage & Perks, 2007). It can be related to "what-if-not" strategies of Brown & Walter (1990), who mentioned the manipulation of the conditions of a problem when posing a new problem. Question modification refers to modification by changing what students are required to answer. We can find an example of question modification in Crespo (2003), who suggested changing the task into a more open-form or one that requires a process of investigation.

The authors of this study classified the modified tasks presented by the prospective teachers into types, and the classifications not agreed on by all of the authors were discussed until we finally reached an agreement. In addition, this study analyzed the types of teacher knowledge that the prospective teachers used for the task analysis and modification, according to the classification of teacher knowledge suggested by Ball et al. (2008). The types of teacher knowledge used by the prospective teachers were also analyzed in the same way that the classification of the modified tasks was analyzed.

Findings

Types of task modification

As aforementioned, the modified tasks presented by the prospective teachers were analyzed by the following criteria: context modification, condition modification, and question modification. Figure 1 shows the number of each modification type. As can be seen in the figure, the conditions of the tasks were the most often modified by the prospective teachers.

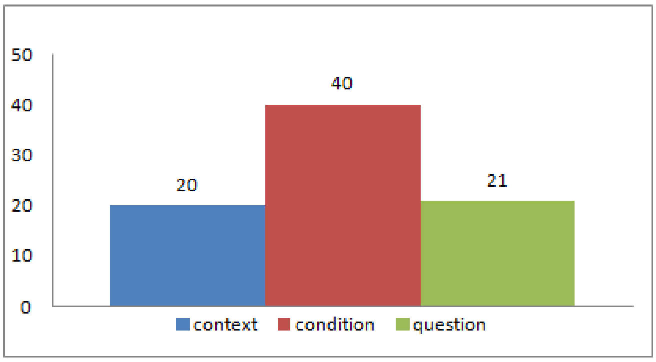


Figure 5 The number of each task modification type made by the prospective teachers

In addition, some prospective teachers changed only one among the three: context, condition, and question; others simultaneously changed more than two for their modification. Therefore, to analyze the results from the prospective teachers' modification types in detail, both cases are described in the following section.

The case of focusing on only one among context, condition, and question

The number of tasks in which only the conditions were modified was 26 out of 64; most of the prospective teachers modified the conditions to correct misconceptions that students might have or to adjust the difficulty level of the tasks. Most prospective teachers who modified the conditions of the tasks to adjust the difficulty level tended to add graphs or pictures to help students' understanding of the tasks. The other prospective teachers noted that students might have misconceptions about mathematical ideas from the tasks in textbooks, so they modified the conditions of the tasks to prevent the misconceptions. For example, Hee (hereafter, all names presented in this study are pseudonyms) mentioned that the task in the introduction of the function unit (Figure 2) might cause some misconceptions. Hee was concerned that students may misunderstand that function as discrete, because the given domain of the task was not continuous as it asked for each moving distance after 1 second, 2 seconds, 3 seconds, and 4 seconds, respectively. Thus, Hee modified the task by changing the condition to 'Show the graph of the moving distance for 4 seconds after

P-waves started moving'. This modification focused only on preventing a misconception that students might have, that might have made it difficult for students to perform a task given in the introduction of the unit.

What is the function?

Let's think

In a certain region, an earthquake's P-wave is traveling about 7km per second. Assume that the P-wave travels 7km every second. Let's find the moving distance of the P-wave after 1 second, 2 seconds, 3 seconds, and 4 seconds.

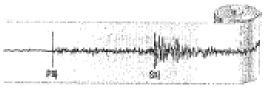


Figure 6 A textbook task to introduce concept of function

The number of tasks in which only the context was modified was 10 out of 64, and in most of the cases, the context was changed to attract students' interests. The prospective teachers who performed context modification claimed that the contexts used in the tasks in the textbooks could not draw students' interest. Thus, they suggested that materials in which students were interested or contexts related to students' real life experiences should be used in tasks. For instance, Nam analyzed an existing task (Figure 3); the context of the task was related to real life but could not engage students' interest. Thus, Nam changed the context by using electronic devices such as MP3 players that students often used. This finding from the context modification indicates that the Korean prospective teachers who participated in the study believed that contexts in mathematical tasks should be related to real life and students' direct experiences, and they tended to use context in a very limited view.

How is the function applied in real life?

Let's think

Jin went to the restroom and saw that the tap was not turned off. To investigate how much water was wasted, he measured the amount of flowing water for a minute and found out it was 2L. Say y is the amount of wasted water for x minutes. Answer to the following questions.

(1) Seek the formula of x and y.

(2) If water is flowing for 10 minutes, what is the amount of wasted water?

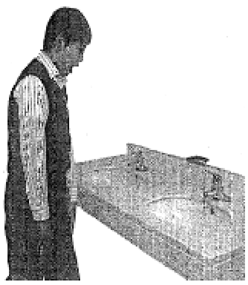
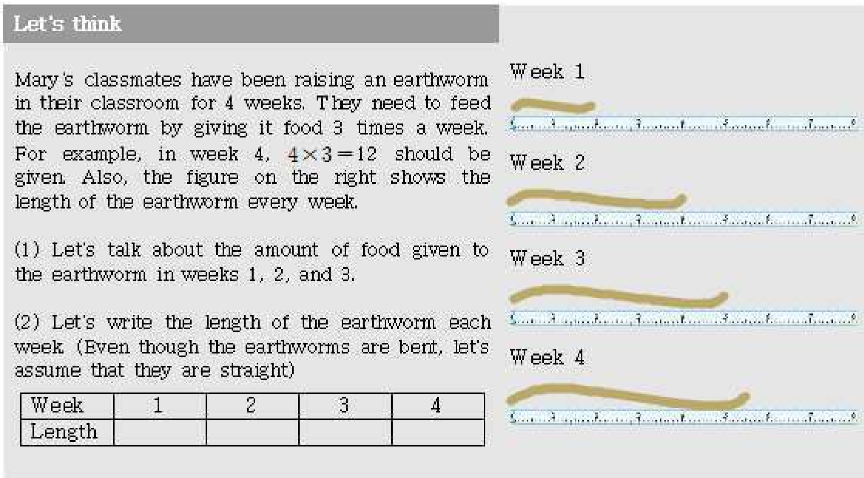


Figure 7 A textbook task to introduce a real life application of function

The case of focusing on more than two among context, condition, and question

There were some prospective teachers who modified the task by focusing on more than two among context, condition, and question. Because condition was the most frequent aspect of change as mentioned above, in this section, we show the responses of prospective teachers for the cases of modifying condition and context, condition and question, or context, condition, and question.

The number of cases in which the prospective teachers modified condition and context at the same time was 2 out of 64. Sol modified the task from the introduction of the function unit (Figure 2). She considered that the task did not show arbitrariness and univalence, so it would lead to the misconception of the function. She thus modified the context by selecting material which shows the function concept properly and provides appropriate difficulty and interest to students, and added a condition by presenting one more new relation. By considering the condition and context at the same time, this prospective teacher presented a much richer and more meaningful task than people who considered only the condition or context. Her modified task is given in Figure 4.



Week	1	2	3	4
Length				

Figure 4 Sol's modified task of the original task in Figure 2

The number of cases in which the prospective teachers modified both the condition and question at the same time was 15 out of 64. Jong modified the task from the introduction of an isosceles triangle in geometry (Figure 5, left). He raised the question of the problem in the textbook being focused only on the shape of the triangle rather than on the properties of the figure. Thus, he changed the condition to focus on the same properties without mentioning the side or the angle by modifying the question so it was an open-ended question asking in general about figures with the same properties (Figure 5, right). In that sense, Jong provides an opportunity for discovering propositions and allowing students to explore the figure, requiring more deep mathematical thinking through changing the question altogether.

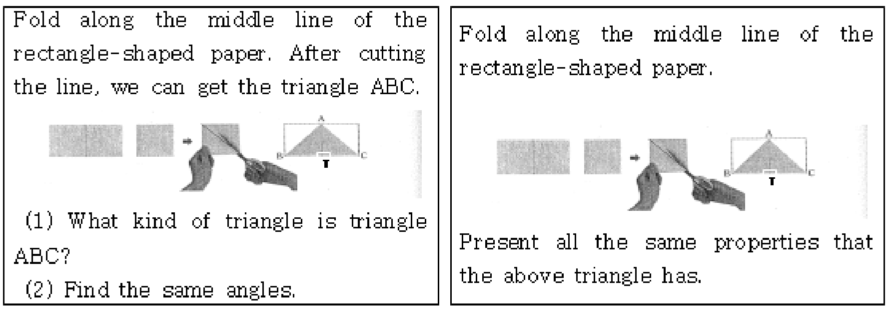


Figure 5 A textbook task to introduce properties of a triangle (left) and Jong's modified task (right)

The number of cases in which the prospective teachers modified context, condition, and question at the same time was 6 out of 64. Young attempted to modify the condition to complement the weak point of the problem (Figure 2), and in that procedure, she also tried to modify to arrive at a proper context and question (Figure 6). Because she pointed out that the problem in the textbook only treated the direct proportion restrictively, she added a condition to extend to the reciprocal proportion. She presented a distribution situation which shows the reciprocal proportion and is familiar to students. In addition, by asking them to share their thinking about the relation of *x* and *y*, she tried to make students more explorative.

Nino has 12 candies. He wants to distribute his candies to his friends.

(1) For example, if he wants to distribute his candies to 2 friends, then each friend would have 6 candies. In this way, fill out the following table.

the number of friends (x)	2	3	4	6	12
the number of candies that each friend would get (y)	6				

(2) Let's talk about the relation between the number of friends (x) and the number of candies that each friend would get (y).

Figure 6 Young's modified task

Teacher knowledge

As aforementioned, teacher knowledge can influence how teachers actually set up tasks in textbooks used in classrooms (Henningsen & Stein, 1997). Similarly, when the prospective teachers modified tasks, teacher knowledge of content and pedagogy might affect the direction and process of modification.

According to what prospective teachers considered when they modified tasks, we classified teacher knowledge. When prospective teachers considered the difficulty of the task or misconceptions and interest of students, they demonstrated KCS. When they focused on what example should be given or the way of teaching, they demonstrated KCT. KCC includes consideration of curriculum or the order of the mathematical content, and SCK includes the good usage of mathematical terms, mathematical ideas, mathematical history, and appropriate examples. CCK is related to general mathematical knowledge, and HCK is related to the relevance of the whole curriculum.

Figure 7 shows types of teacher knowledge used by the prospective teachers in the modification process. As can be seen in Figure 7, the prospective teachers used PCK more frequently than SMK; among PCK, KCS and KCT were more often used than the others. SCK among SMK was more frequently used than CCK and HCK. The SCK used by the prospective teachers was knowledge about mathematical ideas, its historical backgrounds, and various representations.

The fact that the prospective teachers used SCK more frequently than CCK indicates that they considered content knowledge as more important for teaching than using only mathematical content knowledge for their modifications. SCK related to mathematical ideas seemed to have an impact on the identification of misconceptions that students might have, and the prospective teachers tried to modify tasks by suggesting teaching methods or procedures to prevent those misconceptions; in other words, SCK might affect KCS and KCT. The prospective teachers, furthermore, tried to modify tasks by applying the historic-genetic principle or asking students to present different representations on the basis of SCK related to historical backgrounds and various representations. From these findings, it can be clearly seen that content knowledge is closely related to pedagogical content knowledge.

As mentioned earlier, the prospective teachers participating in this study modified the conditions of tasks more frequently than the context or question. The types of teacher knowledge that the prospective teachers used to change the conditions were KCS related to the identification of misconceptions and consideration of difficulty level, KCT related to teaching methods, and SCK related to mathematical knowledge for teaching. As can be seen in Figure 1 and Figure 7, therefore, most of the prospective teachers changed the conditions of existing tasks, and teacher knowledge types such as KCS, KCT, and SCK played an important role in the task modifications.

Considering teacher knowledge utilized in the task modifications in this study, it is noteworthy that the prospective teachers tended to use several types of teacher knowledge simultaneously to modify one task, rather than using only one type of teacher knowledge. It is presumed that Korean teachers' high level of mathematics content knowledge (Park, 2004) might help them activate teacher knowledge, such as KCS relevant to the difficulty of task and misconception, and SCK relevant to various representations, good usage of mathematical terms and mathematical ideas. Consequently, their awareness of this kind of knowledge led them to consider appropriate teaching methods or strategies. Therefore, it can be said that different types of teacher knowledge are closely interconnected and influence each other to activate knowledge. This suggests that an activity, such as task modification, may be a good way to develop teacher knowledge by encouraging teachers to consider different types of knowledge at the same time.

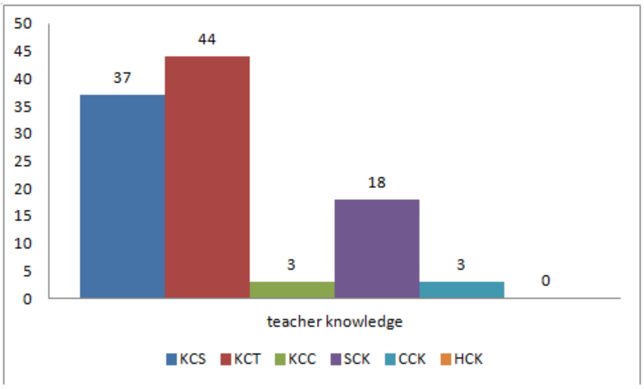


Figure 7 The number of teacher knowledge by types that the prospective teachers used

Conclusion

In this study, we examined the types of task modification conducted by the prospective teachers and the types of teacher knowledge that were activated during the modification process. By classifying the types of modification into context modification, condition modification, and question modification, we found that condition modification was performed at the highest rate. In addition, we found that when prospective teachers focused on more than two among context, condition, and question rather than focusing on only one, they modified the tasks in a more appropriate, richer, and meaningful manner. KCT and KCS were the types of knowledge that prospective teachers used most frequently during modification, and they also used SCK at a high rate. However, other types of teacher knowledge were hardly displayed.

Prospective teachers modified the conditions mostly when they considered students' misconceptions and difficulty level, and intended to provide students with an opportunity to make a discovery. This result is consistent with the highest rate of the usage of the KCT and KCS teacher knowledge types when prospective teachers modified the tasks. Also, because cognitive ability is necessary when modifying the conditions, it can be related to the high rate of usage of SCK.

Prospective teachers who used KCS as their type of teacher knowledge, especially those who considered students' interest, modified the context. So, when they focused only on the context, they usually modified the context on a superficial level by presenting students with a familiar situation. When prospective teachers focused also on the condition or question along with context, they attempted to modify by using their KCS, especially the knowledge of misconceptions, and KCT. In this sense, prospective teachers who focused on more than two aspects at the same time were able to generate a more meaningful task because they activated various factors of teacher knowledge.

The number of prospective teachers who modified the context, conditions, and question all at the same time was not that high, surprisingly. To modify tasks in a meaningful way by considering all three factors, prospective teachers not only need KCT, KCS, and SCK but also need knowledge of the curriculum or attempts to relate to other mathematical concepts. Therefore, the reason that many prospective teachers did not consider all the aspects at the same time is that they possess narrow knowledge rather than balanced teacher knowledge.

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The Lemon Squash Task

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The Lemon Squash Task affords many learning opportunities in connection with algebra and proportionality. In this paper, we show that although opportunities are embedded in the task, they do not necessarily surface when the task is treated in a classroom setting. Depending on the context in which a task is placed and the knowledge and intentions of the teacher, the task may contribute to very different kinds of learning. Modeling real world situations often has its limitations and can, as in this case, make the problem unsolvable unless it is accepted as a 'textbook task' disguised as real but adjusted to the norms of school mathematics.

Keywords: textbooks, tasks, mathematics, proportionality, algebra

Introduction

When a mathematics textbook task is designed for school use, it is generally intended to offer an opportunity to support the learning of some particular mathematical idea. One branch of research about textbook tasks concerns the tasks themselves in relation to, for example, creativity (Lithner, 2008) or cognitive demand (Brändström, 2005), or how tasks appear in the context of assessment such as the TIMSS framework (Mullis, 2005). Quite often, the way tasks are treated in classrooms is absent in this research. Additionally, classroom studies often take textbooks for granted and do not analyse textbook tasks in detail. According to the didactic tradition in the Nordic countries, most studies assume curricula and teaching materials as the basis for their analysis and there is a need for content related classroom studies (Klette, 2007). We argue that few Nordic studies analyse how specific textbook tasks are interpreted in the classroom and we relate this to the affordances and constraints of the tasks per se. What happens with the intended learning opportunities of a textbook task when transposed by the teacher in a classroom setting?

In this paper, we use the term *task* to mean tasks found in mathematics textbooks in line with Chevallards' (2006) definition. Tasks are also seen as cultural artefacts, influenced by the mathematics culture at large as well as by the time and place in which they are created and used. Tasks are a valuable instrument in mathematics education, particularly in a textbook intense teaching culture as the one in Sweden (Johansson, 2006). We seek in the following sections to analyse one particularly interesting task related to algebra and proportionality found in a Swedish grade 6 mathematics textbook and try to answer the following questions:

- What are the affordances and constraints of the chosen task?
- How do these come into play when the task appears in the classroom?

We will also discuss how the results of our analysis could be of use for textbook authors.

Being a teacher and a teacher educator, we take special interest in what types of tasks appear in the classroom and how teachers use the tasks. We are particularly interested in proportionality tasks because proportionality has been a new concept in the Swedish national curriculum for grade 6 since fall 2011 (Lgr11, 2011).

Proportionality and proportional reasoning

Proportionality can be defined in several ways and we chose to define it as a linear relation such that $y = a \cdot x$ where y and x are magnitudes and a is the proportional constant. Miyakawa & Winsløw (2009) call this *dynamic* proportionality, as opposed to *static* proportionality expressed as a multitude of equal ratios $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$. Proportionality can be expressed as internal or external depending on whether it is a relation within or between measure spaces, according to Freudenthal (1983). Proportional reasoning is used to denote reasoning in a system of variables between which there exists a linear functional relationship (Karplus, Pulos, & Stage, 1983).

The Lemon Squash Task studied in this paper involves a mixture of juice, sugar, and water. Tasks involving mixing substances in different proportions have been studied by several researchers (e.g., Mellar, 1991; Noelting, 1980; Nunes, Desli, & Bell, 2003). These are often called mixture problems by Tourniaire and Pulos (1985) and differ from rate problems in that the elements in a mixture create a new object and different quantities are often expressed in the same unit. Tourniaire and Pulos (1985) also describe three context variables that affect the difficulty of a mixture task; firstly the mixture as such, being an object where the different parts are not distinct and therefore more difficult to handle; secondly, it is more easy to visualize discrete quantities than continuous quantities; and finally, if the context is familiar the task is often found to be easier.

The Anthropological Theory of Didactics

The Anthropological Theory of Didactics (ATD) postulates an institutional conception of mathematical activity (Chevallard, 2006). It starts from the assumption that mathematics, like any other human activity, is produced, taught, learned and diffused in social institutions, and designed in terms of mathematical organisations (MO). Knowledge content and form in these activities is a consequence of the didactic transposition process, i.e. a change or adaptation of a selected existing knowledge to 'teachable' knowledge. Mathematics treated in school can be analysed as several types of knowledge (Bosch & Gascón, 2006):

1. Scholarly knowledge;
2. Knowledge to be taught, described in curricular documents;
3. Knowledge actually taught which can be gleaned from the teachers' classroom discourse and the tasks he or she prepares for the students;
4. Knowledge actually learned.

The MO is constituted on two levels, the 'know-how' and the 'know-why' related to a given task, and has a praxeology comprised of four components: type of tasks, techniques, technologies, and theories. The praxis 'know-how' contains types of tasks to be carried out and techniques to do so, technique being considered here in a general sense of 'ways of doing'. The logos or 'know-why' includes technology and theory (Bosch & Gascón, 2006). In this paper, we focus on the 'know-how' of the Lemon Squash Task and the transposition between steps 2 and 3.

Contextualised tasks

In a contextualised task, the realistic touch of the question is supposed to increase the meaning of the task, but it also increases the cultural burden on the student in the solving process, in light of the task being a cultural artefact (Wyndhamn & Säljö, 1997). Everyday contexts will trigger the use of out-of-school knowledge, everyday concepts and common sense, but these may not always be applicable. The Lemon Squash Task is interesting because it entails a switch between everyday concepts and scientific concepts (Vygotskij & Kozulin, 1986). A contextualised textbook problem intended to be a 'realistic' problem is, argues Jablonka & Gellert (2007), neither mathematics nor a real world problem. With reference to the work of Dowling, they discuss the "myth of reference":

> "It is conveyed through problem settings that are constructed mathematically and only retain a trace of non-mathematical significations. It does not remain possible for the learner to evaluate the solution of the problems from a practical point of view."(ibid, p. 2)

The interpretation of a task is also constrained by certain norms of school mathematics, regulated by the so called didactical contract (Brousseau, 1997).

Method and setting

Data for our analysis in this paper is taken from a larger set of video data collected within the project VIDEOMAT (Kilhamn & Röj-Lindberg, 2012), a comparative video study including students from Sweden, Finland, Norway and California about the introduction of algebra in grades 6 and 7. In each country, 4-5 classes were video recorded for 4 consecutive lessons. There was no intervention because the aim was to record authentic instruction and classroom activity concerning introduction of variables. When looking through the Swedish data, the Lemon Squash Task caught our interest because teacher-student interaction around the task appeared seven times in the recordings of one of the grade 6 classrooms. These episodes were selected to form a case study. In addition to the video recorded lessons, a post lesson interview was conducted with each teacher; six months later, a focus group discussion was held with all the teachers where this particular task was discussed.

The choice of analysing the Lemon Squash Task was naturally influenced by our background as teacher and teacher educator (Goodwin, 1994), and our method of analysis can be described as "a whole-to-part inductive approach" (Erickson, 2006). After viewing the seven episodes, they were carefully transcribed, reviewed many times, and also viewed in groups to establish agreement about the phenomenon.

The Lemon Squash Task in our data was found in the algebra unit in a commonly used grade 6 textbook (Carlsson, Liljegren, & Picetti, 2004). With the aim of investigating the transposition of "knowledge to be taught" into "knowledge actually taught", content analysis of the task as it appears in the textbook was first made, focusing on embedded aspects of proportionality and algebra, since it appears in the algebra unit and concerns proportional relationships. The algebra unit in the textbook spans over 26 pages, including 127 tasks of various kinds. It has a strong emphasis on the learning of algebraic symbolic language and the meaning of the terms *equation*, *expression*, and *variable*. There are 6 pages in the teacher's guide on the algebra unit and an additional 8 worksheets. Tutorial notes are given in general terms and no guidance is given in relation to any particular task. The text in the guide concerning the page where the Lemon Squash Task appears is as follows:

> "An expression consists of one or more variables written with letters and sometimes one or more numbers. An expression does not have a fixed value until the value of each variable is known. That is a big difference compared to an equation. The letters in an equation have a fixed value and it is that value you find out by solving the equation. Therefore the letters in an equation are not variables." (Carlsson et al., 2004, p. 64).

The local curriculum, where the core content is explained in more detail, states as learning goals for the unit that students should be able to: interpret and write expressions with variables, work with equalities, and solve simple equations. The teacher in this particular class is a generalist teacher for grades 4-6. She has been teaching the class since grade 4 and said in the post lesson interview that she feels insecure teaching algebra. When she last taught grade 6 three years ago, the national curriculum did not emphasise algebra until grade 7; so she had actually never taught algebra as explicitly as she now does, in spite of her 10 years of teaching experience. The notion of proportionality is part of the national curriculum but not treated as an explicit topic in the textbook. In this classroom, mathematics lessons consist of a short whole class introduction followed by individual deskwork where the teacher walks around helping students when they get stuck. Students work through the textbook unit at their own pace, which means they are seldom working on the same task at the same time, a common practice in Sweden (Carlgren, Klette, Myrdal, Schnack & Simola, 2006).

The Lemon Squash Task

The Lemon Squash Task[24] appears in the textbook under the heading "Expressions". There are two pages where students are asked to write expressions translating from verbal to algebraic representation, followed by one page with tasks where a number is to be inserted in the place of a letter in an expression. The last task is the Lemon Squash Task. A contextualised task like this is a cultural artefact. We believe that the authors of the textbook chose this context because mixing and drinking this type of drink, usually called *saft*, is a common point of reference for Swedish children[25] .

47 You want to mix "Lemon Squash". You have 5 dl lemon juice, $x = 5$ dl. How much do you need of
a) water? b) sugar?

48* How much lemon juice is there in 7dl mixed "Lemon Squash"?

"Lemon Squash"	
x	lemon juice
$2x$	water
$x/2$	sugar

Figure 1: The Lemon Squash Task in the textbook (Carlsson et al., 2004, p. 103), our translation.

[24] We have chosen to use the name "Lemon Squash" although the Swedish name for the task is "Törstsläckare" which literally means "Thirst reliever".

[25] *Saft* is a Swedish sweet drink, which is close to the British *cordial*. Traditionally it is made from juice of berries mixed with sugar and water.

Results

The results will be presented in two parts. First a content analysis of the Lemon Squash Task is made, highlighting its affordances and constraints. The next section presents how this task was treated in a grade 6 classroom in a Swedish school.

Content analysis

The Lemon Squash Task presented here includes tree subtasks. The first two subtasks (47a, b) are both closed questions with a unique solution. Proportions of the three ingredients are given in symbolic representation in the 'recipe', and the student is asked to interpret the algebraic expressions x lemon juice, $2x$ water and $x/2$ sugar. There are two ways of solving these tasks. The interpretation of $2x$ *water* and $x/2$ *sugar* leads to proportional reasoning through a translation into "twice as much water and half as much sugar as the amount of lemon juice." Translating from a symbolic to verbal representation gives x a meaning and makes use of proportional reasoning within the recipe, i.e. how the parts relate to each other. Another solution technique is to stay within the algebraic representation, exchange x for 5 in each expression and interpret water as $2 \cdot 5 = 10$ and sugar as $5/2 = 2.5$. In the second case, it is not certain that x has any meaning outside the fact that is it to be replaced by a number, and no proportional reasoning is necessary. The two tasks could be used as an opportunity to develop proportional reasoning and meaningful interpretations of algebraic expressions, or to practice the procedure of replacing x in an expression to calculate the value. Task 48, however, is much more complex involving the total volume of a mixture of fluids and solid matter. Here we find the first and most striking constraint of the task, namely the fact that mixing these ingredients in reality and the intended mathematics content are not aligned. We will deal more with this constraint in the classroom analysis and the discussion, and for now analyse what we interpret as the intended mathematical content.

Lemon Squash is made of three ingredients: lemon juice, water, and sugar. Task 48 gives the proportions of these ingredients as algebraic expressions and the total amount as a set quantity asking for the amount of water. It can be seen as a proportionality task by considering how the total number of parts (one batch) is related to the total quantity. To find out the number of parts in a batch, the algebraic expressions in the recipe need to be translated into a verbal representation giving x the meaning of one part. One part of lemon juice, two parts of water, and half a part of sugar adds to 3½ parts. We then want to find out how much one part is when 3½ parts are 7 dl. This can be described as an external proportionality where 7 dl:3½ parts = x dl:1 part, or as an internal proportionality where 1 part: 3½ parts = x dl:7 dl. It can be solved using different techniques, for example by using number facts (7 is $2 \cdot 3½$) or a cross multiplication algorithm ($7 \cdot 1 = 3½x$). Proportional reasoning in combination with a trial and error approach is also possible, i.e. if each part is 1 dl the total would be 3½ dl, but 7 is twice as much as 3½ so each part must therefore be twice as much as 1. In this case the proportionality is static.

When Task 48 is interpreted as a linear relation, the total quantity y can be expressed as $y = a \cdot x$. If a is a scalar operator (Vergnaud, 1988) then $f(x) = x \cdot f(1)$. We know that $f(1) = 3½$ and we want to find $f(x) = 7$; we see that $7 = x \cdot 3½$. In this case the meaning of x is the number of times we need to take the whole mixture (all the parts) to get 7. However, if a is a function operator, then $f(x) = 3½ \cdot x$ and the meaning of x is what quantity I need to take 3½ times to get 7.

In this case, the proportionality is dynamic. Whichever way the linear relation is interpreted, a graphic representation could be a useful solution technique.

A more primitive example of proportional reasoning is to consider only the internal proportions among the ingredients and use a trial and error solution technique. If we have 1 dl of lemon juice, we would need 2 dl of water and ½ dl of sugar, which adds to 3½ which is less than 7; so we try with 2 dl of lemon juice.

Despite the way proportional reasoning is used, the translation from algebraic to verbal representation of the expressions presented in the recipe becomes an essential part of the solution technique. The task can be said to afford different types of proportional reasoning and different solution techniques, making use of translation between symbolic and verbal representations.

A different approach to the task is to stay within the algebraic representation and model an equation by adding the expressions: $x + 2x + x/2 = 7$. Again the solution technique can either be one of trial and error, or of formally solving the equation for x to get $x = 2$. In this case the task affords the opportunity for modeling and solving of an equation with one variable. Because the expressions are already given, the modeling part is limited. No switch of representation is needed, although it is possible to translate the algebraic equation into a visual or concrete representation instead of using a formal equation solving procedure.

The Lemon Squash Task involves all the three context variables that often appear in mixture problems according to Tourniare and Pulos (1985). It is a mixture where parts are not distinctly visible, it deals with continuous quantities, and students are familiar with making such mixtures.

Classroom Analysis

The classroom analysis is grounded on the first two episodes where the teacher explains the Lemon Squash Task to individual students (in total she does it 7 times). In the first episode, the teacher first misinterprets the task. She points to the recipe in reasoning proportionally about the amount of water, lemon juice and sugar, and starts discussing doubling and halving 7. The textbook section is about expressions, not equations; suddenly the Lemon Squash Task appears, challenging students to make a mathematical model using an equation. Task 48 seems difficult because there are no clues in the preceding pages on how to model an equation. Our interpretation is that the teacher therefore uses the same technique as in the preceding task. She exclaims "Ah!" when she realises that 7 dl is the total amount, not a part (see excerpt 1).

Excerpt 1:

T: So, then you can first calculate how much, (.)
Ah! it also says seven decilitre of mixed, yes.

[…]

T: and water gives twice as much (.) and it will be (.) mmm in itself. I wonder if sugar also gives some amount there? (.) I have to think, does it really it's no amount of sugar it just ends up in the liquid. Eh, this will be seven together.

S_1: mhm

T: something plus something plus something will be seven.

When the teacher has realised her mistake, she considers the modeling of the task. Knowing that sugar dissolves in water, she contemplates whether it contributes to the total volume. She wants to use the given algebraic expressions in the recipe but her experience tells her that sugar dissolves in water and does not increase the volume. Initially she discards the sugar and considers only the lemon juice and the water, which means, she needs to divide 7 dl into 3 parts. When she realises that 7/3 will be a long decimal number, she states that "they" must have intended the sugar to be treated as if it did contribute to the total volume in the same way as the fluids (see excerpt 2). In fact, the sugar does contribute, but not as much as the fluids. She is avoiding the continuous variable, expecting the task to include only discrete numbers.

Excerpt 2:
T: That was a difficult calculation!
S_1: mm
T: Wasn't it? It would have been easier if it were six (...) decilitres there.
Maybe they count yes eh I think they eh mean that they consider the sugar to give some amount. Because then I can work it out!

After this the teacher solves the task by translating from an algebraic into a visual representation, drawing four lines representing the amount of each ingredient. The lengths of the lines are in proportion to the recipe. One line for lemon juice, two for water, and a line half as long for sugar, adding ½ under the last line for clarification (see figure 2). After drawing the lines, the teacher says she will try out what number is the right one, thus choosing a solution technique using a trial and error approach. She suggests 2 and adds 2 + 2 + 2 + 1 = 7. When leaving to attend another student, the teacher mutters " ... what a complicated task that was!". The technique to represent numbers as lines, she informs us in the post-lesson interview, was something she learned when she was a scholar herself, and she still uses this technique when solving algebra tasks.

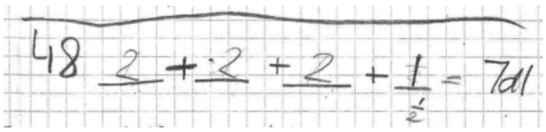

Figure 2: Copy of Task 48 from a student's notebook including the lines drawn by the teacher.

The next episode shows the teacher explaining Task 48 for the second time. Referring to her earlier experience, she tells the student that the task is tricky and explains the mistake made about the sugar contributing to the total (see excerpt 3).

Excerpt 3
T: This is a bit tricky. When I solved this I had to use trial and error.
Do you know why I thought it was difficult to solve? Because I don't think or I guessed that sugar didn't contribute to the volume in lemon squash. But of course it does. This is how I did it.
[T4 takes the pen and paper from the student and starts to draw lines.]
T: If it is seven decilitres mixed lemon squash. Uh, then it is, they write here that, something plus something two times is the secret number *x*. (.) plus half of the secret number. I write like this, so I will remember, it will be seven decilitres. A number plus a number plus a number plus half of the secret number will be seven. Can you try finding the number?

When the teachers in the VIDEOMAT project discussed opportunities for learning in connection to this task, they indicated the task appeared very abruptly in the textbook without sufficient preparation. The solution techniques they suggested for the task were trial-and-error, making a table, or setting up an equation. The teacher in the lesson reported on here commented that the task is about proportionality, meaning the doubling and halving, and that it was an everyday example of the use of simple equations. She said:

> "A trial-and-error technique is not the only possible one but I use it automatically in my head. I try putting 1 in the place of *x* first, checking if that will fit. That's how one solves it in an everyday situation. […] You could do an equation as well but that is not the first thing a student would do." (Focus group discussion)

In the discussion, one teacher assumed a simple proportionality of 1 part sugar, 2 parts lemon juice, and 4 parts water making 7 parts, but without realising that interpreting the $x/2$ as one part is not a trivial matter for students. One of the other teachers said she skipped the task during the algebra unit, using it in a class discussion about recipes later on.

Discussion

When the teacher starts to explain the task, she uses the recipe in the margin because she expects the solution technique to be found in the textbook (Lithner, 2008). The didactical contract says that the task *is* solvable. The teacher considers the amount of sugar and how it will disappear in the total volume, but she continues to use the recipe even if she realises that it is unsolvable. She constructs a model of the recipe using lines with different lengths and successfully solves the task, subsequently repeating this solution technique each time. However, the constraints of her model are never made explicit. The Lemon Squash Task affords many opportunities of learning proportionality, internal and external, dynamic as well as static, but they stay hidden.

The Lemon Squash Task differs from other mixture problems reported in previous research literature (Tourniaire & Pulos, 1985), which are often comparison problems. When making lemon squash, there are three quantities involved (sugar, water and lemon juice), and these ingredients are in a proportional relation to each other. If you add more sugar, the sweetness of the lemon squash will increase proportionally; if you increase lemon juice, the taste will again be less sweet by proportion. The amount of lemon juice is inversely proportional to the taste of sweetness because the juice of a lemon is sour. When the mix is blended with water, again the taste will be weaker. So in order to mix lemon squash, there are both direct and indirect relations between quantities. In this case the within quantity is taste and the between quantity is the amount of sugar, water, and lemon juice. As the task is presented, the sweetness is irrelevant and only the between quantities are focused. Different mixtures are not compared, but the whole idea of mixing water, juice and sugar is an implicit aspect of familiarity relevant as a context variable (Tourniaire & Pulos, 1985) that clearly affects the solution procedure, at least for the teacher.

In this paper we have studied a teacher in the process of transposing the knowledge embedded in a textbook task into knowledge taught. We argue that the teachers' interpretation of the Lemon Squash Task is constrained by the context in which the task is placed and by an existing didactic contract. The task appears in the algebra unit focusing on the use of symbolic notation. Proportional relations, such as static or dynamic proportionality, embedded in the task become a background feature and do not stand out as affordances to the teacher. A relevant solution technique is expected to be found earlier in the unit and difficult calculations are not expected to appear in the algebra unit. These features of the didactical contract lead her to discard her everyday experiences (the sugar does not contribute to the volume) as well as the scientific concept of volume when she says, "I think they mean that they consider the sugar to give some amount". She interprets the task as a mathematics task, not a realistic problem, a common feature in so called 'realistic mathematics' where "The focus is on the mathematics, not the 'realistic' situations from which the mathematisation is hoped to be derived" (Jablonka & Gellert, 2007, p. 4). Although the task appeared in the algebra unit, the teachers found that setting up an equation was too difficult for the students, so they resigned to working it out using trial-and-error. Getting the correct answer became the main goal. Solving it as if it were an everyday situation became the goal instead of some specific mathematics learning.

We have shown that a textbook task can be seen as a cultural artefact greatly dependent on the context in which it is placed. Consequently, a task cannot be judged as good or bad in itself, and the learning opportunities a task designer sees in the task may not stand out as affordances to the teacher. We conclude that textbook authors need to elaborate more on tasks in the teachers' guide, making affordances of a task explicit and include tutorial notes about learning, especially concerning modeling tasks. The Lemon Squash Task becomes exceptionally difficult because the intended mathematical relations are in conflict with both the scientific and the everyday concept of volume when mixing fluids and solid matter. We see it as an unfortunate choice of realistic context and suggest task designers pay a good deal of attention to both everyday experiences and scientific validity of the chosen context.

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Designing Interdisciplinary Tasks in an International Design Community

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A major difference between the world of school and the real world is that, in the former, knowledge application to solve problems is clearly connected to a single 'academic' discipline whilst, in the latter, solving problems requires interdisciplinary knowledge application. The gap between these "worlds" provides a challenge, particularly from a materials design perspective. In this paper, a theoretically based design process of a package of ten interdisciplinary tasks and associated interactive resources is discussed. First, the theoretical foundations behind the design process are introduced. Second, our concept of iterative cycles of improvement, with a structured approach to formative assessment, is presented. Finally, results and outcomes from the analysis and implementation of the package are discussed.

Keywords: interdisciplinary teaching and learning, inquiry-based learning, iterative cycles of improvement.

Background: Interdisciplinary teaching in day-to-day teaching

Mathematics and the sciences in school are often taught as isolated subjects. This, however, does not meet the needs of the world of work, which demands integrated, interdisciplinary solutions to complex problems often requiring the use of technology. In consequence, students should not only be prepared for application of knowledge and skills in single 'academic' domains but learn to solve complex real world problems.

Although in many countries school curricula contain some aspects of interdisciplinary learning, particularly across the sciences[26], in general lessons remain

[26] See e.g., the so-called *Fächerverbund* (Connection of subjects) in Baden-Württemberg, Germany; the new integrated science subject (Advanced Science, Mathematics and Technology) in the Netherlands since 2007; methodological guidelines in the regional curriculum of Andalusia; and new mathematics and science curricula in Cyprus.

mono-disciplinary. In the reality of day-to-day curriculum implementation in schools, there are many obstacles to interdisciplinary approaches to teaching and learning. These include, amongst others, the high degree of organisational work required of teachers (in organising interdisciplinary teaching with their colleagues), training in single disciplinary areas for most teachers, interdisciplinary problems viewed as too complex and time consuming by teachers for their students, and few materials that support mathematics and science from an interdisciplinary perspective. In this paper we present the design work carried out within the EU-project COMPASS[27], which addressed the issues raised above by aiming to provide teachers with a set of challenging interdisciplinary tasks relevant and usable within day-to-day teaching. Fundamental to the project was the development of new approaches to teaching that aim to motivate learning both within and across the disciplines. Pragmatically, given that there is little official demand for such curriculum in most countries, this required careful understanding of how such an innovation might be motivated in ways that ensure current curriculum demands are met.

Our design community and its aims

The COMPASS design community aimed to develop teaching materials connecting the sciences and mathematics with each other and most crucially with the lives of individual students and their communities. We wanted to foster in young people a desire to learn mathematics and science throughout their lives by coming to realize that such learning is essential if they are to be empowered as active citizens with critical scientific inquiry and problem solving skills. In achieving the above aims, the project focused its activities mainly on designing and developing materials to support teachers with their day-to-day teaching.

The design community was comprised of 15 mathematics and science educators from six countries, geographically spread across Europe (the Netherlands, Spain, England, Cyprus, Slovakia, Germany), ensuring a range of complementary expertise. The consortium included design experts in the fields of interdisciplinary tasks, real-life-based tasks, modeling tasks, and ICT mediated tasks, together with partners having considerable experience in the design and implementation of continuing professional development courses. Further, the expertise of the participants was enriched with stakeholders from school authorities and teachers from schools who contributed to the further refinement of the materials during iterative cycles of improvement.

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A theory-based design approach

In this section, the theoretical foundation adopted in the design of the COMPASS tasks is summarized.

Interdisciplinary work

In general terms, "interdisciplinary work" – in relation to science and mathematics – refers to some connection between these 'academic' disciplines. However, there is much debate in the literature about what might be considered interdisciplinary teaching. Many authors base their classification systems on the tightness or looseness of the connection between disciplines in an integrated curriculum. These connections range from no connection at all between disciplines, to some coordination, or to total integration into a new subject (Nikitina, 2006). Geraedts, Boersma and Eijkelhof (2006) mainly consider the context in which interdisciplinary work is established and distinguish between interdisciplinary work implemented at the macro, meso, or micro levels (ranging from interdisciplinary work being demanded by curriculum specification to ad-hoc implementation by teachers in their day-to-day teaching). A totally different way of looking at interdisciplinary approaches is proposed by Nikitina (2006) who does not look at the bonding between disciplines but rather at the concrete focus of the interdisciplinary tasks/teaching units. Here, three different ways of doing interdisciplinary work are identified:

- *Contextualising,* when the material is embedded in a cultural, personal, philosophical or historical context in order to gain a better understanding of the social and cultural development of knowledge.
- *Conceptualizing*, when core content that is central to two or more disciplines (e.g., change, linearity) are considered, with the intention of going, beyond facts, to the underlying concepts. The goal is to understand essential laws of the world and establish a connection among them.
- *Problem-centering,* when an ill-structured real-world problem is used as an axis of connection among disciplines. The aim is to generate possible solutions for this problem, bringing together different disciplines.

From this perspective the COMPASS tasks were developed taking a **problem centred approach (first major design principle)**, starting with a problem that is initially ill-structured but situated in a real-world context and of immediate relevance and importance to European citizens. Contexts such as saving energy, environmental pollution, and biodiversity have been used in developing the COMPASS tasks. Our choice of problem-centering as a design principle had a major impact on task design; we envisage that the tasks would have looked quite different if we had chosen to work within conceptualizing or contextualising perspectives.

Conceptualising approaches are often taken when developing interdisciplinarity in mathematics and science. For example, proportionality and linearity may be explored from mathematical and scientific viewpoints emphasising scientific principles and applications such as Ohm's, Newton's and Hooke's Laws.

Alternatively a *contextualising approach* would have emphasised connections between mathematics and science and the social and historical situation and not necessarily to strong links between mathematics and science. An example might involve exploration using mathematics and science of the impact of the recent catastrophe in Fukushima on the policy for atomic power plants.

Inquiry-based learning

Our second major design principle focused on inquiry-based learning pedagogies that we sought to underpin classroom implementation. *Inquiry-Based Education* (IBE) refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. Students are guided to observe phenomena, ask questions, seek mathematical and scientific ways of answering related questions (for example, carrying out experiments, systematically controlling variables, drawing diagrams, looking for patterns and relationships, making conjectures and generalizations), interpret and evaluate their solutions, and communicate and discuss these effectively (Dorier & Maaß, 2012).

Added value through the optional inclusion of digital-technological tools

Our work builds on the previous wave of research into use of information and communication technologies (ICT) in science and mathematics teaching, which has focused on the design of pedagogical principles for applications such as simulations, dynamic geometry environments, and spreadsheets (Andersen, 2006; Hoyles & Lagrange, 2009; Linn, Davis, & Bell, 2004). These principles are summarized in the following possible activity triads: Predict, test, explain; Tell, explore, check; Construct, validate, prove; Observe, find a pattern, generalize; Explore, validate, document.

Underlying all these principles (which have also been advocated by software developers and implemented in their systems) is the notion that direct manipulation of abstract representations of concrete objects and phenomena can assist students in exploring and testing out their ideas about the natural world in comparison with the theoretical world of science and mathematics (Hoyles & Lagrange, 2009).

Based on these reflections the **third major design principle was that all COMPASS tasks are designed to include ICT** that supports inquiry-based learning.

A library consisting of nine applets has, therefore, been provided as complementary to the COMPASS materials. The applets have been designed as interactive microworlds, offering idealized, dynamic and visual representations of physical phenomena and experiments.

An example of a task (at the end of several cycles of improvement)

During the project, ten interdisciplinary tasks were developed for students age 14 – 16, each comprising six to eight lessons. Each task is introduced by one overarching question placed in a meaningful, problem-centred context. Tasks could be used following a project-based-learning approach, or a more guided series of lessons in both mathematics and science. Each task is divided into sub-tasks, which are all introduced by guiding questions themselves and allocated to the different subjects (mathematics and the sciences) involved. As an example, the following table gives an overview for one of the tasks *Desertec* [28]:

[28] All tasks are available at http://www.compass-project.eu/

Desertec task	
Overarching question: To what extent do solar power plants potentially contribute to European energy needs?	
	Task 1: Science How efficiently is solar energy converted into electricity? Concepts: Solar vs. gas power station, energy production, energy conversion
Task 2: Mathematics Solar power plants consist of mirrors with curved surfaces. These mirrors are constructed with the help of computer technology. Imagine you are a constructor: Find a function that describes the form of the mirrors to be used for computer-based construction. Concepts: Parabolas and their construction, circles, focus point, quadratic functions	Task 3: Science What are the advantages and disadvantages of cylindrical/spherical mirrors as compared to parabolic mirrors? Concepts: Spherical mirrors, parabolic mirrors, reflection of light rays
Task 4 - Mathematics Can a power plant like Desertec really provide 15% of the energy needed in Europe by 2050? What surface area of mirrors would be needed? Concepts: Area, proportional reasoning	Task 5 – Science How can the energy be transported? Concepts: Alternating current, continuous current
Returning to the overarching question: Can solar power stations in the desert – such as the Desertec project – make a meaningful contribution to the energy needs of Europe?	

All materials include pedagogical guidelines and explanations for teachers, in addition to worksheets for students. Moreover, for most of the tasks, interactive software in the form of applets have been designed so that students can inquire about important aspects of the task with a focus on developing important conceptual understanding. For instance, the applet in the *Desertec* task offers a simulation where students can inquiry about how mirrors of different shapes reflect parallel sunlight or beams from focused light sources (see Figure 1).

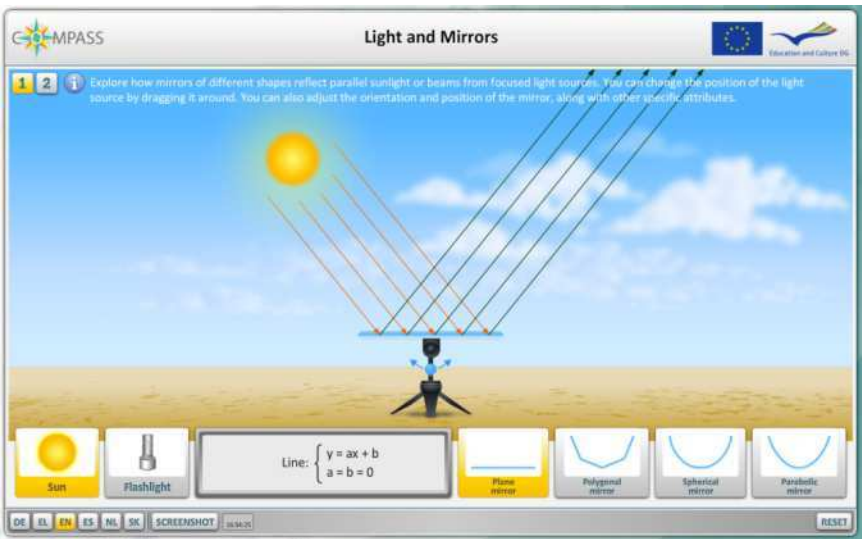

Fig. 1. *Desertec* task: interactive applet

Our concept of iterative cycles of improvement with a structured approach to formative assessment

The design of sustainable and robust materials for use in day-to-day practice was core to the project objectives. A design that works in the laboratory, or can be used by experts, may not work in the average classroom (Walker, 2006). For this reason we studied the resource requirements thoroughly and involved stakeholders in judging the quality of the design (Walker, 2006) during iterative cycles of design and improvement. Our process followed Nieeven's (2009) six steps of formative assessment and improvement to result in the final version of the materials:

a. ***Analysis of the context***: To meet the requirements of the users we carried out an analysis of the context following a systematic approach based on the Anthropological Theory of Didactics (Chevallard, 2006). This analysis ensured careful understanding of the conditions and constraints that might favour or hinder the use of interdisciplinary approaches to the learning of mathematics and science across the different nations of the consortium.
b. ***Screening***: Each national design team identified problem contexts relevant, both at a local community level, and also at a more general European level, that might allow the development of the core concepts of mathematics and science. Each team developed a first draft of its chosen tasks working within constraints agreed by the consortium as a whole, relating to the important characteristics of the tasks and their structure. Following these characteristics, one national team checked the tasks of another team and provided feedback.
c. ***Expert appraisal***: During this first cycle of improvement, the first version was presented to stakeholders from school authorities and schools. They were invited to comment on the tasks and problems posed during a focus group.
d. ***Micro-evaluation***: In each country a small number of teachers (between 3-10) read the materials, piloted them and commented on them raising a number of questions, which lead to a third version of the tasks.
e. ***Try*-out**: In each country, about 20 teachers piloted this version in their classrooms and gave feedback comprising both qualitative and quantitative data to inform the development of the final version of the tasks.
f. ***Evaluation of the dissemination***: Following Kelly (2009) that design research should continue to explore models for diffusion of innovation, we implemented several dissemination activities, including teacher conferences, professional development courses, newsletters, and other information materials, such as posters and flyers. The impact of our dissemination activities was also evaluated, in an attempt to contribute to the design of the materials.

Impact of the steps on the design principles

The design process itself and the feedback that was generated informed a further iteration in the development of design principles which we outline below.

The ***analysis of the context*** showed that competency-oriented curricula have been introduced in most countries of the COMPASS consortium. Most of these also include a clear orientation towards interdisciplinarity, suggesting integrated and/or project oriented approaches. Thus curricula specifications provide favourable conditions for interdisciplinary teaching and learning, although this is still underdeveloped in many countries.

Despite this, there are also several constraints that pertain across the consortium. A major constraining factor is the lack of interdisciplinary tasks in national tests. This testing is particularly important in England and Germany, where these tests have a strong systemic influence, but also in the other countries of the consortium. Testing has a particularly negative effect on the introduction of interdisciplinary tasks within the teaching of mathematics, particularly because assessment is often seen as particularly high-stakes in the curriculum.

In addition, day-to-day school systems and practices, such as the distribution of subjects, the existence of different teachers for mathematics and science, their initial training or the limited collaboration between them, act as barriers for the implementation of interdisciplinary oriented methodologies. Further, data from different studies and reports show that highly *structured teaching practices* are dominant, to the detriment of student-oriented practices (OECD, 2009). These act as difficult obstacles to the introduction of interdisciplinary oriented tasks that follow an inquiry-based learning approach.

In our analysis, a tension (sometimes even a contradiction) between curricular intentions and societal demands, on the one hand, and current school organization and teaching practices, on the other hand, were identified. Teachers can play a crucial role in the reduction of such tensions.

These insights led to the following design principles:

- The materials are designed so as to signal to teachers a clear vision of interdisciplinarity throughout the tasks.
- The materials are well-engineered, carefully planned, realistic in terms of their requirement of time and students' and teachers' efforts, and optimally adapted to each national context, including supporting an adequate use of ICT resources.

The ***screening*** and the ***expert appraisal*** lead to the following more refined principles:

- A clear reference to the official curriculum specification at the beginning of the task should be provided, so as to support teachers in deciding whether or not to use the materials.
- The tasks should offer the opportunity for true interdisciplinary work, but teachers should also be supported to understand how the whole task, which would need several lessons, can be subdivided into interlocking sub-tasks that can be used in separate subjects.
- An overview about the sequence of the required lessons should be provided.
- The tasks should be written in a way that teachers can use them directly in their lessons.
- Interesting and motivating guiding questions for students should be provided for each sub-task.
- The individual tasks should contribute to the solution of the whole overarching question and the links should be made clear.
- A national adaptation to the specific cultural context of the tasks might be needed and this should be signalled to teachers.

The ***micro-evaluation*** (via formative questionnaires, interviews in workshops and written reviews) resulted in further revision and again more detailed design principles, namely:

- A short, yet concrete, theoretical part needs to be included in each activity.

- The materials need to have clear descriptions of the tasks for students, including possible solutions, along with pedagogical guidelines for teachers.
- The materials need to provide a list of mathematical and scientific competencies addressed in the activities (as to align them with the curricula which are competence based).
- The materials need to provide suggestions and guidelines for differentiation.
- Students' worksheets need to be provided separately. The layout of the materials needs to facilitate their use; it must be ensured that teachers can differentiate at first glance between student tasks, background information, solutions, etc.
- The materials need to provide pedagogical alternatives, e.g. for both teachers who want to follow a very open approach (by giving only the overarching questions) and those who want to follow a more structured approach.

Summing up, the initial overarching 'visionary' major design principles guided early drafts of materials; feedback, following trials of these, led to more and more detailed design principles to inform the eventual development of final versions of the materials. A final try-out (step e) suggested that no further improvement seemed necessary, with the design being considered as complete and the final phase of dissemination and exploitation process of the materials (step f) being started.

Results of the try-out and evaluation of the dissemination

For the evaluation of the trial of our materials, we used a mixed method-design, using both qualitative formative questionnaires and quantitative questionnaires in a pre-post design.

The teachers and the students who participated in COMPASS highly valued the developed materials. Further, the majority of teachers reported positive attitudes in using the materials after the end of the project. In the teachers' view, the materials contained motivating and interesting practical activities and dealt with relevant problems for society and realistic questions of sustainability. Furthermore, after testing the COMPASS units, COMPASS teachers showed interest in further materials for interdisciplinary teaching. However, the degree of satisfaction in relation to the practicality of use in day-to-day teaching as well as teachers' opinions about how well the tasks were described was only mediocre. These reactions show what a big step it is for teachers to include interdisciplinary problems in their daily teaching, bringing to the foreground, once again, the systemic conditions that hinder the use of these kinds of activities (mainly through the general organization of schools that supports well established school subjects). Both student and teacher questionnaires revealed that teaching had changed towards a more student-centred and application-orientated way of teaching.

It is difficult to determine the effectiveness of *the dissemination and exploitation* of the materials. However, we can claim that the dissemination activities carried out have been quite successful, although counting participants at an event, subscribers to a newsletter, the number of distributed flyers, and so on, is not a clear indicator of the success. The question remains whether that really gives a reliable insight into the success of the dissemination. For example, more than 250 teachers from across the consortium nations, but mainly from the host nation Germany, attended the final project conference. The project may have an impact on these individual teachers; further 'reach' can be secured by engaging the interest of just one individual, such as the representative of the Baden-Württemberg school authority, responsible for the education of all teacher trainers in the region, who attended the conference and has adopted use of the materials with this group. Altogether, COMPASS ideas and designed tasks and software have reached more than 9500 teachers through dissemination activities, such as conferences, professional development courses, video conferences, etc. Website visits, tasks' and applets' downloads, and positive teacher reactions in workshops and conferences provide us with confidence that COMPASS interdisciplinary tasks have been well adopted and are used in various classrooms in a number of European countries.

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Enabling Education for Values with Mathematics Teaching

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In the fall of 2007, we conducted a workshop for high school teachers that served as a spring board for a 3 year multi-faceted study focusing on the integration of values education in the ordinary teaching of mathematics in high schools in Israel. This paper is focused on the guidelines and principles underlying the development of text-based tasks for the study. Each of them has the potential for contributing slightly to values education, in addition to the learning of some mathematical topic of the mandatory curriculum. The paper also includes a few relevant results of the main part of the study, which included action-research carried out by 12 mathematics teachers at 8th grade who worked with us for a year interweaving in their regular math teaching various tasks of that sort.

Keywords: values and math education, personal values, social values, text-based tasks, action research, thought experiment

Introduction

As in many other countries, the educational system in the state of Israel is based upon the law of mandatory education (1949) and the law of national education (1953, am. 2000, 2003). These laws state that education for values is in the forefront of the national goals of upbringing youth towards becoming literate citizens: some of them are social values like equity, tolerance, social justice; others are personal values, such as being rational, achieving, and exhausting one's intellectual potential. In practice, the study of subject-matter usually takes over, and values education subsides to a negligible corner, in high school even more so than in primary education. The majority of the professional teachers, and particularly high school mathematics teachers, devote their classtime to instruction of their disciplines, and see themselves exempt from values education. Some math teachers may even argue that this combination is in contradiction to the "objective" nature of mathematics.

The mandatory curriculum for high school in Israel is centralized, and includes details about the various subjects to be taught in each grade-school. In the introduction, there is a statement about the general goals for teaching mathematics. These goals set the stage for teaching a large variety of mathematics content, emphasizing that teaching this content should (i) improve students' mathematical thinking, (ii) provide skills and competency in problem solving, (iii) take into consideration differences among students' learning styles and capabilities, avoiding as much as possible students' frustration and failure, and (iv) make students like mathematics and appreciate its beauty (Israel Ministry of Education, 2012). This introduction to the syllabus leaves room for interpretation about the place of values education in mathematics education.

Analysis of formal documents of the ministry of education, the mandatory mathematics curriculum, and introductions to mathematics textbooks shows that there is a gap between the legislators' intentions and the availability of means for their realization. Indeed, it is well known that knowledge impartation, particularly preparation for the matriculation exams, take precedence, in most cases over education for values.

We challenged the possibility for combining values education with mathematics teaching in a multi-faceted research study (Edri, 2010)[29]. This is a further step along the line of research initiated by A. Bishop (1988, 2008), who offered a theoretical model of values in mathematics education, and by Harmin, Kirschenbaum & Simon (1973) and Shechtman (1980) who offered a model of three levels of teaching for Clarifying Values Through Subject Matter. The basic drive for this study stemmed from the a-priori assumption that if a possibility for combining values education with mathematics teaching exists, it is necessary to exhaust it.

The Basic Idea

Basically there are three settings in which each secondary school teacher treats values education:

1. As a role model (not necessarily in class, but also in the courtyard, during a fieldtrip, in a preplanned personal meeting, or a meeting by chance). Teacher's personal norms are the basis for the education s/he gives at such settings.
2. As a classroom manager, regardless of the discipline the teacher is supposed to teach. A mixture of different methods of classroom management and personal norms of the teacher, which usually come from personal background, gives rise to values education at this level, for example, teacher's tolerance to differences in level of achievements among students, teacher's culture of listening to others, teacher's respect to time, etc.
3. As a subject matter teacher, utilizing the curriculum as a lever to education for values. Surely, teacher's personal norms are reflected here, but along with that the discipline itself, in our case mathematics, has a crucial effect on the selection and exploitation of the opportunities that the discipline allows for values education.

The first two settings are common to teachers of all school subjects. The third is dependent on the specific discipline and the opportunities embedded in it for values education. In this paper, we focus on the third setting, addressing opportunities for values education from within mathematics. Needless to say, in practice it is inconceivable to separate the three settings, as they naturally complement each other. Hence, it is a matter of emphasis and priority that we put on the third one.

The study

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A preliminary enquiry through interviews with mathematicians and mathematics educators, and through written questionnaires administered to mathematics teachers, and the top percentile graduates of the educational system, revealed various ways these populations perceive the relevance of values education to mathematics and its learning/teaching. The common thread among these groups, representing an accessible sample of the literate society in Israel, was skepticism about the feasibility of connecting the two. It is noteworthy that many of the high school mathematics teachers considered it important to combine values education with the teaching of *any* subject. However, they found it difficult to connect it to their own subject-matter and cumbersome to intentionally do it in mathematics teaching, beyond the role-model each of them obviously plays as an adult human being who manages a mathematics classroom. This enquiry also revealed that personal values, such as critical thinking, accuracy, and persistence, are mostly perceived as inherent to mathematics instruction, whereas the social ones, such as tolerance, social justice, and empathy, are perceived as not inherent to it.

In the professional literature, we were able to find some less sceptical views: Because mathematics is a discipline that nurtures common sense, it has the potential for education for values (Bishop, 2008; Ernest, 2004; Vinner, 2009) including to social ones (D'Ambrosio, 1990; Gutstein & Peterson, 2006; Osler, 2007; Taplin, 2009; Voss, 2012). Despite the risk, our study originated from the hypothesis that embedded in the mathematics curriculum is a potential not only for personal values, but also for social values education. If this is indeed the case, ways must be found to exhaust this potential. We found some support to this hypothesis in a number of examples given by Mistrik &Thul (1997), Gutstein & Peterson (2006), and later on by Taplin (2009) and by Voss (2012).

The first stage

To start, we conducted a thought experiment, in which an analysis of the syllabus was carried out to identify places in the curriculum suitable for linking to various values. This search yielded "The values-annotated math curriculum". This work was based upon the following guidelines established in a workshop that took place prior to the thought-experiment:

- Replacing the context of 'word problems' (keeping the underlying mathematical model) so that they can yield, beyond the mathematical solution a discussion that has bearing on values, such as environmental conservation and protection, preventing drugs, smoking and drinking, and more.
- Linking mathematical concepts and terms to values (e.g., equality, inequality, true vs. false statement, identity, negative and positive numbers, rational and irrational numbers).
- Assigning higher level tasks and training students to persist in solving harder problems, to join their effort, to collaborate in solving difficult problems, to not give up easily, to seek and experience the fulfilling intellectual satisfaction attained by a solution reached following an investment of effort.
- Providing education for life in a democratic society through mathematics:
    - Axiomatic systems can be compared with The Constitution. Acceptance and follow-up, rather than argue and quarrel.
    - Laws /rules/mathematical theorems can be employed as a springboard for a discussion of values in a more general way – distinguish between lawful and unlawful actions, observing mathematical laws prevents mistakes, and

observing civil laws prevents other trouble. A law in a democratic society is accepted by the majority and can be changed by the majority; in mathematics, there is no democratic vote, there is a proof.

- Connecting mathematical actions to parallel ones outside of mathematics For example, in court there is a demand for a proof for any claim; in court, a proof is only beyond a reasonable doubt. In mathematics, a proof stands on logical basis; once proven, a theorem ("verdict") cannot be reopened for consideration (although there is always room for purification of the proof itself).
- Conflicts and conflict resolution among friends, siblings, rivals or political parties can be brought up following tasks focused around mathematical paradoxes. Emotions, such as frustration, can be reflected. Accepting differences in opinion as a basis of friendship without forcing a consensus, as opposed to the necessity to resolve cognitive conflict and mathematical paradoxes resulted by some hidden errors.
- Encouraging debate in mathematics through tasks involving paradoxes is a challenge in itself. What is right and wrong in mathematics is not determined by a majority vote.
- Freedom of thought with tools and limitations linked to such tasks. There can be distinction between random actions and arbitrary choices, action based on intuition or on rational explanation.
- The ability to reason logically and the power of being rational, rather than emotional or impulsive, in explaining your stance and in drawing conclusions.
- Accountability is another important issue that can be integrated.

• Using a clear and accurate language orally and in writing can be linked to learning mathematics. More precisely, the ability to define clearly a concept, and to sequence logically a chain of arguments which follow one another, can be enhanced through mathematics. This includes:
  - Alternative (equivalent) definitions
  - Finding "holes" in arguments presented by others (in a civilized manner),
  - Responsibility for accurate wording, and liability
  - Elaborating on idioms in the language borrowed from mathematics. For example, "Squaring the circle" (for an impossible mission), "They have a common denominator" (for comparing two issues), Two things are "orthogonal" to each other (meaning cannot go hand in hand) or parallel (go hand in hand but never meet), "This is an axiom" (meaning – one should accept it as true)

• Bringing up human values, such as equity, freedom, determination, and modesty, through interweaving stories about the lives of mathematicians and the history of mathematics that have to do with values. For example:
  - The life story of some female mathematicians, for example, Sophie Germain (1776-1831) who used the pseudonym of a man, Anthony La- Blank, to find her way among the mathematicians of the time and win their respect. Such stories can lead to a short discussion about gender these days, and about "masculine" vs. "feminine" professions.
  - Cartesian coordinate system leads naturally to Rene Descartes. He was a sickly child and his parents let him get up late in the morning and stay home a lot. He used it to think and create mathematics. He is quoted as saying: "I prefer truth over beauty." He dealt with the difficulties and even managed to

create despite the difficulties. He was known as a humble man who lived a Spartan life but never criticized others for having a different life style.

- Addressing matters inherent to teaching, particularly to the teaching of mathematics, that teachers need to be aware of, to emphasize during the teaching process, and to leverage explicitly or implicitly to values education, such as:
  - Self-confidence and lack of it, the freedom to make mistakes and learning from a mistake, acknowledgement of potential ability, patience and tolerance, consistency, persistency, self-control, delayed gratification, intellectual courage.
  - Express feelings of anxiety and fear of failure in mathematics and compare them to similar feelings in students' lives. Hope and despair, frustration and joy of achievement, a sense of power and confidence versus inferiority - how do you deal with them?
  - Exceptions, such as the prohibition of division by zero, can be linked to treating exceptional cases in society.
  - Quantification of the size of an error (in learning statistics) can be linked to the courage to speculate, to guess, to take the risk of mistake. (The freedom to err is the freedom to learn.)
  - Unexpected solutions to a mathematics problem or some other mathematical surprises can be used to talk about situations in life where a problem was solved unexpectedly, the joy of discovery.

The next stages

The second stage was a development of 23 exemplary text-based tasks that teachers would be able to assign to their students while teaching mathematics in the 8th grade. In the main part of the study, 72 experimental lessons were taught in 7 high schools by twelve 8th grade teachers who applied some of the exemplary tasks in their classes and created more tasks inspired by the exemplary tasks and following the guidelines. The rest of this paper is devoted to 3 of the 23 exemplary tasks. An intentional effort was made to keep the part of values education limited to a small part of the mathematics lesson, so that progress in teaching and learning the mandatory mathematics curriculum would not be harmed as a result of assigning these tasks to the students.

Sample Task 1

This task is basically a word problem. At the mathematical level, it is a practice task in basic calculations of percentage rate adapted from a common textbook, by changing the context and adding two parts — no. 1 and 4, which add the values education aspect (see below). At the values education level, the goal is to educate for social equity by raising students' awareness to an affirmative action taken by the government of Israel to promote appropriate representation of minorities in the civil service.

The Task[30]

In 2007, minorities (Arabs, Druze, and Circassians) were one fifth of the population in Israel. Despite this, only 6.2% of all civil service employees in this year were minorities. Over the years, the Israel government has made decisions (in 2004, 2006, and 2007) to promote suitable representation of minorities in the civil service, setting 10% as a target for the percentage of employment of minorities in the civil service.

1. What do you think about the goal that was set by the government?
2. The Ministry of Housing and Construction had 741 employees in 2007. Had the target set by the government been achieved, how many members of minorities would have worked in the Ministry of Housing and Construction?
3. Twelve employees in the Ministry of Housing and Construction were minorities in 2007. What percentage of all employees in the ministry were minorities?
4. What do you have to say about the two results you obtained?

Integrating Education for Social Justice in the Lesson

The answers to Part 1 varied. Some students explained that because minorities were one fifth of all citizens, their proportion in the civil service should be 20% which represented a fifth of the citizens. Other students claimed that people are selected for a job according to their capabilities, and it may be difficult to find suitable people among the minorities for some jobs. They maintained that there is no reason to hire a person who is not qualified only because he belongs to a minority group.

The mathematical solutions of parts 2 and 3 of the task[31] were handled in the common way, starting by independent student-work, followed by a dialogue between the teacher and the students to elucidate the particulars.

Responses to Part 4, which was intended to make students aware of the huge gap between the situation in 2007 and the goal set by the government, indeed gave start to the social-values-oriented discussion that followed.

In the course of the class discussion, the teacher provoked questions about the issue of affirmative action for minorities in Israel. For example, the teacher asked: "Why at all was the government promoting employment of minorities in the civil service?" There were various answers:

- One student said that the principle of equality among people should lead to equality in representation of minorities in all areas.
- Another said that the principle of equity among people should be interpreted as equality in the criterions for hiring for a job.
- Another student said that Arab parliament-members were exerting pressure on the government to take action in this area to improve the economic and social status of the Arabs in Israel.

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[30] The figures in this problem were taken from a report by a parliamentary investigative committee on the subject of accepting Arab employees in the public service, headed by MP Ahmed Tibi (2008).

[31] The mathematical solution to Part 2: 10% of all the employees in the Ministry of Housing should have been members of minorities, i.e. 74 employees.

The mathematical solution to Part 3: 12 employees constitute 1.6% of all the employees in the Ministry of Housing.

- Some others claimed that improving the economic and social status of Arabs in Israel is in our interest, if we wish to live in peace.
- Two girls had a dispute between them: one claimed that promoting equality of Arabs was liable to harm the interests of Jews in Israel which was established as a homeland for the Jewish people. The other girl said that when Jews were a minority in the Diaspora they suffered from injustice; now in Israel, we ought to give equal rights to the minorities in understanding their suffering.

A few clarification issues were also addressed: What is civil service work? What reasons could cause difficulties in implementing the government decision to promote employment of minorities in the civil service?

The task drew students' attention to the question of representation of minorities in The Ministry of Housing and Construction, and in the government in general. Nevertheless, in the course of the discussion about these questions, several related issues came up. One was the kind of work minorities do in Israel. The discussion diverted to other workplaces: schools, hospitals, post offices, the Israel Broadcasting Authority, and more. For example, one student mentioned that in this school an Arab woman worked as an Arabic teacher, and 12 more Arab women worked as cleaners. Another student said that a few Arab doctors can be seen in the hospitals, but many Arabs working there are cleaning staff.

The teacher summarized, acknowledging the importance of the discussion and leaving some of the issues open for further discussion in the future and outside the mathematics classroom. Then the teacher moved to another activity related to percentage rate.

Sample Task 2

At the beginning of the chapter on solving linear equations, students were assigned for homework a Google search or any dictionary/encyclopaedia search of words related to "equation," summarizing the definition of the words they found.

Students came back to class reporting about the definitions of *equator*, *equality*, *equilibrium*, *equalizer*, and more.

The teacher indicated that upon completion of the study of solving mathematical equations, students would be asked to compare their understanding of the processes involved in equation solving with their understanding of the related non-mathematical terms they found.

This final task, which apparently consumes a minimal part of in-class time, was a trigger for summarizing the mathematical aspects of solving linear equations from a new perspective and quite naturally opened the door to the discussion of inequality – gender inequality, racial inequality and even inequality in family relationships. This discussion also served as a nice transition to the next mathematical topic – solving linear inequalities.

Sample Task 3

This task is adapted from *One Equals Zero* (Movshovitz-Hadar and Webb 1998). It is appropriate for presentation either before or after the study of area of a parallelogram, or earlier when the notion of area is introduced.

1. In Figure 1, four strips of width *d* are marked 1, 2, 3, 4. They connect the two parallel lines *PP'* and *QQ'*. Which strip has the largest area? Which one has the smallest area?

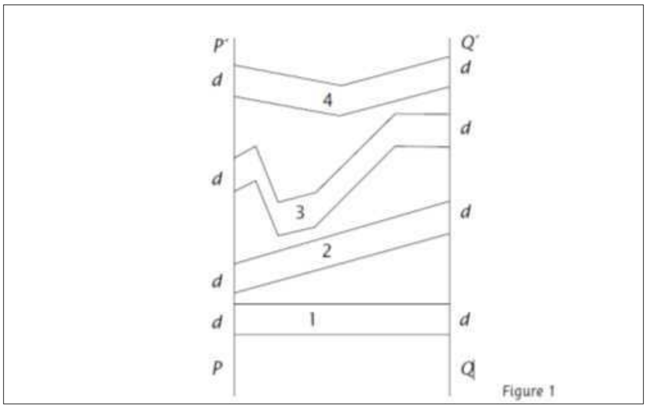

Figure 1

2. Copy the parallelogram (2) so that its left side coincides with the left side of the rectangle (1). Express the area of the parallelogram as a difference between two areas, and reconsider your response. Would you now change your mind?
3. Read the following explanation:

All four strips have the same area. This may not look true, but we can easily verify it.

To show that rectangle 1 and parallelogram 2 have the same area, simply draw the rectangle in such a way that its vertices coincide with the vertices of the parallelogram, as shown in Figure 2. Now consider the area of parallelogram *ABEF* as the difference between the areas of trapezoid *ABCF* and triangle *BCE*. Because triangles *BCE* and *ADF* are congruent, they have the same area. Therefore, the area of the parallelogram is also the difference between the areas of trapezoid *ABCF* and triangle *ADF*, which is exactly the area of rectangle *ABCD*.

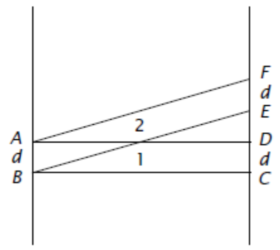

Figure 2

Observe in Figure 3 that the area of polygon 3 is the difference between the areas of polygon *EFGHIJCB* and polygon *E′F′G′H′I′J′CB*. The area of polygon *E′F′G′H′I′J′CB* equals the area of polygon *EFGHIJDA* because the two are congruent. Thus, the area of polygon 3 is exactly the same as the difference between the areas of polygon *EFGHIJCB* and polygon *EFGHIJDA*, which is the area of rectangle *ABCD*.

Using similar reasoning, it is easy to observe that polygon 4 has the same area as the other strips.

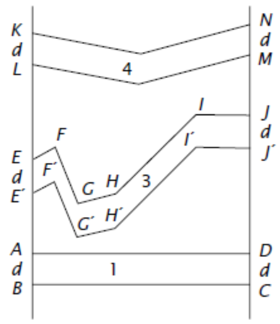

Figure 3

Were you convinced? Did reading this explanation change your mind?

4. Share with your peers the feelings this task gave you. Did you feel that it was unfair to present the task the way it was presented?
5. Would you agree that the freedom to err in mathematics is actually the freedom to learn?

The teacher can stop the individual activity after part 1 and have a vote or even rank order the strips by students' votes.

Many students tend to choose Strip 1 as the smallest in area and Strip 3 as the largest. It would take "Poker-Face Pedagogy" on the teacher's part to let students continue working on the next two parts. (See more about "Poker-Face Pedagogy" as related to tasks, in Movshovitz-Hadar, 2011.) This work might put at stake the students' knowledge about area and hopefully yield a deeper understanding of it.

Nevertheless, some students may feel cheated. Some of them may suspect the teacher is intentionally misleading them, while actually the teacher is intentionally challenging the fragility of their knowledge. (See more about "Knowledge Fragility" as related to tasks, in Movshovitz-Hadar, 1993.) Of course, pedagogically speaking, there is nothing wrong or unethical in presenting this task or others like it. However, it is up to the teacher to handle such tasks with care, so as to prevent possible frustration from creeping in, and exploit the potential opportunity for values education by moving to parts 4 and 5 of the task.

In part 4, students are asked to express their feelings about the task. Have their senses ever tricked them? - Hearing some noise that seemed frightening but proved to be nothing more than a cat; entering a swimming pool and feeling that the water is much colder than it feels after a few minutes; eating a sweet candy after salty French fries. Through tasks of this kind, students can become aware of the power of being rational rather than emotional or impulsive in drawing conclusions. They would gather that they should be careful about their intuition and about taking what they see as a basis for decision making too quickly. The value of trusting one's sense of reason is enhanced as more fundamental than trusting one's intuitive eye-sight.

Emotions such as surprise and frustration can be reflected. This has an embedded potential for connecting to values education as follows. Students can be encouraged to talk about other incidents of frustration following conflicts which they encountered or heard about, such as conflicts among friends, siblings, rivals or political parties. The role of conflict resolution can be discussed and the relief it brings when it is achieved. Some students may even point at a difference between the necessity to resolve a cognitive conflict resulting from some hidden mathematical error or misconception and the unnecessary enforcement of consensus among friends who respect and tolerate each other despite different political views they hold.

Encouraging debate in mathematics paradoxes is a challenge in itself. This task brings up the point that right and wrong in mathematics is not determined by a majority vote. Mathematics has its own tools. Unlike spoken language in which misunderstandings occur because interpretation often depends on the context, and may sometimes lead to conflicts and unfortunate incidents, in mathematics there is very little (if any) room for dispute.

The moral of this task is that the freedom to err yields an opportunity to learn.

Effect of the Experimental Work on the Teachers

The findings of the study (Edri, ibid) indicated that the teachers' involvement in the experimentation contributed to their personal and professional development. Here are a few statements from the reflections of participating mathematics teachers[32]:

> …At first, I was very dubious about integrating values education in my mathematics teaching. Values such as accuracy or intellectual courage, are values that I could see fit in with mathematics, but social values?!? When I heard about the study, I thought that if social values were to be integrated, it would be artificial… Up until now, I never considered integrating values education in the framework of mathematics lessons at all. I didn't think this could really be done. It surprised me that it was possible to integrate values such as "social justice" in mathematics lessons… I thought it could be done only at the margins and only

[32] The study was conducted in Hebrew and the quotes were translated to English by the authors.

> indirectly… When I gave the lessons in statistics concerning violence among students, it opened the way to a lively discussion on the subject… Without my expecting it, students spoke about personal experiences related to violence… During our in-service preparation meetings the examples presented to us started to inspire me… however, when I started employing them in my class, integration of values in teaching mathematics subjects became gradually a second nature to me … the students dared saying to me that my teaching changed, became much more interesting, and relevant to life, and I also enjoyed it a lot… They say that you don't really absorb material until you teach it, and that's exactly what happened to me. Until I experienced activities that combined values education with mathematics in class, I didn't understand how much I could affect the students, and what values discussion could develop around mathematical activity… Today, I make sure to include some task that has to do with values education in almost every lesson I prepare…

Note that all 12 teachers who took part in the study succeeded in integrating personal as well as social values education in their teaching without harming the results of teaching mathematics according to the mandatory curriculum. They attested that their involvement in the experimental teaching contributed to their professional growth and to their personal development as educators.

Summary and relevance to ICMI22 Theme C

In this paper, we focused on a description of sample tasks aimed at integrating values education in teaching subjects from the mandatory mathematics curriculum. The integration is achieved by following the guidelines mentioned at the beginning of the paper into text-based tasks for students, which teachers can adapt to their own classes and teaching style. This was quite a challenge.

Examining the tasks and employing "reverse engineering" analysis[33], the following are suggested as the design principles underlying our work on the development of text-based tasks:

1. The tasks are mathematics-curriculum based assignments, which can be completed within the framework of one class period or one homework session.
2. They include a clearly phrased introduction followed by two kinds of short questions: (i) Mathematical problem solving, preferably of exploratory nature or thought provoking nature; (ii) Dialogue promoting questions intended as a vehicle for values clarification through bridging between the contents and student's feelings, views, and behavior (Harmin, Kirchenvaum & Simon, 1973; Raths, Harmin & Simon, 1978).
   These questions are sometimes provocative, however "neutral" in nature, so as to allow the surfacing of various personal attitudes about human values and avoid any implicit hints as to "the desired" response. These questions may serve as triggers for a short whole-class debate connected to the mathematical work, but not necessarily in a strict way.
3. The text is carefully expressed so as to avoid in as much as possible obstacles in the form of editorial faults (see more in Movshovitz-Hadar, Inbar, & Zaslavsky, 1987).
4. Tasks can be assigned as individual work, or as group work, for class time or for homework, depending upon the teacher's preference. The teacher's manual

[33] Reverse engineering as a tool for validating a goal-oriented task development was introduced in Amit and Movshovitz-Hadar (2011)

includes recommendations for facilitating and handling values-related discussions, which usually are not an ordinary part of mathematics teachers' background.

The amazing fact that values education *can* be integrated even in mathematics, a leading rational field, was confirmed by the team of teachers who participated in the experimentation of the idea. Beyond the feasibility of the integration itself, it was also proven that it is possible to do so *without* lagging behind the mandatory curriculum on the one hand, and on the other hand without harming students' achievements, even improving them in many cases.

As we discussed in an earlier occasion (Edri & Movshovitz, 2009), turning the integration that has been proved possible into a tool regularly used by mathematics teachers is essential, but it will obviously require (i) preparation of suitable tasks for the various age levels and many subjects appearing in the curriculum, and their introduction into textbooks, and (ii) preparing inservice teachers as well as future teachers to use them and possibly invent more of the kind.

The constant search for opportunities to integrate values education in mathematics teaching adds a missionary aspect to the mathematics teacher's work, and justifies the great investment, from which the resulting satisfaction enriches not only the students, but also the teacher.

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Designing Interdisciplinary Curriculum for College Algebra

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Interdisciplinary curriculum, long viewed as a "best practice" for early undergraduates, is usually delivered through team teaching or learning communities. However, the complexity and expense of these approaches preclude wide implementation. This manuscript proposes an approach to designing supplemental, interdisciplinary curriculum for college algebra textbooks that a single mathematics instructor can deliver. Key elements of the model include an essay and disciplinary introduction co-developed by a mathematics instructor, a disciplinary specialist, and a creative writer; explicit learning goals to limit the responsibilities that a teacher feels for expertise in another discipline; and interdisciplinary homework questions. We discuss a sample module that was designed to engage students pursuing social sciences and human services majors.

Keywords: interdisciplinarity, algebra, modeling, undergraduate education

Introduction

This paper proposes an approach to designing interdisciplinary curriculum to supplement college algebra textbooks. Higher educational leaders in the United States have called for increased interdisciplinary undergraduate education for decades, but its implementation has been relatively uncommon (Ellis, 2009; Leskes & Miller, 2006). We outline an approach to designing curriculum for first-year algebra courses that is authentically interdisciplinary, that is, the curriculum supports learning goals that are significant within the partner discipline as well as in algebra.

Our work takes place in college algebra classes that serve students pursuing studies in education and other human services disciplines, such as social work, family social science, sport management, and human resource development. For most of these students, college algebra is their final mathematics class. Teaching students at the end of their formal educational journey in mathematics poses significant dilemmas of responsibility. Within the framework imposed by the institution, how can a teacher help students strengthen their skills and their sense of purpose sufficiently so that mathematics may remain relevant in their academic work and in their lives?

In addressing this dilemma, we focus our curriculum experiment on connections between algebra and the social sciences and the humanities rather than the sciences (Usiskin, 2003). The curriculum module described below takes the unusual step of linking the algebra of finance with a classic theory in psychology, one that students often encounter in an introductory psychology course. Erik Erikson's theory of life stage development holds that people experience identity crises throughout their lives; productive engagement of these crises results in the person's positive psychosocial development. As an uncommon curriculum partner for algebra, psychology provides a useful case for probing the problems and limits of creating interdisciplinary curriculum, more so perhaps, than partnering algebra with a scientific discipline in which the applications are well-understood.

This module is intended to be used as a homework supplement to a college algebra textbook, and so it occupies the curriculum level that Usiskin calls a "lesson" (2003). In this paper, however, we focus primarily on principles of organizing modules so that they support interdisciplinary teaching by a single mathematics instructor. We consider this curriculum as a type of task that is designed according to specific principles and that creates a non-traditional learning environment (Becker & Shimada, 1997).

Staats is an associate mathematics professor in the College of Education and Human Development (CEHD), University of Minnesota, USA, where she teaches college algebra and interdisciplinary seminars, primarily to undergraduates enrolled in education and human services fields of study. One of her research strands involves designing interdisciplinary algebra curriculum and analyzing student reactions to it (Staats, 2005, 2007; Staats & Robertson, 2009). Staats' doctoral degree is in cultural anthropology; her experiences teaching with anthropology curriculum influenced her thinking about interdisciplinary algebra curriculum. Jason Johnson is a graduate student studying educational public policy. Johnson taught mathematics at Henry Sibley High School in Minneapolis, Minnesota for nine years, where he became interested in interdisciplinary curriculum development through founding and coaching the school's debate team.

The theoretical background for this experiment in task design is the pedagogy of mathematical modeling and studies of mathematical discourse. Mathematical modeling involves cycles of simplification, solving, interpretation, validation and generalization, or critical reflection leading to improved solutions and methods (Lesh, et al, 2003; Maaβ, 2006). The module discussed below is assigned within a college algebra class that is dedicated to modeling. Interdisciplinary writing assignments are presented as a particular style of interpreting and validating mathematical work.

Studies of mathematical discourse also inform this work. Gerofsky (1996, 2004) and Morgan (1998) have detailed ways in which mathematical writing and speaking rely on stylistic and grammatical patterns or genres that may influence the responses that students consider appropriate to offer. The genres of mathematical writing thus problematize the goals of interdisciplinary learning. In the curriculum design model discussed here, a creative writer constructs the core essay for the module, helping to establish expectations that are less centered in one discipline.

Interdisciplinary Math in Higher Education

The Curriculum Foundations study by the Mathematical Association of America (1999-2001) identified interdisciplinary, team-written curriculum for early undergraduate mathematics curriculum as a high priority (Ganter & Barker, 2004). However, typical means of organizing interdisciplinary education are complex, expensive, and require an enormous commitment from institutions and instructors (Boix Mansilla, Miller & Gardner, 2000; Burrill & Hernández-Gantes, 2003; Klein, 1990; Lattuca, Voigt, & Fath, 2004; Usiskin, 2003; Wentworth & Davis, 2002). In a team-teaching model, two or more faculty members provide instruction on an interdisciplinary theme to a single class of students. In the learning community model, all the students who enrol in one class also enrol in a class in a different discipline. Although both models ensure that students receive instruction from disciplinary experts, the complexity of scheduling classes, registering students, and achieving an adequate balance between faculty compensation and generated tuition means that the best practices for interdisciplinary education are beyond the means of many higher education institutions.

Strong disciplinary boundaries also make interdisciplinary work challenging. Instructors often feel uncomfortable leading explorations outside their area of expertise, particularly in mathematics. Mathematics curriculum may be defined more rigidly than in other fields (Grossman & Stodolsky, 1995). Mathematics faculties often have limited professional interactions with faculty in other disciplines (Ewing, 1999). At times, even the philosophical basis of mathematics is held in contrast to the integrative goals of interdisciplinary education (McGivney-Burelle, McGivney & Wilburne, 2008; Siskin, 2000). This sense of disciplinary authority and curricular boundedness can impede the adoption of deeply interdisciplinary curricula in mathematics classes (Staats, 2007). When the ideal delivery models for interdisciplinary education are not feasible, carefully-designed, supplemental texts may empower a single mathematics teacher in a standard college algebra class to deliver short, restricted units of interdisciplinary content.

Design Principles

Drake (2010) found that teachers often introduce supplemental materials when their textbooks lack varied activities and when they wish to engage students with far-ranging needs and interests, needs that interdisciplinary curriculum can help fulfill. First, an interdisciplinary curriculum clearly must enable students to demonstrate competency on topics drawn from each discipline. Correspondences between the goals of interdisciplinary education and several contextual approaches to mathematics education can further guide development (Staats, 2007). The modeling approach emphasizes the creative process, critical reflection, and refinement. Realistic mathematics education seeks scenarios that are authentic and relevant to students. Social justice approaches seek to create greater awareness of social issues. Each of these themes — relevance, awareness of social complexity and taking a critical perspective on one's own solution — are key goals of interdisciplinary learning as well. A supplemental interdisciplinary curriculum for college algebra should also be a useable module that a single mathematics teacher can deliver. The following design features help to fulfill these goals:

1. A module is co-authored by a mathematics teacher, disciplinary specialist, and creative writer.
2. The module core is an "essay," written in any genre, which poses questions or presents scenarios that require consideration of algebra and another discipline. The role of the creative writer is to present the interdisciplinary scenario in a lively and engaging way, and to ensure that the module avoids framing questions within the ideology of a single discipline.
3. An introduction can convey disciplinary content in more direct fashion than the core essay. This allows for more expressive latitude for the essay.

4. The essay is followed by a list of explicit learning goals for both algebra and the partner discipline. The learning goals help limit the content from the partner discipline that the mathematics teacher will discuss in class.
5. Homework includes scaffolding questions on algebra and on the partner discipline, and interdisciplinary activities.
6. The final component is a short bibliography to support further reading by either instructor or student.

This structure for curriculum writing is modeled in part after a style of undergraduate curriculum writing that is prevalent in the social sciences. In a typical introductory anthropology class, for example, students use a textbook which conveys the course content directly and a reader with case studies in the subfields of cultural, biological and linguistic anthropology and archaeology. A reading has an introduction and a case study, either one written as curriculum, or an excerpt from a published work. This style of curriculum writing allows instructors to present material that diverges widely from their own specialty. Scientifically-oriented biological anthropologists, for example, regularly teach about linguistics or symbolic analysis, perhaps having taken only a single class in this subfield. The case of anthropology curriculum development and delivery suggests that it is possible for a single instructor to teach across wide epistemological differences.

In the proposed algebra curriculum project, enlisting a creative writer to develop the focal essay helps to ensure interdisciplinary integrity. The creative writer chooses a genre — fiction, life history, poetry, academic prose, nonfictional exposition — that conveys the perspectives, philosophies, and curricular needs of the two disciplines in a lively way. The essays must invite students to demonstrate analytical skills in two disciplines, but they should not privilege one perspective over another. Creating this dynamic tension between different viewpoints and different ways of organizing ideas is at the heart of the writer's craft and, perhaps, of interdisciplinary thinking as well.

Mathematics teachers are not expected to be experts in the linked discipline. It is, however, reasonable to expect mathematics teachers to be able to learn a limited amount of first-year university-level material and to lead a critical, not authoritative, classroom conversation based on this reading. Some mathematics teachers do this when they use newspaper clippings as a basis for mathematics lessons. Some universities assign instructors from across the disciplines to lead discussions on a book that all entering students read. These teaching activities are relatively commonplace situations in which mathematics teachers lead students in discussion of material outside their area of scholarly specialization. They position mathematics instructors as lifelong learners who have developed basic intellectual and civic competencies through the liberal education process that their first-year students are about to commence.

The sample module that connects financial mathematics to Erikson's theory of psychosocial development can be viewed at z.umn.edu/icmi22. The core of the module is a short story titled *Indebted*, in which a young man wrestles with the question of how to pay for his college education. The young man visits his grandfather, who suffers from Alzheimer's disease and lives in a nursing home. The grandfather hoped to contribute to his grandson's education, but instead, he was obligated to use his life's savings for his own health care. The young man considers mathematical scenarios associated with indebtedness, such as rapidly rising college tuition and the per capita value of the national debt. Finally, he uses an elegant writing pen, a gift from his grandfather, to sign his college loan papers.

Relevance in K – 12 Settings

In primary grades, interdisciplinary learning is often experiential, sometimes drawing from the foundation of community knowledge. For example, an elementary school in Arizona explored a wide range of mathematical, environmental and literacy topics by creating a class garden (Kahn & Civil, 2001). These active lessons are sometimes called *predisciplinary curriculum* (Applebee, Adler, & Flihan, 2007, p. 1005). The interdisciplinary activities in this design model could certainly suggest experiential or ethnographic activities. Students could educate peers on financial literacy through social media, record a financial life history of an elder in the family, or interview adults about their methods of reconciling monthly wages and payments.

Still, the project described here focuses on principles of organizing written texts for use by individual students. For this reason, it seems to be most applicable in an upper secondary setting as a bridge experience into university life. The modules could introduce students to a wider range of disciplines than they would encounter in secondary school. They would help students reflect on the theoretical foundation of disciplines, a goal of a recent curriculum reform in Danish schools which calls for the development of *multi-disciplinary* learning activities (in contrast to integrated, interdisciplinary ones) (Andresen & Lindenskov, 2009). Interdisciplinary activities can also support opportunities to write from a variety of perspectives, including research-based, persuasive, fictional, or experiential. Overall, this curriculum approach seems more appropriate for students making the transition to the writing-based academic experiences that are typical of university learning.

In some settings, secondary schools are working towards interdisciplinary learning in ways that could be enhanced by this curriculum. A case study of 11 interdisciplinary teaching teams serving over 500 students in middle and high schools found a variety of levels of curriculum integration, from simply juxtaposing subjects to fully reconstructing the curriculum (Applebee, Adler, & Flihan, 2007). The team that included a mathematics teacher represented relatively little integration. A team of 9th grade teachers developed substantial integration across English, social studies, and art through the theme of teaching respect and understanding for varied cultures and peoples. Team members reported a sense of being co-learners as their own interdisciplinary understanding emerged. They developed interdisciplinary exam questions together, but many student activities were developed "individually out of each teacher's sense of the issues shared by the team as a whole" (Applebee, Adler, & Flihan, 2007, p. 1027).

Portions of the proposed design model would capture some of this intensely interpersonal work so that it could be shared in new settings or with new team members. Artifacts of the team's work that would prove useful to others include learning goals for each discipline, student activities, and disciplinary materials (corresponding to items 4 – 6 in the previous section). The teaching team's reflections on the disciplinary ideas that informed student activity could form the introduction of the module. The focal essay, however, would not develop from the work of the teaching team in a natural way. A creative writer would have to construct a scenario that affords consideration of all the subjects.

The structure of the design model might make interdisciplinary learning more acceptable within the institutionalized curriculum frameworks which are prevalent in K-12 settings. Teaching across disciplines is at times controversial and politicized. In Boyle & Bragg's 2008 study of combining curriculum in the U.K., mathematics and history were taught as single subjects more often than other subjects, even during time periods of increasing cross-curricular teaching. The organization of the proposed curriculum, in particular, listing explicit learning goals for each discipline, may help empower teachers to justify connections to required curriculum.

The *Indebted* Module: Research in Practice

Staats has used this module in her college algebra classes over the last three years, revising components of it at the end of each semester based on the quality of students' work. This section briefly outlines some of the design choices made during this research-in-practice (Schoenfeld, 2009). Revisions were governed by four questions:

1. What format would make this module a usable homework supplement to a college algebra textbook for an instructor who does not work in an interdisciplinary program?
2. What types of interdisciplinary questions encourage students to synthesize ideas from algebra and psychology?
3. What kinds of interdisciplinary questions stimulate students to pose and answer algebra scenarios that are more complicated than textbook word problems?
4. What kinds of interdisciplinary questions do students seem to enjoy?

The financial mathematics section of the class textbook (Harshbarger & Yocco, 2010) was both engaging and frustrating for students. Many students find this topic relevant and interesting, particularly when problems allow them to understand the high accumulated interest costs of major purchases, such as houses and cars. However, many students also find it difficult to decide which equation — compound interest, present or future value of annuities, amortization — to use for different scenarios. Occasionally a student made a wry comment about homework problems that presented overly positive scenarios of parental savings for children's higher education. In the spring of 2009, Staats asked her colleague, Gary Peter, if he would write a short story that would help students develop a critical perspective on the financial mathematics section of the textbook. Gary Peter is a literature and sociology of law instructor who also writes fiction.

The first version of the module, then, was the short story itself. The story begins:

> "For all the important documents in your life," my grandfather said. "Your marriage license, your first mortgage…you know. A young man about to go out into the world should have a really nice pen for those things."

As the story progresses, the young man comes to understand his grandfather's deeply held polarity between hard work and personal debt. He learns of his own father's financial troubles. Trying to grasp the relative sizes of debts, the young man calculates his own share of the U.S. national debt. Near the end of the story, he receives an application for a student loan:

> *Neither a borrower nor a lender be.* Sorry, Grandpa. We all should have listened to you. But it was too late now.
> I picked up the pen, the pen that my grandfather gave me, and signed my name.

During the first semesters of the module's use, several students decided to model tuition increases at the University of Minnesota. In accordance with revision principle 4, we added a small bit of text to the story in response to this student interest:

> Maybe I should just work for a while before college. Save up my own nest egg, pay for tuition straight up. Less debt. But then I remembered the newspaper article last month: University tuition increases 3%. Would it be worse if I waited?

This revision to the story was intended to encourage students to consider modeling debt issues in matters related to their lives.

A mathematical dilemma that has always been present in implementations of this module is that the underlying mathematics of the formulas for this section of the class requires consistent payment and interest values. The algebra of geometric series limits the variety of scenarios that students can mathematize algebraically. Consequently, students most commonly pose a series of financial problems for various life stages. At the end of each life stage, the student reconciles the account and uses the output as the input for another problem. This can assist students' mathematical understanding because it deepens their understanding of how to choose a formula that applies in a particular situation, one of the typical difficulties. However, students typically do not extend the algebraic formula, for example, by deriving a new equation. In the following semester, we plan to use a spreadsheet exercise to encourage students in creating more flexible numerical models to accompany their short stories (revision principle 3).

In the original version of this module, its interdisciplinary character was not clear. Was it economics, sociology of aging, or literature? Establishing the disciplinary partner for this story explicitly was a necessary stage in making the module usable by other teachers (revision principle 1). In the spring of 2010, Staats asked a retired psychology professor (her mother, Sara Staats of Ohio State University!) to read the story and comment on any themes that are addressed in an introductory psychology course. Erik Erikson's model of lifelong identity struggles seemed to provide an academic means of analyzing the positions of the young man and of his grandfather. Staats and Staats developed the introduction to the module so that it became more formally an experiment in linking the disciplines of algebra and psychology. In this module, because the story was written first, all disciplinary content had to be placed in the introduction, but in the future, maintaining these as two separate components will likely prove useful, as well. The introduction provides a space for explaining framing ideas, and this allows the essay to present focused case studies and lively exposition. At this point, the learning goals and the scaffolding questions in algebra and in psychology were also added to the unit. Both the learning goals and the introduction are written to communicate to the instructor as much as to the student, to highlight direct disciplinary content as well as the "intentions and purposes" of the module to aid instructor implementation (Mason & Johnston-Wilder, 2006, p. 28).

Once the introduction and learning goals were added to the module, students could be held accountable for content in psychology. The introductory material made it easier to ask students to think about the emotional stories that lie unspoken behind the assumptions of an equation. Interdisciplinary homework questions were revised to create thinking around life stage dilemmas, personal relationships, financial behaviors, and modeling the results of these behaviors. The interdisciplinary assignments have changed or been refined with almost every semester as Staats learned more about the scholarship of interdisciplinary teaching and learning. For example, one of the original questions was:

> When David's grandfather entered the nursing home, the family didn't have time to develop a different savings plan. What if the grandfather's illness happened earlier and the family did have time to change their savings plan?
> Make up an example in which a family uses two different savings plans to save for college. Try to estimate how much they will be able to save. You can make whatever assumptions about income and interest rate that you want, as long as you state them clearly.

This question prompt was no longer adequate because it did not ask students to engage a psychological perspective. Each semester, there were always quite a few students who seemed to enjoy writing fictional stories, and so new interdisciplinary questions (revision principle 2) asked students to write stories that portray life stage dilemmas:

> Create a character representing a person moving through several of Erikson's life stages: adolescence, young adulthood, middle adulthood, and senior. Discuss the identity conflicts that this person experiences in each stage, and describe the character's financial behaviors, with a sample calculation, for each stage. Can you create a scenario using a single equation that produces a similar financial outcome for this character?

The last question in this prompt, "Can you create a scenario using a single equation that produces a similar financial outcome for this character?" proved to be problematic. The question asks students to reflect on their story and calculations and then to create a single equation that would create the same financial outcome as their series of posed equations. It was inspired by the scholarship of interdisciplinary teaching and learning that suggests that students should take a reflective stance towards their solutions, to recognize that alternative solutions are possible. The question also responded to revision principle 3, as an attempt to create a deeper modeling orientation in students' mathematical work. In practice, though, most students ignored this part of the question, a sign that they didn't understand it or that it was uninteresting to them. They wrote stories that illustrated lifelong identity questions supported by sample calculations, but only a few students produced a final "life equation." This question might be more successful if it were separated from the first one. It could be a follow up question for students to answer after they completed their stories.

Interdisciplinary question prompts also created a dilemma for students who wished to write creatively while at the same time demonstrating objective knowledge of Erikson's theory. Some students resolved this problem by incorporating phrases from Erikson's model, as when a student wrote in her story, "She was restless, and had a very difficult time connecting with people, struggling with intimacy versus isolation." Here, the student draws phrases directly from the essay introduction. Other students portrayed conflict or life choices less directly, through descriptions of the characters' relationships.

> That same lady is on a bus many years later…at the time she was not aware she was pregnant with her first child…people were waiting for her, her parents, brothers and sisters, friends and her best friend…who she knew she loved (but didn't always know it). Sitting in the sun she realized that she was happy to go home and meet up with everyone who was missing her.
> The woman saved money for her child once a year.
> Regular payment =1000, Interest rate = 6%, length of time = 18 years
> [Note: typed version of handwritten mathematical work]

$$1000\left[\frac{\left(1+\frac{0.06}{\cancel{12}\ 1}\right)^{\cancel{216}\ 18}-1}{\frac{0.06}{\cancel{12}\ 1}}\right] = \frac{\cancel{1.936765972}\ \ 1.854339153}{\cancel{.005}\ .06} = \cancel{\$387,353.19}\ 30,905.65$$

Characterizing an adequate interdisciplinary synthesis of both algebra and psychology is an open research question. In this selection, the student poses the character's dilemma subtly, and leaves it to the reader to acknowledge that this represents a resolution of Erikson's young adulthood conflict of balancing intimacy with isolation. The student initially wrote a problem representing monthly deposits of \$1000, and then changed the problem to coordinate with the annual payment schedule that she posed. She avoided a typical difficulty of understanding the relationships among variables in the formulas. While the student has arguably demonstrated disciplinary understanding in both algebra and psychology, the question of whether the response is an interdisciplinary synthesis is no doubt debatable. To some, the narrative itself, in which mathematical examples represent the outcome of a life stage crisis, are evidence of synthesis. Other readers may desire a tighter synthesis. For example, if the student had explained how the character's choice of annual payment was connected to her resolution of her relationship with her partner, it might have achieved a greater synthetic quality.

Conclusion

Textbook word problems on annuities and amortization use geometric series which assume consistent investment patterns — implied behaviours that are unrealistic for many people. Setting financial mathematics problems within a psychological theory of lifelong identity challenges allows students to pose and answer more realistic and humanistic mathematical questions, and it allows them to incorporate mathematics into a more personal vision of a human life-span.

Currently, we are developing additional interdisciplinary modules that align with standard college algebra textbooks, seeking reviewers from disciplines of psychology and mathematics for this module, and seeking grant funding to allow classroom testing of these modules. The long-term goal of this project is to establish a peer-reviewed, open-access website for interdisciplinary algebra curriculum that would help provide professional recognition for university level teacher-designers (Wittman, 1985). Such a website would also allow algebra teachers to incorporate low-cost interdisciplinary teaching into their classes.

The general, liberal education experiences of an undergraduate degree program set students on the pathway of becoming lifelong learners who can evaluate complex social and political questions. Students' instructors, having already completed this training, can be expected to model this perspective. The project is based on the assumption that a mathematics instructor can engage many of the topics of the first-year undergraduate curriculum, not as a disciplinary expert but as a well-educated, thoughtful adult. By asking this of instructors, we can better engage non-STEM majors, the largest university mathematical audience, and make their last formal mathematical training more deeply relevant.

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Assessments Accompanying Published Curriculum Materials: Issues for Curriculum Designers, Researchers, and Classroom Teachers

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Many classroom teachers use the assessment tasks that accompany their adopted curriculum materials, assuming that these assessments align with the curriculum. In this paper, we share results suggesting that these assessment tasks may limit students' opportunities to demonstrate mathematical understanding and we share a framework that curriculum designers, researchers, and teachers could use to inform future development and use of such resources.

Keywords: ancillary resources, mathematical assessment, conceptual understanding

Introduction

Curriculum materials, specifically textbooks, are an integral part of the mathematics classroom throughout the world. The nature of textbooks, including the types of lessons (e.g., inquiry vs. direct instruction) and emphasis of problem sets for student practice (e.g., focus on procedural skills or conceptual understanding), influences the opportunities students have to learn mathematics. In addition, the assessment tasks that accompany those materials potentially provide a variety of insights to teachers about their students' understanding of mathematics (e.g., ability to complete skills or engage with mathematical processes). According to the National Council of Teachers of Mathematics (NCTM) in the United States, assessment that is aligned with the current vision of school mathematics should provide students "with opportunities to formulate problems, reason mathematically, make connections among mathematical ideas, and communicate about mathematics" (1995, p. 11). In particular, assessment should be "an integral part of instruction that encourages and supports further learning" (NCTM, 1995, p. 13).

As noted by Valverde et al. (2002), textbooks translate the intended curriculum of national standards into the potentially implemented curriculum of the classroom. Thus, textbooks provide evidence for teachers of the recommended focus of instruction. In the United States, adopted curriculum materials are typically accompanied by a suite of ancillary materials, including at the minimum a teacher's edition with suggestions on instruction and assignments, supplementary practice activities, and assessment resources (i.e., unit tests). These ancillary materials are often used extensively by classroom teachers. In fact, some researchers (e.g., Delandshere & Jones, 1999; Hunsader, Thompson, & Zorin, 2012b; Senk, Beckmann, & Thompson, 1997) have documented that many classroom teachers use the unit tests accompanying their curricula, often with only minimal adjustments and modifications.

Assessments that occur as summative measures at the end of a unit of learning "should promote valid inferences about [students'] mathematics learning." Given that "an inference about learning is a conclusion about a student's cognitive processes that cannot be observed directly" (NCTM, 1995, p. 19), those assessments need to provide opportunities for students to demonstrate the depth of their mathematical understanding and knowledge. So, an issue for focus in this study conference is whether curricular assessments have been designed in ways that support the instructional goals of the curriculum, and by extension national/state standards. That is, if it is important during instruction for students to engage with mathematical processes (e.g., reasoning and proof, communication, connections, representation), then assessments need to provide opportunities for students to demonstrate performance with those processes. Because tests are often developed at the end of the design cycle, they are not necessarily given the same attention by curriculum designers or researchers as other aspects of the curriculum. Consequently, the nature of such tests may undermine the instructional goals of the curriculum as teachers use them for gauging students' mathematical achievement, and possibly make incomplete inferences about the nature of their students' understanding of the mathematics being studied.

Our perspective on tasks

In this paper, we use the term "task" to refer to the entirety of a unit test designed to assess students' mastery of the concepts, processes, and procedures of the mathematics under study. Clearly, an individual item provides insight into a particular concept, yet no one item can or should be expected to assess knowledge across all aspects of a concept. By focusing on the *set* of items comprising a unit test, we are able to consider what might be lacking from that set of items, an analysis that would not be appropriate at the level of an individual item. Reviewing comparable analyses across a set of unit tests for a particular curriculum then provides insight into the extent to which the assessment tasks for that curriculum focus on important aspects of mathematical understanding, including both content and mathematical processes.

We have not designed the tasks (that is, the unit tests) that are the focus of this paper. Rather, we have developed a means to analyze the tasks that accompany published curricula and have thought about how that analysis can potentially be used with teachers, researchers, and curriculum designers to improve them. Our views and the subsequent issues we discuss are based on our detailed analysis of tests accompanying the elementary grades 3, 4, and 5 (ages 8-10) curriculum from three different publishers in the United States.

Integrating processes into curriculum and instruction

Skemp (1971) provided a means of classifying instructional tasks as focused on instrumental learning (i.e., procedural learning) or relational learning (i.e., conceptual understanding). With the release of the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) in the

United States, curriculum materials began to be designed so that both of these types of learning were explicitly addressed. In addition, the *Standards* identified not only content foci (e.g., number and operations, algebraic thinking), but mathematical processes that needed to be part of both curriculum and instruction, specifically representation, connections, communication, reasoning and proof, and problem solving. These processes are essential to students' demonstration of their conceptual understanding of mathematics. As students explain their thinking, teachers are able to gain greater insight into their level of mathematical understanding than is possible solely from the manipulation of procedural skills.

It is relatively straightforward to determine if specific content recommendations are implemented into the textbooks or ancillary materials, such as tests. Designers, researchers, and teachers can do a simple checklist to indicate whether the content is present because evidence of content is explicit. However, identifying whether the mathematical processes are part of the textbook or the tests is more difficult because, by their nature, evidence of the mathematical processes tends to be implicit. Yet, it is through the mathematical processes that students are able to demonstrate their relational understanding. Without opportunities on assessment tasks to demonstrate understanding through the mathematical processes, it is often difficult for teachers to see beyond a surface level what students know about mathematics.

Just because the curriculum and the instruction based on it provide opportunities for students to engage with the processes, one cannot assume that the task used to assess mastery of that curriculum (i.e., the unit test) also includes opportunities to engage with the processes. Indeed, in the United States the development of ancillary materials is often outsourced to development firms. So, consumers of the materials, the teachers, may assume that the task provides the same opportunities for engagement with the processes as is true with the materials themselves, but this is not necessarily the case. Teachers may also assume that the task will provide them with adequate evidence about their students' conceptual understanding of the content to inform future instruction.

In a survey of 43 teachers, mathematics coaches, and mathematics supervisors in our own state, 81% used the assessments that accompanied their curriculum and 68% of these reported using them because they ensured alignment with the curriculum (Hunsader et al., 2012). Furthermore, when asked to indicate the extent, on a scale of 1 (low) to 5 (high), to which they perceived the tests engaged students with the mathematical processes, only problem solving (mean = 3.4) and representation (mean = 3.2) had a rating over 3. But both the ratings and the rationale for using the assessment tasks accompanying the curriculum are based on perceptions. What evidence can or should exist that these perceptions are on target? What evidence would inform curriculum designers or researchers about future design or research about these tasks? What evidence could be used to persuade teachers to take a critical and objective look at their assessments rather than assuming that these tasks give them the needed insight into students' understanding to guide future instruction? These are the questions we explore in the remainder of this paper.

A framework to guide the analysis and design of mathematical processes on assessment tasks

As a means to analyze assessment tasks in an objective manner, we developed the *Mathematical Processes Assessment Coding Framework* (MPAC Framework) in Figure 1 to determine the extent to which the mathematical processes were embodied in assessment tasks accompanying published curricula (Hunsader, Thompson, & Zorin, in preparation).

Reasoning and Proof

N	The item does not direct students to provide or show a justification or argument for <u>why they gave that response</u>.
Y	The item directs students to provide or show a justification or argument for <u>why they gave that response</u> ('Check your work' by itself is not justification).

Opportunity for Mathematical Communication

N	The item does not direct students to communicate <u>what they are thinking</u> through symbols (beyond a numeral answer), graphics/pictures, or words.
Y	The item directs students to communicate <u>what they are thinking</u> through symbols, graphics/pictures, or words.
V	The item <u>only</u> directs students to record a vocabulary term or interpret/create a representation of vocabulary.

Connections

N	The item is not set in a real-world context and does not explicitly interconnect two or more mathematical concepts (e.g., multiplication and repeated addition, perimeter and area).
R	The item is set in a real-world context outside of mathematics.
I	The item is <u>not</u> set in a real-world context, but explicitly interconnects two or more mathematical concepts (e.g., multiplication and repeated addition, perimeter and area).

Representation: Role of Graphics

N	No graphic (graph, picture, or table) is given or needed.
S	A graphic is given but no interpretation is needed for the response, and the graphic does not explicitly illustrate the mathematics inherent in the problem. (superfluous)
R	A graphic is given and no interpretation is needed for the response, but the graphic explicitly illustrates the mathematics inherent in the problem.
I	A graphic is given and must be interpreted to answer the question.
M	The item directs students to make a graphic or add to an existing graphic.

Representation: Translation of Representational Forms

N	Students are not expected to record a translation between different representational forms of the problem.
SW	Students are expected to record a translation from a verbal representation to a symbolic representation of the problem or vice versa.
GS	Students are expected to record a translation from a symbolic representation to a graphical (graphs, tables, or pictures) representation of the problem or vice versa.
WG	Students are expected to record a translation from a verbal representation to a graphical representation of the problem or vice versa.
TG	Students are expected to record a translation from one graphical representation of the problem to another graphical representation.
A	Students are expected to record two or more translations among symbolic, verbal, and graphical representations of the problem.

Figure 1. Mathematical Processes Assessment Coding Framework.
Source: Hunsader, Thompson & Zorin (in preparation).

The framework focuses on four of the five processes from the *Principles and Standards for School Mathematics* (NCTM, 2000); for representation, we generated two criteria to reflect different aspects of this process standard – the role of graphics and translation between and among representational forms. We were interested in analyzing the assessed curriculum as reflected in written tests, without the need to analyze the intended written curriculum of the textbook or the enacted curriculum of classroom instruction. Hence, the framework does not address the problem-solving standard because whether an item on an assessment requires problem solving or is a routine exercise depends on prior experiences students have had, which would require analyzing the textbook and instruction.

We have applied the framework to analyze over 100 complete unit tests and over 2000 individual items across those tests. Table 1 summarizes the extent to which these assessment tasks (i.e., tests) provide opportunities for students to engage with the mathematical processes.

Process Standard	Minimum %	Maximum %	Median %
Reasoning & Proof	0	23	0
Communication (yes)	0	85	13
Communication (vocabulary)	0	73	0
Connections (R or I)	0	100	44
Representation (role of graphics: M or I)	0	100	22
Representation (translation of forms)	0	67	9

Table 1. Range and Median Percent of Items Per Assessment Task Addressing the Mathematical Processes.
Note: The unit of analysis is an entire test.

With the advent of the *Standards* movement in the United States, most curriculum materials purport to integrate those standards, both content and process. However, the results in Table 1 suggest that there is a wide range in the extent to which the process standards are integrated into the assessment tasks.

How might designers and researchers use this coding framework and related analysis?

The potential value of this framework is that it provides an objective look at assessment tasks – either to confirm perceptions or suggest where changes are needed. Consider the item in Figure 2 cloned from an item on one grade 3 assessment task.

Five friends have 20 pieces of candy to share equally. How many pieces of candy will each friend get?

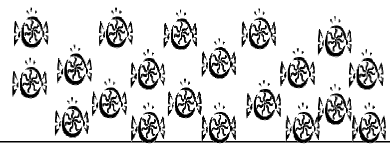


Figure 2. Clone of an Item from a Grade 3 Assessment.

We would classify this item as providing no opportunity for reasoning or communication because students are simply asked to provide a numerical answer but not to explain <u>what</u> they were thinking or <u>why</u> their answer is correct. For the role of graphic, we would classify the item as S because no interpretation of the graphic is needed to answer the item and the graphic does not explicitly illustrate the mathematics inherent in the problem. Note that the student does not need to consider the graphic to solve the problem nor could the student make sense of the problem simply by interpreting the graphic.

A closer analysis of the item raises some interesting issues. Although the graphic is not needed to answer the item, it does not appear to be random. The designers did not use a single candy, or 5 or 10 candies. They used the exact number needed in the problem, and yet, they never explicitly asked students to use the graphic. Is the graphic there to support struggling students or those who are English language learners? Is the graphic there because the curriculum encourages students to use graphics to help solve problems? Is it there simply to liven up the page? Is the graphic possibly distracting to students who do not need it?

The issues raised by this one item provide potential research opportunities. One might ask designers why they chose to include this particular graphic but not to ask students to use it in any way. Researchers or teachers might observe students as they answer the item to determine how many use the graphic without leaving any written record of its use as well as how many actually show their solution on the graphic. Researchers might ask teachers how they perceive their students will attend to the graphic. Another question might be whether such related, but not needed, graphics are a regular part of the curriculum and whether this type of use encourages students to ignore graphics in their mathematics textbook. What might be the unintended consequences later in students' mathematical careers if they become accustomed to ignoring the graphics in their curriculum?

Figure 3 adapts the item from Figure 2 in three different ways. How do these adaptations engage students with the process standards and what further insights into students' thinking do they provide teachers?

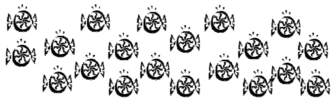
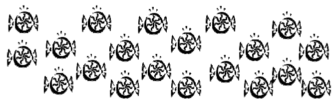
Adaptation 1. Five friends have 20 pieces of candy to share equally. How many pieces of candy will each friend get? Write a number sentence to show how many pieces of candy each friend will get.	
Adaptation 2. Five friends have 20 pieces of candy to share equally. How many pieces of candy will each friend get? Write a number sentence to show how many pieces of candy each friend will get. Use the picture to explain your thinking about the problem.	
Adaptation 3. Five friends have 20 pieces of candy to share equally. Draw a picture to show how many pieces of candy each friend will get. Write a number sentence to represent your picture.	

Figure 3. Adaptations of the Item from Figure 2.
Note: Adaptation 3 from Hunsader, Thompson, & Zorin (2012b).

Adaptation 1 does not resolve the issues about the use of the graphic raised in the discussion of Figure 2. But having students write a number sentence does provide insight into what they were thinking. So, when students write a number sentence to translate the problem from words to symbols as part of the solution, they communicate *what* they were thinking and give teachers more insight about their understanding than simply providing the answer 4.

Adaptation 2 explicitly asks students to use the graphic to explain their answer, thus requiring them to interpret a graphic in the sense that the student must illustrate a solution on the graphic. Hence, this adaptation engages students in both communication and reasoning by having them show *what* they were thinking (e.g., the number sentence) and *why* they gave that response (e.g., the solution on the graphic).

Adaptation 3 also engages students in both communication and reasoning, while also requiring them to make a graphic to illustrate a solution. Thus, Adaptation 3 extends the requirements of Adaptation 2 so students independently demonstrate their ability to connect verbal, symbolic, and visual representations of a problem.

All three adaptations have the potential to provide teachers insight into students' conceptual understanding as well as the procedural understanding expected from the original item. To some extent, there is also a sequence to these items so that researchers could investigate whether achievement varies for different versions of the item and different explicit references to use of the graphic.

If the original item (Figure 2) were the only such item in a set of items comprising an assessment task, one might overlook the apparent haphazard use of graphics or the lack of explicit engagement with the process standards. After all, no one expects every item on an assessment task to integrate one or more of the process standards. However, if such items are the norm on a given assessment or are the norm across assessments, as suggested by the results in Table 1, then consumers of those tasks might rightly question whether the task is providing the insight into students' mathematical thinking and understanding that is expected from an assessment. Thus, the framework and analysis can help curriculum designers ensure that assessments maintain the same focus on processes as they design into their curriculum. Researchers can use the framework and analysis to understand how variations of items might enhance student achievement.

How might teachers and teacher educators use this coding framework and related analysis?

Classroom teachers might be encouraged to use such a framework and the analysis it provides to better understand the nature of the assessment tasks they use with students. As indicated previously, teachers often use the assessment tasks that accompany their curriculum because they assume such assessments are well aligned to the textbook. Thus, having them reflect on the nature of these assessments can be a powerful influence on their future practice.

For instance, after we have worked with practicing or prospective teachers to help them recognize the need for more mathematical discourse in the classroom and for use of the mathematical processes, we have had them analyze several items or tests associated with their curriculum. The results typically lead to a stark reality – there is often little or no integration of the process standards. Although the assessments clearly relate to the content of the related unit, opportunities for reasoning, communication, representation, or connections may be limited. Teachers then consider how they might adapt some of the items comprising the overall task so that opportunities to engage with the processes are more evident. The potential impact of such work with assessment tasks is reflected in the following comments from teachers:

- "The [MPAC] framework reminded me to consider areas of the problem that I sometimes overlook."
- "…actually doing the test item analyses really made me think outside of the box in how I would teach students about math and assess them."
- "[The experience] will greatly impact my formulation of chapter tests in mathematics when I am a teacher."

Conclusion and relation to the study conference

Obviously, the design of assessment tasks to accompany a curriculum can only occur after the curriculum has been designed. At that point, there may be fewer financial resources available for development, or assessments may not have been the intention of the designers but are required for commercial publication. Our analysis of assessment tasks from three different elementary publishers found some variation across publishers, content domains, and grade levels, but the overall pattern was consistent – connections (i.e., real-world problems) were moderately integrated into assessment tasks, but other process standards were less so. Subsequent analysis of tests associated with U. S. middle grades curricula has found a similar pattern.

The issues raised in this paper suggest that more attention needs to be paid to the assessments associated with curriculum, particularly if those assessments are used to judge the effectiveness and quality of a curriculum and to make inferences about the student learning that such a curriculum facilitates. If the curriculum and the assessments are not aligned in terms of the process standards, then accurate information about learning is not obtained. Consequently, teachers may have an incomplete picture of their students' mathematical knowledge as they design future instructional tasks. They may assume students have a depth of knowledge about mathematical concepts that is not supported by the type of performance elicited on the assessment tasks, and thus fail to develop appropriate instruction to extend or enhance mathematical knowledge.

One issue for the study conference to address is how research such as that described here can or should inform future curriculum designers and the next round of curriculum revision. A second issue is how to ensure that all aspects of the curriculum cycle – textbook or material development, instructional implementation, and assessment tasks – are aligned and receive equal attention in the design process.

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Different features of task design associated with goals and pedagogies in Chinese and Portuguese textbooks: The case of addition and subtraction

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This paper presents different features of task design for addition and subtraction in Chinese and Portuguese textbooks and their different goals and pedagogies. Through comparison, we address the question, "what is the effect of different cultural backgrounds on tasks and task design for addition and subtraction", and propose some conjectures that arise from answers to this question.

Keywords: variation theory, example and exemplification, addition and subtraction, Chinese mathematics education, textbook design, Portuguese mathematics education

Introduction

China has a clear textbook-centred tradition. Textbooks play multiple functions in the Chinese mathematical education system, such as tools for teachers` professional development by studying textbooks (e.g. Ma, 1999), self-learning instruments for out-of-school learners, and the main medium for teaching and learning in the classroom. In these ways, textbooks play a central role in shaping students' learning and teachers' teaching through their central role in the interactions of local groups of teachers and academic organizations. Although Chinese textbooks play a critical role in schooling, it is worth noting that Chinese textbooks are designed by local experts based on their experience and local curriculum standards. Principles of task design are generally not explicitly articulated. However, Marton (2008) has identified a specific implicit underlying principle of task design in Chinese culture as expressed in the following words:

> There is a very powerful pedagogical tradition in the Chinese culture. The careful molding of 'the space of learning', the blending of necessary different components of learning and of different ways of supporting it, might occasionally run counter to the ever shifting pedagogical fashion. But what is proven to work well in practice and what is theoretically not only defensible, but even positively implied, should not be replaced, but developed further. …Chinese students do very well when compared to students from other cultures. Teachers spend much more time on planning and reflecting than teachers in other countries, and they develop their professional capabilities by the teaching, in which patterns of

> variation and invariance, necessary for learning (discerning) certain things, usually brought about by juxtaposing problems and examples, illustrations that have certain things in common, while resembling each other in other respects. By such careful composition, the learner's attention is drawn to certain critical features and each problem and example make a unique contribution to the things that the learner will hopefully pay attention to in the future, instead of just going through problems that are supposed to be examples of the same method of solution. (p. 1)

These principles were expressed and illustrated by Liu (2004). More recently, Sun (2007, 2011) indicated that two kinds of problem variation (变式题) could describe these features of example design in China: OPMS (One Problem Multiple Solution methods) and OPMC (One Problem Multiple Changes), both of which are local routine teaching terms,. Liu's examples were concerned with absolute value, and Sun's with division of fractions. In the present paper, we compare textbooks from two countries using the question "what varies?", and ask whether this is helpful to identify differences in the space of learning.

According to variation theory (Marton & Booth, 1997), which emphasizes "simultaneity," the "one-thing-at-a-time" design might miss the chance to discern critical aspects among two or more topics. More importantly, the contemporaneous variation approach, to a greater extent, emphasizes "general relationships" more than others. Thus, problem variations aim to discern, compare the invariant feature of the relationship among concepts and solution methods, and provide opportunities for making connections; comparison is considered the pre-condition to perceive the structures, dependencies, and relationships that may lead to mathematical abstraction. The emphasis on "general relationships" reflects the soul of variation theory, emanating from the work of Marton, who argues that "learning will take place through discernment of variation in simultaneous events"; "variation is a necessary condition for effective discernment" (Bowden & Marton, 1998, p. 35).

As mentioned before, Chinese teachers are required to study textbooks intensively (研究教材) and this is regarded as an important method for professional development. Textbooks, together with their teachers' guidebooks and a professional research group (Yang, 2009), are the predominant force in a teacher`s professional life. This professional development environment is rarely known outside of the Chinese community, so to study the detail of design in textbooks is a strong proxy for studying the way content is shaped through professional development. In this study, we explore how example design, and the associated curriculum standards and reference books, might influence teachers' teaching, and hence, the space of learning... For comparison, a Portuguese textbook and teaching guide are used as a "mirror". The research questions of this study are:

What are the different features of task design in the topics of addition and subtraction within Chinese / Portuguese textbooks?

How might features of task design in textbooks influence the space of learning in Chinese / Portuguese classrooms?

We focus on a Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005) that has been used for over 30 years by the majority of Chinese students from diverse backgrounds. It represents the Chinese national curriculum and is seen, with its teaching guide, as an authoritative guide on what to teach/learn. However, all national examination problems are required to be "from textbooks, but above textbooks"; hence, the material is supposed to extend knowledge beyond the procedures that might be carried out in the textbook. The Portuguese textbook (Gregório, Valente, & Calafate, 2010), with supporting teacher guide (similar to the Chinese teacher guide in that it informs teachers about appropriate goals and pedagogies), reflects typical practices of teaching and learning these topics at this level in Portugal.

Different features of task design for addition and subtraction within Chinese /Portuguese textbooks

The content structure of Chinese textbooks is fixed. The organization is consistent without repetition:

- knowing numbers 1-5 as a foundation, then knowing addition and subtraction from 1-5;
- knowing numbers 6-10, then addition and subtraction from 6-10;
- knowing numbers 11-20, then addition and subtraction from 11-20;
- knowing numbers within 100 (1000,10000), then addition and subtraction algorithms within 100(1000,10000), step by step.

Invariant concept vs. variant concepts embedded in Chinese / Portuguese textbook examples

Addition and subtraction are almost always elicited together in examples of OPMC in the Chinese textbook[34], rather than separated in different chapters as they are in the Portuguese textbook. The two concepts of addition and subtraction are always connected by problems with variation. Here are two typical "prototype" examples of OPMC in the Chinese textbook.

[34] In Chinese, OPMC references "changes"; in English, variation theory would likely use the word "transformations" to mean the same thing.

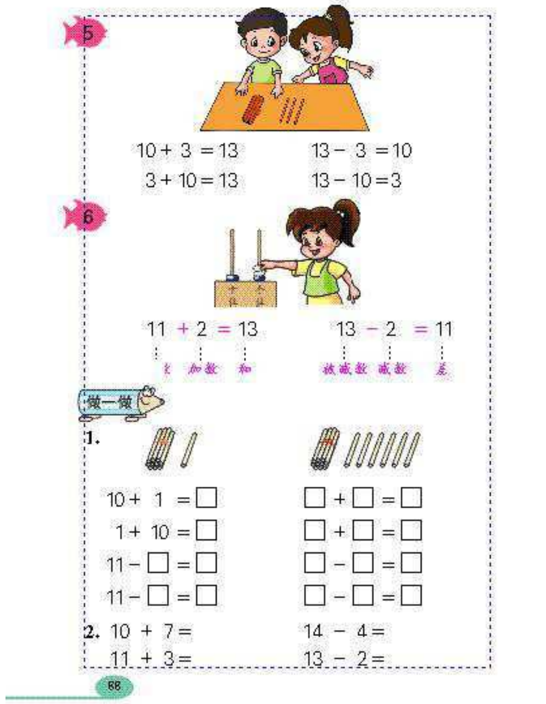

Fig.1 from Mathematics Textbook Developer Group for Elementary School, 2005, 68

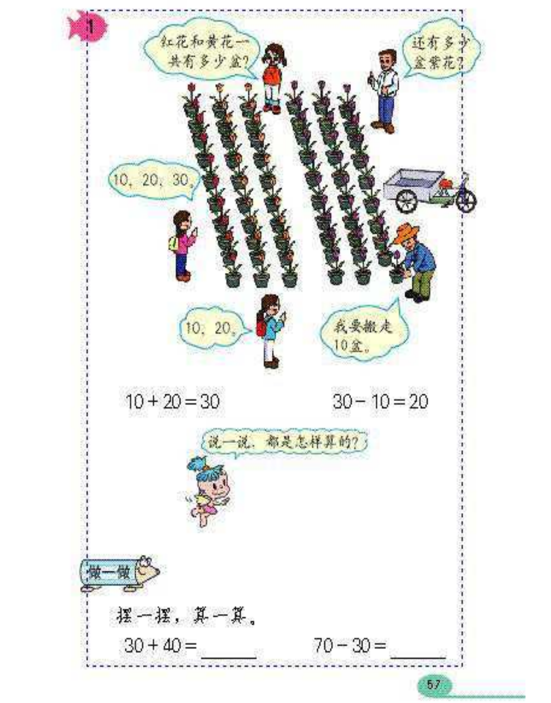

Fig.2 from Mathematics Textbook Developer Group for Elementary School, 2005, 57

It is interesting to note that the Chinese textbook authors did not separate the subtraction concept from the addition concept, even in the first lesson (Sun, 2011). Figure 1 shows a paradigmatic example of problem variation: 10+3=13, 13-3=10, 3+10=13; 13-10=3.The problem set intends to help learners recapitulate the relationship between addition and subtraction, and the meaning of “equal”. The textbook design offers visual models enabling learners to understand the underlying additive relationship.

Every example in the Portuguese textbook introduced the concept of addition and subtraction without any connection, and without a visual model to situate the problem. The pages below are from two separate chapters. Addition and subtraction concepts are introduced as counting in the following examples. While the number line model provides a method of calculating the answers, it is unclear whether it enables learners to use the underlying meaning to connect addition and subtraction.

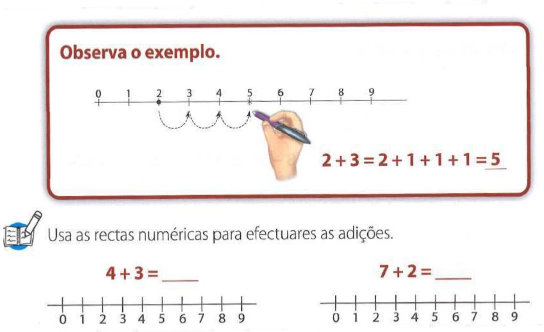

Fig. 3 The example introducing addition concept by counting in one of the Portuguese textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, 2010, p. 75)

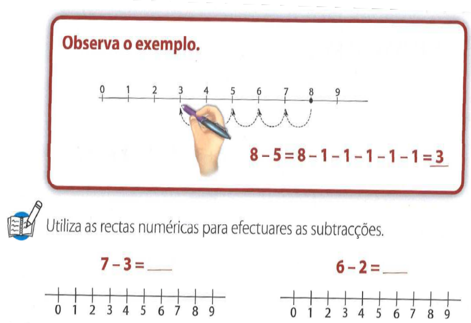

Fig. 4 The example introducing subtraction concept identifying inverse operation in one of the Portuguese textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, 2010, p. 76)

Although the Chinese textbook authors appear to use multiple concepts in every example, the underlying invariant concept is of *part-part-whole* relations and the tool being used is knowledge of numbers. =In contrast, the addition examples in the Portuguese textbook use multiple underlying physical concepts, such as "counting" (as in Figures 3 and 4), "combining", and "adding". The subtraction examples in also use multiple concepts such as, "taking away", "comparing", and "identifying the inverse operation of addition" but they do not relate these to the addition concepts in any direct way. In the Chinese textbooks, the meaning of the additive relation is invariant, but the way it is represented and enacted varies. In contrast, in the Portuguese textbook the meanings of the addition and subtraction concept appear to vary, and they are not connected except by the notion of 'inverse'. Thus, the space of learning (i.e., what is available to be learnt) is different in the two countries.

Single solution vs. multiple solution methods embedded in every Chinese / Portuguese textbook example

In the Chinese textbook, multiple solution methods are almost always elicited together, illustrating the OPMS principle, rather than a *single solution* method as in the Portuguese textbook.

	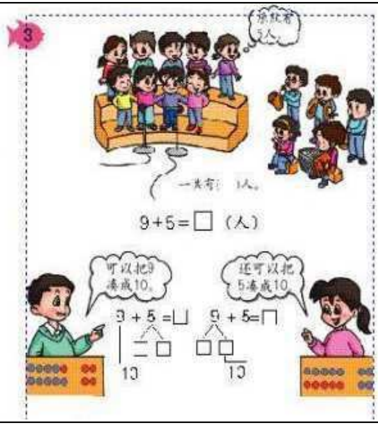
Fig. 5. Three solution methods for variations of 4+1=5 (Mathematics Textbook Deveiopers Group for Elementary School, 2005,1(16))	Fig. 6. Three solution methods for variations of 9+5=14 (Mathematics Textbook Developers Group for Elementary School, 2005, 1(30))

Figure 5 is a typical “prototype” example of OPMS in the Chinese textbook. In the problem variation, 4+1=5 is designed to introduce naturally a solution system of addition. Within the problem set in the example, there are three solution methods given. The first one is that of addition by counting from 1 to 5; the second solution is that of counting from the addend 4 to 5; the third is that of addition by regrouping 5 with 4 and 1. Figure 6 shows the problem variation of OPMS, 9+5=14. The first solution method is that of addition by regrouping with 5 and 5; the second one is regrouping with 9 and 1, both of which highlight the “make-10 method” and the decimal system.

Compared with the Chinese multiple-solution method approach, every example in the Portuguese textbook is intended to be carried out using a single solution method without necessarily connecting that method to other approaches in the book. For example, as shown in Figure 7 a typical design to introduce addition is through *a specific single solution method of “doubles” or “doubles plus 1”*. The concept on each of these two pages does not vary; however, a learner might not connect this method to other models of addition, nor to subtraction, nor be able to choose when to use these methods without juxtaposing the examples, as Marton suggests (2008), with some that are not solvable using ‘doubles’ or ‘doubles plus 1’. The patterns of variation and invariance from which the learner might discern the underlying conceptual relations are unclear.

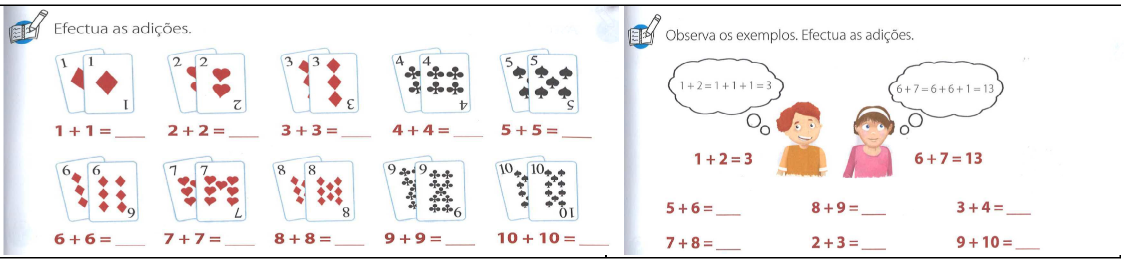

Fig. 7 The example elicits addition by the solution method of “doubles” or “doubles plus 1” in one of the Portuguese textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, 2010, p. 99)

Invariant solution method vs. variant solution methods embedded in Chinese / Portuguese textbook examples

Although Chinese textbook authors use multiple solution methods in every example, the particular methods in the Portuguese textbook, which depend on counting and doubling, are rarely introduced. Only one specific solution method, “make-10”,” is addressed explicitly among all the addition /subtraction examples within the first 6 chapters. In contrast, the addition examples in the Portuguese textbook suggest multiple solution methods, such as “*doubles*”, “*doubles plus 1*”, “*compensation*” (6+8=7+7=14), and “*reference number*” (6+7=5+1+5+2=10+3=13). The subtraction examples use multiple solution methods, such as “*counting back”*, “*the use of tables for the addition to subtraction*”, and “*identifying the inverse operation of subtraction as addition*”.

Different goals and pedagogies in teaching reference books

Different cultural traditions have different educational goals and traditional pedagogies. The Chinese curriculum goal is “two basics”, i.e. knowledge and skills; these are the central aspect of the unified teaching outline of the Ministry of Education (1963, 1991, 2001).The basics include important mathematical facts and experiences, mathematical thinking, and skills for further development. In this case, the basic knowledge is the “part-part-whole” or additive relation as a foundation for algebra, and the “make-10” foundation for place value and the decimal system. The textbook emphasises these as basic knowledge and skills and the examples illustrate how this is done.

In general, to realise the “two basics”, the Chinese develop specific pedagogies, starting from analyzing the focal points of the curriculum content (重点), then identifying the difficulties that students might encounter (难点) and the critical points for learning (▪键点) (as derived from students’ difficulties in learning), and finally building in relevant variations in the teaching that focus on these three points. Because this theory is at too general a level for teachers to enact in every lesson, the concrete goals and pedagogies for every lesson are clearly presented in the teachers’ guide. For example, Figure 8 shows the focal points for knowing numbers 6-10 (EMD, 2005, p. 67).

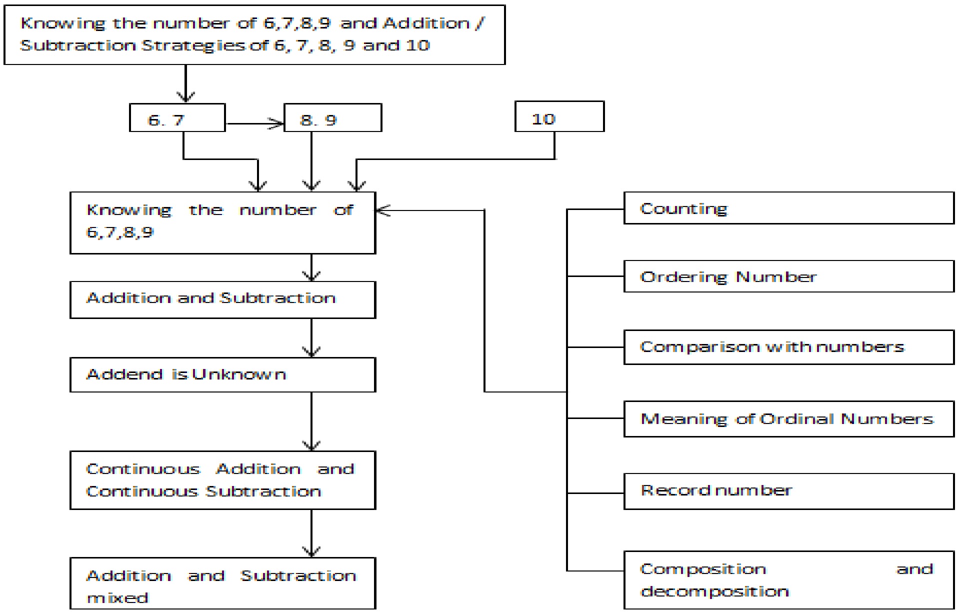

Fig. 8 The concept struture of knowing numbers 6-10 (EMD, 2005, p. 34)

The Chinese example design principle

Based on the Chinese examples and analysis above and knowledge of the teacher guide, we could infer the underlying design principle of 'two basics' by noticing: the foundation of knowing numbers; the connection between addition and subtraction; and the invariant concept/ solution: part-part-whole /make-10 solution.

Portuguese curriculum goals and pedagogies

The Portuguese textbook and supportive guide does not reflect such a coherent knowledge structure (Gregório, Valente, & Calafate, 2010). The initial work about addition and subtraction =focuses =on addition understanding it as combining, and performing it using counting, doubling, compensation, or using and adapting known facts, with subtraction also having several meanings. The underlying design principle is not made explicit; we could infer that it is more fragmented, more focused on learning one method at a time, and less dependent on foundational principles for future work.

Summary

The task design discussed in this paper shows a pedagogical phenomenon in organizing a curriculum with an emphasis on discerning relationships through the variation approach. Many readers may argue that the variation approach may be confusing and that a sequential organization with time gaps ("one-thing-at the-time") should be preferred. In fact, the variation approach might come from different kinds of pedagogical traditions and philosophies developed for centuries. The issue of variations in problem sets directly reflects the old Chinese proverb, "no clarification, no comparison" (沒有比較就沒有鑒別), rather than "to consolidate one topic, or skill, before moving on to another," a notion broadly used in most textbook development (Rowland, 2008) in Europe and throughout the world. In contrast, the "one-thing-at-a-time" design would clearly provide fewer opportunities for making connections compared to those of contemporaneous variation approaches. The "one-thing-at-a-time" design might possibly reflect the hidden belief that learners would naturally make the relevant connections. In this context, the role of the curriculum in presenting connections might be taken for granted and remain implicit.

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Theme D
Principles and Frameworks for task design within and across communities

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Considerations and principles for designing and sequencing tasks depend highly upon the context of the design activities. Various design communities, such as those consisting of researchers, teachers, professional developers and teacher trainers, or textbook writers, have different aims and agendas for task design. Thus, principles for task design vary across the context in which the communal practice is situated. In addition, principles for task design can vary as to whether they are applied to the initial creation of tasks or to the shaping and modification of existing tasks (Remillard, 2005), as well as to whether they are applied to the design of a single task versus a sequence of tasks. Moreover, tasks can be designed not only by members of a singular community but also by groups whose members cut across two or more design communities (see, e.g., the international examples of such efforts (Kieran, Krainer, & Shaughnessay, 2013). For example, recent projects where teachers are regarded as key stakeholders in research (i.e., as (co)producers of professional and/or scientific knowledge) and where they have a significant role to play in the design of tasks have been shown to yield not only rich task designs for mathematical learning, but also make the link between research and practice more fruitful for both sides. In this working group, we address the diversity and the interactions between design principles and communities that are involved in task design and attempt to make explicit those principles of task design within and across design communities that have up to now been largely tacit.

This Working Group has a twofold aim:

- To solicit papers that delineate principles and frameworks for task design within singular design communities so as to illuminate differences and commonalities across the specific contexts of the various communities.
- To solicit papers that delineate principles and frameworks for task design by teams that cut across the various diverse communities so as to illuminate the nature of, and thereby aid in encouraging the further emergence of, such interactive, cross-community approaches to task design.

Papers being submitted to this working group should specify which of the two above aims is the main focus of the paper. Papers being proposed for this group should also address and develop some subset of the following questions, in addition to whatever other issues might be considered relevant to the given theme:

- If you identify yourself as a member of a singular design community, which one is it? Or if you identify yourself as a member of a design group that cuts across communities, which ones are they? If the latter, how did this cross-community come to be formed?
- When you or your group engages in designing tasks, what are you trying to achieve? What are your primary considerations?
- Do the principles applied to task design depend on the nature of the mathematical activity inherent in the tasks (i.e., tasks for exploration, concept development, practicing, generalizing and reflection)? If so, in which ways?
- In which ways do the principles for task design interact with the issue of the time factor, that is, whether a task sequence is to occur across several lessons or within one given lesson?
- Which theoretical, mathematical, pedagogical, technological, cultural, and/or practical aspects are taken into account when designing a task or a task sequence? Which aspects are considered primary?
- Is there a particular framework or theory of learning that is drawn upon in designing a task or task sequence, and how is this framework reflected in the task design?
- What is the extent to which individual/communal value systems and beliefs about how mathematics is to be learned enter into the designing of tasks?
- What is the extent to which the inclusion of digital-technology tools within a task or task sequence is reflected in the principles employed in designing the task or task sequence?
- Are the designed tasks subject to revision in later cycles of the work? If so, what is it that specifically leads to the redesign? On what basis and according to which principles is the redesign carried out?
- What constitutes the main differences and commonalities between design principles for different design communities?
- What constitutes the main differences and commonalities between design principles for different age groups and school levels?

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Task design within the Anthropological Theory of the Didactics: Study and Research Courses for pre-school.

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Abstract. Task design has become a central issue within the Anthropological Theory of Didactics, connected to the necessity of a renewed paradigm in the teaching and learning of Mathematics. It is the aim of this paper to situate "task design" within the Anthropological Theory of Didactics, to introduce how the term "task" is conceptualized (*study and research courses and activities*), to outline key design principles and, finally, to exemplify the design process in the case of a task for pre-school.

Keywords. Anthropological Theory of Didactics, Study and Research Course, Study and research Activities, Pre-school.

Introduction

The Anthropological Theory of the Didactic (ATD, from now on) has been developed over the last 30 years, giving rise to a rich and complex theoretical model. In its evolution, initial constructs have been developed and enriched. The emergence of new problems and the enlargement of initial questioning have led to new developments of the theory, in a continuing process that is still going on.

The origins of the ATD can be found in the reconceptualization of mathematics as a human activity (Chevallard, 1999). This fundamental hypothesis gave rise to the notion of praxeology, which captured the essential components of any (mathematical) activity: *tasks*, *techniques*, *technologies* and *theories*. "Tasks" and "techniques" considered as the "know-how", whilst "technologies" and "theories" interpreted as the "knowledge" that explains and justifies the techniques used to deal with kind of tasks.

> (…) no human action can exist without being, at least partially, "explained", made "intelligible", "justified", "accounted for", in whatever style of "reasoning" such an explanation or justification may be cast. Praxis thus entails logos which in turn backs up praxis (Chevallard, 2006, p. 23)

Therefore, from the very beginning the notion of "task" has been central within the ATD. In this initial approach, the word "task" has to be considered in a very general sense and institutionally situated.

Thanks to the notion of praxeology, and the subsequent development of the theory, researches within the ATD have been able to analyse the teaching and learning of mathematics, to identify didactic phenomena, to formulate didactic problems, to find tentative answers and, of course, to create new didactic infrastructures. Looking backwards, an important part of this research has to do with the design of tasks (for instance, in the doctoral dissertation of García (2005), Barquero (2009), or Ruiz-Munzón (2010)).

It is the aim of this paper to summarize the notion of "task" used in our research within the ATD, linked to a deep understanding of the "didactic system" and to an explicit epistemological and didactic paradigm of school mathematics, to reveal the design principles behind, and, finally, to exemplify them in the case of a study process designed and experimented in preschool.

Towards a new paradigm of mathematics at school

The notion of "task" we will introduce in the next section is strongly connected to a particular paradigm[35] about the teaching and learning of mathematics. This "new paradigm" is the rationale for this kind of "tasks" and, reversely, the design and implementation of this kind of "tasks" shows to what extent this "renewed paradigm" could be possible.

Let consider mathematics as a human activity, institutionally situated, and modelled in terms of *mathematical works* or *praxeologies*, being the mission of school to make possible the encounter and appropriation of some of them by new generations. Then, analysing how school and mathematics are organized in order to make this encounter and appropriation possible, and also developing new ways of making this process more *effective*, could be considered as one of the main aims of the research in the *didactics of mathematics*.

The traditional paradigm of organizing this encounter and appropriation, still dominant in many places, is what Chevallard (2012) called the *paradigm of visiting works*. According to it, mathematics is split in several little works, which students visit at school (like monuments), being the teacher the guide. Considered as cultural and historical works, students are expected to admire and enjoy this visit. Some side effects of this paradigm are students' difficulties to give sense to the works they are visiting (lack of their *raison d'être*), the atomization of mathematics in little works that can be linearly programmed in a school syllabus, or the disconnection and deficient articulation between some of this pieces.

In contrast, Chevallard (2012) pled for a (counter)*paradigm of questioning the world.* If we model the didactic system as a triplet (X, Y, O), being X the set of students, Y the set of teachers (normally restricted to a single person) and O the *didactic stake* (learning outcome), the main change of this paradigm is that students' access to O has to be through meaningful and generative questions Q. It is through the *study* of Q, and the subsequent questions that might arise from it, that students will find O, or at least some elements of O, in order to get to possible answers to Q. This *study process* might need considering, studying, and even evaluating, already existing works O_j as well as already existing answers $A_j^{\diamond}$, created and diffused by other institutions (which form the *didactic milieu M*). At the end, the aim is to get to an answer $A^{\heartsuit}$, which will depend on the path followed by the community of study and

[35] The word paradigm is considered here in the sense of "a set of rules prescribing, however implicitly, what is to be studied—what the didactic stakes O can be—and what the forms of studying them are" (Chevallard, 2012, p. 2)

on the works and answers found in the *journey*. This study process can be synthesized in what Chevallard calls the Herbartian schema:

$$\left[S(X,Y,O) \hookrightarrow M = \left\{A_1^{\diamond}, \dots, A_j^{\diamond}, O_{j+1}, \cdots, O_n\right\}\right] \hookrightarrow A^{\heartsuit}$$

In this paradigm, works are not desirable by themselves, but insofar they contribute in the process of getting to a better answer $A^{\heartsuit}$. They do not arise as meaningless objects but, on the contrary, they get their meaning from the questions they are contributing to get to an answer. The division and isolation of school mathematics is reduced, as meaningful generative questions give rise to a connected activation of several praxeologies, even extramathematical ones. Also because the analysis and evaluation of existing answers, as well as the development and justification of new ones, situates the activity beyond the routine application of techniques, giving place to the *technological dimension* of the mathematical activity which, in turns, originates that students' work is placed, at least, at the level of *local praxeologies*[36].

The *paradigm of questioning the world* offers some clear connections with what it is named as *inquiry-based learning*. Like this one, is still rather marginal in many educational systems. It is beyond the aim of this paper to analyse this issue. However, it is clear that the adoption of this paradigm needs a completely renewed understanding of the "tasks" used at school. And, reversely, only through the design of a totally different kind of tasks will there be a chance for this paradigm to exist.

Study and Research courses and activities

Very briefly, in this section we will introduce the main features of the kind of tasks in accordance with the *paradigm of questioning the world*. Therefore, we make our definition of "task" explicit.

Within the ATD, the term *study and research course and activity* has been coined to refer to tasks in which:

1. The starting point is a *crucial and "alive" question* Q_0. Therefore, the community of study (X, Y) considers that the study of Q_0 is worth it, and not just an excuse to introduce some pieces of mathematics.
2. Q_0 is a *generative* question. The study of this question will lead to new questions Q_i. That might make the study process open and even undermined in advance.
3. There is a collaborative and shared study process, looking for good answers as well as for good questions. In this process, the classical distribution of responsibilities between X (students) and Y (teachers) will be continuously renegotiated. Specifically, Y avoids been the *question-poser*, the unique source of information (*media*) and the ultimate source of validity (*milieu*).
4. The study of the questions Q_i will need to consider intermediate answers. Some if these answers might be already available within the *community of study*, others might be "imported" from other institutions, and, even others

[36] Within the ATD, *punctual praxeologies* are those characterised by a single technique to solve an isolated task (like solving linear equations in grade 9 by the technique of transposing terms from one side to the other). *Local praxeologies* are characterized by a *technology* that articulates and connects several punctual praxeologies (like solving polynomial equations with one variable). Beyond these, *regional praxeologies* are characterized by a unifying *theory* for a set of different *local praxeologies* (like the theory of algebraic equations).

might have to be constructed ad hoc. In particular, extra-mathematical works might have to be considered.

5. The access to intermediate answers will be mediated by the *media*[37] accessible to the *community of study*. As far as these answers might have been developed elsewhere, the study and adoption of some elements of theses answers should include their possible validation and justification against the *didactic milieu* available to the *community of study* (dialectic process between *media* and *milieu*, Chevallard, 2006).
6. The study process will culminate with the development of a possible, sometimes tentative, answer. This answer might include both mathematical and non-mathematical contents, which are the *didactic stake* (Chevallard, 2012). It will be meaningful as it offers an answer to the initial question Q_0, as well as to the subsequent questions Q_i. Therefore, students in X will learn in order to *think about and interact with* the world they live in, and not just about a *world already thought* by others.

The whole process can be visualized under the light of a modelling process. Questions do not live isolated, but embodied in systems (biological, economical, social…). The study of Q_0 involves the study of at least one system and some of its relationships. Answers (already built, or created throughout the study process) entail considering or creating models (not just mathematical ones). Intermediate questions might arise from the original system but also from the progressive models considered. Particularly, the process would lead to the development of praxeologies of increasing complexity (García, 2005). The final answer will depend on how the initial system has been modelled: which relations have been considered and which ones ignored (structuring and simplification), which kind of model has been used (setting up a model), the kind of information produced by the model (working inside the model) and how this information has been interpreted and validated against the starting system.

Within this context, a distinction between *study and research activities* (SRA) and *courses* (SRC) has to be made. In a SRA, the *didactic stake* (learning outcome) has been set in advance. That means that, starting from a generative question, the study process has been designed to ensure that students meet some praxeologies already decided. On the contrary, SRC are undetermined by definition. Starting from a generative question Q_0, it will be the responsibility of the community to decide which path they will follow, which will determine the praxeologies they meet with. Particularly, an SRC might include one or several intermediate SRA in order to build some needed praxeologies to make the study process advance.

Designing study and research courses and activities: principles and process

In accordance with the paradigm of *questioning the world*, the identification of *generative* and *crucial* questions is a critical issue. It is not just finding nice and motivating questions. At least, three conditions must be considered: *mathematical*, *functional* and *social legitimacy* (García, 2005).

The *mathematical legitimacy* implies that studying and looking for answers to the initial question should give rise to the mathematical praxeologies the study process is aiming at, in the case of SRA, or at least to a set of possible mathematical

[37] In the sense of "mass media", like books, press, TV, or Internet.

praxeologies identified in advance (the didactic stake). In that way, mathematical praxeologies will emerge as motivated objects, as far as they are tools to deal with the initial question, as well as with the questions derived from this.

Identifying questions with generative power for some mathematical praxeologies need to carry out a deep work of mathematical engineering. The design process needs to consider and explicitly question all the steps in the didactic transposition process[38]. That is, instead of accepting the dominant and culturally unquestioned transpositions of scholarly knowledge as the unique ones, researchers need get out of school (considered as an institution) and question the meaning of the mathematical objects they are working with, their origin and evolution, their relation with other parts of mathematics, their integration in one or another praxeology. The result will be an explicit epistemological model used as the reference for the design of the SRC/A. Specifically, this mathematical engineering process is crucial to identify the meaning (one or several) of the mathematical praxeologies that will be integrated within the SRC/A[39]. It is this meaning which will inform the kind of generative questions that might be considered.

The *social legitimacy* implies that questions should overcome the limits of mathematics and school, and be connected with society and its problems (*questioning the world*). Many of the questions currently studied at school have a pretended social legitimacy that rapidly vanishes. Extra-mathematical context are used mainly for motivational purposes, and disappear quickly to give rise to a decontextualized intra-mathematical activity.

The *functional legitimacy* means that the questions should be selected so that its study does not lead to a dead end. On the contrary, crucial and generative questions should allow the study community to explore new territories, interesting both from a mathematical and social perspective. From the mathematical side, the epistemological analysis and the construction of the epistemological reference model will be again crucial to determine the *functional legitimacy of* a question. From the extra-mathematical side, the *functional legitimacy* is connected with the *social* one.

The initial design of SRC/A mainly involves researchers in the field of didactics, especially to carry out the mathematical engineering process. But the collaboration of teachers has been quite productive in our research. In particular, to identify and consider restrictions emerging from school that might provoke that the final product does not fit in school. However, the *ecology* of the SRC/A is a main issue that we cannot discuss here.

In line with the *didactic engineering* methodology (Brousseau, 1997), the a-priori design just explained has to be complemented with the piloting of the SRC/A and its a-posteriori optimization. Researchers and teachers normally carry out this step jointly, and it is structured in advance. During the piloting, it is tested both students' mathematical activity (mathematical praxeologies) and teacher's teaching

[38] Scholarly Knowledge → Knowledge to be taught → Taught Knowledge → Learned, available knowledge

[39] For instance, in the SRA designed in García (2005), the epistemological analysis questioned the isolation of proportionality in the Spain curriculum. Proportionality is culturally modelled arithmetically and considered more as a static relation between magnitudes than a functional one. The epistemological reference model we built integrated proportionally in the world of functional relationships, as a possible (and not unique) relation in a variation system. As a consequence, in the SRA designed, students had to work in a variation system (savings plan) in which different variations hypotheses were possible. So, proportionally emerged as a kind of functional relation, not as a privileged one. (García, Gascón, Ruiz-Higueras and Bosch, 2006).

activity (didactic praxeology). In the students' dimension, it is analysed: how the different elements of the intended praxeologies emerge; the creation of techniques to deal with problems; the identification and formulation of new questions; the exploration of mathematical techniques, including determining its validity and limits; how the mathematical activity is described and justified (technological moment); how the mathematical objects are evaluated by the community of study; the institutionalization of the intermediate and final answers. From the teacher's dimension, it is analysed: the "place" of the teacher and students during the study process (*topogenesis*); the management of the didactic time and its consequences in the evolution of the mathematical activity (*chronogenesis*); the management of the learning environment (*mesogénesis*)[40]. Further, piloting also gave evidence about the *ecology* and the *economy of* the designed task.

Study and research course for pre-school: taking care of our silkworms

In this section we will introduce, very briefly, a study and research activity for pre-school (4-6 years old students). It has been designed jointly by researchers and pre-school teachers and it has been implemented several times[41].

From the mathematical perspective, the didactic stake is learning about elementary arithmetic. In this level, students are supposed to develop quantification skills and the cardinal sense of numbers, languages and forms of expression to communicate about quantities, as well as start building the sense of some arithmetic operations (addition and subtraction mainly). Following Ruiz-Higueras (2005), which relies on the work of Brousseau and his colleagues in the 80s and 90s, an epistemological reference model was developed. Briefly, the cardinal sense of numbers will emerge in measuring situation with discrete magnitudes. Numbers emerge as models to express the measure of a set, to verify its conservation, to manage it, to remember the quantity, to reproduce or produce a set of a known quantity and to compare two or more sets. Numbers (as mathematical objects) and codes to express them (numerals) will emerge *naturally* in communicative situations where the aim is not just to measure a discrete set, but to communicate about it so that another person can produce a set similar to the initial one without having access to it (neither visually nor manipulatively)[42].

Having in mind this reference model, many a-didactical situations have been designed elsewhere. However, they are normally isolated and, sometimes, the context is quite fictitious. So, we wanted to embed numbers in the study of a system that would be real and meaningful for pre-school students, and also rich enough to allow a generative activity that might give rise to the emergence of new questions and different mathematical and extra-mathematical praxeologies. This context came up from the collaboration with a pre-school teacher. She suggested using collections of silkworms, considering that it was springtime and students were bringing some of them into the classroom.

The initial question faced by students was: *if we've got N silkworms, how many leaves do we need to feed them?* Firstly, they could go to the playground and

[40] For the SRC described in the next section, a detailed description of the analysis of the mathematical and didactic dimension can be found in Ruiz-Higueras and García (2011).

[41] In fact, some elements of this work have given rise to the publication by the teacher of a textbook for pre-school (Aguilar, 2010).

[42] Definition of a fundamental situation for cardinal numbers (Ruiz-Higueras, 2005).

collect the leaves from a mulberry tree. But after a few days, they had to ask the gardener to collect the leaves for them, using a written message. That provoked the need of using codes to express quantities[43].

In this stage, the mathematical activity is meaningful and significant for pupils, but restricted to elements of a single praxeology. The turning point is when the biological system starts to evolve: silkworms turn into cocoons, then moths arise and, finally, they die. Pupils have to control a heterogeneous collection made of silkworms, cocoons and moths, ruled by a conservative law[44]. This is challenging for pupils:

- Firstly, because they need techniques to record the evolution of the system. In the experimentation, the teacher prepared a table to record and control the evolution of the system (Fig. 1). She introduced a new artefact into the didactic milieu so that students could take the control of the evolution of the system under their own responsibility.

Fig.1. Table to record control system's evolution

- Secondly, because addition, in the sense of "combining", appears and new techniques are also needed. Due to the conservative law of the system, pupils need to control that the total initial amount remains constant.

Additionally, an expected issue emerged from the study of the system. The magnitude "time" was not expected a-priori (undetermined character of an SRC). However, pupils raised the question of how long will it take to a moth to rise since the cocoon was formed. This task involved the need of developing techniques to record and control the time. In the experimentations carried out, pupils used a calendar and their counting skills as a technique to measure time. Starting from an a-priori hypothesis from the *media* (answers $A_j^{\diamond}$, given by their parents[45]), they had the opportunity to evaluate this "already existing answer" against their learning milieu, getting to their "own" answer $A^{\heartsuit}$. Finally they found that a silkworm inside a cocoon needed 12 days to become a moth (later, they discovered a little variation in this time).

Tables like the one in Fig 1. had to be adapted as the system was changing. Moths had to be included and, finally, alive moths and dead moths as well.

At the end, when all the moths died, the system disappeared. However, pupils had lots of information about its evolution. And a final task was proposed: reconstructing the system. From a modelling perspective, the process is inverted. Now, interpreting models make it possible to recover information about the life of system that will never be back again.

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[43] In this early age, many pupils identify written numerals but do not give them a meaning in terms of quantities. Indeed, most of them use self-invented codes (drawings) to communicate about quantities. Gradually, through activities like this one, they start giving some sense to the Arabic numerals and leave their primitive codes.

[44] Each state of the system can be described with the vector *(t, n(t), c(t), m(t))* where *t* is time, *n* is the number of silkworms, *c* is the number of cocoons and *m* is the number of *moths*. A conservative law rules the system: for every *t*, $n(t)+c(t)+m(t)=N$, being *N* the original number of silkworms.

[45] Different parents gave different number of days.


Fig. 3. Reconstructing the system from the model

Conclusions

Task design within the ATD is a complex process. On the one hand, tasks are intrinsically linked to an epistemological conception of mathematics as a human activity, modelled in terms of praxeologies. On the other hand, it is connected with an explicit paradigm about mathematics in school, the so-called *paradigm of questioning the world*, in contrast with the *paradigm of visiting works*, still dominant in many school systems.

Study and research activities and courses within the ATD constitute a promising field of research and design. Affordances of this approach are, among others, that it is aiming at a renewed school epistemology sensible to the world and its problems, opened to a more democratic flow of knowledge between school and society (Chevallard, 2012). Also, it contributes to a better articulation and integration of topics that are normally separated in school mathematics (García et. al., 2006), and even topics from different disciplines.

On the contrary, limitations of the approach are connected with the cognitive dimension. As far as cognition is considered institutionally within the ATD, what remains unexplored is the issue of the cognitive demand of a designed SRC. Further research in this direction would probably clarify and strengthen the scope of this design approach.

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Task design for systemic improvement: principles and frameworks

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Tasks play many roles in mathematics education, some of them unintentional. They come in many forms, spanning many dimensions, including those of mathematical content, processes and practices, length, and modes of working. In this paper we point out the crucial role that tasks play in forwarding or preventing the process of improvement of teaching and learning – and thus of an education system. We argue that multi-dimensional schemes of task classification have a powerful role to play in the design of tasks and task sequences.

Keywords : classification, dimensions, balance, curriculum, assessment, sequences

Tasks in this paper will mean "activities that students are asked to do, in which mathematics has an important role". Students meet tasks in the course of classroom learning, in informal tests or examinations, in the course of their everyday lives, and in other school subjects. Task design for all these roles has been a central concern of the work of Shell Centre team over the last 30 years, which the authors have led (Swan and Burkhardt, 2012). Our mathematical and epistemological perspectives are eclectic, reflecting the priority in the team's research given to direct impact on systemic improvement.

In discussing tasks and their design, exemplification is absolutely essential to clarify the meaning of descriptions. As so often with print, the limitations of length in this paper mean that we cannot give examples here; we shall accept the opportunity to extend the range on-line, with links in the text to examples of various kinds.

1. Tasks and their various roles

The major focus of this Study will be on the nature and use of tasks in teaching and learning in classrooms. From the systemic viewpoint of this paper, however, it is appropriate to describe and discuss their roles starting with the system, and moving through assessment in its various forms to the classroom.

In specifying a curriculum Currriculum[46] specifications are normally in the form of descriptions of principles, amplified by an analytic model of the domain. The latter in particular, vary greatly in length, from a few pages on competencies in the Denmark to long and detailed lists of mathematical content in British or US documents. The one common feature is that they are largely expressed through language. Such descriptions do not, in fact, specify learning or performance goals – for example, lists of mathematical content could be taught and assessed entirely through short tasks on separate elements, or through substantial projects in which the student chooses and uses appropriate elements of content and process for the investigation in hand – or, more sensibly, for a balanced variety of types of performance. Task exemplars can play a crucial role in reducing this ambiguity. We have argued (Burkhardt, 1990) that a curriculum specification needs three different elements: an analytic model of the domain: an exemplar task set, with each task linked to the model; a list of the range of classroom learning activities that should be involved (for a brief example, see Cockcroft Report, 1982, paragraph 243).

In high-stakes examinations In countries that have tests where the results have life consequences, the range and balance of types of task in the tests have a strong influence on the range and balance of classroom learning activities (see e.g. OFSTED 2012). Indeed they often seem to define the *de facto* "implemented curriculum" in most classrooms, whatever the intended curriculum of the last paragraph may say. So high-stakes assessment, and the tasks the tests contain, plays three roles:

- A: to 'measure' performance – ie
  "to enable students to show what they know, understand and can do"

but also, with high-stakes assessment that impacts students' and teachers' lives, inevitably

- B: to exemplify the performance goals – assessment tasks communicate vividly to teachers, students and their parents what is valued by society, and thus
- C: to drive classroom learning activities *(What You Test Is What You Get)*

These roles carry responsibilities for test designers and those who commission tests – responsibilities that are widely ignored. Psychometricians, too, focus on measurement and statistical error, ignoring the systematic error that comes from assessing only a part of what you want students to learn. Ignoring roles B and C, and the systemic responsibility for test design that they imply, is a major source of the mismatch between intentions and outcomes in school systems.

In classroom assessment In some countries including ours, classroom assessment has traditionally reflected the formal tests, with similar task sets. This is a natural way for teachers to check progress towards an important goal. Some teachers, aware of the limitations of the tests, have always used a broader range of task types in the classroom. In the last decade there has been growing awareness of the power of formative assessment, when well done, in forwarding student learning (see Black and Wiliam 1998, 2001). This approach integrates assessment and teaching in a form where the design of *task sequences* plays a crucial role. This is challenging for teachers so there has been work on the design of support, initially through live

[46] By curriculum, we mean the whole set of learning activities that a student experiences in school.

professional development and, more recently, through classroom materials (Swan et al., 2011)

In teaching and learning We expect the roles of tasks in teaching and learning to be the focus of most of the papers in this study – so here we shall be brief. That their range and variety should cover all the learning and performance goals of the intended curriculum is clear. What makes this area rich are the issues of, and principles for, designing *task sequences* that will lead students along the road to the understanding and performance goals. One may view task sequences as the spine around which all teaching is built, whether they be the succession of closely related exercises of "incremental learning" and its behaviourist relatives or, at the other extreme, the mathematical microworlds of "open investigation", where creating a task sequence by posing questions is part of the student's responsibility. We shall say something more in this in Section 3.

2. Task difficulty

The issue of task difficulty is often ignored but is important in all aspects of task classification and design. It is known from research that the difficulty of a task depends on various factors, notably its:

- *complexity* – the number of variables, the variety and amount of data, and the number of modes in which information is presented, are some of the aspects of task complexity that affect the difficulty it presents.
- *unfamiliarity* – non-routine tasks (those which aren□t just like the tasks one has practiced solving) are more difficult than routine exercises.
- *technical demand* – tasks that require more sophisticated mathematics for their solution are more difficult than those that can be solved with more elementary mathematics.
- *student autonomy* – guidance from an expert (usually the teacher), or from the task itself (e.g., by structuring or "scaffolding" it into successive parts) makes a task easier than if it is presented without such guidance.

Assessments of student performance need to take these factors into account. For example, these factors imply that, in order to design a task for a given level of difficulty, a relatively complex non-routine task that students are expected to solve without guidance needs to be technically easier than a short exercise that employs a routine skill. Focusing on technical aspects alone can lead to rich tasks as being dismissed as "below grade".

The difficulty of a task is determined by trialling the task with a random sample of students drawn from the target population. All assessment tasks, whether for use in the classroom or in summative tests, should be developed in this way, establishing their level of difficulty without undermining their validity as good mathematics.

3. Task variety and task classification

Given the range of roles that tasks play, outlined above, it is clear that appropriate forms of task classification may be useful for various purposes. In this section we set out some schemes and ways they have proved useful in supporting various task roles. There is a constructive duality between holistic and analytic dimensions of classification.

Novice, Apprentice and Expert tasks

This simple holistic dimension of classification (Swan et al 2011) has proved useful in drawing attention to the mismatch between widely accepted goals of mathematics education and current practice in both assessment and curriculum[47].

Mathematical skills and practices can be taught and/or assessed partly in isolation, partly under scaffolded conditions, and partly when students face substantial problems without scaffolded support. We call tasks that assess these three different types of performance *novice, apprentice,* and *expert* tasks respectively. More specifically:

- *Expert Tasks*. Experts solve problems as they arise. Expert tasks are rich tasks, each presented in a form in which it might naturally arise in mathematics, science or daily life. They require the effective use of problem solving strategies, as well as concepts and skills. Performance on these tasks indicates how well a person will be able to do and to use mathematics beyond the mathematics classroom. Expertise is the end goal of mathematics education.
- *Novice Tasks.* Novices are learning the tools of the trade. Novice tasks are short items, each focused on a specific concept or skill. Reflecting the high-stakes assessment, mathematics teaching and learning in Britain and the US is mainly focused on novice tasks.
- *Apprentice Tasks.* Apprentices solve problems, but usually carefully structured problems with guidance from an expert. Apprentice tasks are substantial, often involving several aspect of mathematics, and structured so as to ensure that all students have access to the problem. Students are guided through a "ramp" of increasing challenge to enable them to show the levels of performance they have achieved. Because the structure guides the students, the strategic demands and the range of mathematical practices involved are at a comparatively modest level. Apprentice tasks have a role in developing expertise.

A Framework for Balance

Clearly, classification needs to go well beyond this. The NSF-funded project *Balanced Assessment for the Mathematics Curriculum* aimed to design assessment that reflected the goals set out in the NCTM Standards (NCTM, 1989, 2001). We set out two supervening design principles: *Curriculum balance*: a test should be such that a teacher who "teaches to the test" is led to deliver a curriculum balanced in accord with the Standards. *Curriculum value*: doing the assessment tasks should be a worthwhile learning experience. To articulate what this means we developed the *Framework for Balance*, summarised in the table below.

The headings are self-explanatory except, perhaps, for *reasoning length*. This is the time envisaged for the student to work on the longest prompted section of the task – so a 10-minute task that is structured into many equal parts may have a short reasoning length. (Driven by the naive criterion-referencing behind the National Curriculum, this is common in the UK, where tasks often consist of a sequence of

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[47] The reader can find examples of each at http://map.mathshell.org.uk/materials/tasks.php

short items, set in a common context.). The most important features of the Framework for Balance are:

- the classification is multidimensional, addressing the major aspects of performance
- the dimensions are both analytic (content, process, etc.) and holistic (task type, openness, goals, etc.)
- it provides a method of choosing a set of tasks for a tests to meet conditions, particularly balance, for this
- the analytic dimensions are handled semi-quantitative, with the elements of content or process in a task given rough proportions
- an associated "balancing matrix" can be used to ensure that, while every combination of properties cannot be assessed, the main dimensions are samples with appropriate weight.

This approach was first used for balancing collections of classroom materials (Balanced Assessment 1997-99). It also produced a way around a design dilemma: the more constraints you impose on a task designer (such as a cell in a content matrix to assess), the poorer the holistic quality of the tasks that result. The alternative approach is to free designers to design good mathematical tasks, classifying them later and choosing a balanced set for each test.

Framework for Balance

Mathematical Content Dimension

• **Mathematical content** in each task will include some of:

Number and Operations including: number concepts, representations relationships and number systems; operations; computation and estimation.

Algebra including: patterns and generalization, relations and functions; functional relationships (including ratio and proportion); verbal, graphical tabular representation; symbolic representation; modeling and change.

Measurement including: measurable attributes and units; techniques tolls and formulas.

Data Analysis and Probability including: formulating questions, collecting, organizing, representing and displaying relevant data; statistical methods; inference and prediction; probability concepts and models.

Geometry including: shape, properties of shapes, relationships; spatial representation, location and movement; transformation and symmetry; visualization, spatial reasoning and modeling to solve problems.

Mathematical Process Dimension

• **Phases** of problem solving include some or all of:

Modeling and Formulating;
Transforming and Manipulating;
Inferring and Drawing Conclusions;
Checking and Evaluating;
Reporting.

• **Processes** of problem solving, reasoning and proof, representation, connections and communication, together with the above phases will all be sampled.

Task Type Dimensions

• **Task Type** will be one of: design; plan; evaluation and recommendation; review and critique; non-routine problem; open investigation; re-presentation of information; practical estimation; definition of concept; technical exercise.

• **Non-routineness** in: context; mathematical aspects or results; mathematical connections.

• **Openness** –tasks may be: closed; open middle; open end with open questions.

• **Type of Goal** is one of: pure mathematics; illustrative application of the mathematics; applied power over a practical situation.

• **Reasoning Length** is the expected time for the longest section of the task.

Circumstances of Performance Dimensions

• **Task Length**: in these tests most tasks are in the range 5 to 15 minutes, supplemented with some short routine exercise items.

• **Modes of Presentation, Working and Response**: these tests will be written.

This idea has been taken further: Daro and Burkhardt (2012) proposed the development of a "population of tasks" that epitomises the curriculum goals, and from which tests will be drawn as balanced samples.

While we do not present these classification schemes as definitive (though they have worked well for specific purposes), we do see task classification as an important part of task design.

4. **Principles for the design of tasks and task sequences**

We have recently written in some detail[48] on the principles and processes of task design (Swan and Burkhardt, 2012). Here we have space for a bare list of principles. We argue that curriculum and assessment should be built on tasks that:

1. *Reflect the curriculum in a balanced way*. Assessment should be based on a balanced set of tasks that, together, provide students with opportunities to show *all* types of performance that the curriculum goals set out or imply.
2. *Have 'face validity'*. Assessment tasks should constitute worthwhile learning activities in their own right. The tasks should be recognizable as problems worth solving – because they are intriguing and/or potentially useful.
3. *Are fit for purpose*. The nature of the tasks and scoring should correspond to the purposes of the assessment. Individual tasks should assess students' ability to *integrate* as *mathematical practices* their fluency, knowledge, conceptual understanding, and problem solving strategies. These aspects should not be assessed separately.
4. *Are accessible yet challenging*. Tasks should be accessible with opportunities to demonstrate both modest and high levels of performance, so the full range of students can show what they can do (as evidenced by high response rates with a wide range of levels of response).
5. *Reward reasoning rather than results*. Tasks should elicit chains of reasoning, and cover the phases of problem solving (formulation, manipulation, interpretation, evaluation, communication) even though their entry may be scaffolded with short prompts to ensure access.
6. *Use authentic or 'pure' contexts*. Assessment should contain tasks that are 'outward-looking', making connections within mathematics, with other subjects, and to help one to better understand life and the outside world. As in the real world, they may contain insufficient data (where the student makes assumptions and estimates) or redundant data (where the student makes selections). Students may be asked to respond in a given role: e.g. a designer, planner, commentator, or evaluator. Tasks that use contrived contexts should be avoided.
7. *Provide opportunities for students to make decisions*. Tasks should be included that encourage students to select and choose their own methods, allowing them to

[48] See http://www.educationaldesigner.org/ed/volume2/issue5/article19

surprise or delight. Some may be open-ended, permitting a range of possible outcomes.

8. *Are transparent in their demands*. Students should be clear what kinds of response will be valued in the assessment.

Task sequences in curriculum design

More broadly, we see the concept of *task sequence* at the heart of curriculum design. Too large a topic for this paper with its systemic focus, it will be the focus of a forthcoming article. Here we will just reinforce the idea with a few examples.

We have already noted that the “incremental learning” approach to curriculum design, characterised by small steps, is strongly reinforced by the nature of assessment. Instead, we have cotinued developing an approach that is very different (see Swan, 2006).

We distinguish whether task sequences are designed primarily to foster conceptual development or problem solving processes. The focus of the first is on discussing different *interpretations* of mathematical ideas; the second is on the contrasting alternative *approaches* that may be taken. In both cases we begin by seeking to find out students prior knowledge, by asking them to tackle a carefully chosen task individually, unaided. Their responses are assessed by the teacher, outside the classroom, who must then prepare a series of questions (tasks) designed to prompt students’ deeper reflection. We provide a set of questions matched to typical responses and to assist the teacher in this.

In a problem solving lesson, students are then invited respond to these questions and form small groups to produce joint solutions that both combine the best of their individual ideas, and that address the teachers’ questions. A sharing of alternative approaches is then undertaken, akin to the Japanese practice of ‘neriage’.

Often, students do not consider the most powerful problem solving approaches without further prompting. We therefore provide students with some “sample student work”, chosen and collected by ourselves. This work is designed to show more sophisticated attempts at the problem. Students’ task is now to critique, improve, complete and extend suggested solutions – a challenge to their existing thinking.

The concept-focused lessons are similar in structure to the problem solving lessons, but here we identify the different *task genres* that promote concept development and select a rich of that kind. Examples are given in the table below:

Task genres	**Description of tasks**
Classifying and defining	Students devise classifications for mathematical objects, and/or apply classifications devised by others. They discriminate, recognise properties and develop mathematical language and definitions.
Interpreting and translating between multiple representations	Students match cards that show different representations of mathematical objects - words, diagrams, algebraic symbols, tables, graphs. They share interpretations, compare and group the cards in ways that made connections between underlying concepts. The discussion of common 'misconceptions' is encouraged by the inclusion of distracters.
Testing and evaluating mathematical statements and conjectures	Students are given short mathematical statements or generalisations, are asked to make posters that describe their domain of validity and provide examples, counterexamples and explanations to support their decisions.
Creating and solving variants of	Students devise new or variants of existing problems, prepare solutions then challenge other students to solve them. They offer support when the solver

mathematical problems	becomes stuck. This promotes awareness of the structures underlying problems, and focuses attention on the doing and undoing processes in mathematics.
Analysing reasoning and solutions	Students compare different methods for doing a problem, organise solutions and/ or diagnose the causes of errors in solutions. They begin to recognise that there are alternative pathways through a problem, and develop their own chains of reasoning.

These activities are conducted in a collaborative atmosphere, with the teacher acting as a provoker, using the prepared questions to prompt students to argue and refine their interpretations, with a whole class 'neriage' discussion as wrap-up.

Challenges for research and development

Currently, the tasks presented by high stakes examinations and textbooks, (which in the UK are often written by examiners who focus on repetitive practice of examination-type questions) largely determine the types of task that are used within classrooms. We need to challenge this state of affairs at policy level using such classification schemes as we have described above in curriculum documents to describe learning objectives. A vital component, often missing form such documents, is the vivid exemplification that is necessary to show exactly what such tasks might look like.

At a deeper level, further refinement and illustration of the task-types we have described here is needed; in particular, further classroom evidence of their individual impact on teacher and student practices and performances is required.

In addition, research is needed to show how student performances on conceptual and problem solving tasks might be reliably measured and reported. Otherwise examiners and teachers will continue to assess fragments rather than complete performances.

Background and context

While all members of the team contribute to the various aspects of the Shell Centre's work, the authors have played central roles. Malcolm Swan has led the design of tasks and the elicitation of design principles (see e.g Swan and Burkhardt 2012) while Hugh Burkhardt has played a leading role in the development of the analytic frameworks for describing and balancing tasks and the strategic design of tests and their curriculum support (see e.g. Burkhardt 2009). The work has had ongoing national and international support over the past 25 years.

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Task design in a school-based professional development programme

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In this paper we report on two principles of task design arising from our study of a school-based mathematics teacher professional development programme in Shanghai, China. The two principles are: a) developing a 'hypothetical learning structure' for the topic; and b) developing tasks within a web-like structure of knowledge connections. This paper provides an example of each and connects the literature review with the described research design.

Keywords: task design; school-based professional development; expert teachers; Chinese mathematics pedagogy; collective task design.

Introduction

This paper focuses on the issue of how published tasks (sourced from textbooks) are appropriated by teachers for instructional purposes and hence how task design influences mathematics teaching. In particular, we focus on the central aim of the Working Group Theme D (in the ICMI Study 22 Discussion Document) by reporting on the design and implementation of tasks by a team drawn from three communities: academic researchers; the local school district professional developer, who is an expert teacher with considerable skill in school-based teacher professional development; and a group of Chinese teachers from Shanghai Soong Ching Ling School, Shanghai. In doing so, we take the stance highlighted by Kieran, Krainer and Shaughnessy (2013) that teachers are key stakeholders in research and can play a significant role in the design of rich tasks for mathematical learning. To add to this, we argue that the expert teacher/professional developer can also play a significant role in task design and its 'implementation'.

In China, the 2011 version of National Mathematics Curriculum Standards (briefly called Standards in this paper) emphasises the shift from the traditional "Two Basics" (basic knowledge and basic skills) to the current "Four Basics" (adding basic mathematical thinking, and basic activity experiences). As a result, teachers are expected to carefully consider and prepare the teaching of particular mathematical knowledge and skills, in addition to the development of students' individual understandings and thinking, and to building up students' learning activity experiences in mathematics (Shi, 2012). The problem now facing educators and teachers in China is which mathematical tasks, and task sequences, should be selected/considered (for particular topics), and how to work with such tasks, so as to implement effectively the range of intentions laid out by the Standards.

Theoretical framework and key pedagogical features

Simon (1995) proposed the notion of Hypothetical Learning Trajectory (HLT) to help mathematics educators (i.e. teachers, researchers, curriculum developers) to think about the design and use of mathematical tasks to promote mathematical conceptual learning. The HLT comprises of three components (ibid p. 136): the learning goal; the learning activities; and the hypothetical learning process - a prediction of how the students' thinking and understanding might evolve in the context of the learning activities. Simon (1995) further points out that teachers need to develop skills to generate hypotheses about students' understandings (which go beyond soliciting and attending to students' thinking), to generate HLTs, and to engage in conceptual analysis related to the mathematics that they teach.

Ma (1999, p. 97) reports that one feature of Chinese teachers' knowledge is their well-developed "knowledge packages" of the range of ideas needed to teach a topic like arithmetic. Within such a package, there is a certain "key" piece of knowledge that is fundamental to enable students to learn other knowledge. Another kind of key piece of knowledge in the package is the "concept knot" (p. 98), which links to several different concepts of the learning topic. Ma (1999) considers that teachers' knowledge packages reveal their understandings of the longitudinal process of opening-up and developing mathematics of a particular field in students' minds.

Gu, Huang and Marton (2004) identify two types of teaching with variation in the Chinese mathematics classes. One is conceptual variation (CV), concerned with understanding concepts from multiple perspectives; the other is the procedural variation (PV), focused on developing insights into the hierarchical features of mathematical activity. Gu (2012) further interprets that the PV can be used to connect, in a dynamic way, different mathematical processes as a whole process. That is, a teacher can divide the process/es of a mathematical activity into a number of sub-activities, and then use variation as a means of "*pu dian*" (i.e. scaffolding) of knowledge between the sub-activities.

Based on the literature (e.g. Ma 1999; Gu et al. 2004), we identify the following key features of Chinese mathematics pedagogy: 1) understanding concepts from multiple perspectives; 2) designing for individual student's actual learning processes and responses; 3) gradually deepening learning through an orderly-layered teaching procedure (a teaching/learning method called "*Xun Xu Jian Jin*" in Chinese). In our project we utilised a school-based teacher professional development program (TPDP) (as detailed in the next section) as a means to enable researchers, expert teacher and school teachers to work together. In the TPDP, we are aware of Gravemeijer's (2004) notion of a teacher's "local instruction theory" (highlighted in Choppin, 2011), which is concerned with student thinking about complex tasks and how this is situated within a broader instructional 'space'. For more on the mathematical and epistemological basis of our task design study, see Lin (2011).

The task design study

The main study was arranged in Shanghai Soong Ching Ling School located in Qingpu district, a western suburb of the city. The school was an international laboratory school funded by China Welfare Institute with the key mission of launching innovative and experimental educational classroom studies aimed at improving the quality of compulsory education for children in the country.

The reasons for this study are twofold. First, the school planned to launch its own innovative mathematics curriculum to complement the strengths of the Shanghai (SH) mathematics curriculum. Second, an innovative school-based model of curriculum was expected to bridge the gap between curriculum developers' intentions and school teachers' classroom practice.

The methodology of this study is based on a Design-Based Research (DBR) (e.g. Design-Based Research Collective, 2003) approach to study, document and advance a model for sustainable teacher professional development. The approach involved processes such as iteration and feedback loops in such ways that development and research took place through cycles of design, enactment, analysis, and redesign (Cobb et al. 2003).

The tasks in the pilot study were sourced from published material from two countries: the SH official textbooks; and the online professional development resources from New Zealand (NZ):
http://www.nzmaths.co.nz/professional-development
The participant groups of the pilot study were: (1) three researchers (the three authors); (2) an expert teacher (called 'Mr Zhang' in this paper); and (3) three teachers, one of them a teacher-researcher (first author). The three researchers (authors) designed the study and provided the theoretical background (literature, research design, etc.). Next, in the school-based TPDP, the first author (a teacher-researcher) designed the first intervention based on ideas from the NZ online resources. The expert teacher, Mr Zhang, commented on it together with other teachers in the TPDP. Two Grade 4 teachers (one experienced teacher, and one less experienced teacher) then took Mr Zhang's comments into account in their re-designed lessons of the same topic in two other grade 4 classes. Mr Zhang reflected upon both lessons, together with other teachers. Subsequently, the three researchers analysed the data, which consisted of data on the context; observations and videos of the three lessons; discussions/interviews with Mr Zhang after each lesson; and documents, such as curriculum materials and textbooks; also official documents, guidelines for teaching, syllabi, etc..

In this paper we define a 'task' as a learning situation with a specific teaching goal in a single lesson. The main body of the task design is therefore to create a sequence of multi-layered learning situations within a broader instructional plan and to develop a general picture of the students' learning paths. In the pilot study the teachers were actively engaged in the work of how to select, modify, sequence, teach, observe and reflect on a single lesson and on a sequence of lessons.

As laid out in the SH Grade 4 mathematics textbook (term II), the chapter on decimals (30 pages in total) consists of the following sub-topics: decimals in real life; the meanings of decimals; comparing decimals; the properties of decimals; the movement of the decimal point; the addition and subtraction of decimals; and the application of the addition and subtraction of decimals in problem solving. In their teaching teachers were expected to appreciate the key connections of the teaching/learning goals embedded in the SH textbook as follows: (for pupils)

1. to learn/understand the relationship between decimals and fractions (to understand the base-ten place value system of decimals and the meaning of 0.1, 0.01, and 0.001);
2. to learn/understand the units of decimals (e.g., 0.1) (to understand the meanings of decimals with one, two, three digitals; and to know a whole number with a decimal);
3. to learn/understand the concept of place value of decimals (to understand the place value table of decimals).

In what follows, we focus on delineating the two major principles that emerged from the data analysis of the three teachers' lessons on 'place value of decimals' (briefly called 'decimal value') and of Mr Zhang's comments during the post-lesson TPDP.

Findings: two principles of the task design study

Principle 1: Developing a 'hypothetical learning structure' for a particular topic

The first teacher intended to use the NZ online teaching resources to complement the strengths of the SH mathematics textbook and pedagogy. For the lessons of decimal value, the teacher's "local instruction theory" (Gravemeijer, 2004) was to enable her students to accumulate learning experiences with the big idea that the decimal number system is a base-ten place value system. Thus, she used the NZ online tasks to enable students to examine two sub-ideas associated with the base-ten place value system.

1. The places (or columns) in the number system are based around groupings of ten. For example, 10 ones = 1 ten, 10 hundreds = 1 thousand.
2. The decimal point is a convention that indicates the units place.

The teacher started the lesson by presenting the place value of whole number (briefly called whole number value) on the blackboard (see the left half in Figure 1). While students observed it, they recalled previously learnt knowledge such as series, and a 10-to-1 relationship between the values of any two adjacent places (or columns). For instance, some students discussed that 10000 equals 1000 of ten. Others argued that 10000÷10=1000. Still others added that the "Wans" (tens of thousands) are 3 places to the left of "Shi" (tens). This means that there is a 10×10×10 relationship between "Shi" and "Wan".

…	Qian (Thousands)	Bai (Hundreds)	Shi (Tens)	Ge (Ones)	Shi Fen Zhi Yi (Tenths)	Bai Fen Zhi Yi (Hundredths)	Qian Fen Zhi Yi (Thousandths)	…
…	1000	100	10	1	1/10 0.1	1/100 0.01	1/1000 0.001	…

Figure 1. The decimal number system

Next, the teacher drew students' attention to the decimal value (see the right half in Figure 1). For instance, according to the newly learnt knowledge of decimals, students were able to discuss in the class that 10 of 1/100 is 1/10. Some represented such a relationship by decimals (10 of 0.01 is 0.1). The teacher then led students to pay attention not only to a 1-to-10 relationship between adjacent places to the right where the places were getting smaller by a factor of ten, but also to the relationship between any two places in the number system. In observing such relationships as 0.1÷10=0.01, 0.1÷100=0.001, and 0.1÷1000=0.0001, students were also led to think about the movement of the decimal point. Moreover, the teacher posed tasks to enable students to see the connection between the movement of the decimal point and the units of a number. For instance, 6501.4 (in this case the “ones” is assumed),□650.14 tens (in this case the units are tens),□65014 tenths (in this case the units are tenths),□and 6.5014 thousands (in this case the units are thousands).

In the first post-lesson professional activity (PPA), Mr Zhang firstly commented on the teacher’s lesson from three perspectives, namely (1) the HLT of students; (2) the learning methods; and (3) the degree of learning difficulty.

> I found that the (first) teacher dealt with the textbook in her lesson differently from those in our common lessons. ... There is a starter in the HLT. That is, the anchor of knowledge. For instance, what types of knowledge, experience and methods of learning the students already have? And to what degree? The starter of this lesson was at an abstract level, such as the comparison of the place values, enlarge or reduce a number, the relationship of places, etc. … Some students in the class may be lost at this abstract level. … Moreover, the learning method is different. In this lesson, the teacher provided a bit of context of the problems, students then developed a discussion in the class and then solved some points of problems by themselves. It’s considerably random. The logical structure of learning itself is loose. For instance, what is the first step and then the next step of learning? In a whole, what is the general goal of learning? The learning was designed in a macro way.

Next, Mr Zhang drew the teachers’ attention to the HLT from a micro perspective of the “concept knots” (the meaning and the places of decimals) (Ma 1999) in the decimal chapter of the SH textbook.

> To view the learning from a micro perspective, students are expected to master a number of factors from this topic. For instance, can a number be read on the decimal value (dv) table? Can a decimal be put on the dv table? Can the form of the decimal be explained on the dv table? In the last lesson [according to the SH textbook], it is about the structure of the meaning of decimals. In this lesson, a shift is to be made to the structure of the places of decimals. In the end of this lesson, the two structures should be connected.

From such a micro perspective, Mr Zhang also addressed the importance for teachers to recognize the connection of the “key” piece of knowledge (the whole number value) (Ma 1999) and new knowledge (the decimal value) in the HLT.

> When pupils learn the decimal value, their cognitive anchor is on what they have previously learnt of the place value of whole numbers [learned in Grade 4 -Term I]. Teachers thus should create cognitive conflicts of the new topic for their students, to go beyond their previous knowledge of whole numbers. It is because … to ‘fill’ a decimal is something new for students. In so doing, their intellect is challenged and they can be engaged in thinking about how to create a decimal value which is the topic of this new lesson.

In our study, we use the term “hypothetical learning structure” (HLS) to distinguish the Chinese expert teacher’s concept of HLT from Simon’s (1995) HLT, in terms of two considerations: (1) The HLS in our study is not based on the western constructivist theory, but largely on the Chinese expert teacher’s expertise in predicting students’ learning processes and responses through observing the same topic many years during authentic classroom practice. (2) The use-aim of the HLS is not only to address the teacher’s well-developed “knowledge packages” (Ma, 1999) in mathematics, but also to distinguish the learning methods from those of the HLT (Simon, 1995). That is, in Mr Zhang’s view, pupil learning could be more ‘efficient’ (in the sense of ‘whole class learning’) if students do not randomly use their previous knowledge to respond to the teacher’s questions in a set of points, but are engaged in the teacher's well-designed mathematical tasks in a set of 'blocks'. Mr Zhang offered the following 'blocks of tasks' for the decimal value lesson:

1. Write decimals and then write decimals with whole number on the number line;
2. Write units of decimals on the number line (e.g., 0.1, 0.01, and 0.001);
3. Write whole number in the whole number table and then write decimals with whole numbers in the whole number table;
4. Create the decimal table, design the place value in the decimal table and recognize the role of the decimal point;
5. Develop an understanding of the units of decimals in the decimal table;
6. Write the form of decimals in mathematical notation (e.g., 0.23 = ( ) × 0.1 + ( ) × 0.01);
7. Write the form of decimals in word language;
8. Define two types of decimals (decimals and decimals with whole numbers).

Principle 2: Developing tasks within a web-like structure of knowledge connections

Two teachers followed up the conversations with Mr Zhang by carefully planning the second and the third lessons for each of their classes on the same topic of decimal values according to the HLS (outlined above). Their lessons were quite different due to the teachers' different instructional intentions and the different students in each class. The second teacher (with two years teaching experiences in primary mathematics) sought to develop students' understanding of the form of decimals on the concrete geometrical model (the number line), and then their abstract thinking in the decimal table. For instance, in the starter of his lesson, the teacher used a considerable amount of time for students to write decimals on the different sections of the number line (e.g. those between 0 and 1, 0 and 1/10, 0 and 1/100, 0, 1 and 2, and 61, 62 and 63).

The third teacher (with ten years teaching experiences in primary mathematics) did not use the number line, but developed her students' abstract thinking of decimals directly from the form of decimals to the decimal table. For instance, in the starter of her lesson, the teacher posed a number of decimals and a whole number on the blackboard (e.g. 0.23, 0.63, 1.08, 61.52, 88.888, 1045). She then asked students to discuss the form of these numbers. Some of the examples are given below:

0.23 = ( ) × 0.1 + ( ) × 0.01
1.08 = ( ) × 1 + ( ) × 0.1 + ( ) × ( )
61.52 = ( ) × 10 + 1 × ( ) + ( ) × ( ) + ( ) × ( )

In the second PPA, Mr Zhang mainly commented on the two approaches applied by the two teachers, in order to help teachers to develop an insight into selected didactical ideas in mathematics teaching which are advocated in the latest national curriculum reforms. The new curriculum advocates developing students' complex thinking in a web-like knowledge structure, so as to enable students to see the most important feature of how mathematical knowledge is constituted. Mr Zhang said the following:

> Different from the second teacher, the third teacher's lesson was from number to table. In fact, there are a considerable number of concepts in the (decimal) table. For instance, 10-to-1 relationship between the values of any two adjacent places, the name of the units of decimal, the name of the place of decimals, the unit of

the place of decimal, the number on the unit of decimal, the unit number of decimals, etc. The (third) teacher's instruction was clearly concerned about the first two concepts, but did not talk about the other concepts in the lesson. The main 'missing point' of the lesson is the lack of concrete diagrams to support students to understand the connection of the abstract concepts, such as the unit of the decimal place and its meaning (as the second teacher did). The advantages of the combination of decimal table and the number line are as follows: firstly, the infinity in dividing a small segment on the number line can be connected to the concept of number place in the table. At the same time, it is an opportunity for students to develop the concept of infinity. In such a way the teacher helps students to see the connection between the representation of the geometrical model and the mental representation of the decimal table. Consequently, the teacher makes the scaffolding for students to understand the abstract concepts and ideas such as infinity, set, 'approaching'. A lesson like this is likely to be a deep learning lesson. … In the latest pedagogy reforms, 'gathering thinking' is highlighted in teaching. That is, thinking is developed as a web, rather than the linear line/progression of knowledge connections. The 'gathering thinking' is based on a rich web of knowledge and a web of experiences. New knowledge is the result of the richer/more connections. The point is to understand what kinds of web new knowledge is based on. Then teachers could see how to help students to develop multiple connections of knowledge and experiences within the web. The connection of a new point of knowledge to the multiple points of knowledge on the web would enable students to enrich knowledge representations, make knowledge transformation and application, and develop complex thinking, namely the 'gathering thinking' in mathematics.

Mr Zhang then drew two web-like structures of knowledge connections of the decimal table. Figure 2 is a macro-level of the knowledge structure of the decimal table. Figure 3 is a micro-level of the knowledge structure of the decimal table.

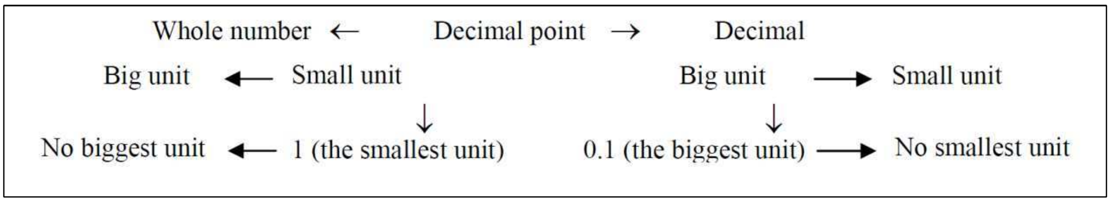

Figure 2. A macro-level of the knowledge structure of the decimal table

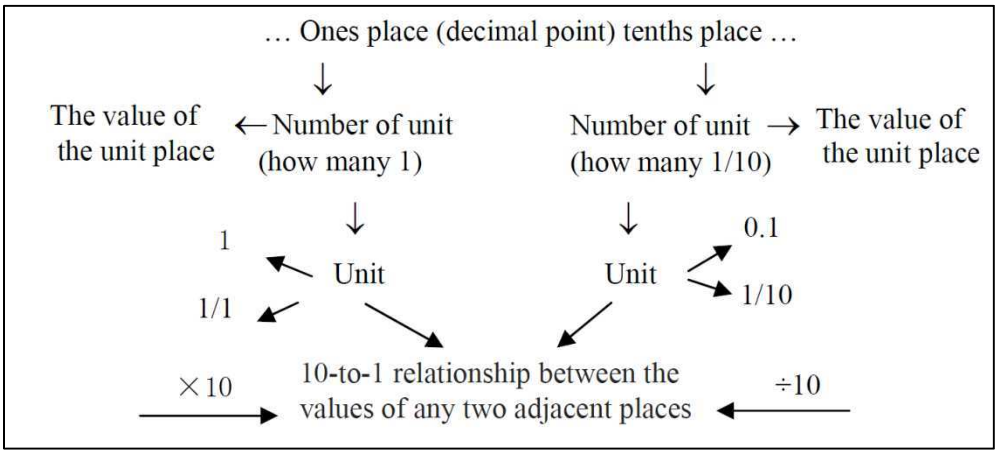

Figure 3. A micro-level of the knowledge structure of the value place of decimal table

The two figures indicate two key theoretical ideas of this expert teacher in helping teachers to develop knowledge and ideas to make the multi-layered task design: first, the conceptual variation (Gu et al., 2004) for understanding concepts from multiple perspectives; second, the "*Xun Xu Jian Jin*" method to develop the complex mathematical thinking within the whole structure of knowledge connections.

Implications

In this paper, we have delineated two principles which had emerged in our pilot study of the school-based tasks and instruction design project in Shanghai. However, there are challenges for the main study that concern the difficulties of the connection between theoretical frameworks and principles for our task design across communities:

(a) It is challenging to situate knowledge from different design communities into the 'living context' – the classroom in a broader culture (in our study the Chinese culture). In particular, if we wish to introduce innovation in teaching, we need to understand more comprehensively the alternative "local instruction theories" (Gravemeijer, 2004) that different teachers (researchers, professional developer and school teachers in our study) appear to hold.

(b) The HLT (Simon, 1995) features a teacher's design decisions based on her/is best guess of how learning might proceed. In our study the HLS addresses the importance for teachers to develop their "knowledge packages" (Ma, 1999) and to try things out in practice. It would be necessary to understand how the expert teacher's HLS may help teachers to make the connection between the unpredictable nature of individual learning and the pedagogic practice/ mathematical-didactic structures (macro and micro) suggested by the expert. Further, it would be necessary to develop new insights into the different teaching/learning methods underlying Simon's (1995) HLT in the US classroom and the HLS in the Chinese classroom. For instance, the distinction of the western construction (i.e., scaffolding) and the Chinese "*Xun Xu Jian Jin*" and "*pu dian*" (i.e., procedural variation, in Gu et al., 2004, p. 340), the theoretical ideas of the 'proper potential distance' (Gu et al., 2004, p. 343), and the micro and macro perspectives of learning viewed by the expert teacher in our study.

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Hybrid Tasks: Promoting Statistical Thinking and Critical Thinking through the Same Mathematical Activities

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Even though statistical thinking and critical thinking appear to have strong links from a theoretical point of view, empirical research into the intersections and potential interrelatedness of these aspects of competence is scarce. Our research suggests that thinking skills in both areas may be interdependent. Given this interconnection, it should be possible to stimulate both forms of thinking through the one task. An exploratory qualitative study has been undertaken into thinking processes when working on tasks encompassing both these areas. This paper explores the implications of this study for the design of tasks that simultaneously stimulate critical thinking and domain-specific thinking.

Keywords: Statistical and Critical Thinking, Task Design, Hybrid Tasks

Background

Mathematics classroom instruction is generally organized around and delivered through students' activities on mathematical tasks (Doyle, 1988). Notably, in all of the seven countries that participated in the TIMSS 1999 Video Study, eighth-grade mathematics was most commonly taught by spending at least 80% of lesson time in mathematics classrooms working on mathematical tasks (Hiebert et al., 2003). For our purposes, complete description of a mathematical task requires specification of the intentions, actions and interpretations of both teacher and student/s, together with details of the context in which the task was undertaken and by whom (Mesiti & Clarke, 2010). In this paper, "task" mostly refers to the written stimulus, although we also provide the details required to understand the performative realisation of the task as a mathematical activity. Classroom activities are coherent actions shaped by the instructional context, in general, and, in particular, by what is taught through the use of tasks (Stodolsky, 1988). The tasks that teachers assign can determine how students come to understand what is taught. In other words, tasks serve as a context for students' thinking, during and after instruction. Doyle argues the point that

> tasks influence learners by directing their attention to particular aspects of content and by specifying ways of processing information (Doyle, 1983, p.161).

To achieve quality mathematics instruction, the role of mathematical tasks to stimulate students' cognitive processes is crucial (Hiebert & Wearne, 1993). Contemporary curricula prioritise more than just procedural knowledge. In the mathematics curricula of countries as culturally and geographically distant as China, Australia and Finland, student thinking is promoted in more sophisticated terms than simply knowledge of facts and procedures. Korea and Singapore, in particular, seek to retain the high level of mathematical competence documented in TIMSS and PISA results, while improving higher order thinking and problem solving expertise. This poses the question as to whether both can be achieved. The possibility is explored in this paper that the same tasks might be employed to realise both goals. The tasks employed came from the research of Kuntze and his colleagues, the actual interviews were conducted by Aizikovitsh-Udi, the analyses of the task responses with respect to Statistical Thinking and Critical Thinking were coordinated by Kuntze and Aizikovitsh-Udi and the task-specific interpretive analysis was done by Clarke.

In summary, the centrality of tasks in mathematics classroom is evident from theoretical perspectives as well as in empirical results from international comparative studies. The role of mathematical tasks provides a key to any attempt to understand teaching and learning in research on classroom practices in mathematics. But can a given task stimulate and promote both discipline-specific thinking and more generic forms of higher-order thinking? To explore this question, we take Statistical Thinking (ST) and Critical Thinking (CT) as our exemplars of the two modes of thought.

Statistical Thinking and Critical Thinking

In a well-known definition of Statistical Literacy by Gal (2004), a "critical stance" is included among the key attitudes for successful statistical thinking (ST) – hence, Gal includes such attitudes in his definition of statistical literacy (cf. also Wallman, 1993; Watson, 1997; Reading, 2002). However, being critical in statistical contexts is not only an attitude. It is possible to describe specific abilities that have to be used in order to critically evaluate statistical data. Two key concepts or overarching ideas in statistical thinking relevant for a critical evaluation of data are *manipulation of data by reduction* (Kröpfl, Peschek & Schneider, 2000) and *dealing with statistical variation* (e.g. Watson & Callingham, 2003). Successfully manipulating data by reduction requires the awareness of such things as that calculating a mean value affords an overview on the original data, but it reduces the initial information. Hence, the resultant statistical value is (only) an indicator corresponding to a specific mathematical model, and we should not forget that it reflects only a part of the information. In order to critically evaluate the data, we might need additional information about the distribution, such as the variance, or information about extreme values.

Critical thinking (CT) skills rely on self-regulation of the thinking processes, construction of meaning, and detection of patterns in supposedly disorganized structures (Ennis, 1989). Critical thinking tends to be complex and often terminates in multiple solutions, each with advantages and disadvantages, rather than a single clear solution. It requires the use of multiple, sometimes mutually contradictory criteria, and frequently concludes with uncertainty. This description of CT already suggests links with ST, such as dealing with uncertainty, contradictions and a critical evaluation of given claims (cf. McPeck, 1981). Dealing critically with information – a crucial aspect for both domains – demands critical/evaluative thinking based on rational thinking processes and decisions (Aizikovitsh-Udi, 2012; Aizikovitsh-Udi & Amit, 2008). Can a single task be used to elicit and promote both forms of thinking?

In-depth analyses are required into *how* CT and ST may interdepend. In order to design the type of hybrid tasks proposed here, it is essential that we understand the connection between the two forms of thinking that provide the specific goal for the use of such tasks in mathematics classrooms and how these connect to task characteristics.

Investigating Hybrid Tasks

In attempting to stimulate particular thinking skills, it might be thought best to target either discipline-specific thinking (eg. Twelve numbers have a mean of 10 and a standard deviation of 2, what might the numbers be?) or generic critical thinking skills in discipline-free contexts (eg. Five people are isolated by flood, under what circumstances would it not be appropriate to share the available food equally?). Hybrid tasks seek to promote both. This paper explores the actions such tasks promote, the rationale for their use, and their design characteristics.

In order to explore thinking processes related to tasks in the domains of both Statistical Thinking and Critical Thinking, individual semi-structured interviews were conducted with mathematics teachers. By using mathematics teachers as subjects, basic content competence can be assumed and it becomes possible to examine their content-related higher order thinking skills, both in terms of statistical thinking and critical thinking. The interviews focused on thinking-aloud when solving tasks and each lasted about 40–50 minutes. Beyond solving the tasks, the interviewees were also free to give their personal views on the tasks.

In the following section, the results from one interview with a single teacher are used to exemplify the sort of data generated and the type of task likely to stimulate Statistical Thinking and Critical Thinking. The analysis concentrated on identifying Statistical Thinking (ST) and Critical Thinking (CT) as employed by the interviewees. Our interest is not just in the capacity of a single task to stimulate both ST and CT, but whether the interaction between ST and CT made them mutually supportive or disruptive. A first analysis was done focusing on ST only (Watson, 1997), then a second analysis employed a CT point of view (Ennis & Millman, 1985). We then carried out a combined interpretative analysis in order to examine relationships between CT and ST elements. In this methodological approach, the analyses were done by two coders working in parallel. Figure 1 shows the first task analysed.

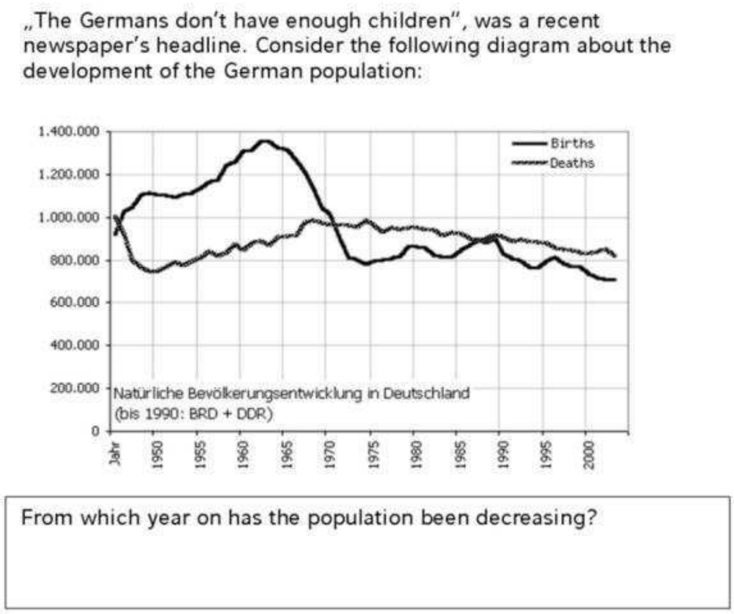

Figure 1: Task "population"
(Kuntze, Lindmeier & Reiss, 2008)

Nena was an experienced U.S. secondary Mathematics teacher with 20 years of mathematics teaching experience. She was dedicated to improving her teaching and had been participating in a year-long professional development program for mathematics teachers at the time this interview occurred. In the interview, Nena was asked to solve the problem in Figure 1, while thinking aloud:

Interviewer:	What do you think? From which year on has the population been decreasing?
Nena:	Well…..the population has been decreasing since 1963 until about 1973, where the population begins to rise a bit..... but it is still down since 1963.....
Interviewer:	Are you confident about it?
Nena:	Of course… I am sure! You can look at this problem mathematically, anywhere there is a positive slope you could say the population is increasing, where there is a negative slope the population is decreasing. But compared to 1963, it's always been down.

Dealing with assumptions is one of the key elements of Critical Thinking (Ennis & Millham, 1985). Nena initially assumed that the graph of births completely determined the population development. Even when asked to reflect on this assumption, Nena did not generate possible counter-arguments for testing her initial assumption, nor did she appear to question this assumption. She tended to seek confirming evidence rather than evidence that might challenge her initial assumption.

Seen from the perspective of Statistical Thinking (ST), Nena chose an inappropriate statistical model for interpreting the data given in the diagram (cf. description of this task in Kuntze, Lindmeier & Reiss, 2008). She appeared to focus on the data related to the births only, and she deduced her conclusions from a mathematical consideration of slopes. Even when encouraged by the interviewer, she did not check this model against the full data given in the diagram.

Nena's answers show deficits both in Critical and Statistical Thinking. Looking at the relationships between CT and ST: At the very beginning, Nena shows only a partial perception of the evidence, focusing on the birth data from 1963 on. This selective focus may have been a result of the headline given in the diagram ('The Germans don't have enough children'). This headline may have triggered

Nena’s misinterpretation of the births as determining the population, from the turning point in 1963. It is interesting that Nena emphasised that it was possible to “look at this problem mathematically”, which suggests that she saw a discrepancy between looking at the situation from the perspective of a mathematical model and looking at it from the perspective of the context (population and children born in Germany). Possibly the mathematical or statistical model is taken as an authority that is used to justify the appropriateness of the assumption instead of questioning the model chosen initially. It is possible to conclude from this combined analysis that the elements of reasoning in both domains interfere and interact. Given this interconnectedness, help in either domain, either in CT and ST, may have had a positive impact on the thinking process as a whole. Moreover, the CT and ST perspectives offer not only simultaneous and parallel ways of interpreting the reasoning process, but, through a combined analysis, can explain how CT and ST can be mutually beneficial, reinforcing related reasoning approaches. Tasks that stimulate the use of both CT and ST are consequently of practical importance.

However, CT and ST are not always interdependent in an obvious way, as the following example suggests (cf. Figure 2 and the corresponding interview section):

Mrs. Blum would like to buy a reliable Laptop, either a C-Pad or an S-Top.
In a computer magazine, 400 laptops of each brand have been tested. In this comparison the C-Pad has turned out to be more reliable.
In the evening she talks to three friends. Two have S-Tops and never had problems. The third had a C-Pad, but had so many hardware problems with it, that he has sold it again immediately.

With which of the following statements do you agree?

- ☐ **Mrs. Blum should buy an S-Top, because the friend with the C-Pad had made bad experiences, whereas the friends are happy with the S-Tops.**
- ☐ **Mrs. Blum should buy a C-Pad, because the test in the computer magazine is based on a high number of computers, not only on one or two.**
- ☐ **No matter how she decides, it can happen that she gets a Laptop that causes problems frequently.**

Figure 2: Task

Interviewer:	So, what do you think, with which of the following statements do you agree?
Nena:	I would agree with the 3rd statement.
Interviewer:	Can you explain, please?
Nena:	Yes... 400 computers is not a large sample when talking about computers so I would go with the 3rd statement and just listening to the comments of her friends and the consumer’s magazine, it is possible that both the computers could be just so.
Interviewer:	Are you sure?
Nena:	Sure. I like this question!

Seen from the perspective of CT, Nena questions not only the experiences of the friends, but also the results from the study with the 400 laptops. On this basis, she expressed agreement only with the third statement, which highlights questioning evidence as a sub-aspect of CT. However, considering Nena’s answers under the lens of ST, she appears not to acknowledge the statistical power of the sample of 400 laptops. Nena remarks that it is not possible to make a prediction on the base of the data, and she appears to compare the number of the 400 laptops to the number of all laptops. She does not reflect in depth on the statistical power of the magazine study.

From a joint perspective, Nena's dominant critical attitude may have blocked her use of elements of ST, e.g. reasoning related to the sample size and representativeness. This example gives insight into how CT and ST may interfere. In this case, CT practice acted to the detriment of ST. Conversely, ST can be dominant over CT, as the following example (associated to the problem in Figure 3) suggests:

The police president shows the following diagram and says: "This diagram shows, that since 2005, the number of crimes in the city center has increased, so that we have to expect that it will further go up in the next years. We need more policemen for patrol in the city center."

In a more extensive press communication, Fred has found the data of the last 7 years:

Year	2000	2001	2002	2003	2004	2005	2006
Number of crimes	504	528	525	499	529	518	538

Do you agree with the interpretation of the police president?
Why or why not?

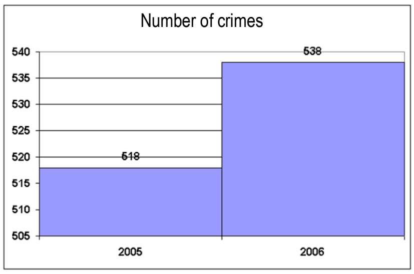

Figure 3: Task "crimes"

Interviewer:	…and here... Do you agree with the interpretation of the police president? Why or why not?
Nena:	I do not see a big difference with the number of crimes for any given year that would warrant extra police force to be hired...
Interviewer:	Can you justify it, please?
Nena:	It looks like the average is approximately 520 which is close to all the numbers so I do not think anything different is really happening from any given year.

According to Ennis' taxonomy (Ennis, 1989) one crucial element of CT is raising questions, having doubts, and exploring key definitions, like "crime". In this task, the nature of the crimes is a key consideration. For example, if we knew that all the crimes were murders, we might decide differently than if the crimes related to paying taxes or fraud. No question about such a definition was raised by Nena and the focus was purely numerical. In this task, Nena employed a restricted set of CT skills.

From the point of view of ST, Nena used an appropriate model and showed an awareness of statistical variation. By these means, she arrived at the conclusion, that, given the variation of the data of the past years, the rise of the crime number is not significant. Consequently, seen from the ST perspective, Nena showed an appropriate understanding of the statistical situation.

Looking at this part of the interview in the joint CT and ST interpretation mode, the analysis suggests that Nena successfully questioned the statement of the police president by using the data given in the problem and a statistical argument. She appeared to remain in the statistical domain, giving more details related to the model she had chosen (distances to the average value). This focus on ST may have hindered her use of any of the CT skills, such as analysing and questioning the definition of 'crime' as the key notion here, questioning the evidence (i.e. the way the data had been collected), etc.

Questioning data plays a role also in the following example related to the problem in Fig. 4:

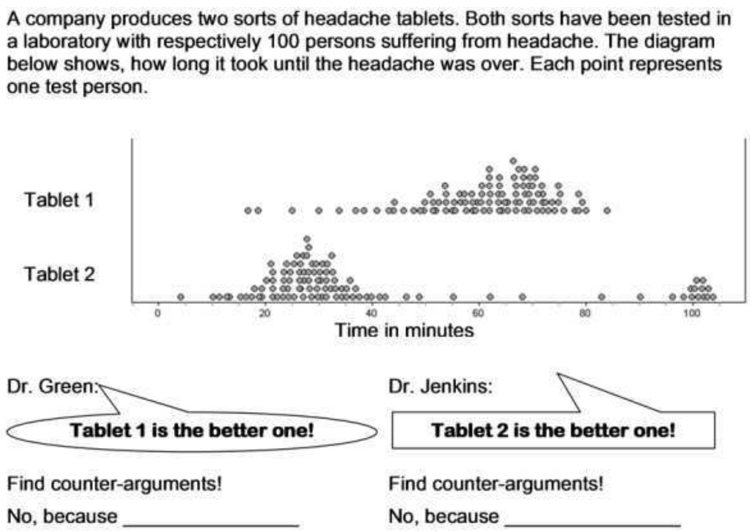

Figure 4: Task "tablets" from Kuntze, Lindmeier & Reiss (2008)

> Nena: And here… I think tablet one takes too long to get rid of the headache. Tablet two seems to get rid of the headache a lot quicker for the majority of the people. You don't really know the age or weight of the people. Many factors play into the reason why a headache might occur so the statistics are poor. Based on the chart, I would have to pick tablet number two in hope of a speedy recovery.

From the perspective of CT, Nena not only evaluates the given statements, but she also shows CT elements when going beyond the data given: She gives examples of relevant influencing factors, and questions the data provided in the diagram ("the statistics are poor").

From the point of view of ST, the analysis of Nena's short answer yields that Nena chose an appropriate model and was aware of the key elements of the problem, even if she did not explicitly discuss the minority of cases with very slow recovery for tablet 2. These considerations led to her personal conclusion to pick tablet number two, as she obviously sees the chance of a "speedy recovery" as more important than the risk of a very slow recovery.

Looking at both CT and ST, the example appears to highlight how elements of CT can contribute to ST, e.g. when evaluating data, its presentation and analysis, planning data collection, etc. In the example, Nena suggests an analysis that takes into account the age or weight of the persons in the study. Conversely, aspects of ST like dealing with statistical variation and uncertainty can contribute to CT, especially when it comes to decisions in non-determinist situations, where full data is unavailable. These examples are intended to illustrate how both ST and CT skills can be evoked by the same task. We suggest that this models authentic and useful thinking practice more effectively than a more closed task that stimulated only statistical thinking and the application of taught procedures. Our question now, is "what are the messages for task design?"

Conclusions: The feasibility, utility and design of hybrid tasks

The tasks used in this exploratory study have certain distinctive characteristics and we would argue that each of these characteristics constitutes a key principle of task design:

- each uses a "real-world" situation as its "figurative context" (see Clarke & Helme, 1998);

- each provides succinct statistical information relevant to that context (Kuntze, Lindmeier & Reiss, 2008);
- the problem is stated very simply;
- some form of evaluation is integral to the problem (Ennis & Millham, 1985);
- the task affords many reasoning approaches.

The exploratory study demonstrates that connections clearly exist between Statistical Thinking and Critical Thinking at the level of individual reasoning practices. In seeking to stimulate both forms of thinking we suggest that an individual employing Statistical Thinking has access to a structured framework of analytical principles that guide and support their reasoning. That is, the relationship between measures of central tendency and variance, for example, structure any consideration of distribution of data that might be invoked in drawing evidence-based conclusions or making evidence-based judgements. On the other hand, the components of Critical Thinking are not related in such a structured fashion and an individual's inclination to employ one strategy (e.g. Questioning Evidence or Questioning Assumptions) can be given expression without any obligation to also invoke other components of Critical Thinking. Some Critical Thinking skills resemble the "heuristics" that were the focus of the enthusiasm for problem solving in the 1980s and 1990s (Clarke, Goos, & Morony, 2007). Catalogues of such heuristics were similarly fragmented.

Ennis and others have catalogued critical thinking skills (Ennis, 1989) and even arranged these categories in a form of hierarchy, but the connection between specific critical thinking skills is under-theorised in comparison with Statistical Thinking. Nonetheless, the forms of Critical Thinking identified in such classificatory schemes are clearly of significance, both as aspects of reasoning and as potential curriculum content. If it were possible to develop a structure for Critical Thinking in which the component elements were not only identified, but also their relationship established, then to invoke one aspect of Critical Thinking would serve to catalyse the use of other related aspects, because the connections between elements would be well known and understood. The question of how best to conceptualise these skills, how to integrate or connect them with other curricular goals, and how best to promote them and nurture their development in the classroom has been a major challenge. An earlier study (Aizikovitsh-Udi, 2012), using similar tasks, has documented efforts to produce CT through a program of instructional immersion in the related topic of probability. In this paper, we argue that particular tasks can stimulate the use, promotion and development of both Statistical and Critical Thinking. We would like to suggest that an instructional program of hybrid tasks could be devised that provides the opportunity to employ Statistical Thinking, while simultaneously introducing students to the practices and structure of Critical Thinking. The design characteristics of such hybrid tasks have been identified.

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Effects of variations in task design using different representations of mathematical objects on learning: A case of a sorting task

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The goal of our study was to examine how variations in task design using tools may affect the process and object of learning. The study focuses on sorting tasks, i.e., learning tasks that require grouping a given set of mathematical items, in as many ways as possible, according to different criteria suggested by the learners. We present an example of a sorting task for which the items to be grouped are related to basic concepts of analytical geometry that are connected to the notion of loci of points. Based on an experimental study[49] with three groups of in-service secondary school mathematics teachers, we report on *intended* and *enacted* objects of learning inherited in three versions of the task. Empirically-based suggestions are drawn about design principles of sorting tasks that potentially to evoke desirable enacted learning.

Keywords: Analytical geometry; classifications; sorting task; variation theory

Introduction

The study aims at examining how variations in task design may affect the process and object of learning. We focus on sorting tasks, i.e., learning tasks that require grouping a given set of mathematical items (e.g., graphical, symbolic or word representations of mathematical concepts), in as many ways as possible, according to different criteria suggested by the learners (Zaslavsky, 2008a; Zaslavsky, Chapman & Leikin, 2003). In particular, we analyse a sorting task based on a set of items related to basic concepts of analytical geometry that are connected to the notion of Loci of Points (we refer to this task as the *LP sorting task*).

The study can be seen as an application of *variation theory* to design of a task aimed at enhancing learners' awareness of mathematics as a connected field of study. Briefly, in variation theory terms (Marton & Booth, 1997; Runesson, 2005),

[49] This study was conducted as part of a Ph.D. dissertation by the third-named author under the supervision of the first- and second- named authors (Dolev, 2012).

awareness of mathematics as a connected field of study was an *intended object of learning*, a particular sorting task was a medium of learning, the number and content of mathematical items included in the task were subjects of didactical manipulations, and the learners' ways of handling the task, which was offered in three different versions, constituted a variation space or an *enacted object of learning*. The latter is the focus of our study.

The study was guided by the following research question: What is the nature of learning that may occur in small groups of mathematics teachers while working on the different versions of a sorting task? Based on the empirical answer to this question, we address at the end of the paper the more general one: What are the characteristics of a set of mathematical objects for sorting that can enhance opportunities for the learners to achieve the intended object of learning and what design principles underlie the construction of such a set?

Theoretical Background

Sorting tasks

Sorting or classification tasks are recognized as a useful teaching/learning tool either in a mathematics classroom or in a teacher professional development workshop. For instance, Sawada (1997) found that students who were not motivated to take part in traditional mathematics lessons were responsive when given a sorting task in the context of geometry and could classify the given items in the variety of ways.

Zaslavsky, Chapman and Leikin (2003) discuss the learning potential of sorting tasks in the context of symbolic and graphical representations of functions. They pointed out the open nature of the task and its potential to evoke different mathematical solutions. Zaslavsky (2008a) analyzed a sorting task based on a set of graphical representations of functions and argued that engaging in the task can draw students' attention to subtle similarities and differences between the items.

Zaslavsky and Leikin (2004) described how a group of mathematics teachers coped with a sorting task based on a set of algebraic equations and inequalities. They found that the participants first suggested ways of sorting based on surface features (i.e., features that can be observed without solving the statements), and only after a while turned to more structural features (i. e., those that may be identified only by solving the statements). A compatible result was found by English and Sharry (1996) for high school students working on a similar sorting task. In their study, the participants tended to classify equations by arithmetic manipulations involved in solving them, but, as a rule, not by deep-level mathematical structure of the equations.

In spite of an increasing attention to the potential of sorting tasks, none of the studies on this type of tasks examined iterative aspects of task design; rather they look at a final version of a task. As a rule, the processes of designing sets of objects for sorting remain salient, though the design processes of some other kinds of open learning tasks have been usefully unpacked (Zaslavsky, 2008b; Liljedahl, Chernoff & Zazkis, 2007). Our study focuses on the task design iterations as well as the differences in the learning experiences inherited in the different versions of the task.

Variation theory of learning

A central premise of a variation theory of learning is that "there is no learning without something being learned" (Runesson, 2005, p. 70), or, in other words, "learning always has an object" (ibid). The theory encompasses three types of objects of learning (Marton & Booth, 1997; Runesson, 2005). A *lived object of learning* denotes what is actually learned from the point of view of a learner. It is difficult to reveal. An *intended object of learning* denotes the capabilities the teacher wants the learners to develop. An *enacted object of learning* is constituted in the interaction between learners and the teacher or between the learners and themselves, and denotes what is *possible* to learn in a particular situation. Intended and enacted objects of learning may or may not coincide.

Analyzing the enacted object of learning requires identifying the affordances and possibilities for developing a certain capability as inherited in the learning situation. In connection to task design, revealing the enacted object of learning means that the design team should critically evaluate, based on empirical evidence, which experiences the task affords and to which extent those are beneficial for co-constructing the intended object of learning.

An additional premise of the variation theory is that objects can be perceived, understood or experienced by a learner or a group of learners in different ways, depending on which aspects of the object one's awareness is directed to and how. Consider an example in the context of sorting tasks. When deciding whether or not two mathematical items belong to the same group, a group of learners can direct their attention to the appearance of the common words or symbols in the items' formulations. Alternatively, they can direct their attention to small differences in the items' formulations and eventually find out that similarly looking formulations describe different mathematical objects. Such possibilities constitute a *space of variations* inherited in the learning situation or task. In variation theory, variations in ways by which a certain object *may* be perceived or experienced by an individual or a group are considered fundamental for learning. Accordingly, our analysis focuses on the diversity of the teachers' ways of experiencing the different versions of the task as well as on their final responses.

The study

The settings

An LP sorting task was developed by the third-named author for the use in 90-minute workshop for in-service secondary school mathematics teachers as learners and potential users of the task in their classrooms. The LP task design was driven by the following learning objective: to facilitate learners' awareness of mathematics as a connected field of study by directing their attention to structural similarities and differences among the basic concepts of analytical geometry and loci of points.

Each workshop had the following structure: a brief presentation of different representations of conic sections – 10 minutes; instructions for working on a sorting task – 5 minutes; small group work on the LP task – 40 minutes; reflective discussion in a whole group – 15 minutes; discussion of those cards that were perceived by the teachers as the most difficult ones – 10 minutes. The instructor refrained from intervening in the main part of the workshop.

Data sources and analysis

Overall, 53 secondary school mathematics teachers participated, as learners, in the study. Each of them took part in one of three 90-minute workshop. Each workshop used a different version of the LP task. Nineteen learners divided into 6 groups took part in the first workshop; 18 learners divided into 5 groups – in the second one and 16 learners divided into 5 groups – in the third one.

The data consisted of transcribed videotape-recorded small-group and whole group discussions, in addition to their sorting actions. For each small group, the completed sorting sheets were collected and analyzed.

The between-group variations in the sorting criteria and the order, in which they appeared in the course of the learners' actions were deduced from the data and considered to be the variation spaces for each version of the task. Patterns of the learners' experiences while working on the task were identified by juxtaposition of the participants' discourse, sorting actions and corresponding sorting sheets. The main two patterns were: (i) making sorting decisions by recognizing the objects as familiar from past study of analytic geometry, and (ii) making sorting decisions by "solving" the item, i.e., by applying algebraic manipulations in order to obtain a familiar symbolic or pictorial representation of the loci of points. The variation spaces of each version of the task were compared to determine to which extent they are compatible with the intended object of learning: to facilitate awareness of structural similarities and differences among the basic mathematical concepts of analytical geometry rather than similarities and differences based on surface features of the items.

Three task iterations

Intended and enacted objects of learning: The first version

Generally speaking, the initial process of the set construction for the LP task was compatible with that described by Watson and Mason (2006) for designing teaching sequences of exercises based on the students' (presumed) perceptions. Specifically, it started from the analysis of how the chosen concepts are represented in textbooks and included decisions about which controlled variations should be considered so that the learners "might observe regularities and differences, develop expectations, make comparisons, have surprises, test, adapt and confirm their conjectures within the exercise" (Watson & Mason, 2006, p.109).

The first version of the LP task consisted of 24 items (see Table 1 in the Supplementary Material (SM) file[50]). It was created so that three types of controlled variations would be maximized. The first type of variation was related to the mathematical objects described in the cards for sorting: a straight line, circles, parabolas, ellipses, hyperbolas and the empty set. Two items disguising the empty set and one item disguising a straight line were included in order to increase the participants' awareness of the internal mathematical features of the mathematical objects under consideration; we refer to these items as "pathological" ones. The second type of variation was related to the type of representation: symbolic, graphical and verbal. Furthermore, the verbal items varied with respect to two factors: the main operation involved in the loci generation and the generating elements given in the descriptions (see Table 2 in SM). The main desirable way of classification was by the names of loci of points. This is because this way of classification required from the learners to unravel structural similarities and differences among the items.

The third type of variation was related to the types of experience needed to handle the task, in relation to the (presumed) mathematical knowledge of the

[50] The Supplementary Material file is available at the following website: http://edu.technion.ac.il/docs/KoichuZaslavskyDolevThemeA_Supplementary_material.pdf

participants. It was hoped that the participants could identify most of the loci of points based on their prior knowledge and on the information given in the introductory presentation, and thus, would have enough time to look for structural similarities and differences. The reality turned to be more complicated than our plans.

First, though all the groups considered several ways of sorting as requested, only two groups wrote more than one way in their sorting sheets and one group did not write in the sheet at all. This finding suggests that the offered format of the task was not comfortable for most of the participants, in spite of (or may be due to) its openness. Second, all the groups started from the most apparent way of sorting, by the types of representations, but only one group wrote it in the sorting sheet. This suggests that most of the participants did not perceive this way as worthwhile sharing. Third, the desirable way of classification, by names of the loci of points, was considered by all the groups only in advanced stages of the work on the task and was documented by 5 out of 6 groups. Forth, the need to "solve" the verbal items manifested itself stronger than we expected. There were 69 attempts to classify 18 verbal items, and in 44 cases (59%) the sorting decisions were made based on algebraic manipulations. Only 21 sorting decisions (36%) were made based on the prior knowledge. This means that the task was much more "technical" for the participants than we expected and that an essential part of the enacted learning was not related to sorting, but to the ways of "solving" the verbal items by means of algebraic manipulations. Finally, it became clear that the overall number of the items was overwhelming. The characteristic assertions on this issue included: "There are too many cards. It is impossible to consider them all!" and "It is a lot of work… we still have 5 cards to sort, but I don't want to do it any more…"

In conclusion, the first version of the LP task, which was designed in accordance with the principle of maximizing the chosen types of variations, resulted in partial success: the desirable way of classification was considered, but a lot of time was devoted to the technical work and to classifying the items by surface features.

Intended and enacted objects of learning: The second version

The second version consisted of 18 items (see Table 1 in SM). The items, which have been approached in all the groups only algebraically, were excluded. (The intended characteristics of the second set of cards are presented in Table 3 in SM).

We found that in spite of a smaller intended variation space, in comparison with the first version of the task, the enacted variation space became richer. The main sorting criteria were: "by the type of representation – symbolic, graphical or word" – 9 appearances; the (desirable) criterion "by the name of locus of points – parabola, hyperbola etc." – 5 appearances; the criterion "by key words in the items descriptions – 'sum of distances', 'ratio' etc." – 7 appearances, and the criterion "by the ability of the group to identify the loci without 'solving the item'" – 3 appearances.

This time all the groups completed the sorting sheets, and 2 out of 5 groups documented in the sheets more than one way of sorting. The desirable way of sorting, by names of the loci of points, was considered by all the groups earlier than in the first workshop and was documented by 4 out of 5 groups. This finding suggests that the teachers were more engaged in looking for structural similarities and differences than in the first version of the task.

The analysis of the teachers' actual experiences with the task supports this suggestion. Fifty-five attempts to classify 11 verbal items were made, and in 29 cases (53%) the sorting decisions were made based on the prior knowledge. In 7 cases

(13%) the decisions were made based on algebraic manipulations (5 decisions were correct), and in 12 cases (22%) algebraic manipulations were used to verify the decision based on prior knowledge (for instance: "I've solved no. 12. It is a straight line. It was for nothing – I could see [from the formulation] that it is about a midline…").

With respect to the teachers' experiences, it is interesting to point out that in 7 cases (13%) the teachers fixed, by means of algebraic manipulation, their earlier wrong decisions based on their prior knowledge. For instance, the "pathological" items were treated in this way and summoned attention and interest of the participants. Overall, it can be concluded that in the second version of the task the use of algebraic manipulations supported the teachers' classification actions, but was not the main experience they gained from the task.

Still, 4 out of 5 groups started from considering the apparent, but not especially instructive criterion "by the type of representation." Moreover, it was evident that the presence of the well-familiar pictorial and symbolic representations in the task postponed and probably hindered the learning experiences related to making sense of the verbal items. For this reason, we decided to leave in the third version of the task only 11 verbal items (see Table 1 in SM).

Intended and enacted objects of learning: The third version

As mentioned, in the third version of the LP task our intention was to supress the appearance of sorting criteria by surface features, in favour of criteria related to identification of structural similarities and differences. The prospective characterization of the set of items can be found in the "verbal representations" part of Table 3 in SM. The findings show that this goal was achieved.

As in the second workshop, all the groups completed the sorting sheets, and 2 out of 5 groups wrote in the sheets more than one way of sorting. As in the second workshop, all 5 groups considered the desirable way of sorting, and 4 out of 5 groups wrote it in the sorting sheets. The overall number of the considered sorting criteria (26) was also compatible with that from the second workshop (30), though the number of items to be classified was considerably smaller, 11 instead of 18.

One essential difference between the second and third versions appeared to be related to the types of experience utilized for making sorting decisions. Out of 55 sorting decisions, 41 (75%) were based on the use of prior knowledge, and only 8 decisions (15%) – on algebraic manipulations. Another difference was related to the use of the "by key words" criterion (15 out of 26 appearances). In the first and the second versions this criterion reflected the learners' attention to certain words in the formulations, but usually without particular attention to their mathematical role in generating the loci. In the third version, the criterion was used more meaningfully and, as a rule, on the way to the desirable criterion "by names of the loci of points."

In closing, the two main enacted subcategories of the "by key words" criterion in the third version of the task – "by the main operation for generating the locus" and "by the main generating elements" – were remarkably close to one of the *intended* types of variation of the LP task.

Discussion: Articulation of the principles underlying design of the LP task

In each successive version of the LP task, the enacted variation space was closer to the intended variation space than in the previous one. In this section, we attempt to pinpoint this phenomenon by extracting from the findings the characteristics of the sets of items for sorting and, in a more general mode, by formulating seven design principles underlying their construction, in addition to the principles formulated in past research.

From only partial success of the implementation of the first version of the LP task, two design principles can be deduced:

1. An amount of work needed to handle the task should be feasible for learners under the given time constraints and their incentives to be engaged in doing the task meaningfully, and not only formally.
2. In order to preserve the central role of the main intended activity (in our case – classifying), the task should not be overloaded with mathematically challenging items that may require "solving" by means of complementary mathematical techniques (in our case – by algebraic manipulations).

The second and the third versions of the task also taught us that:

3. Inclusion of a small number of mathematically challenging items constructed so they can create a feeling of surprise for the learners (in our case, "pathological" items fulfilled this role) is beneficial for learning.

The next design principle, which arises from comparison of all three versions of the task, is in the spirit of the proverb "sometimes less is more:"

4. Well-informed reduction of the intended variation space of a task does not necessarily lead to reduction of the enacted variation space, and, in some cases, may even increase it.

In addition, we observed that:

5. Having items based on different types of representations of the same concepts does not necessarily enhance an experience of looking for *structural* connections between the chosen concepts.

This principle may seem counterintuitive in light of the professional literature, which calls for engaging the students into experiences that require connecting different representations of the same mathematical concepts (e.g., NCTM, 2000). One possible explanation is as follows. Well-familiar symbolic and pictorial representations in the first and the second versions of the task served as *reference points* for looking for ways of how to sort the verbal items by names of the loci of points. By reference points we mean pieces of knowledge the learners hold as true and use as an anchor for planning or monitoring (cf. Harel, Koichu & Manaster, 2006, for the compatible use of a 'reference point' notion in the context of a problem-posing task). Coming back to the LP task, it seems that from the learners' perspective, the verbal items in the first and the second versions should have been "solved" in order to reveal their symbolic or pictorial representations to be then compared with the items presented in symbolic or pictorial forms from the beginning. Symbolic and pictorial representations were unavailable as reference points in the third version, but an idea to sort the items by names of the loci of points was still apparent, given the participants' prior knowledge. Consequently, the participants were forced to look for the subtler reference points in the items' formulations. The main operations for generating the loci and the main generating elements were considered as such.

The next two principles stem from the observed regularities in the order, in which the teachers considered different sorting criteria.

6. Task design should take into account that learners tend to start from easy-to-make decisions on their way to making more effortful (and meaningful) decisions.

7. The easy-to-make decisions can either hinder or support the intended object of learning, thus, the effort should be made to anticipate and organize them so they would serve as reference points on the way to more meaningful decisions.

Simply put, these two principles are in the spirit of the well-known saying "a journey of a thousand miles begins with a single step." A more sophisticated argument is presented below. To recall, we found that in the first and the second versions of the task most of the groups started from sorting the items by the most apparent criterion, "by the types of representation – symbolic, pictorial or verbal". We first observed that in many cases this criterion had been surface-level only for the learners and as such it might hinder the desirable deep-level reasoning around the verbal items. However, when the possibility to consider the types of representations as a sorting criterion was suppressed in the third version, the learners began to consider instead the next most apparent criterion, "by key words". A closer look at the data reveals that this criterion was not always only a surface-feature one for the participants. As we have argued above, in many cases it could be seen as an indicator that the participants looked for the reference points to start with on their way to more mathematically meaningful criteria. Having this idea, we started to reconsider the role of the "by the types of representations" criterion in the first and the second versions. We now are inclined to think that the criterion that appeared at first was not only due to some weakness of the early versions of the task, but also due to some fundamental mechanisms embedded in cognitive activity of classifying or, more generally, in human thinking (cf. Zaslavsky, 2008a, for an elaborated argument).

Back to the LP task, we assume that in virtually any version of the task we can think of, the participants would probably always start from considering some easy-to-see criteria. To recall, English and Sharry's (1996) and Zaslavsky and Leikin's (2004) results indirectly support this assumption. And only after critical evaluation whether the first criterion is interesting enough to be written, the participants might start digging dipper. The pedagogical implication of this simple idea is formulated above as Principle 7. To put it in other words, instead of trying to supress the immediate (as a rule, not especially deep) ideas of the learners coping with an open learning task, the task designers should worry about whether the task provides opportunities for developing these initial ideas into more sophisticated, desirable, ones. The presented case suggests that this goal is feasible, though it is not easy to achieve.

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Principles of task design to foster proofs and refutations in mathematical learning: Proof problem with diagram

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Proving is an essence of mathematical activities, and therefore educational studies should seek effective tasks that foster students' proving activities. Among various aspects of proof and proving in mathematics education, this study aims to develop a set of tasks by which students can experience an authentic process of proofs and refutations. In particular, this paper focuses on proof problems with diagrams, and elaborates task design principles for such problems from a theoretical perspective based on a notion of deductive guessing which Lakatos formulated as one of the heuristic rules. In later cycles of this study, the authors and teachers will jointly scrutinize the design principles from practical perspectives.

Keywords: proof and proving, refutation, Lakatos, deductive guessing, proof problem with diagram

Introduction

Mathematical tasks are one of the most essential tools for promoting students' learning. In particular, what kinds of problems should be posed to students and how teachers should deal with the problems in their classrooms are vital for quality mathematical learning (Henningsen & Stein, 1997; Hiebert & Wearne, 1993). It is also necessary to make clear which mathematical activities are targeted at, because effects of a task depend on the nature of mathematical activities (e.g. conceptual understanding, procedural fluency, mathematical modelling or inquiry, and so on).

This study chooses proof and proving as a main target of mathematical activities, because proving is an essence of mathematics, and therefore it should be a core of students' experiences at all grades (NCTM, 2000). Several mathematics educators have already conducted important researches about tasks related to proof and proving (e.g. Bieda, 2010; Buchbinder & Zaslavsky, 2011; Lin et al., 2012; Stylianides, 2009). For example, Stylianides and Stylianides (2009) develop an instructional sequence which consists of a series of tasks and associated instructor actions so that students can recognize a limitation of empirical arguments and a need for deductive proofs. They deal with one of the most prevailing difficulties that students do not feel a need to learn deductive proofs.

Among various aspects of proof and proving, this study focuses on processes of proofs and refutations that task design researches do not seem to have examined so far. In particular, we intend to develop task design principles for an authentic learning which mirrors mathematical processes that Lakatos (1976, 1978) described. He insisted that "informal, quasi-empirical, mathematics … grow(s) through … the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" (Lakatos, 1976, p. 5). He described processes of mathematical development through rational reconstruction of actual histories about Descartes-Euler conjecture on polyhedra and about uniform convergence. Thus, relying on Lakatos's research could lead to design tasks by which students can experience developmental mathematical activities, which gradually progress through conjectures, proofs and refutations.

As a task which can allow us to realize such authentic learning in regular classrooms in junior or senior high school, this study focuses on "proof problems with diagrams" (the definition and illustration will be provided in the following sections). We first refer to the context of this study briefly, that is, mathematics education in Japan. Japanese students start to learn geometric proofs from the eighth grade (13-14 years old), and eighth and ninth graders learn to prove geometric statements about various properties of triangles, quadrilaterals and circles, using conditions for congruent or similar triangles. In Japan, proof problems with diagrams are standard for students and teachers in the sense that most of the proof problems in school geometry include diagrams that indicate meanings of problem sentences (such problems seem to be common in other countries as well; for example, see Herbst & Brach, 2006). This study attempts to reflect on whether and how an authentic mathematical learning based on proofs and refutations can be realized through such standard problems.

This paper is the first report of a larger study that aims to develop, through collaboration between researchers and teachers, a set of proof problems with diagrams and associated teachers' guidance which prompts students to engage in processes of proofs and refutations. Toward this overall goal, this paper focuses on the first aim of the Working Group on Theme D, that is, to delineate task design principles within a singular community of researchers. Current roles of the authors are to elaborate task design principles for proof problems with diagrams from theoretical perspectives and to develop several tentative tasks, and we address these aspects in this paper. Later on, the authors and teachers will jointly re-examine the principles and tasks from practical perspectives, enact the tasks in classrooms, and refine these principles and tasks from results of the enactment.

Before taking up the main subject, it is better to clarify the definition of "task" in this study. We regard task as a problem such that at once students try to solve it and teachers plant an educational aim of getting students to experience certain mathematical processes during solving it (e.g. Fig. 1 and 3); problems are usually set by teachers or textbooks, but in some cases students may formulate problems by themselves. In task design, it is essential to select or develop problems that have high possibilities to achieve the intended educational aims. In addition, teachers' interaction with students plays a vital role for quality mathematical learning because it is unrealistic to expect that only posing the problems can facilitate students' activities. Therefore, task design in this study involves not only selection or development of problems but also teachers' instructional guidance related to the problems.

Deductive guessing in *Proofs and Refutations*

In elaborating task design principles, this study takes into account a mathematical perspective primarily, in particular a mathematical process which Lakatos (1976) described in *Proofs and Refutations*. Among various methods shown in the literature, this study focuses on "increasing content by deductive guessing" (hereafter, deductive guessing) that has not been sufficiently deliberated in mathematics education literature so far. Deductive guessing indicates that after one proves conjectures and then faces their counterexamples or non-examples, one invents deductively more general conjectures which hold true even for these examples (Lakatos, 1976, p. 76). Lakatos formulated it as one of five *heuristic rules* which summarized several methods for coping with counterexamples. The word "heuristic" seemed to have a normative meaning in his philosophy of mathematics, because he was largely influenced by Polya's mathematical heuristic (Davis & Hersh, 1981; Lakatos, 1976), and Polya (1957) stated that "(m)odern heuristic endeavors to understand the process of solving problems, especially the *mental operations typically useful* in this process" (pp. 129-130, emphasis is original). Thus, Lakatos appreciated deductive guessing as a useful and productive action for advances of mathematical activities.

Since deductive guessing is a mathematical notion, some examination from pedagogical perspectives is also necessary to regard deductive guessing as a central component for task design principles. For a pedagogical rationale of the principles, this study adopts "the intellectual-honesty principle" that Stylianides (2007) indicates. On the principle, he argues that when conceptualizing the notion of proof in school mathematics, it is essential to take into consideration "both the normative aspects of proof in mathematics … and what is known or conceptually accessible to the learners under teacher guidance or collaboration with peers" (Stylianides, 2007, p. 3). Further, he implies that the intellectual-honesty principle should be applied to not only the notion of proof in school mathematics but also other notions or activities.

It is already mentioned in the above that Lakatos regarded deductive guessing as a normative action in mathematical research. In addition, there is some evidence that shows the accessibility of deductive guessing to primary or junior high school students (Komatsu, 2010, 2011). For instance, Komatsu (2011) carried out an experiment in which, after a pair of ninth graders proved their conjecture and then confronted with counterexamples of the conjecture, they could invent a more general statement that held true for the counterexamples. Thus, selecting deductive guessing as a central component for task design principles would enable mathematics teachers and educators to attain an authentic mathematical learning which at once mirrors processes of proofs and refutations and is accessible to students.

Principles of task design for proofs and refutations: Proof problem with diagram

Proof problem with diagram

In order to develop certain kinds of tasks which lead to mathematical processes advocated by Lakatos, this study focuses on "proof problems with diagrams" in geometry. A proof problem with diagrams is a problem in which a statement is described with reference to particular diagrams with symbols (one diagram in most cases) and solvers are required to prove the statement (Fig. 1). In the following, we briefly summarize the nature of such problems and then illustrate that specific kinds of such problems could give students opportunities to find and deal with counterexamples and non-examples.

Not only in Japan but also in many other countries, students encounter proofs within geometry at junior or senior high school, and proofs in school geometry have their origin in Euclidean geometry. In Euclid's *Elements*, propositions are stated in general and abstract words without any symbol or diagram. To prove such proposition, one needs to draw particular diagrams, attach symbols to points, construct deductive arguments according to these diagrams and symbols, and finally examine whether the arguments can demonstrate truth of the given general proposition. In the final phase, it is necessary to see each diagram as "a representative special case" (Polya, 1954) of the general proposition (on this dual nature of diagrams, see Tsujiyama, 2010).

In contrast, in many cases in school geometry, students are given proof problems to which diagrams and symbols are already attached. Mathematics educators in Japan have discussed the educational values of such problems. They argue that the attached diagrams bring some diversity in interpretations of such problems and that the diversity can give students an opportunity to engage in productive inquiry. In particular, Shimizu (1981) discusses that, after students solve such problems, it is important for the students to further inquire "of what (mathematical) relations the given diagram is a representative special case" (Shimizu, 1981, p.36) by utilizing already-obtained proof. He illustrates such inquiry by using a specific kind of proof problems with diagrams as below.

Fig. 1 shows an example of proof problems with diagrams. We consider a case in which one completes the proof of the statement by showing $\Delta BPA \equiv \Delta AQC$, $BP = AQ$ and $AP = CQ$. One might have a doubt on meaning of "in the diagram" or "a straight line l through point A" in the statement, and this doubt will bring an occasion to inquire what may happen if one changes the position of the straight line l. This inquiry leads to encounter non-examples and counterexamples such as the diagrams shown in Fig. 2; line BP does not exist in Fig. 2-a (but, $BP + CQ = PQ$ in a sense), and $BP + CQ \neq PQ$ in Fig. 2-b and c. These examples disclose an implicit assumption in the statement in that "in the diagram" means a case where the straight line l and triangle ABC share only point A. On the other hand, if one observes Fig. 2-b and c with the already-obtained proof, one will find that $|BP - CQ| = PQ$ holds in the diagrams because $\Delta BPA \equiv \Delta AQC$, $BP = AQ$ and $AP = CQ$ are still true. Furthermore, it is possible to organize the relations among segments PQ, BP and CQ in the overall cases in terms of "sum" and "difference" (Shimizu, 1981, pp. 33-36).

In the diagram, angle A is a right angle and AB = AC. To a straight line l through point A, we draw perpendicular lines BP and CQ from points B and C respectively. Then prove that BP + CQ = PQ

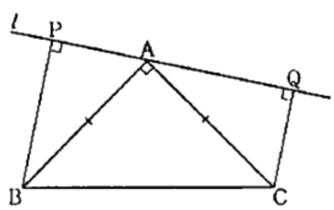


Fig. 1: Proof problem with diagram (Shimizu, 1981, p. 30)

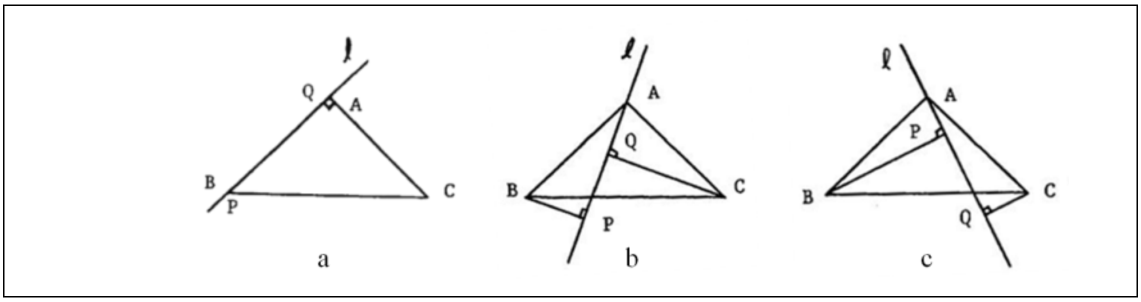


Fig. 2: A non-example and counterexamples of the problem (*ibid.*, pp. 34-35)

This illustration shows the nature of proof problems with diagrams that the attached diagrams sometimes make obscure the domains of the statements because the diagrams may include some implicit assumptions. This feature enables us to practice processes of proofs and refutations which Lakatos advocated, for example, to find counterexamples and non-examples through changing the attached diagrams, to restrict the domains of the statements by articulating the hidden assumptions, and to investigate new conjectures which hold for the counterexamples and non-examples. But, students will need some support from teachers in order to practice these activities successfully, and in next section we elaborate task design principles which would facilitate students' activities.

Principles of task design for proofs and refutations: Proof problem with diagram

According to both deductive guessing in Lakatos's research and the nature of proof problems with diagrams, this study derives three principles of task design which aim to achieve an authentic mathematical learning based on proofs and refutations:

1) Educators and teachers should select or develop a certain kind of proof problems with diagrams where students can find counterexamples or non-examples and engage in deductive guessing through changing the attached diagrams.
2) Teachers should promote their students to change the attached diagrams with keeping the conditions of the statements so that the students can find counterexamples or non-examples of the statements.
3) After students face the counterexamples or non-examples, teachers should plan their instructional guidance with which students can utilize their proofs of initial problems to invent more general statements that hold true for these examples.

The first principle is about selection or development of problems, and it seems to be quite an obvious principle from the intention of this study. Nevertheless, all of proof problems with diagrams do not allow us to find counterexamples or non-examples and engage in deductive guessing by changing the diagrams. Thus, it is necessary for mathematics educators and teachers to select or develop a certain kind of proof problems with diagrams that enables students to practice such activities.

The second is about teachers' instructional guidance for discovery of non-examples and counterexamples. Before deductive guessing, students need to confront with counterexamples or non-examples of the statements, and as shown in Fig. 2, changing diagrams attached to proof problems with keeping conditions of statements leads to finding such examples. However, it would be difficult to expect that students change the attached diagrams spontaneously, because it seems that the diagrams are typically given by teachers or textbooks in ordinary classrooms, and that there are few opportunities where students vary the shapes or places of the diagrams (Herbst &

Brach, 2006). Hence, teachers need to encourage their students to vary the attached diagrams so that the students can find counterexamples or non-examples.

The third is about teachers' instructional guidance for deductive guessing, which indicates to invent deductively more general conjectures which hold for counterexamples or non-examples. Since deductive guessing is usually unfamiliar to students, more active intervention by teachers would be necessary. In fact, Komatsu (2010) analyzed an experiment in which a pair of fifth graders was able to perform deductive guessing successfully, and their success was largely dependent on more specific guidance by the experimenter. Moreover, this guidance by the experimenter was intended to prompt the students to reflect their proof of their primitive conjecture. This experiment illustrates that the proofs of initial problems may become an aid for deductive guessing, and that teachers' guidance would be necessary for students to utilize the proofs. Thus, teachers should plan their instructional guidance with which students can utilize their proofs of initial problems to engage in deductive guessing.

This study has so far developed several proof problems with diagrams according to these principles, and in next section we illustrate these principles with a familiar problem about parallelograms.

An illustration of task design principles

Fig. 3 is another example of proof problems with diagrams, and it has appeared in many textbooks and has been discussed by some researchers (e.g. Chino, 2005; Larson et al., 2007; Okamoto et al., 2012). The reason why we select this problem as an illustration lies in the first design principle of this study. That is, as shown later, this problem allows students to find non-examples of the statement and practice deductive guessing by changing the shape of parallelogram ABCD. This paper re-examines this familiar problem from the task design principles based on processes of proofs and refutations.

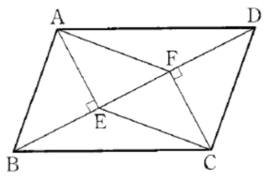
In parallelogram ABCD, we draw perpendicular lines AE and CF to diagonal BD from points A and C respectively. Then prove that quadrilateral AECF is a parallelogram	

Fig. 3: Proof problem about parallelogram (Okamoto et al, 2012, p. 133)

The task will range across two lessons; an initial problem and its proof in the first lesson, non-examples and deductive guessing in the second. Students' actions described in the following are hypothetical, but the hypothetical process is based on an informal pilot study which the authors conducted with undergraduate students.

In order to prove the problem, students would firstly use the fact that angles AEB and CFD are right angles (the hypothesis of the problem), and refer to AB = CD and ∠ABE = ∠CDF (properties of parallelogram). Next, they could deduce ABE CDF because the hypotenuses and acute angles of the two right triangles are congruent with each other. They then could infer that segments AE and CF are equal in length (a property of congruent triangles) and parallel (equality of alternate angles, ∠AEF = ∠CFE), and therefore quadrilateral AECF becomes a parallelogram from one of the conditions for parallelograms.

According to the second design principle of this study, teachers should prompt their students to change the shape of parallelogram ABCD. First of all, students would consider particular parallelograms such as rectangles, rhombuses and squares. Among them, quadrilateral AECF does not exist when quadrilateral ABCD is a rhombus or square, because points E and F coincide with each other in these cases (Fig. 4-a and b). Moreover, students may slant parallelogram ABCD to the left side, and in this case perpendicular lines from point A or C do not intersect with diagonal BD (Fig. 4-c).

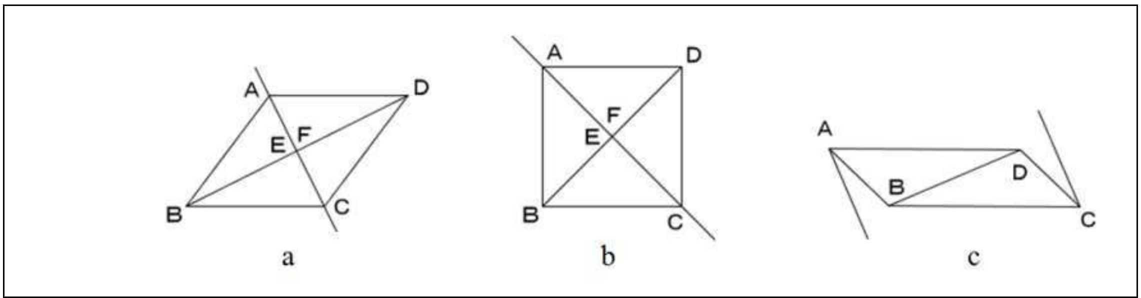


Fig. 4: Non-examples of the problem

The diagrams in Fig. 4 are non-examples rather than counterexamples, because it is assumed in the problem in Fig. 3 that quadrilateral AECF exists or perpendicular lines from point A or C intersect with diagonal BD. However, students may recognize these diagrams as counterexamples. They may also feel that the statement has to be modified by articulating the above assumptions, because these assumptions are embedded implicitly in the problem sentences. Therefore, providing students with opportunities where they can alter the attached diagrams could promote them to find out various non-examples or counterexamples and to become conscious of hidden assumptions of the problems.

The third design principle of this study requires teachers to help their students examine whether they are able to produce more general statements that hold true even for the case of Fig. 4-c. Students may extend diagonal BD so that the perpendicular lines from point A or C intersect with line BD (Fig. 5-a), and investigate whether quadrilateral AECF becomes a particular type or not (Fig. 5-b). In particular, if students draw diagrams by hand, it seems difficult for them to judge the type of quadrilateral AECF from the diagrams because the diagrams would not be clear. Then, it is crucial for teachers to help their students think about whether they can apply their previous proof to this case as well and, if not, how they should vary their proof. In fact, by adjusting slightly the parts in which $\angle$ ABE = $\angle$ CDF and AE // CF were deduced, it is possible to conclude deductively that quadrilateral AECF in Fig. 5-b also becomes a parallelogram. Through this problem and teachers' guidance, students would become able to engage in deductive guessing where they invent more general statements that hold true even for the previous non-examples.

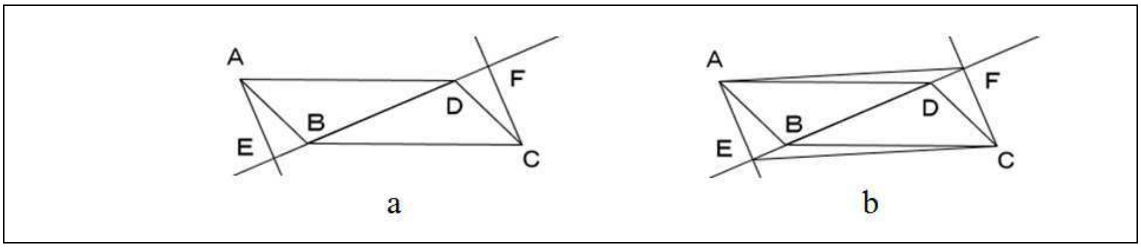


Fig. 5: Deductive guessing about parallelogram

Concluding remarks

This paper elaborated three task design principles for proof problems with diagrams by which students could experience an authentic mathematical process of proofs and refutations, and illustrated these principles with a familiar problem on parallelogram. Here, we refer to implications of this paper on teachers' community, keeping in mind the context of our study. In Japan, as stated earlier, proof problems with diagrams are standard for students and teachers, and learning activities where students inquire into domains of statements by changing the attached diagrams has been already discussed (e.g. Shimizu, 1981) and also may have been practiced in some classrooms. This paper can provide teachers with theoretical supports that show the mathematical authenticity of their practices according to a Lakatosian perspective. In addition, teachers can utilize the principles of this study as a guideline for their task design including necessary instructional guidance when they use proof problems with diagrams to attain mathematical learning based on processes of proofs and refutations.

This paper still remains only a theoretical consideration by a single community of researchers. In order to develop a set of appropriate tasks for regular classrooms, the team of the authors and teachers will later scrutinize the theoretical consideration from practical perspectives. For example, this study plans to re-examine the design principles and tasks shown in this paper from teachers' abundant experiences about teaching and learning of proofs. It is also necessary to enact the tasks in classrooms, and to refine these principles and tasks from results of the enactment.

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The "Language and argumentation" project: researchers and teachers collaborating in task design

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This contribution illustrates the "Language and argumentation" project, carried out since 2008 by the Mathematics Department of the University of Genoa. The project is aimed at designing, experimenting and refining task sequences for a smooth and meaningful approach to proof in lower secondary school. Two examples illustrate the way of working of the team (the cycles of experimentation and refinement) and some special tasks explicitly aimed at promoting students' reflection on processes and products.

Keywords: language, argumentation, proof, low-secondary school, sequence of tasks, students' processes, cycles of experimentation

Introduction

This contribution presents the "Language and argumentation" Project (lower secondary school strand), carried out by the Mathematics Department of the University of Genoa since 2008. The aim of the paper is to describe the structure of the Project team, the theoretical assumptions underlying the task design activity carried out by the team, the special tasks that were created and the way they were progressively refined throughout cycles of experimentation.

The contribution aims at addressing the following questions, as presented in the ICMI Study 22 Discussion Document (Theme D): *If you identify yourself as a member of a design group that cuts across communities, which ones are they? How did this cross-community come to be formed? When you or your group engages in designing tasks, what are you trying to achieve? What are your primary considerations? Which theoretical, mathematical, pedagogical, technological, cultural, and/or practical aspects are taken into account when designing a task or a task sequence? Are the designed tasks subject to revision in later cycles of the work? If so, what is it that specifically leads to the redesign? On what basis and according to which principles is the redesign carried out?*

In reference to the Discussion Document, we take tasks as the mediating tool between teaching and learning, and we illustrate the way in which, in our project, tasks are used in order to achieve specific educational goals (fostering the approach to argumentation and proof).

Background: proof and task design

Scholars agree on the fact that teachers should set up proper actions so as to arouse students' *need* for proof and proving (Zaslavsky et al., 2012), and also point out that teachers need preparation and support in doing that. Indeed, teachers face many challenges when dealing with proof in the classroom (Lin et al., 2012b): they must establish suitable socio-mathematical norms, choose and manage the good tasks, or even create their own tasks, guide the students towards deductive thinking without turning proving into a "ritual" activity. Also teachers' beliefs about proof and its role in the teaching and learning of mathematics play a key role in influencing the effectiveness of the teaching of proof (Furinghetti & Morselli, 2011). For instance, many teachers seem to believe that geometry is the most suitable domain to teach proof and that proof is an advanced mathematics issue, to be taught only in secondary school.

In the literature we can found many examples of task sequences that were created with the explicit aim of improving the teaching and learning of mathematical proof (Stylianides, 2007). Lin et al. (2012a) list a series of principles for task design for conjecturing, proving and the transition between conjecture and proof. Concerning conjecturing, we mention the importance of providing an opportunity to engage in observation, construction, and reflection. Concerning proving, the authors point out the importance of promoting the expression of arguments in different modes of argument representation (verbal arguments, symbolic notations etc.), asking the students to create and share their own proofs and to evaluate proofs produced by the teacher (thus "changing the roles"). Finally, concerning the transition from conjecture to proof, the authors suggest that the teacher should establish "*social norms that guide the acceptance or rejection of participants' mathematical arguments*" (p. 317).

This contribution will illustrate the way the team of the "Language and argumentation" project designed and experimented task sequences aimed at arousing students' "need for proof" in lower secondary school.

The "Language and argumentation" project

In 2004 the Italian Ministry for Instruction, University and Research (MIUR) founded the national Project "Lauree Scientifiche" ("Scientific degrees") (PLS in the following), whose aim was fostering the enrolment in university courses with scientific orientation, stimulating young people's interest in studying sciences and providing a better education in the base sciences. The project had several strands, going from special interventions for "high-achieving students" to pre-university orientation programs. Among them, the so-called PLS Laboratories, that is to say special lessons, performed in the school environment through a collaborative work between university researchers and school teachers.

Within this framework, in 2008 the Mathematics Department of the University of Genoa started the "Language and argumentation " project, a special case of "PLS Laboratory" aimed at designing and experimenting task sequences with a special focus on argumentation and proof.

Three main features characterize the "Language and argumentation" project: 1) task design is a central part of the collaboration between university and school; 2) argumentation and mathematical proof are the core of the task sequences; 3) teachers of different school levels (and not only higher secondary school) are involved, since the project members share the belief that argumentative competence should be developed in a long-term perspective, starting from the very first years of school and throughout all the school levels.

For each school level a team (university + school) was created. The different teams met regularly in order to share theoretical references on argumentation and take advantage of the exchanges and discussions. This contribution specifically refers to the work of the lower secondary school team. The next section illustrates the organization of the teamwork and the theoretical tools that were shared within the team and that helped to perform the didactical and methodological choices.

Task design: the team

Structure of the lower-secondary school team

The lower-secondary school team is currently made up of 7 members: the author FM, a researcher in mathematics education, three teachers with at least 10 years of teaching experience (two of them, MT and EZ, have a university degree in mathematics, one, EQ, has a university degree in chemistry), two teachers with less than 10 years of teaching experience (one, EP, has a degree in mathematics, the other one, GA, has a degree in biology), one retired teacher, AS, with a long experience in collaborative research in mathematics education. The teachers entered the project voluntarily and their participation was strongly supported by the school head. All the teachers (except for AS, retired, and GA, who changed the school after the first year but kept the work in the team) work in the same school.

The team was born in 2008-09: this means that the team just finished its fourth year of work. The first two years were dedicated to the development of a common frame (both in terms of theoretical references and of didactical methodologies). Starting from the third year, task design became a crucial activity for the team. This contribution will especially deal with this part of the project. We also point out that the teachers could design and experiment more than one task sequence for the same group of students[51], throughout the school years from 2008-09 to 2011-12. This means that a sort of mini-curriculum with a focus on argumentation and proof was created, and that students could experience more than one task sequence.

The way of working of the team

The author organized all the team meetings (one scheduled meeting per month, starting from November and until June) and acted as an observer during the class sessions. She also made video recordings of the sessions and collected all the students' written productions. Besides the team meetings, she had individual meetings with each teacher, before and after the class experimentation.

The way of working may be synthesized as it follows: during a preliminary meeting, the researcher proposes the theme of the task sequence and sketches a first draft of the core task. The core task is discussed and the teachers, together with the researcher, set up the sequence of tasks, with a special care in the sequencing of tasks. Afterwards, a first experimentation is carried out. Teacher and researcher perform the analysis of the experimentation immediately after the experimentation; the whole team performs an additional analysis during regular meetings. The analysis may lead to the refinement of the task sequence and, thus, to a new experimentation. Two modalities of experimentation were tested: parallel experimentations of the same sequence, and sequential experimentations. In the first modality, two teachers realized

[51] In Italy, lower secondary school is made up of three years. Usually the teacher teaches the same group of students throughout all the three years.

the task sequence in their classes, almost in the same period. Regular meetings during and after the experimentation allowed a continuous exchange between the two experiences. In particular, students' processes were compared and the actual development of the task sequence in the two classes was analysed and discussed by the whole team. In the second modality, a first experimentation was carried out in one class, afterwards the whole team discussed the way the experimentation was carried out. Possible modifications to the task sequence were discussed, thus leading to a modified sequence to be experimented. In this way, a cycle of planning-experimenting-analyzing-modifying-testing the modified sequence was realized.

In both types of experimentation, the degree of variability left to each teacher is quite high, provided that his/her choices are discussed a priori or analysed a posteriori by the whole team.

In Italy teachers are often involved into research in mathematics education, under the Italian paradigm of the "Research for innovation" (Arzarello & Bartolini Bussi, 1998). Within this paradigm, teachers (who are called "teachers-researchers") collaborate with the researchers in the planning and analysis of the teaching experiments, and theoretical reflections and teaching experiments are performed dialectically, so that the analysis of the teaching experiments may lead to the evolution of the theoretical framework itself. In our case, the teachers were at their first experience of collaboration with researchers. We may say that the "Language and argumentation" project had also the final aim of fostering the professional growth of a new generation of teachers-researchers. Indeed, during the project the teachers did not only receive and implement in their classes the innovative task sequences, rather they were involved in theoretical reflection and a posteriori analysis.

Task design: principles and didactical choices

As regards the level of low secondary school, two educational goals are to be attained: from one side, fostering the development of argumentative and linguistic competences (thus, seeing argumentation as strictly linked to proof, see Durand-Guerrier et al., 2012), from the other side, promoting the first encounter with mathematical proof.

Stylianides (2007) proposes the following definition of proof that can be applied in the context of a classroom community at a given time:

> "*Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: it uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification; it employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; an it is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community". (Stylianides, 2007, p. 291).

Accordingly, the team believes that a smooth and meaningful approach to proof requires the students' progressive acquisition of basic content knowledge, but also the ability to manage (from a logical and linguistic point of view) the reasoning steps and their enchaining (modes of argumentation) and the ability to communicate the arguments in an understandable way. It is important to develop a sort of "argumentative attitude", that is to say being aware of the fact that each choice, opinion, affirmation should be justified by means of a discourse that must be understood and accepted by peers. This is also in line with the idea that learning proof is approaching a form of rationality, as expressed by Morselli & Boero (2009), who proposed an adaptation of Habermas' construct of rationality to the special case of proving, showing that the discursive practice of proving may be seen as made up of three interrelated components:

> "- an epistemic aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning […];
> - a teleological aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product;
> - a communicative aspect: the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture". (Morselli & Boero, 2009, p. 100)

Starting from the theoretical assumptions that were previously sketched, a list of methodological principles were derived and specific task design principles followed. It is important to underline that argumentation is a major educational goal, but also a means to achieve other educational goals, i.e. a better understanding of specific contents: argumentation is the *goal* and also the *means*. Hence, the task sequences are conceived with argumentation as a pervasive activity. We may distinguish between a core task, setting the problem to be worked on (a property to be discovered and justified), and the further tasks. Core tasks are usually proposed as open-ended questions (*What can you tell about…?*) where, according to the socio-mathematical norms of the class, each answer must be justified. The further tasks are conceived so as to foster students' awareness of the epistemic (*this is true because…*), teleological (*I have this goal…*) and communicative (*how could I communicate it in a proper way*?) requirements inherent in the conjecturing and proving process. More specifically, tasks encompass: formulation of conjectures; comparison between different conjectures; justification of conjectures; comparison between individual processes and between individual final products. Didactical methodologies such as group work and mathematical discussions (Bartolini Bussi, 1996) are widely used. The team also explored the importance of having students analyse students' written individual solutions, as it is advocated within the theoretical framework of the fields of experience didactics (Boero & Douek, 2008). We point out that the methodological choices are also in line with the principles listed by Lin et al. (2012a).

In this way, two types of argumentation are fostered: argumentation at content level, as a part of the proving process, and argumentation at meta-level, as a means for fostering reflection on the practices of mathematical proof related to the three components of rationality. Within the task design process, a crucial goal was to create occasions for meta-level argumentations aimed at promoting students' awareness of the epistemic, teleological and communicative requirements of proving. To this aim, specific tasks were created. Some examples are illustrated in the subsequent section.

Some examples

Example 1 – Isoperimetric rectangles

The task sequence "*Isoperimetric rectangles*" was conceived for grade 7 (age of the students: 13-14) and encompassed about 20 hours. The core activity is the conjecture and explanation of the fact that, among all the rectangles with fixed perimeter, the square has the maximum area. That is the task sequence that underwent the most evident changes and refinements throughout the years of experimentation. The first version of the task sequence started with an explorative task in paper and pencil

(“*Draw four rectangles with a perimeter of 20 cm. What can you tell about the areas of the rectangles?*”), followed by a more direct question concerning the maximum area. The students conjectured that the square is the rectangle with maximum area and in some cases they even tried to provide some numeric justifications. Afterwards, the conjecture was proved by means of algebra, under the guidance of the teacher. The starting point of the proof was drawing a generic rectangle and a square (with the same perimeter) and superposing them (see figure 1). Once observed that the rectangle HGDA is part of both the rectangle and the square, the areas of the two figures GFDE and BCGH were expressed algebraically in order to show that the area of the square is greater than that of the rectangle.

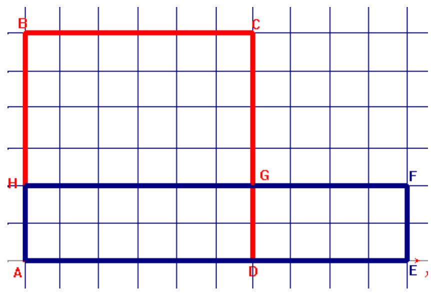

Figure 1 – Isoperimetric rectangles: the starting point for algebraic proof

During the first experimentation, one student, looking at the two superposed figures (see figure 1), raised her hand and proposed to “cut” the rectangle GFDE and place it over the rectangle BCGH, so as to “see” that one surface is bigger than the other one. This fact suggested to the teachers of the team that giving students some cardboard to manipulate (and even “cut”) could be another step before algebraic proof. A crucial point was the place (and role) of the “cardboard” part with respect to the conjecture about the maximum area: was it to be proposed after the drawing phase (within the conjecturing phase), or after the conjecture of the property of the square (so as to promote the idea of superposing figures and pave the way to the algebraic proof)? Further cycles of experimentation allowed for exploring the issue, shedding light on the different roles that the “cardboard phase” may have within the sequence. During the third experimentation, there was also a very rich discussion on the way of drawing rectangles with a given perimeter (*how much are they; how it is possible to create anther rectangle from a given one; is it always possible to draw “couples” of rectangles, that differ only from a rotation of 90 degrees*). This suggested inserting (*in the same experimentation*, after the ingoing analysis of the session) another task of individual reflection, so as to foster the connection between geometrical facts and numerical properties. . The students were asked to write down their reflections about three questions: “*Why is it that adding and subtracting one unit (to the sides) the perimeter doesn’t change? Is it the same if we add and subtract two units? Why is it that changing the basis with the height the perimeter doesn’t change?*”.

At the end of the sequence, there was a “balance” task, conceived in order to foster students’ reflection on the value (and limit) of different approaches: drawing on paper, cutting cardboards, using algebra. A key point is that each approach offers a different perspective on the problem, helps to grasp some aspects and contributes to the whole comprehension. The students were asked to answer individually the following questions:

> Going back to the previous tasks, you may note that we tackled the problem of isoperimetric rectangles by means of different approaches: drawing rectangles in paper & pencil, cutting rectangles on cardboard, using letters.

> What may you say of the different approaches?
> Did the different approaches allow you to understand the same things?
> Were they equally easy to understand?

The "balance" task was aimed at promoting the argumentation at a meta-level on the potentialities and limits of each approach (for instance, drawing may only help the conjecturing phase, but it is not a real "proof"), and on the power of algebra as a proving tool. It encouraged the reflection not only on the correctness (epistemic rationality), but also on the comprehensibility (communicative rationality) and usefulness in relation to the final goal of proving (teleological rationality). Furthermore, the task gave the teachers data for an *a posteriori* evaluation of the task sequence.

Example 2: sum of consecutive numbers

The task sequence "*Sum of consecutive numbers*", conceived for grade 7, encompassed exploration, conjecturing and proving in elementary number theory. The whole sequence lasted about 10 hours. The students were proposed a first task ("*What can you tell about the sum of three consecutive numbers?*"). They worked in small groups and shared and compared the group solutions within a mathematical discussion. Afterwards, the students were given three connected tasks to be solved individually: "*What can you tell about the sum of two consecutive numbers? What can you tell about the sum of four consecutive numbers? What can you tell about the sum of five consecutive numbers?*". As usual for the norms of the class, each answer was accompanied by a justification. The teacher and the researcher analysed all the individual productions and for each task (sum of 2, 4 and 5 numbers respectively) selected three productions to be compared and commented by the students themselves, according to the following task:

> Read the following answers provided by some of your classmates. Compare them and write your reflections. What about the properties they found? What about the explanations they provided?

A mathematical discussion followed. This task fostered a reflection on two connected issues: the truth and comprehensibility of the conjectures, and the validity and comprehensibility of the related explanations (epistemic and communicative components). Students could reflect on the value of numeric examples (for discovery of the conjecture and communication of the property: teleological and communicative component), but also on their limits for justification (epistemic and teleological component). They could also compare justifications in natural language with justifications in algebraic language. In this ways, an argumentation at a meta-level was promoted.

In our opinion this task is an application of the principles listed by Lin et al. (2012) concerning having students produce their own justifications and evaluate justifications presented by others. Furthermore, this task paves the way to the concept of proof as presented by Stylianides (2007) and brings to the fore the importance of three dimensions of rationality, communicative included. We also point out that creating new tasks of reflection starting from students' own productions makes the task sequence very "dynamic", since each implementation must take into account new students' productions.

Discussion and conclusions

The short examples that were described in the previous session show how the model of rationality guided the team in the task design process. The core tasks, aimed at introducing the mathematical content (a property to be discovered and justified), are accompanied by further tasks, aimed at fostering the reflection on the proving process as a rational activity. For instance, the "balance" task (example 1) was aimed at bringing to the fore the role of different methods at epistemic, teleological and communicative level. The "comparison" task (example 2) was aimed at making students reflect on the fact that the same conjecture and proof may be presented in different ways (communicative component) and that different justifications are possible (epistemic and teleological component).

Example 1 also illustrates the cycles of design, experimentation, analysis and refinement that characterize the teamwork. One key feature is that the task sequences are always under refinement. The team analysis may lead to a change in the task formulation or in the sequencing of the tasks.

Furthermore, as evidenced in example 1, additional tasks, especially tasks fostering reflection, may be inserted. Each teacher may suggest modifications to the task sequence.

Finally, it is worth noting that some tasks are "open" and must be set up during the experimentation. Any task sequence cannot be completely set up a priori, because it depends on the students' processes and products. The teacher, with the cooperation of the team, must be able to evaluate *"on the spot"* the emergence of issues to be deepened (as in example 1), and to analyse students' products and promote students' own reflection on productions (as in example 2).

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Designing and using exploratory tasks

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This paper addresses the role of exploratory tasks in mathematics learning. We present the rationale and the main characteristics of these tasks and provide three examples to show how they may support students' reasoning at different school levels. We argue that these tasks, requiring students to model situations and to design their own strategies, drawing on previous mathematical and nonmathematical knowledge, have an important potential for mathematics learning.

Keywords: Exploratory tasks, Generalization, Justification

Introduction

In Portugal, the new mathematics curriculum for basic education (Ponte, Serrazina, Guimarães, Breda, Guimarães, Martins, Menezes, Oliveira, & Sousa, 2007) indicates that teachers must use a variety of tasks in the classroom, stressing in a particular way the value of exploratory tasks. This promoted an important movement of study around such tasks and how to use them in the classroom. At the Institute of Education of the University of Lisbon several teaching experiments have been conducted to suggest ways of teaching specific topics of this new curriculum using exploratory tasks as well as to investigate the possibilities of such tasks for other school levels, with a special attention in the development of students' mathematical reasoning (e.g., Azevedo, 2009; Branco, 2008; Henriques, 2011; Quaresma, 2010). In addition, several collections of tasks, most of which are exploratory, together with teachers' supporting materials, have been designed and made available in the internet (e.g., Ponte, Matos, & Branco, 2009; Ponte, Oliveira, & Candeias, 2009; Ponte, Silvestre, Garcia, & Costa, 2010).

Worthwhile mathematics tasks are central in mathematics teaching (NCTM, 1991). However, tasks appropriate for some teaching purpose may not be for another. In addition, tasks need to be suitable for the students to whom they are proposed. Thus, the teacher needs to decide the purposes regarding students' learning and, taking into account the working conditions of the school, choose suitable tasks for his/her students, regarding mathematical challenge (Potari & Jaworski, 2002), underlying context (Skovsmose, 2001), task structure, classroom organization and time required (Ponte, 2005). Classrooms in which the students do some extended work on tasks, in an autonomous way, solving problems, modelling situations and devising strategies to solve the questions proposed are becoming more common in our country. In some cases, the questions require considerable interpretation and even some reframing to be tackled in a mathematical way. After this initial work, often done in pairs or in small groups, during which students record their representations and solutions, they are asked to present their strategies and solutions to the whole class, to justify them, and to discuss the work of other students. This view of the mathematics classroom fits with what some documents refer to as "reform mathematics teaching" (e.g., NCTM, 2000).

The teaching experiments that we carry out are framed as design research (Cobb, Confrey, diSessa, Lehrer, & Schaube, 2003), with specific hypothesis relating exploratory tasks, use of different representations, and a mode of classroom work that provides opportunities for students' autonomous work and whole class discussions. Data is collected by observing classrooms (usually with video recording), gathering students' written productions, and doing clinical interviews with individual students (see further information on these experiments in the on-line appendix: http://www.ie.ul.pt/portal/page?_pageid=406,1590938&_dad=portal&_schema=PORTAL).

The present paper illustrates this work. Our aim, as a team that cut across diverse communities, is to discuss the characteristics of exploratory tasks, to indicate how they may be used in the classroom, their potential to promote students' learning, their feasibility for regular classroom use, and the critical features that make them suitable for mathematics learning. We begin by presenting the institutional context, definitions and research and design principles of our work and base our argument with snapshot of three examples at different school levels.

Context, definitions and principles

Role of authors, institutional, systemic, and resourcing context of the work

Three of the authors of this paper are classroom teachers involved in research who conducted the implementation of teaching units (Quaresma, grades 5-6, Mata-Pereira, grades 7-9, Henriques, 2nd year university). The first author (Ponte) has mostly a research profile with the role of supervision of the design process. All the four authors participate in the design of tasks and of its classroom use. Usually, the first idea for a task is provided by the classroom teacher and the subsequent refinement is done in interaction with the first author and with other researchers at the Institute who also occasionally discuss particular tasks and transversal issues. This developmental work has been conducted at Institute of Education of the University of Lisbon, within teaching experiments carried out as part of academic degrees. It is based in the close collaboration of two different kinds of expertise – mathematics education research expertise and classroom teaching expertise – that is shared in different levels by all authors. The resourcing (printing copies, audio-visual material for recording classes, etc.) is provided by the Institute, sometimes with external grants or contracts, and also by the schools of the teachers involved in the experimentation.

Mathematical and epistemological perspectives

We regard mathematics as an activity in which students solve problems and produce and justify mathematics statements. In this activity students may use mathematical representations, draw on mathematical concepts, known facts, properties and procedures to design strategies, implement them, and arrive to conclusions. Such strategies include moments of problem posing, conjecturing, making generalizations, testing conjectures (analysing cases, doing computations, experimenting with technological tools, etc.), making connections and establishing relationships. An important part of the mathematical activity is also explaining and justifying solutions and statements. Our epistemological perspective is that mathematics knowledge develops by the process of carrying out mathematical activity and reflecting on such mathematical activity (i.e., carrying out meta-mathematical activity) which includes problem solving and systematization of ideas that is achieved by individual work and social interaction (with colleagues and with the teacher).

Tasks and teaching units

Following Christiansen and Walther (1986), we take a *task* as a statement of a situation to be solved by the students. It is presented to students both in written and oral form, together with indications about the time available to work on the task, the kind of students' production expected, and the mode of organization of the work suggested. A task may correspond to a single question or to a structured sequence of related questions (and possibly subquestions). In our work we strive to develop *exploratory tasks*, that is, tasks that may lead students to exploratory activity, from which they do substantial work and learn new mathematics. In exploratory activity students have to interpret situations in mathematical terms, formulate mathematics questions, and reason in an inductive way, making conjectures and generalizations. This is the basis for symbolizing and formalizing ideas and providing justifications to statements based on known facts and properties, assumed assumptions and definitions.

The key role of such tasks in mathematics teaching has been recognized by mathematics educators such as Sullivan, Bourke and Scott (1997), Boaler (1998), and Skovsmose (2001). We are aware that the kind of classroom work that we have in mind is quite demanding on the teacher. The selection of tasks involves a high level of understanding of the mathematics involved as well as in-depth knowledge about students' abilities and interests. Teachers must know how to introduce such tasks, negotiating the meanings that are critical for the work to carry out, but without providing too many clues that will solve the task for the students. Furthermore, the teachers are called to support the students' autonomous work, while maintaining the cognitive demand of the activity (Stein & Smith, 1998), and to be able to conduct productive discussions during which mathematics ideas are presented, confronted and clarified. In these discussions teachers must provide opportunities for all students to intervene, stimulating moments of controversy and argumentation as well as moments of systematization and formalization of mathematical ideas. In order to allow for a smooth integration in professional practices, we often produce tasks with parts or elements that are not so different from usual tasks, combining them with questions that points towards exploratory activity.

Design principles

Some tasks are intended to provide for the development of new forms of representation, new concepts, and new problem solving strategies. Others are intended to lead students to mobilize and clarify mathematical notions that they already learnt and/or to make connections among different ideas. The structure of a task may vary from seemly simple statements that require considerable interpretation and specification of cases by students to complex statements that include a breakdown of questions in a structured way. However, in this case, often there is a combination of different kinds of questions, beginning with quite straightforward and computational questions to end up with open questions that require significant thinking from the students. The tasks will be ultimately organized in a sequence to be undertaken in several classes in order to promote students' understanding of an important mathematical idea or topic.

Tasks need to be engaging for most (if possible, all) students, not very difficult to get involved in, and lead to the formulation of important mathematical notions (representations, concepts, properties, procedures). In order for the tasks to be engaging for a student they must have an element of challenge, without being too difficult. The tasks involve different kinds of context. Some of them just have a purely mathematical context (usually numbers or geometric figures) whereas others are close to the daily experience of students. However, contexts must not be artificial, must not pose problems for students' understanding, and, of course, must not promote prejudices or stereotypes. It is assumed that, after tasks are introduced, students will spend an extended time working on them, in an autonomous way, and afterwards there is another extended period of whole-class discussion.

In our work, we start with an overall planning of the teaching unit, which includes the formulation of the learning objectives, assumed previous knowledge of students (often based on diagnostic assessment), time available and organization of the schedule (as there are variations among schools). Tasks are later selected to fit the overall planning of the teaching unit, and then there is a dialectic movement of adjustment between the macro level of the unit and the specific level of the task. Next, we present three tasks constructed according to these design principles (See a summary in Table 1).

Design principles	Folding and folding again	Multiplication of integers	Working with intervals
1. The objectives support the development of new representations, concepts, and strategies, lead students to mobilize and clarify mathematical notions, and/or to make connections.	i. To understand and use rational numbers in different meanings. ii. To recognize that they may have different representations, and understand the associated language.	i. To develop new knowledge regarding integers (definition of multiplication). ii. To develop the ability to generalize.	i. To learn concepts and procedures of interval arithmetic (definitions of operations and image of a function). ii. To develop the ability to generalize and to justify.
2. The structure varies from simple to complex statements with different questions (straightforward, computational, open…).	Some questions are quite simple and others require interpreting complex commands ("represent", "compare").	Some questions are simple and others lead to connections and reasoning (generalization and justification).	Open questions, requiring reasoning processes (formulation, testing, generalization and justification).
3. Engaging with an element of challenge.	Challenge in interpreting "represent" and "compare".	Challenge in question 4, especially 4c).	Challenge in both questions.
4. Different kinds of context (purely mathematical and close to the students' experience).	Use of manipulative materials (paper strips).	Mathematical context.	Mathematical context.

Table 1 – Application of design principles to the production of tasks

Example 1: "Folding and folding again"

This exploratory task is included in the supporting materials of the mathematics curriculum (Menezes, Robinson, Tavares & Gomes, 2008). It was proposed in the first class devoted to the study of rational numbers in a teaching experiment at grade 5 focused on comparing and ordering rational numbers and equivalent fractions (Quaresma, 2010). The class, with 22 students, enjoys working in new problems and maintains a productive pace of work. All students show commitment regarding the mathematics work, most of them have good achievement but a smaller group has difficulties in this subject. The task, with two questions, intends to lead students to (i) understand and use rational numbers in the part-whole and measurement meanings, (ii) recognize that simple non-negative rational numbers may represented in the form of fractions, decimals and percent, and (iii) recognize and use the language associated with rational numbers in different representations and meanings. An important aspect of the task is the use of manipulative materials (paper strips) and of written representations, creating many opportunities for oral interactions among the students and the teacher. Question 1 is quite explicit as it asks to do several folds, but then it makes a request to represent them that most students have difficulty in interpreting as an indication to use the decimal representation, favouring an exploratory activity. Question 2 asks students to compare different parts of the strips and to "draw conclusions", which is a quite open indication, that may lead to various solutions.

> **Task. 1.** Find three pieces of paper geometrically equal. Fold them into equal parts: **a)** the first in two parts; **b)** the second in four; and **c)** the third in eight parts. After folding each strip represent the parts that you got indifferent forms.
> **2.** Compare the three parts of the strips that you got. Record your conclusions.

We present some episodes from collective discussions after the students did some extended work on the task. The students work in 6 groups (3 or 4 elements) in an autonomous way, with discreet support of the teacher for about 20 minutes on each question, after which begins the whole class discussion. To support this discussion each group posts their work on the blackboard. Presenting the work of the first group, Diana says: "In [item] b) we wrote: fourth-part, 1 of 4 [1/4]; 1 divided by 4, 25% and 0.4." The other students do not realize the mistake and the teacher decides to return to question later and continues by asking the group of Tiago to present their work:

> *Tiago*: So we have: fourth-part, one of four [$\frac{1}{4}$], one divided by four, 25 and 0.25%.
> *Teacher*: (...) Do you agree Diana?
> *Diana*: Yes...?
> *Class*: No! That is wrong...
> *Teacher*: What is wrong?
> *Rui*: It's 0.25...
> *Teacher*: Why?
> *Rui*: Because it is the fourth part.
> *Daniel*: It is 0.25 because it is a half of the first. The first was 50, if we make half is 25.
> *André*: Oh teacher! I think it is 0.25 because it is the fourth part of 100. Because 25 times 4 gives 100.

The direct questioning of the teacher to Diana suggests that there is something that needs attention. This student becomes confused and her colleagues strive to clarify the situation. After the teacher question "why?", several students provide successively more refined explanations to indicate that the part shall not be represented by 0.4 but by 0.25. The style of questioning the teacher is punctuated by inquiry questions ("What is wrong?", "Why?" ...). The culture of the classroom includes the notion that students can contribute to different responses to disagree and argue with each other.

Working on this task led the students to understand the meaning of representing parts of the strips using different representations of rational numbers, developed their language associated to rational numbers and to compare various parts of the strips. The fact that the first part of question 1 ("folding") is quite explicit and uses manipulative materials, engaged students in the task, while the second part, more open ("representing") prompted students to carry out exploratory activity and to understand the meaning of "to represent". In question 2 (see examples in the appendix) students further established different multiplicative and fractional relationships among quantities.

Example 2: "Multiplication of integers"

This example is taken from a teaching experiment about integer numbers carried out in a grade 7 class with 20 students. This is an exploratory task aiming students to develop new knowledge about properties and concepts regarding integers, namely the definition of multiplication, based on previous knowledge of properties of natural numbers and inciting generalization. The task has four questions with a purely mathematic context. Some of these questions require simple procedures, while others lead students to establish connections and reasoning processes such as generalization and justification. The last question seeks to identify properties of the multiplication of integers, particularly regarding the sign of the product of two integers, based on the connection between multiplication and successive addition of integers:

> **Task. 1.** Find the result of the expressions 3+3+3+3+3+3+3+3+3+3 and 7+7+7+7+7. Explain how you found your answer.
> **2.** Try to formulate a rule that indicates the same result without using addition.
> **3.** Represent the expressions (-3)+(-3)+(-3) and (-4)+(-4)+(-4)+(-4)+(-4) using multiplication. Find the result of these expressions.
> **4.** Try to formulate a rule that indicates the sign of the product of two integers if they are: **a)** Both positive; **b)** One negative and one positive; **c)** Both negative.

The task was included in a set proposed to students on paper in a 90-minute class and discussed in the first 20 minutes of the next class. The students, who sat in pairs or groups of three, were asked to read all questions before starting to solve it. There was a small whole class discussion, widely participated in by students, supporting their interpretation of the task. As students were working, the teacher moved around the classroom, asking them questions to clarify or probe their answers. In the final whole class discussion, in order to construct common meanings, the students were responsible for presenting and justifying their responses and formulating questions regarding the responses of their colleagues.

The students solved questions 1, 2 and 3 with no difficulty. For the last question, most students obtained the intended rules, particularly as regards the sign of the product. Pedro, a student in this class, uses natural language in his written solution to present his generalizations and uses symbolic language (without variables) to present an example for each item. When questioned to justify his answers to items 4.a) and 4.b), he states:

> *Pedro:* The first one was easy, because we had already done it long ago, both positive. I wrote an example, four times four, sixteen. Therefore, if we multiply a positive with a positive would be positive.
> *Teacher:* And one positive and one negative?

Pedro: Negative. I made an example, but then I also thought in the sum. A negative with a positive would be negative. Therefore, minus three minus three would be minus six.

In this question the students justify their answers based on prior knowledge and mathematical properties used in the previous questions. This exploratory task led them to establish generalizations, developing knowledge about the properties of multiplication for integer numbers, based on known properties of natural numbers. As the students still do not know the algebraic language, they justify their responses exemplifying with particular cases. Some of the initial questions required some interpretation but the students easily engaged in the task since it essentially involved known properties or properties used in previous questions, as the relation between multiplication and successive addition of equal addends. Therefore, the structure of the task provided students with the possibility to begin with simple questions and end up dealing with highly demanding questions as question 4. (Other situations in appendix.)

Example 3: "Working with intervals"

This example is taken from another teaching experiment (Henriques, 2011), supported by exploratory tasks conducted during one semester in a numerical analysis course, in order to promote students' experience of doing mathematics and the learning of concepts and procedures. The participants were 36 second year university students. The task has two open questions in a pure mathematical context. Question 1 intends to lead students to deduce and justify the rules of interval arithmetic, which were unknown to them. Students were challenged to formulate, test and justify conjectures involving the exploration of particular cases of known elementary operations (addition, subtraction, multiplication and division) using intervals of real numbers. The second question has similar features and aims to extend the interval arithmetic to functions, requiring the use of prior knowledge on functions and their properties as well different representations to get a solution.

Task. 1. Look at the following situations
[1, 2] + [5, 7] = [6, 9]   [0, 1] + [-5, 2] = [-5, 3]   [-3, -1] + [1, 3] = [-2, 2]
a) Find a rule for adding real number intervals? Do all real number intervals follow that rule? Investigate.
b) Investigate what are the rules for subtraction (X–Y), multiplication (X×Y) and division (X/Y).
2. Consider a function f: $D \subset IR \rightarrow IR$, given by f (X) = X + X and X = $[x_1, x_2] \subset$ D, a real number interval of its domain.
a) What's the image of X = [2, 7]? Explain clearly how do you find your answer.
b) What's the image of a real number interval if now the function f is given by $f(X) = X^2$ or $f(X) = e^X$?

During the exploration of this task, which lasted about 100 minutes, the students worked in small groups of 3 or 4. Afterwards, they presented their work orally to the class, explaining their ideas and strategies, and, prompted by their colleagues and the teacher's questioning, sought for justifications. Outside the classroom, they also wrote a report (WR) aiming their reflection, since it requires students to articulate ideas, at explaining procedures and reviewing the processes used and the results obtained. The work of a student (Gonçalo) is chosen to illustrate a variety of aspects of students' mathematical activity and learning.

Gonçalo begins exploring the task by observing the examples. That observation raises in him some questions as he tries to find a pattern and to understand what was really happening. He is led to a first conjecture about the rule for adding real number intervals: "We concluded that by adding the lower bounds and the upper bounds of the intervals, based on examples" (WR). Later, in an interview, he explains the constructed meaning for the interval obtained from adding two other intervals:

> Before the beginning of this task, I had never seen an addition of intervals and I was really unaware of what it was. But then I realized that considering $[a_1, a_2]$ + $[b_1, b_2] = [c_1, c_2]$, any value between a_1 and a_2, added with any other value between b_1 and b_2, will be into the interval c. (E)

Gonçalo continues the exploration of the task formulating and testing conjectures about the rules for other elementary operations (subtraction, multiplication and division), leading him to further generalizations. In question 2, the student feels the need to use graphical representations to extend the interval arithmetic to functions (We show these aspects of the student reasoning in the appendix.).

In summary, the students were challenged to carry out exploratory tasks very different from the usual ones done at university level. This kind of task, in connection with the classroom work organization, led students to experience mathematics processes such as looking for regularities, posing questions, formulating and testing conjectures, generalizing and justifying them. Authentic mathematics activity was stimulated by exploratory tasks. This work also provided students opportunities to develop their understanding of the concept of interval, the rules of interval arithmetic and to make connections with other mathematical topics, in particular with previous knowledge about properties of numbers and functions. Thus, the results highlight the potential of exploratory tasks to learn numerical analysis concepts and procedures, suggesting that they may be used in university mathematics courses.

Conclusions

The three exploratory tasks reported in this paper demand considerable interpretation by the students and their involvement in mathematical work. The three tasks require students to formulate generalizations and justifications, in one way or another, involving them in mathematical activity. They also were effective in supporting students to achieve a variety of learning objectives related to rational numbers, integer numbers and interval arithmetic. The critical features of these tasks include a combination of structured elements with explicit indications with open questions that require interpretation and the design of strategies. Another critical aspect is the settings that enabled interaction among students and with the teacher in the role of a facilitator. However, the tasks proved to be feasible for a qualified teacher, requiring no special effort. The particular combination of authors in this work – given their complementary roles and their use of a common research framework – seems to be fruitful for the production and experimentation of this kind of tasks and their trial in the classroom. In addition, the experience in Portugal shows that commercial publishers draw on tasks produced in this kind of setting and introduce them in textbooks. Of course, being demanding of teachers' expertise such tasks require a combined effort of task design, classroom implementation studies, curriculum development, and teacher education as we had in Portugal in the last five years.

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Developing Theory for Design of Mathematical Task Sequences: Conceptual Learning as Abstraction

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This paper describes an emerging approach to the design of task sequences and the theory that undergirds it. The approach aims at promoting particular mathematical understandings. Central to this approach is the identification of available student activities from which students can abstract the intended ideas. The approach differs from approaches in which learning to solve the problem posed *is* the intended learning. The paper illustrates the approach through data from a teaching experiment on division of fractions.

Keywords: mathematical tasks, reflective abstraction, learning theory, didactical engineering

Introduction

The “Task Design in Mathematics Education Discussion Document” stated, “We would like to encourage an interest in tasks that have more limited but valid intentions, such as tasks that have a change in conceptual understanding as an aim.” That is the specific aim of this paper and the research program from which it derives.

The research and theoretical work build on several core ideas advanced by Piaget and therefore could be considered constructivist in orientation. As such the paper could be categorized, according to the Discussion Document, as delineating principles and frameworks for task design within a singular design community researchers working from a constructivist perspective). However, our view is that multiple theoretical perspectives are needed to do the work of design of mathematical tasks. Consistent with this view, we see our work as complementary to the work of many other research programs that work from a different core set of theoretical constructs. Further, in our work we use principles from Cobb and his colleagues’ (Cobb & Yackel, 1996) emergent perspective and Davydov and his colleagues’ Russian activity theory (Davydov, 1990). However, it is beyond the scope of this short paper to discuss our use of these different theories.

The emerging task design theory that I present here is narrowly focused on the learning of mathematical concepts and as such offers an approach to addressing difficult to learn concepts and to working with students who are struggling to learn specific concepts. This task design approach does *not* address other important areas of learning mathematics, particularly the important area of mathematical problem solving. Therefore, the approach is meant to complement existing approaches, not replace them. The emerging task design theory is a product of a research program aimed at understanding conceptual learning, particularly the development of abstractions from one's own mathematical activity (activity that occurs in the context of designed sequences of mathematical tasks). Thus our research program involves a spiral approach in which we design task sequences to study learning through student activity, and we use what we come to understand about learning to improve our understanding of task design, and so forth.

We work with a broad definition of task, consistent with the one embraced by the Study Group, as either a question that a student is asked or an objective that they are given to accomplish (e.g., Why is a triangle the only rigid polygon? Measure the length of line segment BC.).

Theoretical basis

The theoretical basis of our research program derives from Piaget's (2001) work on reflective abstraction. DiSessa & Cobb (2004) pointed out,

> Piaget's theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to do so deeply and effectively. (p. 81)

Our research program is aimed at building theory that can inform instructional design.

Two Piagetian-based constructs that are foundational to our work are goal-directed activity and reflection. *Goal-directed activity* includes both physical and mental activity. The notion of *goal-directed* is important, because the learners' goals partially determine both what knowledge they call upon and what learners pay attention to and can notice. *Reflection,* following von Glasersfeld (1995), refers to an innate tendency (often not conscious) to distinguish (associate) commonalities in one's experience.

In our research and theoretical work, consistent with Piaget and others, (c.f., Hershkowitz, Schwarz, & Dreyfus, 2001; Mitchelmore & White, 2008), we consider mathematics conceptual learning as the process of developing new and more powerful abstractions, in particular reflective abstractions. Again following Piaget, we understand *abstractions* to be learned anticipations and reflective abstractions to be those abstractions that result from reflection on one's activity and result in knowledge of logical necessity. (See Simon, 2006) for a discussion of the distinction between reflective abstraction and empirical learning processes.)

Our task design approach builds on this theoretical base and involves specifying hypothetical learning trajectories (Simon, 1995) at multiple levels. In this paper, I focus on the level of design of particular understandings, not the planning of trajectories for larger mathematical topics. A hypothetical learning trajectory consists of three components, (1) a learning goal, (2) a set of mathematical tasks, and (3) a hypothesized learning process. Whereas the specification of the learning goal generally precedes the specification of the tasks and hypothesized learning process, these latter two components necessarily co-emerge. The learning process is at least partially determined by the tasks used and the tasks used must reflect conjectures about the possible learning processes. The design approach outlined here provides a conceptualization of the design process with respect to these two components.

I briefly summarize the theory behind our approach to task design in the following way. (For greater detail, see Simon, et al, 2010; Simon & Tzur, 2004; Simon, Tzur, Heinz, & Kinzel, 2004; Tzur & Simon, 2004.) The pedagogical goal is to promote particular mathematical concepts. These concepts are understood as *learned anticipations*. Given an appropriately designed series of tasks, students can call on available activities, attend to the effects of their activity, and come to an anticipation of the effects of that activity – the development of a new abstraction. In the next section, I provide an example to concretize these ideas and follow that with a discussion of the key steps in the design process.

Instructional Example

In this section, I present an instructional example from a one-on-one teaching experiment. The purpose of the teaching experiment was to promote and analyse students learning of new concepts (making of an abstraction) as they engaged in mathematical activity. Towards this end, we engaged her in a sequence of tasks and restricted the researcher's role to probing thinking and initiating subsequent tasks.

The student, Erin, (one of three that participated in the study) was an undergraduate prospective elementary teacher. Pre-assessment showed she lacked understanding of both the meaning of fraction division and the invert-and-multiply algorithm with which she was familiar. In sessions prior to the one described here, we worked on the meaning of fraction division.

The goal of the session described here can be articulated at two levels of specificity. First, we wanted her to reinvent a common denominator algorithm for division of fractions. More specific in terms of the understanding involved, we wanted her to understand the invariance of division with respect to variation in the common units of the dividend and divisor.

The task sequence, developed by the author, began with division-of-fraction word problems whose dividend and divisors had common denominators. Erin was asked to solve them by drawing a diagram. She was able to solve the first task without difficulty ("I have 7/8 of a gallon of ice cream and I want to give each of my friends a 1/8-gallon portion. To how many friends can I give ice cream?"). The task sequence progressed to word problems in which the dividend and the divisor still had common denominators, but the divisor did not divide the dividend equally and then to similar tasks presented using only as number expressions (e.g., 8/5 ÷ 3/5 =). Erin's solution process can be summarized as follows (actions for 8/5 ÷ 3/5 in parentheses):

1) Draws the dividend (draws 2 whole rectangles divided into fifths, shades 2/5 of one rectangle leaving 8/5 unshaded) (Figure 2).

2) Identifies groups the size of the divisor (circles each 3/5).

4) Counts those groups (counts 2 groups).

5) Identifies the remainder – the ungrouped part of the dividend (r = 2/5).

6) Identifies the fractional part of the quotient by determining the fraction of the divisor represented by the remainder (2/5 is 2/3 of 3/5).

After a number of such tasks, the researcher changed the nature of the task. He gave Erin a task and announced that the numbers were too "messy," that she would not want to draw a diagram. The first such task was 23/25 ÷ 7/25. Erin made it clear that she did not know the answer and the researcher encouraged her to talk through a diagram solution without actually drawing. This she was able to do without difficulty and arrived at the answer. The following task was 7/167 ÷ 2/167. Again, Erin did not know the answer and solved the task by doing a mental run of her diagram drawing strategy. Finally, the researcher gives her 7/103 ÷ 2/103 (same numerators as the previous task[52]). Erin responded with the answer "three and a half"

[52] The fact, that this problem had the same numerators as the previous one, was not pointed out to Erin.

and explained that she did not need to use the denominator. When pushed to justify her claim, she drew a rectangle divided into seven boxes and argued that she could group them into groups of two parts regardless of the size of the parts that were represented. An additional task verified that Erin did not over-generalize to tasks that did not have common denominators and that she could convert to common denominators to use her reinvented algorithm (make common denominators and divide the numerators). Assessment the following week convinced us that she retained her algorithm, she could justify it, and she could use the same reasoning to think about the invariance when composite units were involved (e.g., why 2400 divided by 400 is equal to 24 divided by 4).

Simon et al (2010) gives a more detailed look at the data, analysis, and explanation of Erin's learning. My focus here is the abstraction that Erin made. That abstraction became evident when Erin quickly gave the answer. So what did Erin abstract? Erin abstracted that regardless of the size of the parts (assuming common denominators), the quotient would always be the same and would be equal to the quotient of the numerators.

Readers at this point might be tempted to argue that Erin just saw a number pattern. However, remember that up to this point in the teaching experiment, Erin had *not* seen two tasks with the same pair of numerators. Thus, there was no pattern to observe. The *first* time she encountered the situation she *anticipated* the result.[53] Furthermore, her conclusion was not only about the task in question. She now knew that she could find the quotient of *any* division of fractions involving common denominators solely by dividing the numerators. And, she was able to justify her conclusions using an improvised diagram related to her diagram drawing. These three indications make it clear that the anticipation in question was conceptual in nature and not due to noticing a pattern in the numbers.

The Design Approach

The first two steps in our design approach are the first two steps in most instructional design that is aimed at conceptual learning. We assess student understanding and articulate a learning goal[54] for the students relative to their current knowledge. It is after these first two steps that our approach diverges.

Our third step is to specify an *activity or activity sequence that students currently have available* that can be the basis for the abstraction specified in the learning goal. We attempt to identify an activity that can result in an anticipation that is the particular abstraction intended.[55] The fourth step is to complete the hypothetical learning trajectory, that is, to design a task sequence and related hypothesized learning process. The task sequence must both elicit the intended student activity and lead to the intended anticipation on the part of the students. The hypothesized learning

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[53] Simon (2006) presents a distinction between *reflective abstraction* and *empirical learning processes*. I have argument that Erin's learning was the result of reflective abstraction as there was no empirical pattern to observe.

[54] Articulation of conceptual learning goals is a problematic issue not covered here. It is a theoretical and empirical challenge to specify learning goals in a way and level of specificity that it adequately guides instructional design (as well as instruction and assessment).

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process must account for not only the overt activity of the students, but the mental activities that are expected to accompany those overt activities. I will not focus on steps beyond step 4 (e.g., symbolizing, introducing vocabulary, discussing justification), because again they are common to many approaches.

I will now use the example of the division-of-fractions task sequence to illustrate steps three and four of our task design approach. The consideration of what activity we might elicit begins in a way that is similar to Realistic Mathematics Education (Gravemeijer, 1994), a consideration of students' informal strategies. Whereas RME focuses on developing progressively more formal solution strategies, our approach is focused on developing concepts by developing anticipations from those activities.[56] In many of our design situations, we have a specified learning goal and then endeavor to identify an available activity that could be used. Our example of division of fractions illustrates a design in which there was an interaction between step two, setting the learning goal, and step three, identifying a useful activity. When considering students' informal strategies and considering diagram drawing, we realized that students' informal diagram solutions would lead more naturally to (support development of anticipations related to) a common-denominator algorithm for division of fractions than an invert-and-multiply algorithm. Thus, the goal not only affected the identification of the activity, but the activity available affected the specific goal towards which the design was oriented.[57]

The challenge then was to construct a hypothetical learning trajectory. We began with division of fractions word problems and followed them up with numerical tasks (no context) to elicit a diagram drawing activity sequence like the one that Erin used (outlined above). Once the student is using the intended activity sequence, the task sequence must be designed to provoke the particular anticipation (abstraction). For this purpose we used larger numbers for the denominators and invited mental runs of the diagram drawing activity (which was a change in the activity from the researchers' perspective). The use of mental runs was intended to provoke two types of changes in the student's thinking. First, it is intended to help the student foreground the key quantitative relationships. For example, in the diagram solutions, the student first focused on how to create the number of equal parts in the rectangular unit. At least as much attention was likely to have been paid to that phase as to counting parts in order to circle a divisor shaped group or counting the number of groups. In the mental runs, the student can dispense with the first step by simply announcing, "I would draw twenty-three twenty-fifths." The second intended consequence of shifting to mental runs was to create a need on the part of the student to invoke concepts and mental operations that are critical to the concept being developed. In Erin's mental run she had to use whole number division to find out how many divisor-sized groups are in the dividend, whereas in her diagram solutions, she could simply count the number of groups she had in front of her. At the point at which Erin was doing the mental runs, she was engaging in the activity that could lead to the intended

[56] Although there are often overlaps in what is learned by students using these two approaches, I emphasize here the differences in the primary aim and the theory built to achieve that aim. We definitely build on aspects of RME, particularly their use of *model of becoming model for* (Gravemeijer, 1994).

[57] The target algorithm is not itself a conceptual learning goal. We still needed to specify the understanding involved ("to understand the invariance of division with respect to variation in the common units of the dividend and divisor"). However, I wanted to use the process of coming to the goals to illustrate how the knowledge/activities available can influence the goals that are set.

anticipation, she was comparing multiplicatively the number of parts in the dividend with the number of parts in the divisor. Thus, the diagram drawing activity provided an important step towards this activity, but was unlikely to lead directly to the intended anticipation, because the student did not need to conceptualize the multiplicative comparison.

Doing a couple mental runs did not in itself lead to the intended anticipation. The researcher at that point posed a task with the same numerators as the previous task. The purpose of this particular task was to increase the possibility that the student would see the commonality in her activity and, as a result, anticipate that the answer had to be the same and realize that it was not dependent on the value of the common denominators. So how do we explain the resulting change in thinking?

As mentioned earlier, in doing the mental runs, Erin was able to give minimal attention to establishing the dividend – she merely needed to state that she would draw it. This initial statement established the size of the parts, so that (just as in the diagram drawing) she could focus on the number of parts of that size available and the number of parts of that size in a group. Thus, she naturally, without necessarily making a conscious decision, paid particular attention to how many total parts and how many parts in a group. This *focus of attention* (Simon, et al, 2010) was not because she considered the denominator unimportant, but because she had established the size of the parts and was anticipating her next step, a computation involving these two quantities. In the third task of this set, when Erin was faced with a second task with the same pair of numerators and different common denominators, she realized that she was about to enact the same activity as in the previous task. At that moment, she also realized why the size of the common denominators did not change the quotient. This was an example of Erin's reflection on her (mental) activity. That is, she perceived the commonality in her activity in the two cases that led to an abstraction. This abstraction was an anticipation about fraction division (and division more broadly). She now anticipated that the division of fractions with like denominators is equivalent to the division of the numerators.

Conclusions

Our task design approach is an emerging one. Three are aspects of it that we highlight here and others whose articulation will require additional analyses. I highlight here two features of our task design approach that can be seen in the example provided above. First, the approach provides a theoretically based strategy for promoting specific mathematical understandings. It contrasts with strategies in which students must solve novel problems to progress (or hear the solutions of more able peers). Although mathematics teaching cannot cause learning, it is an approach that involves engineering task sequences so that participating students predictably develop the ability to make the new abstraction. Second, the learning goal is *not* to learn to solve the tasks, as it is in many approaches. The tasks are made to elicit activities that the students already have available. Erin was able to solve all of the tasks prior to making the intended abstraction. Further, she was not consciously trying to find an easier way or to invent an algorithm. Her learning was a product of reflection on her activity across a sequence of tasks.

Let us examine some of the implications of this approach to task design.

1. By using a design experiment (teaching experiment) methodology, involving design and implementations cycles using this design approach, there is an opportunity

to understand in greater depth the interplay among goal-directed activity, reflection, and conceptual learning. (This is discussed in depth in Simon, et al, 2010.)

2. This approach potentially provides a way to design task sequences for concepts that students traditionally do not learn well. These tend to be concepts that many students do not spontaneously reinvent in problem solving situations and concepts of which they do not develop deep understanding by being part of a class discussion with more knowledgeable students. The approach focuses the instructional designers on identifying key activities that are likely to afford the intended abstractions for abstractions that were not readily made.

3. Small group work using task sequences of the kind discussed here can lead to somewhat different class discussions. If students are making the new abstraction as a result of their engagement with the task sequence, discussions can focus more on articulation of the new idea, justification, and establishing the idea as taken-as-shared knowledge.

4. The approach has potential to address issues of equity in two ways. First, many students who have conceptual gaps early on seem to never recover. This design approach provides a methodology for building up the specific experience, based on students' currently available activities, needed to make particular abstractions. Second, success in promoting the new abstractions during small group engagement with the tasks can lead to a greater number of students participating in and benefiting from the class discussions that follows. The underlying hypothesis here is that students who abstract ideas themselves, based on their work with the mathematical tasks, tend to learn the concepts in a more powerful way than those who follow the explanation of their more able peers offered in class discussions.

One final point that was discussed briefly at the beginning of this paper is the relationship of our approach with mathematical problem solving. The approach that I have described and exemplified does not focus on students developing their problem solving abilities. Rather it focuses narrowly on the development of mathematical concepts. Developing problem solving abilities is a key part of mathematics education. One could argue that conceptual understanding and problem solving are the two wings of mathematics education – students cannot fly without effective use of the two together. Students do learn concepts through problem solving lessons. Our approach is in no way intended to minimize the importance of lessons in which that is the case. Rather, our approach provides an additional tool that has the potential for success in areas where mathematics education has been less successful. One open question is how to use this tool in conjunction with the powerful tool of problem solving lessons to maximize the learning of students.

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Appendix

Complete list of instructional task sequence:

Solving Word Problems Using Rectangular Drawings

1. I have seven-eighths of a gallon of ice cream, and I want to give each of my friends a one eighth portion. How many friends can I give ice cream to?
2. A scuba diver has two hours worth of air in her tank. If each dive to the bottom of the bay takes three-eighths of an hour, how many dives can she make with the air she has?
3. Each ticket at an amusement park in France is worth four fifths of a Euro. If a pack of tickets costs four Euros, how many tickets are in a pack?

Solving Context-Free Problems Using Rectangular Drawings

4. 3/4 ÷ ¼ =
5. 7/3 ÷ 2/3 =
6. 8/5 ÷ 3/5 =
7. 5/6 ÷ 4/6 =

Solving Context-Free Problems Using Mental Runs of Rectangular Drawings

8. 23/25 ÷ 7/25 =
9. 7/167 ÷ 2/167 =
10. 7/103 ÷ 2/103 =

An Instructional Design Collaborative in One Middle School

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This article presents the principles for instructional design within collaborative communities in one middle school. During the study, members of the Collaborative engaged in creating and implementing a hypothetical learning trajectory and associated sequence of instructional tasks to teach integers in a middle grades classroom. During implementation of the trajectory, the members of the Collaborative played different roles in task design depending on their background. In this paper, we expand on the varying contributions that teachers, researchers and other educators make during design and implementation of rigorously designed instruction. We also document the artifacts and practices of design that led to increased teacher engagement and learning.

Keywords: Realistic Mathematics Education, Emergent Perspective, Integers, Teacher Research, Hypothetical Learning Trajectory

In 2009 an Instructional Design Collaborative was formed at a middle school (12-14 years old) in suburban Florida, USA. This Collaborative was comprised of a middle school teacher who was a former university professor, a doctoral student from a local university, two middle school mathematics teachers, and one special education teacher. The primary purpose of the Collaborative was to design and implement an instructional sequence for integers that was inspired by Realistic Mathematics Education Design Theory. All five Collaborative members participated daily in the design, implementation, and revision of all instructional tasks.

Design principles and task definition

All tasks designed by the Collaborative used the instructional design theory of Realistic Mathematics Education (RME) as the basis for creation. In RME, the focus is on the development of a *sequence* of tasks, not stand alone mathematical problems. Therefore, tasks are defined as problematic situations that are experientially real for students in that the dilemma that they encounter in the problem can be experienced as real by them. Furthermore, each *task* must be phenomenologically rich in that it lends itself to be mathematized (organized mathematically) by students. RME instructional sequences are guided by the following three design principles:

Heuristic one: Guided reinvention

The sequence of instructional tasks should be designed to encourage students' reinvention of key mathematical concepts. To start developing an instructional sequence, the designer first engages in a thought experiment to envision a learning route the class might invent with guidance of a teacher (Gravemeijer, 2004). It is here that knowledge of the history of mathematics as well as prior research concerning students' invented mathematical strategies can be used. Since students do not have the time to invent the intended mathematics in the same way as mathematicians originally did, the teacher must help students re-invent these ideas in shortened time periods using carefully sequenced problems and tools.

Heuristic two: Sequences should be experientially real for students

The starting points of instructional sequences should be experientially real in that the students are able to engage in personally meaningful activity (Gravemeijer, 2004). Often, this means grounding students' initial mathematical activity in experientially real scenarios (which can include mathematical situations). While many textbook authors see the value of using real-world problems in instruction, RME goes beyond simply situating mathematics in the real world. Rather, instructional tasks draw on realistic situations as a semantic grounding for students' mathematizations and activities are sequenced so that students will organize their activity within the realistic context to re-invent important mathematics.

Heuristic three: Emergent models

Instructional activities should encourage students to transition from reasoning with models of their informal mathematical activity to modeling their formal mathematical activity, also called *emergent modeling* (Gravemeijer & Stephan, 2002). During the transition from informal to formal, the designer/teacher supports students' modeling by introducing or using student created tools, such as physical devices, inscriptions, and symbols that can be shared by students to explain their mathematical reasoning.

Guided by the three heuristics described above, the designer creates an instructional sequence while at the same time envisioning a path that the class may follow as they engage in the tasks (Gravemeijer & Stephan, 2002). The anticipated path has been labelled a *hypothetical learning trajectory* in that the designer makes conjectures about the mathematical route the class, as a community, will travel, including the mathematical goals and tool-use, as they engage with the instructional tasks and anticipates the means by which the teacher can support that route. After implementation, the designer analyzes the collective learning of the class and revises the instructional sequence accordingly. Since the former professor had multiple experiences designing instruction based upon RME, she created an integer HLT for the Collaborative.

Epistemological theory

The authors use a version of social constructivism, called the *emergent perspective* (Cobb and Yackel, 1996), to situate their interpretation of classroom events. The emergent perspective claims that learning is both an individual, psychological process and a social process, with neither taking primacy over another. In other words, a student's mathematical development occurs as he participates in and contributes to the mathematical practices of the classroom.

In particular, to document how the HLT was realized in implementation, the two authors conducted an analysis of the collective learning of the classroom (Stephan & Akyuz, 2012). They found five mathematical practices that emerged as students interacted with the sequence of instructional tasks designed by the Collaborative. T-tests indicated that the students in the first author's class made statistically significant gains in their understanding of integer subtraction, a finding that is rare in integer research.

The Collaborative experiment

Integer concepts and operations is an extremely difficult content area for students to learn with meaning. Additionally, research indicates there is no agreed upon method for teaching integers conceptually. The textbook that had been adopted by the teachers' school did not support students learning integer operations with meaning, so Collaborative members decided to create their own instructional sequence. Their motivation was primarily pragmatic in that their prior students merely memorized the rules for operating with integers. Based on the tenets of RME, they created a first draft of a sequence that used net worth, assets and debts as the realistic context for exploration.

In the first phase of instruction the teachers introduced students to the notion of net worth and posed problems that encouraged students to think about possessing assets and debts, and the effect it has on one's financial situation. Here it is intended that students construct the idea that a net worth is an abstract quantity, not a concrete entity that can be counted like money, but rather the "status" of one's financial value. Net worth is contextually rich for exploring integers in that not only can net worth represent an *object* with positive and negative properties and can be ordered on a number line, but net worth can also be transformed by *actions* (add or subtract) to produce new net worths. These actions and their results can also be inscribed on a vertical number line.

Phase Two envisions students using statements about assets and debts to solve problems in which they find and compare net worths to each other. For example, students are given a listing of the assets and debts of two people, and asked to find and compare their net worths. The goal that the teachers conjecture will become taken-as-shared is to see which person is worth more than the other. Students typically create different strategies for finding net worths, such as 1) finding the *total* assets then *total* debts, and the *difference* between them, and 2) adding assets and debts one at a time until finished. Students' reasoning as they compare two net worths provides the opportunity to introduce a vertical, empty number line (VNL). For example, the teachers pose tasks that asked students to place the net worth of Jackson ($10,000) and Hayden (-$45,000) on a VNL and to find how much they differ by.

Up to this point, the tasks use the words "asset" and "debt" to represent the positive and negative nature of integers, respectively. Phase Three involves vertical mathematization, scaffolding students' symbolizing from reasoning with unsigned to signed integers. To this end, activities are posed that introduce assets and debts with the + and – signs rather than the words "asset" and "debt." Some net worth statements are designed intentionally so that a third strategy might emerge, that of cancelling assets and debts that are equal, and working with the remaining assets and debts.

The Fourth Phase seeks to confront the conceptual stumbling block involving why *two negatives make a positive.* At this point, transaction activities are enacted to help students build on their intuitions to construct an imagistic basis for operations. Students are to judge the effect that various transactions will have on a net worth. These tasks are written to encourage students to create negative and positive signs as both a *process* and *object* where the first sign signifies an action/transformation and the second represents the *status* of the quantity (negative/debt or positive/asset). Thus, - (-100) is read as "taking away a debt of $100" and - (+90) as taking away an asset of $90. Students are asked to judge whether or not the symbols signify what they called a "good decision", one that makes the net worth get better or a "bad decision", one that decreases the net worth.

The Fifth Phase incorporates tasks that encourage students to find the effect that various transactions have on a person's net worth. These tasks are called *transaction tasks* because they involve taking a person's original net worth, performing a transaction on it (like adding a debt) and determining his new net worth. While some students might perform erroneous calculations, teachers can expect "going through zero" to become taken-as-shared. For example, if the task is to find out what happens to Chris' net worth ($1000) if she takes away an asset of $1500, the teachers can re-introduce the empty number line (VNL) to make students' going through zero strategy more visual for the other students (Chris will have to pay off $1000 to get to $0 and she will still be in debt $500). In this way the VNL re-emerges in the class from student thinking as a *model of* transactions on net worths. Later, it might evolve to become a *model for* the formal integer operations of addition and subtraction in less context-dependent problem situations.

In the Sixth Phase, some activities require students to determine the results of various transactions including multiple ones like 100 – 2(-50). At first the problems are posed in context and move towards number sentences. To formalize the rules for integer operations, students are asked to list various transactions that could have taken place to make Chad's net worth go from $10,000 to $12,000. Students will write + (+2000) as the unknown transaction, - (-2000) or others like - (-1000) - (-1000).

Analysis of the varying Collaborative members' roles in task design

The Collaborative is comprised of five members: two mathematics teachers (McManus and Dickey), one special educator (Smith), one doctoral student (Akyuz) and a mathematics teacher with RME experience (Stephan). The Collaborative met one week before instruction began and then almost daily throughout the implementation.

Role 1: Anticipating supportive mathematical imagery

In the first Collaborative planning meeting, the HLT was introduced by Stephan. Since teachers did not have previous experience related with HLTs, the overarching sketch of the intent of the sequence was discussed by Stephan. In doing so, she elaborated the imagery, tool-use and potential discourse that could be supported over time. Even though the teachers learned about the HLT in the first meeting, they made important contributions based on their previous experiences. For example, the imagery of "pay off" came out in the meeting from one of the mathematics teachers, McManus:

> **McManus:** When you say she has lower net worth, it means net worth is worse because she has assets, she can cash them all and **pay off** all debts and be still be in debt...I wonder if there is anything important, mathematically speaking,in **paying it off**?
> **Stephan:** Two things I can come up with. One is when we get to integer operations and let's say we got $600,000 and pretend there is a transaction $900,000 ...she goes into debt 300, 000 dollars.
> **McManus:** Because 600,000 goes to zero on the number line, you only have enough to go to zero but there is still more left over.

Smith: Or I only have this much to pay off but I still owe this!
McManus: That is all related with the number line idea we try to develop later.
Stephan: That whole going under zero that took the math community so long to get.

In this excerpt, the mathematics and special education teacher contributed to the HLT by suggesting that students might employ a "pay off" imagery as they make sense of going below zero. Stephan connected their imagery with the difficulty mathematicians had with negative numbers. The mathematics teacher illustrated his understanding of the intent of the instruction by relating his imagery to the inscriptional device (VNL). This example shows that the mathematics and special education teachers played an important role in suggesting student imagery that might emerge during the course of instruction. Stephan played the role of tying their contributions to the intent of the sequence as well as grounding their notions in the history of the discipline.

Role 2: Creating challenging formative assessments

Another way in which the members of the Collaborative participated in task design involved creating tasks that would cause cognitive conflict in their students as well as cause students to analyze the thinking of others. In one of their planning meetings, the teachers reported that their students were having difficulties with the notation used with the VNL. Ms. Smith was the first to suggest creating what they typically called a "Sam and Sue problem." This type of problem usually listed at least two fictitious students' solutions and the class had to decide which one they agreed with and why. Mr. McManus seized on Smith' suggestion.

McManus: What if we had a couple of number lines with the stuff on it incorrectly and had a story and say "which number line tells the story?"

The teachers worked collaboratively and created the task pictured in Figure 1. The story focused on three imaginary students named Larry, Curly and Mo, who had used a VNL to solve a problem in which a person originally had a net worth of $4000 but added a debt of $8000. The students were asked to decide which students' VNL was the most accurate.

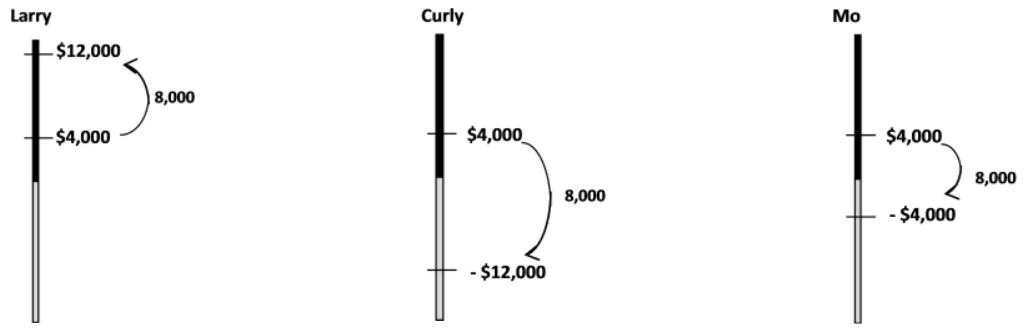

Figure 8: The formative assessment task

This example illustrates the role that the teachers took in planning meetings. In other words, for them, planning meetings consisted of bringing in student data from class that day to discuss next moves in instruction and possibly creating a new task to address issues that had arisen. The task described above was completely motivated by the classroom teachers and incorporated the heuristics of RME.

Role 3. Using their mathematical knowledge to alter the instructional sequence

Sometimes members’ contributions were of a matheamtical nature and questioned gaps in the HLT. In a different planning meeting, Mr. McManus shared a concern that he would like to assess his students’ current understanding of the meaning of a negative net worth and that 0 is not a negative number. In the excerpt below, the Collaborative discusses this issue which then provokes a change in the instructional sequence.

> **McManus:** I want to be sure that when I put a problem up there, all the kids realize that zero is not same as negative numbers. Then they understand the abstract value; you can go below zero and it makes you go in more debt.
> **Akyuz:** Are you going to make them compare negative numbers also?
> **McManus:** Yes… I think you cannot compare negative numbers until they conceptualize net worth as abstract value.
> **Stephan:** Because if you cannot conceptualize anything below zero [as objects in their own right], you will have difficulty with that [ordering negatives]. I am really rethinking the order of sequence now. I think the page after this...we need a number line here. I really think so because even with **concept** of integer, forget **operations** right now, it is a big deal to order those numbers on the number line.
> **Dickey:** I want to see order on the number line. I want to see that - 4000 is below - 1000 and I want them to see that, too.
> **Stephan:** I think we need to have a page asking students to order a bunch of positive and negative numbers. And then we may ask how much Juli’s worth more than Deanna? And then operations. This way, they can start making objects out of distances from zero and they might also start thinking about those pay off ideas again.

During the discussion McManus stated that he wanted to know if his students were able to understand the difference between negative numbers and zero as well as other negative numbers. He stated that he wanted the students to be able to conceptualize negative numberss as abstract objects. In this case, the role that the mathematics teacher played was to bring in his knowledge of mathematics to question a gap in the HLT. For her part, Akyuz suggested that the order of negative integers in relation to positives and zero ought to be assessed. Stephan agreed and contemplated one of the Collaborative’s first changes in the HLT. As a consequence of this interaction, the Collaborative created tasks that asked students to show two people’s net worths on a number line and find out how much more one net worth was than the other (see Figure 2).

Paris’ Net Worth is in the red -$20,000.
Nicole’s Net Worth is in the red -$22,000.
Who is worth more? By how much?

Use the number line to show your solution.

Figure 9: Ordering and comparing net worths

Role 4. Working and revising already created tasks or the sequence of the instruction

Besides creating new tasks, the members of the Collaborative also contributed to the instruction by revising the tasks that had already been created by Stephan before the study started. In other instances, their contributions served as a catalyst for re-ordering certain tasks within the sequence. For example, in one of the meetings we changed the order of the instructional sequence as necessary based on the teachers' reported interactions with the students. Originally, after the "good and bad decision" where the students decided whether a given transaction was good or bad, the instructional sequence followed with the net worth trackers that included vertical number lines and the transactions below them without linking them to the context (see Figure 3).

However, since students were not naturally using the VNL to compare net worths, we decided to pose transaction word problems *before* the Net Worth Trackers (e.g., Donald has a net worth of -$5000. A debt of $3000 is taken away. Is this good or bad? What is his net worth now?). We changed the order because we expected students to get incorrect answers to the word problems and that this would motivate them to want to structure their reasoning using a VNL. Then, students would naturally see the need for reasoning with a more structured tool (VNL) when they worked the word problems.

In summary, the Collaborative members' roles emerged in a variety of ways throughout the implementation of the instructional sequence. We have pointed to at least four different ways that they participated including *anticipating supportive mathematical imagery, creating challenging formative assessments, using their mathematical knowledge to alter the instructional sequence, and working and revising already created tasks or the sequence of the instruction.* Additionally, one of the teachers contributed her knowledge of students with disabilities to the creation of the tasks. The former professor's role included preparing the HLT and supporting instructional tasks. The doctoral student also contributed in these ways, but brought more RME and mathematics education literature into the discussions. It is interesting to note that during the implementation of the instruction, the practitioners began to discuss more theoretical issues (e.g., imagery and abstractness of negative numbers) while the researchers began to think more about teaching practices (e.g., were state standards covered, formative and summative assessments). The work of creating and implementing a sophisticated sequence of mathematical tasks provided them the opportunity to marry theory and practice in a very genuine way.

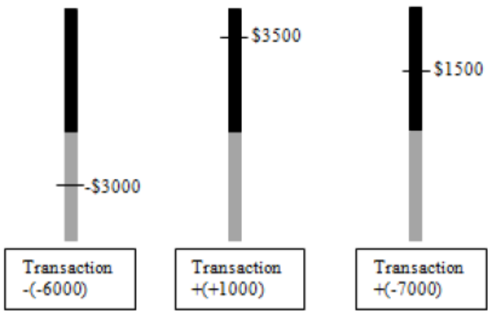

Figure 10: Net Worth Trackers

Artifacts that supported teacher engagement and learning

There are many artifacts that are used or produced in the process of conducting classroom-based research: an HLT (hypothetical learning trajectory), research articles, an instructional sequence, formal and informal assessments, interviews with students, state standards, and instructional frameworks. The artifact that was reported by teachers as most useful in shaping their actions and learning during the collaborative was the instructional sequence itself. They argued that whenever they felt confused about the trajectory of their class, they always had the sequence to go back to as a tool that could re-orient them to their mathematical goals and to the expected reasoning of students. Often, our team meetings would begin by working through the activity sheet that would be used in the next lesson. This activity always led us to discuss students' current and possible future reasoning and how we might support that with discourse, tools and gestures. Interestingly, teachers argued that the *absence* of a teacher's manual was instrumental in their learning since they were forced to create a lesson image for themselves rather than rely on an image from a book.

Of middle importance to teachers was the HLT (the conjectured learning route of students) and the assessments. The HLT was important to them at the outset as a way to orient themselves to the route that we expected students to take; however, teachers did not refer back to the HLT as the experiment progressed. Rather, their conversations always folded back to the activity sheets from the sequence, probably because the sequence was a more concrete manifestation of the trajectory, whereas the HLT is more abstract. On the other hand, both design researchers rated the HLT highly, probably because they were in charge of designing the instructional sequence and utilized the HLT more before and during the experiment as a way to keep in mind the goals of the instruction. With regard to assessments, teachers relied heavily upon assessments, including student classwork and homework, but not as much on the more formal unit test at the end of the sequence. Teachers commented that assessing student thinking and discussing it with peers on a daily basis was an essential practice in supporting the implementation of the instructional sequence.

Teachers rated the state standards, research articles, student interviews, and instructional plans, as having little impact on their implementation of instruction and their learning. Since the instructional sequence was already written, the teachers said they trusted that the designer, Stephan, had incorporated the state and county requirements within the sequence. The articles were not useful for teachers in this instance primarily because there is very little research on how students conceptualize and learn integers. The researchers agreed with this and rated the research on integers very low. In other cases, when the research focuses on how students think about the content or how teachers implement sound instruction, teachers acknowledged that the articles might be more valuable.

Practices that supported teacher engagement and learning

The practice of gathering data on students' conceptions, analyzing it, and using this data to determine the task to be used the next day became a central practice of the teachers in this collaborative. In addition, teachers engaged in the activity of lesson imaging in which we chose the task(s) to be used in class, anticipated the diversity of ways in which students would engage, and imagined how the whole class discussion would ensue, a practice we call *lesson imaging* (Schoenfeld, 2000). In fact, the teachers rated lesson imaging as the second most important activity in which they engaged during the project. They made a distinction between lesson imaging and lesson planning, with the latter being the act of scheduling which activities to do throughout the week. The teachers argued that lesson imaging is a practice that helped them envision how students would be solving problems and how to capitalize on that in order to have powerful mathematical discussions in which objectives were met.

A third practice that emerged in a profound way among our team was *collaboration*. Teachers placed the influence of this practice as number one in their own learning and success in their classroom. They remarked that as powerful as lesson imaging is, it is not as effective in a social vacuum; other teachers are sometimes able to anticipate student strategies that they had not thought of or can think of better ways to engineer the whole class discussion than they would have alone. Additionally, the presence of the experienced researchers in the collaboration was critical to their success according to them. They recognized that they probably would not have engaged in lesson imaging or daily reflection if not for the researchers lending their expertise and leading the meetings.

The effects of participating in the collaborative were sustainable for these teachers as they continued to reflect daily, lesson image and collaborate in years after Stephan's support was directed elsewhere. In addition, the teachers involved in this project have become model mathematics teachers for our school.

Implications of the work on different communities

The work of the Design Collaborative reported in this paper impacted a variety of communities including students, teachers, educators, and administrators. First, their daily reflection, planning and designing led to an increase in both students' state test scores as well as their knowledge of integers more specifically (Stephan & Akyuz, 2012). Designing and testing instructional sequences requires daily formative assessments in order to revise and strengthen the tasks in real time. Thus, with input from a variety of classroom teachers, students received instruction that addressed their immediate needs rather than waiting until the end of the unit and re-teaching.

A second community that was influenced by their work was the teacher population at both the Collaborative's school as well as teachers from around the state. Other mathematics teachers at the Collaborative's school had the opportunity to work as a part of the Collaborative if they desired. Other math teachers joined the team during the last two years as full participants and learned about instructional design processes, analyzing student learning on a daily basis to inform instruction and assessment. Members of the Collaborative were called on by their administrators to lead Lesson Studies with their math colleagues.

Administrators, who held district level positions with supervisory responsibilities across 10 other middle schools, learned of this Collaborative and invited them to present their integer work and collaboration practices to various school groups. In this way, the instructional design and planning practices (including cognitive interviews and daily planning) were made available to district level administrators, principals, assistant principals and teachers from other schools.

Another community that was affected by the Collaborative's work was the teacher education population. At the time that the Collaborative was testing and revising the integer instructional sequence, several mathematics education professors visited the classroom to observe students as they participated in the instruction. In addition, one professor stayed after class occasionally to question one teacher about her rationale for decisions she made during instruction.

Finally, this design work has the potential to impact both teacher educators and researcher/designers by the fact that several articles have been and will continue to be published about their work. Already, the teachers from the Collaborative have published an article that describes the circumstances that supported and constrained their work as a community of teacher learners (Stephan, Akyuz, McManus & Smith, 2010). Akyuz documented the planning and classroom practices of student-centered teachers (Akyuz, 2010). Stephan and Akyuz have published the instructional sequence and the learning of one classroom as they engaged in the instruction in a prominent mathematics education journal (Stephan & Akyuz, 2012). Finally, the work of a mentor teacher in guiding novice student-centered teachers to incorporate inquiry practices into their teaching repertoire will be published within the next year. These articles have the potential to inform researchers, designers, and teacher educators.

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Mind the gap – Task design principles to achieve conceptual change in rational number understanding

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In this paper we focus on the problem of conceptual change in the shift from natural to non-natural numbers. We discuss students' difficulties in this area and present a mathematics textbook analysis to show that this problem is not taken into consideration in instruction. We discuss a number of principles for instruction stemming from the conceptual change perspective to learning and present a number of experimentally tested tasks, designed on the basis of these principles. These tasks were used to investigate and/or to induce conceptual change in the number concept. We argue that such tasks are of value from the point of view of instruction.

Keywords: Rational number understanding, conceptual change

Difficulties in understanding rational numbers

Mastery of the rational numbers represents an important aspect of mathematical literacy. However, learning about rational numbers presents students with many difficulties, mostly when the required reasoning is not in line with their prior knowledge and experience about natural numbers (see Ni & Zhou, 2005, for a review).

In comparing decimals, students judge for instance that longer decimals are larger, thus responding that 2.12 > 2.2 (Resnick et al., 1989); in comparing fractions they think that a fraction gets larger when one of its parts gets larger, resulting in errors such as 2/5 < 2/7 (e.g., Ni & Zhou, 2005). Students also extend the meaning of operations from natural to non-natural numbers. For instance, seeing multiplication of natural numbers as repeated addition leads to the idea that multiplication makes bigger, which has been shown difficult to overcome (Greer, 1994), even in adults (Vamvakoussi, Van Dooren, & Verschaffel, in press). Finally, the dense ordering of rational (and real) numbers is difficult for students to grasp (e.g., Vamvakoussi, Christou, Mertens, & Van Dooren, 2011, Vamvakoussi & Vosniadou, 2004, 2010;)[58].

[58] An order $\leq$ on a set X is dense if, for all x and y in X for which $x < y$, there is a z in X such that $x < z < y$. Unlike the integers, the rational numbers as well as the real number are densely ordered. The real numbers are, in addition, continuous.

Students initially respond that there are no other numbers between two given pseudosuccessive numbers (e.g., 0.005 and 0.006 or 1/2 and 1/3). Later, they refer to some intermediate numbers, but still do not accept that there are infinitely many.

Theoretical framework

Several researchers have argued that many of students' difficulties with rational numbers can be explained from a conceptual change perspective on learning (e.g., Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005). Within this perspective, we adopt a specific theoretical frame, namely the framework theory approach to conceptual change (FTatCC, Vamvakoussi & Vosniadou, 2010), because it proposes a number of specific and testable principles for the design of instruction and tasks (Greer & Verschaffel, 2007; Tsamir & Tirosh, 2007). Originally developed to account for the challenges that students face in regard to science learning, the FTatCC has been fruitfully applied in the past few years in the domain of mathematics learning (e.g., Greer & Verschaffel, 2007; Verschaffel & Vosniadou, 2004). Based on evidence from cognitive-developmental research, the FTatCC assumes (Vosniadou et al., 2008) that young children organize their everyday experiences in the context of lay culture from an early age in domain-specific conceptual structures, termed *framework theories*. These initial theories constitute explanatory frameworks that are generative: They underlie children's predictions and explanations regarding unfamiliar situations in a relatively coherent way. The incompatibility between the background assumptions of students' initial theories and the scientific ideas to which they are exposed mainly via instruction, is assumed to be a major source of misunderstandings and errors for students. Regarding the development of the number concept, the FTatCC assumes that, before they are exposed to rational number instruction, students have formed a rather coherent domain-specific, naïve theory of number—i.e., a complex system of interrelated ideas and beliefs—based on their extensive experiences in the natural number domain. This theory shapes their expectations about what counts as a number and how numbers are supposed to behave. From the students' point of view, numbers are essentially discrete *counting* numbers and are grounded in additive reasoning (Vamvakoussi & Vosniadou, 2011; see also Ni & Zhou, 2005; Smith et al., 2005).

The FTatCC assumption about the structure and content of students' knowledge of numbers—before they are exposed to non-natural number instruction—implies that the shift from natural to non-natural numbers is a slow and gradual process that is difficult to accomplish and requires substantial instructional support. A factor contributing to this difficulty is that students are typically unaware of the background assumptions of their framework theories and thus do not perceive the necessity to re-evaluate or revise them. The FTatCC predicts that, instead, students enrich via the use of additive learning mechanisms their knowledge base with new incompatible information about numbers provided through instruction, thus creating misconceptions such as the ones described above.

Instruction design principles

The conceptual change perspective on learning and instruction has been traditionally associated with the *cognitive conflict* teaching strategy. This strategy has been subject to criticisms and is now acknowledged as a potentially useful approach provided that it is used with caution and only as one out of several other alternatives (Vosniadou, Ioannides, Dimitrakopoulou, & Papademetriou, 2001). Indeed, there is a number of different principles for the design of instruction stemming from the conceptual change perspective on learning (Greer & Verschaffel, 2007; Vosniadou et al., 2001; Vosniadou & Vamvakoussi, 2006; Vosniadou, Vamvakoussi, & Skopeliti, 2008). We will refer here to the ones that are mostly relevant for the purposes of the present paper. We note that, as such, these principles are not unique to the conceptual change perspective on learning, in particular to the FTatCC. We stress, however, that the principles have a very specific focus, namely to address the problems arising from the incompatibility of prior knowledge with the intended new mathematical knowledge. This particular focus has implications also for the implementation of principles in the design of tasks, as we will illustrate in the following sections.

Take students' prior knowledge into consideration

There are several ways for prior knowledge to be taken into consideration in instruction. This principle refers to the necessity to acknowledge the potentially adverse effect of prior knowledge, in cases when it is not compatible with new information coming from instruction. This requires that teachers, curriculum designers, and textbook authors can identify the points where conceptual change is necessary, and that they are informed about students' potential initial understandings. Let us present an example that is related to our discussion in the next section. Prior knowledge and experience about natural numbers can be used to introduce non-natural numbers. In fact, it is commonly used when, for instance, fractions are introduced via their part-whole aspect, or when decimals are presented as whole numbers with a change of units. On the contrary, the differences between natural and non-natural numbers are not explicitly addressed. However, downgrading the differences and focusing on the similarities between natural and non-natural numbers--with a view to build on students' prior knowledge--creates many problems in the long run, as discussed in the first section.

Facilitate students' metaconceptual awareness

As pointed above, students are typically not aware of the background assumptions of their framework theories of numbers (e.g., that numbers are essentially discrete) and this hinders conceptual change learning. It is thus important to create opportunities for students to externalize their ideas, compare them with their peers', and reflect on them. This can be done in learning environments that foster group discussions, particularly when students are engaged in tasks that involve modelling a situation or dealing with external representations. This brings us to the next principle.

Use models and external representations

Again, this principle is not unique to the FTatCC. However, it is important to note that from the FTatCC perspective, taking into consideration students' prior knowledge and how it may influence their interpretations of a situation also applies in the case of models and external representation introduced in instruction. Consider, for example, the (real) number line. It is a powerful representation for numbers, but it is also known to be difficult and even misleading for students. For instance, conceptualizing the number line as a ruler may lead students to believe that there is a finite number of numbers in a given interval.

Foster analogical reasoning

Analogical reasoning, in particular cross-domain mapping, is considered an important mechanism for conceptual restructuring. This is because the comparison between two domains may highlight their common features, reveal unnoticed commonalities, and allow for the projection of inferences from one domain to the other. In the process, re-presentation of one or of both domains may occur to improve the match, which may lead to conceptual restructuring. Consider, for example, the complex interplay between the domain of continuous magnitudes and the domain of number that, in the course of the historical development, resulted in the re-conceptualization of the notion of number, as well as of continuity (Vamvakoussi & Vosniadou, 2012).

In the following, we present a textbook analysis showing that the problem of conceptual change in the number concept is not taken into consideration in instruction. Then we present examples of tasks that are grounded on the above principles and have been experimentally tested with respect to their potential to induce conceptual change learning.

A textbook analysis

A central theme in the task design principles mentioned above is that students need to be pointed explicitly to differences between natural and rational numbers. To see to what extent this currently happens, we did an analysis of the three most frequently used primary school mathematics textbook series in Flanders, Belgium. More specifically, the teachers' manuals from year 2 to 6 were analysed, as these included student materials and several additional clarifications and background information.

The units of analysis were the lines in the teachers' manual that in some way dealt with rational numbers (i.e. fractions, decimals, negative numbers). For every line, it was determined whether and to what extent it made reference to differences between natural and rational numbers, or to similarities between them. In cases when such a difference or similarity was pointed out, it was moreover coded whether this happened in an implicit or in an explicit way. Finally, it was also coded for what aspect (the way to determine the size of a rational number, the effect of operations with rational numbers, the representation of rational numbers, or the density of the rational number system) the difference or similarity was referred to.

The results showed that the textbooks were very comparable in their treatment of rational numbers. With respect to the size of rational numbers, none of the textbooks explicitly referred to the fact that rules that are valid to determine the size of natural numbers do not hold for rational numbers. While most observations referred to differences between both kinds of numbers, they were all implicit. Such an implicit reference is for instance a number line showing the location numbers 0.6, 0.75, and 0.8. Students can derive that even though 75 is larger than 6 and 8, 0.75 is still between 0.6 and 0.8, but it is not explicitly pointed out.

With respect to representations, all textbooks referred to differences merely in implicit ways (such as pointing out that 2/4 = 1/2 without explicitly pointing out that any rational number can have infinitely many different representations).

For the domain of operations, both similarities and differences between natural and rational numbers are pointed out, about two thirds are similarities. An example is that a decimal number like 0.72 is written as 72 tenths in order to do operations with it (such as halving or doubling). Only one textbook explicitly mentions that teachers should explicitly address the idea in students that division will lead to a smaller result, and this happens only at one moment in the fifth year.

Finally, the aspect of the density of the rational numbers is hardly dealt with at all in the three textbook series. In the few cases where it is addressed, this happens in an implicit way, mostly by pointing out that an interval between two given numbers on a number line can be "stretched" after which more numbers can be found, as illustrated in Figure 1. It is however not explicitly pointed out that infinitely many numbers can be found in any interval or that the stretching can be infinitely repeated.

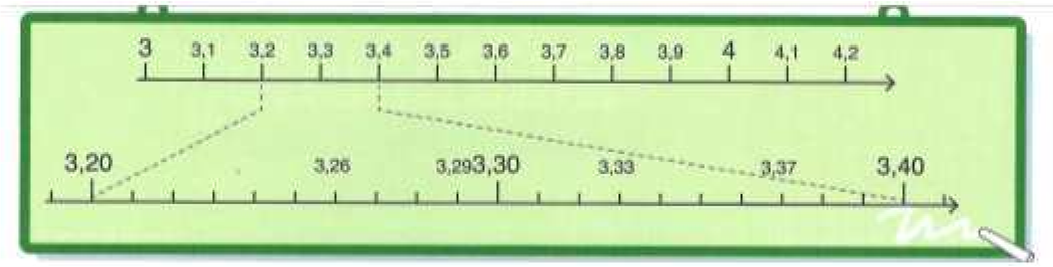


Figure 1. Implicit reference to the density of the number line in a fifth grade textbook

Tasks to investigate and induce conceptual change

In the following we present specific tasks that were employed in experimental settings with a view to investigate secondary students' understandings of the density property of rational (and real) numbers, and to explore possibilities for effective teaching of this counter-intuitive notion. The design of these tasks drew on a series of studies from the FTatCC that focused on students' understanding of density as a paradigmatic case of the problem of conceptual change in the development of the rational number concept (Vamvakoussi & Vosniadou, 2004, 2010; Vamvakoussi et al., 2011). In line with the FTatCC principles, the design of the tasks was informed by empirical evidence about students' pre-existing ideas and typical misconceptions regarding the notion of density in arithmetical, as well as geometrical contexts. This evidence can be summarized as follows: a) the idea of discreteness is robust both in arithmetical and in geometrical contexts, b) students are more inclined to accept that there are infinitely many points on a segment, than that there are infinitely many numbers in an interval, c) accepting that there are infinitely many intermediates (numbers or points) does not imply that one understands that these can never be found one immediately after the other, d) students do not "see" the rational numbers set as a unified system of numbers but rather as consisting of unrelated sets of numbers (e.g., integers, decimals, fractions), which has implications on their judgments about the number, as well as the type of numbers in an interval. In line with the FTatCC principles, the tasks refer to the cross-domain mapping between continuous magnitudes, in particular the straight line, and numbers. This cross-domain mapping is deemed crucial for instruction-induced conceptual change in the number concept (Vamvakoussi & Vosniadou, 2012). It also underlies a representation of numbers, namely the (real) number line that is commonly used in schools settings.

In line with the FTatCC principles, using, evaluating, comparing and constructing representations of numbers and the number line, lie at the heart of the sequence of tasks designed by Vamvakoussi, Kargiotakis, Kollias, Mamalougos, and Vosniadou (2003, 2004) (Table 1). These tasks were experimentally tested in two different settings, both allowing for expressing one's ideas, and discussing and evaluating others' ideas. Specifically, 30 9th graders were split in two groups who worked on the tasks in pairs. Each pair presented their results to their fellow students and they were discussed. One 45-minute session was devoted to each task. Meanwhile, the control group (14 9th graders) worked in their classroom, with paper and pencil, and the results were presented orally and then written on the blackboard by the researcher. The experimental group (16 9th graders) worked in Synergeia, a software designed to support collaborative knowledge building that provides a structured, web-based work space in which documents and ideas can be shared and discussions can be stored. These participants had constant access to their peers' answers, could write comments on them, and respond to comments. Both groups received the same pre- and a post-test with tasks on the density of numbers. They were also interviewed after the intervention. The experimental group improved significantly more in its performance on density tasks than the control group. Moreover, the experimental students displayed greater metaconceptual awareness of the change in their ideas about numbers before and after the intervention. It appears that exchanging ideas on the particular tasks in a structured environment with the features of Synergeia was more profitable for students than the whole class discussion (Vamvakoussi et al., 2003, 2004).

Task	Goal
1. What do you know about the real number line? Describe as good as you can. Read and comment upon the answers of your fellow students.	Express prior knowledge about the real number line
2. We often use the term "the set of real numbers". Suppose someone tries to understand what we mean by that. Could you draw a picture to help him/her understand?	Construct a representation for real numbers
3. We have been talking about two different representations of real numbers: A "formal" one, which we usually use at school, and a second one, which was proposed in our discussion and you seem to find adequate. Could you find a solid reason why we should prefer one over the other?	Compare two different representations
4. Imagine that you can become as small as a point of the number line. Then you could see the other points up close. Suppose that you are on the point that stands for the number 2.3. Can you define what point is the one closest to you? Describe in words or by drawing a picture.	Construct a representation for the number line

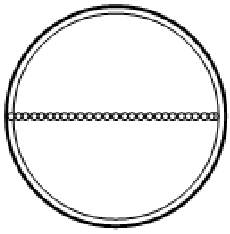
Table 1: Working with representations of numbers and the number line: A sequence of tasks

Vamvakoussi and Vosniadou (2012) further elaborated on the cross-domain mapping between numbers and the line. In line with the FTatCC, they designed a text that a) provided explicit information about the infinity of numbers in an interval, b) made explicit reference to the numbers-to-points correspondence, and c) used a bridging analogy (the number line as a rubber line) to convey the idea that points (and numbers) can never be found one immediately after the other. The excerpt regarding the bridging analogy reads:

> The mathematical number line is a strange object. You can imagine it as a rubber band that never breaks, no matter how much you stretch it. Place numbers between 0 and 1, until it looks like you have used all the available points. If you stretch the rubber band, then you will find out that between the points that looked as if there were the one next to the other, there are more available points, corresponding to more numbers. This procedure can be repeated infinitely many times- don't forget that your imaginary rubber band never breaks!

Vamvakoussi and Vosniadou tested experimentally the value of the "rubber-line" text as compared to two other texts that contained the explicit information and examples of intermediate numbers, or figures illustrating the examples. Six classes of 8th and 11th graders (one experimental class per grade), in total 149 students received a pre-test with density tasks in a arithmetical and a geometrical context, were administered the corresponding text, and then received a post-test containing all the tasks of the pre-test, and 5 additional tasks that examined whether students were able to deal with the no-successor aspect of density (Figure 2). All groups profited from the explicit information about the infinity of numbers presented in the text. However, the experimental group (8th and 11th graders) outperformed the other groups in the "no successor" items of the posttest, and were more consistent in providing correct answers and justifications for their answers.

Kiki says" A line has infinitely many points, which are very close to one another. But if you could magnify a segment of the line at a very high power, then you could see the points one by one, like in the following picture".



➢ **Do you agree with Kiki' s drawing? Yes ❑ No ❑ Explain why:**

Figure 2. Example of a "no-successor" task in a geometrical context

Concluding thoughts

We stress that these tasks come from experimental studies aiming at testing very specific hypotheses, and not in the first place at creating optimal learning environments. As researchers in (the psychology of) mathematics education, being mostly funded to conduct fundamental research, we are not primarily concerned with the development of tasks that can be directly used in classroom teaching, as we mainly aim to analyze and understand students' difficulties in learning particular concepts starting from a certain theoretical stance. Still, we are convinced that our perspective may have an added value for task design. We consider task design an important part of the design of instruction. The tasks presented here were designed on the basis of specific theoretical principles stemming from a conceptual change perspective to learning and instruction. Furthermore, these tasks are empirically tested, also with respect to the conditions under which they can be useful for teachers as well as students.

We suggest that using tasks that prompt students to evaluate, compare, and construct representations (in this case, of numbers) is informative for teachers, in the sense that it provides valuable information on students' thinking. In particular, tasks that were presented as "thought experiments" (e.g., task 4 in Table 1; see also Figure 1) were extremely informative for students' ideas on counter-intuitive notions, which, were not easily accessible via verbal descriptions. On the other hand, the examples presented here indicate that, if such tasks are embedded in a learning environment that is supportive of structured interaction among students, then they may facilitate conceptual change learning. We also provided evidence that even a typical school task, namely extracting information from a text to answer related questions, can lead to conceptual change learning gains, depending on the kind of information that is provided in the text. Specifically, information that bridges between students' initial ideas (e.g., the segment as a "necklace of beads") and the intended mathematical notion (e.g., the segment as a dense array of points) appears to facilitate the grasping of counter-intuitive ideas.

We believe, therefore, that the principles and tasks as elaborated above are interesting from an instructional point of view, and may eventually inspire the development of learning environments. Our textbook analysis – which shows a very large gap with the tasks and principles elaborated above – even strengthens this claim.

Finally, we must stress that the FTatCC is in first instance a cognitively-oriented theory. Of course, students' affect, motivation and beliefs also play an important role in the learning processes to obtain conceptual change, but they are beyond the scope of this paper.

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Theme E
Features of task design informing teachers' decisions about goals and pedagogies

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Based on their mathematical goals for their students, teachers choose or design tasks and sequences of tasks, select media for presenting tasks to students and for students to communicate results, plan pedagogies associated with realising opportunities in tasks, determine the level of complexity of tasks for their students including ways of adapting for them, and anticipate processes for assessing student learning. Each of these decisions is influenced by teachers' understanding of the relevant mathematics, by earlier assessments of the readiness of their students, by the teacher's experience or creativity or access to resources, by their expectations for student engagement, by their commitment to connecting learning with students' lives, and informed by teachers' awareness and willingness to enact the relevant pedagogies. This working group invites contributions from researchers and teachers who have considered such issues from the perspective of task design. The intention is to synthesise what is known about teachers' decision making about tasks, and to offer suggestions about task design for teachers, teacher educators, task designers, text and resource authors, and curriculum developers.

Among the questions that might be considered by authors contributing to the working group and which can be addressed by submitted papers are:

- How do features of design influence teachers' decisions to use particular tasks/sequences, or adapt them, or create their own?
- How do features of tasks/sequences influence teachers' choices about their potential for their class, including the media used for communication about the task?
- How does the design process influence teacher decisions about tasks within sequences?
- How do design considerations facilitate teacher adaptation of tasks/sequences to their students' experiences?
- How does feedback from classroom implementation of tasks/sequences inform future decisions on task design and use?
- How does collaboration between teachers, or between researchers and teachers, influence design of tasks/sequences?
- What are the implications for initial teacher education in task design?
- What is the effect of different cultural backgrounds on teachers' knowledge or belief on tasks and task design?

Teachers and researchers collaborating to develop teaching through problem solving in primary mathematics

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In this paper we focus on collaborative research developed between the staff of a primary school and a university researcher/task developer. The paper draws on findings from the first cycle of research which used a collection of tasks (word problems) and examined how teachers worked with and developed the tasks as springboards for developing students' reasoning in mathematics, as applied to the modelling of additive situations. The tasks were presenting in ways that introduced teachers to the broad principles underpinning the design of the tasks but not the specific intentions behind any one particular problem. This, we argue, encouraged the coming together of the teachers and researcher as a community of learners with teachers adapting and developing the tasks in ways that they saw fit to work with their particular learners. This approach also led, however, to tasks being appropriated in ways that may not have matched with the intentions of the designer: rather than this being an obstacle, it led to rich discussions around the nature of teaching and learning. We argue for the need to consider how tasks are mediated both between task designers and the teachers and between the teachers and learners and to accept that the appropriation at each of these 'interfaces' is open to variation in interpretations, and that working with such variation can lead both the development of pedagogical content knowledge and to student learning. Hence, rather than trying to construct tasks that are 'teacher-proof', working with the fuzziness of appropriation can be a strength.

Keywords: word problem solving, pedagogy, mediation, community of inquiry

Introduction

Problems are key tasks in mathematics education and there is growing support and evidence for the effectiveness of teaching mathematics through problem solving (National Research Council, 2001). This, however, poses challenges to teachers, particularly in primary (elementary) classes. One particular challenge is how teachers can support students to create representations that will help them find problem solutions and the pedagogical decisions involved in using and building upon these representations to make explicit the mathematics inherent in particular problems.

We report here on a teaching experiment in a primary school that involved working with teachers to use problem solving as a starting point through which to help students, aged five to eleven, understand additive reasoning. We examine a double loop of appropriation: the first loop of appropriation of tasks (problems) and representations by teachers when working with a University developer/researcher and the second loop of appropriations that students made in working with the tasks and representations as introduced by the teachers. Through our examination of this double loop of appropriation we aim to answer the key question of:

How were the design features of the tasks appropriated by teachers and in the light of this appropriation what pedagogical decisions were the teachers prompted to make?

In answering this question we seek to examine not only how this type of task design rests on the interplay between artefacts, tools and mathematical knowledge but also on the complex relationship and interface between the task designer/researcher and teachers and between teachers and learners.

In our initial working together (teachers and designer/researcher) several themes emerged that we explore in this paper and that we are looking at in more depth as the work moves into study of developing multiplicative reasoning. The bulk of this paper is jointly constructed, but there are specific observations included from the perspective of the designer/researcher (Mike, first named author) and the teacher initiating the work in the school (Lisa, second named author): where this is the case we present these observations as quotations from one or other of the authors.

Background to the project and the roles of the authors in the design and implementation of the tasks

The work reported here arises from collaboration between a University researcher and the staff of a local primary school. The teaching and learning leader in the school (Lisa) approached the university researcher (Mike) seeking professional development for the school. Although the school's results for mathematics were relatively high, students' scores on assessments of problem solving were consistently lower than scores on items assessing 'pure' computation. The staff had put in place various pedagogies to try and improve students' attitudes towards problem solving and their ability to solve problems, including the explicit use of heuristics but this did not appear to be impacting on standards.

> Lisa: Until recently, teachers have presented learners with word problems and asked them to identify the relevant information and then use some strategy, (draw a picture, make a list, work backwards, etc.) to determine the appropriate calculation to use to solve the question, by being asked, for example, do we add or subtract to determine the answer? We observed that students were getting confused by focusing on the numbers in the problem rather than the context, which in turn was leading to overgeneralisations, such as if the numbers in the problem were, say, 9 and 27, then the answer would be 3 regardless of the question. Students could confidently identify key words such as "more" but again a focus on the numbers given was leading to the assumption to always, say, add. Analysis of assessment results pointed to lower success rates in word problem solving which was in direct contrast to the generally high results in pure number algorithms without context.

The university researcher agreed to work with the school but to treat the approach as one of building a community of joint inquiry into teaching and learning about problem solving, rather than 'delivering' professional development to the staff. At the time of writing one initial cycle of inquiry into learning about problems involving additive reasoning has been completed. The approach taken was relatively informal in terms of data collection as the initial emphasis was on building trust and ways of working together. The cycle of working together has comprised:

- Staff and researcher meeting for a two-hour session after school during which models of working with addition and subtraction problems were introduced and jointly explored and the principles behind the design of the problems (as set out below) shared and discussed.
- Teachers devoting at least one lesson a week to these problems over six weeks.
- Researcher and teaching and learning leader meeting mid-cycle to discuss the progress and to plan the next meeting.
- A half-day second meeting with all the staff to which teachers brought artefacts produced by the children when solving problems and shared how they had incorporated the problems into their teaching.
- The designer/researcher visiting and observing the problem solving lessons.

Task design features

In line with others (Christiansen & Walter, 1986) we use ‘task’ to be what learners are asked to do and ‘activity’ as the subsequent mathematical objects that emerge from engagement in the tasks and through the interaction between participants and resources/representations. Drawing on variation theory (Runesson, 2005) the mathematical object(s) which learner activity enable to emerge can be both direct and indirect. The direct object here was to solve particular word problems, with a first level indirect object of the classification of problems (and thus improve learners’ problem solving) and a second level indirect object of developing understanding of the nature of additive reasoning (and thus improve learners’ understanding of mathematical structure). To promote the emergence of these different mathematical objects the task design features co-ordinated several different aspects, two of which are focused on here: the structure of additive problems, models and representations for solving problems. A third main feature was the choice of contexts through which to frame the word problems. Extensive use of humour and fantasy was made of here, for reasons similar to those set out by Zazkis and Liljedahl (2009) but it is beyond the scope of this paper to discuss this in detail.

Classification of word problems

The word problems were initially designed by adapting the Cognitively Guided Instruction framework (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Problems were written (by the first author) with the intent of exemplifying one of three types of addition/subtraction root problems:

- Change – problems where an initial quantity is increased or decreased;
- Compare – problems with two distinct quantities to compare, one of which is larger or smaller than the other;
- Part-part-whole – problems based around a set comprising two distinct subsets.

All teachers in the school were provided with a collection of these problems, loosely organised into year levels, in a variety of forms, including resources that could be projected onto a whiteboard, or printed off, and versions of the problems that adapted to make similar problems but involving different numbers.

The problems had been previously published (Askew, 2005) and as such were accompanied with booklets providing explicit guidance on the classification and suggested teaching approaches. However it had become clear that although the problems were designed with the above framework in mind, in practice, when discussing problems with teachers and students and discussing where they might fit in the framework did not always result in a particular problem being matched back onto the category that was originally designated. In the light of this inherent ambiguity in the classification of the problems less specific guidance was provided for the teachers in this project: they were left to decide on what the problems exemplified either through discussions with other colleagues or with the students. Thus the intent was that the framework would (a) help teachers make pedagogical decisions about the choice of problems to present to learners and (b) provide a common language for teachers to talk about problems, both amongst themselves and with learners, and also provide a common focus within and across the years.

Models and representations

The work of Carpenter and colleagues has also demonstrated that within this framework for additive reasoning problems the structure of particular problems affects the level of difficult (for example 'change' problem where the 'start' is unknown being cognitively more challenging than those where the final result is unknown). In the light of this a second task design feature was to introduce particular representations to scaffold the move from a word problem into a mathematical model. Rather than have different models for the different types of problems (which would presuppose that the classification of a problem into one of the three types needed to precede the setting up of the model) the use of a 'bar' diagram was chosen as a core representation that could used in all three types of problem. The choice of this model was based in the consideration of not only what artefacts and tools would classroom help learners to solve the specific problems but also, through reasoning about the types of problems, develop awareness of mathematical structure. In other words, the classification of the problems and the setting up of a model were not seen as two distinct, sequential stages but each supporting the other. Artefacts and tools are typically interpreted as having a concrete or virtual 'embodiment', with language regarded as something supporting and developing artefact and tool use. In this project we used two types of artefacts/tools used to support teachers and students. Using a 'bar' diagram where, given three numerical elements in a problem, one of which is unknown, helps the problem solver to decide which is the unknown and to use this not only as a bridge into a solution method but also to think about the structure of the problem and thus became a tool for thinking with (Gravemeijer & Stephan, 2002). For example, given a problem such as:

> Hamsa had 12 muffins. After Hamsa got some more muffins, she had 21 muffins.
> How many muffins did Hamsa get?

A model could be set up thus:

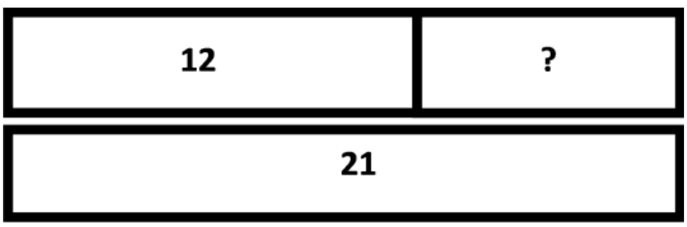

Figure 1: Representing an additive word problem

Emergent findings

Our interest is in examining a ‘double loop’ of appropriation: first, how teachers initially appropriated the classification and representation and the influence of this on the way these were turned into pedagogical tools and second, the subsequent appropriation of the classification and representation by the students. Here we focus in the main on the first of these – teachers’ appropriation.

Teachers’ appropriation of and confidence in working with the tasks.

Our epistemological position is that mathematical understanding and knowledge building arises from meaning making activity - with the emphasis on both the meaning and the making. Meaning does not inhere in words - ‘compare problems’ do not ‘exist’ outside the collection of problems to which this label is attached - and the meaning making comes about through the use of artefacts (classification labels in this case) to think about the problems. Thus the classification of the problems was not seen as a definitive or definite but as a means of provoking discussion about the underlying structure and treating problem solving as a meaning making activity.

> Lisa: The classification of additive problems into the three types was beneficial in two clear ways. The first was in the initial process of identifying the fact that a problem required additive reasoning. Second it enabled students to have a stronger framework upon which to represent the known pieces of information. Representing the second known piece of information in a problem and knowing how this related to the first known piece of information was the challenging step and could lead to some confusion. The framework provided a tool that assisted in the completion of this sense making.

The teaching and learning leader also reported that this approach led to a wealth of discussion between teachers and, in her experience, some of the most extended professional dialogue about the nature and content of the mathematics teaching that she had experienced during her time at the school.

The problems as originally published were organised in groups of three around one of the problem types, with the level of difficulty of the actual calculation being within what might be expected for the age of the pupils. Having solved three problems the expectation was that teachers would discuss with the pupils what types of problems these were. Thus variation (Runesson 2005) in the design of the problems was deliberately based around keeping, initially, the underlying structure of the problems constant and varying the quantities. In the project the teachers became confident in adapting this proposed approach: as they and the students became familiar with thinking about the types of problem, so the teachers felt confident to be flexible:

> Lisa: Initially teachers were using all three of the same type problems from the one page, (all change for example) and students successfully solved these together with teacher direction and also independently in pairs. When instruction moved to another type of problem, this success pattern was continuing. When teachers observed that students were becoming competent in solving the three “types” of problems an assessment was given to review application of this new process. We found, however, that some students tended to apply the process needed to successfully solve the first question to all following problems even if these were of different types rather than selecting appropriate processes for each one. A successful experience in the first question where students knew they had “got it” inadvertently led to using the same reasoning for all following questions. This tendency to “overgeneralise” a known process to different types of problems is typical of what we had noticed previously of our students’ learn one “rule” and

apply it all contexts. And, as before, students weren't always attending to the contextual situation, again manipulating the numbers to get a desired or expected answer regardless of what the question was asking. After much discussion and error analysis of student work in our professional learning teams staff decided to "pick and mix" to encourage the identification or classification of the problem type as well guiding more attention to the context.

Teachers' appropriation of and work with the representations

Assumptions of 'transparency' in the use of representations

After introducing the 'block' diagrams to the teachers, as Lisa explains, these were not initially found to be helpful by all students.

Lisa: The introduction of the block (bar diagram) to students had the intention of providing a means to arrange the relevant information in a relational manner. This visual, as we initially interpreted it, was intended to scaffold student thinking of what we know and how that information related to what we were looking to find out. Teachers modelled the representation by drawing the complete diagram and then inserting given values in appropriate places on the diagram. Students, in pairs, attempted new problems using the bars as a bridging tool to determine the appropriate calculation. This was not totally successful with all students. Some students had difficulty identifying the relationships between the numerical information and using this to place the numbers on the diagram appropriately. What was tending to happen was for them to use the bars as simply another way to lay the numbers out, much as one might be told to write the answer in the box. Thus some learners would write the two given numbers in the short bars and the answer in the third, longest bar, irrespective of the relationship between the numbers. If the underlying calculation was to find the difference between, say, 15 and 40 they would record this as

15	40
25	

Figure 2: Representing (incorrectly) the difference between 15 and 40

Even correct representations were non-proportional making it more challenging to conceptualise the relationships. (For example a problem involving, say, 6 and 24, would show these represented by the same size bar because of the pre-drawing of the whole diagram.)

As the staff discussed what was happening here it became clear that the learners interpretation of the diagram in a 'fill in the boxes' sense could be linked to modelling of the diagram by (some) teachers as a whole rather than building it up sequentially. Following rich professional dialogue between staff in a formal and informal manner, sharing the work samples, (moderation) and further dialogue with Mike, teachers went back to the classrooms to integrate the process of building the diagram one part at a time. Students now were challenged to think about how we could adjust the process of building the visual framework to better represent the problem. Giving the control back to the students to see if they could improve our process enabled them to review and reflect on their own thinking and innovate to assist success rates. Such ownership of the learning enhanced creation of new and shared knowledge. Teachers also noted a significant improvement in student's meta-language used to discuss their thinking processes, thus allowing self recognition of mistakes/successes and the reasoning to support their answers.

Mike: Looking back I can see how the way I introduced the diagrams to the staff would have affected their appropriation of them and in turn how this played out in classroom. I had prepared a 'sorting' activity based on a collection of problems. The collection was based on the Carpenter and Fennema classification with variation in what was unknown creating a dozen or so problems. After sorting and classifying the problems, I had pre-prepared a set of block diagrams to fit with the

> problems. Although we had a rich dialogue around which diagram might go with which problem, I did not engage the teachers with constructing the diagrams and so the points that subsequently emerged in classrooms were not anticipated or addressed.

Thus assumptions of 'transparency' of representations were made in both aspects of the double appropriation loop: assumptions of transparency by the researcher and (both similar and different) assumptions made by the teachers. We argue, however, that rather than this being a difficulty that could have been 'smoothed out' by more attention to detail in setting up the work, that working through the differences resulted in a deeper understanding of the role of the representations by both the teachers and by the students.

> Lisa: The diagram became even more valuable in that it now provided a process through which to present known information in such a way as to illuminate what is required in order to calculate what is not known. The process of drawing these bars, one bar at a time was also valuable as it pushed the students to discuss and to analyse where the second bar of information needed to be placed relative to the first (next to it or below/above and so forth) as well as representing this information in correct proportion to the first piece of unknown information.
> But it also went beyond this in promoting a focus on the proportional relationship between the two pieces of information that were known as well as the relationship between them and what was unknown. For example if one was faced with a problem where 6 apples were combined with 5 apples, the bar that represented the 6 apples needed to be slightly longer than (6/5 of) the bar that represented the 5 apples. And the bar representing the 6 and 5 apples combined together would be as long as the two other bars combined. Thus the representation suggested that this value was going to be in the vicinity of double 5 (10) and double 6 (12). The diagram came to also act as a de-facto 'checking' tool.

Conversions between representations as a linear process

In the discussion in the reporting back meeting several issues emerged but one in particular concerned the interplay between the three aspects of context, representation and numerical equation. It became clear that a popular interpretation was that these were three stages to move through, rather than complementary representations that might be moved back and forth between. Thus once a model had been set up the problem context could be 'set aside' and the model treated as the primary representation that could then be worked with to set up the symbolic. This is consistent with a traditional view of word problems and the contexts therein being regarded as 'immaterial' and 'hollow' vehicles for mathematical calculation: that the context had little to do with the abstract mathematics. In fact, in designing the contexts for the word problems considerable care and effort had been taken to choose contexts that might support mathematising, for example throwing competitions to provoke *comparing*, or homework being destroyed as a context for *change*.

As mentioned, a further design feature was that the problems were not necessarily intended to be 'realistic' or 'relevant' (the realistic-ness or of contexts to individual learners being problematic) but rather to be engaging, particularly through the use of humour and fantasy contexts. Feedback from the staff and children was that everyone had enjoyed the humour thus embodied and a 'playful' approach had been invoked, but this may also have inadvertently reinforced the perception of the contexts only being there as window dressing to be discarded as soon as possible.

> Mike: During my visit to the school one lesson was observed that reinforced this impression of a linear movement from context to diagram to symbols. The context of the problem was two characters involved in a domino stacking competition with questions around the differences in the numbers of dominoes successfully

> stacked. Some of the Grade 5 students working on this were experiencing difficulties over producing an appropriate model for the problem: several children were putting the two quantities together end to end and representing the missing amount as the total rather than setting up a compare model. In the discussion that the teacher had with the class about which of these was the appropriate model, no reference back to the context of actually stacking things up and comparing the heights was made: the discussion was focused around the models and the numbers - the context was not used as a way into setting up the context.

Again this has been treated as a focus of discussion in the meetings between researcher and staff and as the work moves into multiplicative reasoning this is an aspect that explicit attention is being paid to.

Implications of the work reported in the paper on different communities

The first implication is that word problems can provide tasks that teacher and students not only find engaging but also which can give rise to purposeful mathematical dialogue and meaning making.

> Lisa: Initial responses from students have been most positive - they, informally, have reported engagement with the problems, a delight in the humorous nature of them and pleasure in discussing and exploring the possible underlying structures. Whether this has resulted in gains on solving word problems in other situations, most notably national tests, will become clearer as the study unfolds. Teachers involved so far reported a shift in perspectives on the teaching of word problems and an awareness of the meaning making processes.

Second, the introduction of a framework for working on and with the tasks, and treating this as a joint endeavour has promoted teacher inquiry and collegiality

> Lisa: As an integral part of the performance and development culture of the school, staff observe each other's lessons within and across the levels. This direct observation leads to a structured discussion between the staff members about the learning taking place: as all staff have responded to the professional development and focus on problem solving so positively along with a growing relationship of trust, in this context observation and feedback should prove highly beneficial.

A further question is how the publication and presentation of such tasks might be done in ways that provoke similar responses without the presence of a researcher. While publishers might take note of the decision not to supply teachers with the 'key' to what type of problem each word problem was meant to exemplify, in the absence of the discussion around the design principles and the revisiting of these to refine the meaning, then the tasks alone may simply result in a set of 'dull' word problems.

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Conflicts in designing tasks at collaborative groups

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This is a theoretical paper in which some categories are proposed to analyse conflicts in designing tasks in collaborative groups composed of academics and schoolteachers. Based on insights from our own experience in a collaborative group that designs tasks for other teachers, and by using concepts of Basil Bernstein's theory, we propose five arenas of conflicts between academics' and teachers' in collaborative groups while they are producing tasks.

Keywords: Teachers, Academics, Collaboration, Tasks, Conflicts

Introduction

Task design in teacher education is a visible theme in literature (e.g., Zaslavsky & Sullivan, 2011). Let us follow the wide definition proposed in the ICMI Study 22 (ICMI, 2012): "anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something" (p. 10). Some studies suggest tasks do not determine teachers' actions (Remillard, 2005), but tasks provide opportunities and limitations for teachers' and learners' actions (Remillard, 2005; Watson & Mason, 2007; Sullivan, Clarke & Clarke, 2009).

In this paper, we are interested in tasks designed by collaborative groups. This collaborative group may be defined as one which brings together school teachers and academics (Levine & Marcus, 2010). In this case we use the term academics to denote those who work at university institutions conducting research, teaching or developmental programmes. We were convinced by Jaworski's (2011) argument that "teacher educator" is not the best term since we are interested in characterising the partnership between academics and teachers.

Evidence suggests collaborative groups have positive effects on teachers' and students' learning (Levine & Marcus, 2010; Jaworski, 2005, 2011; Ferreira & Miorim, 2011). Despite different understandings of teacher learning (Borko, 2004; Adler & Davis, 2006; Ball, Thames & Phelps, 2008), we wish to be explicit about our point of view: teachers' learning is related to changes in patterns of participation in the classrooms (Borko, 2004). There is not enough space here to develop this topic, so we ask the reader to consider our point of view when we talk about teacher learning throughout the paper.

In collaborative groups, teachers have an active role in choosing what to do. Every participant has an opportunity for mutual learning, i.e., academics may learn from teachers, and vice-versa. Despite the term "collaborative", Jaworski (2005,

2011) and Zaslavsky (2008) identified tensions between teachers' and academics' knowledge in designing tasks as they may present opposing points of view. This finding suggests that collaborative groups are also a terrain for conflict (for now, let us consider the term "conflict" intuitively). The same insight came to us from our own experience in a collaborative group that we discuss below.

Conflicts are likely to emerge in designing tasks in collaborative groups, since it involves the partners making decisions from different perspectives. For instance, researchers' opinions may reflect their theoretical preferences whereas teachers may express their opinions in terms of their own experiences in school contexts. In this paper, we will focus on the role of conflict in designing tasks in collaborative groups. As part of an ongoing research project, our theoretical contribution is based on concepts from Basil Bernstein's theory (Bernstein, 1990, 1996). In order to build our argumentation we mention studies reported in literature and our experience in a collaborative group.

Initially we present a conceptualisation of collaborative groups. Subsequently, we introduce the collaborative group that we were part of and an exemplary task designed by the members of the group. Finally, we suggest five arenas of conflicts in designing tasks inside collaborative groups.

Collaborative groups

Collaborative groups may work in different forms, and also represent different types of learning (Levine & Marcus, 2010). Although not using the term "collaborative", Jaworski (2005) explains the key characteristic of an inquiry community composed of academics and teachers, which is that it requires us all to trust and have confidence in working together. It suggests that members recognise their different expertise and they share them in order to achieve a goal, which is in line with what Wagner (1997) calls *co-learning agreement*. Ferreira and Miorim (2011) make a distinguishing point for collaborative groups:

> [It is] a group in which participation is voluntary, in which all individuals involved search for professional growth, share trust and respect, support the group work, engage in a common purpose, all the while creating and sharing meanings about what they are doing, about their lives and professional practices (p. 138).

These authors state that not every group should be recognised as collaborative. The authors call into question the nature of participation, which is expected to be voluntary. As a result, a compulsory developmental programme should not be understood as collaborative at all, since teachers cannot choose whether they want to take part in or not.

In some sense, collaborative groups are also regulative on what is moved or not to classrooms. Following this view, collaborative groups may be considered part of what Basil Bernstein (1990, 1996) calls *pedagogic recontextualizing field*. The function of this field is to move texts produced in the scientific areas (for instance, Mathematics, Mathematics Education, and so on) and relocate them for instance into classrooms. As collaborative groups focus on ways of improving teaching and learning mathematics, they have a regulative action onto classrooms.

In the above paragraph, we mentioned the term "texts", which is a key concept in Bernsteinian theory. Here, *texts* refer to all communicative actions, which may be oral, written, gestural, and so on (Bernstein, 1990). A task is itself a written text, whereas designing tasks requires participants to use different types of texts.

Collaborative groups establish a pedagogic relationship, but not with fixed positions between who teaches and who learns (using Bernsteinian terminology, between transmitters and acquirers). Both teachers and academics may interchange the position of transmitters and acquirers at different times since the relationship is based on mutual learning. We name this as *floating positions* in collaborative groups. This does not mean that the pedagogic relationship is unstable, but floating positions is its way of being, its nature, if we ensure the principle of mutual learning in the group.

As pedagogic practice, every collaborative group has its own principles that regulate its functioning. Bernstein (1990, 1996) points out two main features of pedagogic practices: classification and framing. The first one regulates what is legitimate to communicate, the second regulates how to communicate. Then by the co-learning agreement, and as long as the collaborative group is going on, participants address the principles that regulate the relationship between them.

As a pedagogic relationship, collaborative groups may also experience conflicts. Jaworski (2005), for instance, reports a conflict between teachers and academics in a collaborative group. While teachers were reporting they use inquiry-based tasks in classrooms as additional to the curriculum, the academics were emphasizing them as being curriculum-related. Piazza, McNeill, and Hittinger (2009) refer to two sources of conflict in teacher communities: beliefs about subject matter and the purpose of education. Conflict is not negative, with some authors viewing it as a source of learning (Jaworski, 2005; Zalavsky, 2008; Piazza, McNeill & Hittinger, 2009). Moreover, conflicts may tell us about the nature of collaborative groups and their internal dynamics.

Through Bernsteinian lenses, we see conflicts in collaborative groups as an expression of the space that insulates original contexts of academics and teachers. Even engaged in mutual learning, the participants are likely to make public their views grounded in either university communities or schools, respectively, be they academics or teachers.

Mutual learning does not remove horizontal hierarchies among the participants. Let us imagine a collaborative group in which a well-respected academic takes part in that. In such a case, the voice of the academic may be stronger in the group than other participants. The same may happen to an experienced teacher who has a higher expertise about anticipating students' actions. Their respective arguments may be more vocal than others, and so reducing the opportunity for others to engage in dialogue.

An example

At the end of 2010, we invited Master and PhD students, prospective and experienced teachers to join with us to form a collaborative group with the purpose of designing mathematical tasks for school students' and teachers' use. Participation was voluntary, and the group was composed of 25 participants in all. It was called *Mathematics Education Watch* (Observatório da Educação Matemática, in Brazilian Portuguese) and is known by the acronym OEM.

The main aim of the OEM was to design a kind of written task referred to as *educative curriculum materials* by Remillard (2005). Educative curriculum materials are those that aim at both students' and teachers' learning (Remillard, 2005). Remillard in turn uses the term “curriculum materials” for those designed only for student learning. The adjective “educative” added to the term means that there is some support for teachers to use the task. Then, the main motivation in organising the OEM was to bring together teachers' and academics' expertise to assure that the materials would be informed by teachers' know-how.

The group focused on the geometry competences recommended by Brazilian official documents (BRASIL, 2008). The whole group was organised in seven subgroups, and each one included, at least one academics (Master and PhD students were also considered academics), one experienced and one prospective teacher.

Figure 1. Group picture taken after a regular meeting.

The group decided on the following phases to design educative curriculum materials:

(1) to review literature on teaching and learning of geometry;

(2) to draft curriculum materials;

(3) to review and refine the curriculum materials;

(4) to use the materials in the OEM teachers' classrooms and to record the implementations;

(5) to design the educative curriculum materials;

(6) to review and to refine them;

(7) to make them available on the Internet for other teachers.

At the time this paper was written, the group was starting phase 5.

Collaborative task design and the associated conflicts

Zaslavsky (2008) has shown that the cycle of designing, implementing and modifying is an effective way of supporting teacher learning. Also this is a rich opportunity for us as academics to learn more about school mathematics and how teachers position themselves, which may ultimately help us to improve the ways of supporting teachers.

Alrø and Skovsmose’s book (2002) was influential within the group, and we decided to design investigation-based tasks. Figure 2 is an example of a task designed by a subgroup, and refined by the whole group. This task is to be used with the software *Geogebra*, which is familiar to the teachers who are involved in the OEM. The task in Figure 2 can be described as curriculum materials, because at this stage it is only focused on supporting student learning.

STUDENT TASK 1

EXPLORING POLYGONS

Today let us study polygons by using the software Geogebra. Follow the guidelines below.

(1) Draw regular or irregular polygons and name their vertices.
(2) Draw all line segments possible between the vertices.
(3) According with your observations, fill in the table:

Number of sides	3	4	5	6	7
Name of the polygon					
Number of diagonals					

(4) What if you have a polygon with 10 sides? How many diagonals? And, if 20 sides? And, if 30 sides?
(5) Do you note any relationship between the number of sides and the number of diagonals?
(6) And, for any number (n) of sides? Can you propose an algebraic representation for the relationship between number of sides and number of diagonals in a polygon?

Figure 2. Task designed at the OEM (translated from Brazilian Portuguese).

The process of producing tasks provides an opportunity for many discussions within subgroups, as well as the group as a whole. From this experience, we noted some conflicting points of views about decisions to be made for the task. In Bernsteinian terms, our interpretation was the conflicting messages were aligned to the different settings participants originally belong to.

In a Bernsteinian analysis, the interest is upon discursive control. In his words: the focus is on how power and control are translated in principles for pedagogic communication (Bernstein, 1996). As previously noted, collaborative groups also represent pedagogic practice based on mutual learning. As a consequence, there are principles that regulate what the participants communicate to each other. It is possible that the control is weaker than other contexts in which transmitter and acquirer have fixed positions. However, it does not mean absence of control, which is operated among and by the participants.

Conflicts about the features of a task should be viewed in terms of agents addressing different principles of pedagogic practice. For instance, the academics might share their theoretical views in the group. By doing this, they select what to say, and also how to build their argumentation to be effective in that context. This is a sort of recontextualization (Bernstein, 1996) operated by academics from scientific field to a setting – in case, a collaborative group – in the pedagogic recontextualizing field. In turn, teachers' arguments might be based on their experience in schools. Also, they select what to say and how to do it as they recognise the setting has a particular set of principles. In Barbosa (2013), it was called *reverse recontextualization*, where legitimate texts move from classrooms to the pedagogic recontextualizing field.

In some sense conflicts in collaborative groups may be framed as the encounters between texts that evoke principles from other two fields (scientific field and classrooms). The same sort of conflict could arise with relation to official pedagogic recontextualizing field, that one represented by state (Bernstein, 1996).

Generally conflicts in designing tasks may refer to different dimensions, all of them related to what to select to be approached in the task and how to do that (a very

Bernsteinian point of view). Let us use the term *arena* to refer to the scope of a conflict, a disagreement, i.e., the subject of the conflict. From our experience in the OEM, and based on Bernsteinian lenses on discursive control, we shall suggest five possible *arenas of conflicts* for task design in collaborative groups: *context*, *use of language*, *structure*, *distribution* and *subjects*. These *arenas of conflicts* are analytic categories proposed to analyse conflicts in collaborative groups when they are engaged in decision making in the design tasks. The categories may be considered part of what Bernstein (1996) calls *languages of description*. The function of the *languages of description* is "to produce a specific text and translate these referential relations into theoretical objects or potential theoretical objects" (p. 133).

The *arena of context* refers to the mathematical context of tasks. Alrø and Skovsmose (2002) theorise three possibilities for mathematical tasks: reference to pure mathematics, semi-reality, and reality. We may consider two extremes – pure mathematics-based and reality-based tasks – and many possibilities between. In this case, the conflict of context arises when participants are discussing the adequacy of the context for a task to achieve its learning goal. In producing the task in Figure 2, for instance, the argument that a reality-based task would be more motivating for students came into conflict with the argument that this sort of context, in the case, could cause difficulties for learners.

The *arena of language* refers to the level of rigour tied to the task. In Figure 2, the rigour is expressed through the use of some terms and, in particular, at question 6, an algebraic representation is required. The level of rigour could be weaker or stronger than that, and it also gives room for conflicting arguments while designing tasks.

The *arena of structure* refers to the degree of openness in tasks. If we have a look at task in Figure 2, we may note that the auxiliary questions follow a sequence that scaffolds students' actions. The task could be much closer, if more auxiliary questions were posed to guide students' actions. Also, the task could be much open, for instance, if the question was only put in terms of investigating the relationship between the number of sides of a polygon and its number of diagonals. Then, those conflicts may be positioned between two extremes: closed-end and opened-end tasks.

The *arena of distribution* refers to what is expected to teach in a task. It refers to selecting content to be focused on tasks. For instance, in the task of Figure 2, a conflict could happen between two possibilities: the task should focus on the relationship between number of sides and diagonals, or on other regularities. The control on what is explored by students while they are approaching an open-ended task is unpredictable. Stein et al. (2000) classified them in high or low levels, as they require memorization or investigations and explorations, respectively.

Finally, the *arena of subjects* refers the way the task positions students and teachers. In a closed-end task, the teacher is expected to keep themselves far from students' doings on the task, once this is very structured. On the other hand, in an opened-end task, teachers are expected to interact more with students, as the task itself does not have many scaffolds. So every task suggests a level of insulation between students and teacher (at least, an expectation).

The five *arenas of conflicts* suggested above are expected to work as tools to analyse conflicts in decision making in the design of tasks within collaborative groups. Note that conflicting arguments in each arena is characterised in terms of a continuum, which means arguments may be positioned at any point of the segment (Figure 3).

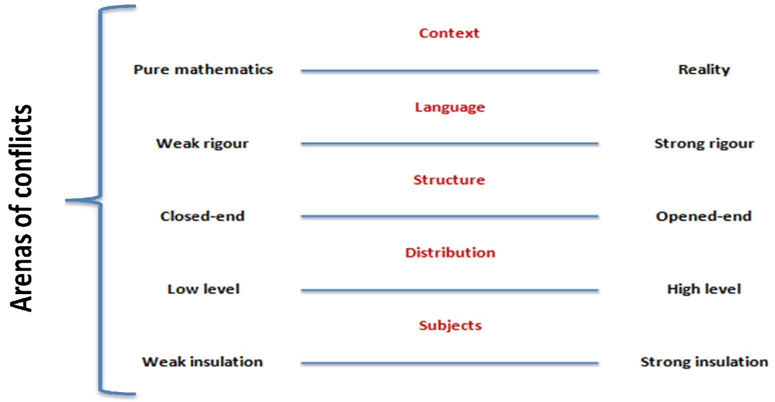

Figure 3. Diagram representing arenas of conflicts in designing tasks inside collaborative groups.

As a result, the resulting task holds the markers of the conflicts, which may be inferred from an analysis on the task (Barbosa, 2013). Then the diagram in Figure 3 may be useful to analyse conflicts in a setting of task design, and also to analyse tasks themselves.

Final remarks and implications

In this paper, we attempted to provide theoretical categories to analyse conflicts in collaborative groups while they are designing tasks. Instead of searching only for agreements within collaborative groups, our argument is that they also hold conflicts represented by opponent points of view, which is line with Jaworski (2005, 2011). We suggest conflicts are results of encounters of texts that follow the logic of the original settings of academics and teachers.

As theoretical description, we propose five arenas of conflicts between academics and teachers while they make decisions about designing tasks. Conflicts must not be understood as violent disputes, which is uncommon for collaborative groups. On the contrary, conflicts are likely to take shape in dialogical conversations and/or harmonious conversations, such as expected for this kind of group (Ferreira & Miorim, 2011).

Whether we are interested in supporting collaborative groups and teacher learning, we need to develop more studies on conflicts in those groups. Particularly, it is necessary to shed light on how academics and teachers deal with and solve the conflicts. Further insights on possible strategies for supporting those groups may be developed.

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A dialogue between cultures about task design for primary school

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The aim of this paper is to exploit intercultural dialogue to analyze two cases of task design (or, better, *re-design*) for word problems in different cultural traditions (the Eastern one, within the Confucian Heritage Culture, CHC, represented here by China and the Western one, represented here by Italy). By means of two paradigmatic examples, one developed in Italy and one in China, we aim at showing, on the one hand, the effects and advantages of intercultural dialogue and, on the other hand, the need to take into account and to respect culturally rooted pedagogies, avoiding uncritical transfer from one culture to another.

Keywords: word problems, problems with variation, primary school, intercultural dialogue, addition/subtraction

Introduction

Word problems are a special kind of tasks, presented all over the world, in mathematics textbooks to link numbers to real life objects and situations. They are the heirs to ancient traditions of mathematical texts, existing in all cultures. In this paper, we consider, as a paradigmatic case, *problems for early childhood classroom to be solved by addition and/or subtraction,* although some observations and examples are more general.

After reviewing some literature about word problems in mathematics education in China and in the West, we present a special kind of word problem taken from Chinese textbooks, i.e. *problems with variation* (*biànshì problems*).

Then we illustrate how the meeting with the Chinese tradition has suggested to a group of Italian researchers (including primary school teachers) to *re-design* (and test in the classrooms) a rich system of problems with variation. On the one hand this redesign is in order to meet needs emerging from Italian school practice, , and to tailor classroom activity to their system of beliefs, on the other hand. Conversely, the meeting with the Western tradition has modified some principles in the Chinese approach to problems with variation, enriching the original concept based structure and offering new criteria for task *re-design*.

Closing remarks summarize authors' contributions to the following questions (concerning themes C and E of the Study):

C1) How do curriculum expectations influence authors' design principles?

C2) How does an intention to promote change influence design?

C3) How do cultural considerations about instruction and pedagogy influence design?

E1) How does collaboration between teachers, or between researchers and teachers, influence design of tasks/sequences?

E2) What is the effect of different cultural backgrounds on teachers' knowledge or belief on tasks and task design?

Some literature about word problems

A short review of Chinese literature

Taoism has had profound influence on Chinese culture. The central Taoism idea of the evolution of events as a change process and the acceptance of the inevitability of change reveal the ideologies of "grasping ways beyond categories"; "categorize in order to unite categories" (以法通類, 以類相從). In China, for 5000 years, mathematics knowledge was elicited by word problems, which stems from the "Shu" (术) spirit (similar to "general methods") in the problem-oriented tradition from Oriental mathematics , "… to produce new methods from word problems, promote them up to the level of general method, generalize them into 'Shu,' and deploy these 'Shu' to solve various similar problems which are more complicated, more important, and more abstruse" (Wu & Li, 1998). Impacted by the idea of "grasping ways beyond categories"; "categorize in order to unite categories", word problems in ancient China were organized into different categories in terms of situations or algorithms. For example, Jiǔzhāng Suànshù（九章算术）, the most classic literature of Chinese mathematics, used 246 word problems to spread mathematical knowledge.

The tradition of categorizing word problems did not go on when Chinese curriculum on mathematics knowledge was totally borrowed from the West in 1878, where word problems, labelled as "application problems" ("应用题"), played a role of knowledge application, not introducting original knowledge. In 1929, the first formal curriculum standard, since new China was founded, primary school arithmetic syllabus (draft) (《小学算术教学大纲(草案》), claimed that "application problems are the important part and should be covered about half of the arithmetic contents" (Wang, 1996), which has remained unchanged as basic requirements of elementary curriculum for 80 years at least. In 1952, application problems in the first official textbook for whole country were categorized into simple application problems and complex application problems. Since 1958, application problems have not been organized by their categories but by their application content for the reason that knowledge system is required as a curriculum framework. Although application problems in textbooks were not organized by their categories again, their teaching organization stressed not a single problem but a group of problems with variation.

Following the thousands-years-traditions "categorize in order to unite categories", one distinctive instruction feature of word problems is to develop the ability to identify the category of word problems (识类) belonging to and discern different categories (归类), namely, discern the invariant elements from the variant elements between problems and recognize the "class" every problem belong to. This pedagogy is generally called as biànshì (变式) in Chinese, where "biàn" stands for "changing" and "shì" means "form", can be translated loosely as "variation" in English (Sun, 2011). Some categories of biànshì are the following (examples follow):

OPMS (*One Problem Multiple Solutions*), where, for instance, the operation to solve the problem is carried out in different ways, with different grouping and ungrouping:

$$8 + 9 = (8+2) + 7;\ 8 + 9 = 7 + (1 + 9) \text{ and so on.}$$

OPMC (*One Problem Multiple Changes*), where in the same situation some changes are introduced.

MPOS (*Multiple Problem One Solution*), where the same operation can be used to solve different problems, as in summary exercises (Sun, 2011).

To sum up, in Chinese mathematics education problem variations aim to discern, compare the invariant feature of the relationship among concepts and solutions and provide opportunities for making connections, since comparison is considered a pre-condition to perceive the structures, dependencies, and relationships that may lead to mathematical abstraction (Sun, Wong, & Lam, 2007).

A short review of Western literature

Different strands (and a different pedagogy) emerge in the literature on word problems. In this paper we mention only some strands.

The *cognitive analysis* of word problems (see different contribution in Carpenter, Moser, & Romberg, 1982) focuses on the difficulties met by students in understanding the problem and looking for effective solution strategies and has produced well established categorizations likely to be employed in educational setting (e.g., the well agreed categorization of additive problems into *combine*, *change* and *compare*).

The *didactical analysis* of word problems (see different contributions in the recent volume by Verschaffel et al., 2009) criticizes the stereotyped and not realistic features, highlights the distance with modelling activity and focuses on the negative effects of word problems on students' sense making capabilities. Actually, there are examples of word problems which suggest uncritical applications of rules. There is the famous case (ibidem, p. xii) concerning the age of captain: "On a boat there are 20 sheep and 6 goats. How old is the captain?" The findings show that many students would respond to such question by adding the numbers of sheep and goats.

Some literature on word problems is strongly related to the early development of algebraic reasoning that may build a bridge between the two cultures (e.g. Ofir & Arcavi, 1992, Cai & Moyer, 2008).

Examples of Chinese problems with variation

The introduction of subtraction in the first grade by problems with variation

Fig 1. Mathematics Textbook Developer Group for Elementary School, 2005, vol. 1

Chinese textbook authors never separated the subtraction concept from addition.

Whenever there is addition there is subtraction (Yang Hui, 1274, in Siu, 2004, p. 164).

Figure 1 shows a paradigmatic example of problem variation: Xiao Ming folds a pink paper crane; xiao li and xiao hua fold two blue paper cranes. How many paper cranes do they fold? There are 3 paper cranes. Xiao ming takes a paper crane. How many paper cranes does he leave? The answers are: 1+2=3, 2+1=3, and 3−1=2. The drawing intends to help learners to recapitulate the relationship of addition and subtraction, and the meaning of “equal” from the problem set 1+2=3, 2+1=3, 3−1=2. The problem sets hinges on exemplifying relationships rather than objects and reflects the mathematical structure underlying the problems in this respect. The addition concept is different from that of subtraction, which belongs to a different category. Yet in the sense of part-part-whole we can combine these two concepts into one category and understand the ancient idea of “grasping ways beyond categories”; “categorize in order to unite categories.

A summary system of problems with variation in second grade

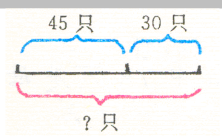
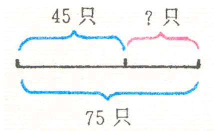
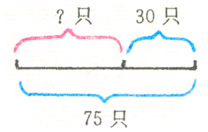
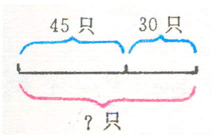
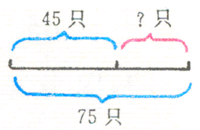
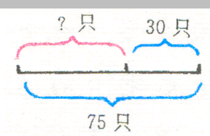
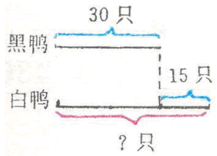
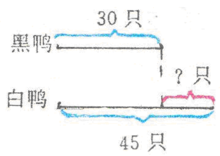
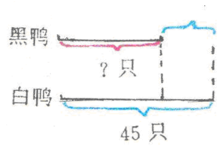
First solve the nine problems below. Then explain why they have been arranged in rows and columns in this way, finding relationships		
(1)**In the river there are 45** white ducks and 30 black ducks. All together how many ducks are there?	(2)**In the river there are white** ducks and black ducks. All together there are 75 ducks. 45 are white ducks**.** How many black ducks are there?	(3)**In the river there are white** ducks and black ducks. All together there are 75 ducks. 30 are black ducks**.** How many white ducks are there?
		
(1) **In the river there is** a group of ducks. 30 ducks swim away. 45 ducks are still there. How many ducks are in the group (at the beginning)?	(2) **In the river there are** 75 ducks. Some ducks swim away. There are still 45 ducks. How many ducks have swum away?	(3) **In the river there are** 75 ducks. 30 ducks swim away. How many ducks are still there?
		
(1) **In the river there are** 30 black ducks. White ducks are 15 more than black ducks (black ducks are 15 less than white ducks). How many white ducks are there?	(2) **In the river there are** 30 black ducks and 45 white ducks. How many white ducks more than black ducks (How many black ducks less than white ducks)?	(3) **In the river there are** 45 white ducks. Black ducks are 15 less than white ducks (white ducks are 15 more than black ducks). How many black ducks are there?
		

Table 1. Beijing education science research institute and Beijing instruction research center for basic education (1996), vol. 4, p. 88.

This is a system of nine problems concerning addition and subtraction, where the organization in rows refers to the already mentioned combine, change, compare categorization and the organization in column refers to the same arithmetic operation (either addition or subtraction, see MPOS above). In each row there is a problem (in the shaded cell) and two variations (see OPMC above). It is taken from a Chinese second grade textbook. The task is very complex and requires the students not only to solve each problem but also to explain why the nine problems have been arranged in this way. Each problem is associated with a graphic scheme, that models on one or two lines the relationship between quantities. Such graphic schemes for both additive and multiplicative problems had been introduced systematically by Davydov (1966) in his algebraic approach to quantitative reasoning and relationships and are used in some countries of the CHC area, such as mainland China, Japan, Singapore. In a study carried out in Italy, we first used it in teacher education as a prompt to challenge teachers' beliefs (Bartolini Bussi et al., 2011; Bartolini Bussi et al., 2012); later a re-designed task was proposed to Italian students (several experiments from grade 2 to grade 5) in order to foster the approach to algebraic reasoning as soon as possible.

Task re-design in Italy and Hong Kong: hint at two case studies

A transposition of problems with variation in Italy

Needs in school practice

In Italy, in spite of the different suggestions of the Standards and Programs (http://www.mathunion.org/icmi/other-activities/database-project/introduction/italy/) where the focus is rather on the sense making of a situation and on modelling (*mathematization*), it is very popular strategy among teachers to suggest "cues" in the problem text in order to detect the operation to be used. For instance, in a popular website (http://www.lannaronca.it/), for additive problems, teachers are instructed to invite students to underline words like "aggiungere" (adding), "in tutto" (in all), for addition, and words like "togliere" (remove), "restare" (remain), "in più" (more), "in meno" (less) and so on for subtraction. Without careful control, this cue might lead randomly to either right or wrong choices. Consider for instance the two texts with the same implicit question: "how many candies does John have?".

> "John has 5 candies and Anna gives him 2 more candies"
> "Anna has 7 candies, 2 more than John".

In spite of the same question and of the same cue, in the former case the direct reference is to addition and in the latter to subtraction. The confusion is reinforced by the fact that in Italian "più" is also the wording of "+" (plus), that is the special sign for addition. The two situations might be related to each other highlighting unknowns:

$$5 + 2 = ? \text{ and } 7 = ? + 2.$$

but teachers are not encouraged to link addition and subtraction. They are rather encouraged to follow the principle one-thing-at-a-time and to practice addition for months before introducing subtraction. A very popular series of exercise books (http://www.erickson.it/Ricerca/Pagine/Results.aspx?k=matematicaimparo&start1=1, *Matematicaimparo*) contains even two different booklets (with different authors), one for addition and one for subtraction problems. The above practice has the effect to produce very poor performances in the solution of arithmetic problems in the national assessment carried out at the end of the second and fifth grades.

Task re-design

Re-design concerns the task of the Table 1. Teachers-researchers who have collaborated in the pilot study have not implemented the same Chinese task, but have *re-designed* it to tailor it to the Italian tradition and to their individual teaching styles and systems of beliefs.

Three main changes were introduced:

1) the single task has been transformed into a set of several tasks;

2) classroom work was organized according to a sequence inspired by the theoretical framework of *semiotic mediation after a Vygotskian approach* (Bartolini Bussi & Mariotti, 2008):

a) individual or small group solution of each row of problems followed by the invention of three problems similar to the given ones, to foster the awareness of the problem structure;

b) collective discussion of the findings, with teacher's orchestration.

Moreover the solving graphic schemes (at the beginning) were removed and introduced later, after thorough exploration and solution of the problems, as students were not familiar with such schemes. In this way the use of a graphic scheme was acknowledged by students as meaningful and not perceived as an automatic answer to a given task. In this way the task (originally developed in China within a teacher centered and textbook centered tradition) was modified to fit a dialogic approach where teaching and learning are considered the two sides of the same collaborative process.

In the same way, multiplicative problems with variation were introduced as from the third grade in several classrooms. (Bartolini Bussi et al., 2011). The project is in progress and involves nearly 100 teachers from September 2012.

A modified approach to problems with variation in Hong Kong

Needs in school practice

Inspired by the Western tradition of phenomenography and pedagogy of variation (Marton & Booth, 1997) and theory on the process of mathematization (National Council of Teachers of Mathematics, 1989), Chinese researchers found that a focus on dealing with concepts and deriving a structure among these concepts is not sufficient; there is also a need to pay attention to two important processes of mathematization, *induction from real life contexts and application to different contexts*, that were introduced into the tradition of biànshì problems.

Task re-design

In a structured set of biànshì problems two types of "contextual variation" were proposed "inductive biànshì" and "application biànshì". These two types of variation should play a role to extend and enrich the use of mathematic (structural) variation by appropriating relevant real life contexts. The use of inductive biànshì begins with a real life context in which the established concepts are carefully embedded and unfolded in a set of problems that leads to the new concept to be established (the deepening biànshì); and using application biànshì provides new contexts (even created by students) for students to connect (or apply) different acquired concepts (the widening biànshì). Examples associated with this approach to the division of fractions are reported by Sun (2007) and Wong, Lam, Sun, & Chan (2009).

Closing remarks

The two cases of the last section have been developed independently by the authors in Italy and Hong Kong before having the occasion to discuss with each other. In both cases a task re-design had been started to answer needs coming from school practice and from theoretical and pedagogical backgrounds, on the one hand, and prompts from the other cultural tradition, on the other hand. In particular, in both cases the needs for change (questions C1 and C2) were in the foreground, because of the awareness that existing practice did not meet the intended curriculum expectations. In both countries (question E1) collaborations between researchers and teachers were realized, although with different modalities (see below).

In general, the Chinese literature on word problems as tasks is focused on *teaching*, e.g. the preparation of good textbooks according to the intended curriculum and the study of effective lessons which are structured around textbooks, whilst the Western literature on word problems as tasks is focused on *learning*, i. e. the analysis of difficulties met by students to interpret the task or to make sense of the problem situations in mathematics. These differences have effects on the researchers' and teachers' conception of task design (and re-design). For instance, Chinese teachers generally have limited space to re-design tasks due to the fact that Chinese curriculum evaluation (exam) is unified by government and curriculum content is required to follow strictly the unified standards and the unified textbooks. In Italy teacher-researchers have more freedom to design tasks and to devote some sessions to pilot teaching experiments like the ones mentioned in this paper. In both cases, the culturally rooted pedagogies were essential to re-design tasks: in Italy teachers disassembled Chinese tasks to introduce collective discussions orchestrated by the teacher; in China induction from real life contexts was put within the variation scheme.

The questions C3 and E2 are surely the trickiest to be addressed. We give some elements to support this claim, although it is not the scope of this paper to address the general comparison of cultural traditions that encompasses the issue of word problems as tasks (for a general discussion, see Xie & Carspecken, 2008).

Word problems have been part of the historical development of mathematics in China and in other parts of the word as well. As Gerofsky (2009) claims, non-realistic word problems are a written and pedagogical genre that expressed generality through exemplification and have played an important role in all the pre-algebraic societies. From these shared roots two cultural traditions developed in different ways. In China the superb development of Algebra (maybe the main contribution of Chinese scholars to mathematics) was not accompanied by a parallel theoretical approach like Euclid's one; in the West the assumption of Euclid's Elements as the paradigm for mathematics development gave for many centuries a different status to practical mathematics (e.g. commercial arithmetic) from where the Western tradition of algebra was later nurtured.

Mathematization became a key point in both traditions, although in different periods. The focus on mathematization started in the West at the beginning of 19th century when the teaching of natural sciences was introduced systematically in schools and gave rise in 1983 to the International Community of Teachers of Mathematical Modelling and Applications that addressed criticisms of stereotyped word problems and influenced the development of mathematics curricula, (http://www.icmihistory.unito.it/ictma.php). The expansion to China is more recent although Chinese educators were exposed to Western influence for decades (e.g. Dewey, Smith, Freudenthal, see Wang, 2013, p. 38 ff.).

Hence, task re-design considered in this paper took place in different cultural traditions. It had in both cultures the advantage of respect the local pedagogical and theoretical roots and to exploit perspectives from the other culture. Our attitude was, in both cases, similar to the one described by Jullien (2008) in the case of philosophical dialogue between China and the West:

> This is not about comparative philosophy, about paralleling different conceptions, but about a philosophical dialogue in which every thought, when coming towards the other, questions itself about its own unthought (Jullien, 2008, p. iii, our translation).

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Scaffolding Tasks for the Professional Development of Mathematics Teachers of English Language Learners

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This article outlines a design framework for classroom exemplars to be used in the professional development of mathematics teachers of English Language Learners. This framework shapes activities built around mathematical practices to scaffold student engagement in interactive tasks that foster their emerging autonomy. Empirical results from applying this framework to design teacher apprenticeship is reported. Data includes both professional development institutes and instructional coaching cycles. Results suggest trajectories for teachers' shifting understanding of conceptual, academic, and linguistic goals as they appropriate a pedagogy of promise that fully develops the potential of all ELLs.

Keywords: English language learners, scaffolding, interaction, lesson design

Introduction: Challenging and Supporting English Language Learners

In the United States, English Language Learners (ELLs) are a rapidly growing population, increasing by 51% in the past decade. Policy at the federal level has positioned ELLs through a deficit lens as "Limited English Proficient". Mainstream pedagogical approaches have remained simplified and simplistic, emphasizing vocabulary terms taught atomistically. Within the high-stakes environment of standardized assessments, simplification and accommodation for ELLs might be necessary and appropriate (e.g., Sato, Rabinowitz, Gallagher, & Huang, 2009). The classroom environment, however, offers broader and more varied opportunities for students to learn important mathematics with use value beyond the classroom and to interact with teachers and their peers. Just as ELLs need support in meeting the challenges of conceptual understanding, procedural fluency, and language proficiency, their teachers also must navigate national shifts in academic needs, emphases, and practices. Rather than lowering the cognitive demands of tasks (Henningsen & Stein, 1997), mathematics teachers must find multiple approaches to provide ELLs with temporary support as they engage with mathematical ideas and with their classmates and they develop their autonomy.

As the Quality Teaching for English Learners (QTEL) intiative at WestEd, we engage teachers in professional development workshops and cycles of instructional coaching through a whole-school model. QTEL employs a pedagogical design framework across multiple disciplines. Our work entails the design of fully articulated lessons which use scaffolding tasks, activities that invite and structure peer support to develop students' independent abilities. Lessons form the basis of site-based workshops for teachers. Teachers then apprentice in cycles of disciplinary coaching. We nurture teachers' growing expertise in setting conceptual, academic, and linguistic goals and their effective planning and implementation of lessons that challenge and support ELLs.

This article is organized as follows. First, I define classroom-based, interactive scaffolding tasks. I then describe the interlocking sets of principles that guide the design and sequencing of these tasks into coherent lessons. Third, I report empirical findings from practitioner research conducted during the first of a three-year partnership at two secondary schools. I conclude with directions for further research and development based upon these design principles.

Defining classroom-based, interactive scaffolding tasks

Classroom-based, interactive scaffolding tasks are drawn from research on second language acquisition. Although these tasks are compatible with mathematical tasks (e.g., Henningsen & Stein, 1997), I focus on meeting the specific needs of ELLs for pedagogical scaffolding and the authentic use of language in interaction.

Ellis (2003) frames the analysis of tasks along five dimensions: 1) goals, 2) input, 3) conditions, 4) procedures, and 5) predicted outcomes as product and process. *Goals* define general purposes and target competencies. *Input* and *conditions* are linked: *input* is the information given including the modality (e.g., oral or written descriptions, or mathematical representations), while *conditions* are how information is either split or shared among students. When information is split there is an information gap (Gibbons, 2009). Students possess or are given pieces of information which they must put together through communicating with one another in order to complete the task. *Procedures* give students discourse moves and participation formats, such as working in pairs, taking explicit turns, or using specified language. *Predicted outcomes* include *products* such as materials students will write or draw and the linguistic or cognitive *processes* the task is intended to engender in students.

The QTEL approach emphasizes not just individual tasks but repeatable task-types with similar structures. Through regular participation, ELLs gain familiarity with the structure of a task-type, and therefore shift focus away from following instructions toward understanding new concepts (Walqui & van Lier, 2010). For example, the Compare and Contrast task-type has the *goal* of identifying similarities and differences between two mathematical objects or situations. As *inputs*, students are given a matrix as a graphic organizer. The two columns are headed by descriptive titles and the three to six rows are labelled with focus questions. The *conditions* are split or shared. One way to split information is to have one student report to another as an expert on a particular case. Alternatively, students could share information, going back and forth as they fill out cells of the matrix. *Procedurally*, students take turns filling out the matrix, orally stating what they are writing down. Once the matrix is complete, students take turns orally pointing out similarities and differences, using the appropriate formulaic expressions such as, "One difference between these two functions is..." Finally, students write a summary of key similarities and differences. The predicted *product* includes a completed matrix and summary statements of key similarities and differences. The predicted *process* includes noticing similarities and differences and expressing them orally as well as in written form.

These five analytic dimensions more fully specify the task-types described by Swan (2007), which emphasize *goals* and *predicted processes*. These task-types include: classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and analyzing reasoning and solutions. A "classifying mathematical objects" task provides students with multiple mathematical objects to sort, either by excluding an "odd one out" or by placing cards containing the objects into a table with given headings for rows and columns. A "multiple representations" task has students match cards containing tables, graphs, and equations. These tasks specify *inputs*.

Focusing attention on the *conditions* and *procedure* ensures that English language learners receive peer support through structured interaction. For example, we use a task-type called "Sort and Label". We stipulate the condition that knowledge of what is written on different cards, while ultimately shared publically, is rationed out sequentially. Information is gradually revealed to the whole group as students take turns drawing one card at a time from a stack, reading out loud or describing what is on that card. Students use targeted formulaic expressions to offer and justify tentative groupings. This procedure distributes participation more evenly. Patterns emerge as cards are placed on the table in an order that is not predetermined. Without this condition, both students and teachers will shuffle around cards wordlessly, with only a few stating reasons after the fact. Further, this task-type can be more cognitively demanding than "classifying mathematical objects" because the categories are not given in advance, and must instead by devised, discussed, and agreed upon by students working as groups.

These five analytic dimensions provide explicit invitations for ELLs to engage with mathematical concepts and procedures as they participate fully in classroom interactions. For the remainder of this article, I use "task" as shorthand for these classroom-based scaffolding tasks that require, specify, and support peer interaction.

Design principles for tasks, lessons, and units

The design of tasks is guided by a framework on three different levels. First, five Principles for Quality Teaching of English Learners address general pedagogical features of the classroom environment. Backwards design emphasizes that all planning must begin by articulating conceptual, academic, and linguistic goals. Finally, an architecture of three moments assists teachers in deconstructing broad goals into connected intermediate objectives that flow together smoothly. This framework thus provides nested layers, including *outcomes* with the Principles, a *process* with backwards design, and an *architecture* with three moments.

Principles of Quality Teaching for English Learners

Five Principles guide the design of instructional experiences for students: 1) academic rigor; 2) high expectations, high support; 3) quality interactions; 4) language focus; 5) quality curriculum (Walqui & van Lier, 2010). In work designed to last three years, each year focuses on different Principles. During the first year, our whole-school coaching model highlighted three Principles: academic rigor, quality interactions, and language focus.

Academic rigor considers the extent to which students acquire deep disciplinary knowledge, use higher order thinking skils, and develop central and generative concepts and skills. This aspect maps well to the construct of the cognitive demand of a mathematical task (Henningsen and Stein, 1997). Within workshops, teachers have engaged in sorting tasks by cognitive demand and then constructing variations that amplify the level of academic rigor.

Quality interactions include both interactions between the teacher and students as well as those between students as peers. These interactions must be sustained and reciprocal so that the teacher is not the sole authority who asks questions and then evaluates students' responses. Rather, students should respond to each other directly, elaborating on their own ideas, qualifying or extending them, and sharing responsibility for negotiating validity. As they co-construct new understandings, students generalize, evaluate, and connect their ideas to each other, reflecting and revising each others' ideas. These quality interactions take place not only in whole-class discussions when students or groups share work, but are infused into students' participation in tasks working in small groups or pairs. This Principle is embedded in the specification of the inputs, conditions, and procedures associated with tasks.

Further, teachers must sustain a *language focus* by providing students with opportunities to use disciplinary language authentically. Therefore, teachers need to have pedagogical content knowledge of language to provide students with clear and purposeful explanations of the metalinguistic knowledge that will assist them in completing a task, such as false cognates or mathematical language functions. Further, language can be viewed as performance, including disciplinary subgenres and language functions such as proving, providing counterexamples, and generalizing. Building on an approach of message redundancy, teachers should not simplify the language associated with a task, but rather amplify through extralinguistic and paralinguistic cues. Finally, in terms of correctness, teachers should judiciously select feedback focusing not on perfect usage or grammar, but on language production that meets the goals of the task (cf Moschkovich, 2012).

Backwards design

Consistent with the Understanding by Design framework (Wiggins & McTighe, 2005), the design of scaffolding tasks begins with identifying goals at the level of the lesson or unit. To meet the needs of English language learners, it is essential to clearly identify not only disciplinary or conceptual goals, but also academic and linguistic goals (Walqui & van Lier, 2010).

Conceptual goals emerge from the discipline of mathematics, and are often associated with the conceptual understanding and procedural fluency that underpin teaching mathematics for understanding (e.g., Kilpatrick, Swafford, & Findell, 2001). Academic goals are generative and span multiple school disciplines. These usually require higher order thinking: generalizing, synthesizing, and comparing and contrasting. These academic goals are aligned with both the National Council of Teachers of Mathematics Process Standards as well as the Standards for Mathematical Practice from the Common Core State Standards in Mathematics. For instance, to "model with mathematics" students need to engage in generating, applying, testing, and revising mathematical representations as they relate to real-world scenarios.

Linguistic goals can be considered on two levels. At a broad level, each unit, lesson, and task has specific language functions or genres as its objectives. For example, comparison and contrast is a language function that applies not only to mathematics but any academic discipline. Providing counterexamples is a language function that is more specific to mathematics. This approach to language views proof, for instance, as a specific genre with its own rules, conventions, and structures about which students need explicit instruction. Further, the genre of proof itself has subgenres: a proof by contradiction reads differently than a constructive proof or an existence proof. These differences can be understood in terms of language functions.

On a narrower level, specific swatches of language are necessary to accomplish various language functions. These formulaic expressions are used not as individual words but flexible grammatical structures. For example, "*9* is *odd*, but not *prime*" is an instance of a formulaic expression useful for giving counterexamples. The expression is like a mathematical formula in that different objects or predicates can be substituted into the positions marked in italics. ELLs in particular need explicit instruction about the formulaic expressions appropriate to mathematical language functions. It would be difficult to devise counterexamples and to communicate them to the classroom community without these linguistic structures.

While it is possible to articulate and define these different goals separately in developing teachers' expertise, in well-designed instruction the goals converge and support each other. Academic goals are the generalizations and transfer of disciplinary goals, and language is a medium for both. Focusing on goals and increasing their challenge is essential in expanding teacher expertise. Setting these goals and objectives for units and lessons is a means of specifying the more general Principles of academic rigor, quality interactions, and language focus.

Lesson architecture in three moments

A typical "traditional" lesson sequence is explanation, example, exercise (Swan, 2007). The teacher gives a general explanation, demonstrates a worked example, and then students engage in repetitive practice of the target procedure. Curricula based upon NCTM Standards such as the Connected Mathematics Project focus on a single, central problem set in a real-world context and follow a "three phase" model: Launch, Explore, Summarize (Lappan et al., 2009). In the Launch phase, students are introduced to the problem and certain key contextual features or mathematical relationships can be explained. Students work in small groups to solve the problem using their own methods in the Explore phase. In the Summarize phase, the teacher orchestrates a whole-class discussion in which different solution methods are publically shared, compared, and contrasted.

By contrast, an architecture in three moments provides a more flexible structure: 1) preparing learners, 2) interacting with the concept, and 3) extending understanding (Walqui & van Lier, 2010). When directed toward a mathematical problem set in a real-world context, Preparing, Interacting, and Extending are compatible with Launch-Explore-Summarize. The more flexible architecture of three moments, however, offers three additional benefits: broader notions of prior knowledge and explicit attention to transitions from everyday to academic language, more flexibility in terms of building students' procedural fluency with embedded opportunities for reflection and interaction, and a clearly delineated, more varied set of options for extending understanding.

The Preparing Learners moment has three possible functions in the lesson. First, it articulates a focus on key understandings for the lesson. Second, the tasks bring to the surface students' prior experiences and knowledge with the objective of narrowing these contributions toward the lesson objective. Finally, the teacher can introduce essential understandings as reflected by key vocabulary terms, presented in context. For example, students can engage in a Think-Pair-Share. The prompt is kept as general as possible to appeal to students' personal experiences rather than their mathematical opinions. For example, students might be asked to tell a story about how they had to balance something, in preparation for a lesson about the arithmetic mean as a balancing point. Students have a few minutes to think individually, before they take turns with a partner sharing responses. The teacher then leads a whole-class discussion, calling individuals to share what *their partners* said. The teacher then summarizes these experiences and connects them explicitly to the mathematical topic of the lesson. In contrast with the Launch phase's focus on a a single mathematical problem set in a real-world context, other tasks, such as ones that involve sorting mathematical objects or representations, function well in the Preparing moment.

As students are Interacting with the Concept, they engage in three processes: deconstructing, reassembling, and connecting different aspects of the central concept of the lesson. Although compatible with the "Explore" phase of investigating a central mathematical problem, this moment can address procedural fluency. For example, students work in groups of four to carry out an Algorithm in Four Steps. In this task-type, students play the roles of different steps of a procedure, such as finding the slope of the line between two points. After completing a case or problem, students rotate roles so that each gets a chance to play each of the four steps. In contrast with the typical "exercise" part of a lesson, students are collaboratively engaged in interaction structured to heighten awareness of the interdependence of steps.

Another suitable Interacting task-type is a Jigsaw Project. Students convene in expert groups to learn about a particular case or solve a problem, becoming experts and answering common focus questions that cut across the different cases. These focus questions cannot just be factual and isolated, and should cohere to require students to negotiate, discuss, and select in expert groups. Students then return to base groups to report their findings. Base groups use a graphic organizer similar to a Compare and Contrast matrix. The focus questions allow the base group to see connections, such as different proofs of the Pythagorean theorem or different real-world instances of unit rate.

Finally, as the lesson moves toward Extending Understanding, students are invited to work in three ways: to apply the concept to novel real-world applications, to connect to other concepts or algorithms previously studied, or to re-present their understanding in new genres and formats. This moment also includes having students create their own problems and solve those created by their peers (cf Swan, 2007). Reflecting on one's own process of thinking and the relative usefulness of different representations is also appropriate in this moment. The Collaborative Poster task-type has students work in groups of four to create a poster, with the condition that each student uses a different color marker. A good prompt requires students to make a choice as a group, such as only choosing one type of representation from among tables, equations, or graphs, in order to compare two different linear functions.

Tracing teacher engagement and growth trajectories

After the first of three years of whole-school coaching and professional development across the disciplines at two secondary schools, three phases in growth among mathematics teachers have begun to emerge. First, are *shifts* in teachers' professed beliefs, priorities, and approaches. Next, teachers *adopt* tasks wholesale during coaching. Third, teachers have begun to *adapt* tasks in planning units and lessons.

When first working with the design framework and tasks, teachers respond most frequently and extensively to three features. First, teachers express appreciation for how tasks specify clearly outlined roles for students and how the interaction is structured. They contrast this approach with the “bare” problems provided in textbooks or other curriculum resources, or generic roles for collaborative work (e.g. recorder, materials manager, etc). Second, teachers respond positively to notions of language which look beyond vocabulary toward language functions and linguistic goals in lesson planning. Many teachers say that they had not been given other tools beyond generic state-based language proficiency standards, or that they have previously focused only on definitions-centered vocabulary. Finally, within the context of planning, teachers focus on the Extending moment and developing flexibility in selecting from the multiple tasks appropriate to that moment. This focus on Extending is particularly important given the lack of closure that is often characteristic of many mathematics lessons in the United States.

Within coaching cycles, initially teachers often place undue focus on task-types as the end goal rather than as a means for achieving outcomes such as quality interactions. This emphasis is perhaps a consequence of previous district-wide mandates which evaluate teachers based upon implementation of specified strategies. Modeling specific tasks in coherent instructional sequences by the coach facilitates both teachers building belief in their students and their technical knowledge for implementing specific tasks and transitions between tasks. A key insight that many teacher reach through practice is that language development is not spontaneous but occurs within the context of planned scaffolding.

Simultaneously, teachers see how the language modeled for students facilitates their conceptual development, and they begin to select and model appropriate formulaic expressions and genres for their students. Teachers begin develop their capacity to enact the Principle of language focus as they become aware of how many mathematical tasks need to be further unpacked for ELLs. A common example is around the prompt to “summarize”. Students typically produce a narrative recount of the procedure, or a laundry list of responses to specific questions, rather than a coherent summary oriented toward goals and methods that generalize. Once they have unpacked the complex processes and structures involved in summarizing, teachers can apply the process to other common but complicated commands, such as “explain” and “justify”.

Teachers’ initial misfires reflect their emerging understanding of the rationale for procedures in task-types as connected to more general goals. Often, teachers create opening prompts that are too narrow, or after students share responses do not efficiently focus students’ contributions toward the key ideas that connect directly with the mathematical topic. Successful openings require both pedagogical content knowledge and implementation skills. For example, in a lesson on solving equations by *undoing*, one teacher gave an example response to a Think-Pair-Share prompt a story of making a mistake with a baking recipe. Because this modelled example would require redoing rather than undoing, many examples students subsequently provided did not move toward the idea of inverse operations, and the intended question of the order in which operations would need to be undone. By engaging in reflection during coaching, teachers produce prompts that start more broadly and focus more narrowly. Through coaching, teachers have the chance to revise their lessons the same day. They thus can examine and reflect upon how changes in the clarity of directions or the inputs or conditions of the task affect student outcomes.

The refinement of teachers’ choices can be traced in the quality of the focus questions that they generate for Jigsaw Projects and Compare and Contrast Matrices. Generic graphic organizers for comparing and contrasting two cases may be organized like a Venn diagram and do not have focus questions. Similarly, teachers often initially misunderstand the rationale for focus questions, omitting these questions, asking questions that are too general or do not apply to individual cases (e.g., "How are they the same?"), or barraging students with recall questions that do draw focus toward key ideas. Over time, teachers have developed questions that are both better phrased individually as well as coherent and well-sequenced as a whole.

Indeed, with more experience with this design framework, teachers begin to engage in a form of "task problematization" (Sierpinska, 2004). Not only are there possible variations on the mathematical question but the other aspects of the design of the task, including inputs, conditions, and procedures. Teachers begin to reflect on the flexibility in choosing similar but subtly different task-types as appropriate to different moments in the lesson and trade-offs between slightly different goals. In particular, teachers gravitate toward the tasks that involve algorithms, whether in the format of a group task as an Algorithm in Four Steps or in Comparing and Contrasting two different algorithms, such as for computing the median. Reflective coaching discussions with teachers about algorithms probe the dual demand for procedural fluency and conceptual understanding. For many teachers, the explicit modeling provided by scaffolding tasks in which students interact as they carry out different steps of procedures is an accessible entry point to providing ELLs with multiple algorithms which they will eventually be able to select from strategically. In this manner, the scaffolding embedded in well-designed tasks allows teachers to increase the academic rigor, or cognitive demand, of classroom activity.

Conclusion: Future directions for design and research

These emerging trajectories for teacher growth suggest three areas for further research and efforts in task design, starting from the level of individual teachers and extending, through coaching relationships, to the level of groups of math teachers working at the same school. 1) How do individual teachers engage with different aspects of the design framework and make connections across different components? 2) How does this design framework function as a coaching tool to foster teachers' development? 3) To what extent can this design framework serve as a common language as teachers collaborate with one another? While the design framework has so far served primarily as a means to design lessons for the purpose of professional development, handover would suggest that teacher-created lessons could also be eventually used for this purpose. Further in depth observational studies of student-to-student interactions would also be appropriate on the way to evaluating the extent to which shifts in teachers' practices around task design and implementation affect student outcomes.

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An experience of teacher education on task design in Colombia

GEMAD

Universidad de los Andes, Colombia

Gemad is a working group of researchers, educators and in-service teachers of mathematics affiliated to the Universidad de los Andes in Bogotá, Colombia. Its members belong to different Colombian and Spanish educational institutions. Their names, in alphabetical order, are F. Y. Arenas, O. J. Becerra, H. M. Becerra, M. Bernal, M. R. Buitrago, S. C. Calderon, M. C. Cañadas, D. P. Castro, P. Cifuentes, L. E. Dimaté, R. A. Gómez, P. Gómez, M. J. González, Y. M. Gutiérrez, G. Y. Hernández, J. L. Lupiáñez, M. Molina, M. Mora, J. F. Morales, E. O. Moreno, E. X. Nieto, A. A. Pinzon, D. L. Polanía, A. M. Rincón , M. Romero, I. M. Romero, L. S. Santoyo, A. Serrano , Y. F. Torres, E. L. Urrutia, J. R. Velasquez, and M. P. Villegas.

We describe an experience in task design within an in-service secondary mathematics teacher education program in Colombia. Following a model known as didactic analysis, a team of researchers, educators, mentors and practicing teachers worked together in designing, implementing, assessing and reformulating secondary school mathematics tasks. We present here the main features of the framework on which the program is based, identify some of the characteristics of the experience lived by trainees, educators and researchers on task design during the first implementation of the program, and analyse the trainees' assessment on their own proposals of tasks and on the contribution of the program on their task design competencies.

Keywords: teacher training; task design; task implementation; cross-communities

Law sets curriculum autonomy in Colombia. Schools and teachers are fully responsible for curriculum design and development in all areas. Schools are expected to produce curriculum planning for each course and academic period and teachers are usually autonomous for designing and implementing the lessons they are in charge of. They often do so by producing what is known as "teaching guides": sets of tasks that they design or copy from different resources, and propose to students. Most pre-service teacher education programs in Colombia do not prepare future teachers in task design nor other practical questions; instead, they are based on theoretical approaches to education.

In this paper, we describe an experience in task design that emerges from an in-service teacher education program in Colombia, known as MAD (Master in Didactic Analysis). It is based on a model—didactic analysis— that enables trainees to design, implement and assess sequences of tasks on specific topics for which a constructivist view of students' learning is assumed (Gómez, 2007). Based on this model, a group of researchers, educators, mentors and in-service mathematics teachers have worked together in MAD. We use the term task as "anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something" (ICMI Study 22, 2012, p. 10).

In what follows, we describe the main features of the framework on which the program is based, identify some of the characteristics of the experience lived by trainees, educators and researchers on task design during the first implementation of MAD, and analyze the trainees assessment on their own proposals of tasks and on the contribution of MAD on their task design competencies. In the final section, we reflect on the role of the different agents in the program.

Framework

MAD is a master's degree in mathematics education for in-service secondary mathematics teachers. We assume a functional view of school mathematics in MAD. This vision puts the focus on the usefulness of the mathematical concepts for solving problems in a variety of contexts. Students are expected to use their mathematical knowledge for this purpose. They are expected to develop their own cognitive strategies, manage different representations of the mathematical concepts, choose the best solution strategies, argue about their decisions and communicate fluently their thinking processes. This functional view of school mathematics is coherent with a constructivist approach to students' learning and can be implemented with different pedagogies. MAD does not explicitly promote any of them, since each trainee and his context impose their own restrictions. Nonetheless, there are some implicit methodological principles: it is considered that the good tasks are those that promote the active implication of students, imply the development of strategic knowledge for problem solving in a diversity of contexts, and require that students make decisions and justify them.

MAD is based on a model known as didactic analysis (Gómez, 2007; Lupiáñez, 2009). This model is a conceptualization of the activities that the teacher has to do in order to design tasks that seek to promote students' learning on a mathematics topic. It is organized around four interrelated analyses: subject matter, cognitive, instruction, and performance analysis. The didactic analysis begins with the identification of the student's knowledge for the topic at hand (see Figure 1). With this information, and taking into account the global planning of his course, the teacher determines the mathematics content he wants to work on and the goals he wants to achieve (Box 1 in Figure 1). The next step involves the subject matter analysis (Box 2), in which the teacher stresses the relationship among concepts, highlights its multiple representations, and distinguishes the phenomena from which they emerge. This information is used in the cognitive analysis, in which the teacher describes his hypothesis about how students construct their knowledge. The cognitive analysis involves the establishment of learning expectations, and the identification of the skills, reasoning, and strategies necessary to achieve those expectations, and of the difficulties, mistakes and obstacles students might face. This information allows the teacher to carry out an instruction analysis: the identification and description of the tasks that can be used in the design of the teaching and learning activities that will compose the instruction in class (Box 3). During the implementation, these tasks should mobilize students' knowledge in order to generate cognitive conflicts and promote the construction of meaning using the materials and resources available (Box 4). In the performance analysis the teacher observes, describes, and analyzes students' performance in order to produce better descriptions of their current knowledge (Box 5). After this process, the teacher can review the planning in order to improve the sequence of tasks for future implementations.

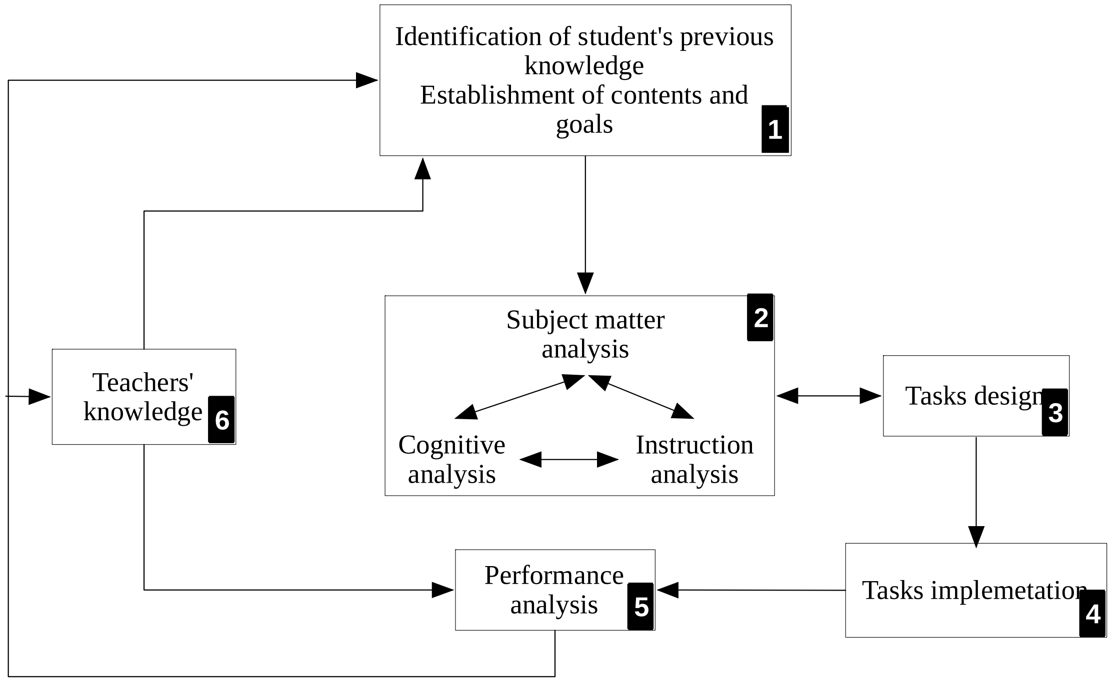

Figure 1. Diagram of a didactic analysis cycle

In this approach, trainees are expected to develop a deep enough knowledge of the topic so that they can support the choices and decisions they make for their lesson plan (Charalambous, 2008). This is a topic-specific knowledge that trainees are expected to develop by performing a series of activities during their training and contributes to the development of their didactic knowledge (Box 6).

Didactic analysis is a cyclic process in which trainees analyze a school mathematics topic with the purpose of designing tasks that provide the learning opportunities required for students to achieve the learning expectations. Trainees make decisions in different moments and with different purposes. When they describe in detail the topic from the mathematics point of view (the concepts and procedures involved, the forms of representing those concepts and procedures and the ways in which the topic organizes the phenomena that give sense to it), they produce and organize information about the topic that allows them to make decisions about those aspects that they consider relevant, about how to formulate and specify the learning expectations, about the capacities that can be used for characterizing those learning expectations, and about the mistakes that students can make when solving tasks related to the topic. The information that trainees produce form those decisions are the basis for further decisions about their anticipations about how students' learning can develop when they solve the tasks (Gómez & González, 2009). This process is based on a procedure that allows trainees to produce the learning paths of the tasks. Tasks' learning paths are a useful tool for assessing the effects of reformulating or extend the original tasks. For instance, trainees can make decisions about the material and resources that can be more effective for achieving the learning expectations, when they analyze the implications of their use in the learning paths of the new tasks. Based on the information about the capacities that characterize the learning expectations, trainees can establish the complexity of the tasks proposed and make decisions about how those tasks align with students' previous knowledge and about how to sequence the tasks. Trainees can also make decisions about the most effective ways of grouping the students and about how to foresight the teachers' performance when students begin solving the tasks and encounter difficulties. In summary, didactic analysis provides trainees with a systematic procedure for analyzing a school mathematics topic and sequentially making decisions that enable them to deepen in the different aspects of the topic and design and assess the tasks with which they pretend to contribute to the achievement of the learning expectations.

Since the information that trainees produce with the didactic analysis is complex and plentiful, when they make decisions, trainees might give priority to some decisions over others. For instance, they might focus on the treatment of the students' mistakes, so that their decisions will focus on ways of facing students with those mistakes and helping them to overcome them. For instance, they might concentrate on the search of representations, resources and possible teachers interventions that can help students overcome their mistakes and difficulties. In the empirical experience that we describe below, we show the concrete decisions that were made by a group of trainees in MAD.

MAD is set up from a social perspective of the trainees' learning (Gómez & González, in press). Trainees are organized in groups of 4 or 5 teachers. Each group has a mentor that accompanies it during the program's two years duration. Each group selects a school mathematics topic on which it will work during the program. The program is composed of 8 modules, two modules per semester. Each module begins with one week of face-to-face instruction in which the educator in charge of the module presents its theoretical basis and introduces the four activities that the groups have to carry out. Each activity spans over two weeks and requires the groups of trainees to analyze or produce information on their topic from a given perspective or with a given purpose. For instance, in an early activity, the groups produce the information concerning the representation systems of their topic. Later in the program, they analyze, in another activity, the role that the teacher had during the implementation of the tasks. For each activity, the groups produce a draft of their work at the end of the first week. They then receive comments on this draft from their mentors, and produce a final version of their work that they present to their pairs, the educators and mentors at the end of the second week.

Experience

In this paper, we report on MAD's first implementation that took place during 2010 and 2011. The 26 teachers that participated in MAD were working in public and private schools of Bogotá (Colombia) and its surroundings. They were organized in 6 groups that worked on the following topics: integers, linear equations (2 groups), straight lines in the plane, and trigonometric ratios (2 groups).

The design, analysis and selection of tasks are processes that span over the whole program. It begins with a first selection of tasks that is refined and improved with new ideas and analysis proposed by educators and mentors. The different activities of the modules structure this process. Once the groups have a tasks sequence that is ready to be taken to class, the groups implement, collect information on its implementation, analyze that information in order to assess the sequence's design and implementation, and make improvement proposals for future implementations. In what follows, we show a summary of the process for the specific case of Group 5, whose topic was trigonometric ratios. This group designed a sequence of five tasks distributed in 12 lessons. The first selection of tasks was guided by activities in which the group set up the learning goals, the contents they expected to cover, the materials they wanted to use and the context (personal, scientific, etc.) in which they wanted to place their proposal. This group focused the design of their tasks in the use of materials and resources, some of which were elaborated by the group itself. The group used the materials and resources for bringing together the tasks characteristics that they considered important for students' learning:

> Materials and resources play an important role in our task sequence because they allow us to motivate students in working with mathematics; they facilitate the achievement of the learning expectations; they promote mathematical communication and the construction of arguments; and they put into play different systems of representation.

Their pedagogical decisions focused in two aspects: (a) the grouping of students and (b) the communication in class. They proposed to use the tasks with different types of students' groupings: heterogeneous groups of three students (with high, medium and low achievement), big group, and, less frequently, individual work. In order to promote the classroom communication, they decided to use the following strategy: at the beginning of each task, the teacher shares its goals; then he induces students to create their solving strategies in small groups; the groups present their strategies and argue in favor of them to the whole group; once the task is finished, the teacher gives students follow-up and feedback on their performance. These decisions were guided and founded on their functional view of school mathematics and on the group's aims of contributing to the development of students' argumentation and justification competencies.

In what follows, we show in detail the design process of one of the tasks, named The streetlight height. In the following excerpt, Group 5 describes the task's features based on the subject matter analysis they have previously realized.

> We expect students to find the streetlight height by using trigonometric ratios, without direct measures. The task covers a conceptual content that includes elements and properties of right triangles and trigonometric ratios. It involves also some procedures: (a) identifying regularities and patterns, (b) formulating equations, (c) using the functional language trigonometric ratios, and (d) situations solving. The task design includes working guides, goniometers made of set squares of 45º and 60º, protractor, calculator and a metric strip. The task refers to a personal situation.

In MAD, once the groups make the first task proposal, educators and mentors introduce new elements of analysis. That is the case, for instance, of considering the concrete capacities that can be activated with the task or the mistakes that students can make when solving the task. The groups characterize the task in terms of these new elements. For example, once Group 5 produced a list of 35 capacities and 12 mistakes for their two learning goals (that we do not have space to include here), they produced a table in which they related the learning goals, the tasks in their sequence, the capacities that each task could activate and the mistakes that students could make when solving each task. Table 1 shows an excerpt of this analysis for Group 5.

Goal	Task	Capacities	Mistakes
2	Streetlight	1, 7, 3, 8, 12, 14, 17, 33, 35	2, 5, 7.3

Table 1. Relationships among learning goals, task, capacities and mistakes

This information led them to modify the task's wording and to determine how to implement the task in class:

> Before starting working with the task, the teacher explains the use of the goniometer. In the first phase, the teacher asks students to go the location in the school where the streetlight is and to calculate its height with the instrument. The goals of this phase of the tasks are twofold: (a) that students recognize the use trigonometric ratios for measuring unreachable distances and (b) that they connect the elements of the instrument with the particular situation at hand in order to represent it and calculate the distance. The students are asked to record their observations in the working sheets provided, including the steps that they took for calculating the length and a graphical representation of the situation. They produce a poster to share their work with the class group.

Once they produced the final design of the tasks sequence, the groups implemented them in class. The following are two excerpts of the balance that Group 5 made of this phase of the process.

> One of the minor changes that we made during the implementation of the tasks sequence concerned the time foreseen for each session. For most tasks, the time required was greater than we expected. We can claim that the Streetlight height task was effective as it was designed because we verified that students activated the capacities that we expected. Furthermore, we observed that they also activated capacity 17 that was not expected by us. On the other hand, we established that capacities 12 and 35 were explicitly activated on a given moment of the task's development, but that they were also present along the task, since students permanently used the trigonometric vocabulary and verified the relevance of their results, when measuring and making calculations.

On the basis of this kind of analysis, Group 5 decided the following improvements for the Streetlight height task:

> (a) to include instructions for the construction of the goniometers; (b) to incorporate an activity for measuring lengths that can be found directly and to use the trigonometric ratios to corroborate the results; (c) to show the diversity of theodolites that can be found, with a brief explanation of each one; (d) to ask students to select the theodolite that they think is best suited for each situation, and (e) to ask questions that can lead students to look for tools that are different to those proposed and to include new strategies of solution.

Trainees' assessment of the program

In the final part of the program, the groups of trainees that participated in MAD performed a SWOT analysis (Strengths, Weaknesses, Opportunities, Threats) of their work and made a personal assessment of the program and their participation in it. We reviewed these analysis looking for the common themes that characterized what the groups of trainees appraised as the most salient features of their work and the most important influences of the program in their capabilities for designing, implementing and assessing tasks. We consider that the trainees' claims represent those features of MAD and their work that highlight the differences between what they usually do in class and what they actually did when planning, implementing and assessing their tasks during the program. We found several themes that were mentioned by most groups. In what follows, we identify and exemplify the most frequent ones.

Features of the work

All groups centred the assessment of their sequence of tasks in terms of its contribution to the achievement of the learning expectations and the overcoming of the students' errors and difficulties. They also claimed that the designed tasks were adapted to the context. The Colombian curriculum guidelines underline the importance of problem solving in context and, hence, of tasks that lead students to solve and interpret mathematical problems in a variety of situations. Trainees claimed that one of the salient features of their tasks was the fact that they were mathematical problems set in varied non-mathematical contexts that contributed to their students' proficiency in problem solving. As Group 1 explained: "the use of tasks in context favored the achievement of the learning expectations proposed because the situations in the tasks were close to the everyday life of the students." Similarly, all group of trainees mentioned the importance of the use of materials and resources (i.e., Cabri, Geogebra, Hands on Equations) in their tasks design and implementation. Their statements make us think that they do not usually introduce those resources in their teaching: "when designing the materials with which we developed our task sequence, we found that they have a great potential for other topics and school grades" (Group 5).

All groups of trainees mentioned the importance in their tasks' design and implementation of using multiple ways of promoting collaborative work in class. They recognized the benefits of having students working in pair or groups and of generating class discussion among them. Some groups of trainees also mentioned the relevance of foreseeing the teachers' reactions to the students' performance in class, particularly to students' mistakes. The assessment made by Group 1 sustains such claim: "We acknowledge the benefits of the groupings proposed for students' work in class. For instance, as a consequence of the interactions produced, the students were able to strengthen their argumentative capacities for validating their results." Similarly, Group 2 claimed that "the tasks sequence involved a methodology that supports constructive learning of individuals and groups because it contributed to create a ZPD and strengthen the establishment of agreements when taking decisions concerning the challenges that were proposed."

The groups of trainees also mentioned some of the problems and deficiencies of their tasks. The most common shortfall referred to their mistakes when foreseeing the time required for implementing the tasks as mentioned by Group 5 above. On the other hand, some groups recognized that their students did not understand properly the wording of some of their tasks or that the tasks did not generate the student's performance that they were expecting. They recognized that, in some cases, they incorrectly assumed that the students had the previous knowledge required to face the tasks. For example, Group 3 acknowledged that "the wording of the instructions in one of the tasks was another weak point [of our task sequence]. This situation affected the time required for the task and the understanding that students developed when solving it." This assessment led them to propose new or modified tasks for a future implementation of the sequence.

Influences of the program

The groups of trainees highlighted the impact of MAD in their competencies for designing, implementing and assessing sequences of tasks. Trainees stated that the program provided them with tools for assessing how the tasks' design could achieve the planned learning expectations and how the tasks' implementation did in fact achieved them. Group 5 claimed that "the didactic analysis procedure lead us to be really conscious of the importance of planning a task sequence and assessing its relevance and effectiveness. We know better about the mathematics that our students should learn and how they should learn them. We recognize now that students' learning depends on the tasks they solve and that the teacher is the main responsible of that learning." In other words, the groups recognized that the program (a) provided them with a better preparation for developing teaching innovations based on a structured method; (b) encouraged them to reflect on their own practice; (c) showed them how to track the results of their lessons; (d) questioned their meanings of knowing and learning mathematics; and (e) motivated them to modify their role as teachers. The groups also reported that the program led them to introduce several data gathering instruments (some of them previously unknown to them—like the students' dairies and observation tables) that allowed them to properly assess the students' performance when solving the tasks. They also recognized that they did not have enough time for performing those procedures and analyze the information gathered.

Across-communities

The design and implementation of the program was a joint venture among researchers, educators, mentors and in-service teachers. Researchers have been working for several years on the development of the didactic analysis model and on a model for conceptualizing the trainees' learning of it. Some of the researchers were also educators in the program. Educators as a team have worked on the design of the program following the models proposed by researchers. They also implemented the program working hand in hand with trainees. This collaborative work was set up through a mentoring process in which mentors (the educators) interacted with their group of trainees weekly. Trainees' work informed researchers on the framework and principles, educators on their teacher education program design and implementation, and mentors on their performance. This joint venture has evolved beyond the program, as this paper shows. Teachers, educators and researchers have created a working group (that signs this paper), which continues working on mathematics task design an implementation. Some of the near-future results of this collaboration is a book with the reports of the groups' work in the program (Gómez, in press), the support of an international publisher for the publication of a set of teaching guides based on the groups' program's work, and the teaching of a course in the Colombian mathematics education congress.

MAD has generated several research studies in which researchers, educators and teachers have also collaborated. That is the case of studies in which we have explored, for instance, the teachers' learning of specific aspects of the program (Gómez & Cañadas, 2012; Suavita, 2012), the role of mentors (Arias, 2011), or the impact of the program in institutional planning (Gómez & Restrepo, 2012).

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Designing Professional Tasks for Didactical Analysis as a research process

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In the present paper, we show a process of designing, assessing and re-designing (following DBR methodology) professional tasks for preservice mathematics teacher training for Secondary School, based on Ontosemiotic perspective for cognition and mathematical instruction (OSA) and the correspondent reflective analysis about associated professional practices. Such a process has been carried out during three consecutive years in the context of the Master's degree for training teachers of mathematics in Spain. The study shows how the successive revisions promote growing depth analysis in the teaching school practices of the future mathematics teachers.

Keywords: Professional tasks, design based research

1. Presentation

In this paper, we show a part of a wider investigation in which we analyze the design of professional tasks in the teachers' formation of Secondary Mathematics Teachers. We focus on the role of design based research (DBR) and teaching experiment (Gravemeijer, 1998) analyzing the planning cycle and redesign of our training process in successive phases, aiming the growing and building knowledge for teaching (Zavlaswski & Sullivan, 2011) by future teachers. We explicitly focus ourselves in recognizing factors that promote the feedback in the design of professional tasks for development of the of didactic analysis competencies of the future mathematics secondary teachers. Our intention is that they can develop sequences of suitable tasks and to be able to re-plan their own designs of school tasks. This work has been carried in a funded Research Project (Assessing and developing professional competencies in mathematics and didactics during initial Secondary Mathematics Teacher Training courses) being the first two authors of this work members of the team who implemented the course. The professional tasks have been evaluated and re-designed by the whole research team during the period 2009-2012.

In our study we call professional task those tasks that we propose to the future teachers in order that they realize didactic analysis and develop their didactic analysis competencies understood as the ability for designing, applying and evaluating sequences of learning by means of didactic analysis techniques and quality criteria. The aim is to establish cycles of planning, implementation, evaluation and proposals for improvement. It is also assumed that one could identify criteria and indicators regarding the development of this competence and how it relates to the other professional competencies required by future secondary school mathematics teachers. This assumption is related to the question: How might professional mathematical tasks being designed in order to make best use of the opportunities for being a teacher as teacher enquirer? (Mason & Johnston-Wilder, 2004).

Our main aim is to investigate how the process of building a sequence of professional tasks (so called formative cycle from now) promotes and generate feedback in the development of the didactic analysis competence of the future teachers within the context of teacher training courses. Such above mentioned development, it is stated when future teachers incorporate and use tools for the description, explanation and process valuation of mathematical school teacher/learning practices. By using our professional tasks design as a design based research cycle, we also want to improve a teacher as teacher-researcher of their own practice. We will show later, examples of professional tasks and reflections of the future teachers on having solved them, that have served to provoke successive feedback that have allowed to improve the sequence of tasks, and to make it increasingly effective.

2. First year design. Building a formative cycle

In the frame of the Project of investigation mentioned, we design and implement diverse formative cycles as teaching experiments for developing transversal competencies as citizenship, digital competency, didactical analysis, and others. In particular, in this presentation we discuss a part of a cycle of formation, named of "Didactic Analysis "articulated across diverse subjects of the courses. The development of the cycle has been based from the beginning in considering six big types of professional tasks:

a) Analysis of practices, objects and mathematical processes.
b) Analysis of didactic interactions, conflicts and norms.
c) Valuation of tasks and classroom episodes using criteria of didactic suitability or quality.
d) Planning and implementation of a didactic unit in their period of practices.
e) Analysis and valuation of the suitability of the didactic implemented unit.
f) Offer of a well-taken improvement of his didactic unit, for a future implementation. This proposal is realized in Master's final Work.

During the first two types of tasks (a - b) it's expected to appear and discuss tools for a descriptive and explanatory analysis that serves to answer "what happens in the classroom and why?" (Font, Planas y Godino, 2010). The analysis and description of the mathematical activity is realized using the theoretical constructs proposed for OSA. In this perspective (Godino, Batanero y Font, 2007), mathematical activity plays a central role and is modelled in terms of systems of operative and discursive practices. From these practices the different types of related mathematical objects emerge building cognitive or epistemic configurations among them (see two internal hexagons in Figure 1). Problem-situations promote and contextualize the activity; languages (symbols, notations, and graphics) represent the other entities and serve as tools for action; arguments justify the procedures and propositions that relate the concepts. Lastly, the objects that appear in mathematical practices and those which emerge from these practices might be considered from the five facets of dual dimensions. Both the dualities and objects can be analyzed from a process-product perspective, a kind of analysis that leads us to the processes shown (decagons in Figure 1).

During the following type of tasks (c - f), we present theoretical tools (suitability criteria, according Godino, Batanero y Font (2007) for a valuated analysis serving to answer "what could we improve?" These criteria are as follows: *Epistemic suitability* refers to the extent to which the mathematics taught are 'good mathematics'. Thus, in addition to the specific content of the curriculum the institutional mathematics on which it is based are also used as a reference. *Cognitive suitability* reflects the degree to which the teaching objectives and what is actually taught are consistent with the students' developmental potential, as well as the match between what is eventually learnt and the original targets. *Interactional suitability* relates to the extent to which the forms of interaction enable students to identify and resolve conflicts of meaning, and promote independent learning. *Mediational suitability* refers to the availability and adequacy of the material and temporal resources required by the teaching/learning process. *Affective suitability* reflects the students' degree of involvement (interest, motivation, etc.) in the study process. *Ecological suitability* refers to the degree of compatibility between the study process and the school's educational policies, the curricular guidelines and the characteristics of the social context, etc.

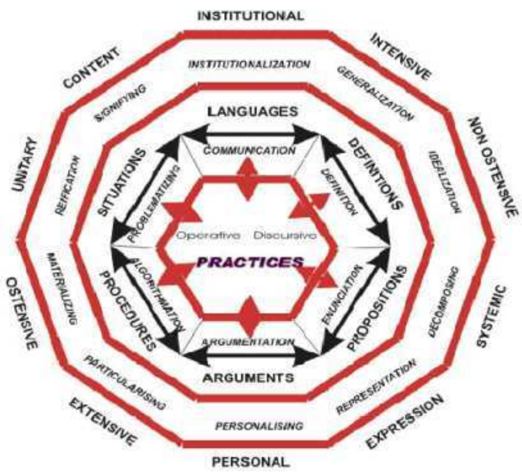


Figure 1. Onto-semiotic representation of mathematical knowledge (Godino, Batanero y Font 2007)

We understand that the study of descriptive and explanatory analysis for a didactical situation is necessary to argue based valuations (Pochulu y Font, 2011).

Methodologically, the research is mainly qualitative in nature as the purpose is to describe the development of competence in didactic analysis among aspiring secondary school mathematics teachers, from the University of Barcelona (Spain). The data were collected from the video recorded observations, sorting sheets produced by the teacher trainers and their reflections at the end of the workshops and using the documentation housed in the Moodle platform (slides, reading material, tasks and the students' responses to them, and questionnaires and the students' responses to them) and printed material. The samples were intentional. During all these academic years, in general, these students vary in the amount of mathematical knowledge they have, and they hold certain conceptual biases regarding the teaching and learning of mathematics.

It was selected an initial task in which students confront a short case study about proportional reasoning, using transcripts of a classroom situation. Such an initial ***tasks*** (type a; non-theoretical), introduce the students for reading and analysing the classroom example, by using their previous knowledge and beliefs of didactic analysis. During the first task, future teachers did naïf comments about a proportion class. It's easy for them to identify mathematical objects but it's difficult for them to recognize all the processes involved in the task. When they analyze interactions, they focus on leadership and teacher interventions (Task b). It's difficult for the future teachers to identify epistemic conflicts and norms. In such first analysis each group of future participant teachers used just implicitly some of the levels of analysis proposed by OSA (described in Font, Planas y Godino, 2010): Analysis of mathematical practices; Analysis of objects and mathematical processes activated by these practices; Analysis of didactical interactions and conflicts; Identification of systems of norms conditioning and making possible instructional process; Valuating didactical suitability of instruction (Font, et al., 2012). In the class debate it was observed that, though every group did not use all these levels of didactic analysis, it is possible to see how the student group as a whole has contemplated the five levels of analysis. The tasks (a-b) were considered fruitful, so they were conserved for new implementations, but they should be completed by means of the 'other voices' technique (Garuti & Boero, 2002).

Later, in the different subjects of the Master, the students realize other analysis of practices, objects activated in the above mentioned professional practices (problem, definition, proposition, representation and argument) and mathematical processes (type a). Observing the analysis realized by the future teachers some difficulties appear: (1) Difficulties to distinguish between concepts and definitions, (2) Duplicity between definitions, propositions and procedures; (3) Duplicity between propositions and thesis of arguments; (4) The description of practices is overlapped by the configuration of objects and by the description of processes, (5) Difficulties to observe and to catalogue mathematical processes, etc.

It also had been designed and implemented tasks (type b), with protocols served to show constructs as cognitive and semiotic conflicts, epistemic obstacle, types of norms, interactive, patterns of models of management, etc. After first year of experience we found that protocols were statics. For the next year it was decided to use videos and corresponding transcripts. After that it was analyzed a class about equations by applying suitability criteria (task type c). The students star by analyzing mathematical practices, objects and processes. Then the teacher develops an example in which it was revised suitability construct. After that the future teachers reflect, improve and refined their analysis by using the notion of epistemic suitability. Nevertheless, it's still difficult for the students to identify some semiotic conflicts.

Next it was proposed a task of planning and further implementation of a didactic unit in their period of practices (task type d). When doing the analysis and valuation of the didactic implemented unit (task type e), future teachers found that their planning was conditioned by the school plans in which they did the practices. As a consequence it was difficult for them to identify the epistemic consideration implicit by the school teacher proposal.

The future teachers had a few autonomy to apply in the design and implementation many learned knowledge. This aspect was considered a difficult problem to solve during redesign process because of institutional framework for the proposal, which did not deal a selection of schools. The tasks type (e) and (f) are considered activities producing the feedback for future teachers and trainers. According to task type e, it was found a superficial use of theoretical tools for valuing teaching practices, due because we had short time to discuss after school practices and a need for more discussion about suitability criteria, and analysis based upon previous experiences. This aspect should be promoted in a redesign.

Positive results and some feedback from the first year are explicitly observed (when analyzing task type e). It was decided to redesign task type a, by emphasizing the analysis of processes. Another aspect to consider when redesign is to find so enough rich episodes which serving to propose different typologies to profit a short time available, instead of using different episodes in each task. It was also observed that some of the final practices' works (task type e) and master's thesis (task type f) were found so rich to be considered as episodes to be incorporated in a later redesign processes.

3. Second year redesign. Improving process analysis

It was decided not to do important changes of the cycle itself for the second year Project. As an important example for the redesign, we consider enlarging task type (a) and (b) by using a new video source. In such a new task (type a) it was proposed the observation of three short ways of introducing perpendicular bisector with 12-13 years old students, by observing three different teachers. The main idea is to present a discussion about the different practices, objects and mathematics processes and to introduce a reflection associated to how each of these classes contribute to introduce different kind of epistemic configurations and objects (see hexagons in figure 2) associated to three different definitions.

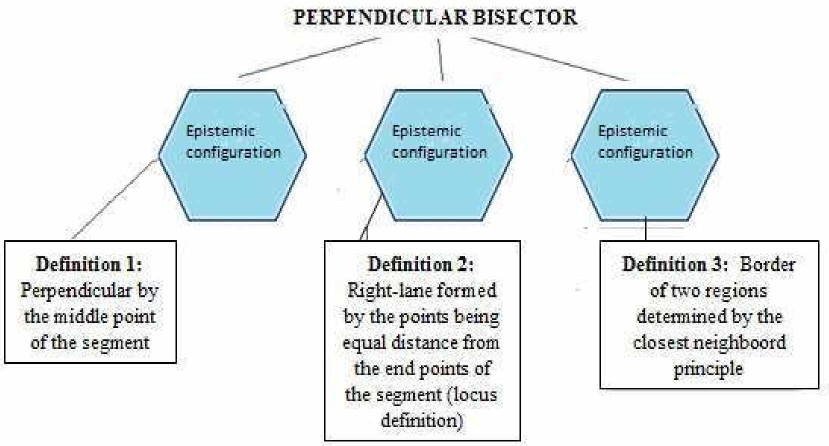


Figure 2. Epistemic Configuration of classroms for teachin g perpendicular bisector

It was observed that both first and second teachers did classical proposals and management about the content and the classroom. The third teacher proposal is innovative not only because of the management but mathematically as a way of changing the regular use of mathematical content as a change of configuration of practices, objects and mathematical processes by using a non-routine task (Tzur, Sullivan, & Zaslavsky, 2008). The class started by presenting a contextualized problem, driving to the division of a desert in a set of regions. Future teachers observed interpretation processes, communication of didactical and mathematical meanings, etc. Furthermore it appears a reflection about distinguishing complex processes from simple processes and also a general reflection about the idea of processes itself.

During the second year, the tasks designed had achieved the effect of improving their analysis of practices, objects and mathematical processes and mainly about processes (Font, et al. 2012). In this improvement, it was judged a crucial role of dynamic videotapes to analyze the visualization of professional didactical processes. On the other hand, they were introduced selected episodes of students' from previous years that were considered as a short distance from prospective teachers' perspectives. We still detect that the future teachers applied epistemic suitability criteria, by means of superficial explanations, short justifications, etc. Therefore, it's needed to improve future teachers' justifications about mathematical and didactical quality of their practices as a basis of the second redesign.

4. Third year redesign. Conectness and representativeness

Epistemic suitability criteria explained for years 1 and 2 were basically sustained in the idea of representativeness, understood as a degree, of representation of learned meanings representing relations to referenced meanings. Due to the superficiality of some students' works during the moment to apply such criteria, it was decided to do an extensive study about how the students have been applied epistemic suitability criteria in their final masters' thesis (to see if they have been used the representativeness criteria, introduced some personal proposals, etc). As a consequence, the changes proposed for the third year were the following: (1) To join the categories for epistemic suitability from OSA with categories from the quality for mathematics instruction given by Hill (2010). In such a way, it was introduced new criteria for valuing mathematical quality as it is: mathematical richness, coherence, errors, etc. (2) To select new case studies from previous years students with more wide and complex explanations than the previous case studies used en year 1 and 2. The aim was to connect echoes and voices to produce more consistent arguments (Garuti & Boero, 2002) to justify mathematical quality of didactical sequences.

A prototypical example of this new task (type c) is a case based analysis upon a student that planned a sequence with 7th grade (13-14 years old students) for Thales theorem. The main idea is to use as a new task, a voice of a previous future teacher M that analyzed her own practice about Thales Theorem after the school practice during the course 2010-2011. It was observed that M did a personal final analysis in which she said "…Additionally, we have tried to establish connections either with the concepts of the unit (relating as an example, Thales with similar triangles; similar triangles with similar figures, and so on) as with other subjects (for example, to compute the measure of a columns with mirrors, Snell's law of refraction, relating phisical concepts to mathematical concepts)... So, in conclusion...my epistemic configuration was right" (St. M; final report of practice and master's thesis, 2011).

The student M did not really a real good mathematical connection (Figure 3). We use such mistake to introduce our new professional task. In such task (type c) we presented three documents: (1) tasks proposed by M to explain Thales theorem in her proposal for school practice; (2) the analysis of epistemic suitability about M proposal, and (3) a textbook in which it was ensured the representativeness of epistemic configurations for Thales Theorem having a coherent connection (see figure 3). When doing the task it was promoted a discussion to understand the idea of representativeness (by using epistemic configurations 1 and 2) and the idea of coherent connection by using triangles in Thales position. The textbook sequence follows the order: Thales, triangles in Thales position, and after that, similarity triangles.

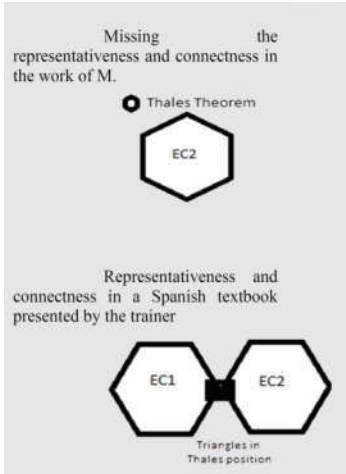
Epistemic Configuration EC1 Thales Theorem	Epistemic Configuration EC2 Similar triangles	
Concepts: point projection; segment projection; Triangles in Thales position **Properties:** **Version 1.** Segments determined by parallel rights over two secants are propotionnal. **Version 2:** Two triangles in Thales position have proportional sides and equal angles.  **Procedures** Find fourth proportion **Argument:** Comprovation of Thales Theorem; justification that triangles in Thales position have proportional sides, by using Thales theorem twice. **Problems** (1) Find one out of four segments determined by parallel rights over two secants (contextualized / decontextualized) (2) Indirect computation of unknown measures, using Thales Theorem	**Concepts/definitions:** similar triangles (similar triangles have proportional sides (def) and equal angles.) **Properties:** similarity criteria **Procedures:** Find fourth proportion **Arguments:** Comprovation of four proportional sides and equal angles	

Figure 3 Left. Two connected configurations in the textbook.
Figure 3 Right. Comparing representativeness and connectness for M and the textbook presented by the trainer.

The aim of this professional task is to recognize a deep level of analysis from such previous prospective teacher's practices (Choppin, 2011). Thus, the future teachers learn from this analysis, the idea of connecting two epistemic configurations. At the end of the third year experiment, we found the students being more carefully presenting their didactical unit as a result of such deeper analysis.

More aspects were observed in this third year, when we analyze final work of future teachers and we found better results than previous years. Just some aspects had been presented in this paper because a lack of space.

5. Conclusions. Perspectives

As a result of our study, we have analyzed in depth what we nominated professional tasks to promote competency of didactical analysis. It was useful for such analysis the levels of didactical analysis proposed by OSA. We assume the power of analyzing case studies based on texts from previous years' students. In fact, it explains the complexity of analysis that the teacher should realize to value his/her own practice to go beyond from narratives and descriptions. We are centered specially in how successive redesign contribute to have better feedback about the analysis of processes and valuing quality using notions of representativeness, connection and coherence. One of our conclusions is that to reflect about mathematical quality it's needed that the future teachers use theoretical powerful elements (Krainer, 1993). We value that some students explicit that by doing master's degree work, "we had been developed our competence of didactical analysis". On the other hand, we recognized the final master degree as the starting point for developing research competency for future teachers. In fact, it gives opportunities for students learning and recognizing problems of their professional context (Giménez, Font, Vanegas y Ferreres, 2012). Following our perspective we intend to see didactical analysis beyond the banality, considering classroom situation as an integral but dynamic system evolving in time, promoting autonomous mathematical thinking and independent validation of its results as future teacher (Laborde, Perrin-Glorian, Sierpinska, 2005).

Our major conjecture in terms of designing didactical sequences of professional tasks for prospective teachers, is that we need epistemic and cognitive analysis not only to criticize each task itself, but to adapt its connections as best as possible to the didactic analysis results. In fact, suitability criteria used for redesigning the tasks (considered as teaching experiments and corresponding case studies) has anticipatory purposes as hypothetical trajectories, but also helps to improve didactic training trajectories. It's important for our task analysis to identify difficulty factors providing frameworks for hypothesizing instructional designs inspired by levels of suitability.

The relevant aspects for our task design proposal are: (1) To understand the redesign process as a teaching experiment, assuming the noticing process (Mason & Wilder 2002) when doing didactic analysis. (2) To use suitability criteria for building and analyzing professional tasks and sequences; (3) To have in mind ethical perspectives of hearing the voice of the prospective teachers as self regulating process. (4) To consider the need for a collaborative research team for redesigning process.

After three years of experience, we assume the difficulties of the future teachers for having a deep reflection upon their proposals (Leikin, 2009) but we consider that our training cycle give opportunities to improve such issues.

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Designing Rich Numeracy Tasks

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In Australia, numeracy is regarded as a general capability to be developed across the whole school curriculum, not just mathematics. This paper draws on a research study that aimed to help teachers in ten schools design numeracy tasks and implement investigative numeracy pedagogies across the middle school (Grades 6-9) curriculum. Teachers were introduced to a rich model of numeracy that gives attention to real-life contexts, application of mathematical knowledge, use of representational, physical, and digital tools, and positive dispositions towards mathematics. These elements are grounded in a critical orientation to the use of mathematics. The paper identifies ways in which collaboration between the researchers and teachers influenced the design of numeracy tasks.

Keywords: Numeracy; Teacher development; Influences on task design

Numeracy is a term used in many English-speaking countries to denote the capacity to deal with quantitative aspects of life. It is often considered to have a similar meaning to terms such as *quantitative literacy* (Steen, 2001) or *mathematical literacy* (OECD, 2004). For example, Steen proposed that the elements of quantitative literacy include: confidence with mathematics; appreciation of the nature and history of mathematics and its significance for understanding issues in the public realm; logical thinking and decision-making; use of mathematics to solve practical everyday problems in different contexts; number sense and symbol sense; reasoning with data; and the ability to draw on a range of prerequisite mathematical knowledge and tools. Some of these elements are visible in the PISA definition of mathematical literacy as:

> an individual's capacity to identify and understand the role mathematics plays in the world, to make well-founded judgments, and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, p. 15)

Steen (2001) argues that, for numeracy to be useful to students, it must be learned in multiple contexts and in all school subjects, not just mathematics. In Australia, support for this challenging notion has come from several sources. A recent national review of numeracy education undertaken by the Australian government recommended that numeracy be recognised as "an across the curriculum commitment" (Council of Australian Governments, 2008, p. 7). In addition, the newly developed Australian Curriculum – the first ever nationally mandated curriculum in this country – identifies numeracy as one of seven general capabilities that apply across all discipline content (Australian Curriculum, Assessment and Reporting Authority, 2012). However, the new curriculum documents do not provide teachers with detailed guidance in recognising the numeracy demands of the subjects they teach, in designing tasks and learning sequences that embed numeracy across the curriculum, or in making decisions about pedagogies that support numeracy learning. This paper draws on data from a one year project that helped teachers to design numeracy tasks and implement numeracy pedagogies across the school curriculum in Grades 6-9. It was conducted before the release of the Australian Curriculum, with the intention of developing and testing a theoretically informed model of numeracy that could guide teachers' curriculum planning, task design, and pedagogical decision making in the context of an existing state-based curriculum framework (South Australian Department of Education and Children's Services, 2005). For the purposes of this project, a "numeracy task" is considered to be an activity that engages students in learning and/or applying some mathematics within a given curriculum context.

This paper aligns with Theme E: Features of task design informing teachers' decisions about goals and pedagogies. It addresses the following question:

- *How does collaboration between researchers and teachers influence the design of rich numeracy tasks?*

Numeracy Model

We developed the model shown in Figure 1 to affirm the value of definitions of numeracy widely accepted in Australia (e.g., Department of Education, Training and Youth Affairs, 2000), while introducing a greater emphasis on tools as mediators of mathematical thinking and action (Sfard & McClain, 2002) and a critical orientation to the ways mathematics is used to support arguments and influence opinions (Jablonka, 2003). The original purpose of the model was to describe the characteristics of a *numerate person*. However, when we worked with teachers we realised the model also provides principles for the design of rich *numeracy tasks*. Thus the elements of the model, elaborated below, were the task design principles that we encouraged teachers to use.

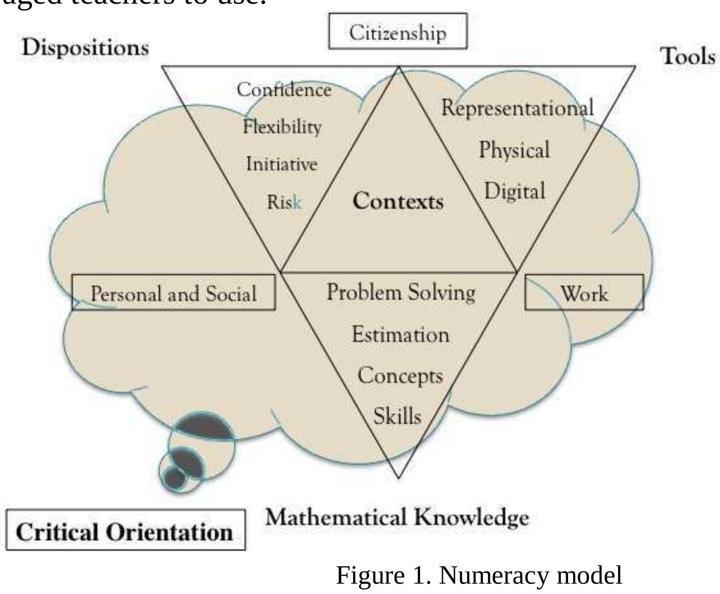

Figure 1. Numeracy model

A numeracy task should require application of *mathematical knowledge*. In a numeracy context, mathematical knowledge includes not only fluency with accessing concepts and skills, but also problem solving strategies and the ability to make sensible estimations (Zevenbergen, 2004).

A numeracy task should promote *positive dispositions* – such as confidence, initiative, and a willingness to apply mathematical knowledge flexibly and adaptively. Affective issues have long been held to play a central role in mathematics learning and teaching (Leder & Forgasz, 2006), and the importance of developing positive attitudes towards mathematics is emphasised in national and international curriculum documents (e.g., National Council of Teachers of Mathematics, 2000; National Curriculum Board, 2009).

A numeracy task should involve using *tools*. Sfard and McClain (2002) discuss ways in which symbolic tools and other specially designed artefacts "enable, mediate, and shape mathematical thinking" (p. 154). In school and workplace contexts, tools may be representational (symbol systems, graphs, maps, diagrams, drawings, tables), physical (models, measuring instruments), and digital (computers, software, calculators, internet) (Noss, Hoyles, & Pozzi, 2000; Zevenbergen, 2004).

Because numeracy is about using mathematics to act in and on the world, numeracy tasks should be embedded in a range of *contexts* (Steen, 2001). These contexts may be drawn from real life or curriculum areas other than mathematics.

Numeracy tasks should develop a *critical orientation* in students since numerate people not only know and use efficient methods, they also evaluate the reasonableness of the results obtained and are aware of appropriate and inappropriate uses of mathematical thinking. Numeracy tasks could ask students to evaluate quantitative, spatial or probabilistic information used to support claims made in the media or other contexts. They could also encourage students to consider how mathematical information can be used to manipulate, disadvantage or shape opinions about social or political issues (Jablonka, 2003).

The design of rich numeracy tasks according to the principles outlined above is not sufficient to enable learning. We argue that teachers also need to adopt *investigative pedagogies* to fully realise the numeracy opportunities that such tasks afford. Diezmann, Watters, and English (2001) define mathematical investigations as "contextualized problem solving tasks through which students can speculate, test ideas and argue with others to defend their solutions" (p. 170). We consider this definition applies equally well to numeracy investigations.

The numeracy model was used in three ways: (1) to analyse the numeracy demands of the South Australian school curriculum (Goos, Geiger, & Dole, 2010); (2) to support teachers' curriculum planning (Goos, Dole, & Geiger, 2011); and (3) to trace changes in teachers' understanding of numeracy (Goos, Geiger, & Dole, 2011). This paper is primarily concerned with (2), and it extends our previously published analyses by focusing on the design of numeracy tasks and implications for pedagogy.

Project Overview

Teachers were recruited from ten schools with diverse demographic characteristics: four primary schools (Kindergarten-Grade 7), one secondary school (Grades 8-12), four small schools in rural areas (Grades 1-12), and one school that combined middle and secondary grades (Grades 6-12). Each school nominated two teachers, thus ensuring that participants could collaborate with a colleague in their own school as well as teachers from the other schools. They included generalist primary school teachers who taught across all curriculum areas as well as secondary teachers qualified to teach specific subjects (mathematics, English, science, social education, health and physical education).

There were three elements to the research plan: (1) an audit of the middle years curriculum to identify the numeracy demands inherent in all curriculum areas; (2) three whole-day professional development workshops at the beginning, middle, and end of the project; and (3) two daylong visits to each school for lesson observations, discussion of planning documents and teaching approaches, and audio-recorded interviews with teachers and students. The overall project design is summarised in Table 1. More details on data collection and analysis methods can be found in Goos, Dole and Geiger (2011).

Time	Researcher activity	Teacher activity
February: Curriculum audit	Identify numeracy demands in all curriculum areas	
March: Workshop #1	Introduce numeracy model; present findings from curriculum audit; provide sample numeracy tasks	Analyse numeracy task design via reference to model; plan for implementation
June: School visits	Observe lessons; provide feedback on planning, task design and pedagogies	Incorporate feedback into planning for further implementation
August: Workshop #2	Provide feedback on first round of school visits; present stimulus materials for task design	Share tasks and strategies tried so far; practise task design with emphasis on critical orientation
October: School visits	Observe lessons; provide feedback on planning, task design and pedagogies	Incorporate feedback into planning for further implementation
November: Workshop #3	Report on student perceptions of numeracy; present stimulus materials for task design	Practise task design; reflect on professional learning trajectories

Table 1. Project Design

Influencing the Design of Numeracy Tasks

We claim that there are several aspects of the project that influenced the design of rich numeracy tasks and enactment of associated investigative pedagogies. The first was the numeracy model itself, the elements of which provided a set of principles for task design. The second was our numeracy audit of the South Australian Curriculum Framework, which identified distinctive numeracy demands for each school subject taught in Grades 6-9. For example, the subject called Society and Environment is organised into four strands: time, continuity and change; place, space and environment; societies and cultures; and social systems. The numeracy audit found that data analysis and spatial sense are the most relevant elements of *mathematical knowledge* for this subject. *Contexts* for numeracy development included the study of social, economic, political and ecological systems. Students were expected to develop *dispositions* enabling them to “to be active citizens who can make informed and reasoned decisions and act on these” (DECS, 2005, p. 291). The use of *tools* such as maps, measuring instruments, online data sources, and spreadsheets for collecting and analysing information was vital to learning in this subject. The goal of enabling students to participate as ethical, active and informed citizens, requires development of a *critical orientation* to viewing information and interpreting data. Presenting the audit findings to teachers was intended to raise awareness of their subject’s numeracy demands.

The third influence on task design was the way in which the research team used the three workshops to (1) immerse teachers in numeracy tasks we had created, (2) model investigative numeracy pedagogies, (3) guide the analysis of tasks using the design principles provided by the numeracy model, and (4) invite the teachers to practise designing their own numeracy tasks. For example, in the first workshop we engaged teachers in cross-curricular numeracy investigations suitable for use with middle years students. These included investigations of Barbie dolls' physical proportion (with links to the health and physical education curriculum), the occurrence of the Golden Rectangle in art, design and nature (linked to the arts and design studies curricula), and planning for participation in the Tour Down Under, a bicycle race similar to the Tour de France (linked to the society and environment curriculum). In the first round of school visits we found little evidence of a critical orientation in the lessons we observed. In interviews with teachers it emerged that they were unsure about how to embed this element of the numeracy model into their planning and practice. Therefore, at the second workshop we presented a range of stimulus materials drawn from print and digital media sources and asked teachers to work together to develop these into tasks that would promote a critical orientation in their students, without losing sight of the other elements of the numeracy model. In both workshops we provided a task design/analysis template that listed each element of the numeracy model, asked how the task developed numeracy with respect to each element, and invited teachers to identify school subjects that could provide a context for using the task.

The fifth influence on task design was the researchers' provision of in situ feedback to teachers during school visits. We were able to suggest ways of modifying tasks used in the lessons we observed to give greater prominence to elements of the numeracy model that appeared to be under-represented. The final influence on task design was the structured sharing of practice by teachers at the second and third workshops. At the second workshop all teachers were asked to bring evidence of one task or lesson sequence they had tried with their class, to describe to the whole group how the task had been implemented and how well (or not) it had worked, explain what they learned from this experience and how they would use this evaluation in their subsequent planning. This workshop provided an opportunity for teachers to see how colleagues in other schools went about designing rich numeracy tasks. At the third and final workshop, and as a result of our analysis of school visit data, we invited four teachers who exemplified different types of professional learning trajectories to report on their experiences. One of these is the teacher whose abbreviated case study is presented below to illustrate how she made decisions about the design of numeracy tasks with an investigative flavour. Although this project emphasised numeracy across the whole school curriculum, the example below shows how numeracy tasks can be designed within mathematics.

Teacher Decision Making about Task Design

Maggie taught mathematics and science at a large secondary school in a rural town. She was in only her second year of teaching. The class with which she worked for this project was a Grade 8 mathematics class.

First school visit

Initially Maggie struggled to come to grips with how to highlight the numeracy within mathematics, but she decided to focus on teaching mathematics in real life contexts that would be of interest to her students. She planned a task based on the television program *The Amazing Race*. Students worked for 3 weeks in the computer laboratory to complete the task, which involved organising an adventure holiday around the world, given an itinerary and a budget of $10,000. They also had to complete a number of challenges for which they earned an additional $2000 each. The challenges, which included *Diving with Sharks* in Cairns, *Skiing* in Switzerland, and visiting *The Roman Colosseum*, had a focus on using directed number in context. In *The Roman Colosseum* challenge students were also required to use formulas in the context of comparing areas of the Colosseum and the Melbourne Cricket Ground, as well as looking at exchange rates and converting between currencies.

Members of the research team observed the second lesson of this unit. Students appeared motivated and well prepared, and they were able to explain the task to us when we questioned them. Maggie noted that some previously disengaged students were interested in the task, while a few others remained aloof. Some students seemed so engaged that they acted as though the task was real; for example, when Maggie asked one boy “Where are you up to?”, he replied “I’m on my way to Paris!”.

This task placed mathematics in the real life *context* of an adventure holiday. It targeted *mathematical knowledge* of directed numbers and operations with integers (money calculations), using digital (internet) and representational (charts, tables) *tools*. We did not observe teacher actions that promoted positive *dispositions* towards numeracy, but students were clearly motivated and confident in tackling the task and trying out different combinations of flights and accommodation bookings that would fit within their budget. A *critical orientation* does not seem to have been built into this task. However, after the lesson we suggested to Maggie that this orientation could be promoted via questioning, such as that we observed when Maggie helped a student compare advantages and disadvantages of booking cheap backpackers’ accommodation. In this way we attempted to show how the numeracy model informed not only task design but also teachers’ pedagogies.

Second school visit

When interviewed before the lesson observation, Maggie said she had given a lot of consideration to the types of tasks she wanted to design for the second research cycle. She was dissatisfied with the length of the *Amazing Race* investigation, as this tended to discourage some students and to make it difficult to complete for any who missed some lessons. As a result of our emphasis in the second workshop on developing a critical orientation, she also decided to give more attention to this element of the numeracy model in designing the next investigation, *Approaches to a Healthy Lifestyle,* which comprised a number of smaller tasks.

In one task, students investigated the relationship between the heights and walking speeds of everyone in the class. The mathematics embedded in the investigation included elements of collecting, representing, reducing and analysing data. In the lesson we observed, students were to make scatter plots using Excel in order to determine whether there was a pattern in the data they had collected. In earlier lessons they had collected height data and calculated the mean, median and mode. In another lesson students had marked out a 40 metre section of a 100 metre running track and then found the time it took to walk this distance. With this information students had calculated their walking speeds in metres per second, metres per minute and kilometers per hour.

Students all appeared engaged with the task and each group or individual produced a scatter plot, although the appearance of the graphs varied depending on the scales chosen or on the choice of variable for the *x* and *y* axes. Most students were able to describe a general trend in the data and use this to make a prediction about what Maggie's walking speed might be, based on her height. Interestingly, many students gave most attention to their own data point within the scatter plot with comments such as "This is me (pointing at the appropriate data point) and "This is how tall I am and how fast I walk". Using personal data seemed to be effective for engaging students with the task. From a student's perspective, the activity was about them and how they compared to the rest of the class.

Students expressed surprise that the scatter plot was not linear, so that taller people did not necessarily walk faster. Maggie challenged them to explain why this should be the case. Some groups suggested that alternative variables – with associated alternative hypotheses – should be explored, including, for example, the relationship between walking speed and leg length or between walking speed and stride rate. One group suggested there might be a stronger relationship between a person's height and their maximum walking pace rather than their natural walking pace.

Maggie chose an engaging *context* that made use of students' personal details to introduce the *mathematical knowledge* that was used in this lesson. The use of personal data encouraged positive *dispositions* towards involvement in and completion of the task. This task required knowledge of how to produce a scatter plot from a data set using Excel and the capacity to make predictions from trends in the data. Maggie asked students to use *representational tools* such as scatter plots and *digital tools* in the form of computers and Excel. By challenging students to explain the variance in their data from the anticipated linear relationship, Maggie introduced a *critical orientation* to the task. A critical orientation was evident in most tasks in the *Approaches to a Healthy Lifestyle* investigation. For example, in the culminating task students were to compare data on the number of overweight and obese Australians in various age groups, and make an argument for whether or not the government needed to introduce a healthy eating policy for South Australian schools.

Implications and Concluding Comments

The work we reported here has implications for teachers, researchers, and curriculum developers. First, it provides some evidence that a focus on task design, when supported by a theoretical model of numeracy that is readily accessible to teachers, can influence teacher learning and development. For example, when Maggie reflected on what she had learned during the course of the project, she identified her readiness to make use of more extended tasks when teaching mathematics. However, she tempered this view by arguing that tasks needed to be made up of self-contained sub-tasks that allowed students to move towards smaller achievable goals. For her, the level of engagement she observed while students were working on numeracy investigations was a compelling case for their inclusion within mathematics classes. Nevertheless, other teachers in the project found it more difficult to decide "how long" a numeracy investigation should be to allow students enough time to explore all aspects of a task without losing interest, and how much guidance to give students in structuring the investigation.

A challenge for researchers is to design larger scale studies that do not simply rely on recruiting more schools or teachers. This is the case for most educational research and is not unique to our project. To tackle these challenges we are currently conducting a follow up study that aims for both scale and sustainability by developing numeracy curriculum leadership within schools, so that the numeracy model and associated task design principles become integrated into schools' planning processes for implementation of the new Australian Curriculum.

The current version of the Australian Curriculum offers some support for recognising the numeracy demands of different school subjects, for example, by providing a numeracy learning continuum together with icons and filters that link numeracy capabilities to relevant curriculum content. However, additional opportunities for developing students' numeracy capabilities are invisible unless one knows how to "see" them and how to design and implement tasks for classroom use. The numeracy model and ways of working with teachers outlined in this paper may prove useful in supporting teachers to fulfil the numeracy intentions of the Australian Curriculum.

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Task design: Supporting teachers to independently create rich tasks

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Teachers want to be able to create opportunities for generalizing and justifying for their students; however, they often lack the skills and the time needed. We attempt to investigate this issue by addressing the question: How can we help teachers use their textbooks to create tasks that provide opportunities for students to justify, and afford teachers opportunities to push students to consider different cases and ultimately engage in generalizing? We define task design to include the activities engaged in with teachers that enable them to independently create tasks that afford students these opportunities. We engage in a design-based methodology with cycles to refine this process.

Keywords: Rich tasks, task design, design cycle, generalizing, justifying

Introduction

Intellectually, teachers, mathematics educators, and mathematicians would agree that it is important for students to generalize and justify. In recent years, teachers have begun to ask students to explain their thinking. Teachers and students often understand this request to mean, *What steps did you use?* This is one type of explanation, but it is not sufficient for helping students to gain mathematical insights that lead to deep understanding. Driscoll (1999) suggested that teachers need to ask questions that prompt students to reflect on the mathematical ideas used in their solution strategies and consider whether the strategies can be generalized. Stein, Smith, Henningsen, and Silver (2000) suggest that teachers need to use rich tasks that maintain cognitive demand by pressing students to make generalizations and justify their solutions. Both of these suggestions assume that teachers have tasks that lend themselves to acts of generalizing and justifying that go beyond finding the solution of a problem. Lampert (1990) highlights the importance of the task itself, “it must be a problem that will engage all students in making and testing hypotheses ... and push them to think about when and whether it holds true in a larger domain.” (p. 40).

Taking time for students to make hypotheses and explore them is a tension for teachers when they are concerned with covering the curriculum and loss of control (Zaslavsky, 2008). The question arises, how can we help teachers use their textbooks to create tasks that provide opportunities to students to justify, and afford teachers the opportunity to push students to consider different cases and ultimately make generalizations?

Making Mathematical Reasoning Explicit (MMRE) is a five-year NSF funded project in which teachers attend three summer institutes and participate in school year professional development. In this context, we examine the transformation of instruction from a model where the goal of learning is to arrive at a correct answer, to instruction with the goal of engaging students in the process of doing mathematics. Lakatos (as cited in Lampert, 1990) described doing math as “the zig-zag path of discovery between conjecture, justification, and revision of conjecture or modification of assumptions that one must necessarily negotiate when engaging in this kind of mathematical activity.” (p. 40). Our first task was to work with teachers so that they better understood the process of doing mathematics where generalizing and justifying were the norms. Second, we gave them two simple routines to use that provided them opportunities to elicit students’ observations, explanations, and justifications. Third, we asked them to create a task that would prompt students to generalize or justify.

This paper examines the fourth phase of our work, which helped them to use their textbook as a source for developing rich tasks that engaged students in this zig-zag path of discovery. Our research question is: What are the essential features of task design that teachers need to attend to when modifying their existing curriculum to engage students in learning new content in ways that require generalization and justification? In this paper, we define task design to include the activities that we engaged in with teachers, so that ultimately, they are able to independently create tasks that afford students the experience of doing mathematics.

Literature Review

Literature on teacher change suggests that for teachers to enact new pedagogy with different mathematical goals, they must rethink and revise their beliefs and practices in light of research on teaching and learning (Cross, 2009). In addition, to make this change, teachers need to consider the instructional goals demanded by the new Common Core Mathematics Standards, create a different learning environment that encourages students to engage in a different kind of learning, establish new classroom norms where argumentation is central to the lesson, and use discourse in which teachers no longer control the discussion through recitation. An additional component of such change is task design.

A review of the literature on mathematics tasks revealed research on characteristics of rich tasks, ways to maintain the cognitive demand of the task, and examples of rich tasks that could be used in classrooms. However, we found scant research that described how to help teachers design tasks. We define a rich task to be one that is complex, non-algorithmic, and non-routine, allowing for multiple strategies and representations and no single pathway to a solution. Any solution to a rich task is not just an answer to be circled, not even simply a description of the strategy or reasoning used to arrive at such an answer. It affords students opportunities to generalize and justify that go beyond finding the answer. It includes the justification for the strategy or reasoning used to arrive at an answer; an explanation of why this particular approach is valid. Our literature review begins with a discussion of the importance of generalizing and justifying and ends with additional comments about rich tasks.

NCTM (2000, 2009) and Common Core State Mathematics Standards (2010) emphasize the importance of reasoning and justification in learning mathematics. Generalization can be thought of as the heart of mathematics (Mason, 1996). There is consensus among mathematicians that generalization is a process that serves to create new knowledge for an individual and for the mathematics community at large (e.g., Polya, 1957; Harel & Tall, 1991;Dreyfus, 1991; Ellis, 2007). In much of reform mathematics, there is an inherent tension between this *process* of doing mathematics and the actual *product* of the mathematical activity. Polya describes the process of doing math as making a tentative generalization from an observation that is tested and revised for different cases. Justification is the tool by which a generalization is verified. Justifications validate claims, provide explanations, create a context for insights or discoveries, and develop structures to systematize knowledge (de Villiers, 1999; Hanna 2000). Clearly, the acts of generalizing and justifying are intertwined and essential to developing mathematics.

The ICME 11 Topic Study Group 34 (2008) describes the nature of tasks and task design as "not merely … 'things to be done' but as crucial in framing subsequent mathematical activity. Identifying relations between task structure, tool use, and mathematical activity can inform the design and analysis of tasks, and also gives insight into the nature of engagement and learning which takes place while tasks are being carried out." (ICME 11 TSG 34, p. 1). Rich tasks are essential in providing opportunities for students to generalize and justify.

Solving a rich task includes the processes of making conjectures, arguments, justfying and generalizing. Only if a teacher establishes these processes as the acceptable socio-mathematical norms (Yackel & Cobb, 1996) for what it means to be doing mathematics in the classroom will the task provide real opportunities for students to engage in these processes. Such tasks require higher-level cognitive demand (Stein et al. 1998) and strategic reasoning to solve (Kilpatrick, Swafford, & Findell, 2001). This high cognitive demand is necessary, but not sufficient, for fostering opportunities for justification, generalization, and conjecturing in the classroom. Teachers must treat the solution to a task not as a signal to stop thinking, but as an occasion to generalize, conjecture, and ask new questions. Doing mathematics in this way requires both students and their teachers to redefine what they believe mathematics to be (Lampert, 1990).

Theoretical Framework

We draw upon sociocultural and psychological frameworks to examine the features of task design that support teachers to reconsider what it means to teach and learn and do mathematics. These perspectives allow us to work with teachers in ways that create opportunities for them to reconsider their deeply held beliefs about the nature of teaching and learning mathematics, while simultaneously providing opportunities to engage students with tasks. Deeply held beliefs about mathematics develop over a lifetime of experiences and are resistant to change. To support teacher change, we rely on concerns based research (Hall & Hord, 2010) to structure a community of practice.

A community of practice is a group of people who share a similar goal; that learning occurs through interactions among the participants (Lave & Wenger, 1991). From this perspective, the learning environment is critical. It is the place where participants share ideas, take risks, and revise their notions. In the MMRE community of practice, participants experience math in new ways, discuss the implications for teaching with different goals, and consider the impact on students' learning. New ideas are developed through dialogue that includes sharing observations, asking questions, and clarifying ideas (Lave & Wenger). This learning is a product of the group and is available for each participant to examine and compare with his/her personal beliefs. When teachers verbalize them, their beliefs are placed in the public domain where they can be examined within the community of practice. The inner and public conversations interact, creating an opportunity for change. Critical to the conversation is the reflection on new experiences and information that may conflict with existing beliefs. With repeated experiences, reflection, and dialogue, we theorize that change is supported.

To make the work of doing mathematics in new ways meaningful, teachers need to examine their beliefs about what is important for teaching and what it means to "do mathematics." Unless they are willing to undertake such change, or at least acknowledge its necessity, it will be difficult if not impossible for them to embrace the type of task design we are advocating. Further, they will be unable to implement the tasks successfully in the classroom, even when such tasks are provided for them. For teachers to teach mathematics in new ways, they need to recognize a new instructional goal, namely, traveling along the zig-zag path that leads to making generalizations and justifications. If is from this perspective that we work with teachers in the MMRE project. We found it was necessary to embark on this preliminary work prior to or concurrently with the task design that we would engage in with teachers.

Methodology

Design-based methodology with cycles frames our work with teachers. We collected baseline data through observations of teachers working in their classrooms to characterize the pedagogy that they used. Data was analyzed using constant comparative methods. We created a rubric from research on types of generalizations and justifications. It was modified as we coded observations and noted classroom examples that contained types of justifications that did not fit into our existing rubric. When we made changes to the rubric, we went back and re-coded all of the observations.

From our research on generalization, justification, pedagogy, and teacher change, we created a blueprint for what we hoped to see if teachers engaged students in generalizing and justifying as tools for learning. Using the concept of backwards design, we created learning trajectories that would move them from their current practice to a new vision of teaching. We theorized that they needed experiences for themselves in order to be able to provide opportunities for their students to generalize and justify, before they would be ready to engage in creating rich tasks themselves.

Context

In our professional development project with teachers of grades 4 through 12, in rural schools in the Pacific Northwest of the United States, our first task was to work with teachers so that they better understood the process of justifying and what was involved in creating and supporting a classroom environment where student justification was the accepted norm. We were supported in this approach by the recent adoption by 45 US states of the Common Core State Standards Initiative (CCSS, 2010). The Standards for Mathematical Practice are arguably the most important part of the CCSS in mathematics. Additionally, they provided the motivation teachers needed in order to embrace our focus on what it would look like to see students “engaged in doing mathematics” in this way. Of the eight Standards for Mathematical Practice, the ones most relevant to our work are that students should: Reason abstractly and quantitatively; construct viable arguments and critique the reasoning of others; and, look for and make use of structure.

Design cycles

We engaged in four cycles in our work to date (see Figure 1). In the first two cycles, we provided teachers with short, easily implementable ways to engage students in justifying: choral counts and strings (Chapin, O’Connor, & Anderson, 2009). Choral counts and strings are examples of pedagogical routines that support students’ learning (Lambert, Beasley, Ghousseini, Kazemi, & Franke, 2010). A choral count is some number pattern, often presented as an array, which is gradually revealed to the class by the teacher. The students are encouraged to say in unison what the next number is in the array. As they begin to see patterns, both recursive and explicit, they discuss what they observe, and make conjectures about what numbers might appear in certain spots in the array, and explain why. A string is a sequence of small problems that leads to the observation of a common property or generalization. These routines were readily understood by teachers and easily incorporated into existing classroom structure. They provided a way for teachers to encourage students to begin making their reasoning explicit. We engaged the teachers in choral count tasks and strings tasks during a professional development session, and then asked them to devise their own choral counts and strings to use in their own classrooms.

Once teachers began to see for themselves that their students were not only capable of justifying, but were enthusiastic about engaging in it, they wanted our help to adapt their existing curriculum materials to provide justifying and generalizing opportunities for their students. During the third cycle, we provided examples for teachers to engage in justifying with; then we discussed how they might use these tasks to move their students towards generalizing and justifying.

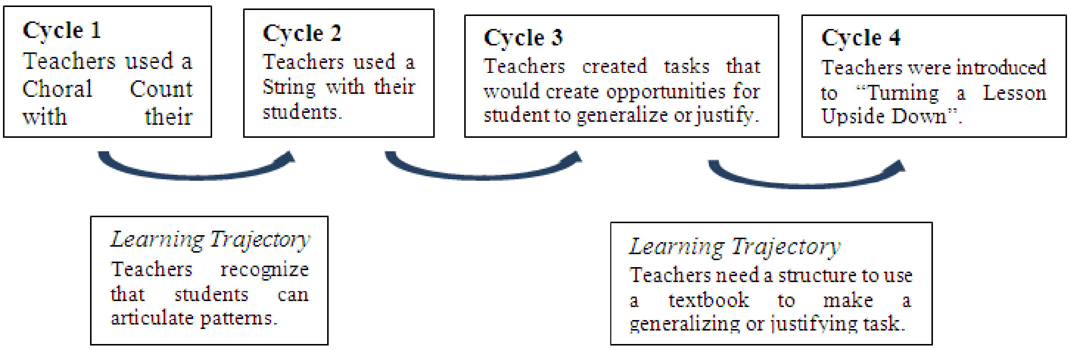

Figure 1: Design Cycles with learning trajectories illustrating the connections between theory and action

In working through these tasks with the teachers as students, we modeled and made explicit for them appropriate classroom strategies – “What do you notice?”; “Tell me why you used that strategy”; not giving confirmation of the correctness of their answers but instead suggesting they discuss it with a classmate - they might use to move their students in this direction. The next step was to work with them to create their own tasks that would lead their students to generalizing and justifying. Not surprisingly, most teachers chose to develop either strings or choral count examples based on their textbook materials, replicating or imitating the work we had done with them.

We devised a strategy we termed "turning a lesson upside down", which would help teachers to start with their existing textbook and modify a task or set of tasks so that the "key mathematical idea" target of the lesson emerged from the students' own engagement with the task. This is in contrast to how their curriculum typically approaches a new concept, by explicitly stating it as an objective of the lesson, and then providing practice problems for students. The focus in this paper is on this fourth cycle – turning a lesson upside down.

Realizing that they needed our support to modify their curriculum, we worked with them in groups during the Summer Institute on "turning a lesson upside down." We made explicit for them the steps involved in turning a lesson upside down which are summarized here:

1. Select a lesson from the textbook and determine the key understanding, method, or concept for the lesson.
2. Write the mathematical idea as a generalization, or a way of thinking that allows one to see general patterns or relationships. Note: Some mathematical idea may not be treated as a generalization.
3. Determine whether the key understanding entails justification or provides a person with a tool for justifying patterns and relationships. Identify representations that can be used to justify the key understanding.
4. Find a task(s) or a sequence of problems that can be used to illustrate or develop the key understandings. Note opportunities for generalization and/or justification that could be pursued, if any.
5. Write questions that you can ask students to make their generalizations or justifications explicit.

Results and discussion

We began cycle four by providing a simple lesson on averaging that was presented in two ways. The first was following a textbook structure in which the procedure of adding and then dividing was defined. In the upside down lesson how to find the idea of averaging emerged from student investigation. The typical textbook lesson would define the average of a set of numbers to be calculated by forming the sum of all the numbers and then dividing by the number of numbers in the set. In contrast, an upside down lesson might contextualize this target idea of finding an average in the following way: Suppose that Bonita has 5 beads, Irv has 14 beads, Jacob has 13 beads and Maria has 8 beads. How could the children redistribute the beads amongst themselves so that each child has the same number of beads?

Later we showed a video that contrasted two model lessons from an elementary school classroom. The first followed a textbook structure and the second a lesson that was turned upside down. The textbook structure dictated the format: the target procedure was made explicit and then students practiced it by solving similar problems. The second lesson was turned upside down: students were given problems that were carefully scaffolded 'finding' the mathematical target, supporting student work towards providing claims, arguments, conjectures and justifications that would help to solve the problems. Both lessons used Cuisenaire rods in which a fractional part was given (the red rod is 1/3) and the problem was to find the whole. Both lessons focused on the relationship between the part and whole. The mathematical target of the lessons was finding the whole when a fractional part was given. We watched the two lessons with the teachers and discussed the actions, observations, and explanations of the students. Not surprisingly, the students in the upside down lesson provided more explanations and seemed more engaged. We intended to compare the structure of the two lessons so that teachers could see the differences in how to construct an upside down lesson with one that followed the structure found in their textbooks. However, we did not examine the lessons' structure because of time constraints and we thought that the differences were obvious. Yes, they were obvious to us, but not necessarily to the teachers.

In addition, one group of the teachers experienced doing mathematics by actually engaging in a lesson that we had turned upside down. We scaffolded the task with a sequence of questions designed to help them find a way through the problem. Once they had spent a considerable amount of time wrestling with the ideas of the problem and sharing their solution processes and strategies with each other, we debriefed the session and made explicit the moves that we had engaged in with them as they investigated the problem. We felt we had prepared them well to begin replicating this process for themselves. But it did not work as well as we had hoped.

Analysis of the upside down lessons written by the teachers revealed that they had difficulty using the process that we had outlined. Many of them wrote key understandings that were too broad. For example, a high school teacher wanted to review division of fractions with her algebra students. The key understanding of the lesson was, "Students will find an algorithm for dividing rational fractions. They will understand why it works." This is a too big a target idea and requires more than one class period to develop. It involves understanding the process of unitizing, partitioning, and different ways to represent fractions. Clearly, we need to help teachers understand the importance of breaking down key understandings into smaller, target components and to focus on one component rather than to try and address all of them at once.

The teacher who was videotaped had anticipated students' responses in her two lessons. She used these responses to help plan questions and interventions that would support students' learning. In the lessons that teachers wrote, this critical piece of working through the lesson and anticipating responses from the student perspective was largely overlooked. The teachers worked collaboratively to create their upside down lessons and may have anticipated students' responses without recording them on the lessons that they turned in. However, we noticed that in some cases the context of the problem interfered with concepts that the teacher wanted students to investigate and working through the problem as a student would have illuminated this disconnect.

A sixth grade teacher identified three key understanding for her lesson: (a) Any two numbers have at least 1 number in common; (b) The greatest common factor is the greatest divisor they share; and (c) You can find the greatest common factor by listing and organizing factors. While all of these understandings are important, there are too many to address in one lesson. The problem required students to construct pages in a scrapbook:

> Annika is placing photos in a scrapbook. She has eight large photos, twelve medium photos, and sixteen small photos. Each page will have only one size of photo. She also wants to place the same amount of photos on each page. What is the greatest number of photos that could be on each page?

She anticipated student responses by describing different ways that they could solve the problem using manipulative (rectangles to represent the photographs), arrays, factor trees, or organized list. However, the problem context does not make sense. Why would anyone want to have the same number of photographs on one page? For example, one could put two pictures on each page, but it would look funny in a scrapbook for one page to have two large photographs and another page to have two small ones. Here, the context of the problem did not help students make sense of the key understandings.

Learning trajectory

To support teachers' learning we need to respect what they know and can do well, and build on it. Teachers were able to identify lessons that had the potential to be turned upside down and they were able to write questions to prompt generalizing and justifying. They will use their upside down lessons with students before we meet again. Thus, the first opportunity to help them focus on anticipating students' responses and articulating a component of the key understanding is through reflection. We will frame our reflection questions around these two ideas.

Summary

This is clearly a work in progress and there will be many more cycles, and many more revisions of these cycles. It is an ongoing, lengthy process. We are excited about sharing the next cycles in this working group as we refine and redesign tasks that support teachers to independently create tasks that provide opportunities for students to justify, and afford teachers the opportunity to press students towards justifying and generalizing.

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The same task? - different learning possibilities

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In this chapter we focus on variation of the design and the implementation of a specific task during three mathematics lesson in the 8th grade in a learning study (Marton & Tsui, 2004; Runesson, 2008). The theme of the lesson was division, with a denominator between 0 and 1. The teachers wanted their students to understand that when dividing with a denominator <1, the quotient is larger than the numerator. Four teachers collaboratively planned, analyzed and revised three lessons in a cyclic process. The study shows that the implementation of the task changed between the lessons. Although the same task was used in the lessons the way it was enacted provided different possibilities to learn.

Keywords: Tasks, variation theory, learning study, mathematics, division

Introduction

In mathematics education tasks play a critical role in the teaching and learning process (e.g., Hiebert & Wearne, 1997; Watson & Mason, 2006; Zaslavsky & Sullivan, 2011). Tasks can mediate important mathematical ideas for the students. In this chapter we address the issue about the implementation of a task. What is made possible to learn from a specific task in different lessons? The task chosen is a sequence of items of non-contextualised arithmetic (see Figure 1). Certainly, implementation of a task can be very different in different classrooms. What is possible for students to learn from a task, may be affected by for instance, how the task is enacted, students' response, students' knowledge, but also what mathematics is made explicit from the task.

It has been suggested that tasks and teaching can be designed with variation in certain dimensions to enhance student learning. For instance, Watson and Mason (2006), who talks about using variation as a tool for designing tasks and the role of the teacher in the this process:

> Constructing tasks that use variation and change optimally is a design project in which reflection about learner response leads to further refinement and precision of example choice and sequence…This process cannot be done by textbook authors working alone under tight publication deadlines but it can be done by teachers for themselves (Watson & Mason, 2006, p. 100).

In this chapter we describe such a design project; how one task was designed, enacted and successively refined by four teachers teaching in the 8th grade. The task was one of several tasks planned collaboratively by the teachers in an iterative process of planning, analysis and revision of a single lesson about division. This form of collaboration, called learning study, have shown that teachers become sensitive to their students' learning and that the way they teach the topic changes due to the insights gained by student learning (Runesson, Kullberg, & Maunula, 2011). The aim is to discuss students' possibilities to learn from the same task enacted in three different ways. How did the teachers implement the task and what was made possible to experience from the task by the learners?

Background of the study

The data used comes from a learning study about division. Learning study (Pang & Marton, 2005) is a version of the Japanese lesson study model (Fernandez, 2005; Lewis, Perry, & Murata, 2006; Stigler & Hiebert, 1999) informed by a learning theory; variation theory (Marton, Runesson, & Tsui, 2004). The aim is to improve students' learning by a careful and systematic inquiry into teaching and learning. The lessons are video recorded and students' learning is mapped by analysing how they solve certain tasks before and after the lessons. The current study lasted one semester with several pre- and post-lesson meetings where the three teachers planned, analysed and revised the lesson plan. After planning the first lesson together, one of the teachers conducted the lesson in his/her class. The video-recorded lesson was analysed and, from what was observed in the lesson and analysis of how the students solved the tasks on the post-tests, changes for the next lesson were agreed about. After revising the lesson, the next teacher taught the lesson to his/her (new) class, and after a second revision, the third lesson was enacted by the third teacher. One of the researchers (third author) had a supportive role in the learning study; she took part in the discussions, recorded the lessons, conducted the tests and informed the teachers about the results.

The teachers wanted their students to learn that in division, when the denominator is a decimal number between 0 and 1 (e.g. 24/0.8=?) the quotient is larger than the numerator. Previous research shows that students' overgeneralisations of rules valid in the domain of natural numbers lead to ideas like 'multiplication always makes bigger and division always makes smaller' (e.g. 45/0.9 equals a quotient <45) (Verschaffel, Greer, & De Corte, 2007, p. 569). The teachers were aware of that the students often had this idea and planned the lesson to overcome this difficulty. Furthermore, they wanted their students to be able to solve divisions like 40/0.2= without having to multiply by ten (400/2=). They believed that this strategy would help the students to solve the problem, however, without understanding the underlying structure of division with decimals. The ideas of 'partitive division' (e.g. 100/20 seen as 100 partitioned into 20 groups) and 'measurement division' (how many groups of 20 go into 100) was introduced in all classes before the specific task was discussed (cf., Greer, 1992). The task chosen to discuss, is a sequence of items of non-contextualised arithmetic designed by the teachers (see Figure 1). The lesson as a whole consisted of several other tasks.

Analysing what is made possible to learn

In order to analyse the implementation of the task from the point of view of what was made possible to learn, variation theory is used. An important idea within variation theory is that learning implies seeing something in a new way by experiencing aspects that you have not experienced previously. To make it possible to notice these aspects it is necessary to experience *variation* (cf., Dienes, 1960; Gibson & Gibson, 1955; Watson & Mason, 2006). The way we experience something, or how we learn to see an object in a particular way, is a function of those aspects we notice or discern at the same time. If different individuals experience 'the same thing' differently, they discern different aspects of the object in question. In order to understand or see a phenomenon or a situation in a particular way one must discern all the critical aspects of the object in question simultaneously. Since an aspect *is noticeable only if it varies against a back-ground in invariance,* the experience of variation is a necessary condition for learning something in a specific way. Four possible *patterns of variation* has been identified; *contrast, generalisation, fusion, and separation* (Marton, Runesson, & Tsui, 2004). In previous studies it has been demonstrated that variation theory can be used to analyze lessons from the point of view of what patterns of variation that are inherent in the lesson. It has also been found that this is reflected in what students actually learn (Kullberg, 2010; Marton & Pang, 2006; Marton & Tsui, 2004). So, identifying the pattern of variation and invariance in the lesson implies to identify what is made possible to learn.

Here we make use of the notion of contrast, since it was shown this pattern of variation was frequently used by the teachers. Marton et al. (ibid.) points to the importance of experiencing contrast for learning;

> "in order to experience something a person must experience something else to compare it with. In order to understand what "three" is, for instance, a person must experience something that is not three: "two" or "four", for example". This illustrates how a value (three, for instance) is experienced within a certain dimension of variation, which corresponds to an aspect (numeriosity or "manyness") (p. 16).

Teaching division with a denominator between zero and one

As can be seen from Figure 1, there is a pattern of variation and invariance built into the task. We can see that *the operations vary*, since there are multiplication as well as division items. This variation makes it possible to discern differences between multiplication and division. Furthermore, *the numbers within each column vary* in a certain way, starting with larger to smaller number, keeping the number 100 invariant, making it possible to discern what happen to the answers when different numbers are multiplied to, or divided from 100. The numbers varied *are positive integers with one or two digits and decimal numbers between zero and one with one decimal*. The numbers were chosen to make it possible to experience differences between divisions with numbers <1. The same numbers are used in the two columns, except for in the answers to the items. With this design the teachers wanted to make it possible for the students to see what happened with the answer when one factor or the denominator changed.

$100{\cdot}20=2000$	$\frac{100}{20}=5$
$100{\cdot}4=400$	$\frac{100}{4}=25$
$100{\cdot}2=200$	$\frac{100}{2}=50$
$100{\cdot}1=100$	$\frac{100}{1}=100$
$100{\cdot}0.5=50$	$\frac{100}{0.5}=200$
$100{\cdot}0.1=10$	$\frac{100}{0.1}=1000$

Figure 1. Items in the planned task.

In the following we show how the design of the task changed as a result of analysis of the revised lessons, the different ways they were implemented in the lessons, and from that draw conclusions about what was made possible to learn from the enacted task.

Differences in implementation of the task

When we analyze how the same task was enacted in the three lessons we can see that different aspects of division are made salient for the learner. Our suggestion is that seemingly the same task offered different learning possibilities for the students. Note, the task was successively written on the whiteboard by the teacher, one item at a time, starting with the first multiplication item, followed by other multiplication items, and thereafter the division items. The order in which the items were discussed in the lesson and what the teacher made explicit by visually pointing out is shown in Figures 2 to 4.

Lesson 1

In lesson 1 the task was enacted in a way, that from the theoretical framework taken, only made it possible for the learners to experience that multiplication can be used to "check" division. The teacher started the discussion about the task in lesson 1 by asking for the answer to the item 100·20. The teacher continued with another item, multiplying one hundred with a smaller factor, 100·4, followed by two division items, 100/20 and 100/4 (Figure 2, see 1). The teacher pointed out that she used the same numbers in the multiplication and division items, however nothing else was discussed. From the set of items it was visually possible to discern that multiplying with a smaller number give a smaller answer, and that dividing with a smaller number give a larger answer, but this was not discussed. The teacher continued with another set of items in multiplication (2) and thereafter division (3). The items were solved, however, with no specific discussion about what happened with divisions with numbers < 1, although this was the intention of the task. After calculating all items the teacher said "How can we connect these rules of arithmetic". The class came to the conclusion that multiplying the quotient with the denominator, for example 20·5 will give the same answer as the numerator, 100 (see A). This relationship was pointed out by the teacher for all division items, namely that the answer to a division can be checked by a multiplication.

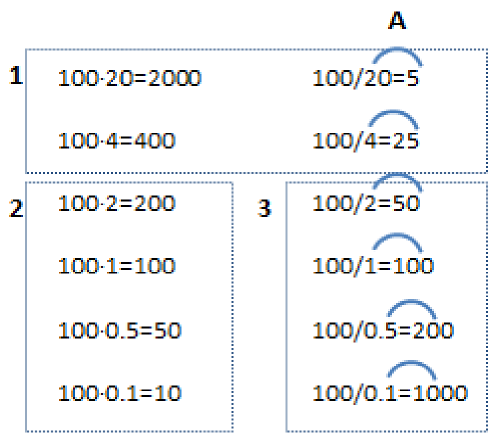

Figure 2. The enacted task in lesson 1. The items are grouped into sections (1 to 3) to show the sequence of how the items were presented on the whiteboard. The lines (A) show what the teacher directed the students' attention to.

As stated earlier the enacted task only made it possible to experience a contrast between multiplication and division by using multiplication to check a division task. At one point the teacher said "the answers get bigger the further down one gets" in the division column. However, we infer that this most likely was insufficient for learning what was intended. Even if the task was designed with the intention to make it possible for the students to discern that the quotient is larger than the numerator when dividing with a number <1, this was not focused upon. Instead, calculations were discussed (A), and no other relationships, patterns or numbers were discussed.

Lesson 2

In contrast to lesson 1, in the second lesson of the cycle, the designed task was handled in a way that made it possible for the learners to experience that when the denominator is a decimal number >0 but <1, the quotient is larger than the numerator. Furthermore, it was made possible to experience that the product become smaller than the larger factor when multiplying by a number <1.

In lesson 2 the teacher started the discussion together with the students about the task by calculating the items in both sections 1 and 2 (See figure 3), starting with the multiplication items (section1) and continuing with the division items in section 2 (Figure 3). After doing the calculations, the teacher asked if the students could see any ‘patterns’ between the items. The students identified two ‘patterns. These were summarized by the teacher:

> – The smaller number multiplied with, the smaller the product and the smaller number divided by, the larger the quotient is.

The teacher explicitly the pointed at the two columns (1 and 2) comparing the products and quotients with one another. Thereafter the teacher pointed at the quotient and the numerator in 100/20=5 and 100/4=25 (See figure 3) and said:

> –**Here** the quotient is a smaller number than the numerator, is it always like that?

This question made it possible for the students to experience a *contrast* between, on the one hand division with numbers < 1, and on the other hand division with 1 ≥. One student said that after zero there was a difference. Other students said that it was not the same for numbers <1, then the quotient became larger than the numerator. The different answers “zero” and “one” made a contrast as to where “it turned”. By comparing 0 and 1, the significant turning point was elucidated. The teacher drew a line under items with a denominator smaller than one (F) so the contrast became visible. Thereafter, the teacher first pointed to the denominators 1 and then to 0.5 and said:

> –When the denominator is smaller than one, the quotient (pointed to the quotient) is larger than the numerator (pointed to the numerator) (F).

Next, the teacher made a contrast between the division item 100/0.5 and the multiplication item 100·0.5, she said “What happens with the multiplication item then?” and the episode ended with the conclusion that the product, 50, is <100 (H). By comparing the items 100/0.5 and 100·0.5 it was made possible to discern that quotient became larger than 100, and the the product smaller.

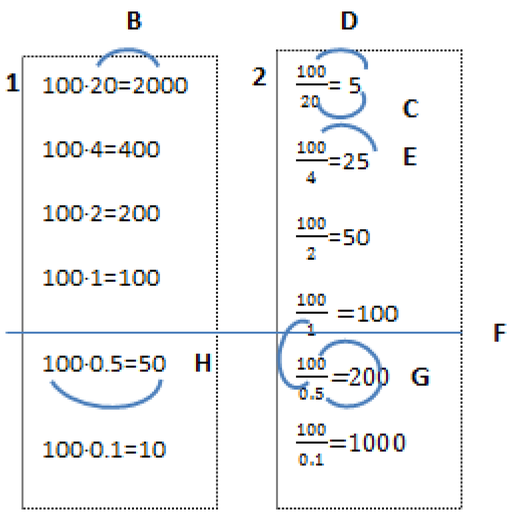

Figure 3. The enacted task in lesson 2. The items are grouped into sections (1 to 2) to show the sequence of how the items were presented on the whiteboard. The lines (B-H) show what the teacher pointed to and hence, directed the students' attention to.

The analysis demonstrates that, in lesson 2, relationships between the quotient, denominator and numerator were made possible to learn by means of a pattern of variation that elucidated how changes in the denominator affected the quotient and the numerator.

Lesson 3

In the last lesson, the task was, in the same way as in lesson 2, handled in a way that made it possible for the learners to experience that when dividing with a number <1, the quotient is larger than the numerator. However, more attention was paid to the items with the denominator <1. This was done by grouping the discussion into three sequences (See Figure 4). One difference was the sequence the items (see Figure 4). The difference in sequence was that in both multiplication and division, items with the number 0.1, was systematically compared with the items with 0.5. This comparison made it possible to generalise both multiplication and division items with a denominator <1.

In lesson 3 initially the discussion was about the multiplication items in section 1 (Figure 4) starting with the item 100·50 followed by 100·5, 100·1 and 100·0.5, followed by the items in section 2; the divisions (2). After that, multiplication and division was discussed separately. The teachers said:

-Look at the multiplication column first, do you see any pattern, anything that is the same or different?

The students said that the smaller number we calculate with the less zeros there are [in the product], and the smaller number there is [the product]. The teacher said:

-What are we used to get as an answer when we calculate a multiplication?

The class said that usually it is a larger number. The teacher continued:

-Is it always like that? When do we not get a larger answer?

The students answered "After one". The teacher pointed at the decimal number 0.5 and the product, and again at the product and 100 to show that the product in this case was smaller than 100 (I). The teacher asked if that was true for all decimals (<1), and introduced the item 100·0.1, as another example with multiplication with a number <1 (3). Thereafter the teacher asked if the students could see patterns in the division column. The teacher said:

-What are we used to get as an answer when we calculate a division?

The students said that they were used to get a smaller answer when they divided. The teacher asked the students if it always is like that. The teacher said:

- Do you see where it turns? Here we get a larger answer (quotient) [the teacher pointed at 100/0.5=200 (J)], when do we not get a larger number (quotient)? Is this true for all decimals? (J).

The teacher introduced another division item, 100/0.1. (J), to compare with 0.5.

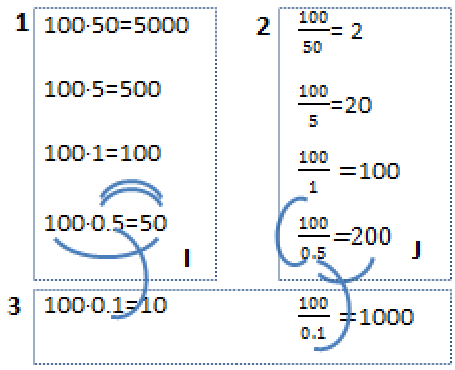

Figure 4. The enacted task in lesson 3. The items are grouped into sections (1 to 3) to show the sequence of how the items were presented on the whiteboard. The lines (I-J) show what the teacher pointed to and hence, directed the students' attention to.

In the end of the lesson, not in the discussion of the specific task analyzed here, division with two and three decimals were also introduced (10/0.02=500, 10/0.002= 5000), as well as multiplication with two decimals (0.02·500=10).

Different possibilities to learn from the same task

The case study reported is an example of a design project where teachers' reflection on their teaching and the learners' responses can lead to a refinement of the task design (cf., Watson & Mason, 2006), but also to a greater accuracy and clarity about what to point out and make discernible to the learners. The task used in the lessons was designed by the teachers. The teachers decided a specific pattern of variation in the task to elicit aspects of division with a denominator <1. In the task the same number (100) was divided or multiplied systematically with a smaller number, so the learner easier would see how the quotient or the product changed. This was the principle behind how the task was designed. We have shown that, when the (same) task was enacted in classrooms, the items were handled differently as regards to sequencing, as well as aspects of the content made explicit by the teacher, and aspects juxtaposed and contrasted. This was a result of the difference of that which was varied and kept invariant between the three lessons. We will suggest that these, rather subtle, differences affected what was made possible for the students to learn. So, what might be worth considering from this example is whether it is possible to say that the learners encountered the same task in the three different lessons or not. In one sense it was the same task, but what the learners encountered in the lesson was different. Our interpretation is that, only in lesson 2 and 3 the teachers' intentions were brought out. As researchers and mathematics teachers we ask ourselves if we sufficiently make use of the possibilities that are inherent in the task (regardless of who the designer is). A lesson is indeed interactive in nature; for instance could the teacher's and the learners' ways of questioning and responding affect how the content is handled. So, a lesson might not turn out to be what we planned. Even if we think we have clearly brought out that which we intended to, this may not be the case. One thing we think could be learned from this study is that, if students' fail to learn, one reason could be that it was not made possible to learn this in the lesson. What we assume the task will mediate, might not be possible for the learners to see. Implications of this study on teachers' pedagogy, we suggest, is that teachers need to take an active role in how the task is enacted in the classroom. Since, we cannot take for granted that the learner discerns from the task what the teachers have in mind. In this study the teacher in lesson 1 wanted the students to see connections between, for instance the quotient and the numerator, however this was never discussed or made possible for students to become aware of. In lesson 2 and 3, on the other hand, the teachers directed their students' attention towards this. We suggest that participation in learning study gave the teachers time for reflection over the content and tasks used, and contributed to the teachers' awareness of the implementation of the task.

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On doing the same problem – first lessons and relentless consistency

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In this paper, we illustrate, through stories from practice, the view that in designing and implementing tasks, teachers have, as a base for decision-making, the classroom cultures they have already established with their students. These cultures are developed over time from the first lessons with a new group. Teachers, and curriculum developers and researchers working with teachers could be seen as leaders of change, the teachers of the learning of their students. In the leadership of change literature, Fullan (2008) has developed what he calls 'six secrets of change' that shift the focus away from detailed planning to learning through reflective action. In Secret 4, 'Learning is the work', he talks about addressing 'core goals and tasks with relentless consistency' (p. 76). So, once a culture in a classroom has been established, through the relentless consistency of practices, and children know what to do to support their learning, this, according to Fullan, frees 'up energy for working on innovative practices' (p. 79). Evidence for establishing classroom practices through relentless consistency that support the continued learning of both students and teachers is given through observations of and interviews with teachers discussing the first lessons of a topic or with a new group of students. For these first lessons, the teachers often use tasks that they are familiar with and have used over many years for their first lesson with a new group to establish ways of working in their classrooms.

Keywords: Teacher change, mathematics education, task design, teacher decision-making, first lessons

Introduction and Background

After studying for a mathematics degree, Laurinda completed a one-year post-graduate teacher education course (PGCE) and began teaching at a secondary school (11-18 year-old students) where, 14 years later, she was head of the mathematics department. An opportunity arose to have a year's sabbatical leave working as a curriculum developer (in fact the job title was Mathematics Editor at the Resources for Learning Development Unit (RLDU)). Her job was working with groups of practising teachers to develop resources either in response to government initiatives (a top-down approach) or in response to suggestions of need from teachers (a bottom-up approach). One such working group was focused on the algebraic activities that would be useful for students to experience before they used them in the new examination courses (GCSEs), where extended investigations were going to be used for the first time. What was important for the editor to be aware of whilst working with the group was the publication that would emerge that would be distributed throughout the local education authority's schools and be on sale nationally and internationally. However, the real task of the editor was supporting the professional development of the teachers. Nevertheless, the resources produced, whilst being quite old now and out of print, are still considered to be some of the best resources available (for example Laurinda has recently been approached by the National STEM Centre in the UK with a view to publishing *Addendum to Cockcroft* on-line and *6th dimension, Developing Teaching Styles in A Level Mathematics* (Brown, 1988) is cited in an e-book (*Risps: Rich Starting Points for A Level Core Mathematics* http://webfronter.com/waltham-forest/CFSMaths/mnu5/Rich_Starting_Points/images/risps_ebook_apr_07.pdf) as 'A wealth of rich and open possibilities for inquisitive mathematicians.').

Following on from the RLDU work, Laurinda became a teacher educator and still works with the one-year PGCE course for prospective teachers. She and Alf Coles have researched and taught together since 1995, Alf having recently completed his PhD and joined Laurinda teaching on the PGCE course at the University of Bristol.

Focus on professional learning or change

The job of editor at the RLDU was illustrative of a subset of Gee's (2004) principles for task design listed in Sullivan (2011, p. 32):

- learners to take roles as 'active agents' with control over goals and strategies
- skills to be developed as strategies for doing something else rather than as goals in themselves.

As editor working with the group of teachers, the role was to keep the focus of the teachers on the publication. It was important that we shared ideas and also that we tried out suggestions for tasks in the different classroom cultures of members of the group (typically not more than 10 teachers and six meetings over eighteen months) since the tasks that made it through to publication were able to be adapted into different classroom cultures. For instance, for the 'function game' activity (shared as part of the algebraic activities group mentioned above, see http://nrich.maths.org/5531 for a short video of Alf introducing this activity to a class as a first lesson for a block of work on functions and graphs), in the final publication *Developing Algebra*, each member of the working group wrote up how they had used the task and these write-ups were printed on a non-white paper and spread throughout the publication. The teachers were 'active agents' in this process because in their discussions they appreciated the power of the activity in being able to be used in many different ways and wanted the activity to be a major part of the final publication. Laurinda coined the phrase 'spurious purposes' (Brown, 1995) to describe the strategy, consonant with Gee's second principle above, of having an agreed endpoint, for example, a publication, when, for the group leader, the work is actually in the process. Spurious purposes work as well in the classroom and are a powerful design principle (e.g., we are going into a primary school to help the students with their mathematics, as a purpose to get slightly older students reflecting on what they have learnt or as motivation for low achieving older students).

Working as editor at the RLDU taught Laurinda about the importance of teacher decision-making in adapting tasks for their own classroom. She had the privilege of visiting the classrooms of the teachers she worked with and got into the habit of interviewing them to gain a sense of how their different classroom cultures had developed. The technique was honed down to interviewing teachers after they had taught their first lessons of the year. How did they choose the tasks to do? How did they set up their classroom cultures? The rest of this paper will initially share some stories of practice, including ones from time Laurinda spent at Monash University, Australia (August, 2012). Rather than searching for new and exciting problems, what seems to be important for establishing cultures is that the teachers worked with their students on familiar tasks that were designed to introduce their students to ways of working that they would continue to use in future lessons. As Alf commented on the NRich website in relation to working with the problem functions and graphs,

> In the end, what I like most about this problem [function game/functions and graphs] is its familiarity; I have worked on it with every class I have ever taught. Rather than this becoming monotonous, it seems to mean that each new class goes further than the one before, as I become more and more attuned to what students say and to what possibilities there are for exploration. And I imagine this is true for everyone's 'favourite problems' - in the end it does not matter so much what they are as what use you have made of them. (http://nrich.maths.org/5531, 2006)

So, Alf's pedagogical decision-making, focused on the meaning making of his students within mathematics, is supported by his history of experiences of using, in this case, the function game. His attention is freed to work contingently with what the students bring.

After some more stories, there will be an interlude discussing Fullan's (2008) 'six secrets of change'. If we want our teachers to develop and the children to learn mathematics we are leaders of change and insights from the leadership of change literature can support our awarenesses of decision making in planning and in action. Fullan's idea of *relentless consistency* will then be used to reflect on the stories and, finally, some insights into classroom practices from first-lesson interviews will be presented.

Laurinda's Stories – narratives of practice

Story 1 (Australia): I asked a group of master's students to come to the session prepared to tell me what they thought 'curriculum' meant and what they thought 'mathematics' was. After a discussion of how she planned what she did in the classroom, one woman, the first to speak, said 'People Maths'. She had found Mike Ollerton's book 'Getting the buggers to add up' and now, whatever the system or structure in her school or in the new Australian National Curriculum, she turned her lessons into what she called 'People Maths'. She did a couple of examples actively with the group and I followed with a couple of active loci (stand same distance from me; with two fixed points, stand twice as far from one point as to the other). I then asked the group what 'People Maths' was and they came up with some thoughts that were effectively design principles for any content.

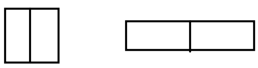
Comment: What 'People Maths' as a design principle does is focus the beginning of the lesson on students being active. Again, the teacher has then to work contingently with what the children do. In working actively on the second locus, stand twice as far from one fixed point as another, questions emerge from the group, is the locus an ellipse? A circle? These questions, force the teacher to make decisions of how to respond, given the particular group being worked with. The more familiar the actual problem and the more familiar, in this case, the teacher is with doing ‘People Maths’, the more extensive the range of options that lead to a decision being taken. The consistency of offer of ‘People Maths’ also serves to set up a culture in the classroom where the students expect to do something active and make meaning from those actions. In such a case, within an established classroom culture, the students might set their own goals, the teacher simply acting as facilitator.

Story 2 (Australia): In a primary classroom that I visited to observe, the teacher picked out one of the problems from Peter Sullivan’s (2004, second edition) *Open-ended Maths Activities* book, to work with her students on. It was from a section entitled ‘area’:

> My granny bought a square rug and each side measured 1 m. When she got it home it would not fit into the hallway so she cut the rug up and joined the pieces together again to make a shape that would fit. What might her rug look like now? (p. 64)

- Who can read that problem out for us? [A volunteer reads.]
- OK, so what are the key words?
~ 1 metre length
~ Square
- What are we trying to do?
~ Cut up the square to fill the hallway. (from my field notes, August, 2012)

At this point all the students started to work, beginning in different ways. Nobody seemed upset by not being given dimensions for the hallway. There were some heated arguments. For example, between two boys about the solutions they had. Both started with Granny’s rug being one square on their paper. One boy filled a hallway that was one half of a square by two squares showing this drawing in his book:



The other boy filled a hallway that was one half of a square by one and a half squares. The first boy had actually cut up Granny’s rug. The other boy argued theoretically, ‘Granny’s rug has 4 1 m sides so I need a shape with 8 half metre length sides. What would you say next?

During sharing time, the children were used to saying what they had done and others listened or asked questions. One boy lost us all:

~ I doubled the first and halved the second .4 times 2.5, .8 times 1.25.

The teacher, myself and the other children look a bit bemused. .8 x 1.25? Surely that can’t be 1? How can we check using what they know?

It was natural for me to offer to work on this with the children. I write .2 x 5 on the board and someone says, ‘that’s 2 tenths’.

- What can I do with 2 tenths?
~ 1 fifth. [Lots of the students look confused.]
- What’s a half of 2? [I invite them to chant, all reply together.] ~ 1
- What’s a third of 3? ~ 1
- What’s a quarter of 4? ~ 1
- What’s a fifth of five? ~ 1
- What’s a millionth of a million? ~1

We’re all smiling now. Those of us who knew the answer began chanting and, gradually, more of the children joined in.

- What about .4 by 2.5?
~ .4 is 2 fifths.
~ 2.5 is 2 and a half.
- How many halves in one? ~ 2 [chanting again. I did not ask this time]
- How many halves in 2? ~ 4
- How many halves in two and a half?

This seems OK. I write one half of two over one on the board without speaking. They will be working on fractions, decimals and percentages soon. I don't want to get into algorithms. They see a pattern and recognize it.

- 2 fifths of 5 halves? ~ 1

There are giggles and the energy release that I recognize of having learnt something. Not everyone is there. However, it's the end of the lesson. The teacher says she will use doubling and halving as the introduction to fractions and decimals. We would do .8 x 1.25 next. The boy who started all this comes up to me before I go for lunch with the teacher and says he's done .8 times 1.25 that way and it works.

Comment: I can remember being amazed by the beginning of this lesson. The question was, indeed open, and the children were well used to this way of working because no child made a comment like "the question doesn't tell us the shape of the corridor". There was an established process by which the children made meaning of the question, reading it out and then identifying keywords, they then all started exploring possibilities. The activity allowed the children to work in groups and we, as teachers simply circulated, learning the children's responses. This led the primary teacher to her decision, when the boy lost us, to invite me to work directly with the group. As a secondary teacher, I was aware that these students would not have algorithms to cope with these difficult multiplications so had to make my offerings contingent upon what they offered me. The offer, by one child, of the fraction for the decimal was key to me realising that the pattern could be used without the need for multiplication of fractions in that moment. The class teacher then realised that this experience could be motivation for a focus on fraction and decimals so that their awarenesses could be applied. It would have been impossible for me to work in this way with a group of students unused to tackling open questions.

Story 3 (UK, after return from travels, August 30th, 2012): I had a tutorial with a Master's student who is submitting her dissertation in early September. Sitting with her, talking, I am aware that, here it is again. In the Epilogue to her work she is discussing her development through doing the project and articulates what are now design principles for her teaching. I ask her if I can quote from this writing and she agrees:

> In the year since completing my research project I have further grown as a teacher, and I am sure I will continue to do so. There are many ways in which this research has affected my teaching directly and consciously, and I believe many ways indirectly and possibly subconsciously. […]
> With all the tasks I made sure I gave clear instructions, but only enough instructions to understand the task. What I found crucial, was not telling the students what to do! My year 7 class this year always wanted me to tell them if they had finished, I never did! Each time a student asked I would ask them back "Well, is there anything else you could do?" whatever answer they gave I said "OK". Even when they said "no" they somehow knew they had something left to finish so would ask their partner, or would just figure it out. Something this simple, transformed the class, they were engaged and took ownership of their

> work. They were completing tasks for themselves, not under my instruction. They had independence! […]
>
> [T]he progress I have made this year, with all my classes, and in particular one boy has amazed me! I do not think I have done anything miraculous, mainly consistency and praise. I have always known these two things, however, the students eventually wore me down. This year, I have remained positive with my students and myself within an inclusive classroom. Simple things like saying hello to the student who I had to send out of the lesson just an hour before, meant I was building that bridge faster and more effectively. […]
>
> I will continue to challenge students to approach problems, systematically and logically, in order to help prepare them for life. (McCarthy, 2012)

Comment: The Masters in Mathematics Education course at the University of Bristol is predicated on students (who are all teachers) developing their own practice through researching an issue in their teaching. Jen, above, describes how the process of exploration has impacted on her own pedagogy. There seem to be some design principles emerging, one of which, common to the primary classroom in Australia and the secondary classroom in the UK, is 'only enough instructions to understand the task' and 'don't tell them what to do'. It would seem also that, for the secondary teacher in Australia , in getting her class to do 'People Maths' there would be minimal instructions to set up the task. Given the engagement of the students, however, the teachers know that they are setting out to challenge their students in various ways and achieve this through sharing time and not simply giving instructions, supporting their students' independence.

At a different level, from my experience as a curriculum developer and teacher, I was able to offer, without prior planning, activities linked to 'People Maths'; chanting; and, support for Master's students and PGCE student teachers. Chanting fractions in this way, for instance, was a familiar activity. Gee's (2004, quoted in Sullivan, 2011) principles are in operation here since, in chanting, students are placed as 'active agents' and it is clear that the purpose of doing the work is to support students' engagement in a wider activity. Although I had never been in exactly these situations before, what was 'the same' was working contingently with the ideas of the group and individuals and collecting strategies linked to a label. The teacher participants are supported in becoming more aware of their own pedagogical principles and in recognising the power of such principles.

Fullan's secrets of change and relentless consistency

The ideas within Fullan's leadership of change literature have been developed through the process of implementing often large-scale change. Having been a consultant on the less-than-successful implementation of the National Numeracy Strategy in the UK, he has since been instrumental in leading teachers in Ontario, Canada to embrace change, raising performance in literacy in the process. The purpose of 'improving literacy in schools in Ontario' was identified, by top managers, as a focus for development in the whole region. The challenge was, how could teachers take forward that agenda as their own. A structure for redoing and rethinking was centred around a meeting, once a year, called the 'Learning Fair'. In the first year of the fair, when teachers from schools were encouraged to share what they had learnt about literacy, many presented something from a book, say. By the third year of the Fair, having picked up ideas from other schools and networked, there was a noticeable growth in the motivation of the teachers whose confidence had built and they were creatively sharing their practices. In schools, headteachers were encouraged to form learning communities for sharing practice where they worked alongside the teachers, not as expert but as learner.

Fullan's six secrets of change, distilled from these experiences, place attention not on the creation of a body of knowledge about teaching and learning, but on processes related to supporting change. There is an acceptance that we learn through doing when this is related to a purpose or purposes, such as implementing change through new materials, new behaviours/practices or/and new beliefs/understandings, in a cyclical manner. Findings are at a meta-level to the actual content of teaching and learning. For instance, one of the secrets is 'Learning is the work'. In Fullan's theory-in-action, evaluations of effectiveness are built in to a process of redoing and rethinking to implement change. Learning is a process and the system can be set up to learn through the interaction of doing and thinking, because 'you are more likely to behave your way into new ways of thinking than you are to think your way into new ways of behaving' (2008, see web-page). Fullan discusses the importance of 'relentless consistency' within the system, not to dampen creativity but to allow the rethinking and redoing cycle that seems so important. In his work with teachers, 'snapshot views' are used to support them becoming aware of their own learning. The system supports the teachers in observing themselves.

Reflections

In neither Story 1 nor Story 2 had Laurinda planned to use the tasks that she offered (in Story 1, the People Maths and in Story 2, the multiplications). She had, however, as Mike Askew said in a recent conversation with Laurinda during a seminar, faith in them that they would be productive. In Story 1, the teacher does not start with the task when planning, she starts with an intention, People Maths. She then adapts any given suggested task, from the syllabus or the national curriculum, to include the practice she has faith in. As a teacher educator working with a group of Master's students, Laurinda did not prepare a set of slides but created a space for discussion. She has used the invitation to share what is meant by curriculum before and is always struck by how experienced teachers have these intentions and their students know what to do in their classroom. In Jen McCarthy's case, she is able to reflect on her journey to be able to create the classroom culture she wants. She is now in her fifth year of teaching and not only does she have an intention that her students will be independent, but she also, over time, illustrates how she has made learning her task and now has the skills to achieve this. She is clear that she will continue the process of learning. Jen is relentlessly consistent now in not telling the students what to do.

In Story 2, Laurinda and the teacher had not planned for a lot of the mathematical ideas that arose. The focus in the book of activities had been area, particularly conservation of area. The level of difficulty was beyond what was expected and the content was not focused only on area, but particularly on fractions and decimals. The children were 'active agents' in this process. The teacher, similarly to Jen McCarthy, did not go beyond supporting her children's understanding of the problem when introducing it to the group. The children were well used to the sequence of prompts: the invitation for someone to read the text; identification of keywords; and finally, identification of the problem. Here is the relentless attention to consistency of practice that seems to hasve opened the way for creativity. It is as if, not having to pay much attention to the routine whilst introducing the activity, a first lesson of a sequence frees the attention of the teacher to learn her students and the attention of the students to work on the activity.

In any curriculum innovation there is the issue of interpretation. In Peter Sullivan's book, the *Granny's rug* problem is defined as an open one. Given the variety of responses, this seems clear. However, another definition of an open task might be that it will extend people's thinking. Was it the task itself that did this in this teacher's classroom, or the procedure established through which the children began to engage with the activity? Another example provided by Barbara Clarke in a seminar discussing these ideas at Monash University was that an activity was open if the children have choices. Having watched a video of the beginning of the lesson where Alf was teaching using the function game, there were at two contrasting views of the task. One that it was closed because the teacher had an answer in mind and secondly that it was open because the children had choice, they decided what the next starting number would be, for instance. At the start of *Granny's rug* problem there was no defined task to do, in that no measurements for the hallway are given. In this teacher's classroom there were no signs of protest from the children that they did not know what to do. So, the task remained open for the children and we, as teachers, had to adapt to what was offered by them. What we seemed to be doing was closely observing what the children were doing, not necessarily intervening. In the sharing time, the move into chanting was comfortable, because it is a process that Laurinda is able to use creatively, adapting to what is offered. In this case, the focus is on multiplication of fractions. Laurinda could also have turned the fraction focus into a People Maths idea. How do we make decisions about what to do as teachers? In the moment we simply act out of all of our past experiences. So, you give me a task, no matter how well designed and, as a teacher, I will work with it. Similarly, when the culture of the classroom is established, even a page of calculations from a textbook will be tackled by children such as those in Story 2, through opening up the questions and discussing alternative strategies.

In conclusion, intentions and first lesson tasks by way of illustration

Laurinda has spent a lot of her life interviewing teachers about or running sessions related to first lessons. First lessons both as the first mathematics lessons taught at the start of the academic year and first lessons as starting a new area or topic of work. Experienced teachers 'do the same problem' (see Brown, Reid and Zack, 1997) with their groups to establish a culture in which they and their students are comfortable to act. In teaching the first lesson with a new group, established teaching practices of teachers, which are often invisible and implicit in classrooms later in the year because they have become part of the culture, are made visible both by the teacher commenting on how they want the students to behave explicitly, or in telling what happened afterwards in interviews.

Intention – story – Teacher C: I like to give things a story because I like to give the children a natural language as a parallel to the mathematical language. I think it allows enabling people to enter the world of maths you are talking about, then, if you have got a story, if it's amusing or catchy in any way they might get interested in the first place, but it does provide short simple language with which they can converse with one another. So, it allows for group work, which is something else I think.

First lesson – activity – Teacher B: The calculation side of it is sort of trivial in a way but the maths of it is looking at the structure, looking for a particular result, trying to understand how the numbers arrange themselves to give you the answer that you want. It's called difference of 3 although I'll probably run it as difference of 4. What you do is put 4 numbers into a two by two grid. And you multiply the two top numbers together and the two bottom numbers together. Add them together and write down the total. Do the same with the columns. Add them to get a total. What's the difference between the two totals? Can they get a difference of 4? 7? Whatever. I'd like to see them asking each other 'What if you try this?' type questions. It's all part of modelling, it's what happens when you change something.

It seems to be the case that when a teacher has an intention such as 'People Maths' or 'Story', then the actual activities used are not so important. There is a relentless consistency in the style of teaching. Not every teacher would be comfortable using these intentions all the time, but what does seem important is that, over time, in the journey from prospective teacher to experienced, not only do teachers find *how* they can work that allows their children to become enthusiastic mathematicians but also that they share with other teachers so that they *extend* their range of possible intentions, for example, not only story but also group work. In the moment, any teacher simply acts contingently in response to their students' actions within the culture that has been developed. Any task, no matter how well designed, cannot be teacher proof, but is open to interpretation through the individual teacher's and their students' cultural lens.

There is relentless consistency that allows the cultures to be established, but also the creative space for both the children and teachers to continue to learn. The teachers have a faith that these ways of working will productive. In talking about the first lessons, the behaviours of the students that are valued by the teacher are explicitly mentioned and would be mentioned in the classroom, as metacomments. The ease of the arithmetic and the teacher's familiarity with the problem (adapting a first offer of 3 to 4 suggesting consistency of problem with creative adaptation) allowed the teacher to focus their attention on structures within doing mathematics that they could stress during this first experience of students working with them problem solving.

So, to sum up, a couple of design principles for teacher decision making in relation to tasks:

- Do something that you are comfortable with that you have done many times before to allow you to adapt and set up the rules of the community with relentless consistency. You can then begin to learn the students. This can be comfortable either at the level of intention or using a familiar problem, making decisions about how to introduce the task, adapting or learning from previous experiences.
- Keep attention on the process and meta-comment about routines.

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Reflections of Nigerian teachers on a learning experience designed to develop pupils' multiplicative reasoning

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This paper presents the reflections of Nigerian teachers on a learning experience which was designed to develop pupils' multiplicative reasoning. The teachers' views indicate that the learning experience was of benefit to the pupils and teachers.

Keywords: International education, mathematics learning, instructional aids, mathematics tasks

The purpose of this paper is to report the reflections of a few Nigerian teachers on a learning experience which was designed to develop pupils' multiplicative reasoning. This paper addresses how features of design influence teachers' decisions.

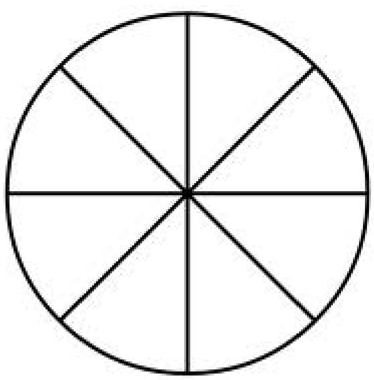
Description of the tasks

An outline of the tasks which were given to the teachers to implement is shown below. The tasks were designed by a mathematics teacher (the author). The purpose of the learning experience was to help develop pupils' multiplicative reasoning through an approach which would be enjoyable and interesting for young children. The tasks were written in a directed manner, in part because of the costs involved in contacting the teachers if further clarification was required.

Exploring proportional reasoning with pineapples
Targeted year groups – Year 2&3
Duration: 30-40 minutes
Group the children in teams of 4 or 6
Materials: A4/A3 paper, scissors, ruler, and a pineapple

Prepare circles on A4 paper, about 12cm radius; divide each into eight equal sectors.

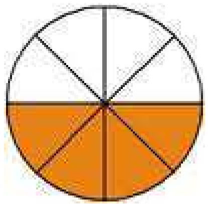
Key words: double, half, quarter, triple, third, eighth, equivalent fraction, etc.



Opener: Display one circle and present question 1

Q.1. Tami, Usman and Abby share a **circular** pineapple slice. It's cut into eight **equal** pieces, through the middle, like this (show the picture) Tami gets one piece, Usman three and Abby four. Danladi says that Abby got half of the pineapple slice. Show your group whether he is right or wrong, and why.

Give one circle to each group of children. Guide children to cut out the sectors, and arrange the four pieces Abby got – it should be evident, even without cutting, that four is half of eight, and that the ratio of Abby's portion to the whole slice is 1:2, otherwise expressed as "Abby gets half of the pineapple slice" Point out to the pupils that four eighths and one half are **equivalent fractions**. If quite a large number of pupils are struggling at this initial point, it might be necessary to do a whole class demonstration at the board, with a group of pupils as helpers to arrange the pieces over a full circle.



Emphasize that the words in bold below are important to preserve the definition of the word fraction – **Equal** portions are necessary. A pineapple slice representation that isn't perfectly **circular** wouldn't fit the definition- hence the choice of paper circles to represent the slices. We can also ignore thickness of the slice, since the paper is uniformly thick. Explain that other shapes can be split into equal portions, e.g., a Kit Kat chocolate bar. Draw a rectangle, split into eight equal sections, shade four of them. Draw one of exactly the same size and shade half of it. Again, point out to the pupils that four eighths and one half are **equivalent fractions**.

Main activity

Guide the groups of children similarly, by arranging pieces, to compare Tami's portion to the whole; and then, each portion with the others. For each portion, draw a rectangular representation of the fraction, so that the children get used to multiple representations of the same fraction. Encourage the children to discuss in groups, to write their completed statements, and draw pictures for each fraction on a shared A4/A3 piece of paper. As the children get used to working with less direction, introduce some open questions.

Examples of statements for pupils to complete - (You could modify these and add more statements & open questions)

T:W
1:8
Tami's portion is *one eighth* of the whole slice.
Tami gets *one eighth* of the pineapple slice
The whole slice is *eight times* as large as T's portion

A:W
4:8
A's portion is . . . .of the whole slice.
A gets ….. of the pineapple slice
The whole slice is 2 times/twice as large as/double . . A.'s portion

These are examples of the solutions one would expect from the pupils' comparison of each child's portion to the others:

A:T 4:1 A's portion is four times as large as T's,
T:A 1:4 T's portion is one quarter of A's
U:T 3:1 U's portion is thrice/ three times as large as T's

T:U 1:3 T's portion is one third of U's
A:U 4:3 (slightly more difficult) U's portion is three quarters of A's

Plenary

Have each group come up to the board for a few minutes to share their pictures and completed statements with the whole class as a small demonstration of what they have learned. Recap the important points, check for understanding of the keywords and clear up any misconceptions.

Snack time

Have the pineapple cut into even slices. Have one slice for each group, and guide each group of children to share their assigned slice equally, with a grown-up at hand to help with cutting the slices into equal sectors.

Extra Notes

Q. 2 below might be used in a follow-on lesson to reinforce what has been learned. Cross-curricular learning opportunities in science can also be explored. It might be beneficial to plan a linked science lesson for use alongside this activity. Health science-nutritional value of fruits & Agricultural science-types of plants are possible linked-lesson ideas.

Q2. Kore, Gbenga and Titi share a **circular** pineapple slice cut into twelve **equal** pieces. They receive two, four, and six pieces respectively. Guide the children to do similar comparisons, using circles split into twelfths.

A definition of tasks which encouraged looking beyond tasks to the type of mathematical thinking and activity stimulated in the pupils (Watson et al, 2008) was adopted. It seems that a teacher's natural intuition, influenced to some degree by the different texts read and previous training received, informed most, if not all of the task design decisions, and that the decisions are supported by the theories in the texts below. The following literature regarding the sorts of mathematical thinking and activity that good mathematics tasks are aimed at generating were consulted: (Piggott, 2004) (Piggott & Back 2004) (Watson & Mason, 2004) (Ingram & Ward-Penny, 2010) (Breen & O'Shea 2010).

Retrospectively, the links between the literature and the design of the tasks can be identified: Ingram and Ward-Penny (2010) explain that it is possible for students to be 'engaged' with a task, busy and seemingly enthused, without actually engaging with the mathematics. A feature of this task that guards against the pitfall described is the requirement for pupils to articulate their mathematical thinking both in writing and in speech, so that the teacher is able to check for engagement with the mathematics.

Piggott (2004) identifies mathematical thinking strategies, some of which apply to the tasks. They include: conjecturing/theorising, testing ideas- guessing and testing (hypothesizing), representing information and reflecting on experience. Piggott and Back (2004) explain a concept termed 'teaching for problem solving' – involving children in solving real problems. In the tasks, the sharing of the pineapple slice enables pupils to apply mathematics to a real-life situation which is relevant to children their age. Watson and Mason (2004) explain the importance of explicitly requiring learners to reflect on their learning experience, noting that developing reflective practices among learners is key. In the tasks, both verbal and written reflection are encouraged.

Another definition of tasks which encouraged looking beyond tasks to learning experiences -with tasks considered as just one element of design- was also adopted (Swan, 2008a). One aim was to create a learning experience that modelled a collaborative orientation of learning (Swan, 2008b). Hence for the tasks, organising the children in groups of 4 or 6 was instructed. Williams (2008) suggests that an optimal group size for interventions is a carefully chosen pair or three, and suggests that larger groups might allow group members to get by without actively engaging with the learning. Thus retrospectively, groups of 2-3 may have been better, depending on practicalities such as total class size.

The learning experience was structured to encourage group discussion and foster active student engagement, based on recommendations from studies on mathematics teaching in Nigeria (Salman, 2009) and beyond (Krainer, 1993; Henningsen & Stein,1997). The improvement of mathematics teaching and learning in the country has been noted as essential by researchers (Eniayeju, 2008; Popoola, Ogini & Ojo 2011). The use of paper circles to model fractions are in line with recommendations to encourage the use of locally available concrete materials and other learning aids in mathematics classrooms (Afolabi and Adeleke, 2010). The use of circle sectors to represent fractions of a whole is not new: Thompson and Saldanha (2003), for instance, in their discussion of fractions and multiplicative reasoning, give a pizza fraction example, but the context and the approach used in the tasks were original.

The learning experience aided the pupils in developing mental images for concepts and encouraged the use of multiple representations (Swan, 2008b). It enabled testing by pupils of conjectures (Pratt and Noss 2010). It also aligned with the definitions of purpose and utility proposed by Ainley and Pratt (2005), in terms of a meaningful outcome for the pupils, and helping them gain an appreciation of situations in which it might be appropriate to apply the skills they were developing.

Excerpts of the reflections of teachers from four schools are outlined below. The reflections were sent in writing-the teachers were not interviewed. In retrospect, it would have been useful to speak to the teachers and confirm whether they carried out all the suggestions outlined in the task.

> S(one pupil) said that R(another pupil) has one out of the eight divisions of the apple she brought. Relating fractions to actual experience or to the use of what goes on around us has really helped in making the topic easier to explain to the pupils.
>
> At first it looked strange to them but after proper explanation, they really understood and enjoyed it. . . The method helps the children to understand fraction(s) better as they were responding well to the questions asked. I used three pupils to represent Tami, Usman and Abby. One represented Tami and I gave him one sector of the pineapple. One represented Usman and I gave him three sectors of the pineapple. One represented Abby and I gave him four sectors of the pineapple which meant half of the whole pineapple.
>
> Learning fraction(s) at this level is difficult for children but with cutting and sharing, the pupils make fun with it and enjoyed the lesson. Using resources at hand and engaging the pupils to contribute to the lesson facilitated the class exercise and it enable(d) the pupils to understand the topic with(out) stress; the method is good but expensive to implement.
>
> Demonstration and practical aspect can be introduced in teaching maths for effective learning because not all of the pupils have interest in mathematics. The lesson was quite interesting.
>
> The lesson was interesting. They (the children) enjoyed the lesson because the pineapple was a motivational material for the pupil's learning. The cutting of the circle into eight part(s) and giving the pupils (the) puzzle to tackle on the board was quite interesting and new. It was quite an interesting class. The class was more interesting seeing the children chanting and cheering themselves after answering questions. It was even more fun after the whole learning period when they had to share the pineapple amongst themselves.

From the teachers' feedback, the following link between task design features and pedagogical choices of the teachers can be inferred (Fig 3):

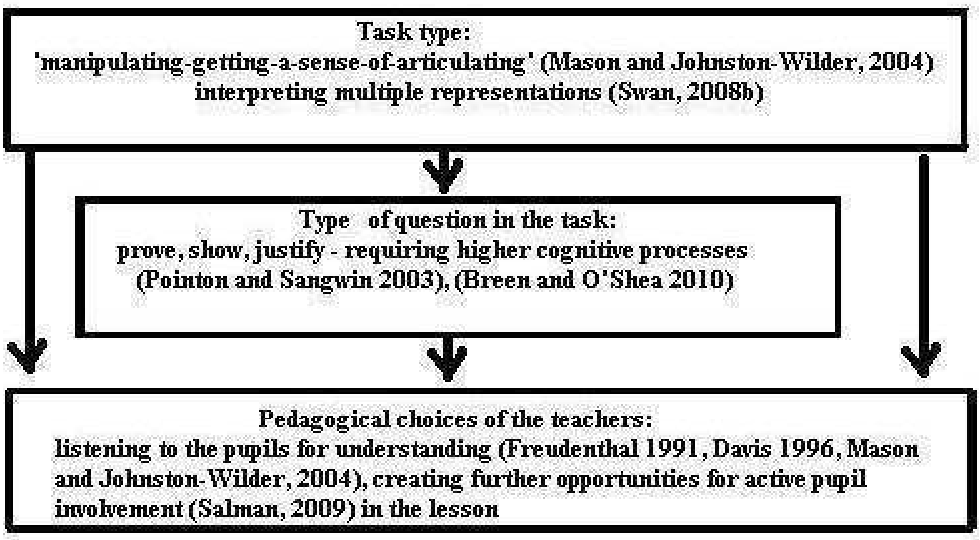


Figure 3

The task type closely relates to one referred to as *'manipulating-getting-a-sense-of-articulating'* (Mason and Johnston-Wilder 2004). A pedagogical decision of a teacher was to listen (Freudenthal 1991, Davis 1996) to the children *articulating* their understanding of the mathematics, while another decision of a teacher was to increase opportunities for the pupils to actively use the concrete objects, *manipulating* these *to get a sense* of the mathematics that they were learning (Salman 2009).

From the reflections, it appears that the learning experience was useful to the pupils and teachers.

It is hoped that the process outlined has the potential to contribute to the improvement of mathematics teaching and learning in Nigerian classrooms.

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Designed to facilitate learning: Simple problems that run deep

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The article focuses on the design and implementation of simple tasks in a teacher education course for mathematics teachers. It discusses the rationale for this design including prospective teachers' difficulty in task analysis and in-depth understanding of task specifications. Then the effect of simple tasks is tested and exhibited through two detailed case studies in which the implementation of two tasks is observed. The tasks are different in nature, one is an algebraic traditional task and the other is a geometric modeling task. Still, both share design similarity, afford many learning opportunities, and can facilitate task design by these future teachers.

Keywords: Task design, teacher education, modeling

Introduction

The design of good effective tasks is crucial in teacher education for two related reasons. The teacher educator wants to use effective tasks to convey the many goals of teacher education, and the prospective teachers need to experience good task examples that would facilitate their own future task design.

In this article we suggest that effective tasks do not have to be complex. We support this claim with data from the second author's doctoral research supervised by the first author. The study informs us of the power of simple tasks through observations and analysis of class implementation of tasks that were designed and presented by the first author in a mathematics teacher education course. The simple design of these tasks is intended to serve as a relatively easy model supporting teacher pedagogical decision making in class,

Although the design of tasks is going to be a major part in their practice, prospective teachers cannot be expected to be good tasks designers without getting some support through specially designed instruction. This has become evident to us when we asked our students in a teacher education course to compare two tasks. One of the tasks involved cutting greeting cards (this task is described further on in the first case study) and the second involved pouring beer from a container into cans. The main and relevant difference between the problems involves the rigidity of the cardboard versus the "flexibility" of liquid. This feature results in different types of "remainders", as the rigid material does not allow remaining scraps to be put together (unlike a situation such as cutting cookies with "flexible" dough). This difference leads to fitting very different mathematical models. Yet, instead of identifying this feature, the students suggested some minor characteristics such as the fact that one problem is two dimensional and the other is in three dimensions.

This task analysis difficulty strengthened our motivation to design simple tasks and to make our design principles explicit to our students. It can be said that the nature of our tasks became a part of the didactical contract in the course.

Further support for the need to stay simple involves the type of change we try to achieve through the tasks. Often, as teacher educators, we aim at changing prospective teachers' conceptions. With some of our goals such changes can be more easily acquired when task context involves simple situations. Peled and Balacheff (2011) discuss characteristics of task examples that aim to promote a shift from a problem solving perspective to a modeling perspective. They use simple word problems in an effort to promote teacher understanding of the meaning of fitting mathematical models in different types of situations. The authors claim that the use of these simple problems instead of complex modeling problems helps avoid teacher resistance to problems that are very different from what they use traditionally, and that it helps focus on the main current goal.

Theoretical background

As mentioned in the introduction, the design of tasks in teacher education has to take into account prospective teacher resistance to change. This is not surprising when we become aware of the strength of teacher beliefs investigated in many studies in teaching mathematic and other subjects as well (Stuart & Thurlow, 2000; Kagan, 1992; Brown, Cooney, & Jones, 1990).

According to Piaget (1985), a change in knowledge and new learning can occur through the creation of discomfort (disequilibrium, in Piaget's terms) with existing knowledge or beliefs, leading to an effort to make some adjustments that would restore the state of equilibrium. Following this recognition, Zaslavsky (2005) designed tasks that were aimed to create uncertainty. Similarly, in our own work (Peled & suzan, 2011), we describe the effect of a task that was designed to create a cognitive conflict, a discomfort that was supposed to lead to abandon intuitive knowledge for the sake of using formal knowledge. However, we found that creating a conflict is one thing and resolving it is something else. That is, as suggested and identified by Limon (2001), the design of good tasks has to obey certain conditions in order to create a meaningful conflict leading to conflict resolution.

When teacher educators use tasks that create cognitive conflict to facilitate desired changes in prospective teachers' conceptions, they can, at the same time, acquire two other important goals. The students, the prospective teachers, can learn that tasks that lead to a cognitive conflict can be effective means for facilitating their own future students' learning. In addition, as described by Peled and Suzan (ibid), becoming aware of the power of intuitive knowledge over their own performance, teachers might become more sensitive to their students' difficulties. As analyzed and discussed by Jaworski (1992), developing this sensitivity is a central teacher education goal.

As can be seen, cognitive conflict tasks can aim at different types of knowledge. They might have mathematical, pedagogical or psychological goals. Peled and Balacheff (2011) discuss the use of such tasks to acquire epistemological goals in the case of learning and understanding issues of modeling.

Modeling, as defined by Peled (2010) in more relaxed terms, is the process of fitting a mathematical model to a situation described in a given problem. This process includes the analysis and organization of the given situation before any mathematization is done. Most of the studies on modeling (for example: English & Watters, 2005; Lesh & Doerr, 2003) involve quite complex problems some of which are termed MEA (model eliciting activities). These problems involve an extensive situation analysis and their main purpose is developing modeling skills, i.e. the ability to analyze, represent, and make autonomous decisions and choices of representations and mathematical tools.

Following a theoretical analysis of modeling (Peled and Bassan, 2005) and with the goal of developing teacher insights about the process of fitting a mathematical model rather than developing skills, Peled and Balacheff (2011) discuss (as mentioned in the introduction) the rationale for using problems that are not as complex as MEA.

In this study we observe the effect of simple tasks that have been designed to make a change in different types of teacher knowledge.

Procedure

The study was conducted in a university teacher education course taken as a part of a mathematics teacher certificate program. The 24 students participating in the course were in the third and last year of their undergraduate studies towards an academic degree in mathematics.

The first author, a researcher and teacher educator, was the course instructor. She designed the course with the purpose of promoting student meta-cognitive perspective of teaching and learning and with the pedagogical approach of "not telling" in the spirit of Lampert (1990). Using a design experiment methodology, as described and used by Cobb et al. (2003), she has kept reflecting on the effect of tasks through several years of conducting the course, redesigning new tasks or modifying existing tasks accordingly.

The second author participated in all class sessions, recorded them and took notes. Student homework and their final course assignment were also collected and analyzed. She also met with the instructor and took notes of the instructor's explanations for task goals and task design.

The implementation of each of the course tasks included a sequence of activities. Each task started with a problem that the students were asked to solve in class, and sometimes solve again at home. This activity was followed by class discussion and by individual reflection that was given as a homework activity.

Results

The findings presented in this article are a part of the findings reported in the doctoral dissertation. We focus on two of the course tasks, where each task includes a sequence of activities, as mentioned above. The research questions refer to the nature of the tasks and to their effect. We analysed the learning opportunities created by each of the tasks, and within each task we identified the added value of the reflection activities that followed the problem. We also analyzed the nature of the tasks and identified some general design similarities.

The two problems that start each activity sequence are a geometric problem, involving a print-shop where the workers are cutting greeting cards from a cardboard, and an algebraic problem where show tickets of different prices are purchased.

The first problem, the greeting cards problem presented in Table 1, originated in a graduate course where students interviewed print-shop workers on the situation described in the problem. While the students thought that they should divide the area of the cardboard by the area of the greeting card, the print-shop workers used a different mathematical model depicted in Figure 1.

How many small 6 cm X 8 cm greeting cards can be cut from a 65 cm X 85 cm piece of cardboard?

Table 1: The Greeting Cards Problem

Using a symbolic formulation, the mathematical model used by the workers involved taking the maximum of the value of [L/l]x[W/w] and the value of , [L/w]x[W/l], while the students used an area division model, [S/s], which is much less appropriate in this situation.

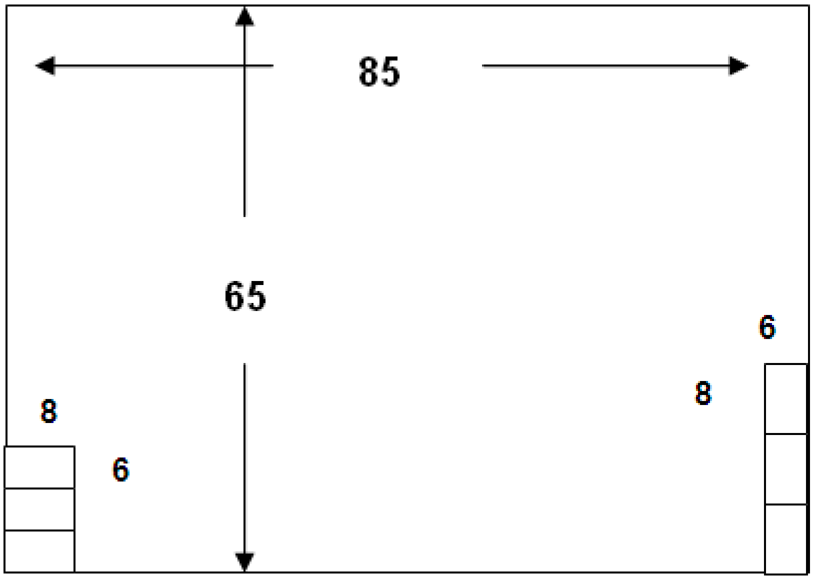

Figure 1. Print-shop worker options for cutting greeting cards.

The goal of this relatively simple modeling situation was to develop understanding of the idea of fitting a mathematical model in a given situation. The implementation of the problem resulted in the construction of a wide range of models. About 40% of the students suggested a model that was close or identical to the area division model. A solution of this type is, most likely, an indication of an automatic identification of the problem as an area problem without making an effort to represent and analyze the situation by, for example, making some drawing of the cardboard and cards.

In the following activity students were asked to generalize their models in order to increase their awareness of the different parameters that were involved. As a result about a quarter of the students shifted to a better model. The discussion that followed these two activities, problem solution and problem generalization, created many learning opportunities that cannot be detailed in this short paper. They include awareness of problem features (simple and yet eliciting different models that can be compared), mathematical knowledge (use of mathematical representations, concept of area), pedagogical and psychological knowledge (misconception of over-linearity), and epistemological (developing meta-understanding of fitting a mathematical model).

The second problem, the show tickets problem presented in Table 2, originated as a result of an error made by very few students when asked to solve a similar problem without a request for multiple solutions. Most of the students chose x to stand for the value that was the described reference, and solved the problem correctly. The few who failed chose x to stand for the less convenient value. As a result of their choice, they had to convert the given relationship between the values. In making this conversion they wrongly asserted that: If A is by K percent smaller than B then B is by K percent bigger than A. Obviously, this is a wrong generalization of "regular" number relations.

Mary took her daughter and her neighbour's daughter to a show. Her neighbour wanted to pay for the show's ticket but Mary did not remember the exact price. She knew the total sum was 143 IS and that a child's ticket was 20% cheaper than an adult's ticket. Help Mary figure out the child's ticket price. You are requested to do it in two ways, each time using a different value (adult's ticket price, child's ticket price) to be represented by the variable x.

Table 2: The Show Tickets Problem.

The original purpose of the modified task was to "not tell" students of their mistake but give them a chance to become aware of it on their own. However, further experience with the multiple solutions version made us aware of the many goals this problem together with its following activities can achieve.

While a conventional problem version resulted in avoidance of the need to make a percentage relation conversion, the multiple solutions version forced all the students to face the conversion difficulty. As a result, 20 out of 24 (83%) solved the non-routine and less convenient alternative incorrectly, asserting that if the child's ticket is 20% cheaper than the adult's ticket, then the adult's ticket is 20% more than the child's ticket. Since this alternative led to an incorrect solution while the convenient route resulted in a correct solution, these students realized that they have two different solutions and therefore something must be wrong.

Thus, most of the class experienced a cognitive conflict situation. However, as it turned out, conflict resolution did not occur spontaneously. A class discussion that immediately followed resulted in partial improvement. As depicted in Figure 2, only about a third (6 out of 20) of those who faced a conflict, managed to resolve it.

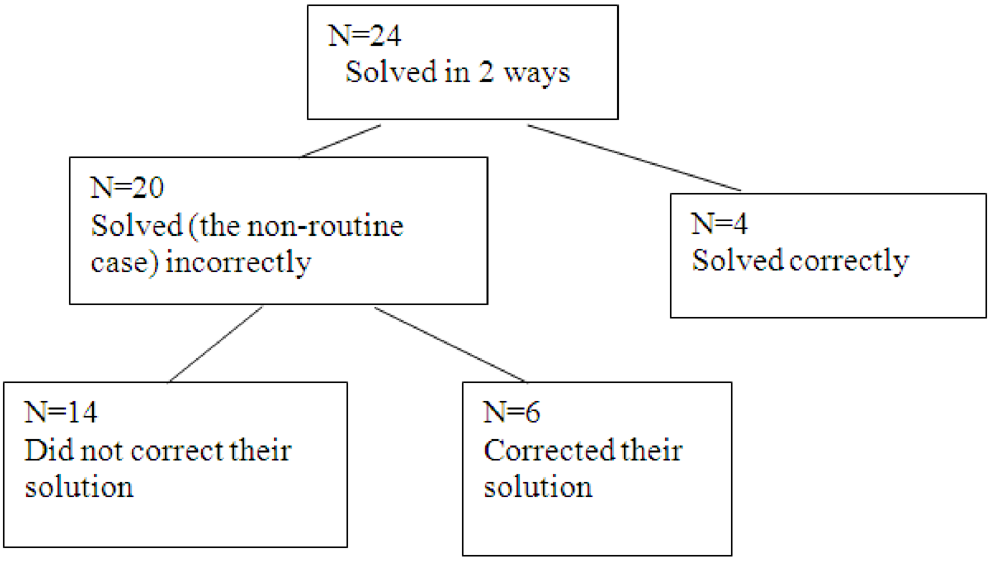

Figure 2: Conflict and (very partial) conflict resolution.

The analysis of learning opportunities created during the activity sequence and especially during the discussion reveals the many goals that can be promoted through this one simple task. Similarly to the first task, the learning opportunities consisted of mathematical, psychological, and pedagogical goals. The teacher educator seized the opportunity to talk about cognitive conflict in general, making the students aware of what they experienced. Since they also experienced difficulty in conflict resolution, this pedagogical and problematic issue could be demonstrated through their own work and thus better understood.

The discussion brought up student realization of the strong power of intuitive knowledge. Some students explicitly expressed their increased sensitivity and understanding towards their future students' mistakes in view of the fact that in spite of their mathematical knowledge they themselves failed to resist incorrect intuitions. As stated by one of the students:

> I personally felt uneasy because I am supposed to know this subject very well and solve this problem easily. Suddenly I got stuck and the solution wasn't so easy and fluent as I expected. Now I realize that if I, who master this subject, got confused, it can certainly happen to my own students too. We, as teachers, shouldn't be upset with them when it happens.

In addition to discussing the task as an example of a cognitive conflict task, students were also made aware of its simple design. The modification of a conventional algebraic problem into an effective interesting task required only a simple addition of a request to solve the problem in two ways.

Conclusions

Our study suggests that prospective teachers can benefit from experiencing simple tasks. This assertion is tested and confirmed in the article through the observations made in a teacher education course conducted in the spirit of a "not telling" approach. Through specially designed tasks, prospective teachers were expected to undergo knowledge change in their mathematical, pedagogical, psychological and epistemological knowledge.

The two case studies presented in the study involve two different tasks. The analysis of learning opportunities in both tasks has shown that not only did each task achieve its main goal, but also that in spite of the relatively simple formulation of the problems, many additional goals could be achieved. This fact was explicitly discussed with the prospective teachers. The intention was that they would experience examples of pedagogical decisions in task design and would become more confident that the task of task construction is within their limits. Of further help was the extensive discussion of the use of cognitive conflict to promote learning. The self experience of change and learning as a result of working on a simple task that triggers the emergence and resolution of a cognitive conflict is expected to make these task design principles a part of their teaching repertoire.

Thus, we believe that the exposure to simple problems and the explicit discussion of problem features created an opportunity for developing task design skills. Although this skill was not tested, students' awareness of task design characteristics was exhibited in their discussions. As mentioned in the introduction, the design of course tasks using explicit design principles became a part of the course didactical contract. As a result, when a new task was presented, students could tell that even though it looked simple, they could expect it to run deep leading to interesting and significant learning.

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Engaging teachers in the web of considerations underlying the design of tasks that foster the need for new mathematical concepts or tools: The case of calculus

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This paper reports on a design experiment, within which the design principles emerged in an iterative way, through examples, expert comments, reflections, and revisions. The focus for this part is on the design considerations of tasks that problematize the use of students' existing mathematical tools and set the grounds for learning about new (for the students) tools. Having gained insights ourselves, we built on our experiences to involve teachers in similar considerations. It turned out to be a way of thinking that teachers had not experienced before, nor were they aware of the possible web of considerations that underlie design issues related to specific choices of examples for tasks that aim at preparing the grounds for a new mathematical concept. We argue that this kind of thinking is not sufficiently attended to, yet should be part of teachers' knowledge for teaching mathematics.

Keywords: task design, teacher education, calculus.

Introduction

Typically, the process of task design is often a 3-stage hierarchical process that includes: 1. Stating goal(s) and connecting the task to the goal(s); 2. Designing a generic task that addresses these goals; and then (when applicable) 3. Carefully choosing the specific examples to "plug in" the generic task. Teachers usually are not engaged in this full process, and mostly make choices about existing tasks in textbooks, rarely with some (minor) modifications.

In this process, the specific examples can change dramatically the quality of the task, its connection to the goals, its target audience, how it may unfold in the class, etc. Analysing the subtle anticipated differences between various versions of the same generic task that only differ along the specifics chosen, and making judicious choices of the specifics of a task, is an important element of desired teacher knowledge and practice that our study aims at developing. In this paper we focus on a particular goal for teaching mathematics, namely, preparing the grounds for introducing a new mathematical concept or tool. More specifically, we illustrate this approach and way of working with teachers through the concept of the derivative as a tool for detecting the behaviour of a graph of a function.

Throughout the school years, teachers are often faced with the task of presenting mathematical concepts that are new to their students. To do this meaningfully, it is important to create the need for the new concepts that are introduced. We base our stand on the necessity-based approach to learning (Harel, 2007). That is, it is important to help teachers learn to problematize the situation for their students. One way to do this is with the use of carefully designed tasks.

This constitutes a challenge for teachers and teacher-educators, because on the one hand, we often want to point to the limitation of existing tools for a particular purpose, while we need to maintain the usefulness and merits of the existing tools for other purposes (otherwise students will believe that much of what they learn will need to be abandoned in the future). We do not want students to feel that what they have learned so far becomes useless as they progress in the study of mathematics. Thus, the progression from existing tools/concepts to new tools/concepts should lead to an extended mathematical ‘toolbox’.

From a design perspective, with respect to tools, we would like teachers to be able to design tasks that foster discussion of the merits and limitation of existing as well as new tools. For example, using a function table to sketch graphs (of functions) is useful for certain functions (e.g., linear functions) and certain properties. However, when we move to the study of more advanced functions (e.g., higher degree polynomial functions), we need to introduce the concept of derivative. But even then, tables still have a significant role. Clearly, derivatives can help detect significant turning points in a graph, which could get lost using an arbitrarily constructed table. However, derivatives may also serve as a tool to construct a more useful function table, thus, they support the value of function tables, so that students do not have to abandon this tool (function table) when more advanced tools (derivatives) are introduced.

In our study, the design principles emerged in an iterative way, through examples, expert comments, reflection, and revisions. Interestingly, we experienced a certain degree of tension in attempting to satisfy some guiding principles. Having gained insights ourselves, we built on our experiences to involve teachers in similar considerations. It turned out to be a way of thinking that they had not experienced before, nor were they aware of the possible web of considerations that underlie design issues related to specific choices of examples for tasks that aim at preparing the grounds for a new mathematical concept/tool. We argue that this kind of thinking is not sufficiently attended to, yet should be part of teachers’ knowledge for teaching mathematics (Ball & Hill, 2009; Hill, Ball, & Schilling, 2008; Silverman & Thompson, 2008).

We turn to a detailed example from calculus, that illustrates the web of considerations that teachers should be able to employ – some more general, and some very specific and detailed. We then move to a report on teachers’ actual thinking and experiences related to this example.

An analysis of the design process of a task for students

Initial Assumptions and Decisions

As the first step of the design process, aimed at introducing the concept of derivative, we adapted a generic task that relies on students’ prior experiences with function value tables. Variations of this type of task can be found in some textbooks. The structure of the task that we developed is presented in Figure 1. Note that we build on students' prior experience in using symmetrical tables for sketching graphs. To turn this into a task for students, a teacher would need to choose the specific function *f*.

The next level of design requires a choice of the type of function *f* that would be used for this task. It makes sense to use a cubic function, assuming students do not have any prior image of how the graph of a cubic function should look. Choosing a quadratic function for this purpose would not be equally appropriate for the stated goal, as the behaviour of its graph can be deduced by other considerations (e.g., symmetry). Actually, it is possible to sketch a rather accurate graph of any quadratic function with the use of a table and the special symmetrical properties of the function (that is one of the reasons that many secondary mathematics curricula include the quadratic function as part of algebra while the cubic function is part of calculus). We assume that after dealing with quadratic functions, students are familiar with the concept of an extreme point of a function, at least at an intuitive level. At this stage, by an accurate sketch of a graph of a function we mean a graph that captures all its extreme points, as well as its increasing and decreasing domains (clearly, moving further on, we can increase the accuracy, by detecting inflection points and concave and convex domains as well. At that stage, a similar approach can be applied to the introduction of the second derivative).

Given the following function: $f(x)=$ $\boxed{}$

1. Complete the missing values in the following Table and mark the corresponding points in a Cartesian coordinate system:

x	-1	0	1
$f(x)$			

2. Do you think the given function increases in the domain $-1<x<1$?
3. Do you think the graph of the function in the domain $-1<x<1$ is a straight line?
4. Complete the values in the following Table, mark the corresponding points, and answer questions 2 & 3 for the domain $-2<x<2$.

x	-2	-1	0	1	2
$f(x)$					

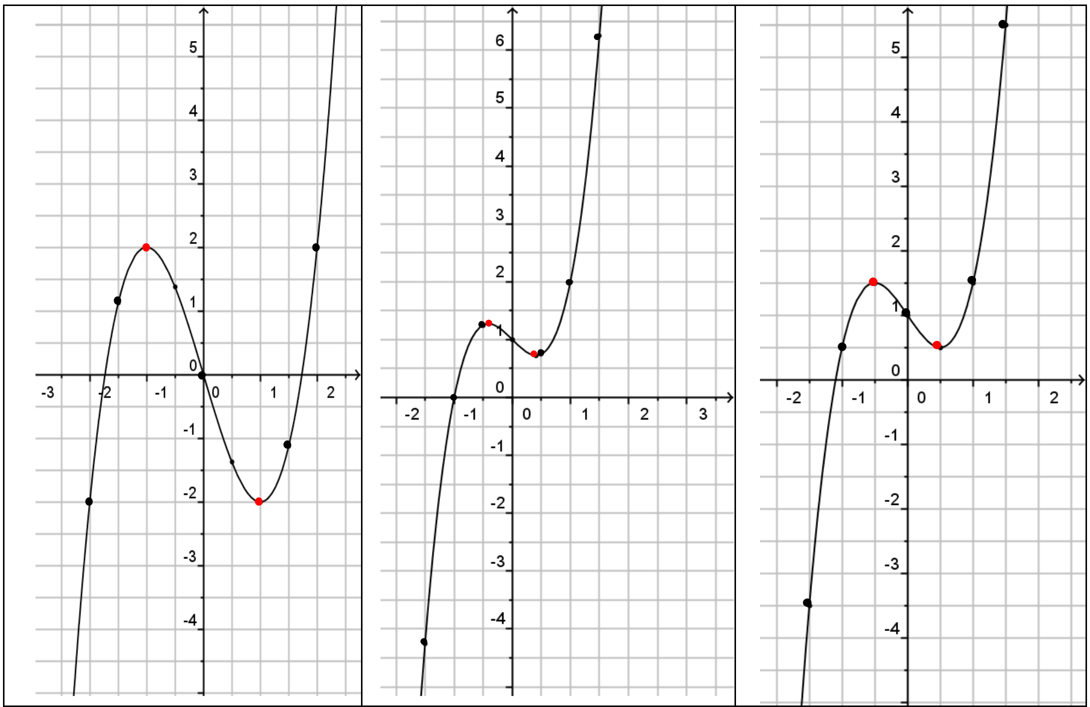
5. Add to your Table the values $f(-1.5)$, $f(-0.5)$, $f(0.5)$, $f(1.5)$, and mark the corresponding points on the graph that you began sketching.
6. Now answer again questions 2 & 3 for the domain $-2<x<2$.
7. Can you know for sure from the values in your table where the function increases and where it decreases? Does the function have any extreme points?

Figure 1: A generic task for students aimed at evoking some limitations of a function value table

The teachers with whom we worked, who had not experienced careful design considerations before, maintained that any cubic function could serve equally well for this task. However, this level of design calls for more refined considerations. In other words, once we decided to use a cubic function of the form: $f(x)=ax^3+bx^2+cx+d$, $a\neq 0$, the question remained: what specific values would be good to choose for the parameters a,b,c,d given the goals of the task? This level of detail is not commonly addressed and was completely unfamiliar to all the teachers in our study. We turn to the description of some of these refined considerations.

What specific function should we choose?

For brevity, we present only part of the web of considerations that could and should be employed. After several deliberations and iterations, we presented teachers with the following three possible functions that could serve for the task (Figure 2).



(1) $f(x) = x^3 - 3x$	(2) $f(x) = 2x^3 - x + 1$	(3) $f(x) = 2x^3 - 1.5x + 1$

Figure 2: Three cubic functions to consider for the generic task for students

All three functions belong to the following sub-set of cubic functions: $f(x)=ax^3+cx+d$, $a\neq 0$. This choice was guided by our knowledge of the properties of such functions. In particular, we considered the following two properties significant from a design standpoint:

1. **Although the function is not linear, its specific parameters should give students, at first, the (wrong) impression that it is linear, when they use familiar tools**.
   All three functions have an inflection point on the *y*-axis.
   For all of them, any choice of symmetrical values for the table, of the form $x=-k$, 0, k ($k>0$), results in 3 co-linear points on the graph that could create an illusion of linearity for students who complete the first function-value table in Figure 1.
2. **The graph of the function should be 'manageable' for students to sketch, in the sense that the parts containing critical points can fit easily in a regular notebook page.**
   For all three functions, the critical values to which students need to pay attention are small enough so that the graphs can be sketched in students' notebooks in a homogenous coordinate system; no need to worry about scaling. The possibility of sketching the graphs in a homogenous coordinate system eliminates distractions by technical aspects of graphing.

Moreover, being able to sketch the graphs in a homogenous coordinate system is important in the context of developing the initial notion of derivative, so that examining the slopes at different points on the graph can be done visually without the need for any calculations.

Figure 3 presents an analysis of the three functions, from mathematical and pedagogical perspectives. It conveys the complexity of considerations surrounding the choice of a specific function for the task (in Figure 1). It raises the difficulty to recommend one answer to the question: Which function (of the three) should we use for the above task? The answer depends amongst others on the level of the students, and on the teacher's goals and intentions beyond this task.

(1) $f(x)=x^3-3x$	(2) $f(x)=2x^3-x+1$	(3) $f(x)=2x^3-1.5x+1$
The function has 3 intersection points with the *x*-axis, all of which can easily be found algebraically. • For strong students, this may not be a good choice, since they could immediately see a way to detect the entire behavior of the graph by solving the equation: x^3-3x=0 and making intuitive inferences. • For weaker students, who are not likely to be able to solve this equation, it could be better to begin by asking them to find where the function is positive and where it is negative. This builds on what they are able to do, and the need for new tools arises when asked about extreme points of the function. • Choosing this function for the task can be an advantage if the teacher intends to use it later to illustrate a full analysis of the graph of a function that includes finding the (exact) intersection points with the axes. • In terms of the goal of the task, the fact that the extreme points are detected by the function value table may leave students with the impression that there is no real need for new tools.	**At this stage, students have no way to find the intersection point of the function with the *x*-axis.** • An advantage for choosing this function is that students are not likely to anticipate that there are extreme points, thus, this finding will surprise them and reinforce the need for new tools. • Although this function can serve to raise the need for new tools, it would not be a good one for illustrating later a full investigation of the graph of the function that includes finding the extreme points of the function. • An advantage of this function is that students are not likely to find the extreme points by checking various (arbitrarily chosen) values in the table, as the *x*-coordinates of the extreme points are irrational. **Using the values -1,0,1 (or actually any table with just integers) gives the impression that the function is an increasing function.** • This may help achieve the goal of the task. • This function lends itself to a gradual realization of its properties: first students see that it is not linear, then – that it is not an increasing function, and later – that points that seem as extreme points are not.	**At this stage, students have no way to find the intersection point of the function with the x-axis.** • An advantage for choosing this function is that students are not likely to anticipate that there are extreme points, thus, this finding will surprise them and reinforce the need for new tools. **The *x*-coordinates of the extreme points are rational numbers that can easily be detected when extending the values in the table (to halves).** • An advantage of this function is that the values in the extended table are all integers (or other easy-to-handle numbers); this makes the calculations easier and reduces distractions that complicated calculations may cause. • In terms of the goal of the task, the fact that the extreme points are detected by the function value table may leave students with the impression that there is no real need for new tools. **Using the values -1,0,1 (or actually any table with just integers) gives the impression that the function is a decreasing function.** • This may help achieve the goal of the task.

Figure 3: An analysis of the three cubic functions under consideration

In terms of teacher education and teacher knowledge, it is not about finding the right function, but about helping teachers make judicious decisions that are based on careful analysis of affordances and limitations that are entailed in the choice of a specific function. Yet, weighing all the considerations for each of the three functions (Figure 3), it seems that all round function (2) may be more advantageous than the other two.

Teachers thinking about the design of such tasks

The setting and research instrument

A group of experienced secondary mathematics teachers were given a written questionnaire, in two parts. Some were also interviewed about their responses. The first part of the questionnaire presented the task (Figure 1) with function (2), and prompted teachers to examine the merits of this task, analyse its advantages and disadvantages, and reflect on their own practice with respect to the use of similar tasks that have the same goals. This part of the questionnaire also included a question about desirable considerations the teachers would (or actually do) take into account when choosing a function for this or for a similar task.

For the second part of the questionnaire we collected the considerations teachers wrote in the first part, and prepared a list of these considerations; for each consideration the teachers were asked to state whether they agree that it is desirable and why. In addition, we had conversations with some of the teachers, to better understand their thinking.

Main findings

Most of the teachers' responses indicated that they had not come across such subtle considerations prior to our study. The questions evoked their awareness to the need to address the issue of limitation of a table for sketching graphs of functions in a more sophisticated and rich way than they encountered both in their practice and in professional development frameworks.

The considerations the teachers suggested included properties of a function that they thought would make it a good choice for the task, e.g.: 'a function that is unfamiliar to the students'; 'a function with more than one extreme point'; 'a function that is not symmetrical'; 'a function that is not special'; 'a function that the coordinates of its extreme points do not appear in the table'; 'a function for which it is hard to detect immediately its intersection points with the *x*-axis and thus its extreme points cannot be inferred instantly'. Interestingly, none of the teachers mentioned at all consideration 1 and 2 above, which could be viewed as critical.

There were teachers who stated that they do not use such tasks (as the one in Figure 1 above). The main claim was that although this is a significant activity, curriculum constraints do not permit spending time on such a task. Some maintained that it is sufficient to build on students' (drill and) practice, because when you practice enough problems with derivatives, then students gain understanding that the derivative is zero at an extreme point. Clearly, this implies that for some teachers the necessity principle is not part of their thinking.

Several teachers claimed that they rely on textbooks and that it does not matter what function is chosen. The three specific examples (of the cubic functions) in our study appeared to most teachers the same, for the purpose of the task; analysing the subtle differences and how they impact the task was totally new to them. Through drawing their attention to the differences between the three functions in terms of their specific affordances in addressing the goal of the task, and to the merits and limitations of each example in terms of its learning potential for their students, they felt they were able to make more judicious pedagogical decisions/choices, while before - they were not aware of or able to see such distinctions.

Conclusion

The mathematical and pedagogical analysis presented above point to the wealth and complexity of considerations underlying what teachers' decisions with respect to task design for their students could and should be, in general, and for tasks that address the need for advanced tools for sketching graphs, in particular. Based on some empirical evidence, our study establishes the need to work with teachers in ways that develop and expand their pedagogical toolbox and encourage them to build on it when they design and make choices regarding specific tasks for certain purposes. In this paper we focused on the purpose to establish the need to expand new tools. This approach can be applied to other purposes and topics. While the scope of the study may seem narrow, it can be generalized, to other similar situations (e.g., other tools/concepts that need to be introduced) as well as to a more general level that deals with working with teachers on the significant of choice of the specific examples for a task, or in other words - how going into details could make a huge difference.

One of the goals of teacher education is to prepare teachers to consider alternatives (Zaslavsky and Sullivan, 2011). We view task design as a fruitful site for considering alternatives. It is not enough to support a specific choice. There is value in considering other possible choices, and by that – crystalizing and refining the criteria for such choices.

The approach we describe moves from finding the 'right' choice to realizing that it is a give and take situation. Each choice has its merits and its limitations. Articulating these merits and limitations is part of what teachers should experience and what teacher education programs should foster.

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