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On the distortions of Archimedean copulas : Application to the non-parametric estimation of their generators

Elena Di Bernardino*, Didier Rullière†

Abstract

We study the impact of some distortions for Archimedean copulas. We give some admissibility conditions for these distortions, and define some equivalence classes for both distortions and generators of Archimedean copulas. We investigate some impacts of the distortions on the tails of the distorted copula. We extend the r -fold composition of the diagonal section of a copula, from $r \in \mathbb{N}$ to $r \in \mathbb{R}$. This extension, coupled with results on equivalence classes, gives us new expressions of distortions and generators. Estimators deriving directly from these expressions are proposed and their convergence is investigated. We provide confidence bands for the estimated generators. Numerical illustrations show the empirical performance of these estimators.

Keywords: Distortions, Archimedean copula, auto-nested copula, non-parametric estimation, tail dependence.

1 Introduction

1.1 Basic notions and preliminaries

Assume that we have d underlying risks described by a d -dimensional nonnegative real-valued random vector $\mathbf{X} = (X_1, \dots, X_d)$. Denote its multivariate distribution function by $F : \mathbb{R}_+^d \rightarrow [0, 1]$ with univariate margins $F_i(x_i) = P(X_i \leq x_i)$, for $i = 1, \dots, d$. Sklar's Theorem (1959) is a well-known result which states that for any random vector \mathbf{X} , its multivariate distribution function has the representation

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)),$$

where C is called the *copula* function. Effectively, it is a distribution function on the d -cube $[0, 1]^d$ with uniform margins and it links the univariate margins to their full multivariate distribution. In the case where we have a continuous random vector, we know that $U_i = F_i(X_i)$ is an uniform random variable so that we can write

$$C(u_1, \dots, u_d) = F(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)),$$

to be the unique copula associated with \mathbf{X} , with quantile functions F_i^{-1} defined by:

$$F_i^{-1}(p) = \inf\{x \in \mathbb{R} : F_i(x) \geq p\}, \quad \text{for } p \in (0, 1).$$

In this paper, we mainly consider Archimedean copulas, which are copulas that can be written

$$C_\phi(u_1, \dots, u_d) = \phi(\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d)), \quad (1)$$

where the function ϕ is called the generator of the Archimedean copula C_ϕ . The generator is a continuous and decreasing function satisfying some supplementary assumptions that will be discussed hereafter. They are symmetrical copulas, that is $C_\phi(u_1, \dots, u_d) = C_\phi(u_{\sigma(1)}, \dots, u_{\sigma(d)})$ for any permutation σ of the set

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$\{1, \dots, d\}$. Such copulas play a central role in the understanding of dependencies of multivariate random vectors. A good introduction is given in [Nelsen \(1999\)](#).

Distortions of Archimedean copulas are a simple way to generate new copulas from initial ones. Among advantages of such distortions, we may cite the possible improvement of the fit of an initial copula, the easy development of iterative distortion schemes, and some properties that may ease the estimation of the distorted copula (for further details see for instance [Di Bernardino and Rullière \(2013\)](#)).

Then, following [Valdez and Yugu \(2011\)](#) (Definitions 3.2 and 3.6), [Charpentier et al. \(2006\)](#), [Di Bernardino and Rullière \(2013\)](#), we now recall the definition of a distorted copula \tilde{C} , of an initial copula C_0 , using a distortion function T , i.e.,

$$\tilde{C}(u_1, \dots, u_d) = T \circ C_0(T^{-1}(u_1), \dots, T^{-1}(u_d)), \quad u_1, \dots, u_d \in [0, 1]. \quad (2)$$

The function $T : [0, 1] \rightarrow [0, 1]$ is a continuous and increasing function on the interval $[0, 1]$, with $T(0) = 0$, $T(1) = 1$, with supplementary assumptions that will be chosen to guarantee that \tilde{C} is also a copula, detailed hereafter. Such distortions are presented for instance in [Durrleman et al. \(2000\)](#). A particular class of distortion is constituted by distortions defined in [Bienvenüe and Rullière \(2011\)](#) with the form $T_f : [0, 1] \rightarrow [0, 1]$ such that

$$T_f(u) = \begin{cases} 0 & \text{if } u = 0, \\ \text{logit}^{-1}(f(\text{logit}(u))) & \text{if } 0 < u < 1, \\ 1 & \text{if } u = 1, \end{cases} \quad (3)$$

where f any bijective increasing function, $f : \mathbb{R} \rightarrow \mathbb{R}$. Function f is said to be a *conversion function*. These distortions help working in the logit-scale, so that we only need to study composition of increasing functions from \mathbb{R} to \mathbb{R} . The main advantage of T_f , with adequate conversion functions f , is to lead to simple analytic expressions for inverse distortions and for level curves of the associated multivariate distribution function. Developments using distortions in (3), with hyperbolic conversion function f , are given in [Bienvenüe and Rullière \(2011\)](#), [Bienvenüe and Rullière \(2012\)](#), [Di Bernardino and Rullière \(2013\)](#).

1.2 Some problematic points

Among problems generated by distortions of Archimedean copulas, one can point out, in particular

- i) The problem of uniqueness: distortions of a given initial copula leading to a given target copula are not unique. This raises some problems for the analysis of the convergence of estimators of the distortion. This also causes problems to compare distortions and to understand their impact, in terms of changing of dependence structure. A further analysis shows that also a generator of an Archimedean copula is not unique, causing the same kind of problems.
- ii) The tail problem: the impact on the tail of the distorted copulas are partially known (see for instance [Durante et al. \(2010\)](#)). In practice this impact has to be investigated. In particular the relationship between the asymptote of the conversion function f in Equation (3) and the regular variation of the distorted tails represents an open interesting point. A good understanding of the tail behavior is indeed required to estimate the shape of the distortion near 0 and 1, in extreme quantiles where there is a lack of data.
- iii) The estimation problem: we aim here at finding non-parametric estimators of a distorted copula, when no parametric shape is assumed for the generator of the distorted copula. This kind of non-parametric estimation of distorted copulas has been treated by using level curves properties and an iterative algorithm in [Di Bernardino and Rullière \(2013\)](#). However, the convergence of this algorithm is not yet demonstrated, and properties of the obtained estimator are not easy to get.

We try to provide, in the following, some answers to these problems in the case of Archimedean families of copulas.

The determination of sufficient and necessary conditions in order to obtain admissible distortions T is fundamental to propose tractable distortions in operational problems. The definition of equivalence classes

for both distortions and generators is also necessary to select some standardized forms for practical use, for the comparison and the interpretation of obtained distribution functions. To our knowledge, despite relying on elementary calculations, the problem of equivalence classes and the selection of functions among equivalence classes is not detailed in the literature.

Distorted copulas permit to introduce, in a more flexible way, families of copulas exhibiting different behaviour in the tails. The tail behavior of a distorted copula can be assessed by determining the tail coefficients of distorted copulas, or by distorting some existing models like the one of [Ledford and Tawn \(1996\)](#). Much of the recent literature focuses on how the tail dependence properties are modified under distortions (see e.g. [Durante et al. \(2010\)](#)). Results about the tail dependence coefficients of an Archimedean copulas are given by [Juri and Wüthrich \(2002\)](#)-[Juri and Wüthrich \(2003\)](#) and [Charpentier and Segers \(2007\)](#) in terms of regularly varying properties of the additive generator. In this paper we propose some tail properties in terms of the asymptotic behavior of the distortion function T . In particular, we will focus on the Ledford and Tawn's model (see [Ledford and Tawn \(1996\)](#)).

At last, the construction of non-parametric estimators are of great interest for practical studies. There is a huge literature concerning the estimation of copula structures, see for example [Genest and Rivest \(1993\)](#), [Joe \(2005\)](#), [Autin et al. \(2010\)](#), [Hernández-Lobato and Suárez \(2011\)](#). A comparison of different parametric and non-parametric methods for estimating a copula is given, for example, in [Kim et al. \(2007\)](#). Here, we will try to base our estimation on the *diagonal section of a copula*, which is central for Archimedean copula (see, e.g., [Nelsen et al. \(2008\)](#)), and to exploit results on equivalence classes of distortions and generators.

1.3 Organization of the paper

In [Section 2](#), we give properties of both distortions and generators. In particular, we detail admissibility conditions for distortions and generators ([Section 2.1](#)). We characterize equivalence classes for these two functions ([Section 2.2](#)), and we investigate tail coefficients in terms of the asymptote of the conversion function f in [\(3\)](#) and using a Ledford and Tawn's distorted model ([Sections 2.3](#) and [2.4](#)).

In [Section 3](#), we show the importance of the diagonal section of an Archimedean copula, and define auto-nested copulas, which are extensions of k -fold diagonal sections of a copula when k belongs to the whole real line ([Section 3.1](#)). Then we express distortions and generators using these auto-nested copulas functions ([Section 3.2](#)).

In [Section 4](#), we use properties of auto-nested copulas to develop some non parametric estimators of distortions and generators of Archimedean copula. We propose some convergence properties of these estimators ([Section 4.1](#)). Confidence bands are given for auto-nested copulas and for estimated generators ([Section 4.2](#)). At last, we show the empirical behavior of these estimators through numerical illustrations ([Section 4.3](#)).

Finally exact analytical formulas for standardized generators, their inverses and theoretical auto-nested copulas, in the case of most popular Archimedean families of copulas, are postponed in the Annex.

2 Properties of distortions and generators

2.1 Admissibility conditions

Remark 1 (Generator of a distorted copula). Let C_0 the initial Archimedean copula with associated generator ϕ . If $\tilde{\phi} = T \circ \phi$ then $\tilde{C}(u_1, \dots, u_d) = T \circ C_0(T^{-1}(u_1), \dots, T^{-1}(u_n))$. So that $\tilde{\phi}$ is the generator of the distorted copula \tilde{C} .

From [Theorem 2.2](#) in [McNeil and Nešlehová \(2009\)](#) $C_\phi(u_1, \dots, u_d) = \phi(\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d))$ is a d -dimensional copula if and only if its generator ϕ is d -monotone on $[0, \infty)$, where the d -monotony definition is recalled hereafter.

Definition 2.1 (*d*-monotone function). A real function f is called *d*-monotone in (a, b) , where $a, b \in \mathbb{R}$ and $d \geq 2$, if it is differentiable there up to the order $d - 2$ and the derivatives satisfy

$$(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, \dots, d - 2$$

for any $x \in (a, b)$ and further if $(-1)^{d-2} f^{(d-2)}$ is non-increasing and convex in (a, b) . For $d = 1$, f is called 1-monotone in (a, b) if it is nonnegative and non-increasing there.

If f has derivatives of all orders in (a, b) and if $(-1)^k f^{(k)}(x) \geq 0$, for any $x \in (a, b)$, then f is called completely monotone.

It follows some admissibility conditions for a distortion T .

Definition 2.2 (Admissible distortions and distorted copula). Let $T : [0, 1] \rightarrow [0, 1]$ be a continuous and increasing function on the interval $[0, 1]$, with $T(0) = 0$, $T(1) = 1$. Let C_0 an initial copula. We say that T is an admissible distortion if

$$\tilde{C}_{T, C_0}(u_1, \dots, u_d) = T \circ C_0(T^{-1}(u_1), \dots, T^{-1}(u_d)) \quad (4)$$

is a also copula.

In the following result we provide a specific characterization for an admissible distortion T , starting from a *d*-variate initial independent copula C_0 .

Proposition 2.1 (Admissibility conditions for the distortion). Let T be a bijection such that $T : [0, 1] \rightarrow [0, 1]$. Let C_0 be the *d*-variate initial independent copula, i.e., $C_0(u_1, \dots, u_d) = \prod_{i=1}^d u_i$, and \tilde{C} the associated distorted dependence structure as in (4). Then the formula (4) yields a copula if and only if

$$\sum_{r=1}^n \alpha_r^n x^{n-1} T^{(n)}(x) \geq 0, \quad \forall n = 1, \dots, d, \quad (5)$$

with $\alpha_1^n = 1$, $\alpha_n^n = 1$ and $\alpha_r^n = r \alpha_r^{n-1} + \alpha_{r-1}^{n-1}$, for $2 \leq r \leq n - 1$.

Proof: We prove this proposition by induction. We first remark that the distortion of an Archimedean copula is still an Archimedean copula, so that \tilde{C} is an Archimedean copula. From [McNeil and Nešlehová \(2009\)](#), \tilde{C} is a copula if and only if this distorted generator $\tilde{\phi} = T \circ \phi$ is a *d*-monotone function. This means that $(-1)^k \tilde{\phi}^{(k)} \geq 0$ for $k = 0, 1, \dots, d - 2$. This condition implies a specific characterization for our admissible distortion T .

Firstly, we show that the statement of Proposition 2.1 holds for $d = 2$. In particular in the case of a bivariate independent copula the distorted generator $T(e^{-t})$ has to be a 2-monotone function. Since T is increasing, this means $T^{(1)}(x) + x T^{(2)}(x) \geq 0$, for all $x \in [0, 1]$. This is exactly Equation (5) in the case $d = 2$.

For $d = n$, it holds that

$$\tilde{\phi}^{(n)} = [T(e^{-t})]^{(n)} = (-1)^n \sum_{r=1}^n \alpha_r^n e^{-rt} T^{(r)}(e^{-t}) = (-1)^n \sum_{r=1}^n \alpha_r^n x^r T^{(r)}(x),$$

with

$$x = e^{-t} \quad \text{and} \quad \alpha_r^n = \sum_{(i_1, \dots, i_r) \in I_r^n} \frac{n!}{\prod_{k=1}^r i_k! \prod_{k=1}^n j_k!}$$

where $I_r^n = \{(i_1, \dots, i_r) \in \mathbb{N}^*, i_1 \geq \dots \geq i_r, i_1 + \dots + i_r = n\}$, and where the coefficients j_k are defined as the numbers of branches of size k in the tree-representation of the composed derivative (using theory of rooted trees, see for instance [Chomette \(2003\)](#)). Then $(-1)^n \tilde{\phi}^{(n)} \geq 0$ if and only if $\sum_{r=1}^n \alpha_r^n x^r T^{(r)}(x) \geq 0$. Alternative expressions of coefficients α_r^n can be obtained using a combinatoric approach derived by Faà di Bruno's formula. The interested reader is referred for instance to [Hardy \(2006\)](#).

For $d = n + 1$,

$$\begin{aligned}
\tilde{\phi}^{(n+1)} &= [T(e^{-t})]^{(n+1)} = (-1)^n \sum_{r=1}^n \alpha_r^n [e^{-rt} T^{(r)}(e^{-t})]^{(1)} \\
&= (-1)^{n+1} \sum_{r=1}^n r \alpha_r^n e^{-rt} T^{(r)}(e^{-t}) + \sum_{r=1}^n \alpha_r^n e^{-(r+1)t} T^{(r+1)}(e^{-t}) \\
&= (-1)^{n+1} \sum_{r=1}^{n+1} \alpha_r^{n+1} e^{-rt} T^{(r)}(e^{-t})
\end{aligned}$$

with $\alpha_r^{n+1} = r \alpha_r^n + \alpha_{r-1}^n$ for $r \leq n$, $\alpha_{n+1}^{n+1} = \alpha_n^n = \dots = \alpha_1^1 = 1$ and $\alpha_0^n = 0$, $\forall n$. Then, for $d = n + 1$, $(-1)^{n+1} \tilde{\phi}^{(n+1)} \geq 0$ if and only if $\sum_{r=1}^{n+1} \alpha_r^{n+1} x^r T^{(r)}(x) \geq 0$.

Hence the result. \square

A discussion on the class of reachable copulas by distorting an initial copula is available in [Di Bernardino and Rullière \(2013\)](#).

2.2 Equivalent distortions and generators

Definition 2.3 (Invariant class for Archimedean generator). *Let ϕ be a generator of an Archimedean copula C_ϕ , i.e., $C_\phi(u_1, \dots, u_d) = \phi(\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d))$. Then a generator ψ of a copula C_ψ is said to belong to the same invariance class of ϕ if and only if $C_\phi = C_\psi$. We denote this class \mathcal{I}_ϕ and we write $\psi \in \mathcal{I}_\phi$. A generator ψ belonging to \mathcal{I}_ϕ will be said to be equivalent to generator ϕ .*

Analogously to Definition 2.3 we introduce the two following invariance classes respectively for the distortion and the conversion functions.

Definition 2.4 (Invariant class for distortions). *Let \tilde{C}_{T_1, C_0} and \tilde{C}_{T_2, C_0} two distorted copula using distortions T_1 and T_2 respectively and with the same initial copula C_0 (see Equation (4)). Then the distortion T_2 is said to belong to the same invariance class of T_1 if and only if $\tilde{C}_{T_1, C_0} = \tilde{C}_{T_2, C_0}$. We denote this class \mathcal{I}_{T, C_0} and we write $T_2 \in \mathcal{I}_{T_1, C_0}$. A distortion T_2 belonging to \mathcal{I}_{T_1, C_0} will be said to be equivalent to T_1 starting from initial copula C_0 .*

Definition 2.5 (Invariant class for conversion functions). *Let C_0 be an initial copula. Let f_1 be a conversion function associated to the distortion T_1 as in (3). Then a conversion function f_2 is said to belong to the same invariance class of f_1 if and only if $C_{T_{f_1}, C_0} = C_{T_{f_2}, C_0}$. We denote this class \mathcal{I}_{f, C_0} and we write $f_2 \in \mathcal{I}_{f_1, C_0}$. A conversion function f_2 belonging to \mathcal{I}_{f_1, C_0} will be said to be equivalent to f_1 starting from initial copula C_0 .*

Proposition 2.2 (Equivalent generator). *Let C_0 an initial Archimedean copula with generator ϕ . Consider the distorted function $\tilde{\phi}$, then the distorted copula is unchanged with respect to C_0 ,*

$$\tilde{\phi} \in \mathcal{I}_\phi \text{ if and only if } \tilde{\phi} = \phi \circ L,$$

where L is a linear function, i.e. $L(x) = ax$, for some $a \in \mathbb{R} \setminus \{0\}$. The function $\tilde{\phi}$ in the case of $a > 0$ is a generator (in the sense of Lemma 4.1.2. in [Nelsen \(1999\)](#)). The generator $\tilde{\phi}$ is thus equivalent to ϕ since it leads to the same distorted copula.

Proof: Assume that ϕ and $\tilde{\phi}$ are continuous and invertible functions. Let

$$\tilde{\phi}(\tilde{\phi}^{-1}(u_1) + \dots + \tilde{\phi}^{-1}(u_d)) = \phi(\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d)).$$

It follows that

$$\phi^{-1} \circ \tilde{\phi}(\tilde{\phi}^{-1}(u_1) + \dots + \tilde{\phi}^{-1}(u_d)) = \phi^{-1}(u_1) + \dots + \phi^{-1}(u_d),$$

and setting $u_1 = \tilde{\phi}(s_1)$, ..., $u_d = \tilde{\phi}(s_d)$, we get for any s_1, \dots, s_d :

$$\phi^{-1} \circ \tilde{\phi}(s_1 + \dots + s_d) = \phi^{-1} \circ \tilde{\phi}(s_1) + \dots + \phi^{-1} \circ \tilde{\phi}(s_d). \quad (6)$$

Remark that if $\phi^{-1} \circ \tilde{\phi}$ is the linear function L then Equation (6) is satisfied. Furthermore if (6) is satisfied, then the function $\phi^{-1} \circ \tilde{\phi}$ is additive. Since $\phi^{-1} \circ \tilde{\phi}$ is continuous, this implies that $\phi^{-1} \circ \tilde{\phi}$ is a linear function (with in particular $L(0) = 0$). Hence the result. \square

Proposition 2.3 (Equivalent distortions). *Let C_0 the initial Archimedean copula with associated generator ϕ , and denote by L a linear function. Let T_1 and T_2 be two distortions respectively associated to copulas C_1 and C_2 (see Equation (4)). If $\psi_1 = T_1 \circ \phi$, $\psi_2 = T_2 \circ \phi$, then*

$$\psi_2 \in \mathcal{I}_{\psi_1} \text{ if and only if } T_2 = T_1 \circ \phi \circ L \circ \phi^{-1}.$$

Then $T_2 = T_1 \circ \phi \circ L \circ \phi^{-1} \in \mathcal{I}_{T_1, C_0}$. The distortion T_2 is said to be equivalent to distortion T_1 , starting from the initial copula C_0 , since they lead to the same distorted copula.

The proof of Proposition 2.3 comes down trivially from Proposition 2.2.

Proposition 2.4 (Equivalent conversion functions). *Let C_0 the initial Archimedean copula with associated generator ϕ . Let f_1 and f_2 be two conversion functions respectively associated to distortions T_{f_1} and T_{f_2} , i.e., $T_{f_1} = \text{logit}^{-1} \circ f_1 \circ \text{logit}(x)$ and $T_{f_2} = \text{logit}^{-1} \circ f_2 \circ \text{logit}(x)$ (from Equation (3)). Then*

$$C_{T_{f_1}, C_0} = C_{T_{f_2}, C_0} \text{ if and only if } f_2 = f_1 \circ \tau, \quad \text{with } \tau = \text{logit} \circ \phi \circ L \circ \phi^{-1} \circ \text{logit}^{-1}.$$

Then $f_2 = f_1 \circ \tau \in \mathcal{I}_{f_1, C_0}$. The conversion function f_2 is said to be equivalent to conversion function f_1 , starting from the initial copula C_0 , since they lead to the same distorted copula.

This result comes down from Proposition 2.3.

Proposition 2.5 (Equivalent distortion passing through a given point). *Let C_0 the initial Archimedean copula with associated generator ϕ . Let T_1 and T_2 be two distortions of this initial copula, respectively associated to copulas C_1 and C_2 (see Equation (4)). If $L(x) = ax$, $x \in \mathbb{R}$, with*

$$a = \frac{\phi^{-1} \circ T_1^{-1}(y_0)}{\phi^{-1}(x_0)} \quad \text{and} \quad T_2 = T_1 \circ \phi \circ L \circ \phi^{-1},$$

then $T_2 \in \mathcal{I}_{T_1, C_0}$ and $T_2(x_0) = y_0$, for any given point $(x_0, y_0) \in (0, 1)^2$. The distortion T_2 is an equivalent distortion of T_1 , starting from initial copula C_0 , passing through the point (x_0, y_0) .

Corollary 2.1. *Let C_0 be the independent copula. Let*

$$T_2(x) = T_1(x^a), \quad x \in [0, 1], \quad \text{with} \quad a = \frac{\ln(T_1^{-1}(y_0))}{\ln(x_0)},$$

then $T_2 \in \mathcal{I}_{T_1, C_0}$ and $T_2(x_0) = y_0$, for any given point $(x_0, y_0) \in (0, 1)^2$.

Proposition 2.5 and Corollary 2.1 can be useful in order to ensure the uniqueness of the distortion T among the invariant class for distortions. In an iterative procedure of estimation the uniqueness of the distortion is essential in order to permit the convergence of the procedure. These results will be useful later in the estimation procedure of the distortion and generator functions (see Sections 3.2 and 4.1).

2.3 Tail coefficients of bivariate distorted copulas

The diagonal section of a bivariate copula C satisfying regular conditions can be also used to study the *tail dependence* of the random pair (U, V) (see Nelsen (1999)). Indeed the upper and lower tail dependence parameters λ_U and λ_L , which are defined as

$$\lambda_U = \lim_{u \rightarrow 1^-} \mathbb{P}[V > u | U > u] \quad \text{and} \quad \lambda_L = \lim_{u \rightarrow 0^+} \mathbb{P}[V \leq u | U \leq u],$$

(if the limits exist), can be computed as follows:

$$\lambda_U = 2 - \lim_{u \rightarrow 1^-} \frac{d}{du} \mathcal{C}(u) \quad \text{and} \quad \lambda_L = \lim_{u \rightarrow 0^+} \frac{d}{du} \mathcal{C}(u),$$

where $\mathcal{C}(u) = C(u, u)$ (see, e.g., [Nelsen et al. \(2008\)](#)).

In this section we are interested in the upper (resp. lower) tail of a bivariate distorted distribution function. Assume that the conversion function f has an asymptote $\bar{f}(x) = ax + b$, for large (resp. small) values of x , with $a > 0$ and $b \in \mathbb{R}$. We denote the associated distortion $T_{\bar{f}} = \text{logit}^{-1} \circ \bar{f} \circ \text{logit}(x)$ (see [\(3\)](#)).

From Propositions 4.2 and 4.3 in [Durante et al. \(2010\)](#), we obtain the following result.

Proposition 2.6 (Upper and lower tail coefficients for logit-linear distorted copulas). *Let C_0 the initial bivariate Archimedean copula with associated generator ϕ such that $\lambda_L(C_0)$ exists. We consider the distortion $T_{\bar{f}} = \text{logit}^{-1} \circ \bar{f} \circ \text{logit}(x)$, with $\bar{f}(x) = ax + b$, for some $a > 0$ and $b \in \mathbb{R}$ and the associated distorted copula $\tilde{C}_{T_{\bar{f}}, C_0}$, as in [\(4\)](#). It holds that*

$$\begin{aligned} \text{if } \lambda_L(C_0) \text{ exists,} & \quad \text{then } \lambda_L(\tilde{C}_{T_{\bar{f}}, C_0}) = (\lambda_L(C_0))^a, \\ \text{if } \lambda_U(C_0) \text{ exists,} & \quad \text{then } \lambda_U(\tilde{C}_{T_{\bar{f}}, C_0}) = 2 - (2 - \lambda_U(C_0))^a. \end{aligned}$$

Proof: Since $T_{\bar{f}} = (1 + (\frac{x}{1-x})^{-a} e^{-b})^{-1}$, then $\lim_{x \rightarrow 0^+} \frac{T_{\bar{f}}(x)}{x^a} = c$, with $c > 0$. From Proposition 4.2 in [Durante et al. \(2010\)](#), $\lambda_L(\tilde{C}_{T_{\bar{f}}, C_0}) = (\lambda_L(C_0))^a$. Furthermore, since $T_{\bar{f}} = (1 + (\frac{x}{1-x})^{-a} e^{-b})^{-1}$, then $\lim_{x \rightarrow 1^-} \frac{1 - T_{\bar{f}}(x)}{(1-x)^a} = c$, with $c > 0$. From Proposition 4.3 in [Durante et al. \(2010\)](#), we obtain that $\lambda_U(\tilde{C}_{T_{\bar{f}}, C_0}) = 2 - (2 - \lambda_U(C_0))^a$. Hence the result. \square

Remark 2 (Admissible values of slope a). Since $\lambda_U(\tilde{C}_{T_{\bar{f}}, C_0}) \in [0, 1]$, one can obtain from Proposition [2.6](#) that

$$a \in \left(0, \frac{\ln(2)}{\ln(2 - \lambda_U(C_0))}\right].$$

Then Proposition [2.6](#) restricts the range of values for the parameter a . In particular when the initial copula C_0 is the bivariate independent copula, this means that $a \in (0, 1]$. Obviously if $a = 1$, $\lambda_U(\tilde{C}_{T_{\bar{f}}, C_0}) = \lambda_U(C_0)$ and $\lambda_L(\tilde{C}_{T_{\bar{f}}, C_0}) = \lambda_L(C_0)$.

Remark 3. From Proposition [2.6](#), we prove that there exists a particular function that satisfies the assumption of Propositions 4.2-4.3 in [Durante et al. \(2010\)](#), i.e. $T_{\bar{f}} = \text{logit}^{-1} \circ \bar{f} \circ \text{logit}(x)$. Furthermore we obtain the value of the link-coefficient between the initial and the distorted tail coefficient, i.e. the slope a of the asymptote of the conversion function f . Finally Proposition [2.6](#) restricts the range of values for the parameter a .

2.4 Tail dependence in distorted Ledford and Tawn's model

Ledford and Tawn propose a multivariate extreme value threshold model for joint tail estimation which overcomes the problems encountered with existing techniques when the variables are asymptotically independent (e.g., see [Ledford and Tawn \(1996\)](#); [Ledford and Tawn \(1997\)](#)). They propose a very flexible and broadly applicable model on the diagonal of the bivariate distribution such that:

$$\mathbb{P}[Z_1 > r, Z_2 > r] \sim \mathcal{L}(r) r^{-\frac{1}{\eta}}, \quad \text{as } r \rightarrow \infty, \quad (7)$$

where Z_1 and Z_2 are two with unit Fréchet marginal distributions, $\eta \in (0, 1]$ is a constant, and $\mathcal{L}(r)$ is a slowly varying (SV) function, i.e. $\mathcal{L}(tr)/\mathcal{L}(r) \rightarrow 1$, as $r \rightarrow \infty$, for all fixed $t > 0$. We denote $\bar{F}(r, r) = \mathbb{P}[Z_1 > r, Z_2 > r]$.

The parameter η characterizes the nature of the tail dependence. It is a constant that effectively determines the decay rate of $\bar{F}(r, r)$, for large r . Since the marginal variables are standardized (unit Fréchet), the parameter η provides a measure of the dependence between the marginal tails. Accordingly, [Ledford and Tawn \(1996\)](#) termed η the *coefficient of tail dependence*. For example, the marginal variables Z_1 and Z_2 are

- asymptotically independent and positively associated, if $\frac{1}{2} < \eta < 1$,
- asymptotically independent and negatively associated, if $0 < \eta < \frac{1}{2}$,
- perfectly independent, if $\eta = \frac{1}{2}$ and $\mathcal{L}(r) = 1$,
- asymptotically dependent, if $\eta = 1$ and $\mathcal{L}(r) \rightarrow 0$, $r \rightarrow \infty$,
- perfectly dependent, if $\eta = 1$ and $\mathcal{L}(r) = 1$.

Proposition 2.7 (Logit-linear distorted Ledford and Tawn's model). *Let C_0 the initial bivariate independent copula and Z_1 and Z_2 are two with unit Fréchet marginal distributions. We consider the distortion $T_{\bar{f}} = \text{logit}^{-1} \circ \bar{f} \circ \text{logit}(x)$, with $\bar{f}(x) = ax + b$, for some $a > 0$ and $b \in \mathbb{R}$. Following the Ledford and Tawn's model in (7), $\bar{F}(r, r) = C_0(\bar{F}_{Z_1}(r), \bar{F}_{Z_2}(r)) \sim r^{-2}$, i.e. $\eta = \frac{1}{2}$ and $\mathcal{L}(r) = 1$. Then the externally distorted survival distribution*

$$\tilde{\bar{F}}(r, r) = T \circ C_0(\bar{F}_{Z_1}(r), \bar{F}_{Z_2}(r)) \quad (8)$$

also satisfies the Ledford and Tawn's model in (7), with $\tilde{\eta} = \frac{\eta}{a}$ and $\tilde{\mathcal{L}}(r) = \frac{r^{2a}}{1+e^{-b}(r^2-1)^a}$.

Proof: Since

$$\tilde{\bar{F}}(r, r) = T \circ C_0(\bar{F}_{Z_1}(r), \bar{F}_{Z_2}(r)) = T \circ ((1 - e^{-\frac{1}{r}}) \cdot (1 - e^{-\frac{1}{r}})) = \frac{1}{1 + \left(\frac{(1 - e^{-\frac{1}{r}})^2}{1 - (1 - e^{-\frac{1}{r}})^2} \right)^{-a} e^{-b}},$$

for r large, since $\eta = \frac{1}{2}$, we obtain $\tilde{\bar{F}}(r, r) \sim \tilde{\mathcal{L}}(r) r^{-2a} = \tilde{\mathcal{L}}(r) r^{-1/\tilde{\eta}}$, where $\tilde{\mathcal{L}}(r) = \frac{r^{2a}}{1+e^{-b}(r^2-1)^a}$. Remark that $\tilde{\mathcal{L}}(r)$ is a SV function. Hence the result. \square

Remark 4 (Discussion of possible asymptotic dependence structure using logit-linear distortions). In the following we discuss the different dependence structures in the joint tail that one can generate using Equation (8). In particular, since C_0 in Proposition 2.7 is the independent copula, from Remark 2, we consider the admissible values of slope a , i.e. $a \in (0, 1]$. Then,

- If $a = 1$ and $b = 0$, trivially $T \equiv Id$, then $\tilde{\mathcal{L}}(r) = 1$, $\tilde{\eta} = \eta = \frac{1}{2}$ and the distorted random variables are again perfectly independent.
- If $a = \frac{1}{2}$ and $b \neq 0$, then $\tilde{\eta} = 1$ and $\tilde{\mathcal{L}}(r) \rightarrow e^b \neq 0$, as $r \rightarrow \infty$. Then the distorted random variables are asymptotically dependent.
- If $a = \frac{1}{2}$ and $b = 0$, then $\tilde{\eta} = 1$ and $\tilde{\mathcal{L}}(r) = 1$, the distorted random variables are perfectly dependent.
- If $a \in (\frac{1}{2}, 1)$, then $\tilde{\eta} \in (\frac{1}{2}, 1)$ and the distorted random variables are asymptotically independent and positively associated.

Remark that if $a \in (0, \frac{1}{2})$, then $\tilde{\eta} > 1$ and the Ledford and Tawn's model is no more available.

Then, using Ledford and Tawn's distorted model in (8), starting from an initial independent copula C_0 and $a \in (0, \frac{1}{2}]$, we obtain a dependence range from perfect independence to perfect positive dependence.

3 Auto-nested copulas

3.1 Definition and properties

In the following, we define *auto-nested copula* functions. We have chosen this terminology in reference to the nested copulas (e.g., see Hofert and Pham (2012)). These functions will be built exclusively from the diagonal of a copula. They will be essential for the non-parametric estimation of a distortion or the

non-parametric estimation of an Archimedean copula generator.

Some assumptions are needed to build these functions. In particular, we will say that an Archimedean copula satisfy *regular conditions* if for all $u \in [0, 1]$, $C(u, \dots, u)$ is a continuous and strictly increasing function of u , and if its generator ϕ is such that $\lim_{t \rightarrow +\infty} \phi(t) = 0$.

We first remark that the diagonal of an Archimedean copula is essential to describe the copula. Some constructions of copulas starting from the *diagonal section* are given for example in [Nelsen et al. \(2008\)](#) and [Wysocki \(2012\)](#).

Remark 5 (Identity of Archimedean copulas). *Let C_1 and C_2 be two Archimedean copulas satisfying regular conditions. The two copulas are identical if and only if*

$$C_1(u, \dots, u) = C_2(u, \dots, u), \quad \text{for all } u \in [0, 1].$$

Proof: From Lemma 1 of [Wysocki \(2012\)](#), we get the identity of generators and thus of the copulas. \square

Remark that the diagonal section of a copula C has several probabilistic interpretations; for instance is the restriction to $[0, 1]$ of the distribution function of $\max(U_1, \dots, U_n)$ whenever (U_1, \dots, U_n) is the random vector distributed as C . The interested reader is referred to [Nelsen et al. \(2008\)](#).

Definition 3.1 (Discrete auto-nested copula). *Consider a copula C satisfying regular conditions. The discrete auto-nested copula of C at order k , $k \in \mathbb{Z}$, is the function \mathcal{C}_k such that for all $u \in [0, 1]$,*

$$\begin{cases} \mathcal{C}_k(u) &= \mathcal{C}_{k-1} \circ \mathcal{C}_1(u), & k \in \mathbb{N} \\ \mathcal{C}_{-k}(u) &= \mathcal{C}_{-k+1} \circ \mathcal{C}_{-1}(u), & k \in \mathbb{N} \\ \mathcal{C}_0(u) &= u, \end{cases}$$

where $\mathcal{C}_1(u) = C(u, \dots, u)$ and \mathcal{C}_{-1} is the inverse function of \mathcal{C}_1 , so that $\mathcal{C}_1 \circ \mathcal{C}_{-1}$ is the identity function.

Discrete auto-nested copulas correspond to the k -fold composition of the diagonal section \mathcal{C}_1 of the copula (see [Wysocki \(2012\)](#)). They are defined for $k \in \mathbb{Z}$ (hence justifying the prefix *discrete*). For a family of discrete auto-nested copulas $\{\mathcal{C}_k\}_{k \in \mathbb{Z}}$, one can easily check that for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}$, for all $u \in [0, 1]$,

$$\mathcal{C}_{j+k}(u) = \mathcal{C}_j \circ \mathcal{C}_k(u).$$

A function of a family satisfying this proposition for all $j, k \in \mathbb{R}$ will be called an extended auto-nested copula, or simply an *auto-nested copula*. The following definition aims at defining the r -fold composition of the diagonal section \mathcal{C}_1 of the copula when $r \in \mathbb{R}$ is not a relative integer.

Definition 3.2 (Auto-nested copulas). *Functions of a family $\{\mathcal{C}_r\}_{r \in \mathbb{R}}$ are called (extended) auto-nested copulas of a copula C , if $\mathcal{C}_k(u)$ is the discrete auto-nested copula of C at order k , for all $k \in \mathbb{Z}$, and if furthermore*

$$\mathcal{C}_{r_1+r_2}(u) = \mathcal{C}_{r_1} \circ \mathcal{C}_{r_2}(u), \quad \forall r_1, r_2 \in \mathbb{R}, \forall u \in [0, 1].$$

The study of auto-nested copulas is thus relying on the study of a family of univariate functions. *Extended auto-nested copulas* can be seen as cumulative distribution functions of some indexed random variables X^{or} , $r \in \mathbb{R}$, distributed on $[0, 1]$, such that for all $r_1, r_2 \in \mathbb{R}$, for all $x \in [0, 1]$,

$$\mathbb{P} \left[X^{o(r_1+r_2)} \leq x \right] = \mathbb{P} [X^{or_1} \leq \mathbb{P} [X^{or_2} \leq x]],$$

with $\mathbb{P} [X^{or_1} \leq x] = \mathcal{C}_r(x)$, and in particular X^{o0} uniformly distributed on $[0, 1]$. A further study of properties of such a family (moments, etc.) could bring some new enlightenments on copulas and multivariate analysis.

Proposition 3.1 (Auto-nested copula of an Archimedean copula). *If C is an Archimedean copula associated with a generator ϕ , then the auto-nested copula of C at order r is*

$$\mathcal{C}_r(x) = \phi(d^r \cdot \phi^{-1}(x)), \quad r \in \mathbb{R}.$$

Proof: We notice that $\mathcal{C}_1(u) = \phi(d \cdot \phi^{-1}(u))$, so that $\mathcal{C}_2(u) = \mathcal{C}_1 \circ \mathcal{C}_1(u) = \phi(d^2 \cdot \phi^{-1}(u))$, and we can show by induction that $\mathcal{C}_k(u) = \phi(d^k \cdot \phi^{-1}(u))$ for all $k \in \mathbb{Z}$. For any $r \in \mathbb{R}$, we can easily check that setting $\mathcal{C}_r(x) = \phi(d^r \cdot \phi^{-1}(x))$ is a discrete auto-nested copula for any $r \in \mathbb{Z}$, and that $\mathcal{C}_{r_1+r_2} = \mathcal{C}_{r_1} \circ \mathcal{C}_{r_2}$ for any $r_1, r_2 \in \mathbb{R}$. \square

Remark 6 (Some expressions of auto-nested copulas). We give here some expressions of auto-nested copulas for some classical copulas that will be considered in numerical illustrations

- If C is the independence copula of generator $\phi(t) = \exp(-t)$, $\mathcal{C}_r(u) = u^{(d^r)}$.
- If C is a Gumbel copula of generator $\phi(t) = \exp(-t^{1/\theta})$, $\mathcal{C}_r(u) = u^{(d^{(r/\theta)})}$, $\theta \geq 1$.
- If C is a Clayton copula of generator $\phi(t) = (1 + \theta t)^{-1/\theta}$, $\mathcal{C}_r(u) = (1 + d^r(t^{-\theta} - 1))^{-1/\theta}$, $\theta \in \mathbb{R}^*$.

Proposition 3.2 (Interpolation of auto-nested copulas). *Let C be an Archimedean copula with generator ϕ . For any real $r \in [k, k+1]$, $k \in \mathbb{Z}$, the auto-nested copula of C satisfies:*

$$\mathcal{C}_r(x) = \phi \left((\phi^{-1} \circ \mathcal{C}_k(x))^{1-\alpha} (\phi^{-1} \circ \mathcal{C}_{k+1}(x))^\alpha \right), \quad x \in [0, 1],$$

with $\alpha = r - [r]$ and $k = [r]$, where $[r]$ denotes the integer part of r .

Proof: Consider an Archimedean copula C and the associated auto-nested copulas \mathcal{C}_r , for $r \in \mathbb{R}$. By Proposition 3.1, $\mathcal{C}_r(x) = \phi(d^r \cdot \phi^{-1}(x))$. Define $g_r(x) = r \ln d - \ln \phi^{-1} \circ \mathcal{C}_r(x)$. One can easily check that for all $r \in \mathbb{R}$, $g_r(x) = -\ln \phi^{-1}(x)$ does not depend on r , so that in particular for any $k_1, k_2 \in \mathbb{Z}$ and $\alpha \in [0, 1]$,

$$g_r(x) = (1 - \alpha)g_{k_1}(x) + \alpha g_{k_2}(x). \quad (9)$$

When $(1 - \alpha)k_1 + \alpha k_2 = r$, this is equivalent to

$$\ln \phi^{-1} \circ \mathcal{C}_r(x) = (1 - \alpha) \ln \phi^{-1} \circ \mathcal{C}_{k_1}(x) + \alpha \ln \phi^{-1} \circ \mathcal{C}_{k_2}(x), \quad (10)$$

and the result holds for any $k_1, k_2 \in \mathbb{Z}$ and $\alpha \in [0, 1]$ such that $(1 - \alpha)k_1 + \alpha k_2 = r$.

In practice, the interpolation in Proposition 3.2 aims at being used even when $g_k(x)$ is not a constant function of k (e.g. if g_k is estimated, or if the copula is not Archimedean) or when ϕ is approximated. For this reason we present it in the particular case where $k_1 = [r]$ and $k_2 = [r] + 1$. The choice of $\alpha = r - [r]$ follows from the condition $(1 - \alpha)k_1 + \alpha k_2 = r$, and also ensures that interpolations (9) and (10) are correct for any $r \in \mathbb{Z}$, even if $g_r(x)$ is not a constant function of r . \square

Corollary 3.1 (Interpolation in the Gumbel or Independence case). *If C is a Gumbel copula with generator $\phi(t) = \exp(-t^{1/\theta})$, then \mathcal{C}_r can be expressed as a function of \mathcal{C}_k and \mathcal{C}_{k+1} , and this function does not depend on the parameter θ of the copula:*

$$\mathcal{C}_r(x) = \exp \left(- \left(-\ln \mathcal{C}_k(x) \right)^{1-\alpha} \left(-\ln \mathcal{C}_{k+1}(x) \right)^\alpha \right), \quad x \in [0, 1],$$

with $\alpha = r - [r]$ and $k = [r]$, where $[r]$ denotes the integer part of r . This result includes also the case of the independent copula, i.e. the Gumbel copula with parameter $\theta = 1$.

In a further estimation section we will use interpolation functions (see Section 4). The interpolation functions satisfying interpolation properties of Proposition 3.2 or Corollary 3.1 will be called *perfect interpolation functions*, as stated in the following definition.

Definition 3.3 (Perfect interpolation functions). *Let C be an Archimedean copula with generator ϕ . A function z is said to be a perfect interpolation function for the copula C if for all $r \in \mathbb{R}$,*

$$\mathcal{C}_r(x) = z \left((z^{-1} \circ \mathcal{C}_k(x))^{1-\alpha} (z^{-1} \circ \mathcal{C}_{k+1}(x))^\alpha \right), \quad x \in [0, 1],$$

with $\alpha = r - [r]$ and $k = [r]$, where $[r]$ denotes the integer part of r . As an example, from Proposition 3.2, $z(x) = \phi(x)$ and $z(x) = \phi(x^a)$, $a \in \mathbb{R}^*$ are perfect interpolation functions. If C is an Gumbel copula, from Corollary 3.1, $z(x) = \exp(-x)$ is a perfect interpolation function which does not depend on the parameter of the copula.

3.2 Expression of distortions and generators using auto-nested copulas

Proposition 3.3 (All points of distortion T). *Let C be an initial Archimedean copula and $\tilde{C}(u_1, \dots, u_d) = T \circ C(T^{-1}(u_1), \dots, T^{-1}(u_d))$ a distorted copula, C and \tilde{C} satisfying regular conditions. Let \mathcal{C}_r and $\tilde{\mathcal{C}}_r$, $r \in \mathbb{R}$, be the respective auto-nested copulas of C and \tilde{C} . If $T(x_0) = y_0$, then $T(x_r) = y_r$ for all $r \in \mathbb{R}$, with*

$$\begin{cases} x_r &= \mathcal{C}_r(x_0), \\ y_r &= \tilde{\mathcal{C}}_r(y_0). \end{cases}$$

Proof: Denote by ϕ and $\tilde{\phi}$ the respective generators of C and \tilde{C} , where $\tilde{C}(u, \dots, u) = T \circ C(T^{-1}(u), \dots, T^{-1}(u))$. If C is an Archimedean copula, then $\tilde{\mathcal{C}}_r(u) = \tilde{\phi}(d^r \tilde{\phi}^{-1}(u))$. Since $\tilde{\phi} = T \circ \phi$, we have $\tilde{\mathcal{C}}_r(u) = T \circ \phi(d^r \phi^{-1} \circ T^{-1}(u))$, so that for all $u \in [0, 1]$,

$$T^{-1} \circ \tilde{\mathcal{C}}_r(u) = \mathcal{C}_r \circ T^{-1}(u).$$

Then, setting $u = y_0$, we get $T^{-1} \circ \tilde{\mathcal{C}}_r(y_0) = \mathcal{C}_r(x_0)$ since $T^{-1}(y_0) = x_0$, and T is passing through the point $(\mathcal{C}_r(x_0), \tilde{\mathcal{C}}_r(y_0))$ for any $r \in \mathbb{R}$. \square

Proposition 3.4 (Distortion T using auto-nested copulas). *Consider Archimedean copulas C and \tilde{C} satisfying regular conditions and the associated auto-nested copulas \mathcal{C}_r and $\tilde{\mathcal{C}}_r$, $r \in \mathbb{R}$. If T is defined by $T(0) = 0$, $T(1) = 1$ and for all $x \in (0, 1)$,*

$$T(x) = \tilde{\mathcal{C}}_{r(x)}(y_0),$$

with $r(x)$ such that $\mathcal{C}_{r(x)}(x_0) = x$,

then the distorted copula using distortion T is equal to \tilde{C} : for all u_1, \dots, u_d ,

$$\tilde{C}(u_1, \dots, u_d) = T \circ C(T^{-1}(u_1), \dots, T^{-1}(u_d)),$$

where $(x_0, y_0) \in (0, 1)^2$ can be arbitrarily chosen. In the case where C is the independence copula,

$$r(x) = \frac{1}{\ln d} \ln \left(\frac{-\ln x}{-\ln x_0} \right),$$

Proof: From Proposition 3.3 and Remark 5 we get $\tilde{C}(u, \dots, u) = C(u, \dots, u)$ for all $u \in (0, 1)$, and the result holds using Remark 5. \square

Proposition 3.5 (Generator $\tilde{\phi}$ using auto-nested copulas). *Consider an Archimedean copula \tilde{C} satisfying regular conditions, and the associated auto-nested copulas $\tilde{\mathcal{C}}_r$, for $r \in \mathbb{R}$. Assume that the copula \tilde{C} is reachable by distorting an Archimedean copula C , and denote by \mathcal{C}_r , $r \in \mathbb{R}$, the auto-nested copulas of C and by ϕ its generator. A generator $\tilde{\phi}$ of \tilde{C} is defined for all $t \in \mathbb{R}^{*+}$ by*

$$\tilde{\phi}(t) = \tilde{\mathcal{C}}_{\rho(t)}(y_0),$$

with $\rho(t)$ such that $\mathcal{C}_{\rho(t)}(x_0) = \phi(t)$,

where $(x_0, y_0) \in (0, 1)^2$ can be arbitrarily chosen. In the particular case where C is the independent copula, then

$$\rho(t) = \frac{1}{\ln d} \ln \left(\frac{t}{-\ln x_0} \right)$$

Proof: Directly comes from Proposition 3.4 and from $\tilde{\phi} = T \circ \phi$. \square

In particular, the suitable generator $\tilde{\phi}$ is passing through the points

$$\{(t_r, \varphi_r)\}_{r \in \mathbb{R}} = \left\{ (\phi^{-1} \circ \mathcal{C}_r(x_0), \tilde{\mathcal{C}}_r(y_0)) \right\}_{r \in \mathbb{R}}.$$

If \tilde{C} is distorted from an independent copula, the suitable generator $\tilde{\phi}$ is passing through the points

$$\{(t_r, \varphi_r)\}_{r \in \mathbb{R}} = \left\{(-d^r \ln x_0, \tilde{C}_r(y_0))\right\}_{r \in \mathbb{R}}.$$

Furthermore, if \tilde{C} is an independent copula, $\tilde{C}_1(u) = u^d$ and $\tilde{C}_r(u) = u^{(d^r)}$, so that we can easily retrieve

$$\tilde{\phi}(t) = \exp\left(-\left(\frac{\ln y_0}{\ln x_0}\right)t\right),$$

which is an equivalent generator of the independence generator $\phi(t) = \exp(-t)$.

4 Non-parametric estimation

4.1 Estimators of distortions and generators

We aim here at finding non-parametric estimators of a distorted copula, when non-parametric shape is assumed for the associated generator. Starting from results of Section 3 for Archimedean families of copulas, we provide some straightforward estimators and some convergence properties of these estimators.

In this section, we consider a copula C satisfying *regular conditions*. We assume that an estimator of the diagonal of the copula $\mathcal{C}_1(u) := C(u, \dots, u)$ and an estimator of the inverse function \mathcal{C}_{-1} of \mathcal{C}_1 are available. We denote respectively $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_{-1}$ these estimators.

Remark that some possible consistent estimators for \mathcal{C}_1 and \mathcal{C}_{-1} are available in the literature. [Deheuvels \(1979\)](#) investigated the consistency of the empirical copula \hat{C} and [Deheuvels \(1980\)](#) obtained the exact law and the limiting process of $\sqrt{n}(\hat{C} - C)$ when the two margins are independent. [Fermanian et al. \(2002\)](#) extended these results by proving the weak convergence of the process in a more general case.

In the following, we show how to build estimators of a whole family of auto-nested copulas $\{\mathcal{C}_r\}_{r \in \mathbb{R}}$, using these two estimators $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_{-1}$.

Proposition 4.1 (Estimation of nested-copulas). *Consider a copula C satisfying regular conditions. Let $\hat{\mathcal{C}}_1$ be an estimator of \mathcal{C}_1 , and $\hat{\mathcal{C}}_{-1}$ be an estimator of the inverse function \mathcal{C}_{-1} . An estimator of \mathcal{C}_k can be obtained for any $k \in \mathbb{Z}$ by setting*

$$\begin{cases} \hat{\mathcal{C}}_k(u) &= \hat{\mathcal{C}}_{k-1} \circ \hat{\mathcal{C}}_1(u), & k \in \mathbb{N} \setminus \{0\} \\ \hat{\mathcal{C}}_{-k}(u) &= \hat{\mathcal{C}}_{-k+1} \circ \hat{\mathcal{C}}_{-1}(u), & k \in \mathbb{N} \setminus \{0\} \\ \hat{\mathcal{C}}_0(u) &= u. \end{cases}$$

At any order $r \in \mathbb{R}$, an estimator $\hat{\mathcal{C}}_r$ of \mathcal{C}_r is

$$\hat{\mathcal{C}}_r(x) = z\left(\left(z^{-1} \circ \hat{\mathcal{C}}_k(x)\right)^{1-\alpha} \left(z^{-1} \circ \hat{\mathcal{C}}_{k+1}(x)\right)^\alpha\right), \quad x \in [0, 1],$$

with $\alpha = r - [r]$ and $k = [r]$, where $[r]$ denotes the integer part of r , and where z is a strictly monotone function driving the interpolation, ideally the generator of the considered copula C or any other perfect interpolation function (see [Definition 3.3](#)). In particular, z is such that for any $x \in [0, 1]$, $z(x) \geq 0$. Note that several interpolation functions may lead to the same interpolation, e.g. $z_1(x)$ and $z_2(x) = z_1(x^\alpha)$, $\alpha \in \mathbb{R}^*$ are both involving the same interpolation. Such interpolators will be called equivalent interpolators.

This estimation is a plug-in estimation relying on [Definition 3.1](#) and [Proposition 3.2](#). The function z drives the interpolation of \mathcal{C}_r , for $r \in \mathbb{R}$, knowing values of \mathcal{C}_k , for $k \in \mathbb{Z}$. If known, the best choice is the generator ϕ of the copula C , i.e. $z(x) = \phi(x)$. Otherwise, the identity function $z(x) = x$ (linear interpolation) could be possible, for $x \in [0, 1]$. However we recommend, in case of positive dependence, the interpolator $z(x) = \exp(-x)$, $x \in (0, 1]$, since it is the best choice for any independence or Gumbel

copula, whatever the parameter of the copula, as a consequence of Proposition 3.1. Another natural choice could be any estimator of the generator of the copula. Finally, remark that this function z does not change values of any \mathcal{C}_k , for $k \in \mathbb{Z}$. Then the global shape of \mathcal{C}_r , as a function of $r \in \mathbb{R}$, is not heavily impacted by the choice of z .

Proposition 4.2 (Non-parametric estimation of a distortion T). *Consider two Archimedean copulas C and \tilde{C} satisfying regular conditions and their respective auto-nested copulas \mathcal{C}_r and $\tilde{\mathcal{C}}_r$, for $r \in \mathbb{R}$. Assume that \tilde{C} is the distorted copula using distortion T and initial copula C . Denote by $\hat{\mathcal{C}}_r$ an estimator of $\tilde{\mathcal{C}}_r$, for $r \in \mathbb{R}$. A non-parametric estimator of T is defined by $T(0) = 0$, $T(1) = 1$ and for all $x \in (0, 1)$ by*

$$\begin{aligned} \hat{T}(x) &= \hat{\mathcal{C}}_{r(x)}(y_0), \\ &\text{with } r(x) \text{ such that } \mathcal{C}_{r(x)}(x_0) = x, \end{aligned}$$

where $(x_0, y_0) \in (0, 1)^2$ can be arbitrarily chosen. In the case where the initial copula C is the independence copula, then

$$r(x) = \frac{1}{\ln d} \ln \left(\frac{-\ln x}{-\ln x_0} \right).$$

In particular, the estimator \hat{T} is passing through the points

$$\{(x_k, y_k)\}_{k \in \mathbb{Z}} = \left\{ (\mathcal{C}_k(x_0), \hat{\mathcal{C}}_k(y_0)) \right\}_{k \in \mathbb{Z}}.$$

Remark that no interpolation function z is needed to get (x_k, y_k) , for $k \in \mathbb{Z}$.

Proposition 4.3 (Non-parametric estimation of a generator $\tilde{\phi}$). *Consider an Archimedean copula \tilde{C} satisfying regular conditions, and associated auto-nested copulas $\tilde{\mathcal{C}}_r$, for $r \in \mathbb{R}$. Assume that the copula \tilde{C} is reachable by distorting a given initial Archimedean copula C with associated generator ϕ . Denote by \mathcal{C}_r the auto-nested copulas of C and by $\hat{\mathcal{C}}_r$ the estimator of $\tilde{\mathcal{C}}_r$, for $r \in \mathbb{R}$. A non-parametric estimator $\hat{\phi}$ of $\tilde{\phi}$ is defined, for all $t \in \mathbb{R}^+$, by*

$$\begin{aligned} \hat{\phi}(t) &= \hat{\mathcal{C}}_{\rho(t)}(y_0), \\ &\text{with } \rho(t) \text{ such that } \mathcal{C}_{\rho(t)}(x_0) = \phi(t), \end{aligned}$$

where $(x_0, y_0) \in (0, 1)^2$ can be arbitrarily chosen. In the case where the initial copula C is the independence copula, then

$$\rho(t) = \frac{1}{\ln d} \ln \left(\frac{t}{-\ln x_0} \right).$$

In particular, the estimator $\hat{\phi}$ of $\tilde{\phi}$ is passing through the points

$$\{(t_k, \varphi_k)\}_{k \in \mathbb{Z}} = \left\{ (\phi^{-1} \circ \mathcal{C}_k(x_0), \hat{\mathcal{C}}_k(y_0)) \right\}_{k \in \mathbb{Z}},$$

where ϕ is the given initial generator. Remark that no interpolation function z is needed to get (t_k, φ_k) , for $k \in \mathbb{Z}$.

For a given Archimedean copula, there is a whole family of equivalent generators leading to this copula. As stated in Proposition 2.2, generators $\phi_1(t)$ and $\phi_2(t) = \phi_1(at)$ lead to the same copula function, whatever the choice of $a > 0$. Then two different generators, ϕ_1 and ϕ_2 , which lead to the same copula may have very different graphical shapes, so that a graphical comparison of these generators would have no sense. For these reasons, in Remark 7, we give formulas in order to force a generator to pass through an arbitrarily chosen point (t_0, φ_0) . After this ‘‘standardization procedure’’ we will be able to graphically compare different generators.

Remark 7 (Equivalent estimated generator passing through (t_0, φ_0)). If one chooses

$$\begin{cases} x_0 &= \exp(-t_0), \\ y_0 &= \varphi_0, \end{cases}$$

then the estimator of the generator $\tilde{\phi}$ in Proposition 4.3 is such that $\hat{\phi}(t_0) = \varphi_0$.

Remark 8 (Equivalent theoretical generator passing through (t_0, φ_0)). Let $(t_0, \varphi_0) \in \mathbb{R}^* \times (0, 1)$. Let ϕ be a generator of an Archimedean copula. If one set for all $t \in \mathbb{R}$

$$\bar{\phi}(t) = \phi(at) \quad \text{with} \quad a = \frac{\phi^{-1}(\varphi_0)}{t_0}$$

then $\bar{\phi}$ is an equivalent generator of ϕ such that $\bar{\phi}(t_0) = \varphi_0$. This equation is equivalent to $\bar{\phi}(t) = \mathcal{C}_{r(t)}(\varphi_0)$, with $r(t)$ such that $d^{r(t)} = t/t_0$.

As an example, we give here some standardized generators passing through a given point (t_0, φ_0) :

- Standardized Gumbel generator: $\bar{\phi}(t) = \varphi_0^{(t/t_0)^{1/\theta}}$, $\theta \geq 1$. If $(t_0, \varphi_0) = (1, e^{-1})$, $\bar{\phi}(t) = \exp(-t^{1/\theta})$.
- Standardized independence generator: $\bar{\phi}(t) = \varphi_0^{(t/t_0)}$. If $(t_0, \varphi_0) = (1, e^{-1})$, $\bar{\phi}(t) = \exp(-t)$.
- Standardized Clayton generator: $\bar{\phi}(t) = \left(1 + (\varphi_0^{-\theta} - 1) \frac{t}{t_0}\right)^{-1/\theta}$, $\theta \in \mathbb{R}^*$. If $(t_0, \varphi_0) = (1, e^{-1})$, $\bar{\phi}(t) = (1 + (e^\theta - 1)t)^{-1/\theta}$.

Exact analytical formulas for standardized generators, their inverses and theoretical auto-nested copulas \mathcal{C}_r , in the case of most popular Archimedean families of copulas, are postponed in the Annex.

In numerical applications (see Section 4.3) we will consider generators passing through $(t_0, \varphi_0) = (1, e^{-1})$. Applying Remark 7, this corresponds to $x_0 = y_0 = e^{-1}$. In this case, applying Remark 8, standardized independence and Gumbel generators correspond to the usual Gumbel-generator (see Nelsen (1999)), and standardized Clayton generator becomes $\bar{\phi}(t) = (1 + (e^\theta - 1)t)^{-1/\theta}$ which is an equivalent generator of the usual generator $\phi(t) = (1 + \theta t)^{-1/\theta}$.

4.2 Confidence bands

In this section our goal is to quantify the estimation error of the estimated generator $\hat{\phi}$ in terms of the error of the estimation of $\hat{\mathcal{C}}_1$. To this aim, we proceed in the following way. Firstly, we assume to be able to quantify the estimation error of $\hat{\mathcal{C}}_1$ (see Assumption 4.1). From this assumption we derive the estimation error on any $\hat{\mathcal{C}}_r(u)$, for $r \in \mathbb{R}$ (see Proposition 4.4). Finally, we use this last result to control the estimation error of $\hat{\phi}$ (see Proposition 4.5). Illustrations of these results, in the particular case of a Gumbel copula, are postponed in the numerical illustrations section (see Section 4.3).

So, we consider the following assumption on the estimation error of $\hat{\mathcal{C}}_1$.

Assumption 4.1 (Estimation error on $\hat{\mathcal{C}}_1$). *For a copula \tilde{C} satisfying regular conditions, denote $\tilde{C}(u) = \tilde{\mathcal{C}}_1(u) = \tilde{C}(u, \dots, u)$ and $\hat{\mathcal{C}}(u) = \hat{\mathcal{C}}_1(u)$ an estimator of \tilde{C} . There exists two nonnegative reals ε^- and ε^+ and a continuous and strictly monotone function h , from $[0, 1]$ to $X \subset \mathbb{R}$, such that for any $u \in [0, 1]$,*

$$h^{-1} \circ L_{\varepsilon^-} \circ h \circ \tilde{C}(u) \leq \hat{\mathcal{C}}(u) \leq h^{-1} \circ L_{\varepsilon^+} \circ h \circ \tilde{C}(u), \quad (11)$$

where $L_\varepsilon(u) = \varepsilon u$.

This kind of assumption allows a large variety of bounding of the quantity $\hat{\mathcal{C}}(u)$, for example:

- $h(x) = \ln(x)$ leads to assuming $\tilde{C}(u)^{\varepsilon^-} \leq \hat{\mathcal{C}}(u) \leq \tilde{C}(u)^{\varepsilon^+}$, where obviously $\varepsilon^+ \leq 1 \leq \varepsilon^-$.
- $h(x) = x$ leads to assuming $\tilde{C}(u) \cdot \varepsilon^- \leq \hat{\mathcal{C}}(u) \leq \tilde{C}(u) \cdot \varepsilon^+$, where obviously $\varepsilon^- \leq 1 \leq \varepsilon^+$.
- $h(x) = \exp(x)$ leads to assuming $\tilde{C}(u) + \ln \varepsilon^- \leq \hat{\mathcal{C}}(u) \leq \tilde{C}(u) + \ln \varepsilon^+$, where obviously $\varepsilon^- \leq 1 \leq \varepsilon^+$.

Since this assumption may not be fulfilled in every possible situation, we consider in the following the probability that this assumption is fulfilled.

Proposition 4.4 (Estimation error on $\widehat{\mathcal{C}}_r$, for $r \in \mathbb{R}^+$). Consider a copula $\tilde{\mathcal{C}}$ with generator $\tilde{\phi}$, satisfying regular conditions. Denote by $\widehat{\mathcal{C}}$ an estimator of $\tilde{\mathcal{C}}$. Denote by $\tilde{\mathcal{C}}_r$ (resp. $\widehat{\mathcal{C}}_r$) the auto-nested copula of $\tilde{\mathcal{C}}$ (resp. $\widehat{\mathcal{C}}$). Assume that $\widehat{\mathcal{C}}_r$ is interpolated with a perfect interpolation function in Proposition 4.1. If the probability that $\widehat{\mathcal{C}}$ satisfies Assumption 4.1, for the function $h = \tilde{\phi}^{-1}$, is greater than a given threshold $\eta \in [0, 1]$, i.e., if there exists reals δ^- and δ^+ such that

$$\mathbb{P} \left[\tilde{\mathcal{C}}_{\delta^-} \circ \tilde{\mathcal{C}}(u) \leq \widehat{\mathcal{C}}(u) \leq \tilde{\mathcal{C}}_{\delta^+} \circ \tilde{\mathcal{C}}(u), \forall u \in [0, 1] \right] \geq \eta, \quad (12)$$

then it holds for any $r \in \mathbb{R}^+$ that

$$\mathbb{P} \left[\tilde{\mathcal{C}}_{r\delta^-} \circ \tilde{\mathcal{C}}_r(u) \leq \widehat{\mathcal{C}}_r(u) \leq \tilde{\mathcal{C}}_{r\delta^+} \circ \tilde{\mathcal{C}}_r(u), \forall u \in [0, 1] \right] \geq \eta. \quad (13)$$

Proof: Assume that there exists a real ε and such that for all $u \in [0, 1]$,

$$\widehat{\mathcal{C}}(u) \leq h^{-1} \circ L_\varepsilon \circ h \circ \tilde{\mathcal{C}}(u). \quad (14)$$

By Proposition 3.1, $\tilde{\mathcal{C}}(u) = \tilde{\phi} \circ L_d \circ \tilde{\phi}^{-1}(u)$, with $L_d(u) = d \cdot u$. It follows

$$\widehat{\mathcal{C}}(u) \leq h^{-1} \circ L_\varepsilon \circ h \circ \tilde{\phi} \circ L_d \circ \tilde{\phi}^{-1}(u),$$

and in the case where $h = \tilde{\phi}^{-1}$,

$$\widehat{\mathcal{C}}(u) \leq \tilde{\phi} \circ L_{\varepsilon \cdot d} \circ \tilde{\phi}^{-1}(u).$$

Since Equation (14) holds for any $u \in [0, 1]$ then in particular for $u = \widehat{\mathcal{C}}(u_1)$

$$\widehat{\mathcal{C}} \circ \widehat{\mathcal{C}}(u_1) \leq \tilde{\phi} \circ L_{\varepsilon \cdot d} \circ \tilde{\phi}^{-1} \circ \widehat{\mathcal{C}}(u_1) \leq \tilde{\phi} \circ L_{(\varepsilon \cdot d)^2} \circ \tilde{\phi}^{-1}(u_1).$$

And, by induction for any $k \in \mathbb{N}^*$,

$$\widehat{\mathcal{C}}_k(u_k) \leq \tilde{\phi} \circ L_{(\varepsilon \cdot d)^k} \circ \tilde{\phi}^{-1}(u_k)$$

holds for any value u_k such that $\widehat{\mathcal{C}}_k(u_k) = u$ with $u \in [0, 1]$, that is for all $u_k \in [0, 1]$. Then

$$\left[\widehat{\mathcal{C}}(u) \leq \tilde{\phi} \circ L_\varepsilon \circ \tilde{\phi}^{-1} \circ \tilde{\mathcal{C}}(u), \forall u \in [0, 1] \right] \implies \left[\widehat{\mathcal{C}}_k(u) \leq \tilde{\phi} \circ L_{(\varepsilon \cdot d)^k} \circ \tilde{\phi}^{-1}(u), \forall u \in [0, 1] \right]. \quad (15)$$

Setting δ^+ such that $d^{\delta^+} = \varepsilon$, from Proposition 3.1, we obtain $\tilde{\phi} \circ L_{(d^{\delta^+} \cdot d)^k} \circ \tilde{\phi}^{-1}(u) = \tilde{\mathcal{C}}_{k\delta^+ + k}$ and

$$\left[\widehat{\mathcal{C}}(u) \leq \tilde{\mathcal{C}}_{\delta^+} \circ \tilde{\mathcal{C}}(u), \forall u \in [0, 1] \right] \implies \left[\widehat{\mathcal{C}}_k(u) \leq \tilde{\mathcal{C}}_{k\delta^+} \circ \tilde{\mathcal{C}}_k(u), \forall u \in [0, 1] \right]. \quad (16)$$

Proceeding the same way for both inequalities, checking the result is obvious when $k = 0$, result in (16) holds for any $k \in \mathbb{N}$. Now assume that $z(x)$ is a perfect interpolation function (see Definition 3.3), $z(x)$ and $\tilde{\phi}(x)$ are equivalent interpolation functions, and both $\widehat{\mathcal{C}}_r$ and $\tilde{\mathcal{C}}_r$ are interpolated with the same interpolation function. Without loss of generality, assume $z(x)$ and $z^{-1}(x)$ are decreasing functions of x (would they be increasing, there exists decreasing equivalent interpolation functions). Assume now that for any $k \in \mathbb{N}$, and for all $u \in [0, 1]$, $\tilde{\mathcal{C}}_{k\delta^-} \circ \tilde{\mathcal{C}}_k(u) \leq \widehat{\mathcal{C}}_k(u) \leq \tilde{\mathcal{C}}_{k\delta^+} \circ \tilde{\mathcal{C}}_k(u)$. Since \mathcal{C}_r and $\tilde{\mathcal{C}}_r$ are interpolated by the same perfect interpolation function $z(x)$, then for any $\alpha \in [0, 1]$, recalling $z^{-1}(x) \geq 0$ for any $x \in [0, 1]$ as in Proposition 4.1,

$$\begin{aligned} \left(z^{-1} \circ \tilde{\mathcal{C}}_{k\delta^-} \circ \tilde{\mathcal{C}}_k(u) \right)^{1-\alpha} &\geq \left(z^{-1} \circ \widehat{\mathcal{C}}_k(u) \right)^{1-\alpha} \geq \left(z^{-1} \circ \tilde{\mathcal{C}}_{k\delta^+} \circ \tilde{\mathcal{C}}_k(u) \right)^{1-\alpha} \\ \left(z^{-1} \circ \tilde{\mathcal{C}}_{(k+1)\delta^-} \circ \tilde{\mathcal{C}}_{k+1}(u) \right)^\alpha &\geq \left(z^{-1} \circ \widehat{\mathcal{C}}_{k+1}(u) \right)^\alpha \geq \left(z^{-1} \circ \tilde{\mathcal{C}}_{(k+1)\delta^+} \circ \tilde{\mathcal{C}}_{k+1}(u) \right)^\alpha \end{aligned}$$

By Proposition 3.2, we get for any $\delta \in \mathbb{R}$, as in the proof of Proposition 3.2, if $(1 - \alpha)k + \alpha(k + 1) = r$,

$$z \left(\left(z^{-1} \circ \tilde{\mathcal{C}}_{k\delta^+} \circ \tilde{\mathcal{C}}_k(u) \right)^{1-\alpha} \left(z^{-1} \circ \tilde{\mathcal{C}}_{(k+1)\delta^+} \circ \tilde{\mathcal{C}}_{k+1}(u) \right)^\alpha \right) = \tilde{\mathcal{C}}_{(1-\alpha)(k\delta^+ + k) + \alpha((k+1)\delta^+ + k + 1)}(u) = \tilde{\mathcal{C}}_{r\delta^+ + r}(u). \quad (17)$$

Finally, setting $k = \lfloor r \rfloor$, and since z is assumed to be decreasing, we get

$$\tilde{\mathcal{C}}_{r\delta^-} \circ \tilde{\mathcal{C}}_r(u) \leq \widehat{\mathcal{C}}_r(u) \leq \tilde{\mathcal{C}}_{r\delta^+} \circ \tilde{\mathcal{C}}_r(u)$$

and the result holds. If $z(x)$ is not an equivalent interpolator as $\tilde{\phi}$, one easily check that the result still holds for integer values $r \in \mathbb{N}$. \square

From Proposition 4.4, if all values of $\widehat{\mathcal{C}}(u)$, $u \in [0, 1]$ are in a confidence band with a given confidence level η (see (15)), then all values of $\widehat{\mathcal{C}}_r(u)$, $u \in [0, 1]$ will be in a (larger) confidence band (see (16)), for $r \in \mathbb{R}^+$.

These last results may be extended to the case where $r \in \mathbb{Z}^-$ or $r \in \mathbb{R}^-$ starting from a bounding assumption for $\widehat{\mathcal{C}}_{-1}$. For the sake of simplicity, these extensions are omitted here. Using Proposition 4.4, we quantify in the following result the error for the estimated generator $\widehat{\phi}$.

Proposition 4.5 (Estimation error on $\widehat{\phi}$). *Assume that the interpolation function $z(x)$ in Proposition 4.1 is a perfect interpolation function (as defined in Definition 3.3). If there exists some constants δ^- , δ^+ , γ^- , γ^+ such that*

$$\begin{cases} \mathbb{P} \left[\tilde{\mathcal{C}}_{\delta^-} \circ \tilde{\mathcal{C}}(u) \leq \widehat{\mathcal{C}}(u) \leq \tilde{\mathcal{C}}_{\delta^+} \circ \tilde{\mathcal{C}}(u), \forall u \in [0, 1] \right] \geq \eta, \\ \mathbb{P} \left[\tilde{\mathcal{C}}_{\gamma^-} \circ \tilde{\mathcal{C}}_{-1}(u) \leq \widehat{\mathcal{C}}_{-1}(u) \leq \tilde{\mathcal{C}}_{\gamma^+} \circ \tilde{\mathcal{C}}_{-1}(u), \forall u \in [0, 1] \right] \geq \eta, \end{cases} \quad (18)$$

then

$$\begin{cases} \mathbb{P} \left[\tilde{\mathcal{C}}_{\rho(t)\delta^-} \circ \tilde{\phi}(t) \leq \widehat{\phi}(t) \leq \tilde{\mathcal{C}}_{\rho(t)\delta^+} \circ \tilde{\phi}(t) \right] \geq \eta, & \text{if } \rho(t) \geq 0, \\ \mathbb{P} \left[\tilde{\mathcal{C}}_{\rho(t)\gamma^-} \circ \tilde{\phi}(t) \leq \widehat{\phi}(t) \leq \tilde{\mathcal{C}}_{\rho(t)\gamma^+} \circ \tilde{\phi}(t) \right] \geq \eta, & \text{if } \rho(t) < 0, \end{cases} \quad (19)$$

with $\rho(t)$ such that $\mathcal{C}_{\rho(t)}(x_0) = \phi(t)$, with $\phi(t)$ the generator of the initial non-distorted copula and \mathcal{C}_r the auto-nested copulas of the initial copula. In the case where the initial copula is the independence copula, and if $x_0 = y_0 = \exp(-1)$, we get $\rho(t) = \ln t / \ln d$.

Proof: As a direct consequence of the Equation (16) in the proof of Proposition 4.4, in all cases where $\tilde{\mathcal{C}}_{\delta^-} \circ \tilde{\mathcal{C}}(u) \leq \widehat{\mathcal{C}}(u) \leq \tilde{\mathcal{C}}_{\delta^+} \circ \tilde{\mathcal{C}}(u)$, $\forall u \in [0, 1]$, we get $\tilde{\mathcal{C}}_{k\delta^-} \circ \tilde{\mathcal{C}}_k(u) \leq \widehat{\mathcal{C}}_k(u) \leq \tilde{\mathcal{C}}_{k\delta^+} \circ \tilde{\mathcal{C}}_k(u)$, $\forall u \in [0, 1]$. We can show that the same property holds for $k \in \mathbb{R}^+$. If $\rho(t) > 0$, then in particular for $k = \rho(t)$ and $u = y_0$, we show that $\tilde{\mathcal{C}}_{\delta^-} \circ \tilde{\mathcal{C}}(u) \leq \widehat{\mathcal{C}}(u) \leq \tilde{\mathcal{C}}_{\delta^+} \circ \tilde{\mathcal{C}}(u)$, $\forall u \in [0, 1]$ implies $\tilde{\mathcal{C}}_{\rho(t)\delta^-} \circ \tilde{\phi}(t) \leq \widehat{\phi}(t) \leq \tilde{\mathcal{C}}_{\rho(t)\delta^+} \circ \tilde{\phi}(t)$. Proceeding the same way when $\rho(t) < 0$, we get the final result. \square

Remark 9. Remark that if in Proposition 4.5, the condition on the interpolation function z does not hold, the result is still available for any t such that $\rho(t) \in \mathbb{Z}$.

This last property gives direct confidence bounds for $\widehat{\phi}$, depending on some constants δ^- , δ^+ , γ^- , γ^+ . One should notice that if the distribution of the process $\{\widehat{\mathcal{C}}(u)\}_{0 \leq u \leq 1}$ is known, and if the family of targeted copula is known, then δ^- and δ^+ can be computed at least numerically, e.g. by simulating paths of the process $\{\widehat{\mathcal{C}}(u)\}_{0 \leq u \leq 1}$. If the family of targeted copulas is unknown, constants δ^- and δ^+ and final confidence bounds can be estimated by replacing $\tilde{\mathcal{C}}_r$, $r \in \mathbb{R}$, by their estimators. For example using results of Deheuvels (1980) and Fermanian et al. (2002), i.e. using the law and the limiting process of $\sqrt{n}(\widehat{C} - \tilde{C})$, one can get suitable constants δ^- , δ^+ and γ^- , γ^+ for a given confidence level η , and thus confidence bounds for $\widehat{\phi}$.

In the following, we apply Proposition 4.5 in the case of a Gumbel copula.

Corollary 4.1 (Estimation errors in the Gumbel case). *Consider a Gumbel copula \tilde{C} with generator $\tilde{\phi}(t) = \exp(-t^{1/\theta})$, and set $z(x) = \exp(-x)$ as interpolation function. We take as initial non-distorted copula the independent copula, and $x_0 = y_0 = \exp(-1)$. If there exist some reals α^- , α^+ , β^- , β^+ such that $\widehat{\mathcal{C}}_1$ and $\widehat{\mathcal{C}}_{-1}$ satisfies*

$$\mathbb{P} \left[\tilde{\mathcal{C}}(u)^{\alpha^-} \leq \widehat{\mathcal{C}}(u) \leq \tilde{\mathcal{C}}(u)^{\alpha^+}, \forall u \in [0, 1] \right] \geq \eta,$$

and

$$\mathbb{P} \left[\tilde{\mathcal{C}}_{-1}(u)^{\beta^-} \leq \widehat{\mathcal{C}}_{-1}(u) \leq \tilde{\mathcal{C}}_{-1}(u)^{\beta^+}, \forall u \in [0, 1] \right] \geq \eta,$$

then this implies the following bounding for $\widehat{\phi}$,

$$\mathbb{P} \left[\tilde{\phi}(t)^{(t^{\lambda^-})} \leq \widehat{\phi}(t) \leq \tilde{\phi}(t)^{(t^{\lambda^+})} \right] \geq \eta, \quad \text{if } t \geq 1, \quad (20)$$

and

$$\mathbb{P} \left[\tilde{\phi}(t)^{(t^{\mu^-})} \leq \widehat{\phi}(t) \leq \tilde{\phi}(t)^{(t^{\mu^+})} \right] \geq \eta, \quad \text{if } t < 1, \quad (21)$$

with $\lambda^- = \frac{\ln \alpha^-}{\ln d}$, $\lambda^+ = \frac{\ln \alpha^+}{\ln d}$ and with $\mu^- = \frac{\ln \beta^-}{\ln d}$, $\mu^+ = \frac{\ln \beta^+}{\ln d}$.

Proof: By direct application of Proposition 4.4, setting $\alpha^- = d^{(\delta^-/\theta)}$ and $\alpha^+ = d^{(\delta^+/\theta)}$, and using Remark 6, that gives in the Gumbel case $\tilde{\mathcal{C}}_r(u) = u^{(d^{(r/\theta)})}$, we obtain (e.g. when $k > 0$)

$$\mathbb{P} \left[\tilde{\mathcal{C}}_k(u)^{(\alpha^-)^k} \leq \widehat{\mathcal{C}}_k(u) \leq \tilde{\mathcal{C}}_k(u)^{(\alpha^+)^k}, \forall u \in [0, 1] \right] \geq \eta, \quad k \in \mathbb{N}. \quad (22)$$

The bounding on $\widehat{\phi}$ holds by application of Proposition 4.5. In the case where C is an independent copula and $x_0 = y_0 = e^{-1}$, $\rho(t) = \ln t / \ln d$, so that $\mathcal{C}_{\rho(t)\delta^+} = u^{(t^{\delta^+/\theta})}$, and $t^{\delta^+/\theta} = t^{\ln \alpha^+ / \ln d}$. Hence the result. \square

As expected, there is no uncertainty when t is in a neighbourhood of $t_0 = 1$, since distortions are here chosen such that $(x_0, y_0) = e^{-1}$, implying that $\phi(t_0) = \varphi_0$ with $(t_0, \varphi_0) = (1, e^{-1})$.

4.3 Numerical illustrations

In this section we provide some numerical illustrations of the proposed non-parametric estimation procedure for the distortion T (Proposition 4.2) and the generator $\tilde{\phi}$ (Proposition 4.3). The impact of the choice of the function z driving the interpolation is also analyzed (see Proposition 4.1). In this section, we will estimate the diagonal of the copula $\mathcal{C}_1(u) := C(u, \dots, u)$ and its inverse function \mathcal{C}_{-1} using the consistent empirical copula \widehat{C} in Deheuvels (1979).

Firstly, using Proposition 4.3, we illustrate the finite sample properties of the non-parametric estimation of the generator for an Archimedean copula. We take the independence initial copula C . Then $\widehat{\phi}(t) = \widehat{\mathcal{C}}_{\rho(t)}(y_0)$ where $\rho(t) = \frac{1}{\ln d} \ln \left(\frac{t}{-\ln x_0} \right)$ and d is the dimension of the problem. We have chosen here $x_0 = y_0 = e^{-1}$, and in this case

$$\widehat{\phi}(t) = \widehat{\mathcal{C}}_{(\ln t / \ln d)}(e^{-1}).$$

The values of $\widehat{\mathcal{C}}_r$, $r \in \mathbb{R}$ are interpolated from values of $\widehat{\mathcal{C}}_k$, $k \in \mathbb{Z}$. As a consequence, in the dimension $d = 2$, for $t \in [1000^{-1}, 1000]$, $\widehat{\phi}(t)$ does only depend on $\widehat{\mathcal{C}}_k$, with $k \in \{-10, \dots, 10\}$. For $t \in [30^{-1}, 30]$, $\widehat{\phi}(t)$ does only depend on $\widehat{\mathcal{C}}_k$, with $k \in \{-5, \dots, 5\}$. In practice, we thus only need to compute values of $\widehat{\mathcal{C}}_k$ for a small range of values of k .

4.3.1 Simulated data illustration

In Figure 1, we generate two bivariate samples of size $n = 150$ and $n = 1500$ from a Gumbel copula with parameter $\theta = 3$ and we draw the estimated generator on these two different samples. We compare the obtained $\hat{\phi}(t)$ with the theoretical standardized Gumbel-generator, i.e., $\bar{\phi}(t) = \exp(-t^{1/\theta})$, since $(t_0, \varphi_0) = (1, e^{-1})$. In this case, we take as function z driving the interpolation, $z(x) = \exp(-x)$, $x \in (0, 1]$, since it is the best choice for any independence or Gumbel copula, whatever the parameter of the copula, as a consequence of Proposition 3.1.

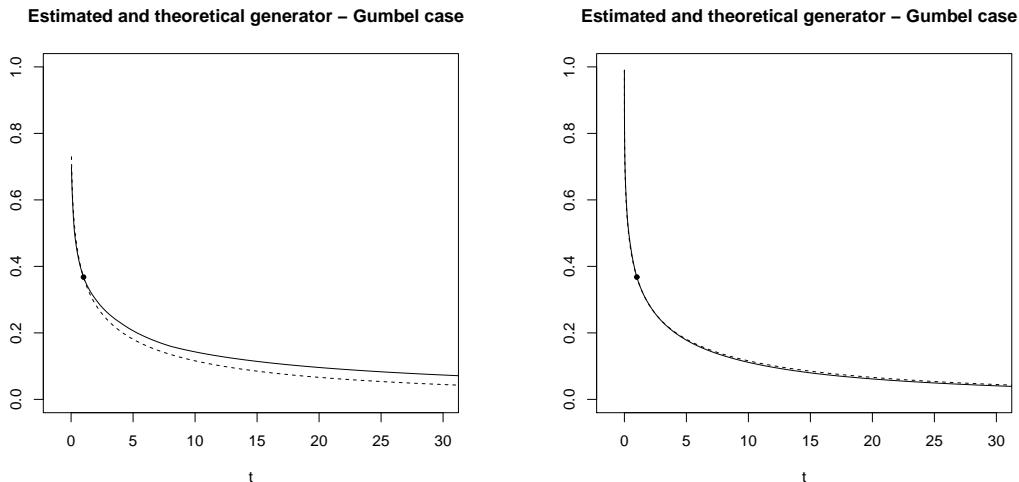


Figure 1: Estimated versus theoretical Gumbel-generator with parameter $\theta = 3$. Size of simulated samples $n = 150$ (left) and $n = 1500$ (right). Estimated $\hat{\phi}(t) = \hat{\mathcal{C}}_{\rho(t)}(y_0)$ as in Proposition 4.3 (full line). The theoretical standardized Gumbel-generator, i.e., $\bar{\phi}(t) = \exp(-t^{1/\theta})$, is drawn using a dashed line. We force the generators to pass through the point $(t_0, \varphi_0) = (1, e^{-1})$ (black point).

Analogously, in Figure 2, we generate two sample of size $n = 150$ and $n = 1500$ from a Clayton copula with parameter $\theta = 6$ and we draw the obtained $\hat{\phi}(t)$. We compare the obtained $\hat{\phi}(t)$ with the theoretical standardized Clayton-generator, i.e., $\bar{\phi}(t) = (1 + (e^\theta - 1)t)^{-1/\theta}$, since $(t_0, \varphi_0) = (1, e^{-1})$. Also in this case we take as interpolation function $z(x) = \exp(-x)$, $x \in (0, 1]$.

Since in these estimations we use the consistent empirical copula \hat{C} in Deheuvels (1979), then, as expected, the greater n is, the better the estimations are (see in Figures 1-2 the quality of the estimation in the plots on the right-hand, for $n = 150$, with respect to that on the left-hand, for $n = 1500$).

Following Proposition 4.2, in Figure 3 we draw the non-parametric estimation for the distortion T starting from the independence initial copula C , i.e. $\hat{T}(x) = \hat{\mathcal{C}}_{r(x)}(y_0)$, with $r(x) = \frac{1}{\ln d} \ln \left(\frac{-\ln x}{-\ln x_0} \right)$. We choose $x_0 = y_0 = 0.5$. We generate two samples of size $n = 1500$ from a Clayton (Figure 3, right) and a Gumbel (Figure 3, left) copula with different parameters θ . In both cases we take as interpolation function $z(x) = \exp(-x)$, $x \in (0, 1]$.

In Figures 4-5 we provide an illustration of the estimation of a nested-copula (see Proposition 4.1). In Figure 4 (resp. Figure 5) we generate a sample of size $n = 1500$ from a Clayton (resp. Gumbel) copula with parameter $\theta = 6$ (resp. $\theta = 3$). We consider $k = -3, -2, -1, 0, 1, 2, 3$ and we estimate the auto-nested copula $\hat{\mathcal{C}}_k(u)$, for $u \in [0, 1]$.

In order to evaluate the impact of the interpolation function z in the evaluation of $\hat{\mathcal{C}}_r$, $r \in \mathbb{R}$, we define

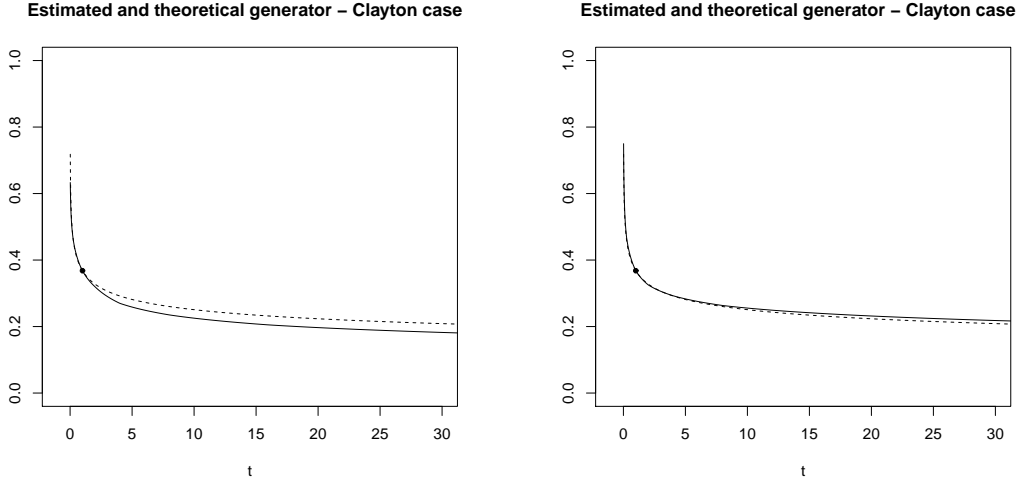


Figure 2: Estimated versus theoretical Clayton-generator with parameter $\theta = 6$. Size of simulated samples $n = 150$ (left) and $n = 1500$ (right). Estimated $\hat{\phi}(t) = \hat{C}_{\rho(t)}(y_0)$ as in Proposition 4.3 (full line). The theoretical standardized Clayton-generator, i.e., $\bar{\phi}(t) = (1 + (e^\theta - 1)t)^{-1/\theta}$, is drawn using a dashed line. We force the generators to pass through the point $(t_0, \varphi_0) = (1, e^{-1})$ (black point).

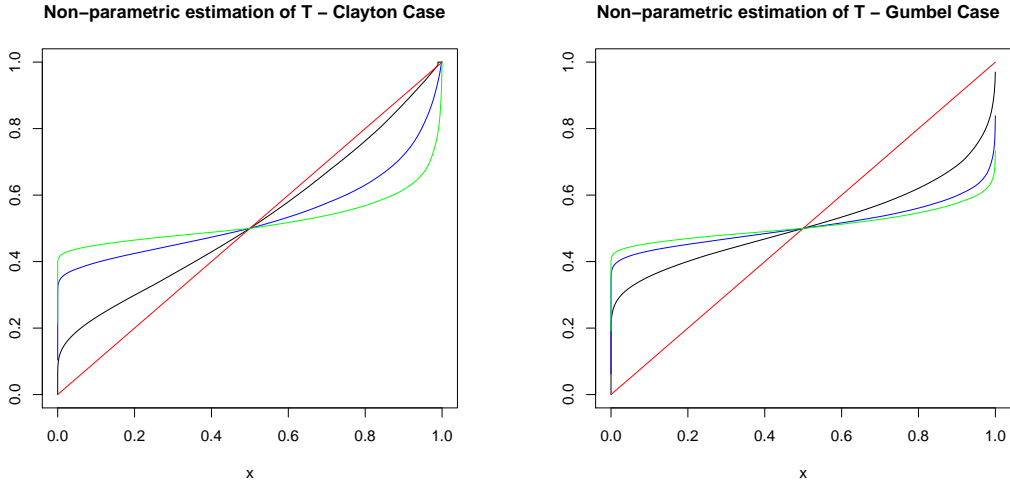


Figure 3: Non-parametric $\hat{T}(x)$ as in Proposition 4.2 estimated on a sample of size $n = 1500$. (Left) Clayton-case with parameter $\theta = 1$ (black line), $\theta = 6$ (blue right), $\theta = 10$ (green line). (Right) Gumbel-case with parameter $\theta = 3$ (black line), $\theta = 6$ (blue right), $\theta = 10$ (green line). The red line represents the bisectrix of the quadrant.

the theoretical auto-nested copula using a (possibly wrong) interpolator z as

$$\mathcal{C}_r^z(x) = z \left((z^{-1} \circ \mathcal{C}_k(x))^{1-\alpha} (z^{-1} \circ \mathcal{C}_{k+1}(x))^\alpha \right), \quad x \in [0, 1] \quad (23)$$

where $k = \lfloor r \rfloor$ and $\alpha = r - \lfloor r \rfloor$.

In Figure 6 we analyse the impact of the choice of the function z . Indeed this function drives the interpolation of \mathcal{C}_r , for $r \in \mathbb{R}$, knowing values of \mathcal{C}_k , for $k \in \mathbb{Z}$ (see Proposition 4.1). By Proposition 3.2, if known, the best choice for z is the generator ϕ of the copula C .

However we illustrate the error obtained by using another interpolation function. In particular, we denote

- $\mathcal{C}_r^{\text{Id}}$ theoretical auto-nested copula in (23) where z is the identity function $z(x) = x$ (linear interpolator),
- $\mathcal{C}_r^{\text{Gu}}$ theoretical auto-nested copula in (23) where $z(x) = \exp(-x)$ (Gumbel interpolator),

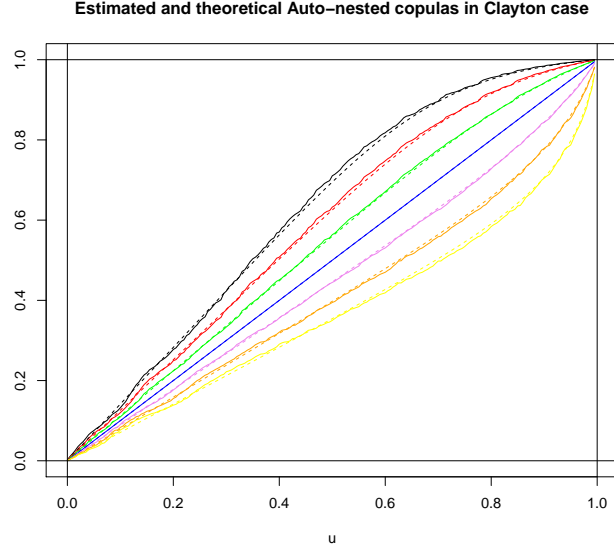


Figure 4: Estimation of auto-nested copula $\widehat{\mathcal{C}}_k(u)$ as in Proposition 4.1 in the Clayton-case with parameter $\theta = 6$ for $k = -3, -2, -1, 0, 1, 2, 3$. The estimated $\widehat{\mathcal{C}}_k(u)$ are represented using full lines, the theoretical one's using dotted lines. The black upper curve corresponds to $k = -3$, the yellow lower curve to $k = 3$.

- $\mathcal{C}_r^{\text{Cl}}$ theoretical auto-nested copula in (23) where $z(x) = (1 + (e^\theta - 1)x)^{-1/\theta}$ (Clayton interpolator).

In Figure 6 we consider a Clayton copula with parameter $\theta = 1$. In this case, by Proposition 3.2, the true theoretical auto-nested copulas in (23) are $\mathcal{C}_r^z = \mathcal{C}_r^{\text{Cl}}$, for $r \in \mathbb{R}$. We draw the theoretical errors $|\mathcal{C}_r^{\text{Cl}}(u) - \mathcal{C}_r^{\text{Id}}(u)|$ (Figure 6, left) and $|\mathcal{C}_r^{\text{Cl}}(u) - \mathcal{C}_r^{\text{Gu}}(u)|$ (Figure 6, right), for $u = 0.5$, as a function of $r \in [-15, 15]$. Trivially for $r = k \in \mathbb{N}$ the error is null since there is no interpolation procedure. For $r \in \mathbb{R} \setminus \mathbb{N}$ this error is not zero but however it is really small (< 0.01). In all cases, the induced relative error is less than 1.5%.

As a consequence, there are no visual differences in graphical representations of $\widehat{\phi}$ if using an interpolator or another (and such figures are omitted here). It should be noticed that, even if interpolation error is small, it can be easily reduced, if necessary, by replacing z by a previous estimation of $\widehat{\phi}$ at a step ν , then giving an estimation of $\widehat{\phi}$ at a step $\nu + 1$, $\nu \in \mathbb{N}$.

At last, we are looking for theoretical confidence bands for the estimated generator, in the Gumbel case, as detailed in Corollary 4.1. Let $\widetilde{\mathcal{C}}$ be a Gumbel copula of parameter $\theta = 2$. Corresponding estimators $\widehat{\mathcal{C}}_1$ and $\widehat{\mathcal{C}}_{-1}$ were build as previously, using a bivariate sample of size $n = 2000$. We just aim here at showing the shape of the confidence bands, so that we did not estimate constants $\alpha^-, \alpha^+, \beta^-, \beta^+$ such that

$$\begin{cases} \mathbb{P} \left[\widetilde{\mathcal{C}}(u)^{\alpha^-} \leq \widehat{\mathcal{C}}(u) \leq \widetilde{\mathcal{C}}(u)^{\alpha^+}, \forall u \in [0, 1] \right] \geq \eta, \\ \mathbb{P} \left[\widetilde{\mathcal{C}}_{-1}(u)^{\beta^-} \leq \widehat{\mathcal{C}}_{-1}(u) \leq \widetilde{\mathcal{C}}_{-1}(u)^{\beta^+}, \forall u \in [0, 1] \right] \geq \eta, \end{cases}$$

We have chosen for these constants some values $\alpha^- = 1.05$, $\alpha^+ = 0.9$, $\beta^- = 1.05$, $\beta^+ = 0.9$. For these chosen constants, the confidence bands for $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}}_{-1}$ are given in Figure 7 (left). This figure gives one path of $\widehat{\mathcal{C}}(u)$ (resp. for $\widehat{\mathcal{C}}_{-1}$) and band $[\mathcal{C}(u)^{\alpha^-}, \mathcal{C}(u)^{\alpha^+}]$ (resp. $[\mathcal{C}_{-1}(u)^{\alpha^-}, \mathcal{C}_{-1}(u)^{\alpha^+}]$) for chosen constants α^- and α^+ (resp. β^- and β^+). The resulting theoretical confidence bands for $\widehat{\phi}$ using Equations (20) and (21) are given in Figure 7 (right). Obviously, the confidence band around $\widehat{\phi}(t)$ gets narrow when t is close to $t_0 = 1$, since $\widehat{\phi}(t)$ is the chosen equivalent generator passing through $(t_0, \varphi_0) = (1, e^{-1})$.

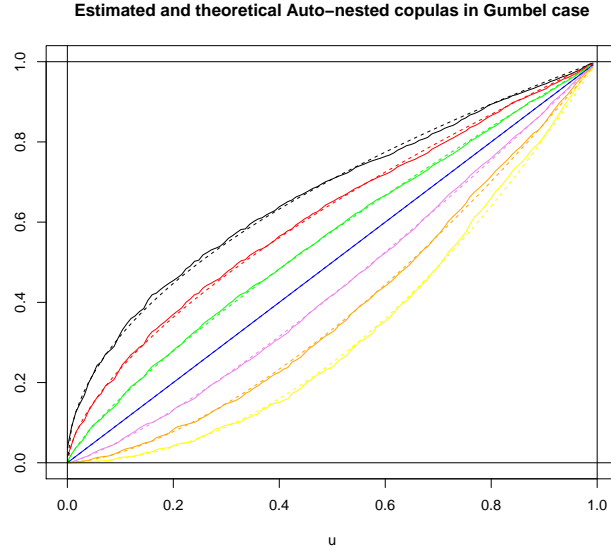


Figure 5: Estimation of auto-nested copula $\widehat{\mathcal{C}}_k(u)$ as in Proposition 4.1 in the Gumbel-case with parameter $\theta = 3$ for $k = -3, -2, -1, 0, 1, 2, 3$. The estimated $\widehat{\mathcal{C}}_k(u)$ are represented using full lines, the theoretical one's using dotted lines. The black upper curve corresponds to $k = -3$, the yellow lower curve to $k = 3$.

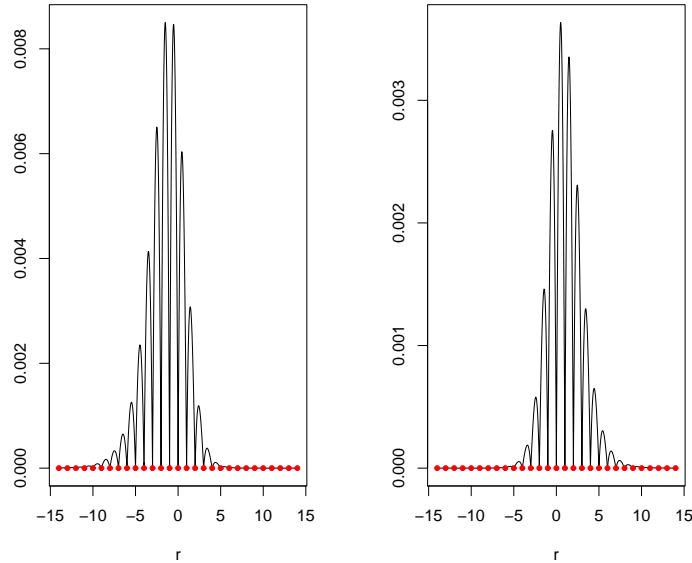


Figure 6: Theoretical errors $|\mathcal{C}_r^{\text{Cl}}(u) - \mathcal{C}_r^{\text{Id}}(u)|$ (left) and $|\mathcal{C}_r^{\text{Cl}}(u) - \mathcal{C}_r^{\text{Gu}}(u)|$ (right), for $u = 0.5$, as a function of $r \in [-15, 15]$. For $r = k \in \mathbb{N}$ the error is null (red points) since there is no interpolation procedure.

4.3.2 Real data illustration

We now propose the non-parametric estimation $\widehat{\phi}(t)$ using two real-data set (see Proposition 4.3). Firstly, we consider the **Loss-ALAE data** (for details see Frees and Valdez (1998)). The data size is $n = 1500$. Each claim consists of an indemnity payment (the loss, X) and an allocated loss adjustment expense (ALAE, Y). Examples of ALAE are the fees paid to outside attorneys, experts, and investigators used to defend claims.

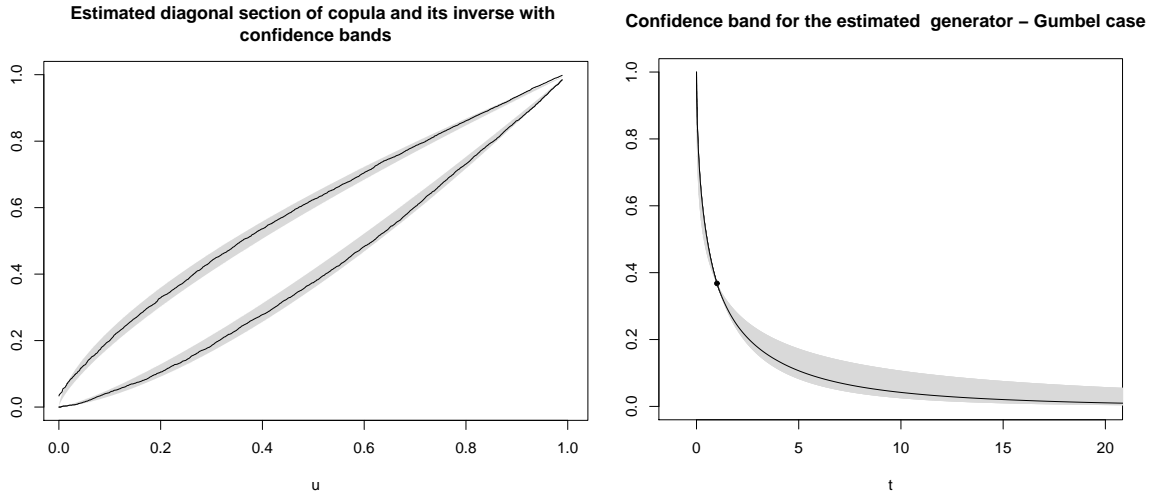


Figure 7: (Left) Confidence bands for \widehat{C}_1 and \widehat{C}_{-1} for chosen parameters $\alpha^- = \beta^- = 1.05$, $\alpha^+ = \beta^+ = 0.9$. (Right) Resulting confidence band for $\widehat{\phi}$. \widehat{C} is a Gumbel copula of parameter $\theta = 2$, the size of generated sample is $n = 2000$.

We take the independence initial copula C , $x_0 = y_0 = e^{-1}$ and $z(x) = \exp(-x)$ (Gumbel interpolator). The obtained non-parametric generator $\widehat{\phi}(t)$ is represented in Figure 8 (left). Different authors, in the recent literature, agree that a satisfying fit on these data can be represented by the Gumbel-Hougaard copula with parameter $\theta = 1.453$ (see for instance Frees and Valdez (1998) and Genest et al. (2009)). Then the standardized Gumbel generator with parameter $\theta = 1.453$ is also represented in Figure 8 in order to exhibit the quality of our non-parametric estimation.

Secondly, we consider a subset of the Framingham Heart study data (<http://www.framingham.com/heart/>). We focus on the dependence structure underlying the diastolic (DBP) and the systolic (SBP) blood pressures (in mm Hg) measured on 663 male subjects at their first visit (see Qu and Yin (2012)). Lambert (2007) proposed a ratio approximation of the Archimedean copula generator and he found that the Gumbel copula was appropriate for this data without being fully satisfactory. The estimated parameter of this Gumbel copula, $\theta = 2.11$, is given in Qu and Yin (2012). Then, in Figure 8 (right), we represent our estimation $\widehat{\phi}(t)$ and the standardized Gumbel generator with parameter $\theta = 2.11$. As we can see the non-parametric generator has a different form (in particular a different concavity) with respect to the analytical function $\phi(t) = \exp(-\frac{t}{2.11})$.

Conclusions

We describe some properties on distortions of Archimedean copulas, among which the characterization of an equivalence class for both distortions and generators. This characterization is necessary to build distortions and generators as function of what we call *auto-nested copulas* functions. Using their properties we propose a non-parametric estimator for the auto-nested copula functions, as well as for the distortions and the generators. This estimation is straightforward and does not rely on any optimization procedure. Then we can easily get convergence properties of such estimates. Numerical illustrations showed the simplicity of these estimators, the good fit to theoretical values in simulated example, the good fit to literature parametric adjustments in real-data problems. Furthermore, in the present work we start the investigation of the tail behavior of distorted Archimedean copulas. We prove some results that may be understood as constraints relying on the distortions, in order to obtain some desired tail coefficients for example. Such a study may help developing a further work on the estimation of the distortions around values 0 and 1, corresponding to extreme quantiles of the distorted distribution.

Some perspectives are the following ones: using results in Di Bernardino and Rullière (2013), we can

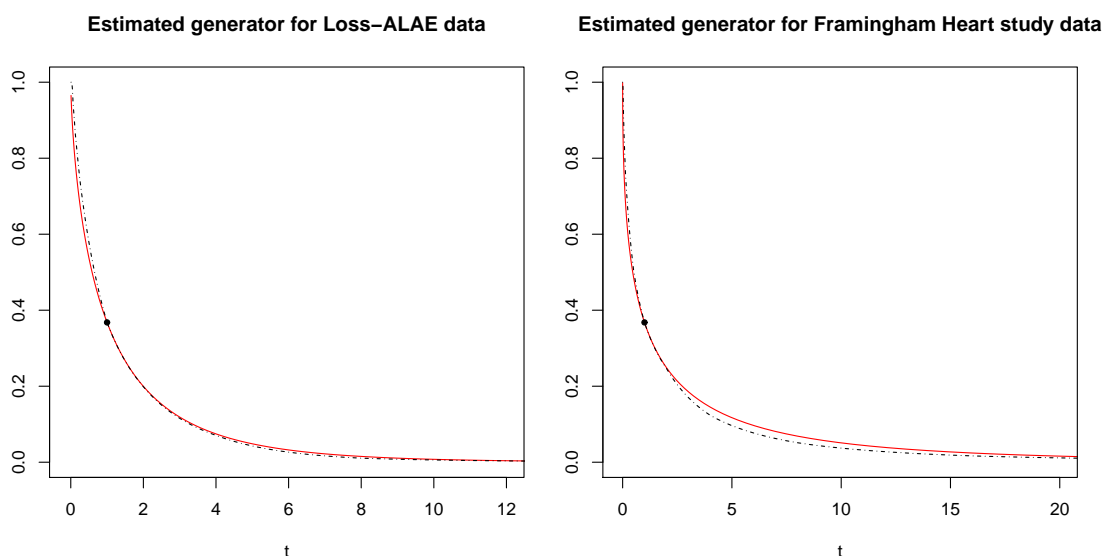


Figure 8: Non-parametric estimation of $\widehat{\phi}(t)$. (Left) Loss-ALAE data (black dotted line) and standardized Gumbel generator with parameter $\theta = 1.453$ (red line). (Right) Framingham Heart study data (black dotted line) standardized Gumbel generator with parameter $\theta = 2.11$ (red line). We force the generators to pass through the point $(t_0, \varphi_0) = (1, e^{-1})$ (black point).

get easily a whole parametric copula estimation, with a tunable number of parameters and without optimization procedures. The simplicity of the estimators may help the construction of nested copulas or to obtain complex dependence structures with many parameters. Such development may also ease the inversion and smoothing of the empirical copula as well as its tail estimation. At last, the measure of the goodness of fit and the construction of specific tests, based on the non-parametric estimated generator of a copula, are interesting perspectives.

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Annex

In this Annex we give the analytical formulas for standardized generators ($\bar{\phi}(t)$), their inverses ($\bar{\phi}^{-1}(t)$) and theoretical auto-nested copulas (\mathcal{C}_r), in the case of most popular Archimedean families of copulas.

In Table 1 we present some classical generators and their associated inverses (see Equation (1)), well known in the literature (see for instance Nelsen (1999)).

Copula	$\phi(t)$	$\phi^{-1}(t)$	parameter θ
Ali-Mikhail-Haq	$\frac{1-\theta}{\exp(t)-\theta}$	$\ln\left(\frac{1-\theta+\theta t}{t}\right)$	$\theta \in [0, 1)$
Clayton	$(1 + \theta t)^{-1/\theta}$	$\frac{1}{\theta}(t^{-\theta} - 1)$	$\theta \in (0, \infty)$
Frank	$-\frac{1}{\theta} \ln(1 - (1 - \exp(-\theta))e^{-t})$	$-\ln\left(\frac{\exp(-\theta t)-1}{\exp(-\theta)-1}\right)$	$\theta \in (0, \infty)$
Gumbel	$\exp(-t^{1/\theta})$	$(-\ln(t))^\theta$	$\theta \in [1, \infty)$
Independence	$\exp(-t)$	$(-\ln(t))$	none
Joe	$1 - (1 - \exp(-t))^{1/\theta}$	$-\ln(1 - (1 - t)^\theta)$	$\theta \in [1, \infty)$

Table 1: Classical generators and their associated inverses in the case of most popular Archimedean families of copulas.

Following Remarks 7 and 8, we give in Table 2 the equivalent theoretical generators associated to those presented in Table 1 and the associated inverses. In particular, let $(t_0, \varphi_0) \in \mathbb{R}_+^* \times (0, 1)$ and ϕ be a classical generator of an Archimedean copula as in Table 1. Then the standardized generator $\bar{\phi}$ is an equivalent generator of ϕ such that $\bar{\phi}(t_0) = \varphi_0$. We remark that :

$$C(u_1, \dots, u_d) = \bar{\phi}(\bar{\phi}^{-1}(u_1) + \dots + \bar{\phi}^{-1}(u_d)),$$

for all $(t_0, \varphi_0) \in \mathbb{R}^{+*} \times (0, 1)$.

In Table 2 we also provide the expressions for the theoretical auto-nested copulas $\mathcal{C}_r(u)$, for $r \in \mathbb{R}$ and $u \in (0, 1)$. Remark that $\mathcal{C}_0(u) = u$.

Copula	$\mathcal{C}_r(u)$	$\bar{\phi}(t)$	$\bar{\phi}^{-1}(t)$
Ali-Mikhail-Haq $\theta \in [0, 1)$	$\frac{1-\theta}{\left(\frac{1-\theta+\theta u}{u}\right)^{d^r} - \theta}$	$\frac{1-\theta}{\left(\frac{1-\theta+\theta \varphi_0}{\varphi_0}\right)^{t/t_0} - \theta}$	$t_0 \frac{\ln\left(\frac{1-\theta+\theta t}{t}\right)}{\ln\left(\frac{1-\theta+\theta \varphi_0}{\varphi_0}\right)}$
Clayton $\theta \in (0, \infty)$	$(1 + d^r (u^{-\theta} - 1))^{-1/\theta}$	$\left(1 + \frac{t}{t_0} (\varphi_0^{-\theta} - 1)\right)^{-1/\theta}$	$t_0 \left(\frac{t^{-\theta}-1}{\varphi_0^{-\theta}-1}\right)$
Frank $\theta \in (0, \infty)$	$-\frac{1}{\theta} \ln\left((1 - e^{-\theta}) \left(\frac{e^{-\theta u}-1}{e^{-\theta}-1}\right)^{d^r}\right)$	$-\frac{1}{\theta} \ln\left((1 - e^{-\theta}) \left(\frac{e^{-\theta \varphi_0}-1}{e^{-\theta}-1}\right)^{t/t_0}\right)$	$t_0 \frac{\ln\left(\frac{1-\exp(-\theta t)}{1-\exp(-\theta)}\right)}{\ln\left(\frac{1-\exp(-\theta \varphi_0)}{1-\exp(-\theta)}\right)}$
Gumbel $\theta \in [1, \infty)$	$u^{(d^r/\theta)}$	$\varphi_0^{((t/t_0)^{1/\theta})}$	$t_0 \left(\frac{\ln t}{\ln \varphi_0}\right)^\theta$
Independence	$u^{(d^r)}$	$\varphi_0^{(t/t_0)}$	$t_0 \frac{\ln t}{\ln \varphi_0}$
Joe $\theta \in [1, \infty)$	$1 - (1 - (1 - (1 - u)^\theta)^{d^r})^{1/\theta}$	$1 - (1 - (1 - (1 - \varphi_0)^\theta)^{t/t_0})^{1/\theta}$	$t_0 \frac{\ln(1 - (1 - t)^\theta)}{\ln(1 - (1 - \varphi_0)^\theta)}$

Table 2: Standardized generators $\bar{\phi}$, such that $\bar{\phi}(t_0) = \varphi_0$, their associated inverses and theoretical auto-nested copulas in the case of most popular Archimedean families of copulas.