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Dead-beat functional observers for discrete-time LPV systems with unknown inputs

Mirko Fiacchini*, Gilles Millerioux#

***Abstract*— This paper deals with functional observers for discrete-time Linear Parameter Varying (LPV) systems. More specifically, dead-beat observers for reconstructing linear combinations of the state are proposed. The system inputs are assumed to be unknown and the observers reduce to functions of the output over a finite number of time-steps. The existence of such observers is proven by resorting to the notion of inverse system together with the concepts of maximal robust invariant subspaces and nilpotent semigroups. A constructive approach to derive the explicit equations of the observer is provided. An example illustrates the efficiency and the computational aspects of the method.**

I. Introduction

Reconstructing the state vector of a system from its accessible outputs is an important issue in automatic control. In various contexts, the inputs of the system may be unknown, either because the control is not accessible as in some decentralized control setups or because the inputs actually stand for disturbances and faults, unknown by definition. Functional observers aim at reconstructing a particular (often linear) function of the state (possibly of the input as well). Their use is motivated by several arguments. Determining a particular set of states can be especially interesting for diagnosis, fault detection and isolation problems or for supervision. Indeed, in this context, one can focus on a restricted set of state variables. Furthermore, partial state reconstruction can be well suited to face computational and real-time stringent constraints, as often required for embedded or large-scale systems. Finally, functional observers are useful to directly estimate a feedback control without estimating each component of the state.

Functional observers for linear systems have been widely addressed since the pioneering work [18]. A complete framework providing necessary and sufficient conditions for convergence and design procedure has been presented in [12]. The books [10] and more recently [30] present a state of the art of the research in this field. Functional observers have been investigated for some classes of nonlinear systems with unknown inputs. For continuous-time systems, local considerations are used in [19], sliding mode techniques in [20] or LMI-based approaches in [30] taking into account the Lipschitz nature of the nonlinearities. The scarce consideration of nonlinear discrete-time systems with unknown inputs motivates the present work. In this paper we propose a dead-beat delayed functional observer for discrete-time LPV systems with unknown inputs. Let us notice that different terms are used in the literature to define the same concept: dead-beat, finite-time, fast or non-asymptotic.

Closely related works can be cited with their discrepancies. The paper [17] addresses the estimation of a nonlinear function of the state of nonlinear discrete-time systems but the inputs are supposed to be known. Delayed functional observers for discrete-time systems was one of the issues treated in [25] and [29] but the works exclusively deal with linear systems. Finite-time unknown inputs functional observers have been proposed in [15] for singular systems. Finite-time state reconstruction is also closely related to algebraic observability. Concerning this issue, the reader should refer to [13] [2] for continuous nonlinear systems and [16] [11] for linear and polynomial discrete-time systems. However, these works consider the problem of full state reconstruction and partial state reconstruction is not addressed. Algebraic observability was previously addressed in [24] in the context of optimal control as "perfect observability".

The approach proposed in this paper for the design of dead-beat functional observers is based on results of set-theory for control, in particular the concept of invariance. Invariance of a set or a subspace of the state space is related to many classical topics in control, such as stability, Lyapunov theory, constrained control. Invariance and set-theoretic methods in control appeared at the end of the sixties, see the pioneering work [6], and they raised an increasing interest in the last decades, see in particular the recent monograph [7] and references therein for an exhaustive overview. The characterization of invariant subspaces, strongly related also with the properties of controllability and observability, has been treated in [3]–[5]. More recently, invariant subspaces have been used in [1] for similar purposes in the context of LPV systems.

The outline of the paper follows. Section II is devoted to the problem statement. In Section III, a condition based on the notion of inverse system for the dead-beat linear functional observer to exist is given. Two approaches to check the condition in a tractable way are presented in Section IV. The concepts of invariant subspaces and nilpotent semigroups are used. The computational issues are addressed in Section V and an illustrative example is given in Section VI.

Notation: $\mathbb{N}$ is the set of natural numbers. Given $n \in \mathbb{N}$ denote $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$ and $\bar{\mathbb{N}}_n = \mathbb{N}_n \cup \{0\}$. Denote with $x^{(i)}$ the i-th entry of the vector x, with x_k the realization of the time-dependent vector x at time k. Denote with $\mathbf{I}_m$ the identity matrix of dimension m and with $\mathbf{0}_{p\times q}$ the matrix in $\mathbb{R}^{p\times q}$ whose entries are zero. Denote with P_i the i-th element of a set of matrices. We assume the convention that $\prod_{i=0}^{K} P_i = P_K \dots P_0$. For a matrix X, $X^\dagger$ stands for the classical Moore-Penrose generalized inverse of X.

* GIPSA-lab, Grenoble Campus, 11 rue des Mathématiques, BP 46, 38402 Saint Martin d'Hères Cedex, France. mirko.fiacchini@gipsa-lab.fr

Research Center for Automatic Control of Nancy (CRAN), Université de Lorraine, Nancy, France. gilles.millerioux@univ-lorraine.fr

II. Problem statement

Consider the discrete-time Linear Parameter Varying (LPV) system obeying

$$\begin{cases} x_{k+1} = A(\theta_k)x_k + B(\theta_k)u_k, \\ y_k = C(\theta_k)x_k + D(\theta_k)u_k, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input assumed to be unknown, $y_k \in \mathbb{R}^p$ is the output vector. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ depend on the parameter vector $\theta_k \in \mathbb{R}^s$ belonging to a known set Ω_θ, possibly unbounded.

We are interested in the following issue.

Problem 1: Given $G \in \mathbb{R}^{d\times n}$, provide a condition for the existence of a parameter-dependent function f_θ and $t_1, t_2 \in \mathbb{Z}$ such that

$$Gx_k = f_\theta(y_{k+t_1}, \ldots, y_{k+t_2}),$$

for every $x_k \in \mathbb{R}^n$ and every possible realization of the parameter $\theta_k \in \Omega_\theta$.

As defined in [28], the quantity Gx_k is called a rank-d linear functional. The function f_θ acts as a dead-beat delayed linear functional observer for reconstructing this functional. Having in mind the different applications mentioned in the introduction, it is noteworthy to point out that this problem is rather general. Indeed, every linear combination of the state can be selected by properly choosing G and then partial and full state reconstruction can be addressed.

The proposed approach to solve Problem 1 rests on the notion of inverse system of (1) which is detailed in the following definition.

Definition 1: An inverse system for (1) is a system for which, when driven by a sequence of output y_k of (1), the trajectory of its state vector $\hat{x}_k$ coincides with the trajectory of x_k (possibly up to a delay $r \geq 0$ referred to as inherent delay, by analogy with the term used in [26] for linear systems) whenever both state vectors share the same initialization.

Define $\mathscr{O}(\theta_{k:k+i}) = C(\theta_k)$ if $i = 0$ and

$$\mathscr{O}(\theta_{k:k+i}) = \begin{bmatrix} C(\theta_k) \\ C(\theta_{k+1})A(\theta_k) \\ \vdots \\ C(\theta_{k+i})\prod_{l=0}^{i-1} A(\theta_{k+l}) \end{bmatrix}, \qquad \text{if } i > 0,$$

the vectors

$$y_{k:k+i} = \begin{bmatrix} y_k^T, & y_{k+1}^T, & \ldots, & y_{k+i}^T \end{bmatrix}^T,$$
$$\theta_{k:k+i} = \begin{bmatrix} \theta_k^T, & \theta_{k+1}^T, & \ldots, & \theta_{k+i}^T \end{bmatrix}^T,$$

and the matrix $\bar{I} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0}_{m\times m\cdot(r-1)} \end{bmatrix}$. Define also $M(\theta_{k:k+i})$ in the following recursive way

$$M(\theta_{k:k+i}) = \begin{bmatrix} D(\theta_k) & \mathbf{0}_{p\times m\cdot(i-1)} \\ \mathscr{O}(\theta_{k:k+i})B(\theta_k) & M(\theta_{k+1:k+i}) \end{bmatrix},$$

with $M(\theta_{k:k}) = D(\theta_k)$. The following proposition is a straightforward extension of the result given in [27] for switched systems.

Proposition 1: If there exists an integer $r \geq 0$ fulfilling for every possible realization of the parameter $\theta_k \in \Omega_\theta$

$$rank\ M(\theta_{k:k+r}) - rank\ M(\theta_{k+1:k+r}) = rank \begin{bmatrix} B(\theta_k) \\ D(\theta_k) \end{bmatrix}, \tag{2}$$

then the system

$$\hat{x}_{k+r+1} = P_\theta(\theta_{k:k+r})\hat{x}_{k+r} + Q_\theta(\theta_{k:k+r})y_{k:k+r}, \tag{3}$$

with

$$\begin{aligned} P_\theta(\theta_{k:k+r}) &= A(\theta_k) - Q(\theta_{k:k+r})\mathscr{O}(\theta_{k:k+r}), \\ Q_\theta(\theta_{k:k+r}) &= B(\theta_k)\bar{I}M^\dagger(\theta_{k:k+r}), \end{aligned} \tag{4}$$

is an inverse system for (1).

Proof: Define $e_k = x_k - \hat{x}_{k+r}$ as the state reconstruction error. Analogously to [27], it can be shown that, from (1) and (3) and provided that (2) is satisfied, the dynamics of the error reads

$$e_{k+1} = P_\theta(\theta_{k:k+r})e_k, \tag{5}$$

Hence, if $e_0 = 0$, that is $\hat{x}_r = x_0$, then $e_k = 0$ and so $\hat{x}_{k+r} = x_k$ for every $k > 0$, which is in accordance with Definition 1. ■

III. Direct approach

The following result yields to characterize a solution to Problem 1.

Proposition 2: If there exists a positive $K \in \mathbb{N}$ and $P_\theta(\theta_{k+l:k+l+r})$ with $l \in \bar{\mathbb{N}}_K$ are such that

$$G\prod_{l=0}^{K} P_\theta(\theta_{k+l:k+l+r}) = 0, \tag{6}$$

for all $\theta_{k+l:k+l+r} \in \Omega_\theta^{r+1}$, for every $l \in \bar{\mathbb{N}}_K$, then $G\hat{x}_{k+K+r+1}$ is a function of the outputs y_{k+j} and the parameters θ_{k+j} with $j \in \bar{\mathbb{N}}_{K+r}$.

Proof: From (3) we have

$$\begin{aligned} \hat{x}_{k+K+r+1} = &\prod_{l=0}^{K} P_\theta(\theta_{k+l:k+l+r})\hat{x}_{k+r} + Q_\theta(\theta_{k+K:k+K+r})y_{k+K:k+K+r} \\ &+ \sum_{l=0}^{K-1}\prod_{t=l+1}^{K} P_\theta(\theta_{k+t:k+t+r})Q_\theta(\theta_{k+l:k+l+r})y_{k+l:k+l+r}, \end{aligned}$$

and then, whenever (6) holds, $G\hat{x}_{k+K+r+1}$ does not depend on the initial state $\hat{x}_{k+r}$ and reads

$$\begin{aligned} G\hat{x}_{k+K+r+1} &= GQ_\theta(\theta_{k+K:k+K+r})y_{k+K:k+K+r} \\ &+ G\sum_{l=0}^{K-1}\prod_{t=l+1}^{K} P_\theta(\theta_{k+t:k+t+r})Q_\theta(\theta_{k+l:k+l+r})y_{k+l:k+l+r}. \end{aligned} \tag{7}$$

■

From the definition of inverse system, condition (6) implies the following result.

Proposition 3: If K and $P_\theta(\theta_{k+l:k+l+r})$ with $l \in \bar{\mathbb{N}}_K$ are such that (6) holds then $G\hat{x}_{k+K+r+j} = Gx_{k+K+j}$ for all $j > 0$.

Proof: From (5) and (6), we have that $Ge_{k+K+1} = G\prod_{l=0}^{K} P_\theta(\theta_{k+l:k+l+r})e_k = 0$ for all $e_k \in \mathbb{R}^n$, which implies $G\hat{x}_{k+K+r+j} = Gx_{k+K+j}$ for all $j > 0$. ■

The following proposition is a direct consequence of Proposition 2, in particular considering the equation (7), and Proposition 3.

Proposition 4: If K and $P_\theta(\theta_{k+l:k+l+r})$ with $l \in \bar{\mathbb{N}}_K$ are such that (6) holds then Problem 1 is solved with f_θ given by

$$\begin{aligned} Gx_k = G\hat{x}_{k+r} = GQ_\theta(\theta_{k-1:k+r-1})y_{k-1:k+r-1} \\ +G\sum_{l=0}^{K-1}\prod_{t=l+1}^{K} P_\theta(\theta_{k+t-K-1:k+t+r-K-1}) \\ \cdot Q_\theta(\theta_{k+l-K-1:k+l+r-K-1})y_{k+l-K-1:k+l+r-K-1}, \end{aligned}$$

and the integers fulfill $t_1 = -K-1$ and $t_2 = r-1$.

In the remaining part of the paper we provide an approach to check whether the condition (6) is fulfilled and to reformulate Proposition 4 accordingly. First, notice that the parametric dependence of the matrices in (3) and (4) with respect to $\theta_{k:k+r} \in \Omega_\theta^{r+1}$ is not linear in general. However, given the set Ω_θ, there always exists a function p such that P_θ depends linearly on the parameter ρ_k defined as $\rho_k = p(\theta_{k:k+r})$. Consider for instance p determined by the functions of θ_{k+j}, with $j \in \bar{\mathbb{N}}_r$, appearing in the entries of $P_\theta(\theta_{k:k+r})$. A set $\Omega_\rho \subseteq \mathbb{R}^q$ such that $\rho_k \in \Omega_\rho$ if $\theta_{k:k+r} \in \Omega_\theta^{r+1}$ can always be determined as well. To simplify the notation, we define

$$P(\rho_k) = P_\theta(\theta_{k:k+r}), \quad Q(\rho_k) = Q_\theta(\theta_{k:k+r}), \quad \bar{y}_k = y_{k:k+r}.$$

Following the considerations above, $P(\rho_k)$ can be expressed as

$$P(\rho_k) = \sum_{i=1}^{q} P_i \rho_k^{(i)}, \tag{8}$$

for all $\rho_k \in \Omega_\rho$ and with Ω_ρ subset of $\mathbb{R}^q$ and Proposition 4 is reformulated as follows.

Theorem 1: There exists a solution to Problem 1 with $t_1 = -K-1$ and $t_2 = r-1$ if there is a positive $K \in \mathbb{N}$ and P_i, for $i \in \mathbb{N}_q$, are such that

$$G\prod_{k=0}^{K} P_{i_k} = 0, \tag{9}$$

for all $\mathscr{I} = [i_0, \ldots, i_K]^T \in \mathbb{N}_q^{K+1}$. The function f_θ satisfies

$$\begin{aligned} Gx_k = G\hat{x}_{k+r} = G\ Q(\rho_{k-1})\ \bar{y}_{k-1} \\ +G\sum_{j=0}^{K-1}\sum_{\mathscr{I}\in\mathbb{N}_q^{j+1}}\prod_{l=K-j}^{K} \rho_{k+l-K-1}^{(i_l)} P_{i_l} Q(\rho_{k-j-2})\ \bar{y}_{k-j-2}. \end{aligned} \tag{10}$$

Proof: By definition we have

$$\begin{aligned} G\hat{x}_{k+r+1} &= G\sum_{i_0\in\mathbb{N}_q} \rho_k^{(i_0)} P_{i_0}\hat{x}_{k+r} + GQ(\rho_k)\bar{y}_k, \\ G\hat{x}_{k+r+2} &= G\sum_{\mathscr{I}\in\mathbb{N}_q^2}\prod_{l=0}^{1} \rho_{k+l}^{(i_l)} P_{i_l}\hat{x}_{k+r} + GQ(\rho_{k+1})\bar{y}_{k+1} \\ &\quad +G\sum_{\mathscr{I}\in\mathbb{N}_q^1} \rho_k^{(i_0)} P_{i_0}\ Q(\rho_k)\,\bar{y}_k, \\ &\cdots \\ G\hat{x}_{k+r+K+1} &= G\sum_{\mathscr{I}\in\mathbb{N}_q^{K+1}}\prod_{l=0}^{K} \rho_{k+l}^{(i_l)} P_{i_l}\hat{x}_{k+r} + GQ(\rho_{k+K})\,\bar{y}_{k+K} \\ &\quad +G\sum_{j=0}^{K-1}\sum_{\mathscr{I}\in\mathbb{N}_q^{j+1}}\prod_{l=K-j}^{K} \rho_{k+l}^{(i_l)}\ P_{i_l}\ Q(\rho_{k+K-1-j})\ \bar{y}_{k+K-1-j}, \end{aligned}$$

where $\mathscr{I} = [i_0, \ldots, i_k]^T \in \mathbb{N}_q^{k+1}$ for all $k \in \bar{\mathbb{N}}_K$. If for every $\mathscr{I} \in \mathbb{N}_q^{K+1}$ condition (9) holds, then

$$\begin{aligned} G\hat{x}_{k+r+K+1} = GQ(\rho_{k+K})\,\bar{y}_{k+K} \\ +G\sum_{j=0}^{K-1}\sum_{\mathscr{I}\in\mathbb{N}_q^{j+1}}\prod_{l=K-j}^{K} \rho_{k+l}^{(i_l)}\ P_{i_l}\ Q(\rho_{k+K-1-j})\ \bar{y}_{k+K-1-j}. \end{aligned}$$

Finally, based on Propositions 3 and a suitable shift, we get (10) which completes the proof and shows that f_θ involves the output y_{k+l} (and the parameter θ_{k+l}) for $-K-1 \le l \le r-1$. ■

The point is that checking condition (9) can be computationally unaffordable since it requires to test the product of many sequences of $P(\rho_k)$, whose number can be exponential in K. In the following, two approaches are presented to obtain tractable conditions for (9) to hold.

IV. Invariant set-based approach

The concept of invariant subspaces is recalled here, see the monograph [5] for details.

Definition 2 (Invariant subspace): Given the matrix $A \in \mathbb{R}^{n\times n}$, a subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is an invariant if $A\mathscr{V} \subseteq \mathscr{V}$. Given the set of matrices $A(\rho) \subseteq \mathbb{R}^{n\times n}$, defined by $A : \mathbb{R}^q \to \mathbb{R}^{n\times n}$ for $\rho \in \Omega_\rho$, a subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is a robust invariant if $A(\rho)\mathscr{V} \subseteq \mathscr{V}$, for all $\rho \in \Omega_\rho$. Given a subspace $\mathscr{C} \subseteq \mathbb{R}^n$, the maximal invariant (resp. robust invariant) subspace contained in $\mathscr{C}$ is given by the sum of all invariant (resp. robust invariant) subspaces contained in $\mathscr{C}$.

In other words, $\mathscr{V}$ is an invariant (resp. robust invariant) subspace if every state trajectory generated by the map A (resp. $A(\rho)$) starting inside it, remains confined within it. The set of invariant (resp. robust invariant) subspaces contained in $\mathscr{C}$ admits a supremum. We recall here a property of the invariant subspaces, see [5].

Property 1 ([5]): Given a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, a subspace $\mathscr{V} \subseteq \mathbb{R}^n$ with matrix basis $T \in \mathbb{R}^{n\times g}$ is invariant if and only if there exists a matrix $X \in \mathbb{R}^{g\times g}$ such that

$$AT = TX. \tag{11}$$

Clearly the kernel of A, denoted $ker\,(A)$, is an invariant subspace. Indeed, if T is the basis matrix of $ker\,(A)$, it follows that $AT = 0$, and then (11) holds merely posing $X = 0$. Alternatively seen, if $x \in ker\,(A)$ then $Ax = 0 \in ker\,(A)$, and hence $ker\,(A)$ is invariant.

Definition 3: Given $P(\rho_k)$ as in (8), define

$$\mathscr{P} = \bigcap_{i=1}^{q} ker\left(P_i^T\right).$$

Moreover, denote with $M \in \mathbb{R}^{n\times g}$ a basis matrix of $\mathscr{P}$.

Notice that the definition of $\mathscr{P}$ above involves the transpose of the matrices P_i for all $i \in \mathbb{N}_q$.

Proposition 5: The set $\mathscr{P}$ as in Definition 3 is an invariant subspace for the LPV system given by $z_{k+1} = P_T(\rho_k)z_k$ with

$$P_T(\rho_k) = \sum_{i=1}^{q} \rho_k^{(i)} P_i^T = P(\rho_k)^T. \tag{12}$$

Proof: Since M is a basis matrix of $\mathscr{P}$, then $P_i^T M = 0$ for all $i \in \mathbb{N}_q$, which implies that $P_T(\rho_k)M = 0$. From Property 1, $\mathscr{P}$ is invariant for the system whose dynamics is given by (12) if and only if there exists X such that $P_T(\rho_k)M = MX$, that holds posing $X = 0$. ■

A. Kernel-based approach

A sufficient condition for Theorem 1 to hold, then for Problem 1 to admit a solution, follows.

Proposition 6: Consider the system (3), the set $\mathscr{P}$ and the matrix M as in Definition 3. If $G \in \mathbb{R}^{d\times n}$ is such that the columns of G^T are in $\mathscr{P}$, i.e. there is $X \in \mathbb{R}^{d\times g}$ such that $G = XM^T$, then there exists a solution to Problem 1 with $K = 0$.

Proof: From the definition of $\mathscr{P}$, if the rows of G are in $\mathscr{P}$ then $GP_i = 0$, for all $i \in \mathbb{N}_q$. Thus condition (9), sufficient for solving Problem 1, holds with $K = 0$. ■

The result in Proposition 6 is implicitly based on the fact that $\mathscr{P}$ is an invariant for the system (3) as proved in Proposition 5. On the other hand, this condition requires that the state of (3) reaches the invariant subspace in one step, i.e. $G\hat{x}_{k+r+1} = 0$. This is obviously restrictive but provides an interesting insight on the problem and a clue on its possible alternative solution.

B. Invariance-based approach

The definition introduced below is central for what follows.

Definition 4: Given the system (3) with $P(\rho_k)$ as in (8), denote: with $\mathscr{G}$ the maximal robust invariant subspace for $\{P_i\}_{i=1}^q \subseteq \mathbb{R}^{n\times n}$ contained in $ker\,(G)$; with g the dimension of $\mathscr{G}$; with $\bar{T} \in \mathbb{R}^{n\times g}$ its basis matrix and with $\hat{T} \in \mathbb{R}^{n\times h}$ a matrix such that $T = [\bar{T}, \hat{T}] \in \mathbb{R}^{n\times n}$ is nonsingular.

Based on the results presented in [5], the dynamics on the robust invariant subspace and on its complement can be characterized as follows.

Proposition 7: Given the system (3) with $P(\rho_k)$ as in (8), consider $\mathscr{G}$ and T as in Definition 4. Then $\mathscr{G}$ is invariant for $P(\rho_k)$ and there exist $\bar{P}(\rho_k) \in \mathbb{R}^{g\times g}$, $\tilde{P}(\rho_k) \in \mathbb{R}^{g\times h}$ and $\hat{P}(\rho_k) \in \mathbb{R}^{h\times h}$, linear in ρ_k and such that:

$$P(\rho_k) = T \begin{bmatrix} \bar{P}(\rho_k) & \tilde{P}(\rho_k) \\ \mathbf{0}_{h\times g} & \hat{P}(\rho_k) \end{bmatrix} T^{-1}, \tag{13}$$

for every $\rho_k \in \Omega_\rho$.

Proof: First notice that, by definition of invariant subspaces, if $\mathscr{G}$ is a robust invariant for $\{P_i\}_{i=1}^q$ then it is invariant for every single P_i. This implies that there exist $\bar{P}_i \in \mathbb{R}^{g\times g}$, $\tilde{P}_i \in \mathbb{R}^{g\times h}$, $\hat{P}_i \in \mathbb{R}^{h\times h}$ such that

$$P_i = \begin{bmatrix} \bar{T} & \hat{T} \end{bmatrix} \begin{bmatrix} \bar{P}_i & \tilde{P}_i \\ \mathbf{0}_{h\times g} & \hat{P}_i \end{bmatrix} \begin{bmatrix} \bar{T} & \hat{T} \end{bmatrix}^{-1},$$

for all $i \in \mathbb{N}_q$, being $\bar{T}$ a basis of $\mathscr{G}$. Then (13) holds by posing

$$\bar{P}(\rho_k) = \textstyle\sum_{i=1}^q \bar{P}_i \rho_k^{(i)},\ \tilde{P}(\rho_k) = \sum_{i=1}^q \tilde{P}_i \rho_k^{(i)},\ \hat{P}(\rho_k) = \sum_{i=1}^q \hat{P}_i \rho_k^{(i)},$$

for all $\rho_k \in \Omega_\rho$. Thus $\mathscr{G}$ is invariant for $P(\rho_k)$, from (11). ■

Based on Proposition 7, the following theorem, considered as the central theorem of the paper, can be stated as an alternative to Theorem 1.

Theorem 2: There exists a solution to Problem 1 with $t_1 = -K-1$ and $t_2 = r-1$ if there is a positive $K \in \mathbb{N}$ and P_i, for $i \in \mathbb{N}_q$, are such that

$$\prod_{k=0}^{K} \hat{P}_{i_k} = 0, \tag{14}$$

for all $\mathscr{I} = [i_0, \ldots, i_K]^T \in \mathbb{N}_q^{K+1}$. The function f_θ reads as (10).

Proof: By construction, $\bar{T}$ is a basis of the invariant subspace $\mathscr{G}$ that is contained in $ker\,(G)$ then $G\bar{T} = 0$. Hence, for all $\mathscr{I} = [i_0, \ldots, i_K]^T \in \mathbb{N}_q^{K+1}$, we have

$$\begin{aligned} G\prod_{k=0}^{K} P_{i_k} &= G\prod_{k=0}^{K} T \begin{bmatrix} \bar{P}_{i_k} & \tilde{P}_{i_k} \\ \mathbf{0}_{h\times g} & \hat{P}_{i_k} \end{bmatrix} T^{-1} \\ &= \prod_{k=0}^{K} G[\bar{T}, \hat{T}] \begin{bmatrix} \bar{P}_{i_k} & \tilde{P}_{i_k} \\ \mathbf{0}_{h\times g} & \hat{P}_{i_k} \end{bmatrix} T^{-1} = \begin{bmatrix} \mathbf{0}_{d\times g}, & G\hat{T}\prod_{k=0}^{K} \hat{P}_{i_k} \end{bmatrix} T^{-1} = 0. \end{aligned}$$

Thus condition (9) is satisfied for all $\mathscr{I} = [i_0, \ldots, i_K]^T \in \mathbb{N}_q^{K+1}$, which is sufficient for the existence of a solution to Problem 1. Finally, the expression of f_θ is obtained by following along the same lines as in the proof of Theorem 1. ■

The function f_θ stands for the expected dead-beat delayed linear functional observer written in terms of $P(\rho_k)$ and $Q(\rho_k)$. Finally, we are in position for giving a tractable approach for checking (14). To this end, let us recall the concept of nilpotent semigroups.

Definition 5 (Semigroup): A semigroup $\mathscr{S}$ is a set together with an associative internal law.

It is said to be finite if it has a finite number of elements. If $\mathscr{S}$ is a set of matrices, the associative internal law is the matrix multiplication and there exists an absorbing element which is the null matrix. In such a case, we can define a nilpotent semigroup of matrices.

Definition 6 (Nilpotent semigroup): A semigroup $\mathscr{S}$ of matrices with the multiplication law is said to be nilpotent if any product involving a finite number $t \in \mathbb{N}$ of its elements (possibly the same element) is always equal to the null matrix. The smallest integer t is called the class of nilpotency of $\mathscr{S}$.

Then, the relation (14) in Theorem 2 means that $\hat{P}_i$ with $i \in \mathbb{N}_q$ generate a nilpotent semigroup of class $t = K+1$. As a direct consequence, the following computation-oriented result follows.

Corollary 1: If $\hat{P}_i$ for all $i \in \mathbb{N}_q$, as defined in Proposition 7, generate a nilpotent semigroup of class t then Problem 1 is solved with $K = t-1$. The explicit function f_θ is given by (10) with $t_1 = -t$ and $t_2 = r-1$.

Remark 1: The class of nilpotency t is smaller than or equal to h, that is the dimension of the subspace complementary to $\mathscr{G}$ in $\mathbb{R}^n$. Then, the bigger the dimension of $\mathscr{G}$, the smaller such a bound, in general. Moreover, notice that for lower values of d, number of rows of G, the dimension of $\mathscr{G}$ can be higher and then the dimension of matrices $\hat{P}_i$ is smaller. This is reasonable, as the smaller is the partial state to be estimated through the inverse system, the smaller is the invariant subspace, on which the state projection must converge to 0 in finite time.

V. Computational issues

In this section, we first give some insights on the computational procedures for obtaining the maximal robust invariant subspace and the matrices $\hat{P}_i$ with $i \in \mathbb{N}_q$ of Proposition 7 involved in Theorem 2. Then some considerations on the condition for the matrices $\hat{P}_i$ with $i \in \mathbb{N}_q$ to generate a nilpotent semigroup, as stated in Corollary 1, are provided.

According to Definition 2, a subspace $\mathscr{V}$ is invariant for a linear system if its image through the linear transformation is contained in $\mathscr{V}$. Well established results on how to generate

invariant subspaces have been presented in literature, see in particular [5]. Interestingly for our purpose, the algorithm for computing invariant subspaces for LPV systems are directly obtained from those concerning linear systems, see [1], [5]. Every robust invariant subspace (then also the maximal one) contained in $\mathscr{V}$ for $\{A_i\}_{i=1}^q$ is a robust invariant subspace also for $A(\Omega_\rho)=\{A(\rho)=\sum_{i=1}^q A_i\rho^{(i)}:\rho\in\Omega_\rho\}$ for all $\Omega_\rho\subseteq\mathbb{R}^q$, as proved in Proposition 7. Given $\mathscr{A}\subseteq\mathbb{R}^{n\times n}$, every robust invariant for any $A(\Omega_\rho)$ such that $\mathscr{A}\subseteq A(\Omega_\rho)$ is invariant also for $\mathscr{A}$. Clearly, the smaller is the set $A(\Omega_\rho)$ containing $\mathscr{A}$, the closer are the sets of invariant subspaces for $A(\Omega_\rho)$ and for $\mathscr{A}$. The following algorithm provides a robust invariant subspace contained in $\mathscr{V}$ for every $\mathscr{A}\subseteq\mathbb{R}^{n\times n}$ whose elements are linear combinations of the matrices A_i with $i\in\mathbb{N}_q$.

Algorithm 1: Given the subspace $\mathscr{V}\subseteq\mathbb{R}^n$ and the set of matrices $\{A_i\}_{i=1}^q\subseteq\mathbb{R}^{n\times n}$, the maximal robust invariant subspace for $\{A_i\}_{i=1}^q$ contained in $\mathscr{V}$ is $\mathscr{J}_n$ provided by:

$$\begin{aligned}&\mathscr{J}_0=\mathscr{V},\\&\mathscr{J}_{k+1}=\mathscr{J}_0\cap\bigcap_{i\in\mathbb{N}_q}A_i^{-1}\mathscr{J}_k,\qquad k=0,\dots,n-1.\end{aligned}\tag{15}$$

Remark 2: It can be proved that the same sequence of subspaces $\mathscr{J}_k$ with $k=0,\dots,n$ is generated by the following iteration

$$\mathscr{J}_{k+1}=\mathscr{J}_k\cap\bigcap_{i\in\mathbb{N}_q}A_i^{-1}\mathscr{J}_k,\qquad k=0,\dots,n-1.\tag{16}$$

Remark 3: Notice that, with a slight abuse of notation, the meaning of the symbol A_i^{-1} does not denote the inverse of matrix A_i, but it is the operator that associates to a subspace its inverse image. That is, given the subspace $\mathscr{J}_k\subseteq\mathbb{R}^n$ one have $A_i^{-1}\mathscr{J}_k=\{x\in\mathbb{R}^n:A_ix\in\mathscr{J}_k\}$, which is defined also for singular and nilpotent matrices.

Remark 4: Algorithm 1, either implementing the iteration (15) or (16), is based, substantially, on the computation of the intersection between subspaces and of the inverse image of a subspace. We provide hereafter a sketch of the procedure. Consider two subspaces $\mathscr{V}\subseteq\mathbb{R}^n$ and $\mathscr{U}\subseteq\mathbb{R}^n$, whose basis matrices are $V\in\mathbb{R}^{n\times v}$ and $U\in\mathbb{R}^{n\times u}$. The subspace $\mathscr{V}\cap A^{-1}\mathscr{U}$ is given by the vectors $x\in\mathbb{R}^n$ that can be expressed as a linear combination of the columns of V and their image through A as a linear combination of those of U. Then, in practice, $x\in\mathscr{V}\cap A^{-1}\mathscr{U}$ if there exist $y\in\mathbb{R}^v$ and $z\in\mathbb{R}^u$ such that $x=Vy$ and $Ax=Uz$, that means such that $AVy=Uz$. Hence, $\mathscr{V}\cap A^{-1}\mathscr{U}$ can be obtained by computing the kernel of $[AV,\ U]$.

As for the computational aspects regarding the nilpotency property, involved in Corollary 1, we must recall the Levitsky's theorem (Theorem 2.1.7 stated in [23]).

Theorem 3 (Levitsky's theorem): Any semigroup of nilpotent matrices can be triangularized.

Thus, all the matrices of a same nilpotent semigroup can be triangularized by means of a common change of basis. The triangularization can be performed for example with the algorithm given in [14] and it leads to a substantial reduction of the computational complexity, see [21] or a detailed investigation in [22].

VI. Illustrative example

The example is devoted to the problem of rewriting a nonlinear discrete-time systems as an LPV system with accessible time-varying parameter θ_k. Such a problem has been investigated in [9] and called "exact LPV description". It is noteworthy to stress that LPV systems permit to model a wide class of nonlinear systems. Indeed, it is sufficient, for instance, that the nonlinear dynamical and output functions, respectively denoted f and h, are differentiable and such that $f(0,0)=0$ and $h(0,0)=0$ for having equivalence between the two models, see [8]. In case that the parameter is not accessible, the LPV model can be considered as a Linear Difference Inclusion (LDI) system. When the input is assumed to be unknown as it holds here, the problem is the search for a suitable function θ_k which can be expressed in terms of the output over a finite number of time-steps. In general, several such functions may exist and a possible approach is to resort to a functional observer, as illustrated in the following example.

Consider the controlled nonlinear system given by

$$\begin{cases}x_{k+1}^{(1)}&=&-4x_k^{(1)}+x_k^{(2)},\\x_{k+1}^{(2)}&=&3(x_k^{(1)})^2+2x_k^{(2)}x_k^{(1)}+x_k^{(3)}+u_k,\\x_{k+1}^{(3)}&=&-2x_k^{(1)}+u_k,\\y_k&=&x_k^{(1)},\end{cases}\tag{17}$$

where $u_k\in\mathbb{R}$ is the control supposed to be unknown. First, we choose $\theta_k=x_k^{(1)}$. The system can be rewritten as (1) with

$$A(\theta_k)=\begin{bmatrix}-4&1&0\\3\theta_k&2\theta_k&1\\-2&0&0\end{bmatrix},\ B(\theta_k)=\begin{bmatrix}0\\1\\1\end{bmatrix},\ C(\theta_k)=\begin{bmatrix}1&0&0\end{bmatrix},\ D(\theta_k)=0.$$

Clearly, the problem of exact LPV description of (17) is trivially solved by such a function θ_k since $\theta_k=x_k^{(1)}=y_k$ and so is accessible from the output.

Alternatively, choosing $\theta_k=3x_k^{(1)}+2x_k^{(2)}$, the system (17) results as in (1) with

$$A(\theta_k)=\begin{bmatrix}-4&1&0\\\theta_k&0&1\\-2&0&0\end{bmatrix},\ B(\theta_k)=\begin{bmatrix}0\\1\\1\end{bmatrix},\ C(\theta_k)=\begin{bmatrix}1&0&0\end{bmatrix},\ D(\theta_k)=0.\tag{18}$$

Now, we must check whether θ_k is accessible from the output. To this end, we search for a dead-beat linear functional observer to estimate θ_k, if it exists. The quantity Gx_k with $G=[3,2,0]$ is the rank-1 linear functional to be estimated. It turns out that (2) is fulfilled with $r=2$. Hence, the related inverse system (3) is characterized (see (4)) by the parameter-dependent matrix

$$P_\theta(\theta_{k:k+2})=\begin{bmatrix}-4&1&0\\-16&4&0\\-14-\theta_k&4&-1\end{bmatrix},$$

that is affine in $\theta_k\in\mathbb{R}$. Here, the function p such that the matrix $P_\theta(\theta_{k:k+2})$ depends linearly on a parameter ρ_k is merely $\rho_k=p(\theta_{k:k+2})=[1,\theta_k]^T$. The matrices P_i, as in (8), are

$$P_1=\begin{bmatrix}-4&1&0\\-16&4&0\\-14&4&-1\end{bmatrix},\quad P_2=\begin{bmatrix}0&0&0\\0&0&0\\-1&0&0\end{bmatrix}.$$

Consider Problem 1 with $G=[3,2,0]$, whose kernel basis is given by the vectors $[-2/3,1,0]^T$ and $[0,0,1]^T$. To check whether Theorem 2 is fulfilled, we compute the matrices $\hat{P}_i$,

as explained in Section V. Algorithm 1 generates a sequence of subspaces $\mathscr{J}_i$ and provides the maximal robust invariant subspace $\mathscr{J}_n$ in $ker(G)$, whose basis is $\bar{T} = [0,0,1]^T$. Considering the canonical basis matrix T, we obtain the matrices $\hat{P}_i$ as defined in Proposition 7

$$\hat{P}_1 = \begin{bmatrix} -4 & 1 \\ -16 & 4 \end{bmatrix}, \quad \hat{P}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Corollary 1 is fulfilled since $\hat{P}_1$ and $\hat{P}_2$ are both nilpotent (necessary condition) and generate a nilpotent semigroup with class of nilpotency $t = 2$. Hence, Problem 1 is solved with $K = 1$ and the explicit solution reads

$$Gx_k = G\hat{x}_{k+2} = 11y_k + 2y_{k+1}. \tag{19}$$

As the expression (19) does not depend on θ_k, it is a dead-beat delayed linear functional observer suitable to provide an exact LPV description of (17), replacing θ_k by $11y_k + 2y_{k+1}$ in (18).

VII. Conclusions

An approach for designing a dead-beat functional observer with unknown inputs for discrete-time LPV systems has been presented. Existence conditions have been derived. They are based on the notion of inverse system, invariant subspaces and nilpotent semigroups. The proof is constructive and an explicit formulation of the observer is provided. The results concern a large class of problems in control, including set-membership analysis or worst-case robust design.

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