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# CONDITIONED RANDOM WALKS FROM KAC-MOODY ROOT SYSTEMS

CÉDRIC LECOUCVEY, EMMANUEL LESIGNE AND MARC PEIGNÉ

ABSTRACT. Random paths are time continuous interpolations of random walks. By using Littelmann path model, we associate to each irreducible highest weight module of a Kac Moody algebra  $\mathfrak{g}$  a random path  $\mathcal{W}$ . Under suitable hypotheses, we make explicit the probability of the event  $E$ : “ $\mathcal{W}$  never exits the Weyl chamber of  $\mathfrak{g}$ ”. We then give the law of the random walk defined by  $\mathcal{W}$  conditioned by the event  $E$  and proves this law can be recovered by applying to  $\mathcal{W}$  the generalized Pitman transform introduced by Biane, Bougerol and O’Connell in [1]. This generalizes the main results of [15] and [10] to Kac Moody root systems and arbitrary highest weight modules. Moreover, we use here a completely new approach by exploiting the symmetry of our construction under the action of the Weyl group of  $\mathfrak{g}$  rather than renewal theory and Doob’s theorem on Martin kernels.

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## 1. INTRODUCTION

The purpose of the paper is to study conditionings of random walks using algebraic and combinatorial tools coming from representation theory of Lie algebras and their infinite-dimensional generalizations (Kac-Moody algebras). We extend in particular some results previously obtained in [15], [16], [1], [10] and [11] to random paths in the weight lattice of any Kac-Moody algebra  $\mathfrak{g}$ . To do this, we consider a fixed  $\mathfrak{g}$ -module  $V$  in the category  $\mathcal{O}_{int}$  (a convenient generalization of the category of Lie algebras finite dimensional representations). It decomposes as the direct sum of its weight spaces, each such space being parametrized by a vector of the weight lattice of  $\mathfrak{g}$ . The transitions of the random walk associated to  $V$  are then the weights of  $V$ .

The prototype of the results we obtain appears in the seminal paper [15] by O’Connell where it is shown that the law of the one-way simple random walk  $W$  in  $\mathbb{Z}^n$  conditioned to stay in the cone  $\mathcal{C} = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 \geq \dots \geq x_n \geq 0\}$  and with drift in the interior  $\overset{\circ}{\mathcal{C}}$  of  $\mathcal{C}$ , is the same as the law of a Markov chain  $H$  obtained by applying to  $W$  a generalization of the Pitman transform. This transform is defined via an insertion procedure on semistandard tableaux classically used in representation theory of  $\mathfrak{sl}_n(\mathbb{C})$ . The transition matrix of  $H$  can then be expressed in terms of the Weyl characters (Schur functions) of the irreducible  $\mathfrak{sl}_n(\mathbb{C})$ -modules. Here the transitions of the random walk  $W$  are the vectors of the standard basis of  $\mathbb{Z}^n$  which corresponds to the weights of the defining representation  $\mathbb{C}^n$  of  $\mathfrak{sl}_n(\mathbb{C})$ . In addition to the insertion procedure on tableaux and some classical facts about representation theory of  $\mathfrak{sl}_n(\mathbb{C})$ , the main ingredients of O’Connell’s result are a Theorem of Doob on Martin boundary together with the asymptotic behavior of tensor product multiplicities associated to the decompositions of  $V^{\otimes \ell}$  in its irreducible components.

We consider in [10] more general random walks  $W$  with transitions the weights of a finite-dimensional irreducible  $\mathfrak{g}$ -module  $V$  where  $\mathfrak{g}$  is a Lie algebra. The law of  $W$  is constructed so that the probabilities of the paths only depend of their lengths and their ends. We then show that the process  $H$  obtained by applying to  $W$  a generalization of the Pitman transform introduced in [1] is a Markov chain. When  $V$  is a minuscule representation (i.e. when the weights of  $V$  belong to the same orbit under the action of the Weyl group of  $\mathfrak{g}$ ) and  $W$  has drift in the interior  $\overset{\circ}{\mathcal{C}}$  of the cone  $\mathcal{C}$  of dominant weights, we prove that  $H$  has the same law as  $W$  conditioned to never exit  $\mathcal{C}$ . Similarly to the result of O’Connell, this common law can be expressed in terms of the Weyl characters of the simple  $\mathfrak{g}$ -modules. Nevertheless the methods differ from [15] notably because there was no previously known asymptotic behavior for the relevant tensor multiplicities in the more general cases we study. In fact, we deduce the convergence of the Martin kernel in Doob’s Theorem from a quotient renewal theorem that we establish for random walks conditioned to stay in a cone. In particular, the asymptotic behavior of the tensor multiplicities appears as a consequence of our

renewal theorem. When  $W$  is not defined from a minuscule representation, we also show that the transition matrix of  $H$  cannot be obtained as the Doob  $h$ -transform of the restriction of the transition matrix of  $W$  to  $\mathcal{C}$ . Therefore, the law of  $W$  with drift in  $\mathring{\mathcal{C}}$  conditioned to never exit  $\mathcal{C}$  does not coincide in this case with that of  $H$ .

In [11], we use the renewal theorem of [10] and insertion procedures on tableaux appearing in the representation theory of the Lie superalgebras  $\mathfrak{gl}(m, n)$  and  $\mathfrak{q}(n)$  to extend the results of [15] to one way simple random walks conditioned to never exit a cone  $\mathcal{C}'$  for examples of cones  $\mathcal{C}'$  different from  $\mathcal{C}$ .

In view of the results of [10], it is natural to ask whether the Markov chain  $H$  is related to a suitable conditioning of  $W$  in the non minuscule case. Also what can be said about the law of  $W$  conditioned to never exit  $\mathcal{C}$ ? In the sequel, we will answer both questions (partially for the second) not only for random walks defined from representations of Lie algebras but, more generally, for similar random walks with transition the set of weights of a highest weight module  $V(\kappa)$  in the category  $\mathcal{O}_{int}$  associated to a Kac-Moody algebra  $\mathfrak{g}$  of rank  $n$  (see [6] for a detailed exposition on Kac-Moody algebras).

By using Littelmann path model [13], one can associate to  $V(\kappa)$  a set of piecewise continuous linear paths  $B(\pi_\kappa)$  in the weight lattice  $P \subset \mathbb{R}^n$  of  $\mathfrak{g}$  such that all the paths in  $B(\pi_\kappa)$  have the same finite length. These paths (called elementary in the sequel) are regarded as functions  $\pi : [0, 1] \rightarrow \mathbb{R}^n$  such that  $\pi(0) = 0$  and  $\pi(1) \in P$ . The weights of  $V(\kappa)$  are then the elements  $\pi(1), \pi \in B(\pi_\kappa)$ . The set  $B(\pi_\kappa)$  depends a priori on the choice of a path from 0 to  $\kappa$  such that  $\pi(t) \in \mathcal{C}$  for any  $t \in [0, 1]$ . Here  $\mathcal{C}$  is the Weyl chamber of  $\mathfrak{g}$ . It is also endowed with the structure of a colored and oriented graph, and Littelmann shows in [13] that this graph structure is independent of the choice of the path  $\pi_\kappa$ . It is also known that this graph is isomorphic to the crystal graph of  $V(\kappa)$  as defined by Kashiwara from the representation theory of the quantum group  $U_q(\mathfrak{g})$ .

Once we have fixed the path  $\pi_\kappa$ , we use the crystal graph structure on  $B(\pi_\kappa)$  to endow it as in [10] with a probability density  $p$ . This yields a random variable  $X$  defined on  $B(\pi_\kappa)$  with probability distribution  $p$ . Let  $(X_\ell)_{\ell \geq 1}$  be a sequence of i.i.d. random variables with the same law as  $X$ . We then define a random path  $\mathcal{W}$  such that for any  $t \geq 0$ ,  $\mathcal{W}(t) = X_1(1) + \dots + X_{\ell-1}(1) + X_\ell(\ell - t)$  for any  $t \in [\ell - 1, \ell]$ . The sequence  $W = (W_\ell)_{\ell \geq 0}$  defined by  $W_\ell = \mathcal{W}(\ell)$  is then a random walk with transitions the weights of  $V(\kappa)$ . The main result of the paper is that, when  $W$  has drift in  $\mathring{\mathcal{C}}$ , the law of its conditioning by the event  $E = (\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0)$  coincides with the law of the Markov chain  $H$  obtained as the image of  $W$  by the generalized Pitman transform. This generalized Pitman transform is defined as in [1]. It sends a path in  $\mathbb{R}^n$  on a path in  $\mathcal{C}$  by using the crystal structure on paths. In particular, the law of  $H$  does not depend on the path  $\pi_\kappa$  chosen. Nevertheless our methods differ from [15] and [10]. Indeed, we do not use here any Doob theorem neither any renewal theorem. Our strategy is rather to exploit the symmetry of the representations with respect to the Weyl group  $W$  of  $\mathfrak{g}$  and study simultaneously a family of random paths  $\mathcal{W}^w$  indexed by the elements  $w \in W$ . Using the Kac-Weyl character formula, we are able to compute the harmonic function  $\psi$  defined on  $\mathcal{C} \cap P$  by  $\psi(\mu) = \mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0)$ . Our main result then follows from the computation of the transition matrix of the Markov chain  $H$ . Note that there are complications due to the fact that the representations we consider are infinite dimensional when  $\mathfrak{g}$  is not a Lie algebra. We have to deal with infinite series of reals but we prove that, under simple hypotheses, these series have tractable convergence properties.

The paper is organized as follows. In Section 2, we introduce the notion of random walk and random path used in the paper. Section 3 recalls the necessary background on Kac-Moody algebras and their representations and summarize some important results on Littelmann's path model. The

random path  $\mathcal{W}$  and the random walk  $W$  associated to  $V(\kappa)$  are introduced in Section 4 together with the generalized Pitman transform and the Markov chain  $H$ . In Section 5, we use a process of symmetrization to define the random paths  $\mathcal{W}^w, w \in W$  from  $\mathcal{W} = \mathcal{W}^1$ . This allows us to give an explicit expression of the harmonic function  $\psi$  in Section 6 and prove our main theorem. Finally Section 7 is devoted to additional results: we give asymptotic behavior of tensor power multiplicities and also compare the probabilities  $\mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C})$  for any  $t \geq 0$  and  $\mathbb{P}_\mu(W_\ell \in \mathcal{C})$  for any  $\ell \geq 0$ .

## 2. RANDOM PATHS

**2.1. Background on Markov chains.** Consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a countable set  $M$ . A sequence  $Y = (Y_\ell)_{\ell \geq 0}$  of random variables defined on  $\Omega$  with values in  $M$  is a *Markov chain* when

$$\mathbb{P}(Y_{\ell+1} = \mu_{\ell+1} \mid Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) = \mathbb{P}(Y_{\ell+1} = \mu_{\ell+1} \mid Y_\ell = \mu_\ell)$$

for any  $\ell \geq 0$  and any  $\mu_0, \dots, \mu_\ell, \mu_{\ell+1} \in M$ . The Markov chains considered in the sequel will also be assumed time homogeneous, that is  $\mathbb{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu) = \mathbb{P}(Y_\ell = \lambda \mid Y_{\ell-1} = \mu)$  for any  $\ell \geq 1$  and  $\mu, \lambda \in M$ . For all  $\mu, \lambda$  in  $M$ , the transition probability from  $\mu$  to  $\lambda$  is then defined by

$$\Pi(\mu, \lambda) = \mathbb{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu)$$

and we refer to  $\Pi$  as the transition matrix of the Markov chain  $Y$ . The distribution of  $Y_0$  is called the initial distribution of the chain  $Y$ .

In the sequel, we will assume that  $M$  is a subset of the euclidean space  $\mathbb{R}^n$  for some  $n \geq 1$  and that the initial distribution of the Markov chain  $Y = (Y_\ell)_{\ell \geq 0}$  has full support, i.e.  $\mathbb{P}(Y_0 = \lambda) > 0$  for any  $\lambda \in M$ . In [10], we have considered a nonempty set  $\mathcal{C} \subset M$  and an event  $E \in \mathcal{T}$  such that  $\mathbb{P}(E \mid Y_0 = \lambda) > 0$  for all  $\lambda \in \mathcal{C}$  and  $\mathbb{P}(E \mid Y_0 = \lambda) = 0$  for all  $\lambda \notin \mathcal{C}$ ; this implied that  $\mathbb{P}(E) > 0$ , we could thus define the conditional probability  $\mathbb{Q}$  relative to this event:  $\mathbb{Q}(\cdot) := \mathbb{P}(\cdot | E)$ . For example, we considered the event  $E := (Y_\ell \in \mathcal{C} \text{ for any } \ell \geq 0)$ . In the present work we will study more general situations, this involves to introduce some generalities about continuous time Markov processes.

A *continuous time Markov process*  $\mathcal{Y} = (\mathcal{Y}(t))_{t \geq 0}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $\mathbb{R}^n$  is a family of random variables defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  such that, for any integer  $k \geq 1$ , any  $0 \leq t_1 < \dots < t_{k+1}$  and any Borel subsets  $B_1, \dots, B_{k+1}$  of  $\mathbb{R}^n$ , one gets

$$\mathbb{P}(\mathcal{Y}(t_{k+1}) \in B_{k+1} \mid Y(t_1) \in B_1, Y(t_2) \in B_2, \dots, Y(t_k) \in B_k) = \mathbb{P}(Y(t_{k+1}) \in B_{k+1} \mid Y(t_k) \in B_k).$$

This is the Markov property, that we will use very often. In the following, we will use a version of this property concerning more complicated Borel sets which is a consequence of the above. One can indeed show that for any  $T \geq 0$  and any Borel sets  $A \subset (\mathbb{R}^n)^{\otimes [0, T]}$ ,  $B \subset \mathbb{R}^n$  and  $C \subset (\mathbb{R}^n)^{\otimes [T, +\infty[}$ , one gets

$$\mathbb{P}((\mathcal{Y}(t))_{t \geq T} \in C \mid (\mathcal{Y}(t))_{0 \leq t \leq T} \in A, Y(T) \in B) = \mathbb{P}((\mathcal{Y}(t))_{t \geq T} \in C \mid Y(T) \in B).$$

In the sequel, we will assume the two following conditions:

(1) for any integer  $\ell \geq 0$ , one gets

$$(1) \quad Y_\ell := \mathcal{Y}(\ell) \in M \quad \mathbb{P}\text{-almost surely}$$

It readily follows that the sequence  $Y = (Y_\ell)_{\ell \geq 0}$  is a  $M$ -valued Markov chain.

(2) for any  $0 \leq s \leq t$  and any Borel subsets  $A, B \in \mathbb{R}^n$

$$(2) \quad \mathbb{P}(\mathcal{Y}(t+1) \in B \mid \mathcal{Y}(s+1) \in A) = \mathbb{P}(\mathcal{Y}(t) \in B \mid \mathcal{Y}(s) \in A).$$

Combining this condition with the Markov property, one checks that for any  $T \geq 1$  and  $x \in \mathbb{R}^n$ , the conditional distribution of the process  $(\mathcal{Y}(t+1))_{t \geq T}$  with respect to the event  $(\mathcal{Y}(T+1) = x)$  is equal to the one of  $(\mathcal{Y}(t))_{t \geq T}$  with respect to  $(\mathcal{Y}(T) = x)$ .

In the sequel, we will assume that the initial distribution of the Markov process  $(\mathcal{Y}(t))_{t \geq 0}$  has full support, i.e.  $\mathbb{P}(\mathcal{Y}(0) = \lambda) > 0$  for any  $\lambda \in M$ . We will also consider a nonempty set  $\mathcal{C} \subset \mathbb{R}^n$  and will assume that the probability of the event  $E := (\mathcal{Y}(t) \in \mathcal{C} \text{ for any } t \geq 0)$  is positive; the conditional probability  $\mathbb{Q}$  relative to  $E$  is thus well defined.

**Proposition 2.1.** *Let  $(\mathcal{Y}(t))_{t \geq 0}$  be a continuous time Markov process with values in  $\mathbb{R}^n$  satisfying conditions (1) and (2) and  $\mathcal{C} \subset \mathbb{R}^n$  such that the event  $E := (\mathcal{Y}(t) \in \mathcal{C} \text{ for any } t \geq 0)$  has positive probability measure. Then, under the probability  $\mathbb{Q}(\cdot) = \mathbb{P}(\cdot | E)$ , the sequence  $(Y_\ell)_{\ell \geq 0}$  is still a Markov chain with values in  $\mathcal{C} \cap M$  and transition probabilities given by*

$$(3) \quad \forall \mu, \lambda \in \mathcal{C} \cap M \quad \mathbb{Q}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu) = \Pi^E(\mu, \lambda) \frac{\mathbb{P}(E \mid Y_0 = \lambda)}{\mathbb{P}(E \mid Y_0 = \mu)}$$

where  $\Pi^E(\mu, \lambda) = \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [\ell, \ell + 1] \mid Y_\ell = \mu)$ . We will denote by  $Y^E$  this Markov chain

*Proof.* By definition of the probability  $\mathbb{Q}$ , for any  $\ell \geq 1$  and any  $\mu_0, \dots, \mu_\ell, \lambda \in \mathcal{C}$ , one gets

$$\begin{aligned} \mathbb{Q}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) &= \frac{\mathbb{Q}(Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0)}{\mathbb{Q}(Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0)} \\ &= \frac{\mathbb{P}(E, Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0)}{\mathbb{P}(E, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0)} =: \frac{N_\ell}{D_\ell}. \end{aligned}$$

We first have, using the Markov property

$$\begin{aligned} N_\ell &= \mathbb{P}(\mathcal{Y}(t) \in \mathcal{C} \text{ for } t \geq 1, Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\ &= \mathbb{P}(\mathcal{Y}(t) \in \mathcal{C} \text{ for } t \geq \ell + 1 \mid Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell + 1], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\ &\quad \times \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell + 1], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\ &= \mathbb{P}(\mathcal{Y}(t) \in \mathcal{C} \text{ for } t \geq \ell + 1 \mid Y_{\ell+1} = \lambda) \\ &\quad \times \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell + 1], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\ &= \mathbb{P}(\mathcal{Y}(t) \in \mathcal{C} \text{ for } t \geq 0 \mid Y_0 = \lambda) \\ &\quad \times \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell + 1], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \end{aligned}$$

with

$$\begin{aligned} &\mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell + 1], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\ &= \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [\ell, \ell + 1] \mid \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\ &\quad \times \mathbb{P}(\mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\ &= \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [\ell, \ell + 1] \mid Y_\ell = \mu_\ell) \times \mathbb{P}(\mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0). \end{aligned}$$

We therefore obtain

$$\begin{aligned} N_\ell &= \mathbb{P}(E \mid Y_0 = \lambda) \times \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [\ell, \ell + 1] \mid Y_\ell = \mu_\ell) \\ &\quad \times \mathbb{P}(\mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0). \end{aligned}$$

A similar computation yields

$$D_\ell = \mathbb{P}(E \mid Y_\ell = \mu_\ell) \times \mathbb{P}(\mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [0, \ell], Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0).$$

Therefore

$$\begin{aligned} \mathbb{Q}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) &= \\ &= \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [\ell, \ell + 1] \mid Y_\ell = \mu_\ell) \times \frac{\mathbb{P}(E \mid Y_0 = \lambda)}{\mathbb{P}(E \mid Y_0 = \mu_\ell)}. \end{aligned}$$

□

In the sequel, to simplify the notations we will denote  $\mathcal{C}$  the set  $\mathcal{C} \cap M$  as soon as we will consider the Markov chain  $(Y_\ell)_{\ell \geq 0}$  and  $\Pi^E = (\Pi(\mu, \lambda))_{\mu, \lambda \in \mathcal{C}}$  the “restriction” of the transition matrix  $\Pi$  to the event  $E$  defined by: for any  $\mu, \lambda \in \mathcal{C}$

$$\Pi^E = (\Pi^E(\lambda, \mu))_{\lambda, \mu \in \mathcal{C}} \text{ where } \Pi^E(\lambda, \mu) = \mathbb{P}(Y_{\ell+1} = \lambda, \mathcal{Y}(t) \in \mathcal{C} \text{ for } t \in [\ell, \ell + 1] \mid Y_\ell = \mu).$$

A *substochastic matrix* on the countable set  $M$  is a map  $\Pi : M \times M \rightarrow [0, 1]$  such that  $\sum_{y \in M} \Pi(x, y) \leq 1$  for any  $x \in M$ . If  $\Pi, \Pi'$  are substochastic matrices on  $M$ , we define their product  $\Pi \times \Pi'$  as the substochastic matrix given by the ordinary product of matrices:

$$\Pi \times \Pi'(x, y) = \sum_{z \in M} \Pi(x, z) \Pi'(z, y).$$

A function  $h : M \rightarrow \mathbb{R}$  is *harmonic* for the substochastic transition matrix  $\Pi$  when we have  $\sum_{y \in M} \Pi(x, y) h(y) = h(x)$  for any  $x \in M$ . Consider a (strictly) positive harmonic function  $h$ . We can then define the Doob transform of  $\Pi$  by  $h$  (also called the  $h$ -transform of  $\Pi$ ) setting

$$\Pi_h(x, y) = \frac{h(y)}{h(x)} \Pi(x, y).$$

We then have  $\sum_{y \in M} \Pi_h(x, y) = 1$  for any  $x \in M$ . Thus  $\Pi_h$  can be interpreted as the transition matrix for a certain Markov chain.

An example is given in formula (3): the state space is now  $\mathcal{C}$ , the substochastic matrix is  $\Pi^E$  and the harmonic function is  $h_E(\mu) := \mathbb{P}(E \mid Y_0 = \mu)$ ; the transition matrix  $\Pi_{h_E}^E$  is the transition matrix of the Markov chain  $Y^E$ .

**2.2. Elementary random paths.** Consider a  $\mathbb{Z}$ -lattice  $P$  with finite rank  $d$ . Set  $P_{\mathbb{R}} = P \otimes_{\mathbb{Z}} \mathbb{R}$  so that  $P$  can be regarded as a  $\mathbb{Z}$ -lattice of rank  $d$  in  $\mathbb{R}^d$ . An *elementary path* is a piecewise continuous linear map  $\pi : [0, 1] \rightarrow P_{\mathbb{R}}$  such that  $\pi(0) = 0$ . Two paths  $\pi_1$  and  $\pi_2$  are considered as identical if there exists a piecewise, surjective continuous and nondecreasing map  $u : [0, 1] \rightarrow [0, 1]$  such that  $\pi_2 = \pi_1 \circ u$ .

The set  $\mathcal{F}$  of continuous functions from  $[0, 1]$  to  $P_{\mathbb{R}}$  is equipped with the norm  $\|\cdot\|_{\infty}$  of uniform convergence: for any  $\pi \in \mathcal{F}$ , on has  $\|\pi\|_{\infty} := \sup_{t \in [0, 1]} \|\pi(t)\|_2$  where  $\|\cdot\|_2$  denotes the euclidean norm on  $\mathbb{R}^d$ . Let  $B$  be a *countable set of paths with the same norm* and fix a probability distribution  $p = (p_{\pi})_{\pi \in B}$  on  $B$  such that  $p_{\pi} > 0$  for any  $\pi \in B$ . Let  $X$  be a random variable defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and with distribution  $p$  (in other words  $\mathbb{P}(X = \pi) = p_{\pi}$  for any  $\pi \in B$ ). Since the paths in  $B$  have the same norm, the variable  $X$  admits a moment of order 1 : namely  $\mathbb{E}(\|X\|) < +\infty$ , the series of functions  $\sum_{\pi} p_{\pi} \pi$  thus converges uniformly on  $[0, 1]$  and the function

$$m := \mathbb{E}(X) = \sum_{\pi \in B} p_{\pi} \pi$$

is continuous on  $[0, 1]$ .

The concatenation  $\pi_1 * \pi_2$  of two elementary paths  $\pi_1$  and  $\pi_2$  is defined by

$$\pi_1 * \pi_2(t) = \begin{cases} \pi_1(2t) & \text{for } t \in [0, \frac{1}{2}], \\ \pi_1(1) + \pi_2(2t - 1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

In the sequel,  $\mathcal{C}$  is a closed convex cone in  $P_{\mathbb{R}}$  with interior  $\overset{\circ}{\mathcal{C}}$  and we set  $P_+ = \mathcal{C} \cap P$ .

**2.3. Random paths.** Let  $B$  be a set of elementary paths and  $(X_\ell)_{\ell \geq 1}$  a sequence of i.i.d. random variables with law  $X$  where  $X$  is the random variable with values in  $B$  introduced in 2.2. We define the random process  $\mathcal{W}$  as follows: for any  $\ell \in \mathbb{Z}_{>0}$  and  $t \in [\ell, \ell + 1]$

$$\mathcal{W}(t) := X_1(1) + X_2(1) + \cdots + X_{\ell-1}(1) + X_\ell(t - \ell).$$

The sequence of random variables  $W = (W_\ell)_{\ell \geq 0} := (\mathcal{W}(\ell))_{\ell \geq 0}$  is a random walk with set of increments  $I := \{\pi(1) \mid \pi \in B\}$ .

For any  $\ell \geq 1$ , let  $\psi_\ell$  be the map defined by

$$\forall \mu \in \mathcal{C} \quad \psi_\ell(\mu) = \mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, \ell])$$

so that  $\psi_\ell(\mu)$  is the probability that  $\mathcal{W}$  starting at  $\mu$  remains in  $\mathcal{C}$  for any  $t \in [0, \ell]$ . As  $\ell \rightarrow +\infty$ , the sequence of functions  $(\psi_\ell)_{\ell \geq 0}$  converges to the function  $\psi$  defined by

$$\forall \mu \in \mathcal{C} \quad \psi(\mu) = \mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0).$$

**Proposition 2.2.** *Assume there exists a path  $\pi \in B$  such that  $\text{Im } \pi \subset \mathring{\mathcal{C}}$ . For any  $\mu \in \mathcal{C}$ , we then have  $\psi(\mu) > 0$  when  $m(1) \in \mathring{\mathcal{C}}$  and  $\psi(\mu) = 0$  otherwise.*

*Proof.* Observe that  $\psi(\mu) = \mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0) \leq \mathbb{P}_\mu(W_\ell \in \mathcal{C} \text{ for any } \ell \geq 0)$ . By a straightforward application of the strong law of large numbers (see [10] for more details), we have  $\mathbb{P}_\mu(W_\ell \in \mathcal{C} \text{ for any } \ell \geq 0) = 0$  when  $m(1) \notin \mathring{\mathcal{C}}$ . Thus  $\psi(\mu) = 0$  when  $m(1) \notin \mathring{\mathcal{C}}$ . Since  $B$  contains a path  $\pi_0$  such that  $\text{Im } \pi_0 \subset \mathring{\mathcal{C}}$ , we have, setting  $\nu = \pi_0(1) \in \mathring{\mathcal{C}}$

$$\forall k \geq 1 \quad \mathbb{P}_\mu(\mathcal{W}(k) = k\nu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, k]) > 0.$$

Let  $r > 0$  be such that  $\mathcal{B}(m(1), r)$  is included in  $\mathcal{C}$ ; for any integer  $\ell \geq 1$  we thus have  $\mathcal{B}_\ell := \mathcal{B}_\ell(\ell m(1), \ell r) \subset \mathcal{C}$  and since all the paths in  $B$  have the same norm and  $m(1) \in \mathring{\mathcal{C}}$ , one gets, for  $k$  large enough

$$\left( \mathcal{W}(k) = k\nu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq k \right) \iff \left( \mathcal{W}(k) = k\nu, \mathcal{W}(\ell) \in \mathcal{B}_\ell \text{ for any } \ell \geq k \right)$$

and so, by the strong law of large numbers

$$\mathbb{P}\left( \mathcal{W}(k) = k\nu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq k \right) \geq \mathbb{P}\left( \mathcal{W}(k) = k\nu, \mathcal{W}(\ell) \in \mathcal{B}_\ell \text{ for any } \ell \geq k \right) > 0.$$

Consequently, by the Markov property

$$\psi(\mu) \geq \mathbb{P}_\mu\left( \mathcal{W}(k) = k\nu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, k] \right) \times \mathbb{P}_{k\nu}\left( \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq k \right) > 0.$$

□

### 3. REPRESENTATIONS OF SYMMETRIZABLE KAC-MOODY ALGEBRAS

**3.1. Symmetrizable Kac-Moody algebras.** Let  $A = (a_{i,j})$  be a  $n \times n$  generalized Cartan matrix of rank  $r$ . This means that the entries  $a_{i,j} \in \mathbb{Z}$  satisfy the following conditions

- (1)  $a_{i,j} \in \mathbb{Z}$  for  $i, j \in \{1, \dots, n\}$ ,
- (2)  $a_{i,i} = 2$  for  $i \in \{1, \dots, n\}$ ,
- (3)  $a_{i,j} = 0$  if and only if  $a_{j,i} = 0$  for  $i, j \in \{1, \dots, n\}$ .

We will also assume that  $A$  is *indecomposable*: given subsets  $I$  and  $J$  of  $\{1, \dots, n\}$ , there exists  $(i, j) \in I \times J$  such that  $a_{i,j} \neq 0$ . We refer to [6] for the classification of indecomposable generalized Cartan matrices. Recall there exist only three kinds of such matrices: when all the principal minors of  $A$  are positive,  $A$  is of *finite type* and corresponds to the Cartan matrix of a simple Lie algebra over  $\mathbb{C}$ ; when all the proper principal minors of  $A$  are positive and  $\det(A) = 0$  the matrix  $A$  is said of *affine type*; otherwise  $A$  is of *indefinite type*. For technical reasons, from now on, we will restrict



ourselves to *symmetrizable* generalized Cartan matrices i.e. we will assume there exists a diagonal matrix  $D$  with entries in  $\mathbb{Z}_{>0}$  such that  $DA$  is symmetric.

The root and weight lattices associated to a generalized symmetrizable Cartan matrix are defined by mimic the construction for the Lie algebras. Let  $P^\vee$  be a free abelian group of rank  $2n - r$  with  $\mathbb{Z}$ -basis  $\{h_1, \dots, h_n\} \cup \{d_1, \dots, d_{n-r}\}$ . Set  $\mathfrak{h} := P^\vee \otimes_{\mathbb{Z}} \mathbb{C}$  and  $\mathfrak{h}_{\mathbb{R}} := P^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ . The weight lattice  $P$  is then defined by

$$P := \{\gamma \in \mathfrak{h}^* \mid \gamma(P^\vee) \subset \mathbb{Z}\}.$$

Set  $\Pi^\vee := \{h_1, \dots, h_n\}$ . One can then choose a set  $\Pi := \{\alpha_1, \dots, \alpha_n\}$  of linearly independent vectors in  $P \subset \mathfrak{h}^*$  such that  $\alpha_i(h_j) = a_{i,j}$  for  $i, j \in \{1, \dots, n\}$  and  $\alpha_i(d_j) \in \{0, 1\}$  for  $i \in \{1, \dots, n-r\}$ . The elements of  $\Pi$  are the *simple roots*. The free abelian group  $Q := \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$  is the *root lattice*. The quintuple  $(A, \Pi, \Pi^\vee, P, P^\vee)$  is called a *generalized Cartan datum* associated to the matrix  $A$ . For any  $i = 1, \dots, n$ , we also define the *fundamental weight*  $\omega_i \in P$  by  $\omega_i(h_j) = \delta_{i,j}$  for  $j \in \{1, \dots, n\}$  and  $\omega_i(d_j) = 0$  for  $j \in \{1, \dots, n-r\}$ .

For any  $i = 1, \dots, n$ , we define the simple reflection  $s_i$  on  $\mathfrak{h}^*$  by

$$(4) \quad s_i(\gamma) = \gamma - h_i(\gamma)\alpha_i \quad \text{for any } \gamma \in P.$$

The *Weyl group*  $W$  is the subgroup of  $GL(\mathfrak{h}^*)$  generated by the reflections  $s_i$ . Each element  $w \in W$  admits a reduced expression  $w = s_{i_1} \cdots s_{i_r}$ . One can prove that  $r$  is independent of the reduced expression considered so the signature  $\varepsilon(w) = (-1)^r$  is well-defined.

**Definition 3.1.** *The Kac-Moody algebra  $\mathfrak{g}$  associated to the quintuple  $(A, \Pi, \Pi^\vee, P, P^\vee)$  is the  $\mathbb{C}$ -algebra generated by the elements  $e_i, f_i, i = 1, \dots, n$  and  $h \in P$  together with the relations*

- (1)  $[h, h'] = 0$  for any  $h, h' \in P$ ,
  - (2)  $[h, e_i] = \alpha_i(h)e_i$  for any  $i = 1, \dots, n$  and  $h \in P$ ,
  - (3)  $[h, f_i] = -\alpha_i(h)f_i$  for any  $i = 1, \dots, n$  and  $h \in P$ ,
  - (4)  $[e_i, f_j] = \delta_{i,j}h_i$  for any  $i, j = 1, \dots, n$ ,
  - (5)  $ad(e_i)^{1-a_{i,j}}(e_j) = 0$  for any  $i, j = 1, \dots, n$  such that  $i \neq j$ ,
  - (6)  $ad(f_i)^{1-a_{i,j}}(f_j) = 0$  for any  $i, j = 1, \dots, n$  such that  $i \neq j$ ,
- where  $ad(a) \in \text{End } \mathfrak{g}$  is defined by  $ad(a)(b) = [a, b] := ab - ba$  for any  $a, b \in \mathfrak{g}$ .

Denote by  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  the subalgebras of  $\mathfrak{g}$  generated by the  $e_i$ 's and the  $f_i$ 's, respectively. We have the triangular decomposition  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$  and  $\mathfrak{h}$  is called the Cartan subalgebra of  $\mathfrak{g}$ . For any  $\alpha \in Q$ , set

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for any } h \in \mathfrak{h}\}.$$

The algebra  $\mathfrak{g}$  then decomposes on the form

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$$

where  $\dim \mathfrak{g}_\alpha$  is finite for any  $\alpha \in Q$ . The roots of  $\mathfrak{g}$  are the nonzero elements  $\alpha \in Q$  such that  $\mathfrak{g}_\alpha \neq \{0\}$ . We denote by  $R$  the set of roots of  $\mathfrak{g}$ . Set  $Q_+ := \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$ ,  $R_+ := R \cap Q_+$  and  $R_- = R \cap (-Q_+)$ . Then one can prove that  $R = R_+ \cup R_-$  and  $R_- = -R_+$  as for the finite dimensional Lie algebras. For any  $\gamma = \sum_{i=1}^n a_i\alpha_i \in Q_+$ , we set

$$ht(\gamma) := \sum_{i=1}^n a_i.$$

We have the decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha.$$

For any  $\alpha \in R_+$ , we set  $\dim \mathfrak{g}_\alpha = m_\alpha$  the multiplicity of the root  $\alpha$  in  $\mathfrak{g}$ . The set  $R_+$  is infinite as soon as  $A$  is not of finite type; the multiplicity  $m_\alpha$  may be greater than 1 but is always bounded from above as follows (see [6] § 1.3):

$$(5) \quad m_\alpha \leq n^{ht(\alpha)} \quad \text{for any } \alpha \in R_+.$$

When  $A$  is not of finite type, the Weyl group  $W$  is also infinite and there exists roots  $\alpha \in R$  which do not belong to the orbit  $W\alpha_i, i = 1, \dots, n$  of the simple roots; these roots are called *imaginary roots* in contrast to *real roots* which belong to the orbit of a simple root  $\alpha_i$ .

The root system associated to a matrix  $A$  of finite type is well known (see for instance [2]) and are classified in four infinite series ( $A_n, B_n, C_n$  and  $D_n$ ) and five exceptional systems ( $E_6, E_7, E_8, F_4, G_2$ ). In contrast, few is known on the root system associated to a matrix of indefinite type. In the intermediate case of the affine matrices, there also exists a finite classification which makes appear seven infinite series and seven exceptional systems. The root system can be described as follows. First, the rows and columns of  $A$  can be ordered such that the submatrix  $A^\circ$  of size  $(n-1) \times (n-1)$  obtained by deleting the row and column indexed by  $n$  in  $A$  is the Cartan matrix of a finite root system  $R^\circ$ . The kernel of  $A$  has dimension 1; more precisely, there exists a unique  $n$ -tuple  $(a_1, \dots, a_n)$  of positive relatively prime integers such that  $A^t(a_1, \dots, a_n) = 0$  and the vector  $\delta = \sum_{i=1}^n a_i \alpha_i$  then belongs to  $R$ . The sets of real roots, of imaginary roots, of positive real roots and positive imaginary roots can be completely described in terms of roots in  $R^\circ$  and  $\delta$ . We refer to [6] p. 83 for a complete exposition and only recall the following facts we need in the sequel. In particular, we do not need the complete description of the sets  $R_+^{re}$  which strongly depends on the affine root system considered. We have

$$R_+^{re} \subset \{\alpha + k\delta \mid \alpha \in R^\circ, k \in \mathbb{Z}_{>0}\} \cup R_+^\circ$$

except for the affine root system  $A_{2n}^{(2)}$  in which case

$$R_+^{re} \subset \{\alpha + k\delta \mid \alpha \in R^\circ, k \in \mathbb{Z}_{>0}\} \cup \left\{ \frac{1}{2}(\alpha + (2k-1)\delta) \mid \alpha \in R^\circ, k \in \mathbb{Z}_{>0} \right\} \cup R_+^\circ.$$

We also have in all affine cases

$$(6) \quad R_+^{im} = \{k\delta \mid k \in \mathbb{Z}_{>0}\} \quad \text{and} \quad R_+ = R_+^{re} \cup R_+^{im}.$$

The multiplicities of the positive roots verify (see [6] Corollary 8.3).

$$(7) \quad m_\alpha = 1 \text{ for } \alpha \in R_+^{re} \text{ and } m_\alpha \leq n \text{ for } \alpha \in R_+^{im}.$$

**3.2. The category  $\mathcal{O}_{int}$  of  $\mathfrak{g}$ -modules.** Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody algebra. We now introduce a category of  $\mathfrak{g}$ -modules whose properties naturally extend those of the finite-dimensional representations of simple Lie algebras.

**Definition 3.2.** *The category  $\mathcal{O}_{int}$  is the category of  $\mathfrak{g}$ -modules  $M$  satisfying the following properties:*

- (1) *The module  $M$  decomposes in weight subspaces on the form*

$$M = \bigoplus_{\gamma \in P} M_\gamma \quad \text{where } M_\gamma := \{v \in M \mid h(v) = \gamma(h)v \text{ for any } h \in \mathfrak{h}\}.$$

- (2) *For any  $i = 1, \dots, n$ , the actions of  $e_i$  and  $f_i$  are locally nilpotent i.e. for any  $v \in M$ , there exists integers  $p$  and  $q$  such that  $e_i^p \cdot v = f_i^q \cdot v = 0$ .*

For any  $\gamma \in P$ , let  $e^\gamma$  be the generator of the group algebra  $\mathbb{C}[P]$  associated to  $\gamma$ . By definition, we have  $e^\gamma e^{\gamma'} = e^{\gamma+\gamma'}$  for any  $\gamma, \gamma' \in P$  and the group  $W$  acts on  $\mathbb{C}[P]$  as follows:  $w(e^\gamma) = e^{w(\gamma)}$  for any  $w \in W$  and any  $\gamma \in P$ .

The irreducible modules in the category  $\mathcal{O}_{int}$  are the irreducible highest weight modules, they are parametrized by the *integral cone of dominant weights*  $P_+$  of  $\mathfrak{g}$  defined by

$$P_+ := \{\lambda \in P \mid \lambda(h_i) \geq 0 \text{ for any } i = 1, \dots, n\}.$$

The irreducible highest weight module  $V(\lambda)$  of weight  $\lambda \in P_+$  decomposes as  $V(\lambda) = \bigoplus_{\gamma \in P} V(\lambda)_\gamma$ ; observe that  $\dim V(\lambda)$  is infinite when  $\mathfrak{g}$  is not of finite type, nevertheless the weight space  $V(\lambda)_\gamma$  is always finite-dimensional and we set  $K_{\lambda,\gamma} := \dim(V(\lambda)_\gamma)$ . Furthermore, we have  $\dim V(\lambda)_\lambda = 1$  and  $e_i(v) = 0$  for any  $i = 1, \dots, n$  and  $v \in V(\lambda)_\lambda$ ; the elements of  $V(\lambda)_\lambda$  thus coincide up to a multiplication by a scalar and are called the *highest weight vectors*.

The character  $s_\lambda$  of  $V(\lambda)$  is defined by  $s_\lambda := \sum_{\gamma \in P} K_{\lambda,\gamma} e^\gamma$ ; it is invariant under the action of the Weyl group  $W$  since  $K_{\lambda,\gamma} = K_{\lambda,w(\gamma)}$  for any  $w \in W$ . Observe that the orbit  $W \cdot \gamma$  intersects  $P_+$  exactly once when  $K_{\lambda,\gamma} > 0$ .

From now on, we fix a weight  $\rho \in P$  such that  $\rho(h_i) = 1$  for any  $i = 1, \dots, n$ ; we have the Kac-Weyl character formula :

$$\textbf{Theorem 3.3.}$$
 *For any  $\lambda \in P_+$ , we have  $s_\lambda = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})^{m_\alpha}}$ .*

The category  $\mathcal{O}_{int}$  is stable under the tensor product of  $\mathfrak{g}$ -modules. Moreover, every module  $M \in \mathcal{O}_{int}$  decomposes as a direct sum of irreducible modules. Given  $\lambda^{(1)}, \dots, \lambda^{(k)}$  a sequence of dominant weights, consider the module  $M := V(\lambda^{(1)}) \otimes \dots \otimes V(\lambda^{(k)})$ . Then  $\dim M_\gamma$  is finite for any  $\gamma \in P$ , the character of  $M$  can be defined by  $\text{char}(M) := \sum_{\gamma \in P} \dim M_\gamma e^\gamma$  and we have

$$\text{char}(M) = s_{\lambda^{(1)}} \cdots s_{\lambda^{(k)}}.$$

Each irreducible component of  $M$  appears finitely many times in this decomposition, in other words there exist nonnegative integers  $m_{M,\lambda}$  such that

$$M \simeq \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus m_{M,\lambda}} \text{ or equivalently } \text{char}(M) := \sum_{\lambda \in P_+} m_{M,\lambda} s_\lambda.$$

Consider  $\kappa, \mu \in P_+$  and  $\ell \in \mathbb{Z}_{\geq 0}$ . We set

$$(8) \quad V(\mu) \otimes V(\kappa)^{\otimes \ell} = \sum_{\lambda \in P_+} V(\lambda)^{\oplus f_{\lambda/\mu}^{\kappa,\ell}} \text{ and } m_{\mu,\kappa}^\lambda = f_{\lambda/\mu}^{\kappa,1}.$$

In the sequel, we will fix  $\kappa \in P_+$  and write  $f_{\lambda/\mu}^{\kappa,\ell} = f_{\lambda/\mu}^\ell$  for short. Observe that in the previous decomposition all but a finite number of coefficients  $f_{\lambda/\mu}^\ell$  are equal to 0.

**3.3. Littelmann path model.** The aim of this paragraph is to give a brief overview of the path model developed by Littelmann and its connections with Kashiwara crystal basis theory. We refer to [12], [13], [14] and [8] for a detailed exposition. Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody algebra associated to the quintuple  $(A, \Pi, \Pi^\vee, P, P^\vee)$  where  $A$  is a  $n \times n$  symmetric generalized Cartan matrix with rank  $r$ . In the following, it will be convenient to fix a nondegenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{h}_{\mathbb{R}}^*$  invariant under  $W$ . For any root  $\alpha$ , we set  $\alpha^\vee = \frac{\alpha}{\langle \alpha, \alpha \rangle}$ . We have seen that  $P$  is a  $\mathbb{Z}$ -lattice with rank  $d = 2n - r$ . We define the notion of elementary piecewise linear paths in  $P_{\mathbb{R}} := P \otimes_{\mathbb{Z}} \mathbb{R}$  as we did in § 2.2. Let  $\mathcal{P}$  be the set of such elementary paths having *only rational turning points* (i.e. whose inflexion points have rational coordinates) and ending in  $P$  i.e. such that  $\pi(1) \in P$ . The Weyl group  $W$  acts on  $\mathcal{P}$  as follows: for any  $w \in W$  and  $\eta \in \mathcal{P}$ , the path  $w[\eta]$  is defined by

$$(9) \quad \forall t \in [0, 1] \quad w[\eta](t) = w(\eta(t))$$

and the weight  $\text{wt}(\eta)$  of  $\eta$  is defined by  $\text{wt}(\eta) = \eta(1)$ .

We now define operators  $\tilde{e}_i$  and  $\tilde{f}_i$ ,  $i = 1, \dots, n$ , acting on  $\mathcal{P} \cup \{\mathbf{0}\}$ . If  $\eta = \mathbf{0}$ , we set  $\tilde{e}_i(\eta) = \tilde{f}_i(\eta) = \mathbf{0}$ ; when  $\eta \in \mathcal{P}$ , we need to decompose  $\eta$  into a union of finitely many subpaths and reflect some of these subpaths by  $s_{\alpha_i}$  according to the behavior of the map

$$h_\eta : \begin{cases} [0, 1] & \rightarrow \mathbb{R} \\ t & \mapsto \langle \eta(t), \alpha_i^\vee \rangle. \end{cases}$$

Let  $m_\eta$  for the minimum of the function  $h_\eta$ . Since  $h_\eta(0) = 0$ , we have  $m_\eta \leq 0$ .

If  $m_\eta > -1$ , then  $\tilde{e}_i(\eta) = \mathbf{0}$ . If  $m_\eta \leq -1$ , set  $t_1 := \inf\{t \in [0, 1] \mid h_\eta(t) = m_\eta\}$  and let  $t_0 \in [0, t_1]$  be maximal such that  $m_\eta \leq h_\eta(t) \leq m_\eta + 1$  for any  $t \in [t_0, t_1]$  (see figure 1). Choose  $r \geq 1$  and  $t_0 = t^{(0)} < t^{(1)} < \dots < t^{(r)} = t_1$  satisfying the following conditions: for  $1 \leq a \leq r$

- (1) either  $h_\eta(t^{(a-1)}) = h_\eta(t^{(a)})$  and  $h_\eta(t) \geq h_\eta(t^{(a)})$  on  $[t^{(a-1)}, t^{(a)}]$ ,
- (2) or  $h_\eta$  is strictly decreasing on  $[t^{(a-1)}, t^{(a)}]$  and  $h_\eta(t) \geq h_\eta(t^{(a-1)})$  on  $[0, t^{(a-1)}]$ .

We set  $t^{(-1)} = 0$  and  $t^{(r+1)} = 1$  and, for  $0 \leq a \leq r+1$ , we denote by  $\eta_a$  the elementary path defined by

$$\forall u \in [0, 1] \quad \eta_a(u) = \eta(t^{(a-1)} + u(t^{(a)} - t^{(a-1)})) - \eta(t^{(a-1)}).$$

Observe that  $\eta_a$  is the elementary path whose image translated by  $\eta(t^{(a-1)})$  coincides with the restriction of  $\eta$  on  $[t^{(a-1)}, t^{(a)}]$ ; the path  $\eta$  decomposes as follows

$$\eta = \eta_0 * \eta_1 * \dots * \eta_r * \eta_{r+1}.$$

For  $1 \leq a \leq r+1$ , we also set  $\eta'_a = \eta_a$  in case (1) and  $\eta'_a = s_{\alpha_i}(\eta_a)$  in case (2). For  $i \in \{1, \dots, n\}$ , we set

$$\tilde{e}_i(\eta) = \begin{cases} \mathbf{0} & \text{if } h_\eta(1) < m_\eta + 1, \\ \eta_0 * \eta'_1 * \dots * \eta'_r * \eta_{r+1} & \text{otherwise.} \end{cases}$$

To define the  $\tilde{f}_i$ , we first propose another decomposition of the path  $\eta$ . If  $h_\eta(1) < m_\eta + 1$ , then  $\tilde{f}_i(\eta) = \mathbf{0}$ . Otherwise ( $h_\eta(1) \geq m_\eta + 1$ ), set  $t'_0 := \sup\{t \in [0, 1] \mid h_\eta(t) = m_\eta\}$  and let  $t'_1 \in [t'_0, 1]$  be minimal such that  $h_\eta(t) \geq m_\eta + 1$  for  $t \in [t'_1, 1]$  (see figure 1). Choose  $r \geq 1$  and  $t'_0 = t^{(0)} < t^{(1)} < \dots < t^{(r)} = t'_1$  satisfying the following conditions: for  $1 \leq a \leq r$

- (3) either  $h_\eta(t^{(a-1)}) = h_\eta(t^{(a)})$  and  $h_\eta(t) \geq h_\eta(t^{(a-1)})$  on  $[t^{(a-1)}, t^{(a)}]$ ,
- (4) or  $h_\eta$  is strictly increasing on  $[t^{(a-1)}, t^{(a)}]$  and  $h_\eta(t) \geq h_\eta(t^{(a)})$  on  $[t^{(a)}, 1]$ .

We set  $t^{(-1)} = 0$  and  $t^{(r+1)} = 1$  and, for  $0 \leq a \leq r+1$ , we notice  $\eta_a$  the elementary path defined by

$$\forall u \in [0, 1] \quad \eta_a(u) = \eta(t^{(a-1)} + u(t^{(a)} - t^{(a-1)})) - \eta(t^{(a-1)}).$$

As above, the path  $\eta$  decomposes as  $\eta = \eta_0 * \eta_1 * \dots * \eta_r * \eta_{r+1}$ ; for  $1 \leq a \leq r+1$ , we thus set  $\eta'_a = \eta_a$  in case (3) and  $\eta'_a = s_{\alpha_i}(\eta_a)$  in case (4) and the operator  $\tilde{f}_i$ ,  $1 \leq i \leq n$ , is defined by

$$\tilde{f}_i(\eta) = \begin{cases} \mathbf{0} & \text{if } h_\eta(1) < m_\eta + 1, \\ \eta_0 * \eta'_1 * \dots * \eta'_r * \eta_{r+1} & \text{otherwise.} \end{cases}$$

**Remarks:** 1. Observe that the paths  $\eta$ ,  $\tilde{e}_i(\eta)$  and  $\tilde{f}_i(\eta)$  have the same length.

2. When  $\tilde{e}_i(\eta)$  is computed, the segments of  $\eta$  which are replaced by their symmetric under  $s_{\alpha_i}$  correspond to intervals where  $h_\eta$  is strictly decreasing. This implies that  $h_\eta(t) \leq h_{\tilde{e}_i(\eta)}(t)$  for any  $t \in [0, 1]$ . Similarly, we have  $h_\eta(t) \geq h_{\tilde{f}_i(\eta)}(t)$  for any  $t \in [0, 1]$ .

The operators  $\tilde{e}_i$  and  $\tilde{f}_i$  satisfy the following properties :

**Proposition 3.4.**

- (1) Assume  $\tilde{e}_i(\eta) \neq \mathbf{0}$ ; then  $\tilde{e}_i(\eta)(1) = \eta(1) + \alpha_i$  and  $\tilde{f}_i(\tilde{e}_i(\eta)) = \eta$ .



(2) for any  $i = 1, \dots, n$  we have

$$(10) \quad \tilde{e}_i(\eta * \pi) = \begin{cases} \eta * \tilde{e}_i(\pi) & \text{if } \varepsilon_i(\pi) > \varphi_i(\eta) \\ \tilde{e}_i(\eta) * \pi & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{f}_i(\eta * \pi) = \begin{cases} \tilde{f}_i(\eta) * \pi & \text{if } \varphi_i(\eta) > \varepsilon_i(\pi) \\ \eta * \tilde{f}_i(\pi) & \text{otherwise.} \end{cases}$$

In particular,  $\tilde{e}_i(\eta * \pi) = \mathbf{0}$  if and only if  $\tilde{e}_i(\eta) = \mathbf{0}$  and  $\varepsilon_i(\pi) \leq \varphi_i(\eta)$  for any  $i = 1, \dots, n$ .

(3)  $\tilde{e}_i(\eta) = \mathbf{0}$  for any  $i = 1, \dots, n$  if and only if  $\text{Im } \eta$  is contained in  $\mathcal{C}$ .

The following theorem summarizes crucial results of Littelmann (see [12], [13] and [14]).

**Theorem 3.6.** Consider  $\lambda, \mu$  and  $\kappa$  dominant weights and choose arbitrarily elementary paths  $\eta_\lambda, \eta_\mu$  and  $\eta_\kappa$  in  $\mathcal{P}$  such that  $\text{Im } \eta_\lambda \subset \mathcal{C}$ ,  $\text{Im } \eta_\mu \subset \mathcal{C}$  and  $\text{Im } \eta_\kappa \subset \mathcal{C}$  and joining respectively 0 to  $\lambda$ , 0 to  $\mu$  and 0 to  $\kappa$ .

(1) We have  $B(\eta_\lambda) := \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \eta_\lambda \mid k \in \mathbb{N} \text{ and } 1 \leq i_1, \dots, i_k \leq n\} \setminus \{\mathbf{0}\}$ .

In particular  $\text{wt}(\eta) - \text{wt}(\eta_\lambda) \in Q_+$  for any  $\eta \in B(\eta_\lambda)$ .

(2) The graph  $B(\eta_\lambda)$  is contained in  $\mathcal{P}_{\min \mathbb{Z}}$ .

(3) If  $\eta'_\lambda$  is another elementary path from 0 to  $\lambda$  such that  $\text{Im } \eta'_\lambda$  is contained in  $\mathcal{C}$ , then  $B(\eta_\lambda)$  and  $B(\eta'_\lambda)$  are isomorphic as oriented graphs i.e. there exists a bijection  $\theta : B(\eta_\lambda) \rightarrow B(\eta'_\lambda)$  which commutes with the action of the operators  $\tilde{e}_i$  and  $\tilde{f}_i$ ,  $i = 1, \dots, n$ .

(4) The crystal  $B(\eta_\lambda)$  is isomorphic to the Kashiwara crystal graph  $B(\lambda)$  associated to the  $U_q(\mathfrak{g})$ -module of highest weight  $\lambda$ .

(5) We have

$$(11) \quad s_\lambda = \sum_{\eta \in B(\eta_\lambda)} e^{\eta(1)}.$$

(6) For any  $i = 1, \dots, n$  and any  $b \in B(\eta_\lambda)$ , let  $s_i(b)$  be the unique path in  $B(\eta_\lambda)$  such that

$$\varphi_i(s_i(b)) = \varepsilon_i(b) \quad \text{and} \quad \varepsilon_i(s_i(b)) = \varphi_i(b)$$

(in other words,  $s_i$  acts on each  $i$ -chain  $\mathcal{C}_i$  as the symmetry with respect to the center of  $\mathcal{C}_i$ ).

The actions of the  $s_i$ 's extend to an action of  $\mathbb{W}$  on  $\mathcal{P}$  which stabilizes  $B(\eta_\lambda)$ . In particular, for any  $w \in \mathbb{W}$  and any  $b \in B(\eta_\lambda)$ , we have  $w(b) \in B(\eta_\lambda)$  and  $\text{wt}(w(b)) = w(\text{wt}(b))$ .<sup>1</sup>

(7) For any  $b \in B(\eta_\lambda)$  we have  $\text{wt}(b) = \sum_{i=1}^n (\varphi_i(b) - \varepsilon_i(b)) \omega_i$ .

(8) Given any integer  $\ell \geq 0$ , set

(12)  $B(\eta_\mu) * B(\eta_\kappa)^{* \ell} = \{\pi = \eta * \eta_1 * \cdots * \eta_\ell \in \mathcal{P} \mid \eta \in B(\eta_\mu) \text{ and } \eta_k \in B(\eta_\kappa) \text{ for any } k = 1, \dots, \ell\}$ .

The graph  $B(\eta_\mu) * B(\eta_\kappa)^{* \ell}$  is contained in  $\mathcal{P}_{\min \mathbb{Z}}$ .

(9) The multiplicity  $m_{\mu, \kappa}^\lambda$  defined in (8) is equal to the number of paths of the form  $\mu * \eta$  with  $\eta \in B(\eta_\kappa)$  contained in  $\mathcal{C}$ .

(10) The multiplicity  $f_{\lambda/\mu}^\ell$  defined in (8) is equal to cardinality of the set

$$H_{\lambda/\mu}^\ell := \{\pi \in B(\eta_\mu) * B(\eta_\kappa)^{* \ell} \mid \tilde{e}_i(\pi) = \mathbf{0} \text{ for any } i = 1, \dots, n \text{ and } \pi(1) = \lambda\}.$$

Each path  $\pi = \eta * \eta_1 * \cdots * \eta_\ell \in H_{\lambda/\mu}^\ell$  verifies  $\text{Im } \pi \subset \mathcal{C}$  and  $\eta = \eta_\mu$ .

**Remarks:** 1. Combining assertion (2) of Proposition 3.4 together with assertions (1) and (5) of the Theorem 3.6, one may check that the function  $e^{-\lambda} s_\lambda$  is in fact a polynomial in the variables  $T_i = e^{-\alpha_i}$ , namely

$$(13) \quad s_\lambda = e^\lambda S_\lambda(T_1, \dots, T_n)$$

<sup>1</sup>This action should not be confused with that defined in (9) which does not stabilizes  $B(\eta_\lambda)$  in general.

where  $S_\lambda \in \mathbb{C}[X_1, \dots, X_n]$ . Observe also that the quantity  $S_\infty := \prod_{\alpha \in R_+} \frac{1}{(1-e^{-\alpha})^{m_\alpha}}$  is a formal power series in the variables  $T_1, \dots, T_n$ . M. Kashiwara proved (see for instance [5] § 20.7) that the crystal  $B(\lambda)$  admits a projective limit  $B(\infty)$  when  $\lambda$  tends to infinity and that

$$\text{char}(B(\infty)) = \sum_{b \in B(\infty)} e^{\text{wt}(b)} = S_\infty.$$

Now, since  $B(\lambda)$  can be embedded in  $B(\infty)$  up to a translation by the weights by  $\lambda$ , we have

$$(14) \quad S_\lambda(T_1, \dots, T_n) \leq S_\infty(T_1, \dots, T_n);$$

in other words  $S_\infty(T_1, \dots, T_n) = S_\lambda(T_1, \dots, T_n) + \sum_{\mu \in P} a_\mu e^\mu$  where the coefficients  $a_\mu$  are nonnegative integers.

2. Using assertion (1) of Theorem 3.6, we obtain  $m_{\mu, \delta}^\lambda \neq 0$  only if  $\mu + \delta - \lambda \in Q_+$ . Similarly, when  $f_{\lambda/\mu}^{\delta, \ell} \neq 0$  one necessarily has  $\mu + \ell\delta - \lambda \in Q_+$ .

3. A minuscule weight is a dominant weight  $\kappa \in P_+$  such that the weights of  $V(\kappa)$  are exactly those of the orbit  $W \cdot \kappa$ . In this case, if we take  $\eta_\kappa : t \mapsto t\kappa$ , the crystal  $B(\eta_\kappa)$  contains only the paths  $\eta : t \mapsto t\kappa$ . In particular, these paths are lines.

4: Given any path  $\eta_\lambda$  such that  $\text{Im } \eta_\lambda \subset \mathcal{C}$ , the set of paths  $B(\eta_\lambda)$  is in general very difficult to describe (even for in the finite types). Nevertheless, this can be done by using a particular choice of  $\eta_\lambda$  related to appropriate notion of tableaux as explained in the following paragraph.

**3.4. Example of crystals.** When  $\mathfrak{g}$  is a simple classical Lie algebra (i.e.  $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$  or  $\mathfrak{so}_{2n}$ ) or  $\mathfrak{g}$  is of type  $G_2$ , there exists a simple description of crystals in terms of tableaux. The root system of  $\mathfrak{g}$  is realized in  $\mathbb{R}^n$  whose standard basis is denoted  $(\varepsilon_1, \dots, \varepsilon_n)$ . A partition of length  $n$  is a decreasing sequence  $\lambda_1 \geq \dots \geq \lambda_n$ ; its dominant weight  $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$  belongs to  $P_+$ . The Young diagram associated to  $\lambda$  is the juxtaposition of rows of lengths  $\lambda_i, i = 1, \dots, n$ . Each rows of length  $\lambda_i$  is divided in  $\lambda_i$  boxes and it is common to identify  $\lambda$  with its Young diagram. A tableau is a filling of the boxes in  $\lambda$  by integers taken in a range depending on  $\mathfrak{g}$  and verifying a list of combinatorial properties. Each integer used labels a weight of a particular irreducible representation, called the *vector representation*, with corresponding partition  $\lambda = (1, 0, \dots, 0) \in \mathbb{Z}^n$ ; for instance, in the case of a Lie algebra of  $N \times N$  matrices, this representation is nothing but the space  $\mathbb{C}^N$ . We list below the crystals  $B$  of the vector representation  $V$  for each classical type.

- $\mathfrak{sl}_n : 1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n$
- $\mathfrak{so}_{2n+1} : 1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} 0 \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \rightarrow \bar{2} \xrightarrow{1} \bar{1}$
- $\mathfrak{sp}_{2n} : 1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \rightarrow \bar{2} \xrightarrow{1} \bar{1}$
- $\mathfrak{so}_{2n} : 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \begin{array}{c} \nearrow n \\ \searrow n-1 \end{array} \begin{array}{c} \bar{n} \\ \overline{n-1} \\ \nearrow n \end{array} \xrightarrow{n-2} \cdots \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$

In the simplest case of  $\mathfrak{g} = \mathfrak{sl}_n$ , the integers to consider are  $1, \dots, n$ . Then each row of a tableau should weakly increase from left to right and each column should strictly decrease from top to bottom (semistandard tableaux). For  $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$  or  $\mathfrak{so}_{2n}$  the description of the relevant

tableaux is similar but we use the integers  $1, \dots, n$ , their negatives  $\bar{1}, \dots, \bar{n}$  and more complicated rules to fill the partition  $\lambda$ . We refer to [9] and [7] for a detailed exposition.

Let us now explain how to obtain the crystal structure on tableaux. We first introduce a crystal structure on the words of length  $\ell$  by identifying each such word  $w = x_1 \cdots x_\ell$  with the vertex  $x_1 \otimes \cdots \otimes x_\ell \in B^{\otimes \ell}$ . Here the crystal structure on  $B^{\otimes \ell}$  is obtained by using recursively the same rules as in (10). Define the reading  $\text{wt}(T)$  of a tableau  $T$  as the word (see the example below) obtained by reading its columns from left to right and top to bottom (Japanese reading). The crystal structure on the set of tableaux is then deduced from the crystal structure on the set of their readings.

To associate an elementary path to a tableau, we first associate to each vertex  $k \in \{1, \dots, n\}$  (resp.  $\bar{k}$  with  $k \in \{1, \dots, n\}$ ) in the above crystals the elementary path  $t \mapsto t\varepsilon_k$  (resp.  $t \mapsto -t\varepsilon_k$ ). To the vertex 0 is associated the path  $t \mapsto t\varepsilon_n$  for  $t \in [0, \frac{1}{2}]$  and  $t \mapsto (1-t)\varepsilon_n$  for  $t \in [\frac{1}{2}, 1]$ . The path associated to a tableau  $T$  of reading  $w(T) = x_1 \cdots x_\ell$  is then the concatenation  $x_1 * \cdots * x_\ell$ .

**Examples 3.7.** We explicit here several examples of tableaux and crystals, using the rules presented in Proposition 3.5.

- (1) The semistandard tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & 5 \\ \hline 2 & 4 & 4 & \\ \hline 4 & 5 & & \\ \hline \end{array} .$$

of shape  $\lambda = (4, 3, 1)$  admits the reading  $\text{wt}(T) = 534345124$ . The associated path in  $\mathbb{R}^5$  is  $\varepsilon_5 * \varepsilon_3 * \varepsilon_4 * \varepsilon_3 * \varepsilon_4 * \varepsilon_5 * \varepsilon_1 * \varepsilon_2 * \varepsilon_4$ . The dominant weight associated to  $\lambda$  is  $4\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3$ .

- (2) The crystal of the irreducible representation of  $\mathfrak{sl}_2$  associated to the partition  $\lambda = (3, 0)$  is

$$\boxed{1 \ 1 \ 1} \xrightarrow{1} \boxed{1 \ 1 \ 2} \xrightarrow{1} \boxed{1 \ 2 \ 2} \xrightarrow{1} \boxed{2 \ 2 \ 2}$$

corresponding to the path  $\eta_\lambda = \varepsilon_1 * \varepsilon_1 * \varepsilon_1, \varepsilon_2 * \varepsilon_1 * \varepsilon_1, \varepsilon_2 * \varepsilon_2 * \varepsilon_1$  and  $\varepsilon_2 * \varepsilon_2 * \varepsilon_2$  in  $\mathbb{R}^2$ .

- (3) The crystal of irreducible representation of  $\mathfrak{so}_5$  associated to the partition  $\lambda = (1, 1)$  is

$$\begin{array}{ccccccc} \boxed{\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}} & \xrightarrow{2} & \boxed{\begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline \end{array}} & \xrightarrow{2} & \boxed{\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}} & \xrightarrow{1} & \boxed{\begin{array}{|c|} \hline 2 \\ \hline 2 \\ \hline \end{array}} & \xrightarrow{1} & \boxed{\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}} \\ & & \downarrow 1 & & \downarrow 2 & & & & \\ & & \boxed{\begin{array}{|c|} \hline 2 \\ \hline 0 \\ \hline \end{array}} & \xrightarrow{2} & \boxed{\begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array}} & \xrightarrow{2} & \boxed{\begin{array}{|c|} \hline 0 \\ \hline 2 \\ \hline \end{array}} & \xrightarrow{1} & \boxed{\begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array}} & \xrightarrow{2} & \boxed{\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}} \end{array}$$

corresponding to the paths  $\eta_\lambda = \varepsilon_1 * \varepsilon_2, \varepsilon_1 * \varepsilon_0, \varepsilon_1 * -\varepsilon_2, \varepsilon_2 * -\varepsilon_2, \varepsilon_2 * -\varepsilon_1, \varepsilon_2 * \varepsilon_0, \varepsilon_0 * \varepsilon_0, \varepsilon_0 * -\varepsilon_2, \varepsilon_0 * -\varepsilon_1, -\varepsilon_1 * -\varepsilon_2$ .

- (4) The crystal of the irreducible representation of  $\mathfrak{sp}_4$  associated to the partition  $\lambda = (2, 0)$  is

$$\begin{array}{ccccccc} \boxed{1 \ 1} & \xrightarrow{1} & \boxed{1 \ 2} & \xrightarrow{2} & \boxed{1 \ \bar{2}} & \xrightarrow{1} & \boxed{1 \ \bar{1}} \\ & & \downarrow 1 & & & & \downarrow 1 \\ & & \boxed{2 \ 2} & \xrightarrow{2} & \boxed{2 \ \bar{2}} & & \boxed{2 \ \bar{1}} \\ & & & & \downarrow 2 & & \downarrow 2 \\ & & & & \boxed{\bar{2} \ \bar{2}} & \xrightarrow{1} & \boxed{\bar{2} \ \bar{1}} \\ & & & & & & \downarrow 1 \\ & & & & & & \boxed{\bar{1} \ \bar{1}} \end{array}$$

corresponding to the paths,  $\eta_\lambda = \varepsilon_1 * \varepsilon_1, \varepsilon_2 * \varepsilon_1, -\varepsilon_2 * \varepsilon_1, -\varepsilon_1 * \varepsilon_1, \varepsilon_2 * \varepsilon_2, -\varepsilon_2 * \varepsilon_2, -\varepsilon_1 * \varepsilon_2, -\varepsilon_2 * -\varepsilon_2, -\varepsilon_1 * -\varepsilon_2, -\varepsilon_1 * -\varepsilon_1$  in  $\mathbb{R}^2$ .



- (5) *The crystal of the spin representation of  $\mathfrak{so}_8$  with highest weight  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  contains the four paths  $\gamma_1 = \frac{1}{2}\varepsilon_1 * \frac{1}{2}\varepsilon_2 * \frac{1}{2}\varepsilon_3$ ,  $\gamma_2 = \frac{1}{2}\varepsilon_1 * -\frac{1}{2}\varepsilon_3 * -\frac{1}{2}\varepsilon_2$ ,  $\gamma_3 = \frac{1}{2}\varepsilon_2 * -\frac{1}{2}\varepsilon_3 * -\frac{1}{2}\varepsilon_1$  and  $\gamma_4 = \frac{1}{2}\varepsilon_3 * -\frac{1}{2}\varepsilon_2 * -\frac{1}{2}\varepsilon_1$ . The corresponding crystal is*

$$\gamma_1 \xrightarrow{3} \gamma_2 \xrightarrow{1} \gamma_3 \xrightarrow{2} \gamma_4.$$

**Remarks:** 1. Observe that each operators  $\tilde{e}_i$  or  $\tilde{f}_i$  modifies only one letter in the tableaux of the previous example. This is in fact a general property of this construction.

2. As far as we are aware, a simple description of the crystal  $B(\eta_\lambda)$  where  $\eta(t) = t\lambda$  for any  $t \in [0, 1]$  is not known for any  $\lambda \in P_+$  even for  $\mathfrak{g} = \mathfrak{sl}_n$ .

#### 4. RANDOM PATHS AND SYMMETRIZABLE KAC-MOODY ALGEBRAS

**4.1. Probability distribution on elementary paths.** Consider  $\kappa \in P_+$  and a path  $\pi_\kappa \in \mathcal{P}$  from 0 to  $\kappa$  such that  $\text{Im } \pi_\kappa$  is contained in  $\mathcal{C}$ . Let  $B(\pi_\kappa)$  be the connected component of  $\mathcal{P}$  containing  $\pi_\kappa$ . We now endow  $B(\pi_\kappa)$  with a probability distribution  $p_\kappa$ , which will be characterized by the datum of a  $n$ -tuple  $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}_{>0}^n$  (each  $\tau_i$  can be regarded as attached to the positive simple root  $\alpha_i$ ). For any  $u = u_1\alpha_1 + \dots + u_n\alpha_n \in Q$ , we set  $\tau^u = \tau_1^{u_1} \dots \tau_n^{u_n}$ . Let  $\pi \in B(\pi_\kappa)$ : by assertion (1) of Theorem 3.6, one gets

$$\pi(1) = \text{wt}(\pi) = \kappa - \sum_{i=1}^n u_i(\pi)\alpha_i$$

where  $u_i(\pi) \in \mathbb{N}$  for any  $i = 1, \dots, n$ . We have  $S_\kappa(\tau) := S_\kappa(\tau_1, \dots, \tau_n) = \sum_{\pi \in B(\pi_\kappa)} \tau^{\kappa - \text{wt}(\pi)}$ .

**Proposition 4.1.** *For any  $\kappa \in P_+$ ,*

- (1) *if  $A$  is of finite type then  $0 < S_\kappa(\tau) < \infty$  for any  $\tau \in \mathbb{R}_{>0}^n$ ,*
- (2) *if  $A$  is of affine type then  $0 < S_\kappa(\tau) < \infty$  for any  $\tau \in ]0, 1[^n$ ,*
- (3) *if  $A$  is of indefinite type then  $0 < S_\kappa(\tau) < \infty$  for any  $\tau \in ]0, \frac{1}{n}[^n$ .*

*Proof.* The inequality  $S_\kappa(\tau) > 0$  is immediate since  $\tau_i > 0$  for any  $i = 1, \dots, n$ .

When  $A$  is of finite type, the crystal  $B(\pi_\kappa)$  is finite, so that  $S_\kappa(\tau) < \infty$ . When  $A$  is not of finite type, we have by (14)

$$S_\kappa(\tau) \leq S_\infty(\tau) = \prod_{\alpha \in R_+} \frac{1}{(1 - \tau^\alpha)^{m_\alpha}}$$

and it suffices to prove that  $S_\infty(\tau)$  is finite.

- Assume first that  $A$  is of affine type different from  $A_{2n}^{(2)}$ . By (6) and (7), we have

$$\prod_{\alpha \in R_+} \frac{1}{(1 - \tau^\alpha)^{m_\alpha}} \leq \left( \prod_{\alpha \in R_+^\circ} \frac{1}{1 - \tau^\alpha} \right) \left( \prod_{k=1}^{+\infty} \frac{1}{(1 - \tau^{k\delta})^n} \right) \prod_{\alpha \in R^\circ} \left( \prod_{k=1}^{+\infty} \frac{1}{1 - \tau^{\alpha + kr\delta}} \right)$$

since  $0 < \tau^\alpha < 1$  for any  $\alpha \in R_+$  and  $R^\circ$  is finite. We have to prove that the infinite products in the above expression are finite. Let  $\tau_0 = \max(\tau_i, i = 1, \dots, n)$ . Since  $ht(\delta) \geq n$ , we have  $\tau^\delta \leq \tau_0^n$ ; moreover  $\alpha + kr\delta \in Q_+$  for any  $k \geq 1$  and  $\alpha \in R^\circ$ . We therefore get

$$\prod_{k=1}^{+\infty} \frac{1}{(1 - \tau^{k\delta})^n} \leq \left( \prod_{k=1}^{+\infty} \frac{1}{1 - \tau_0^{kn}} \right)^n < +\infty$$

since the series  $\sum_{k=1}^{+\infty} \ln(1 - \tau_0^{kn})$  converges for  $\tau_0^n \in ]0, 1[$ . Similarly, since  $\tau_0^{rn} \in ]0, 1[$  one gets

$$\prod_{k=1}^{+\infty} \frac{1}{1 - \tau^{\alpha+kr\delta}} \leq \prod_{k=1}^{+\infty} \frac{1}{1 - \tau^\alpha \tau_0^{krn}} < +\infty.$$

The case  $A_{2n}^{(2)}$  is obtained by the same arguments.

- Secondly, assume that  $A$  is of indefinite type. By (5), we have  $S_\kappa(\tau) \leq \prod_{\alpha \in R_+} \left( \frac{1}{1 - \tau^\alpha} \right)^{n^{ht(\alpha)}}$ .

Moreover, since  $0 < \tau^\beta < 1$  for any  $\beta \in Q_+$  and  $R_+ \subset Q_+$ , we have also

$$S_\kappa(\tau) \leq \prod_{\substack{\beta \in Q_+ \\ \beta \neq 0}} \frac{1}{(1 - \tau^\beta)^{n^{ht(\beta)}}} = \prod_{k=1}^{+\infty} \prod_{\substack{\beta \in Q_+ \\ ht(\beta)=k}} \frac{1}{(1 - \tau^\beta)^{n^k}}$$

with

$$\prod_{\substack{\beta \in Q_+ \\ ht(\beta)=k}} \frac{1}{(1 - \tau^\beta)^{n^k}} \leq \prod_{\substack{\beta \in Q_+ \\ ht(\beta)=k}} \frac{1}{(1 - \tau_0^{ht(\beta)})^{n^k}} \leq \left( \frac{1}{1 - \tau_0^k} \right)^{(k+1)^n n^k}$$

since  $\#\{\beta \in Q_+ \mid ht(\beta) = k\} \leq (k+1)^n$ . We thus get

$$S_\kappa(\tau) \leq \prod_{k=1}^{+\infty} \frac{1}{(1 - \tau_0^k)^{n^k (k+1)^n}} < +\infty$$

using the fact that the series  $\sum_{k=1}^{+\infty} n^k (k+1)^n \ln(1 - \tau_0^k)$  converges for  $\tau_0 \in ]0, \frac{1}{n}[$ .  $\square$

**From now on**, we write  $\mathcal{T}$  for the set of  $n$ -tuples  $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}_{>0}^n$  such that

- $\tau_i \in ]0, 1[$  for  $1 \leq i \leq n$  when  $A$  is of finite or affine type,
- $\tau_i \in ]0, \frac{1}{n}[$  for  $1 \leq i \leq n$  when  $A$  is of indefinite type.

**Corollary 4.2.** *For any  $\mu \in P_+$  and  $w \in \mathbb{W}$ , the weight  $\mu + \rho - w(\mu + \rho)$  belongs to  $Q_+$ ; moreover, for  $\tau \in \mathcal{T}$ , one gets*

$$\left| \sum_{w \in \mathbb{W}} \varepsilon(w) \tau^{\mu + \rho - w(\mu + \rho)} \right| \leq \sum_{w \in \mathbb{W}} \tau^{\mu + \rho - w(\mu + \rho)} < +\infty.$$

*Proof.* By the Weyl-Kac character formula, one gets

$$e^{-\mu} s_\mu = \frac{\sum_{w \in \mathbb{W}} \varepsilon(w) e^{w(\mu + \rho) - \rho - \mu}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})^{m_\alpha}}.$$

Since  $e^{-\mu} s_\mu$  and  $\prod_{\alpha \in R_+} (1 - e^{-\alpha})^{m_\alpha}$  are polynomial in  $e^{-\beta}$  with  $\beta \in Q_+$ , we have  $\mu + \rho - w(\mu + \rho) \in Q_+$  for any  $w \in \mathbb{W}$ . Now observe that  $\mu + \rho$  belongs to  $P_+$  and is the dominant weight of  $V(\mu + \rho)$ , each  $w(\mu + \rho)$  is thus also a weight of  $V(\mu + \rho)$ . Therefore, the coefficients of the decomposition of  $s_{\mu + \rho} - \sum_{w \in \mathbb{W}} e^{w(\mu + \rho)}$  on the basis  $\{e^\beta \mid \beta \in P\}$  are nonnegative; in other words  $\sum_{w \in \mathbb{W}} e^{w(\mu + \rho)} \leq s_{\mu + \rho}$  which readily implies that  $\sum_{w \in \mathbb{W}} e^{w(\mu + \rho) - \mu - \rho} \leq e^{-\mu - \rho} s_{\mu + \rho}$ . By specializing  $e^{-\alpha_i} = \tau_i$ , one gets  $\sum_{w \in \mathbb{W}} \tau^{\mu + \rho - w(\mu + \rho)} \leq S_{\mu + \rho}(\tau) < +\infty$ .  $\square$

**Definition 4.3.** *We define the probability distribution  $p$  on  $B(\pi_\kappa)$  setting  $p_\pi = \frac{\tau^{\kappa - \text{wt}(\pi)}}{S_\kappa(\tau)}$ .*

**Remark:** By Assertion 3 of Theorem 3.6, for  $\pi'_\kappa$  another elementary path from 0 to  $\kappa$  such that  $\text{Im } \pi'_\kappa$  is contained in  $\mathcal{C}$ , there exists an isomorphism  $\Theta$  between the crystals  $B(\pi_\kappa)$  and  $B(\pi'_\kappa)$  and one gets  $p_\pi = p_{\Theta(\pi)}$  for any  $\pi \in B(\pi_\kappa)$ . Therefore, the probability distributions we use on the graph  $B(\pi_\kappa)$  are invariant by crystal isomorphisms.

Let  $X$  a random variable with values in  $B(\pi_\kappa)$  and probability distribution  $p$ ; as a direct consequence of Proposition 4.1, we get the

**Corollary 4.4.** *The variable  $X$  admits a moment of order 1. Moreover the series of functions*

$$m = \sum_{\pi \in B(\pi_\kappa)} p_b \pi$$

converges uniformly on  $[0, 1]$ .

*Proof.* Since the operators  $\tilde{f}_i$  and  $\tilde{e}_i$ ,  $1 \leq i \leq n$ , preserve the length of the paths, all the paths  $\pi$  in  $B(\pi_\kappa)$  have the same length  $L_\kappa$ ; consequently, for any  $t \in [0, 1]$ , we have  $\|\pi(t)\|_2 \leq L_\kappa$  with  $\sum_{\pi \in B(\pi_\kappa)} p_b L_\kappa = L_\kappa < +\infty$ . This proves that the series  $m$  converges uniformly on  $[0, 1]$ .  $\square$

**4.2. Random paths of arbitrary length.** We now extend the notion of elementary random paths. Assume that  $\pi_1, \dots, \pi_\ell$  a family of elementary paths; the path  $\pi_1 \otimes \dots \otimes \pi_\ell$  of length  $\ell$  is defined by: for all  $k \in \{1, \dots, \ell - 1\}$  and  $t \in [k, k + 1]$

$$(15) \quad \pi_1 \otimes \dots \otimes \pi_\ell(t) = \pi_1(1) + \dots + \pi_k(1) + \pi_{k+1}(t - k).$$

Let  $B^{\otimes \ell}(\pi_\kappa)$  be the set of paths of the form  $b = \pi_1 \otimes \dots \otimes \pi_\ell$  where  $\pi_1, \dots, \pi_\ell$  are elementary paths in  $B(\pi_\kappa)$ ; there exists a bijection  $\Delta$  between  $B^{\otimes \ell}(\pi_\kappa)$  and the set  $B^{*\ell}(\pi_\kappa)$  of paths in  $\mathcal{P}$  obtained by concatenations of  $\ell$  paths of  $B(\pi_\kappa)$ :

$$(16) \quad \Delta : \begin{cases} B^{\otimes \ell}(\pi_\kappa) & \longrightarrow & B^{*\ell}(\pi_\kappa) \\ \pi_1 \otimes \dots \otimes \pi_\ell & \longmapsto & \pi_1 * \dots * \pi_\ell \end{cases}.$$

In fact  $\pi_1 \otimes \dots \otimes \pi_\ell$  and  $\pi_1 * \dots * \pi_\ell$  coincide up to a reparametrization and we define the weight of  $b = \pi_1 \otimes \dots \otimes \pi_\ell$  setting

$$\text{wt}(b) := \text{wt}(\pi_1) + \dots + \text{wt}(\pi_\ell) = \pi_1(1) + \dots + \pi_\ell(1).$$

We now endow  $B^{\otimes \ell}(\pi_\kappa)$  with the product probability measure  $p^{\otimes \ell}$  defined by

$$(17) \quad p^{\otimes \ell}(\pi_1 \otimes \dots \otimes \pi_\ell) = p(\pi_1) \dots p(\pi_\ell) = \frac{\tau^{\ell\kappa - (\pi_1(1) + \dots + \pi_\ell(1))}}{S_\kappa(\tau)^\ell} = \frac{\tau^{\ell\kappa - \text{wt}(b)}}{S_\kappa(\tau)^\ell}.$$

In particular, for any  $b, b'$  in  $B^{\otimes \ell}(\pi_\kappa)$  such that  $\text{wt}(b) = \text{wt}(b')$ , one gets

$$p^{\otimes \ell}(b) = p^{\otimes \ell}(b').$$

Write  $\Pi_\ell : B^{\otimes \ell}(\pi_\kappa) \rightarrow B^{\otimes \ell-1}(\pi_\kappa)$  the projection defined by  $\Pi_\ell(\pi_1 \otimes \dots \otimes \pi_{\ell-1} \otimes \pi_\ell) = \pi_1 \otimes \dots \otimes \pi_{\ell-1}$ ; the sequence  $(B^{\otimes \ell}(\pi_\kappa), \Pi_\ell, p^{\otimes \ell})_{\ell \geq 1}$  is a projective system of probability spaces. We denote by  $(B^{\otimes \mathbb{N}}(\pi_\kappa), p^{\otimes \mathbb{N}})$  its injective limit; the elements of  $B^{\otimes \mathbb{N}}(\pi_\kappa)$  are infinite sequences  $b = (\pi_\ell)_{\ell \geq 1}$  and by a slight abuse of notation, we will also write  $\Pi_\ell(b) = \pi_1 \otimes \dots \otimes \pi_\ell$ .

Now let  $X = (X_\ell)_{\ell \geq 1}$  a sequence of i.i.d. random variables with values in  $B(\pi_\kappa)$  and probability distribution  $p$ ; the random path  $\mathcal{W}$  on  $(B^{\otimes \mathbb{N}}(\pi_\kappa), p^{\otimes \mathbb{N}})$  are thus defined by

$$\mathcal{W}(t) := \Pi_\ell(X)(t) = X_1 \otimes X_2 \otimes \dots \otimes X_{\ell-1} \otimes X_\ell(t) \text{ for } t \in [\ell - 1, \ell].$$

By (15), the path  $\mathcal{W}$  coincides with the one defined in § 2.3.

**Proposition 4.5.**

(1) For any  $\beta, \eta \in P$ , one gets

$$\mathbb{P}(W_{\ell+1} = \beta \mid W_\ell = \eta) = K_{\kappa, \beta - \eta} \frac{\tau^{\kappa + \eta - \beta}}{S_\kappa(\tau)}.$$

(2) Consider  $\lambda, \mu \in P^+$  we have

$$\mathbb{P}(W_\ell = \lambda, W_0 = \mu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, \ell]) = f_{\lambda/\mu}^\ell \frac{\tau^{\ell\kappa + \mu - \lambda}}{S_\kappa(\tau)^\ell}.$$

In particular

$$\mathbb{P}(W_{\ell+1} = \lambda, W_\ell = \mu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [\ell, \ell + 1]) = m_{\mu, \kappa}^\lambda \frac{\tau^{\kappa + \mu - \lambda}}{S_\kappa(\tau)}.$$

*Proof.* 1. We have

$$\mathbb{P}(W_{\ell+1} = \beta \mid W_\ell = \eta) = \sum_{\pi \in B(b_\pi)_{\beta - \eta}} p_\pi$$

where  $B(b_\pi)_{\beta - \eta}$  is the set of paths in  $B(b_\pi)$  of weight  $\beta - \eta$ . We conclude noticing that all the paths in  $B(b_\pi)_{\beta - \eta}$  have the same probability  $\frac{\tau^{\kappa + \eta - \beta}}{S_\kappa(\tau)}$  and  $|B(b_\pi)_{\beta - \eta}| = K_{\kappa, \beta - \eta}$ .

2. By Assertion 7 of Theorem 3.6, we know that the number of paths in  $B(\pi_\mu) * B^{*\ell}(\pi_\kappa)$  starting at  $\mu$ , ending at  $\lambda$  and remaining in  $\mathcal{C}$  is equal to  $f_{\lambda/\mu}^\ell$ . Since the map  $\Delta$  defined in (16) is a bijection, the integer  $f_{\lambda/\mu}^\ell$  is also equal to the number of paths in  $B(\pi_\mu) \otimes B^{\otimes \ell}(\pi_\kappa)$  starting at  $\mu$ , ending at  $\lambda$  and remaining in  $\mathcal{C}$ . Moreover, each such path has the form  $b = b_\mu \otimes b_1 \otimes \cdots \otimes b_\ell$  where  $b_1 \otimes \cdots \otimes b_\ell \in B^{\otimes \ell}(\pi_\kappa)$  has weight  $\lambda - \mu$ . Therefore we have  $p_b = \frac{\tau^{\ell\kappa + \mu - \lambda}}{S_\kappa(\tau)^\ell}$ .  $\square$

**4.3. The generalized Pitman transform.** By Assertion 8 of Theorem 3.6, we know that  $B^{\otimes \ell}(\pi_\kappa)$  is contained in  $\mathcal{P}_{\min \mathbb{Z}}$ . Therefore, if we consider a path  $b \in B^{\otimes \ell}(\pi_\kappa)$ , its connected component  $B(b)$  is contained in  $\mathcal{P}_{\min \mathbb{Z}}$ . Now, if  $\eta \in B(b)$  is such that  $\tilde{e}_i(\eta) = 0$  for any  $i = 1, \dots, n$ , we should have  $\text{Im } \eta \subset \mathcal{C}$  by Assertion 3 of Proposition 3.5; Assertion 1 of Theorem 3.6 thus implies that  $\eta$  is the unique path in  $B(b) = B(\eta)$  such that  $\tilde{e}_i(\eta) = 0$  for any  $i = 1, \dots, n$ . This permits to define the *generalized Pitman transform* on  $B^{\otimes \ell}(\pi_\kappa)$  as the map  $\mathfrak{P}$  which associates to any  $b \in B^{\otimes \ell}(\pi_\kappa)$  the unique path  $\mathfrak{P}(b) \in B(b)$  such that  $\tilde{e}_i(\eta) = 0$  for any  $i = 1, \dots, n$ . By definition, we have  $\text{Im } \mathfrak{P}(b) \subset \mathcal{C}$  and  $\mathfrak{P}(b)(\ell) \in P_+$ .

Let  $\mathcal{W}$  be the random path of § 4.2. We define the random process  $\mathcal{H}$  setting

$$(18) \quad \mathcal{H}(t) = \mathfrak{P}(\Pi_\ell(\mathcal{W}))(t) \text{ for any } t \in [\ell - 1, \ell].$$

For any  $\ell \geq 1$ , we set  $H_\ell := \mathcal{H}(\ell)$ ; one gets the

**Theorem 4.6.** *The random sequence  $H := (H_\ell)_{\ell \geq 1}$  is a Markov chain with transition matrix*

$$(19) \quad \Pi(\mu, \lambda) = \frac{S_\lambda(\tau)}{S_\kappa(\tau)S_\mu(\tau)} \tau^{\kappa + \mu - \lambda} m_{\mu, \kappa}^\lambda$$

where  $\lambda, \mu \in P_+$ .

*Proof.* Consider  $\mu = \mu^{(\ell)}, \mu^{(\ell-1)}, \dots, \mu^{(1)}$  a sequence of elements in  $P_+$ . Let  $\mathcal{S}(\mu^{(1)}, \dots, \mu^{(\ell)}, \lambda)$  be the set of paths  $b^h \in B^{\otimes \ell}(\pi_\kappa)$  remaining in  $\mathcal{C}$  and such that  $b^h(k) = \mu^{(k)}$ ,  $k = 1, \dots, \ell$  and  $b^{(\ell+1)} = \lambda$ . Consider  $b = b_1 \otimes \cdots \otimes b_\ell \otimes b_{\ell+1} \in B^{\otimes \ell+1}(\pi_\kappa)$ . We have  $\mathfrak{P}(b_1 \otimes \cdots \otimes b_\ell)(k) = \mu^{(k)}$  for any  $k = 1, \dots, \ell$  and  $\mathfrak{P}(b)(\ell+1) = \lambda$  if and only if  $\mathfrak{P}(b) \in \mathcal{S}(\mu^{(1)}, \dots, \mu^{(\ell)}, \lambda)$ . Moreover, by (17), for any

$b^h \in S(\mu^{(1)}, \dots, \mu^{(\ell)}, \lambda)$ , we have  $\mathbb{P}(b \in B(b^h)) = \sum_{b \in B(b^h)} p_b = \sum_{b \in B(b^h)} \frac{\tau^{(\ell+1)\kappa - \text{wt}(b)}}{S_\kappa(\tau)^{\ell+1}}$ ; combining (11)

and (13), one obtains  $\mathbb{P}(b \in B(b^h)) = \frac{\tau^{(\ell+1)\kappa - \lambda} S_\lambda(\tau)}{S_\kappa(\tau)^{\ell+1}}$ , which only depends on  $\lambda$ . This gives

$$\begin{aligned} bP(H_{\ell+1} = \lambda, H_k = \mu^{(k)}, \forall k = 1, \dots, \ell) &= \sum_{b^h \in S(\mu^{(1)}, \dots, \mu^{(\ell)}, \lambda)} \sum_{b \in B(b^h)} p_b \\ &= \sharp S(\mu^{(1)}, \dots, \mu^{(\ell)}, \lambda) \frac{\tau^{(\ell+1)\kappa - \lambda} S_\lambda(\tau)}{S_\kappa(\tau)^{\ell+1}}. \end{aligned}$$

By assertion 9 of Theorem 3.6 and an easy induction, we have also

$$\sharp S(\mu^{(1)}, \dots, \mu^{(\ell)}, \lambda) = \prod_{k=1}^{\ell-1} m_{\mu^{(k)}, \kappa}^{\mu^{(k+1)}} \times m_{\mu, \kappa}^\lambda.$$

We thus get

$$\mathbb{P}(H_{\ell+1} = \lambda, H_k = \mu^{(k)}, \forall k = 1, \dots, \ell) = \prod_{k=1}^{\ell-1} m_{\mu^{(k)}, \kappa}^{\mu^{(k+1)}} \times m_{\mu, \kappa}^\lambda \frac{\tau^{(\ell+1)\kappa - \lambda} S_\lambda(\tau)}{S_\kappa(\tau)^{\ell+1}}.$$

Similarly

$$\mathbb{P}(H_k = \mu^{(k)}, \forall k = 1, \dots, \ell) = \prod_{k=1}^{\ell-1} m_{\mu^{(k)}, \kappa}^{\mu^{(k+1)}} \frac{\tau^{\ell\kappa - \mu} S_\lambda(\tau)}{S_\kappa(\tau)^\ell},$$

this readily implies

$$\begin{aligned} \mathbb{P}(H_{\ell+1} = \lambda \mid H_k = \mu^{(k)}, \forall k = 1, \dots, \ell) &= \frac{\mathbb{P}(H_{\ell+1} = \lambda, H_k = \mu^{(k)}, \forall k = 1, \dots, \ell)}{\mathbb{P}(H_k = \mu^{(k)}, \forall k = 1, \dots, \ell)} \\ &= \frac{S_\lambda(\tau)}{S_\kappa(\tau) S_\mu(\tau)} \tau^{\kappa + \mu - \lambda} m_{\mu, \kappa}^\lambda. \end{aligned}$$

□

## 5. SYMMETRIZATION

In § 4.1, we have chosen a probability distribution  $p$  on a crystal  $B(\pi_\kappa)$  where  $\kappa \in P_+$  and  $\pi_\kappa$  is an elementary path from 0 to  $\kappa$  remaining in the cone  $\mathcal{C}$ . This distribution depends on  $\tau \in \mathbb{R}_{>0}^n$  and Proposition 4.1 gives a sufficient condition to ensure that  $S_\kappa(\tau)$  is finite. Since the characters of the highest weight representations are symmetric under the action of the Weyl group, it is possible to define, starting from the distribution  $p$  and for each  $w$  in the Weyl group  $\mathbb{W}$  of  $\mathfrak{g}$ , a probability distribution  $p_w$  which reflects this symmetry.

**5.1. Twisted probability distribution.** Recall  $\tau = (\tau_1, \dots, \tau_n) \in \mathcal{T}$  is fixed. Given any  $w \in \mathbb{W}$ , we want to define a probability distribution on  $B(\kappa)$  for each  $w \in \mathbb{W}$ . Recall that  $w(\alpha_i)$  is a (real) root of  $\mathfrak{g}$  for any  $w \in \mathbb{W}$  and any simple root  $\alpha_i$ ; this root is neither simple or even positive in general. By general properties of the root systems, we know that  $w(\alpha_i)$  can be decomposed as follows

$$w(\alpha_i) = \begin{cases} \alpha_{k_1} + \dots + \alpha_{k_r} \\ \text{or} \\ -(\alpha_{k_1} + \dots + \alpha_{k_r}) \end{cases}$$

where  $\alpha_{k_1}, \dots, \alpha_{k_r}$  are simple roots depending on  $w$ . Let us define the  $n$ -tuple  $\tau^w = (\tau_1^w, \dots, \tau_n^w) \in \mathbb{R}_{>0}^n$  setting

$$\tau_i^w = \begin{cases} \prod_{s=1}^r \tau_{k_s} & \text{if } w(\alpha_i) = \alpha_{k_1} + \dots + \alpha_{k_r}, \\ \prod_{s=1}^r \tau_{k_s}^{-1} & \text{if } w(\alpha_i) = -(\alpha_{k_1} + \dots + \alpha_{k_r}), \end{cases}$$

that is

$$(20) \quad \tau_i^w = \tau^{w(\alpha_i)}.$$

More generally for any  $\bar{u} = u_1\alpha_1 + \dots + u_n\alpha_n \in Q$ , we have

$$(\tau^w)^{\bar{u}} = (\tau_1^w)^{u_1} \dots (\tau_n^w)^{u_n} = \tau^{w(\bar{u})}.$$

Observe also that  $\tau^w \notin \mathcal{T}$  in general; indeed we have the following

**Lemma 5.1.**  $\tau(w) \in \mathcal{T}$  if and only if  $w = 1$ .

*Proof.* It suffices to show that for any  $w \in W \setminus \{Id\}$  distinct from the identity, there is at least a simple root  $\alpha_i$  such that  $w(\alpha_i) = -(\alpha_{k_1} + \dots + \alpha_{k_r}) \in -Q_+$ . Indeed, we will have in that case  $\tau_i^w = \frac{1}{\tau_{k_1} \dots \tau_{k_r}} > 1$  since  $\tau(w) \in \mathcal{T}$ . Consider  $w \in W \setminus \{Id\}$  such that  $w(\alpha_i) \in Q_+$  for any  $i = 1, \dots, \ell$ . Let us decompose  $w$  as a reduced word  $w = s_{i_1} \dots s_{i_\ell}$ ; by lemma 3.11 in [6], we must have  $w(\alpha_{i_i}) \in -Q_+$ , hence a contradiction. This means that  $w = 1$ .  $\square$

Consider  $\kappa \in P_+$ . Recall that we have by definition  $s_\kappa = e^\kappa S_\kappa(T_1, \dots, T_n)$  where  $T_i = e^{-\alpha_i}$ . Since  $s_\kappa$  is symmetric under  $W$ , we have  $s_\kappa = e^{w(\kappa)} S_\kappa(T_1^w, \dots, T_n^w)$  with  $T_i^w = e^{-w(\alpha_i)}$  for any  $i = 1, \dots, n$ . Therefore

$$S_\kappa(T_1^w, \dots, T_n^w) = e^{\kappa - w(\kappa)} S_\kappa(T_1, \dots, T_n) \text{ for any } w \in W.$$

Since  $\kappa - w(\kappa)$  belongs to  $Q$ , we can specialize each  $T_i$  in  $\tau_i$ . Then  $T_i^w$  is specialized in  $\tau_i^w$  and we get

$$(21) \quad S_\kappa(\tau^w) = \tau^{w(\kappa) - \kappa} S_\kappa(\tau),$$

in particular, it is finite.

**Definition 5.2.** For any  $w \in W$  and any integer  $\ell \geq 1$ , let  $p^w$  be the probability distribution on  $B(\pi_\kappa)^{\otimes \ell}$  defined by: for any  $b \in B(\pi_\kappa)^{\otimes \ell}$

$$p_b^w := \frac{(\tau^w)^{\ell\kappa - \text{wt}(b)}}{S_\kappa(\tau^w)^\ell} = \frac{\tau^{\ell w(\kappa) - \text{wt}(w(b))}}{S_\kappa(\tau^w)^\ell}$$

where  $w(b)$  is the image of  $b$  under the action of  $W$  (see Assertion 6 of Theorem 3.6). In particular,  $p^1 = p$  coincides with the probability distribution (17).

The following lemma states that the probabilities  $p^w$  and  $p$  coincide up to the permutation of the elements in  $B(\pi_\kappa)^{\otimes \ell}$  given by the action of  $w$  described in Assertion 6 of Theorem 3.6.

**Lemma 5.3.** For any  $w \in W$  and any  $b \in B(\pi_\kappa)^{\otimes \ell}$ , we have  $p_b^w = p_{w(b)}$ , where  $w(b)$  is the image of  $b$  under the action of  $W$  (see Assertion 6 of Theorem 3.6).

*Proof.* Recall that  $\text{wt}(w(b)) = w(\text{wt}(b))$ ; therefore  $p_{w(b)} = \frac{\tau^{\ell\kappa - \text{wt}(w(b))}}{S_\kappa(\tau)^\ell}$ . On the other hand, by (21)

we have  $p_b^w := \frac{\tau^{\ell w(\kappa) - \text{wt}(w(b))}}{S_\kappa(\tau^w)^\ell} = \frac{\tau^{\ell w(\kappa) - \text{wt}(w(b))}}{\tau^{\ell w(\kappa) - \ell\kappa} S_\kappa(\tau)^\ell}$  and the equality  $p_b^w = p_{w(b)}$  follows.  $\square$

**5.2. Twisted random paths.** Let  $w \in \mathbb{W}$  and denote by  $X^w$  the random variable defined on  $(B(\pi_\kappa), p^w)$  with law given by:

$$\mathbb{P}(X^w = \pi) = p_\pi^w = p_{w(\pi)} \quad \text{for all } \pi \in B(\pi_\kappa).$$

Set  $m^w := \mathbb{E}(X^w)$  and  $m := m^1$ .

**Proposition 5.4.** *Assume  $\tau \in \mathcal{T}$ . One gets*

- (1)  $m(1) \in \mathring{\mathcal{C}}$ ,
- (2)  $m^w = w^{-1}(m)$ ,
- (3)  $m^w(1) \in \mathring{\mathcal{C}}$  if and only if  $w$  is equal to the identity.

*Proof.* 1. By definition of  $\mathring{\mathcal{C}}$ , we have to prove that  $h_i(m(1)) > 0$  for any  $i = 1, \dots, n$ . Recall that  $m = \sum_{\pi \in B(\pi_\kappa)} p_\pi \pi$ ; observe that the quantity

$$c_i = h_i(m(1)) = \sum_{\pi \in B(\pi_\kappa)} p_\pi h_i(\pi(1))$$

is well defined since  $\sum_{\pi \in B(\pi_\kappa)} p_\pi \|\pi(1)\|_2 < +\infty$ . We can decompose the crystal  $B(\pi_\kappa)$  in its  $i$ -chains, that is the sub-crystal obtained by deleting all the arrows  $j \neq i$ . When  $\mathfrak{g}$  is not of finite type, the lengths of these  $i$ -chains are all finite but not bounded. The contribution to  $c_i$  of any

$i$ -chain  $C : a_0 \xrightarrow{i} a_1 \xrightarrow{i} \dots \xrightarrow{i} a_k$  of length  $k$  is equal to  $c_i(C) = \sum_{j=0}^k p_{a_j} h_i(\text{wt}(a_j))$ . Since  $\tilde{e}_i(a_0) = 0$

and  $\tilde{f}_i^{k+1}(a_0) = 0$ , we obtain  $h_i(\text{wt}(a_0)) = k$ . By definition of the distribution  $p$  and Proposition 3.4, we have the relation  $p_{a_j} = \tau_i^j p_{a_0}$ . Finally, we get

$$c_i(C) = p_{a_0} \sum_{j=0}^k \tau_i^j (k - 2j) = p_{a_0} \sum_{j=0}^{\lfloor k/2 \rfloor} (k - 2j) (\tau_i^j - \tau_i^{k-j}).$$

In particular the hypothesis  $\tau_i \in ]0, 1[$  for any  $i = 1, \dots, n$  implies that  $c_i(C) > 0$  for any  $i$ -chain of length  $k > 0$ ; one thus gets  $c_i > 0$  noticing that  $B(\pi_\kappa)$  contains at least an  $i$ -chain of length  $k > 0$ , otherwise the action of the Chevalley generators  $e_i, f_i$  on the irreducible module  $V(\pi_\lambda)$  would be trivial.

2. By Lemma 5.3, we can write

$$m^w = \sum_{\pi \in B(\pi_\kappa)} p_{w(\pi)} \pi = \sum_{\pi' \in B(\pi_\kappa)} p_{\pi'} w^{-1}(\pi') = w^{-1} \left( \sum_{\pi' \in B(\pi_\kappa)} p_{\pi'} \pi' \right) = w^{-1}(m)$$

where we use assertion 7 of Theorem 3.6 in the third equality.

3. Since  $m^w = w^{-1}(m)$  and  $m(1) \in \mathring{\mathcal{C}}$ , one gets  $m^w(1) \notin \mathcal{C}$  because  $\mathcal{C}$  is a fundamental domain for the action of the Weyl group  $\mathbb{W}$  on the tits cone  $\mathcal{X} = \cup_{w \in \mathbb{W}} w(\mathcal{C})$  (see [6] Proposition 3.12).  $\square$

Now let  $X^w = (X_\ell^w)_{\ell \geq 1}$  be a sequence of i.i.d. random variables defined on  $B(\pi_\kappa)$  with probability distribution  $p^w$ . The random process  $\mathcal{W}^w = (\mathcal{W}_t^w)_{t > 0}$  is defined by: for all  $\ell \geq 1$  and  $t \in [\ell - 1, \ell]$

$$\mathcal{W}^w(t) := \Pi_\ell(X^w)(t) = X_1^w \otimes X_2^w \otimes \dots \otimes X_{\ell-1}^w \otimes X_\ell^w(t).$$

By (15), the random walk  $W^w$  is defined as in § 2.3 from  $\mathcal{W}^w$ . For any  $\ell \in \mathbb{Z}_{\geq 0}$ , we also define the function  $\psi_\ell^w$  on  $P_+$  setting

$$\psi_\ell^w(\mu) := \mathbb{P}_\mu(\mathcal{W}^w(t) \in \mathcal{C} \text{ for any } t \in [0, \ell]).$$

The quantity  $\psi_\ell^w(\mu)$  is equal to the probability of the event “ $\mathcal{W}^w$  starting at  $\mu$  remains in the cone  $\mathcal{C}$  until the instant  $\ell$ ”. We also introduce the function

$$\psi^w(\mu) := \mathbb{P}_\mu(\mathcal{W}^w(t) \in \mathcal{C}, t \geq 0).$$

For  $w = 1$ , we simply write  $\psi$  and  $\psi_\ell$  instead of  $\psi^1$  and  $\psi_\ell^1$ .

The following proposition is a consequence of the previous lemma and Proposition 2.2.

**Proposition 5.5.**

- (1) We have  $\lim_{\ell \rightarrow +\infty} \psi_\ell^w(\mu) = \psi^w(\mu)$  for any  $\mu \in P_+$ .
- (2) If  $w \neq 1$ , then  $\psi^w(\mu) = 0$  for any  $\mu \in P_+$ .
- (3) If  $w = 1$ , then  $\psi(\mu) > 0$  for any  $\mu \in P_+$ .

Similarly to Proposition 4.5 and using (21), we obtain the

**Proposition 5.6.**

- (1) For any weights  $\beta$  and  $\eta$ , one gets

$$\mathbb{P}(W_{\ell+1}^w = \beta \mid W_\ell^w = \eta) = K_{\kappa, \beta - \eta, \frac{\tau^{w(\kappa + \eta - \beta)}}{S_\kappa(\tau^w)}} = K_{\kappa, \beta - \eta, \frac{\tau^{\kappa + w(\eta) - w(\beta)}}{S_\kappa(\tau)}}.$$

- (2) For any dominant weights  $\lambda$  and  $\mu$ , one gets

$$\mathbb{P}(W_\ell^w = \lambda, W_0^w = \mu, \mathcal{W}^w(t) \in \mathcal{C} \text{ for any } t \in [0, \ell]) = f_{\lambda/\mu}^\ell \frac{\tau^{w(\ell\kappa + \mu - \lambda)}}{S_\kappa(\tau^w)^\ell} = f_{\lambda/\mu}^\ell \frac{\tau^{\ell\kappa + w(\mu) - w(\lambda)}}{S_\kappa(\tau)^\ell}.$$

In particular

$$\mathbb{P}(W_{\ell+1}^w = \lambda, W_\ell^w = \mu, \mathcal{W}^w(t) \in \mathcal{C} \text{ for any } t \in [\ell, \ell + 1]) = m_{\mu, \kappa}^\lambda \frac{\tau^{w(\kappa + \mu - \lambda)}}{S_\kappa(\tau^w)} = m_{\mu, \kappa}^\lambda \frac{\tau^{\kappa + w(\mu) - w(\lambda)}}{S_\kappa(\tau)}.$$

## 6. THE HARMONIC FUNCTION $\psi$

By Assertion 2 of the previous proposition, we can write

$$(22) \quad \psi_\ell^w(\mu) = \mathbb{P}_\mu(\mathcal{W}^w(t) \in \mathcal{C} \text{ for any } t \in [0, \ell]) = \sum_{\lambda \in P_+} f_{\lambda/\mu}^\ell \frac{\tau^{\ell\kappa + w(\mu) - w(\lambda)}}{S_\kappa(\tau)^\ell}$$

where  $f_{\lambda/\mu}^\ell$  is the number of highest weight vertices in the crystal

$$(23) \quad B(\mu) \otimes B(\kappa)^{\otimes \ell} \simeq \bigoplus_{\lambda \in P_+} B(\lambda)^{\oplus f_{\lambda/\mu}^\ell}.$$

By interpreting (23) in terms of characters, we get

$$(24) \quad s_\mu \times s_\kappa^\ell = \sum_{\lambda \in P_+} f_{\lambda/\mu}^\ell s_\lambda$$

where all but a finite number of coefficients  $f_{\lambda/\mu}^\ell$  are equal to 0. The Weyl character formula

$$s_\lambda = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})^{m_\alpha}}$$



yields

$$\prod_{\alpha \in R_+} (1 - e^{-\alpha})^{m_\alpha} s_\mu \times s_\kappa^\ell = \sum_{\lambda \in P_+} f_{\lambda/\mu}^\ell \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho} \text{ and}$$

$$\prod_{\alpha \in R_+} (1 - \tau^\alpha)^{m_\alpha} S_\mu(\tau) = \sum_{\lambda \in P_+} f_{\lambda/\mu}^\ell \sum_{w \in W} \varepsilon(w) \frac{e^{-\ell\kappa+w(\lambda+\rho)-\rho-\mu}}{e^{-\ell\kappa} s_\kappa^\ell}.$$

Setting  $\tau_i = e^{-\alpha_i}$ , we get

$$\prod_{\alpha \in R_+} (1 - \tau^\alpha)^{m_\alpha} S_\mu(\tau) = \sum_{\lambda \in P_+} f_{\lambda/\mu}^\ell \sum_{w \in W} \varepsilon(w) \frac{\tau^{\ell\kappa+\rho+\mu-w(\lambda+\rho)}}{S_\kappa(\tau)^\ell}.$$

The sum indexed by  $\lambda$  is over finitely many terms, we can thus flip the two sums in the right hand side, getting

$$(25) \quad \prod_{\alpha \in R_+} (1 - \tau^\alpha)^{m_\alpha} S_\mu(\tau) = \sum_{w \in W} \varepsilon(w) \sum_{\lambda \in P_+} f_{\lambda/\mu}^\ell \frac{\tau^{\ell\kappa+\rho+\mu-w(\lambda+\rho)}}{S_\kappa(\tau)^\ell}.$$

We now need the following lemma.

**Lemma 6.1.** *For any  $w \in W$  and  $\mu \in P_+$ , set  $\Pi_\ell^w(\mu) := \sum_{\lambda \in P_+} f_{\lambda/\mu}^\ell \frac{\tau^{\ell\kappa+\rho+\mu-w(\lambda+\rho)}}{S_\kappa(\tau)^\ell}$ .*

*We then have  $\lim_{\ell \rightarrow +\infty} \Pi_\ell^w(\mu) = 0$  when  $w \neq 1$  and the series  $\sum_{w \in W} \varepsilon(w) \Pi_\ell^w(\mu)$  converges uniformly in  $\ell$ .*

*Proof.* Using (22), one gets

$$(26) \quad \Pi_\ell^w(\mu) = \tau^{\rho-w(\rho)+\mu-w(\mu)} \sum_{\lambda \in P_+} f_{\lambda/\mu}^\ell \frac{\tau^{\ell\kappa+w(\mu)-w(\lambda)}}{S_\kappa(\tau)^\ell} = \tau^{\rho-w(\rho)+\mu-w(\mu)} \psi_\ell^w(\mu).$$

Fix  $w \neq 1$ . Since  $\tau^{\rho-w(\rho)+\mu-w(\mu)}$  does not depend on  $\ell$  and  $\lim_{\ell \rightarrow +\infty} \psi_\ell^w(\mu) = 0$  by Proposition 5.5, we derive  $\lim_{\ell \rightarrow +\infty} \Pi_\ell^w(\mu) = 0$  as desired (recall the sum is over finitely many terms).

Now, we have obviously  $0 \leq \psi_\ell^w(\mu) \leq 1$  and the series  $\sum_{w \in W} \tau^{\rho-w(\rho)+\mu-w(\mu)}$  converges by Corollary 4.2. The uniform convergence in  $\ell$  of the series  $\sum_{w \in W} \varepsilon(w) \Pi_\ell^w(\mu)$  thus follows from the inequality  $|\varepsilon(w) \Pi_\ell^w(\mu)| \leq \tau^{\rho-w(\rho)+\mu-w(\mu)}$ , which is a direct consequence of (26).  $\square$

Consequently, we have

$$\prod_{\alpha \in R_+} (1 - \tau^\alpha)^{m_\alpha} S_\mu(\tau) = \sum_{w \in W} \varepsilon(w) \Pi_\ell^w(\mu) = \Pi_\ell^1(\mu) + \sum_{w \neq 1} \varepsilon(w) \Pi_\ell^w(\mu)$$

with  $\Pi_\ell^1(\mu) = \psi_\ell(\mu)$ , by (26). Letting  $\ell \rightarrow +\infty$ , the previous lemma finally gives

$$\psi(\mu) = \prod_{\alpha \in R_+} (1 - \tau^\alpha)^{m_\alpha} S_\mu(\tau).$$

We have established the following theorem, which is the analogous in our context of Corollary 7.4.3 in [10]:

**Theorem 6.2.** *For any  $\mu \in P_+$ , we have*

$$\psi(\mu) = \mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0) = \prod_{\alpha \in R_+} (1 - \tau^\alpha)^{m_\alpha} S_\mu(\tau).$$

In particular, the harmonic function  $\psi$  does not depend on the dominant weight  $\kappa$  considered.

**Corollary 6.3.** *The law of the random walk  $W$  conditioned by the event*

$$E := (\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0)$$

is the same as the law of the Markov chain  $H$  defined as the generalized Pitman transform of  $W$  (see Theorem 4.6). In particular, this law only depends on  $\kappa$  and not on the choice of the path  $\pi_\kappa$  such that  $\text{Im } \pi_\kappa \subset \mathcal{C}$ .

*Proof.* Let  $\Pi$  be the transition matrix of  $W$  and  $\Pi^E$  its restriction to the event  $E$ . We have seen in § 2.1 that the transition matrix of  $W$  conditioned by  $E$  is the  $h$ -transform of  $\Pi^E$  by the harmonic function

$$h_E(\mu) := \mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0).$$

By the previous theorem, we have  $h_E = \psi$ . It also easily follows from Theorem 4.6 that the transition matrix of  $H$  is the  $\psi$ -transform of  $\Pi^E$ . Therefore both  $H$  and the conditioning of  $W$  by  $E$  have the same law.  $\square$

**6.1. Random walks defined from non irreducible representations.** For simplicity we restrict ourselves in this paragraph to the case where  $\mathfrak{g}$  is a (finite-dimensional) Lie algebra with (invertible) Cartan matrix  $A$ . Assume  $\tau = (\tau_1, \dots, \tau_n) \in \mathcal{T}$ . Then  $A$  is the transition matrix between the weight and root lattices. In particular, each weight  $\beta \in P$  decomposes on the basis of simple roots as  $\beta = \beta'_1 \alpha_1 + \dots + \beta'_n \alpha_n$  where  $(\beta'_1, \dots, \beta'_n) \in \frac{1}{\det A} \mathbb{Z}^n$  and we set  $\tau^\beta = \tau_1^{\beta'_1} \dots \tau_n^{\beta'_n}$ .

Let  $M$  be a finite dimensional  $\mathfrak{g}$ -module with decomposition in irreducible components

$$M \simeq \bigoplus_{\kappa \in \mathcal{X}} V(\kappa)^{\oplus a_\kappa}$$

where  $\mathcal{X}$  is a finite subset of  $P_+$  and  $a_\kappa > 0$  for any  $\kappa \in \mathcal{X}$ . For each  $\kappa \in \mathcal{X}$  choose a path  $\eta$  in  $P$  from 0 to  $\kappa$  contained in  $\mathcal{C}$ . Write  $H(M)$  for the set of highest paths so introduced. Let  $B(\mathcal{X})$  be the set of paths obtained by applying the operators  $\tilde{e}_i, \tilde{f}_i, i = 1, \dots, n$  to the paths in  $H(M)$ ; this set is a realization of the crystal of the  $\mathfrak{g}$ -module  $\bigoplus_{\kappa \in \mathcal{X}} V(\kappa)$  (without multiplicities) and we have

$$B(\mathcal{X}) = \bigsqcup_{\eta \in H(M)} B(\eta).$$

To define a probability distribution on  $B(\mathcal{X})$  compatible with its weight graduation and taking into account the multiplicities  $a_\kappa$ , we first set

$$\Sigma_M(\tau) = \sum_{\pi \in B(\mathcal{X})} a_\kappa \tau^{-\text{wt}(\pi)} = \sum_{\kappa \in \mathcal{X}} a_\kappa \tau^{-\kappa} S_\kappa(\tau).$$

We define the probability distribution  $p$  on  $B(\mathcal{X})$  setting  $p_\pi = a_\kappa \frac{\tau^{-\text{wt}(\pi)}}{\Sigma_M(\tau)}$  for any  $\pi \in B(\mathcal{X})$ . Then two paths of  $B(\mathcal{X})$  with the same weight have the same probability. It is then possible to extend our results to the random path  $\mathcal{W}$  and its corresponding random walk  $W$  obtained from the set of elementary paths  $B(\mathcal{X})$ . We have then

$$\mathbb{P}(W_{\ell+1} = \beta \mid W_\ell = \gamma) = \frac{K_{M, \beta - \gamma} \tau^{\gamma - \beta}}{\Sigma_M(\tau)}$$

for any weights  $\beta$  and  $\gamma$  where  $K_{M, \beta - \gamma}$  is the dimension of the space of weight  $\beta - \gamma$  in  $M$ . We thus have  $K_{M, \beta - \gamma} = \sum_{\kappa \in \mathcal{X}} a_\kappa K_{\kappa, \beta - \gamma}$  where  $K_{\kappa, \beta - \gamma}$  is the number of paths  $\eta \in B(\kappa)$  such that

$\eta(1) = \beta - \gamma$ . Given  $\lambda$  and  $\mu$  two dominant weights, we also get

$$\mathbb{P}(W_{\ell+1} = \lambda \mid W_\ell = \mu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [\ell, \ell + 1]) = \frac{m_{M,\mu}^\lambda}{\Sigma_M(\tau)} \tau^{\mu-\lambda}$$

where  $m_{M,\mu}^\lambda$  is the multiplicity of  $V(\lambda)$  in  $M \otimes V(\mu)$ . We then define similarly the generalized Pitman transform and the Markov chain  $H$  as in § 4.3. Our arguments can then be easily extended and we obtain the same harmonic function  $\psi : \mu \rightarrow \mathbb{P}(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0)$  together with the following theorem. Indeed, we do not need the fact that  $M$  is irreducible in our proofs.

**Theorem 6.4.** *The law of the random walk  $W$  conditioned by the event*

$$E := (\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0)$$

*is the same as the law of the Markov chain  $H$  defined as the generalized Pitman transform of  $W$  (see Theorem 4.6). The associated transition matrix  $\Pi^E$  verifies*

$$(27) \quad \Pi^E(\mu, \lambda) = \frac{S_\lambda(\tau)}{S_\mu(\tau)\Sigma_M(\tau)} m_{M,\mu}^\lambda \tau^{\mu-\lambda}.$$

**6.2. Example: random walk to the height closest neighbors.** We now study in detail the case of the random walk in the plane with transitions 0 and the height closest neighbors. The underlying representation is not irreducible and does not decompose as a sum of minuscule representations. So the conditioning of this walk cannot be obtained by the methods of [10].<sup>2</sup>

The root system of type  $C_2$  is realized in  $\mathbb{R}^2 = \mathbb{R}\varepsilon_1 \oplus \mathbb{R}\varepsilon_2$ . The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

The simple roots are then  $\alpha_1 = \varepsilon_1 - \varepsilon_2$  and  $\alpha_2 = 2\varepsilon_2$ . We have  $P = \mathbb{Z}^2$ . The fundamental weights are  $\omega_1 = \varepsilon_1$  and  $\omega_2 = \varepsilon_1 + \varepsilon_2$ . We have  $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x \geq y \geq 0\}$  and  $P_+ = \{\lambda = (\lambda_1, \lambda_2) \mid \lambda_1 \geq \lambda_2 \geq 0\}$ , the set of partitions with two parts. Choose  $\tau_1 \in ]0, 1[$ ,  $\tau_2 \in ]0, 1[$ . For  $\lambda = (\lambda_1, \lambda_2) \in P_+$ , we have  $\lambda = \lambda_1\alpha_1 + \frac{\lambda_1+\lambda_2}{2}\alpha_2$ . Thus  $\tau^\lambda = (\tau_1^{\lambda_1}(\sqrt{\tau_2})^{\lambda_1+\lambda_2})^{-1}$ .

Consider the  $\mathfrak{sp}_4(\mathbb{C})$ -module  $M = V(1)^{\oplus m_1} \oplus V(1, 1)^{\oplus m_2}$ . The elementary paths in  $B(\varkappa)$  can be easily described from the highest weight paths

$$\pi_1 : t \mapsto t\varepsilon_1 \text{ and } \gamma_{12} : \begin{cases} 2t\varepsilon_1, t \in [0, \frac{1}{2}] \\ \varepsilon_1 + 2(t - \frac{1}{2})\varepsilon_2, t \in ]\frac{1}{2}, 1] \end{cases} \text{ in } \mathcal{C}.$$

We obtain  $B(\varkappa) = B(\pi_1) \oplus B(\gamma_{12})$  where

- (1)  $B(\pi_1) : \pi_1 : t \mapsto t\varepsilon_1, \pi_2 : t \mapsto t\varepsilon_2, \pi_{\bar{1}} : t \mapsto -t\varepsilon_1$  and  $\pi_{\bar{2}} : t \mapsto -t\varepsilon_2$  with  $t \in [0, 1]$
- (2)  $B(\gamma_{12}) :$

$$\begin{aligned} \gamma_{12} &: \begin{cases} 2t\varepsilon_1, t \in [0, \frac{1}{2}] \\ \varepsilon_1 + 2(t - \frac{1}{2})\varepsilon_2, t \in ]\frac{1}{2}, 1] \end{cases} & \gamma_{1\bar{2}} &: \begin{cases} 2t\varepsilon_1, t \in [0, \frac{1}{2}] \\ \varepsilon_1 - 2(t - \frac{1}{2})\varepsilon_2, t \in ]\frac{1}{2}, 1] \end{cases} \quad , \\ \gamma_{2\bar{2}} &: \begin{cases} 2t\varepsilon_2, t \in [0, \frac{1}{2}] \\ \varepsilon_2 - 2(t - \frac{1}{2})\varepsilon_2, t \in ]\frac{1}{2}, 1] \end{cases} \\ \gamma_{2\bar{1}} &: \begin{cases} 2t\varepsilon_2, t \in [0, \frac{1}{2}] \\ \varepsilon_2 - 2(t - \frac{1}{2})\varepsilon_1, t \in ]\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma_{\bar{2}\bar{1}} &: \begin{cases} -2t\varepsilon_2, t \in [0, \frac{1}{2}] \\ -\varepsilon_2 - 2(t - \frac{1}{2})\varepsilon_1, t \in ]\frac{1}{2}, 1] \end{cases} \quad . \end{aligned}$$

<sup>2</sup>The results of [10] permit only to study the random walk *in the space*  $\mathbb{R}^3$  with transitions  $\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$  corresponding to the weights of the spin representation of  $\mathfrak{g} = \mathfrak{so}_9$ .

The crystal  $B(\varkappa)$  is the union of the two following crystals

$$\begin{array}{ccccccc} \pi_1 & \xrightarrow{1} & \pi_2 & \xrightarrow{2} & \pi_{\bar{2}} & \xrightarrow{1} & \pi_{\bar{1}} \\ \gamma_{12} & \xrightarrow{2} & \gamma_{1\bar{2}} & \xrightarrow{1} & \gamma_{2\bar{2}} & \xrightarrow{1} & \gamma_{2\bar{1}} & \xrightarrow{2} & \gamma_{\bar{2}\bar{1}} \end{array}$$

Observe that for the path  $\gamma_{2\bar{2}}$ , we have  $\gamma_{2\bar{2}}(0) = \gamma_{2\bar{2}}(1) = 0$ . The other transitions correspond to the 8 closest neighbors in the lattice  $\mathbb{Z}^2$ .

We now define the probability distribution  $p$  on the set  $B(\pi_1)^{\oplus m_1} \oplus B(\gamma_{12})^{\oplus m_2}$ . We have

$$\Sigma_M(\tau) = m_1 \frac{1 + \tau_1 + \tau_1\tau_2 + \tau_1^2\tau_2}{\tau_1\sqrt{\tau_2}} + m_2 \frac{1 + \tau_2 + \tau_1\tau_2 + \tau_1^2\tau_2 + \tau_1^2\tau_2^2}{\tau_1\tau_2}.$$

The probability  $p$  is defined by

$$\begin{aligned} p_1 &= \frac{m_1}{\Sigma_M(\tau)\tau_1\sqrt{\tau_2}}, & p_2 &= \frac{m_1}{\Sigma_M(\tau)\sqrt{\tau_2}}, & p_{\bar{2}} &= \frac{m_1\sqrt{\tau_2}}{\Sigma_M(\tau)}, & p_{\bar{1}} &= \frac{m_1\tau_1\sqrt{\tau_2}}{\Sigma_M(\tau)} \\ p_{12} &= \frac{m_2}{\Sigma_M(\tau)\tau_1\tau_2}, & p_{1\bar{2}} &= \frac{m_2}{\Sigma_M(\tau)\tau_1}, & p_{2\bar{2}} &= \frac{m_2}{\Sigma_M(\tau)}, & p_{2\bar{1}} &= \frac{m_2\tau_1}{\Sigma_M(\tau)}, & p_{\bar{2}\bar{1}} &= \frac{m_2\tau_1\tau_2}{\Sigma_M(\tau)}. \end{aligned}$$

The set of positive roots is

$$R_+ = \{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\} \text{ and } \rho = (2, 1).$$

The action of the Weyl group on  $\mathbb{Z}^2$  yields the 8 transformations which preserves the square of vertices  $(\pm 1, \pm 1)$ . For any partition  $\mu = (\mu_1, \mu_2) \in P_+$ , we obtain from the Weyl character formula

$$\begin{aligned} S_\mu(\tau_1, \tau_2) &:= \frac{\sum_{w \in W} \varepsilon(w) \tau^{w(\mu+\rho) - (\mu+\rho)}}{(1-\tau_1)(1-\tau_2)(1-\tau_1\tau_2)(1-\tau_1^2\tau_2)} = \\ &= \frac{1 - \tau_1^{\mu_1 - \mu_2 + 1} - \tau_2^{\mu_2 + 1} + \tau_1^{\mu_1 + \mu_2 + 3} \tau_2^{\mu_2 + 1} + \tau_1^{\mu_1 - \mu_2 + 1} \tau_2^{\mu_1 + 2} - \tau_1^{2\mu_1 + 4} \tau_2^{\mu_1 + 2} - \tau_1^{\mu_1 + \mu_2 + 3} \tau_2^{\mu_1 + \mu_2 + 3} + \tau_1^{2\mu_1 + 4} \tau_2^{\mu_1 + \mu_2 + 3}}{(1-\tau_1)(1-\tau_2)(1-\tau_1\tau_2)(1-\tau_1^2\tau_2)} \end{aligned}$$

and  $S_\mu(\tau_1, \tau_2) \in \mathbb{Z}_{\geq 0}[\tau_1, \tau_2]$ . By Theorem 6.2

$$\mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C}, t \geq 0) = (1-\tau_1)(1-\tau_2)(1-\tau_1\tau_2)(1-\tau_1^2\tau_2)S_\mu(\tau_1, \tau_2).$$

Moreover, the law of the random walk  $W$  conditioned by the event

$$E := (\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0)$$

is the same as the law of the Markov chain  $H$  defined as the generalized Pitman transform of  $W$  (see Theorem 4.6). To compute the associated transition matrix  $M$ , we need the tensor product multiplicities  $m_{\mu, M}^\lambda = m_1 m_{(1,0), \mu}^\lambda + m_2 m_{(1,1), \mu}^\lambda$ . We have for any partitions  $\lambda$  and  $\mu$  with two parts

$$m_{(1,0), \mu}^\lambda = \begin{cases} 1 & \text{if } \lambda \text{ and } \mu \text{ are equal or differ by only one box} \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_{(1,1), \mu}^\lambda = \begin{cases} 1 & \text{if } \lambda \text{ and } \mu \text{ are equal or differ by two boxes in different rows} \\ 0 & \text{otherwise.} \end{cases}$$

We thus have for any  $\lambda, \mu \in P_+$

$$\Pi^E(\mu, \lambda) = \frac{S_\lambda(\tau)}{S_\mu(\tau)\Sigma_M(\tau)} \left( m_1 m_{(1,0), \mu}^\lambda + m_2 m_{(1,1), \mu}^\lambda \right) \tau_1^{\mu_2 - \lambda_2} (\sqrt{\tau_2})^{-(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)}.$$

## 7. SOME CONSEQUENCES

In the remaining of the paper, we assume  $\mathcal{W}$  is constructed from an the irreducible  $\mathfrak{g}$ -module  $V(\kappa)$  in the category  $\mathcal{O}_{int}$ .

**7.1. Asymptotics for the multiplicities  $f_{\lambda/\mu}^\ell$ .** Consider  $\lambda, \mu \in P_+$  and  $\ell \geq 1$  such that  $f_{\lambda/\mu}^\ell > 0$  and  $f_\lambda^\ell > 0$ . Then, we must have  $\ell\kappa + \mu - \lambda \in Q_+$  and  $\ell\kappa - \lambda \in Q_+$ . Therefore  $\mu \in Q$  and it decomposes as a sum of simple roots. In the sequel, we will assume the condition  $\mu \in Q \cap P_+$  is satisfied.

We assume the notation and hypotheses of Theorem 6.2. Consider a sequence  $\lambda^{(\ell)}$  of dominant weights such that  $\lambda^{(\ell)} = \ell m(1) + o(\ell)$ . By Assertion 2 of Proposition 4.5, we have  $\mathbb{P}(W_\ell = \lambda^{(\ell)}, W_0 = \mu, \mathcal{W}(t) \in \mathcal{C} \forall t \in [0, \ell]) = f_{\lambda^{(\ell)}/\mu}^\ell \frac{\tau^{\ell\kappa + \mu - \lambda}}{S_\kappa(\tau)^\ell}$ . Since  $m(1) \in \mathring{\mathcal{C}}$ , the strong law of large numbers implies that the sequence  $(\frac{1}{\ell} W_\ell)_{\ell \geq 0}$  tends to  $m(1)$  almost surely. We get

$$\lim_{\ell \rightarrow +\infty} \frac{\mathbb{P}_\mu(W_\ell = \lambda^{(\ell)}, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, \ell])}{\mathbb{P}_0(W_\ell = \lambda^{(\ell)}, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, \ell])} = \lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}/\mu}^\ell \tau^\mu}{f_{\lambda^{(\ell)}}^\ell}.$$

On the other hand,  $\lim_{\ell \rightarrow +\infty} \mathbb{P}_\mu(W_\ell = \lambda^{(\ell)}, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, \ell]) = \psi(\mu)$ . So we obtain for any sequence of dominant weights of the form  $\lambda^{(\ell)} = \ell m(1) + o(\ell)$

$$\lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}/\mu}^\ell \tau^\mu}{f_{\lambda^{(\ell)}}^\ell} = \frac{\psi(\mu)}{\psi(0)} = S_\mu(\tau)$$

where the last equality follows from Theorem 6.2 for  $S_0(\tau) = 1$ . We have proved the following Corollary of Theorem 6.2.

**Corollary 7.1.** *For any  $\mu \in Q \cap P_+$ , and any sequence of dominant weights of the form  $\lambda^{(\ell)} = \ell m(1) + o(\ell)$ , we have  $\lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}/\mu}^\ell}{f_{\lambda^{(\ell)}}^\ell} = \tau^{-\mu} S_\mu(\tau)$ .*

**7.2. Probability that  $W$  stay in  $\mathcal{C}$ .** By Theorem 6.2, we can compute  $\mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, \ell])$ . Unfortunately, this does not permit to make explicit  $\mathbb{P}_\mu(W_\ell \in \mathcal{C} \forall \ell \geq 1)$ . Nevertheless, we have the immediate inequality

$$\mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0) \leq \mathbb{P}_\mu(W_\ell \in \mathcal{C} \text{ for any } \ell \geq 1).$$

In the remaining of this paragraph, we will assume that  $\mathfrak{g}$  is of finite type. Then, each crystal  $B(\pi_\kappa)$  is finite. For any  $i = 1, \dots, n$ , write  $m_0(i) \geq 1$  the maximal length of the  $i$ -chains appearing in  $B(\pi_\kappa)$ . Set  $\kappa_0 = \sum_{i=1}^n (m_0(i) - 1)\omega_i$ . Observe that  $\kappa_0 = 0$  if and only if  $\kappa$  is a minuscule weight.

**Lemma 7.2.** *Assume  $W_k \in \mathcal{C}$  for any  $k = 1, \dots, \ell$ . Then  $\kappa_0 + \mathcal{W}(t) \in \mathcal{C}$  for any  $t \in [0, \ell]$ .*

*Proof.* Since  $\kappa_0$  is a dominant weight, we can consider  $\pi_{\kappa_0}$  any path from 0 to  $\kappa_0$  which remains in  $\mathcal{C}$ . First observe that the hypothesis  $W_k \in \mathcal{C}$  for any  $k = 1, \dots, \ell$  is equivalent to  $\kappa_0 + \mathcal{W}(k) \in \kappa_0 + \mathcal{C}$  for any  $k = 1, \dots, \ell$ . We also know by Assertion 8 of Theorem 3.6 that  $B(\pi_{\kappa_0}) \otimes B(\pi_\kappa)^{\otimes \ell}$  is contained in  $\mathcal{P}_{\min \mathbb{Z}}$  for any  $\ell \geq 1$ . Set  $\mathcal{W}(\ell) = \pi_1 \otimes \dots \otimes \pi_\ell$ . By Assertion 3 of Proposition 3.4, we have to prove that  $\tilde{e}_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \dots \otimes \pi_\ell) = 0$  for any  $i = 1, \dots, n$  and any  $\ell \geq 1$  providing  $\text{wt}(\pi_{\kappa_0} \otimes \pi_1 \otimes \dots \otimes \pi_\ell) = \pi_{\kappa_0} \otimes \pi_1 \otimes \dots \otimes \pi_\ell(1) \in \kappa_0 + P_+$  for any  $\ell \geq 1$ . Fix  $i = 1, \dots, n$ . Set  $\kappa_0(i) = m_0(i) - 1$ . We proceed by induction.

Assume  $\ell = 1$ . Since we have  $\tilde{e}_i(\pi_{\kappa_0}) = 0$ , it suffices to prove by using Assertion 2 of Proposition 3.5 that  $\varepsilon_i(\pi_1) \leq \varphi_i(\pi_{\kappa_0})$ . By definition of the dominant weight  $\pi_{\kappa_0}$ , we have  $\varphi_i(\pi_{\kappa_0}) = \kappa_0(i)$ . Assertion 7 of Theorem 3.6 and the hypothesis  $\text{wt}(\pi_{\kappa_0} \otimes \pi_1) \in P_+$  permits to write

$$(28) \quad \text{wt}(\pi_{\kappa_0})_i + \text{wt}(\pi_1)_i = \text{wt}(\pi_{\kappa_0} \otimes \pi_1)_i \geq \kappa_0(i).$$

Recall that  $\pi_1$  belongs to  $B(\pi_\kappa)$ . So  $\varepsilon_i(\pi_1) \leq \kappa_0(i) + 1$  because  $\varepsilon_i(\pi_1)$  gives the distance of  $\pi_1$  from the source vertex of its  $i$ -chain. When  $\varepsilon_i(\pi_1) < \kappa_0(i) + 1$  we are done. So assume  $\varepsilon_i(\pi_1) = \kappa_0(i) + 1$ . This means that  $\pi_1$  satisfies  $\varphi_i(\pi_1) = 0$ . Therefore,  $\text{wt}(\pi_1)_i = -\kappa_0(i) - 1$ . But in this case, we get by (28)

$$\text{wt}(\pi_{\kappa_0} \otimes \pi_1)_i = \kappa_0(i) - (\kappa_0(i) + 1) = -1 \geq \kappa_0(i)$$

hence a contradiction.

Now assume  $\tilde{e}_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1}) = 0$  for any  $k = 1, \dots, \ell - 1$ . Observe that  $\text{wt}(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1})_i = \varphi_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1}) \geq \kappa_0(i)$  since  $\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1} \in \kappa_0 + P_+$  and  $\tilde{e}_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1}) = 0$ . We also have

$$(29) \quad \text{wt}(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1} \otimes \pi_\ell)_i = \text{wt}(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1})_i + \text{wt}(\pi_\ell)_i \geq \kappa_0(i).$$

We proceed as in the case  $\ell = 1$ . Assume first  $\varepsilon_i(\pi_\ell) \leq \varphi_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1})$ . Then by Proposition 3.5 and the induction equality  $\tilde{e}_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1}) = 0$ , we will have  $\tilde{e}_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_\ell) = 0$ .

Now assume  $\varepsilon_i(\pi_\ell) > \varphi_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1})$ . Since  $\varphi_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1}) \geq \kappa_0(i)$  and  $\pi_\ell \in B(\pi_{\kappa_0})$ , we must have  $\varepsilon_i(\pi_\ell) = \kappa_0(i) + 1$  and  $\varphi_i(\pi_{\kappa_0} \otimes \pi_1 \otimes \cdots \otimes \pi_{\ell-1}) = \kappa_0(i)$ . Therefore, we get  $\text{wt}(\pi_\ell)_i = -\kappa_0(i) - 1$ . Then (29) yields yet the contradiction

$$-1 \geq \kappa_0(i).$$

□

**Remark:** In general the assertion  $W_k \in \mathcal{C}$  for any  $k = 1, \dots, \ell$  is not equivalent to the assertion  $\kappa_0 + \mathcal{W}(t) \in \kappa_0 + \mathcal{C}$  for any  $t \in [0, \ell]$ . This is nevertheless true when  $\kappa$  is a minuscule weight since  $\kappa_0 = 0$  in this case.

We deduce from the previous lemma the inequality

$$\mathbb{P}_\mu(W_\ell \in \mathcal{C} \text{ for any } \ell \in [0, \ell]) \leq \mathbb{P}_{\mu+\kappa_0}(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, \ell]).$$

When  $\ell$  tends to infinity, this yields

$$\mathbb{P}_\mu(W_\ell \in \mathcal{C} \text{ for any } \ell \geq 1) \leq \mathbb{P}_{\mu+\kappa_0}(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0).$$

By using Theorem 6.2, this implies the

**Theorem 7.3.** *Assume  $\mathfrak{g}$  is of finite type. Then, for any  $\mu \in P_+$  we have*

$$\prod_{\alpha \in R_+} (1 - \tau^\alpha) S_\mu(\tau) \leq \mathbb{P}_\mu(W_\ell \in \mathcal{C} \text{ for any } \ell \geq 1) \leq \prod_{\alpha \in R_+} (1 - \tau^\alpha) S_{\mu+\kappa_0}(\tau).$$

*In particular, we recover the result of Corollary 7.4.3 in [10] :*

$$\mathbb{P}_\mu(W_\ell \in \mathcal{C} \text{ for any } \ell \geq 1) = \prod_{\alpha \in R_+} (1 - \tau^\alpha)^{m_\alpha} S_\mu(\tau)$$

*when  $\kappa$  is minuscule.*

**Remark:** The inequality obtained in the previous theorem can also be rewritten

$$1 \leq \frac{\mathbb{P}_\mu(W_\ell \in \mathcal{C} \text{ for any } \ell \geq 1)}{\mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, +\infty])} \leq \frac{S_{\mu+\kappa_0}(\tau)}{S_\mu(\tau)}.$$

When  $\mu$  tends to infinity, we thus have  $\mathbb{P}_\mu(W_\ell \in \mathcal{C} \forall \ell \geq 1) \sim \mathbb{P}_\mu(\mathcal{W}(t) \in \mathcal{C} \text{ for any } t \geq 0)$  as expected.

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