Alternative Financial Solutions seeking via a Genetic Like Algorithm

Emmanuel Frenod, Jean-Philippe Gouigoux, Landry Touré

Alternative Financial Solutions seeking via a Genetic Like Algorithm

Emmanuel Frénod[1] Jean-Philippe Gouigoux[2] and Landry Touré[1,2]

1 Introduction

For local communities, political decisions with heavy financial consequences need rigorous and detailed studies. The purpose of those studies is to offer Decision-Makers forecasts and projections of their financial circumstances to come, for various sequences of projects responding to political goals and various ways to finance them.

Tackling those forecasts and projections is a delicate task for experts. Indeed, on the one hand factors constraining the projects of a local community are essentially laws, management rules which fluctuate and public opinion which is fickle. On the other hand, the way that those constraints are perceived by a local community Decision-Makers is also time-varying.

Often, local community calls on experts in order to achieve this task. Practically an expert works in straight collaboration with Decision-Makers of the local community he is engaged with in order to take into account all the targets and perceive constraints of every projects of the local community. Its work consists in building financial plans, or, say, Prospective Budgets, which are in some sense optimized, consistent with the capacity of investment of the local community and, of course, complying with the political goals of the Decision-Makers. For each prospective budget the state of various indicators, relevant of the resulting financial health of the local community in future years, are given. As a result of this work, viable scenarios satisfying partially the political goals are proposed among which, in an ideal situation, Decision-Makers may make a choice. Unfortunately in a large number of non-ideal situations constraints and goals cannot be satisfied together. In those cases the set of viable scenarios influences the evolution of the political goals and the constraint perception in order to begin a new iteration of the work process.

The existing tools are a little limited to help this iterating work process. The goal of this paper is to set out a new tool to contribute to filling this gap in a specific context we shall describe now.

[1]Université Européenne de Bretagne, LMBA (UMR 6205), Université de Bretagne-Sud, Centre Yves Coppens, Campus de Tohannic, F-56017, Vannes

[2]MGDIS, Parc d'Innovation de Bretagne Sud, F-56038 Vannes

The local communities usually need visibility on their budget over a time period of several years linked to the characteristic duration of political mandate. The main strict constraint, generally imposed by current legislation is that the difference between the receipts and the expenditures cannot be negative. That makes-up the balanced budget rule. This balanced budget is in most countries shared into sub-budgets which are not necessary balanced. However, each expenditure or receipt clearly belongs to a unique sub-budget. For instance, French local communities share their budget into an investment budget and an operating budget. Positive credit balance amount from the operating budget can be transferred to the investment budget. Our work joins in the French model of local communities' management but the tackled questioning and the tool we set out are clearly more general.

In order to explain the goal of the tool we build, we restrict ourself to the particular case of a Prospective Budget building where, among the political goals, two objectives are to be reached. With software environment avalaible nowadays, a Prospective Budget may be figured out with the first objective achieved. In particular, the consequences on the factors involved in the second objective may be quantified. Of course, the same can be done exchanging the roles of the two objectives. Nonetheless, generally speaking, it is not possible to satisfy both objectives, since the constraints are too numerous. Schematically, it may be said that it is possible to bring out two Prospective Budgets S^1 and S^2, where S^1 satisfies the first objective, which is symbolized by the fact that indicator V_1 reaches a targeted value $\widetilde{V_1}$. Prospective Budget S^2 satisfies the second objective which is translated by $V_2 = \widetilde{V_2}$ for a targeted value $\widetilde{V_2}$. In Prospective Budget S^1, $V_2 \neq \widetilde{V_2}$ but is determined by the budget building process which takes constraints into account and which is, in some sense, optimized. In a similar way, in Prospective Budget S^2, $V_1 \neq \widetilde{V_1}$.

Having those Prospective Budgets on hand, the next step of the game consists in finding several alternative ones which, may be, are such that neither $\widetilde{V_1}$ is reached by V_1 nor $\widetilde{V_2}$ by V_2, but which satisfy the constraints and are more satisfactory. If the game is played by an expert, the building of those alternative Prospective Budgets uses one more time the tool after having let the targeted values evolve, influenced by its knowledge and the interaction with Decision-Makers.

Yet, the new tool which is described in this paper has the ambition to automatically generate a collection of alternative Prospective Budgets and to introduce them in a usable way, so that Decision-Makers can choose the one which best fits their goals. It is based on Genetic Algorithms.

For a exhaustive review on them, we refer to Goldberg [5] and Davis [3]. We also refer to Chen [10] for their use in the context of financial optimization.

The financial modeling is very active on the market finance sector (see for instance Goodman & Stampfli [6], Ilinski [7] and Fama [4]). It is much less productive for applications in public sector (see for instance Musgrave [8] and Rosen [9]). Mathematical modeling for finance of local community seems to be very poor (see Tiebout [11]). And this work is also a contribution to that field.

Let us now explain in a simplified manner the formalism and the method on which the tool is based. It can be considered that any given Prospective Budget is characterized

uniquely by the two values V_1 and V_2. In other words, indicators V_1 and V_2 become variables on which Prospective Budget depends. To simplify the purpose, V_1 and V_2 are both supposed to be n-dimensional, so that it can be assumed that $(V_1, V_2) \in \mathbb{R}^{2n}$. Prospective Budget associated with values V_1 and V_2 writes $S(V_1, V_2)$. Of course, for some values of the variables, say (V_1^f, V_2^f), Prospective Budget $S(V_1^f, V_2^f)$ does not satisfy the constraints. Then, constraints may be seen as defining a sub-domain of the space $\mathbb{R}^{2n}$ in which the variables lie. Within this framework, the Prospective Budget S^1 described above writes $S(\widetilde{V_1}, V_2^c)$ where V_2^c is computed by the software environment. In the same way, Prospective Budget S^2 writes $S(V_1^c, \widetilde{V_2})$.

The method explores, in $\mathbb{R}^{2n}$ – the space where the variables lie, the intersection of a box containing the two points $(\widetilde{V_1}, V_2^c)$ and $(V_1^c, \widetilde{V_2})$, associated with the budgets already on hand, and of the sub-domain where constraints are satisfied in order to identify a shape joining $(\widetilde{V_1}, V_2^c)$ and $(V_1^c, \widetilde{V_2})$ around which Prospective Budgets well fit what Decision-Makers are waiting for, are in some sense optimized and satisfy the constraints. The box is built by considering in $\mathbb{R}^{2n}$ the middle point of $(\widetilde{V_1}, V_2^c)$ and $(V_1^c, \widetilde{V_2})$ and by building in this point an orthonormal frame whose first vector is the normalization of vector $\overrightarrow{(\widetilde{V_1}, V_2^c)(V_1^c, \widetilde{V_2})}$ – joining $(\widetilde{V_1}, V_2^c)$ to $(V_1^c, \widetilde{V_2})$. The other vectors of the frame are exhibited by the mean of the Gram-Schmidt routine.

The method consists in defining a fitness function F which integrates the Decision-Makers' political goals. We also have to build the sub-domain on which budgets satisfy the constraints. We define a method to encode the variables in the considered box. This coding calls, among others, on a sub-product of the Gram-Schmidt routine. Then, we use a Genetic Like Algorithm which consists first in generating a collection of N values $(V_1^{l^0}, V_2^{l^0})_{l=1,\dots,N}$ which are within the box and which satisfy the constraints. For each value, Prospective Budget $S(V_1^{l^0}, V_2^{l^0})$ and its fitness $F(S(V_1^{l^0}, V_2^{l^0})) = F(V_1^{l^0}, V_2^{l^0})$ can be computed. By crossover, mutation and constraint management methods, usually used in Genetic Algorithms, a new collection $(V_1^{l^1}, V_2^{l^1})_{l=1,\dots,N}$ (lying in the box and satisfying the constraints) is then generated. Going further finally leads to the k^{th} generation $(V_1^{l^k}, V_2^{l^k})_{l=1,\dots,N}$ which may be close to the sought shape in the intersection of the box and the sub-domain where constraints are satisfied.

2 Description of the Alternative Financial Solutions seeking problem

This section is devoted to the description of the kind of financial problems we tackle with our method and tool. We begin by introducing, with a systemic point of view, the French local community yearly budget working. Then, we explain the problematic of seeking alternative Multi-Year Prospective Budgets.

2.1 Local community Yearly Budget System working

In Figure 1 is depicted the schematic working of French local community yearly budget. To explain this working, we adopt a systemic point of view allowing us to give a global and macroscopic description, without going into technical or semantical details, of what we call

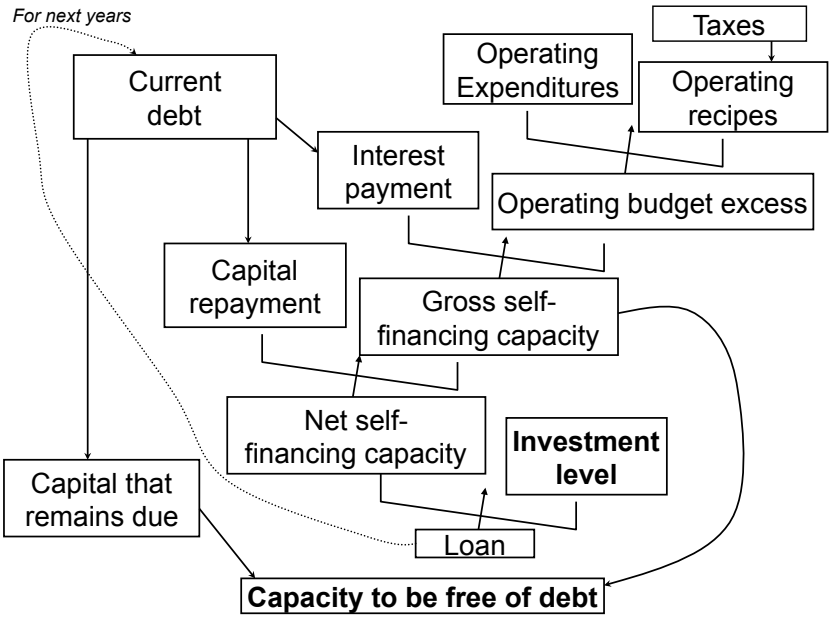

Figure 1: French local community Yearly Budget System working.

in the following: the Yearly Budget System.
For readers interested in French local community finance system, we refer to [2] and [1].

Among incomes contributing to local community operating budget, there are essentially state allocations and local "Taxes". Local community cannot influence the state allocation level, however the setting of local Tax Level is part of its own competences. As a consequence, we consider Taxes as an input of the Yearly Budget System. They lie at the top-right of the figure.

The other inputs of this system are linked with the "Current Debt". They are: the capital associated to this debt that remains due, the capital that needs to be repaid this year and the interests that have to be paid. Those amounts are defined by loan contracts of previous years. Those inputs are placed on the left-hand side of the figure. Of course, local community cannot have a direct effect on those inputs, but acts on their values in the next years by contracting or not new loans. This is symbolized by the dash line in the figure.

Generally, local community plans to get operating recipes that allow it to face all operating expenditures and debt interests, and, once those expenditures realized, that leave a remaining sum that can be used for investment. This remaining sum is called "Gross Self-Financing Capacity". This "Gross Self-Financing Capacity" is used to repay the capital that needs to be, and after that, the remainder, which is called "Net Self-Financing Capacity" contributes to the investment budget with the goal to top up subventions and loans to reach the Investment Level wanted by the community. This System generates a

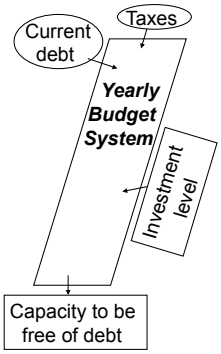

Figure 2: Synthetic Yearly Budget System

balanced budget.

At the bottom of the figure, the "Capacity to Be Free of Debt" is mentioned. This indicator is computed from the capital that remains due and from the Gross Self-Financing Capacity. It is, by definition, the time (generally expressed in years) for the community to repay all the capital of its debt, if no other loans are contracted and if the Gross Self-Financing Capacity remains constant over the next years. This indicator is seen here as the output of the system. A generally-accepted maximum figure for the Capacity to Be Free of Debt is 15 years

The Yearly Budget System is presented in Figure 2 in a synthetic diagram. This diagram illustrates that Current Debt, Taxes and Investment Level are seen as acting on the Yearly Budget System. Since Current Debt cannot be influenced directly, only Taxes and Investment Level are considered as active Inputs of the system. Considering this makes Taxes and Investment Level the variables on which the Yearly Budget depends and makes the Capacity to Be Free of Debt a result of the Yearly Budget, or in other words, an Output of the system.

2.2 Multi-Year Prospective Budget Systems

From the Yearly Budget System, it may be built a multi-year budget. Since those kinds of multi-year budgets are intended to explore possible futures under several assumptions, we call them Multi-Year Prospective Budgets. The functioning of such a Multi-Year Prospective Budget is depicted in figure 3. On the left-hand side of this picture is drawn the Budget System of the first year. This diagram is the synthetic one with arrows coming from the Current Debt box, the Tax box and the Investment Level box and an arrow going to the Capacity to Be Free of Debt box. Loans that are contracted during this first year have a consequence on the debt of the next years. This is symbolized by the arrow going from the Budget System of the first year to the Current Debt box of the second year. Going on, following the arrows, it is possible to create a Multi-Year Prospective Budget System for an arbitrary number of years.

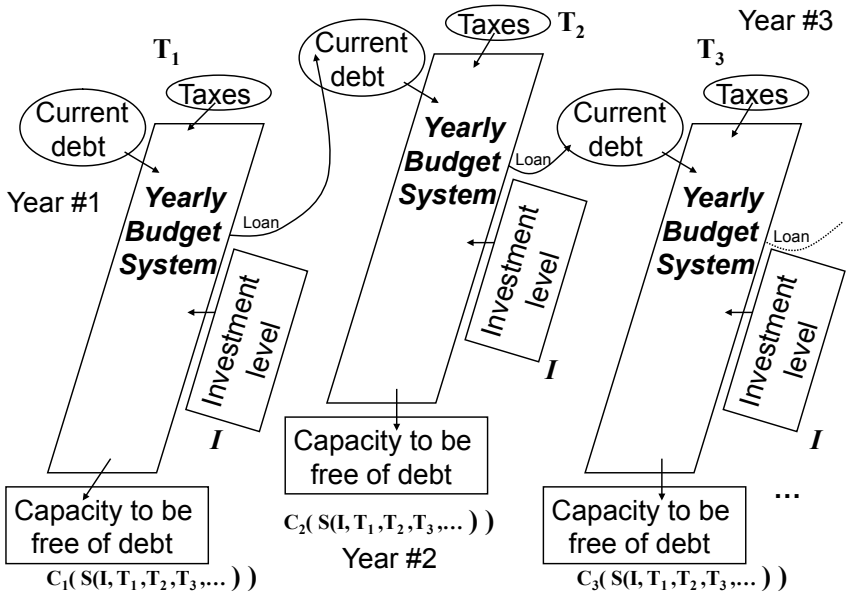

Figure 3: Multi-Year Prospective Budget System working.

To formalize a bit, the investment leads to the realization of a set of projects, which belong to the list of all projects which are desired to be carried out. Then, Investment Level may be described using a list of numbers, the cardinal of which being the number of projects. Each number indicates if its associated project will be done or not, and possibly, if it is done, how close or far from its targeted date it is carried out.
Then, if for instance the Prospective Budget is considered over five years and if five projects are considered, every possible Multi-Year Prospective Budget depends on ten values $(I_1, I_2, I_3,\ I_4, I_5, T_1, T_2, T_3, T_4, T_5) = ((I_i)_{i=1,...,5}, (T_i)_{i=1,...,5})$; five numbers $(I_i)_{i=1,...,5}$ given information on project realization and then indicating the Investment Level, and five Tax Levels: $(T_i)_{i=1,...,5}$, one for each year. Of course there are structural constraints on the variables: the T_i cannot be negative. The Prospective Budget corresponding to a given value $(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \bar{I}_5, \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{T}_4, \bar{T}_5) = ((\bar{I}_i)_{i=1,...,5}, (\bar{T}_i)_{i=1,...,5})$ of the Investment Level and Tax Levels is seen as a solution $S(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \bar{I}_5, \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{T}_4, \bar{T}_5) = S(((\bar{I}_i)_{i=1,...,5}, (\bar{T}_i)_{i=1,...,5}))$. The five Capacities to Be Free of Debt are seen as the image associated with a Prospective Budget by a mapping. They read: $(C_k(S(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \bar{I}_5, \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{T}_4, \bar{T}_5)))_{k=1,...,5} = (C_k(S((\bar{I}_i)_{i=1,...,5}, (\bar{T}_i)_{i=1,...,5}))_{k=1,...,5}$.

2.3 Alternative Prospective Budget seeking problematic

Having on hand this formalism, we can insert in it the political goals of Decision-Makers which are generally unreachable. In a first place, we present how a part of the political goals may be incorporated to create two Multi-Year Prospective Budgets which are only partially

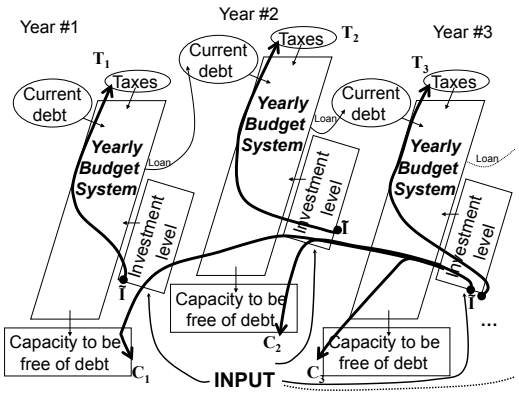

Figure 4: Multi-Year Prospective Budget computed considering Investment Level as input and Tax Levels and Capacities to Be Free of Debt as outputs.

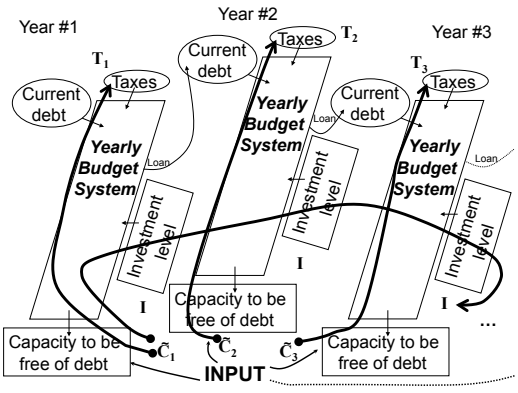

Figure 5: Multi-Year Prospective Budget computed with Capacities to Be Free of Debt as inputs and Tax Levels and Investment Levels as outputs.

satisfactory. Then we explain how from two Budgets, Alternative Prospective Budgets may be sought. Of course, those Budgets are also only partially satisfactory, but among the generated collection of Prospective Budgets, one can be preferred and then chosen.

We resume on the example of the five years Prospective Budgets involving five projects, but of course what is explained in the following is true for any number of years and projects.

Always being schematic to the extreme, we consider here that a Political Goal is a given collection $(\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3, \widetilde{I}_4, \widetilde{I}_5, \widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4, \widetilde{C}_5) = ((\widetilde{I}_i)_{i=1,...,5}, (\widetilde{C}_i)_{i=1,...,5})$ of targeted Investment Level and targeted Capacities to Be Free of Debt translating into financial terms the projects the Decision-Makers plan to be realized and the level of financial sanity they want to reach. In addition, the Political Goal is provided by a Tax Evolution Pattern expressing at what time Decision-Makers accept tax increases and at what time they prefer stability of tax levels.

Then with the help of a software environment properly programmed, it is possible to compute Prospective Budget $S((\widetilde{I}_i)_{i=1,...,5}, (\widetilde{T}_i)_{i=1,...,5})$, whose every Yearly Budget is balanced and which satisfies the requested Political Goal, i.e. such that $C_k(S((\widetilde{I}_i)_{i=1,...,5}, (\widetilde{T}_i)_{i=1,...,5})) = \widetilde{C}_k$ for $k = 1, ..., 5$.

However this view is much too naive, since Politic Goal $((\widetilde{I}_i)_{i=1,...,5}, (\widetilde{C}_i)_{i=1,...,5})$ generally requires Tax Levels incompatible with rules and regulations or which are not acceptable to Decision-Makers.

Then the work of Decision-Makers, assisted by public-finance experts consists in degrading Politic Goal $((\widetilde{I}_i)_{i=1,...,5}, (\widetilde{C}_i)_{i=1,...,5})$ to be reached with acceptable Taxes Levels. This is done with the help of a software environment. For instance, SOFI software, edited

by MGDIS[1], provides solutions to this question in considering two problems. The first one, depicted in Figure 4, consists in considering as input the targeted Investment Level $(\widetilde{I}_i)_{i=1,...,5}$ and, having set constraints on Tax Levels, in computing a Multi-Year Prospective Budget $S((\widetilde{I}_i)_{i=1,...,5}, (T_i^c)_{i=1,...,5})$ such that, for $k = 1, ..., 5$, the Capacities to Be Free of Debt $C_k(S((\widetilde{I}_i)_{i=1,...,5}, (T_i^c)_{i=1,...,5}))$ are as close as possible (in a given sense) to the Goals $\widetilde{C}_k$ and with Tax Levels $(T_i^c)_{i=1,...,5}$ that satisfy the constraints and whose every Yearly Budget is balanced. The second one, represented in Figure 5, takes the targeted Capacities to Be Free of Debt $(\widetilde{C}_k)_{k=1,...,5}$ as inputs and computes Multi-Year Prospective Budget $S((I_i^c)_{i=1,...,5}, (\widetilde{T}_i)_{i=1,...,5})$ such that $(\widetilde{T}_i)_{i=1,...,5}$ satisfies the constraints on Tax Levels, $C_k(S((I_i^c)_{i=1,...,5}, (\widetilde{T}_i)_{i=1,...,5}) = \widetilde{C}_k$ for $k = 1, ..., 5$ and Investment Levels $(I_i^c)_{i=1,...,5}$, are as close as possible to the Goal $(\widetilde{I}_i)_{i=1,...,5}$.

Once those two solutions are set out, they may be evaluated by Decision-Makers, in regards, among others, of the Tax Evolution Pattern.

Other Multi-Year Prospective Budgets may then be built by modifying values within the Politic Goal $((\widetilde{I}_i)_{i=1,...,5}, (\widetilde{C}_i)_{i=1,...,5})$, after financial and political discussions between Decision-Makers and experts. This generation of Alternative Financial Solutions or Alternative Multi-Year Prospective Budgets can be fastidious and long. Thus, only a small number can be generate.

The topic of the method and tool we propose here is to automate this generation of Alternative Financial Solutions or Alternative Multi-Year Prospective Budgets and to set out a relatively wide number of them, provided with indicators of their quality in order for Decision-Makers to be able to make a choice between them.

Roughly speaking we can consider that Prospective Budgets $S((\widetilde{I}_i)_{i=1,...,5}, (T_i^c)_{i=1,...,5}))$ and $S((I_i^c)_{i=1,...,5}, (\widetilde{T}_i)_{i=1,...,5}))$ are associated with two points in a 10-dimensional space and that the possible Alternative Financial Solutions or Alternative Multi-Year Prospective Budgets are gathered around a geometrical object joining those two points and that have to be sought and identified.

3 Genetic Like Algorithm

We go on treating the case when the number of years and the number of projects are both five.

3.1 Dimensionless problem setting

In order to manage variables and results which are dimensionless and of order one we first rescale the problem. For this we introduce a characteristic Investment Level describer value I, a characteristic Capacity to Be Free of Debt C and a characteristic Tax Level T. For

[1] http://www.mgdis.fr/

instance we can chose

$$\begin{aligned}
\text{I} =& \frac{1}{10}\big(\widetilde{I}_1+\widetilde{I}_2+\widetilde{I}_3+\widetilde{I}_4+\widetilde{I}_5+I_1^c+I_2^c+I_3^c+I_4^c+I_5^c\big)=\frac{1}{10}\sum_{i=1}^{5}\widetilde{I}_i+I_i^c,\\
\text{T} =& \frac{1}{10}\big(\widetilde{T}_1+\widetilde{T}_2+\widetilde{T}_3+\widetilde{T}_4+\widetilde{T}_5+T_1^c+T_2^c+T_3^c+T_4^c+T_5^c\big)=\frac{1}{10}\sum_{i=1}^{5}\widetilde{T}_i+T_i^c, \qquad (1)\\
\text{C} =& \frac{1}{10}\sum_{k=1}^{5} C_k(S((\widetilde{I}_i)_{i=1,...,5},(T_i^c)_{i=1,...,5}))+C_k(S((I_i^c)_{i=1,...,5},,(\widetilde{T}_i)_{i=1,...,5})),
\end{aligned}$$

which are the mean values of values reached by the two Prospective Budgets we have on hand. Then we define the dimensionless variables and results

$$\mathcal{I}_i=\frac{I_i}{\text{I}},\quad \mathcal{T}_i=\frac{T_i}{\text{T}} \quad \text{and} \qquad (2)$$

$$\mathcal{C}_k(\mathcal{S}((\mathcal{I}_i)_{i=1,...,5},(\mathcal{T}_i)_{i=1,...,5}))=\frac{C_k(S((\text{I}I_i)_{i=1,...,5},(\text{T}T_i)_{i=1,...,5}))}{\text{C}}. \qquad (3)$$

On those variables there are organic constraints:

$$\mathcal{C}_k(\mathcal{S}((\mathcal{I}_i)_{i=1,...,5},(\mathcal{T}_i)_{i=1,...,5}))\geq 0, \qquad \text{for} \quad k=1,...,5, \qquad (4)$$

which translate the fact that the Capacity to Be Free of Debt is a duration. There are also constraints linked with rules and regulation and what is politically admissible. These read:

$$\mathcal{T}_i\leq \mathcal{T}_i^{\max}(\mathcal{T}_1,\ldots,\mathcal{T}_{i-1}), \quad \text{for} \quad i=1,...,5, \qquad (5)$$

$$\mathcal{C}_k(\mathcal{S}((\mathcal{I}_i)_{i=1,...,5},(\mathcal{C}_i)_{i=1,...,5}))\leq \mathcal{C}^{\max}, \qquad \text{for} \quad k=1,...,5. \qquad (6)$$

Those constraints, when expressed in dimensionless variables, involve maximum values $\mathcal{C}^{\max}$ and $(\mathcal{T}_i^{\max})_{i=1,...,5}$ which essentially do not depend on the size of the concerned local community. Inequalities (6) express the fact that, at each year, the Capacity to Be Free of Debt is limited by common rules. The $\mathcal{T}_i^{\max}$ in (5) depend on the Tax Levels of the previous years and are both imposed by law, which restricts tax evolution, and prescribed by what community Decision-Makers exclude.

Remark - The question of knowing if the fact that every Yearly Budget of every Multi-Year Prospective Budget needs to be balanced has to enter into the constraint collection may be addressed. The answer is that we work under the assumption that any Multi-Year Prospective Budget, for instance computed using SOFI, from any variable collection $((\mathcal{I}_i)_{i=1,...,5},(\mathcal{T}_i)_{i=1,...,5})$ generates loans and consequently Capacity to Be Free of Debt set $\mathcal{C}_k(\mathcal{S}((\mathcal{I}_i)_{i=1,...,5},(\mathcal{T}_i)_{i=1,...,5}))$ insuring the balanced character of all its Yearly Budgets.

Within the dimensionless variables, the dimensionless Political Goals and other quantities are expressed as

$$\widetilde{\mathcal{I}}_i=\frac{\widetilde{I}_i}{\text{I}},\quad \widetilde{\mathcal{C}}_i=\frac{\widetilde{C}_i}{\text{C}},\quad \mathcal{I}_i^c=\frac{I_i^c}{\text{I}} \quad \text{and} \quad \mathcal{C}_i^c=\frac{C_i^c}{\text{I}}, \qquad (7)$$

and we have on hand two dimensionless Prospective Budgets $\mathcal{S}((\widetilde{\mathcal{I}}_i)_{i=1,...,5},(\mathcal{T}_i^c)_{i=1,...,5}))$ and $\mathcal{S}((\mathcal{I}_i^c)_{i=1,...,5},(\widetilde{\mathcal{T}}_i)_{i=1,...,5})$ which are associated with two points in a 10-dimensional space, located not so far from the origin.

3.2 Fitness choice

Among the criteria that may enter the fitness definition, they are the targeted values $\widetilde{\mathcal{I}}_i$ and $\widetilde{\mathcal{C}}_i$ and the Tax Evolution Pattern.

We begin by explaining how easy it is to define a model of Tax Evolution Pattern within dimensionless variables. In the case when the number of years is 5, it is a collection of 5 non negative values $(\mathcal{A}_k)_{k=1,\dots,5}$ such that

$$\sum_{k=1}^{5} \mathcal{A}_k = 1, \tag{8}$$

and which has the property that $\mathcal{A}_k = \mathcal{A}_{k+1}$ in the case of a desired stability of Tax Level between years number k and number $(k+1)$ and the property that $\mathcal{A}_k < \mathcal{A}_{k+1}$ in the case of a planned increase. Then, a way to measure how far from the Tax Evolution Pattern a given Multi-Year Prospective Budget $\mathcal{S}((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5})$ reduces to computing

$$F_{\mathrm{T}}((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5})) = \phi_{\mathrm{T}}\left(\sum_{k=1}^{5} \left| \frac{\mathcal{T}_k}{\sum_{i=1}^{5} \mathcal{T}_i} - \mathcal{A}_k \right| \right), \tag{9}$$

where ϕ_{T} is a non-increasing function such that $\phi_{\mathrm{T}}(0) = 1$ and $\lim_{x\to+\infty} \phi_{\mathrm{T}}(x) = 0$. In this definition, the division by $\sum_{i=1}^{5} \mathcal{T}_i$ allows us to insure that the values which are compared with the $\mathcal{A}_k$ range between 0 and 1.

Ways to measure how far from the Politic Goals a Multi-Year Prospective Budget is, are the computations of

$$F_{\mathrm{I}}((\mathcal{I}_i)_{i=1,\dots,5}) = \phi_{\mathrm{I}}\left(\sum_{k=1}^{r} \left| \mathcal{I}_k - \widetilde{\mathcal{I}}_k \right| \right) \text{ and}$$

$$F_{\mathrm{C}}((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5})) = \phi_{\mathrm{C}}\left(\sum_{k=1}^{5} \left| \mathcal{C}_k(\mathcal{S}((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5})) - \widetilde{\mathcal{C}}_k \right| \right), \tag{10}$$

where ϕ_{I} and ϕ_{C} have similar definition as ϕ_{T}.

With these three functions F_{T}, F_{I} and F_{C}, defining three non negative constants γ_{T}, γ_{I} and γ_{C}, having a sum not too far from 1, we chose the following Fitness Function

$$F((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5}) =$$
$$\gamma_{\mathrm{T}} F_{\mathrm{T}}((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5}) + \gamma_{\mathrm{I}} F_{\mathrm{I}}(\mathcal{I}_i)_{i=1,\dots,5}) + \gamma_{\mathrm{C}} F_{\mathrm{C}}((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5}), \tag{11}$$

and the largest $F((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5}))$ the best the solution $\mathcal{S}((\mathcal{I}_i)_{i=1,\dots,5}, (\mathcal{T}_i)_{i=1,\dots,5}))$.

With materials we built, we can reformulate the question of seeking Alternative Financial Solution as follows: we want to exhibit a collection of N points $((\mathcal{I}_i^{l\&})_{i=1,\dots,5}, (\mathcal{T}_i^{l\&})_{i=1,\dots,5})$ such that $\mathcal{S}((\mathcal{I}_i^{l\&})_{i=1,\dots,5}, (\mathcal{T}_i^{l\&})_{i=1,\dots,5})$ satisfied constraints (5), (4) and (6) and with Fitness worth $F((\mathcal{I}_i^{l\&})_{i=1,\dots,5}, (\mathcal{T}_i^{l\&})_{i=1,\dots,5})$ as large as possible.

3.3 Frame building by Gram-Schmidt routine

In the 10-dimensional vector space, for two vectors $\mathcal{W} = ((\mathcal{J}_i)_{i=1,\dots,5}), (\mathcal{D}_i)_{i=1,\dots,5}))$ and $\mathcal{W}' = ((\mathcal{J}'_i)_{i=1,\dots,5}), (\mathcal{D}'_i)_{i=1,\dots,5}))$, the following inner product and norm naturally exist:

$$\langle \mathcal{W}, \mathcal{W}' \rangle = \sum_{i=1}^{5} \mathcal{J}_i \mathcal{J}'_i + \mathcal{D}_i \mathcal{D}'_i \quad \text{and} \quad \|\mathcal{W}\| = \sqrt{\langle \mathcal{W}, \mathcal{W} \rangle}. \tag{12}$$

Moreover, it is provided with its canonical basis:

$$\begin{array}{llll} \mathbf{e}_1 = ((1,0,0,0,0),(0,0,0,0,0)), & & \mathbf{e}_2 = ((0,1,0,0,0),(0,0,0,0,0)), \\ \mathbf{e}_3 = ((0,0,1,0,0),(0,0,0,0,0)), & \dots, & \mathbf{e}_{10} = ((0,0,0,0,0),(0,0,0,0,1)). \end{array} \tag{13}$$

On the one hand, from the points $((\widetilde{\mathcal{I}}_i)_{i=1,\dots,5}, (\mathcal{T}_i^c)_{i=1,\dots,5})$ and $((\mathcal{I}_i^c)_{i=1,\dots,5}, (\widetilde{\mathcal{T}}_i)_{i=1,\dots,5})$, we can build the first vector of the frame by normalizing the vector linking those two points. This vector is:

$$\mathbf{g}_1 = \frac{\breve{\mathbf{g}}_1}{\|\breve{\mathbf{g}}_1\|}, \quad \text{where} \quad \breve{\mathbf{g}}_1 = ((\widetilde{\mathcal{I}}_i - \mathcal{I}_i^c)_{i=1,\dots,5}, (\mathcal{T}_i^c - \widetilde{\mathcal{T}}_i)_{i=1,\dots,5}). \tag{14}$$

On the other hand, we look for index i_{b} which is such that the absolute value of the inner product of $\mathbf{g}_1$ by $\mathbf{e}_{i_{\mathrm{b}}}$ is as large as possible, i.e. such that

$$\langle \mathbf{g}_1, \mathbf{e}_{i_{\mathrm{b}}} \rangle = \max_{i=1,\dots,10} \{\langle \mathbf{g}_1, \mathbf{e}_i \rangle\}. \tag{15}$$

The basis is then built by induction: Once j orthonormal vectors are obtained, the $(j+1)^{\text{th}}$ is gotten by removing from $\mathbf{e}_{(i_{\mathrm{b}}+j \mod 10)}$ its projection onto every vector of the new basis already computed and by renormalization, or in other words, by computing

$$\mathbf{g}_{j+1} = \frac{\breve{\mathbf{g}}_{j+1}}{\|\breve{\mathbf{g}}_{j+1}\|}, \quad \text{where} \quad \breve{\mathbf{g}}_{j+1} = \mathbf{e}_{\eta(i_{\mathrm{b}}+j)} - \sum_{p=1}^{j} \langle \mathbf{e}_{\eta(i_{\mathrm{b}}+j)}, \mathbf{g}_p \rangle \mathbf{g}_p, \tag{16}$$

where $\eta(i) = i$ if $1 \leq i \leq 10$ and $\eta(i) = i - 10$ if $10 \leq i \leq 20$.

Once all the $(\mathbf{g}_j)_{j=1,\dots,10}$ are obtained they make an orthonormal basis of the vector space whose first vector is born by the straight line linking the two points associated with the two dimensionless Prospective Budgets we have on hand.

With the help of this basis, we will build the box in which we will seek the targeted geometrical object and the coding of Prospective Budgets.

Let $\mathbf{B}$ be the 10×10 matrix such that if $\mathcal{W} = ((\mathcal{J}_i)_{i=1,\dots,5}, (\mathcal{D}_i)_{i=1,\dots,5})$ is a vector expressed in the canonical frame

$$\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_{10}) = \mathbf{B}\mathcal{W}, \tag{17}$$

gives its coordinates within frame $(\mathbf{g}_j)_{i=1,\dots,10}$. The i^{th} column of $\mathbf{B}$ is made of the coordinates of $\mathbf{e}_i$ within the new frame and the i^{th} column of $\mathbf{B}^{-1} = \mathbf{B}^T$ is made of the coordinates of $\mathbf{g}_i$ within the canonical frame. In the following we will consider that $\mathbf{B}$ and its inverse matrix $\mathbf{B}^{-1}$ are known.

3.4 Box building and coding

The geometrical object will be looked after within a box containing the points $((\widetilde{\mathcal{I}}_i)_{i=1,...,5}$, $(\mathcal{T}_i^c)_{i=1,...,5}))$ and $((\mathcal{I}_i^c)_{i=1,...,5}, (\widetilde{\mathcal{T}}_i)_{i=1,...,5}))$ associated with the two dimensionless Prospective Budgets we have on hand.

The box we chose is the cube centered in the middle point $\mathcal{M}$ of $[((\widetilde{\mathcal{I}}_i)_{i=1,...,5}, (\mathcal{T}_i^c)_{i=1,...,5}))$, $((\mathcal{I}_i^c)_{i=1,...,5}, (\widetilde{\mathcal{T}}_i)_{i=1,...,5}))]$, whose coordinates are

$$\mathcal{M} = \left(\left(\frac{\mathcal{I}_i^c + \widetilde{\mathcal{I}}_i}{2} \right)_{i=1,...,5}, \left(\frac{\mathcal{T}_i^c + \widetilde{\mathcal{T}}_i}{2} \right)_{i=1,...,5} \right), \tag{18}$$

with edges being $\{2\|\breve{\mathbf{g}}_1\|\mathbf{g}_1, (\|\breve{\mathbf{g}}_1\|\mathbf{g}_i)_{i=2,...,10}\}$, where $\breve{\mathbf{g}}_1$ is defined by formula (14).
Two opposite faces of this hypercubic box are orthogonal to the straight line linking the points $((\widetilde{\mathcal{I}}_i)_{i=1,...,5})$, $(\mathcal{T}_i^c)_{i=1,...,5}))$ and $((\mathcal{I}_i^c)_{i=1,...,5}), (\widetilde{\mathcal{T}}_i)_{i=1,...,5}))$.

Another (and more usable) way to characterize the box is to say that it is the range of $[-\frac{1}{2}, \frac{1}{2}] \times [-1, 1]^9$ by the mapping

$$\mathcal{P} \mapsto \mathcal{M} + \|\breve{\mathbf{g}}_1\| \, \mathbf{B}^{-1}\mathcal{P}, \tag{19}$$

whose inverse is

$$\mathcal{R} \mapsto \frac{1}{\|\breve{\mathbf{g}}_1\|}\mathbf{B}(\mathcal{R} - \mathcal{M}), \tag{20}$$

where $\mathcal{M}$ is given by (18), $\breve{\mathbf{g}}_1$ by (14) and matrix $\mathbf{B}$ by (17).

Moreover this transformation gives a coding of any solution $\mathcal{S}(\mathcal{R})$ by a point in $[-1, 1] \times [-\frac{1}{2}, \frac{1}{2}]^9$. Hence, without any supplementary effort we have two codings at our disposal: a solution $\mathcal{S}(\mathcal{R})$ may be coded by its directly interpretable values $(\mathcal{R}_1, \ldots, \mathcal{R}_{10}) = ((\mathcal{I}_i)_{i=1,...,5})$, $(\mathcal{T}_i)_{i=1,...,5})$ or by the collection of values $(\mathcal{P}_1, \ldots, \mathcal{P}_{10})$ that are the coordinates of point $\mathcal{P} = (1/\|\breve{\mathbf{g}}_1\|) \, \mathbf{B}(\mathcal{R} - \mathcal{M}) \in [-1, 1] \times [-\frac{1}{2}, \frac{1}{2}]^9$.

Generically, in the following we will denote the coding by $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_{10})$. It will designate the coding by $\mathcal{P}$ or $\mathcal{R}$ or any other.

3.5 Initial Prospective Budget collection

Fixing the number of members of the collection, and denoting this number by N, a collection of N points $\mathcal{P}^{l\,0} = (\mathcal{P}_1^{l\,0}, \ldots, \mathcal{P}_{10}^{l\,0})$, for $l = 1, ..., N$, of $[-1, 1] \times [-\frac{1}{2}, \frac{1}{2}]^9$ is generated randomly.

The initial collection of solutions is then $\mathcal{R}^{l\,0} = (\mathcal{R}_1^{l\,0}, \ldots, \mathcal{R}_{10}^{l\,0})$, for $l = 1, ..., N$, where $\mathcal{R}^{l\,0} = \mathcal{M} + \|\breve{\mathbf{g}}_1\| \, \mathbf{B}^{-1}\mathcal{P}^{l\,0}$.

We assume that every dimensionless Prospective Budget $\mathcal{R}^{l\,0}$ satisfies all the constraints (4), (5) and (6). This means that the random generation has to run until N dimensionless Prospective Budgets satisfying the constraints are obtained.

Being Generic, the coding of this initial Prospective Budgets will be denoted by $\mathcal{Q}^{l\,0} = (\mathcal{Q}_1^{l\,0}, \ldots, \mathcal{Q}_{10}^{l\,0})$.

3.6 Constraint management

For a collection made of $2N$ individuals, we will manage constraints by integrating them in the Fitness Function. This consists in adding to Fitness Function defined by (11), the following quantity, or a quantity of this kind,

$$-\phi\Bigg(-\sum_{k=1}^{5}\min(\mathcal{C}_k(\mathcal{S}((\mathcal{I}_i)_{i=1,...,5},(\mathcal{T}_i)_{i=1,...,5})),0)) \\ +\sum_{k=1}^{5}\max(\mathcal{C}_k(\mathcal{S}((\mathcal{I}_i)_{i=1,...,5},(\mathcal{T}_i)_{i=1,...,5}))-\mathcal{C}^{\max},0)+\sum_{k=1}^{5}\max(\mathcal{T}_k-\mathcal{T}_k^{\max},0)\Bigg), \quad (21)$$

for a function non decreasing function ϕ such that $\phi(0)=0$ and $\lim_{x\to+\infty}\phi(x)=1$, multiplied by a factor γ relatively large in front of 1. This penalization makes the value of the fitness of Prospective Budgets not respecting the constraints to decrease and then diminishes their chance to pass the selection to come.

3.7 Algorithm to produce next generation

Having on hand the $m-$th collection of Prospective Budgets $\mathcal{S}(\mathcal{R}^{l\,m})$, for $l=1,...,N$, and their coding $\mathcal{Q}^{l\,m}$, a new collection $\mathcal{S}(\mathcal{R}^{l\,m+1})$, with coding $\mathcal{Q}^{l\,m+1}$ is generated by a usual Genetic Algorithm Like routine we now briefly describe.

In a first step, couples of codings of the Prospective Budgets are randomly formed. Then for any couple $(\mathcal{Q}^{l\,m},\mathcal{Q}^{k\,m})=((\mathcal{Q}_1^{l\,m},\dots,\mathcal{Q}_{10}^{l\,m}),(\mathcal{Q}_1^{k\,m},\dots,\mathcal{Q}_{10}^{k\,m}))$, an integer i_a is randomly chosen among $\{1,2,\dots,10\}$ and the two codings $(\mathcal{Q}_1^{l\,m},\dots,\mathcal{Q}_{i_a-1}^{l\,m},\mathcal{Q}_{i_a}^{k\,m},\dots,\mathcal{Q}_{10}^{k\,m})$ and $(\mathcal{Q}_1^{k\,m},\dots,\mathcal{Q}_{i_a-1}^{k\,m},\mathcal{Q}_{i_a}^{l\,m},\dots,\mathcal{Q}_{10}^{l\,m})$ are generated. At the end of this first step, we have on hand $2N$ points: the $\mathcal{Q}^{l\,m}$, for $l=1,...,N$ and all the ones generated as described previously that are denoted $\mathcal{Q}^{l\,m}$ for $l=N+1,...,2N$.

The second step consists in making some codings to mutate. For this, a small integer i_b, ranging, say, between 0 and $N/50$ is randomly generated. Then, i_b codings are randomly chosen among the codings generated in the first step, i.e. among the $\mathcal{Q}^{l\,m}$ with $l=N+1,...,2N$. For each of them, an integer i_c is randomly chosen among $\{1,2,\dots,10\}$, a number ν ranging between -1 and 1 is also randomly generated and the i_c-th component of the concerned coding is incremented by ν.

In the third step the Fitness Function is evaluated on every coding $\mathcal{Q}^{l\,m}$, for $l=1,...,2N$, which results from the three first steps. In order to do so, it is necessary to determine the $\mathcal{R}^{l\,m}$, for $l=1,...,2N$ associated with the $\mathcal{Q}^{l\,m}$, for $l=1,...,2N$, the Prospective Budgets $\mathcal{S}(\mathcal{R}^{l\,m})$, for $l=1,...,2N$, and finally, the Fitnesses (penalized by constraints) $F(\mathcal{R}^{l\,m})$ for $l=1,...,2N$.

The target of the fifth step is to randomly select N codings among all the codings $\mathcal{Q}^{l\,m}$, for $l=1,...,2N$ that were brought out by the previous steps with the principle that the higher the fitness of a coding, the more likely to be selected. In addition, we can use an

elitism routine which consists in deterministically choosing the N_{elit} codings best scoring the Fitness Function.

In practice, to implement the routine just described we used the library "Aforge.Genetics". We conducted a study to measure the impact of the values of "CrossRate" and "MutationRate" parameters of this library. This study showed that the default values were convenient. Hence we used them.

3.8 Resizing of obtained solutions

After several iterations of the algorithm just described, we have on hand a collection of N codings. $\mathcal{Q}^{l\&}$, for $l = 1, ..., N$. Then, the $\mathcal{R}^{l\&} = ((\mathcal{I}_i^{l\&})_{i=1,...,5}, (\mathcal{T}_i^{l\&})_{i=1,...,5}))$, for $l = 1, ..., N$ are deduced. If the coding is the directly interpretable one there is nothing to do. If the coding is based on points $\mathcal{P} \in [-1, 1] \times [-\frac{1}{2}, \frac{1}{2}]^9$ one needs to apply transformation (19).

Then, for $l = 1, ..., N$, the resized values $((\mathcal{I}_i^{l\&})_{i=1,...,5}, (T_i^{l\&})_{i=1,...,5}))$, are computed by inverting formula (2) and real Prospective Budgets $S((\mathcal{I}_i^{l\&})_{i=1,...,5}, (T_i^{l\&})_{i=1,...,5}))$, are also computed.

4 Tests

The method is now tested on several examples. First, it is tested on one-dimensional problems in order to set out its capability to exhibit the point where the maximum of the fitness is located, and, in the case when the fitness shows a plateau with value of which is its maximal, to produce a population which is essentially located on the interval which range is the plateau. Secondly it is tested on the example of local community finances that is described in the previous sections.

4.1 Test on a one-dimensional problem with a Fitness Function having one maximum

The Fitness Function considered here is a function with one maximum, which is the sum of two quadratic ones. More precisely, defining

$$h_1(x) = \frac{1}{2}\max\left(1 - 30\,(x - 0.45)^2, 0\right), \quad h_2(x) = \frac{1}{2}\max\left(1 - 30\,(x - 0.55)^2, 0\right), \tag{22}$$

which are given in Figure 6 at the top and in the middle, the Fitness Function is

$$F = h_1 + h_2, \tag{23}$$

defined on $[0, 1]$ and drawn at the bottom of Figure 6. A simplified version of the method described above was implemented on this example, with the maxima of $h_1(x)$ and $h_2(x)$ playing roles analogous to those played by points $((\widetilde{\mathcal{I}}_i)_{i=1,...,5}, (\mathcal{T}_i^c)_{i=1,...,5})$ and $((\mathcal{I}_i^c)_{i=1,...,5}$, $(\widetilde{\mathcal{T}}_i)_{i=1,...,5})$ in the above described method.

On this example, the method works and gives after 500 generations a collection of points which is very concentrated around $x = 0.5$, which is the argument of the maximum of the fitness function.

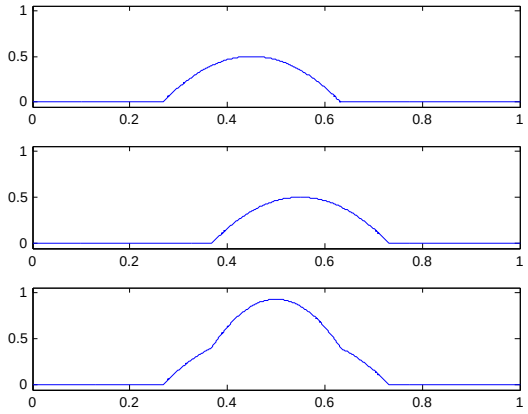

Figure 6: Fitness Function (bottom) with one maximum, which is the sum of two functions (top and middle).

Nonetheless its efficiency was compared to optimization methods using a similar Genetic Like Algorithm, but not involving the two points around which the argument of the maximum is sought. The method built here was not more efficient. This seems to lead to the conclusion that the contribution of this method is not to be sought in this direction.

4.2 Test on a one-dimensional problem with a Fitness Function having a maximum plateau

The real original capability of the method described in this paper is that it can give a good representation of the argument of the maximum of the fitness function when it is an interval. To illustrate this capability, a fitness function will be built which is the sum of two other functions (that have both one maximum) and which has a plateau whose span is smaller than the interval defined by the maximum localizations of the two functions. For this, in a first place, functions f_1 and f_2 defined by

$$f_1(x) = 1 - 10\,|x - 0.45| \text{ and } f_2(x) = 1 - 10\,|x - 0.55|, \tag{24}$$

and drawn at the top of Figure 7 are considered. Functions

$$g_1(x) = \min\big(1 - 10\,(x - 0.48), 1\big) \text{ and } g_2(x) = \min\big(1 - 10\,(0.52 - x), 1\big), \tag{25}$$

are also considered. Finally, the Fitness Function is

$$F = l_1 + l_2, \tag{26}$$

with

$$l_1 = 0.7 \max\big(0.5 f_1 + 0.5 g_1, 0\big) \text{ and } l_2 = 0.7 \max\big(0.5 f_2 + 0.5 g_2, 0\big). \tag{27}$$

Functions l_1 and l_2 are drawn in Figure 8 on the top and in the middle and Fitness Function F is given in the bottom of this Figure and detailed in Figure 9. Its maximum makes up

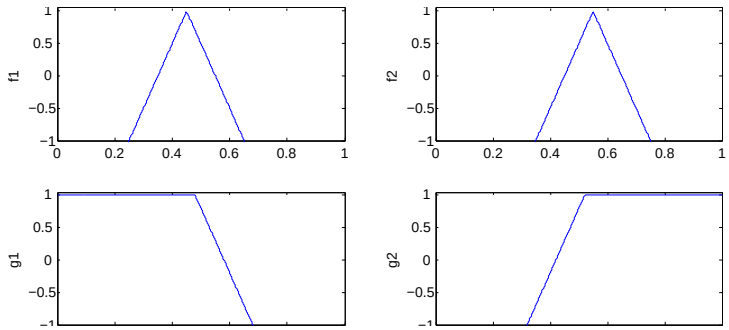

Figure 7: functions f_1 (top left), f_2 (top right), g_1 (bottom left) and g_2 (bottom right).

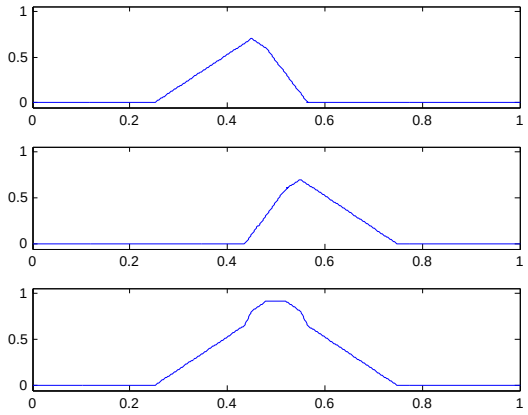

Figure 8: The Fitness Function (bottom) with maxima on a plateau which is strictly included in the interval made by the argument-of-the-maximum localizations of the two functions (top and middle) it is the sum of.

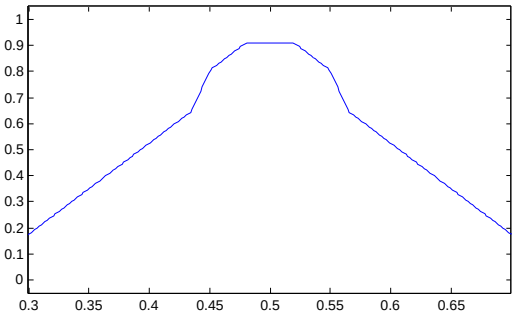

Figure 9: Detail of the fitness with maximum on a plateau which is strictly included in the interval made by the argment-of-the-maximum localizations of the two functions it is the sum of.

a plateau which is the range of an interval ($[0.475, 0.525]$) strictly included in the interval ($[0.45, 0.55]$) made by the maximum localizations of the two functions l_1 and l_2.

Here again a simplified version of the method described above was implemented on this example, with the maxima of $l_1(x)$ and $l_2(x)$ in the role of points $((\widetilde{\mathcal{I}}_i)_{i=1,\dots,5}, (\mathcal{T}_i^c)_{i=1,\dots,5})$ and $((\mathcal{I}_i^c)_{i=1,\dots,5}, (\widetilde{\mathcal{T}}_i)_{i=1,\dots,5})$ in the more general method. The collection, at each generation is made of 35 points. The resulting collection after 500 generations is given in Table 1. The collection is well distributed on the interval ($[0.475, 0.525]$) whose range is the plateau. In particular, it does not undergo concentrations that could orientate by mistake interpretation toward concluding that the Fitness Function has isolated maxima.

This capability of the method is very important for reaching operational problems. This is what is done in the last test.

Table 1: The collection of points after 500 generations when the Fitness Function is given by (26).

x	0.489	0,491	0,491	0,500	0,492	0,489	0,487
F	0,910	0,910	0,910	0,910	0,910	0,910	0,910
x	0,506	0,492	0,493	0,490	0,491	0,497	0,497
F	0,910	0,910	0,910	0,910	0,910	0,910	0,910
x	0,494	0,493	0,489	0,496	0,489	0,494	1,471
F	0,910	0,910	0,910	0,910	0,910	0,910	0,00
x	-0,475	0,464	0,462	0,485	0,492	0,501	0,488
F	0,000	0,910	0,910	0,910	0,910	0,910	0,910
x	0,489	0,492	0,496	0,492	0,435	0,534	0,484
F	0,910	0,910	0,910	0,910	0,646	0,860	0,910

4.3 Test on an operational problem

The method set out in section 3 is now tested in a realistic situation.

The experience uses a software product dedicated to the optimization of local communities' budgets, with a French interface. Using this application allows to pick two possible solutions. The first one, presented in Figure 10, is quite liberal, with all projects being realized, and the taxes increased at the maximum rate of 7% in the first three years (see the second line of the second part of the Table in Figure 10). The Capacity to Be Free of Debt ratio remains in the acceptable range (see the "Capacité de désendettement" line at the bottom of the table in Figure 10).

The second solution picked is a much more careful, with only the top priority projects being done, and a very limited increase of tax applied so that the capacity to be free of debt ratio remains below 15 years, which is the prudential limit. Figure 11 shows the values for this second solution.

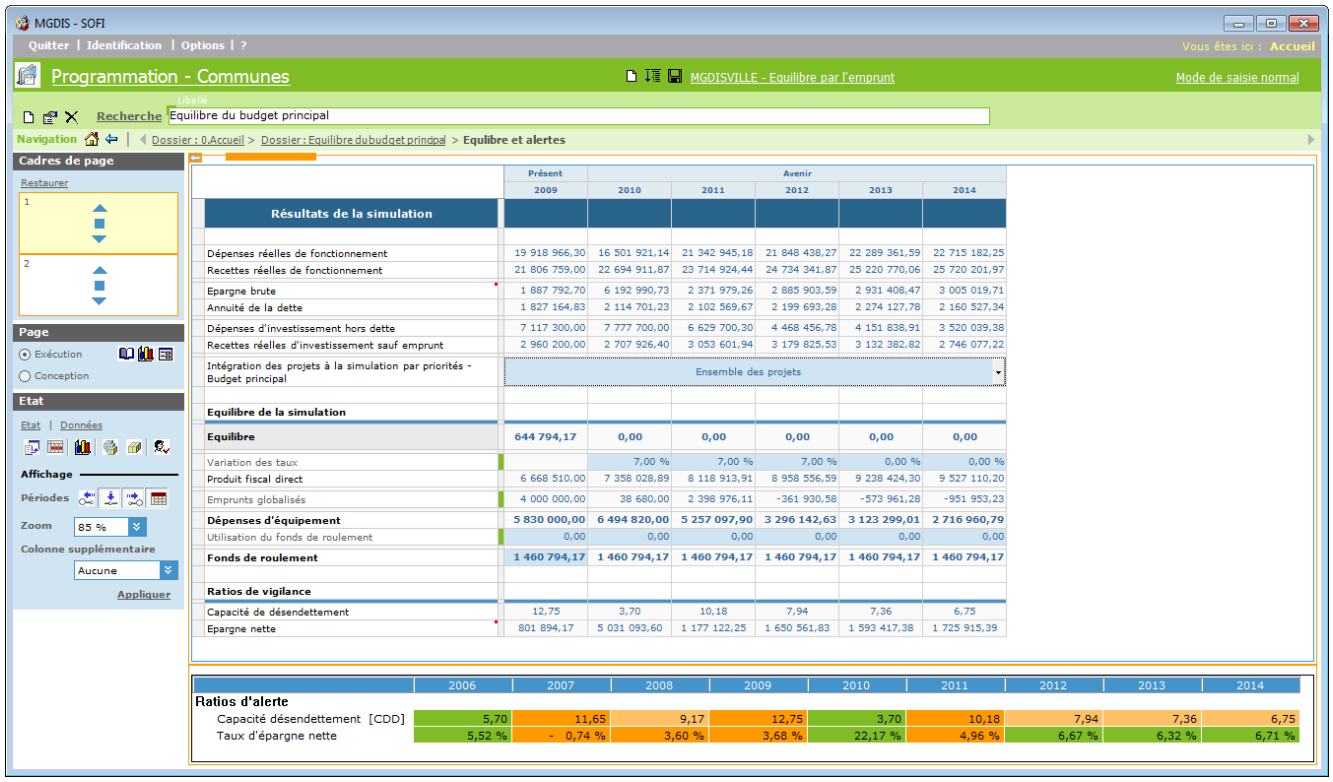

Figure 10: The quite liberal solution. (The software has a French interface; the translations are: Capacité de désendettement = Capacity to Be Free of Debt, Variation des taux = Tax Increase, Produit fiscal direct = Taxes, Dépenses réelles de fonctionnement = Operating Expenditures, Recettes réelles de fonctionnement = Operating Recipes, Epargne brute = Operating Budget Excess.)

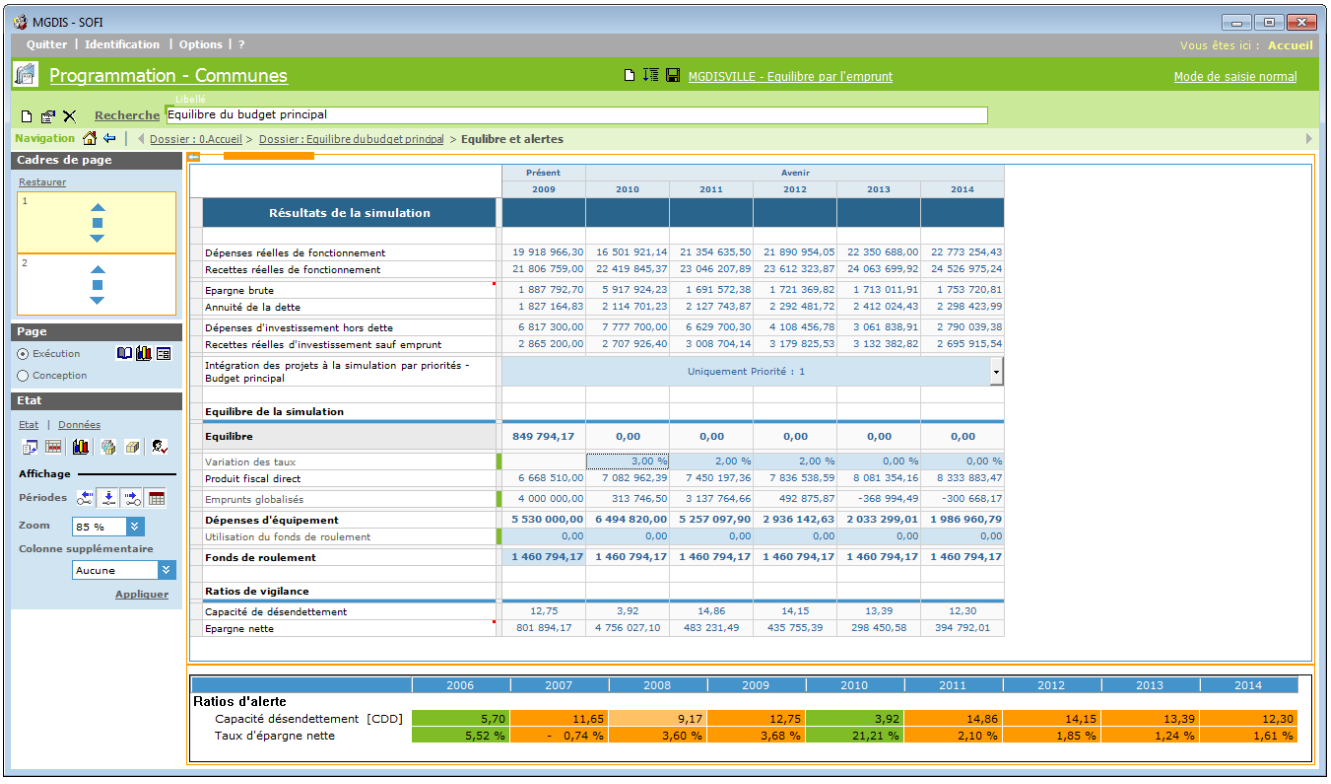

Figure 11: The careful solution.

Figure 12 shows a representation of the proceedings of the projects for the careful solution. A color code helps spotting the priorities of the projects (from hight priority: red, orange, yellow, blue: low priority). It should be noted that, in our example, priority one projects account for the vast majority of the budget.

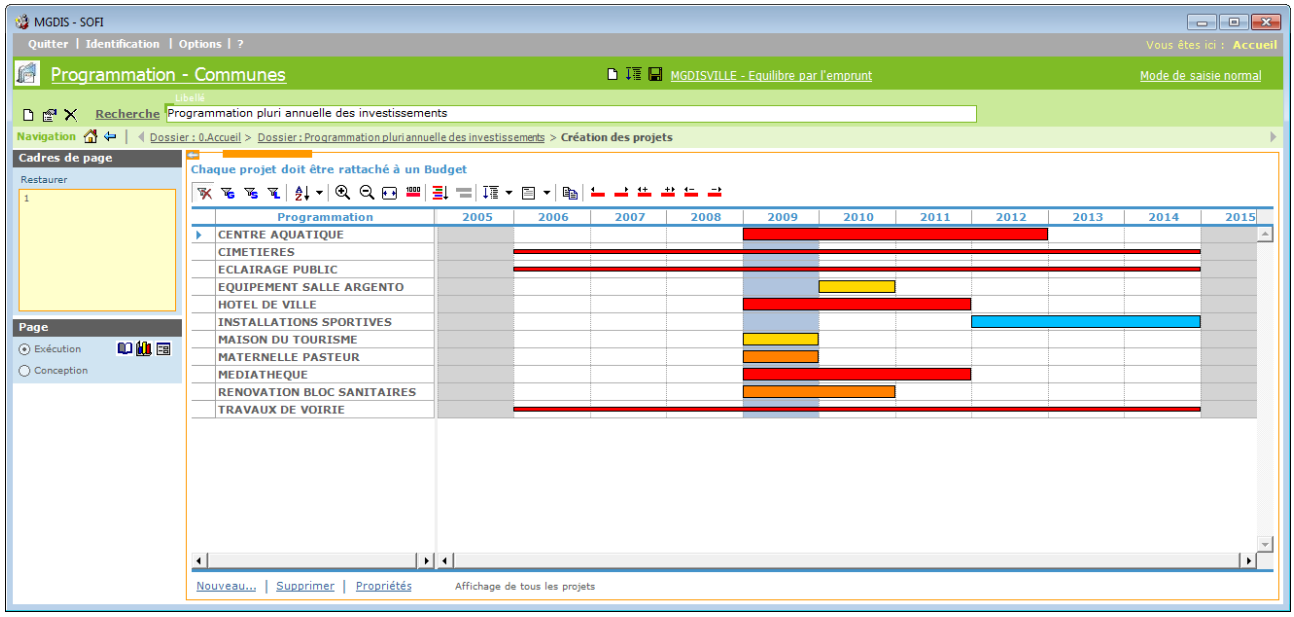

Figure 12: The proceedings of the projects for the careful solution.

The coding follows such rules:

- The first five genes code the five less-than-one-priority projects (the priority-one projects are always active).
- The coding is equally distributed for each unit: below half, the project is inactive, above half, the project is active.
- The next five genes contain the evolution of tax, written as a double.
- We consider the limits to be 0% at the minimum, and 7% at the maximum. Thus, the decoding will realize a modulus 0.07 function on the values.

The fitness function that was used works as follows:

- The number of projects brings a linear satisfaction, and accounts for 25% of the final fitness.
- The evolution of the tax is best at 0% and worst at 7%.
- Its average evolution accounts for 10% of the global fitness, and its evolution in the last two years accounts for 5%.

- The capacity to be free of debt is optimal at 0 year and worst at 15 years and more. This accounts for 25% of the final grading of the solution.
- The capacity to spare money is optimal at 5%, and accounts for 25% of the global fitness.
- No variation at all (in the tax evolution) gives the best results, and this part of the grade accounts for the remaining 10% of the global fitness.

The language used is C#, and the genetic algorithm framework is the one from AForge, which is an Open Source project. The definition of the two vectors corresponding to the solution points are programmed as:

```
Vector v1 = new Vector(new double[] { 0.75, 0.75, 0.75, 0.75, 0.75,
            0.07, 0.07, 0.07, 0.00, 0.00 });
Vector v2 = new Vector(new double[] { 0.25, 0.25, 0.25, 0.25, 0.25,
            0.03, 0.02, 0.02, 0.00, 0.00 });
```
(28)

By constraining the solutions into a hypercube using the method described in this paper, we reach an optimum which has been validated by a finance professional as a reasonable solution for the community budget. The corresponding coding is:

```
[1,03949069282959, 0,19207769961155, 0,51186133809657,
0,205107769055541, 0,785367264162938, -0,254824609597842,
0,378497610225784, 1,04175330250962, 0,590217152232071,
-11,383992284572].
```
(29)

These values correspond to:

Project 1: OFF, Project 2: OFF, Project 3: ON, Project 4: OFF, Project 5: ON, Tax evolution : 3.48%, 2.85%, 6.18%, 3.02%, 3.39%. (30)

The prudential ratios are respected, as the solution in the graphic simulator of Figure 13 shows (Capacity to Be Free of Debt, abbreviated as CDD in French, must remain under 15 years). The fitness obtained is 59%, and the 500 generations took 17 minutes and 32 seconds to be simulated on the reference machine. Interestingly, an independent computation with only 100 generations, which took 3 minutes and 32 seconds, showed a final fitness of 58%, so the convergence is quite quick for this particular example. The corresponding chromosome was:

```
[1,11266759532518, 0,0838990704498006, 0,565754259647956,
0,440107396614116, 0,813652694225311, 0,642521321773529,
0,575349082741285, 0,447334636593206, 0,488786454202292,
0,990373758336907]
```
(31)

In terms of budget, this means:

Project 1: OFF, Project 2: OFF, Project 3: ON, Project 4: OFF, Project 5: ON, Tax evolution : 1.25%, 1.53%, 2.73%, 6.88%, 1.04%. (32)

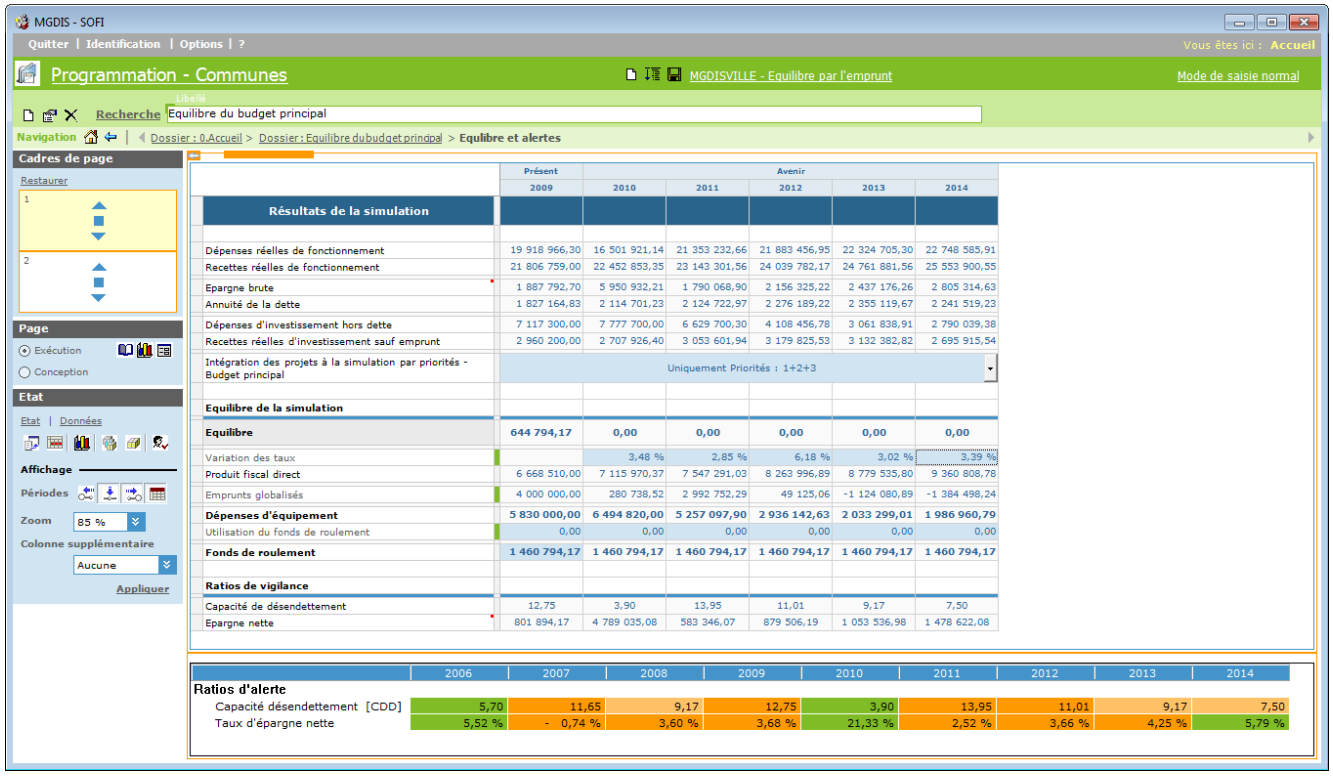

Figure 13: The optimal solution.

One will notice that the activation of the different projects is the same as the other solution, whereas the choice of Tax Evolution pattern is quite different. A quick conclusion would be that the fitness is more dependent on the projects activation than on the tax evolution, but this would need a robustness analysis, and this was not the subject of the present study. The most interesting part of the result is that, along the generations, the solutions found are quite concentrated, as shown in Figure 14. This should be compared to the initial genetic

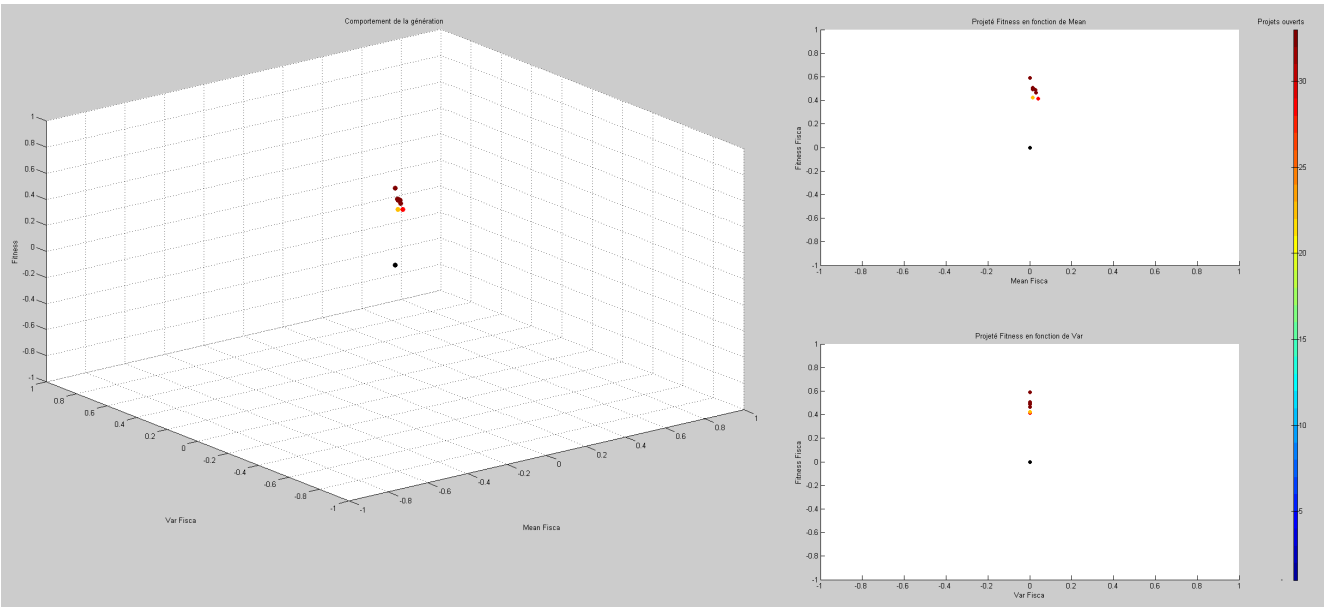

Figure 14: The final population.

algorithm optimization without Gram Schmidt projection.

The effect of using this particular technique is that the solutions are found into a constrained set of solution, without the user needing to explain how it is constrained, but by letting one propose two solutions surrounding the searched one. The effect on the rapidity on the convergence was expected as an additional result of the study, but we could not demonstrate any noticeable or provable effect on this factor. Further studies need to be done, with a high volume of test, in order to determine whether the two points approach helps the convergence of the genetic algorithms or not, and depending on which conditions on the fitness of the coding of the chromosomes.

5 Conclusion

In this paper, a method based on Genetic Algorithm to build a collection of Financial Solutions from two acceptable ones is set out, and explored.
The way to tackle that the collection is sought in a neighborhood of the two acceptable solutions calls on a Gram-Schmidt routine to comfortably build a box surrounding them. This routine brings also a way of coding the solution that can be used in the Genetic Like Algorithm.

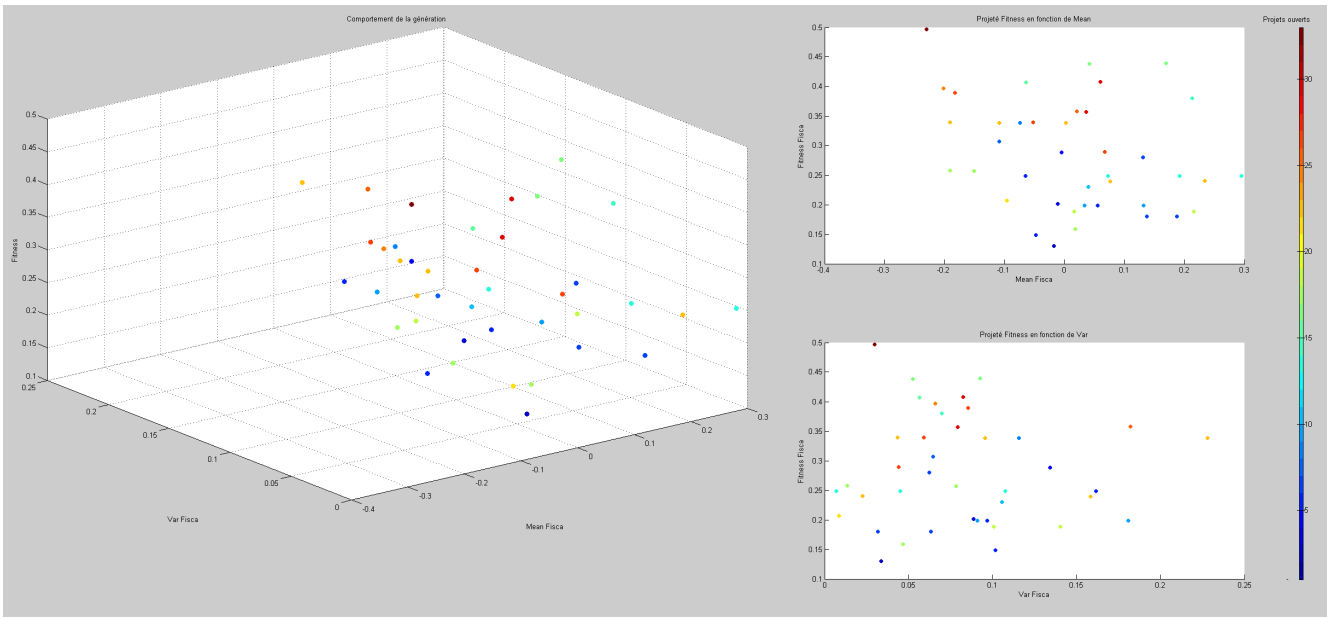

Figure 15: The initial population.

The method is then tested on simplified one-dimensional problems to exhibit that it has the capability to locate the argument of the maximum of a Fitness Function and to generate a collection of solutions which is distributed over the set of all the arguments of the maximum when the Fitness Function has a plateau as a maximum. This last capability is the important one in view of the targeted operational applications which concerns the financial strategy of local communities.
Finally, the method is tested on an example of the targeted operational applications and gives interesting results which is promising. it seems to be a potential alternative or a support to the heavy protocol (involving many meetings with experts and decision makers) to set out a suitable Financial Solution for local communities.

References

[1] Annexe num 1: Plan de comptes développé des communes de 500 habitants et plus au 1ier janvier 2009. Technical report, Plan M14 comptabilité.

[2] La qualité comptable au service d'une gestion performante des collectivités locales - guide des bonnes pratiques num 18. Technical report, Académie des sciences et techniques comptables financières.

[3] L. Davis. *Handbook of Genetic Algorithms*. Van Nostrand Reinhold, New York, 1991.

[4] E.F. Fama. Market efficiency, long-term returns, and behavioral finance. *Journal of Financial Economics*, 49(3):283 – 306, 1998.

[5] D. E. Goldberg. *Genetic Algorithms in Search, Optimization, and Machine Learning*. Addison-Wesley Professional, 1 edition, January 1989.

[6] V. Goodman and J. G. Stampfli. *The mathematics of finance: modeling and hedging*. American Mathematical Society, 2001.

[7] K. Ilinski. *Physics of finance : gauge modelling in non-equilibrium pricing*. Wiley, 2001.

[8] R.A. Musgrave. *The theory of public finance : a study in public economy*. McGraw-Hill, 1959.

[9] H. S. Rosen. Public finance. In *Readings in Public Choice and Constitutional Political Economy*, pages 371–389. Springer US, 2008.

[10] Chen S.-H., editor. *Genetic Algorithms and Genetic Programming in Computational Finance*. Kluwer Academic Publishers, 2002.

[11] C.M. Tiebout. A pure theory of local expenditures. *Journal of Political Economy*, 64(5):416–424, 1956.