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QUADRATIC POLYNOMIALS, MULTIPLIERS AND EQUIDISTRIBUTION

XAVIER BUFF AND THOMAS GAUTHIER

Abstract. Given a sequence of complex numbers ρ_n, we study the asymptotic distribution of the sets of parameters $c \in \mathbb{C}$ such that the quadratic maps z^2+c has a cycle of period n and multiplier ρ_n. Assume $\frac{1}{n}\log|\rho_n| \to L$. If $L \leq \log 2$, they equidistribute on the boundary of the Mandelbrot set. If $L > \log 2$ they equidistribute on the equipotential of the Mandelbrot set of level $2L - 2\log 2$.

Introduction

In this article, we study equidistribution questions in the parameters space of the family of quadratic polynomials

$$f_c(z) := z^2 + c,\ c \in \mathbb{C}.$$

We denote by K_c the filled-in Julia set of f_c and by J_c the Julia set:

$$K_c := \left\{ z \in \mathbb{C} \mid \left(f_c^{\circ n}(z)\right)_{n\in\mathbb{N}} \text{ is bounded} \right\} \quad \text{and} \quad J_c := \partial K_c.$$

The Mandelbrot set M is the set of parameters $c \in \mathbb{C}$ such that $0 \in K_c$.

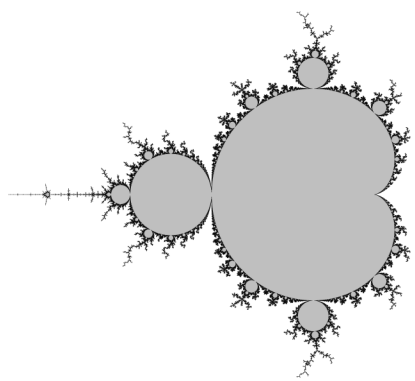

Figure 1. The Mandelbrot set M.

The Green functions $g_c : \mathbb{C} \to [0, +\infty)$ and $g_M : \mathbb{C} \to [0, +\infty)$ are defined by

$$g_c := \lim_{n\to+\infty} \max\left(\frac{1}{2^n}\log\left|f_c^{\circ n}(z)\right|, 0\right) \quad \text{and} \quad g_M(c) := g_c(c).$$

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The bifurcation measure μ_{bif} is defined by

$$\mu_{\text{bif}} := \Delta g_M.$$

The support of the measure μ_{bif} is the boundary of the Mandelbrot set M.

Let ρ be a complex number with $|\rho| \leq 1$. For $n \geq 1$, denote by X_n the set of parameters $c \in \mathbb{C}$ such that the quadratic polynomial f_c has a cycle of period n with multiplier ρ. Let ν_n be the probability measure

$$\nu_n := \frac{1}{\text{card}(X_n)} \sum_{x \in X_n} \delta_x.$$

Bassanelli and Berteloot [BB] proved that as $n \to +\infty$, the sequence of measures ν_n converges to the bifurcation measure μ_{bif} (in fact, they prove a more general result, valid in families of polynomials of arbitrary degree). In this note, we prove that the result holds without the assumption that $|\rho| \leq 1$ and we study cases where the multiplier is not fixed.

First, the set of parameters we will consider may fail to equidistribute on the boundary of the Mandelbrot set. They may equidistribute on an equipotential, i.e. a level curve of the function g_M. For $\eta \geq 0$, we let μ_η be the probability measure defined by

$$\mu_\eta := \Delta \max(g_M, \eta).$$

For $\eta > 0$, the support of μ_η is exactly the equipotential $\{c \in \mathbb{C} \mid g_M(c) = \eta\}$.

Second, we are interested in parameters $c \in \mathbb{C}$ for which f_c has a cycle of period n with multiplier ρ. When $|\rho| > 1$, we may have to count such parameters c with multiplicities. We proceed as follows. If f_c has no parabolic cycle, we set:

$$R_n(c, \rho) := \prod_{C \text{ cycle of } f_c \text{ of period } n} \big(\rho - \rho(C)\big)$$

where $\rho(C)$ is the multiplier of C as a cycle of f_c. According to Bassanelli and Berteloot [BB], this defines a polynomial $R_n \in \mathbb{C}[c, \rho]$ which is therefore defined even when f_c has parabolic cycles.

Examples. The fixed point of f_c are the roots of $z^2 - z + c$ and the multiplier of f_c at a fixed point z is $\rho = 2z$. The resultant of $z^2 - 2z + c$ and $\rho - 2z$ as polynomials in z is $\rho^2 - 2\rho + 4c$:

$$R_1(c, \rho) = \rho^2 - 2\rho + 4c.$$

The periodic points of period 2 are the roots of $z^2 + z + c + 1$ and the multiplier at a periodic point z is $\rho = 4z(z^2 + c)$. The resultant of $z^2 + z + c + 1$ and $\rho - 4z(z^2 + c)$ as polynomials in z is $(\rho + 4c + 4)^2$:

$$R_2(c, \rho) = \rho + 4 + 4c.$$

Similarly, one computes

$$R_3(c, \rho) = \rho^2 - 16\rho + 64 - 8\rho c + 64c + 128c^2 + 64c^3.$$

The degree d_n of $R_n(c,\rho)$ as a polynomial of the variable c does not depend on ρ. It may be defined recursively by

$$d_1 = 1 \quad \text{and} \quad d_n = 2^{n-1} - \sum_{\substack{m \text{ divides } n \\ m \neq n}} d_m.$$

As $n \to +\infty$ we have $d_n \sim 2^{n-1}$. For $n \geq 1$ and $\rho \in \mathbb{C}$, we set

$$u_{n,\rho} := \frac{1}{d_n} \log\big|R_n(c,\rho)\big| \quad \text{and} \quad \nu_{n,\rho} := \Delta u_{n,\rho}.$$

The measure $\nu_{n,\rho}$ is a probability measure and its support is the set of parameters c such that f_c has a cycle of period n and multiplier ρ (except when $\rho = 1$, in which case the support also contains parameters of period k dividing n whose multiplier is a n/k-th root of unity).

Our result is the following.

Theorem. *Let $(\rho_n)_{n\geq 1}$ be a sequence of complex numbers such that*

$$\lim_{n\to+\infty} \frac{1}{n} \log|\rho_n| = L \in [-\infty, +\infty).$$

As $n \to +\infty$, the sequence $(\nu_{n,\rho_n})_{n\geq 1}$ converges to μ_η with $\eta := \max(0, 2L - 2\log 2)$.

The note is organised as follows. In Section 1, we establish the Theorem relying on two Lemmas. The first one, which is proved in Section 2, is concerned with the asymptotic behavior of the sequence (u_{n,ρ_n}) outside of the compact set $\{c \in \mathbb{C} \mid g_M(c) \leq \eta\}$. The second one is a direct application of a general comparison lemma for subharmonic functions on $\mathbb{C}$ which we establish in Section 3.

1. Strategy of the proof of the Theorem

Let us set

$$v := g_M + 2\log 2 : \mathbb{C} \to [0, +\infty).$$

Let $(\rho_n)_{n\geq 1}$ be a sequence of complex numbers such that

$$\lim_{n\to+\infty} \frac{1}{n} \log|\rho_n| = L \in [-\infty, +\infty)$$

and set

$$\eta := \max(0, 2L - 2\log 2).$$

Our proof is based on the following lemmas.

Lemma 1. *If $c \in \mathbb{C} - M$ satsifies $g_M(c) > \eta$, then*

$$\lim_{n\to+\infty} u_{n,\rho_n}(c) = v(c).$$

Lemma 2. *Any subharmonic function $u : \mathbb{C} \to [-\infty, +\infty)$ which coincides with v outside M coincides with v everywhere.*

The proof is then completed as follows. Extracting a subsequence if necessary, we may assume that the sequence of probability measures $\nu_{n_k,\rho_{n_k}}$ converges to some limit ν. According to Lemma 1, the sequence $u_{n_k,\rho_{n_k}}$ then converges in L^1_{loc} to a limit u which satisfies

$$\Delta u = \nu \quad \text{and} \quad u(c) = v(c) \text{ if } g_M(c) > \eta.$$

In addition, for every $c \in \mathbb{C}$, we have

$$u(c) \geq \limsup_{n \to +\infty} u_{n_k, \rho_{n_k}}(c)$$

with equality outside a polar set. According to Lemma 1, the subharmonic functions u and v coincide in the region $\{c \in \mathbb{C} \mid g_M(c) > \eta\}$. If $L \leq \log 2$, i.e. if $\eta = 0$, Lemma 2 implies that $u = v$ and so

$$\nu = \Delta u = \Delta(g_M + 2\log 2) = \Delta g_M = \mu_{\text{bif}}.$$

So, let us consider the case $L > \log 2$, or equivalently $\eta > 0$. Note that for $n \geq 2$ the cycles of $f_0(z) = z^2$ of period n all have multiplier 2^n. There are k_n/n such points, where k_n is the number of periodic points of period n which may be defined recursively by

$$k_1 := 1 \quad \text{and} \quad k_n = 2^n - \sum_{\substack{m \text{ divides } n \\ m \neq n}} k_m.$$

As $n \to +\infty$, we have $k_n \sim 2^n$. Since $L > \log 2$, $2^n = \mathrm{o}(\rho_n)$ and so:

$$u_{n,\rho_n}(0) = \frac{k_n/n}{d_n} \log|2^n - \rho_n| \sim \frac{2^n/n}{2^{n-1}} \log|\rho_n| \to 2L.$$

As a consequence, $u(0) \geq 2L$. According to [R, Theorem 3.8.3],

$$\limsup_{\substack{g_M(c) \to \eta \\ g_M(c) < \eta}} u(c) = \max_{g_M(c) = \eta} u(c) = \limsup_{\substack{g_M(c) \to \eta \\ g_M(c) > \eta}} u(c) = 2L.$$

The Maximum Principle for subharmonic functions implies that $u(c) = 2L$ as soon as $g_M(c) \leq \eta$. So,

$$u = \max(g_M + 2\log 2, 2L) = \max(g_M, \eta) + 2\log 2 \quad \text{and} \quad \nu = \mu_\eta.$$

2. Outside the Mandelbrot set

We now study the behaviour of the multipliers of f_c when c is not in M. Our aim in the present section is to prove Lemma 1.

First, when $c \in \mathbb{C} - M$, the Julia set J_c of f_c is a Cantor set and $f_c : J_c \to J_c$ is conjugate to the shift on 2 symbols. In addition, when c varies outside M, the Julia set moves locally holomorphically. However, it is not quite true that the holomorphic motion is parameterized by $\mathbb{C} - M$: if we start with $c \in (1/4, +\infty)$ and follow the Julia set as c turns around the Mandelbrot set, every point comes back to its complex conjugate. After 2 turns, the points come back to their initial location. See [BDK] for details. To avoid the monodromy problems, it is more convenient to pass to a cover of degree 2 of $\mathbb{C} - M$.

Let $\phi_M : \mathbb{C} - M \to \mathbb{C} - \overline{\mathbb{D}}$ be the conformal representation which sends $(1/4, +\infty)$ to $(1, +\infty)$ and for $\lambda \in \mathbb{C} - \overline{\mathbb{D}}$, set

$$\mathfrak{c}(\lambda) := \phi_M^{-1}(\lambda^2).$$

Then, the holomorphic map $\mathfrak{c} : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - M$ is a covering map of degree 2.

Set $I := \{0, 1\}^{\mathbb{N}}$ and let $\sigma : I \to I$ be the shift. A point $\mathbf{i} := (i_0, i_1, i_2, \ldots) \in I$ is called an itinerary. There is a map $\psi : (\mathbb{C} - \overline{\mathbb{D}}) \times I \to \mathbb{C}$ such that

- for all $\lambda \in \mathbb{C} - \mathbb{D}$, the map $\psi_\lambda : \mathbf{i} \mapsto \psi(\lambda, \mathbf{i})$ is a bijection between I and $J_{\mathfrak{c}(\lambda)}$ conjugating σ to $f_{\mathfrak{c}(\lambda)}$:

$$\psi_\lambda \circ \sigma = f_{\mathfrak{c}(\lambda)} \circ \psi_\lambda$$

- for all $\mathbf{i} \in I$, the map $\psi_\mathbf{i} : \lambda \mapsto \psi(\lambda, \mathbf{i})$ is holomorphic.

We may choose the map ψ so that for $\lambda \in (1/4, +\infty)$, the map ψ_λ sends itineraries for which $i_0 = 0$ in the upper half-plane and itineraries for which $i_0 = 1$ in the lower half-plane. Then,

$$\psi_\mathbf{i}(\lambda) \underset{\lambda \to +\infty}{\sim} \sqrt{-1} \cdot \lambda \text{ if } i_0 = 0 \quad \text{and} \quad \psi_\mathbf{i}(\lambda) \underset{\lambda \to +\infty}{\sim} -\sqrt{-1} \cdot \lambda \text{ if } i_0 = 1.$$

Next, if $\mathbf{i}$ is periodic of period n for σ, let $\rho_\mathbf{i}(\lambda)$ be the multiplier of $\psi_\mathbf{i}(\lambda)$ as a fixed point of $f_{\mathfrak{c}(\lambda)}^{\circ n}$. Note that

$$\rho_\mathbf{i}(\lambda) := \prod_{k=0}^{n-1} 2 \cdot \psi_{\sigma^{\circ k}(\mathbf{i})}(\lambda) \underset{\lambda \to +\infty}{\sim} \varepsilon_\mathbf{i} \cdot (2\lambda)^n \quad \text{with} \quad \varepsilon_\mathbf{i} = \pm \cdot (\sqrt{-1})^n.$$

Since $\rho_\mathbf{i}$ has local degree n at ∞, we may write $\rho_\mathbf{i} = \varepsilon_\mathbf{i} \cdot \sigma_\mathbf{i}^n$ for some holomorphic map $\sigma_\mathbf{i} : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - \overline{\mathbb{D}}$ which satisfies

$$\sigma_\mathbf{i}(\lambda) \underset{\lambda \to +\infty}{\sim} 2\lambda \quad \text{and} \quad \sigma_\mathbf{i}(\omega\lambda) = \omega \cdot \sigma_\mathbf{i}(\lambda) \text{ if } \omega^n = 1.$$

The maps $\sigma_\mathbf{i}$ take their values in $\mathbb{C} - \overline{\mathbb{D}}$ and so, form a normal family.

Let $(\mathbf{i}_n)$ be a sequence of periodic itineraries of period n. Let $\sigma : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - \overline{\mathbb{D}}$ be a limit value of the sequence $(\sigma_{\mathbf{i}_n} : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - \overline{\mathbb{D}})$. Then, σ is tangent to $\lambda \mapsto 2\lambda$ at infinity and commutes with rotations. Thus, $\sigma(\lambda) = 2\lambda$. Now, if $c \in \mathbb{C} - M$, then $c = \mathfrak{c}(\lambda)$ with $\log|\lambda| = \frac{1}{2} g_M(c)$. As $n \to +\infty$,

$$\log\big|\sigma_{\mathbf{i}_n}(\lambda)\big| - \frac{1}{n} \log|\rho_n| \underset{n \to +\infty}{\longrightarrow} \log|2\lambda| - L = \frac{g_M(c) - \eta}{2}.$$

So, if $g_M(c) > \eta$, then there is an $\varepsilon > 0$ such that for any periodic itinerary $\mathbf{i}$ of high enough period n, we have

$$\left|\frac{\rho_n}{\rho_\mathbf{i}(\lambda)}\right| = \left|\frac{\rho_n}{\sigma_\mathbf{i}(\lambda)^n}\right| \leq \mathrm{e}^{-n\varepsilon}.$$

In that case

$$\begin{aligned} u_{n,\rho_n}(c) &= \frac{1}{d_n} \sum_{\substack{\mathbf{i} \text{ periodic itinerary} \\ \text{of period } n}} \frac{1}{n} \log\big|\rho_\mathbf{i}(\lambda) - \rho_n\big| \\ &= \frac{1}{d_n} \sum_{\substack{\mathbf{i} \text{ periodic itinerary} \\ \text{of period } n}} \log\big|\sigma_\mathbf{i}(\lambda)\big| + \frac{1}{n} \log\left|1 - \frac{\rho_n}{\rho_\mathbf{i}(\lambda)}\right| \\ &\underset{n \to +\infty}{=} \frac{k_n}{d_n} \log|2\lambda| + \mathrm{o}\left(\frac{k_n}{d_n}\right) \sim 2\log|2\lambda| = g_M(c) + 2\log 2, \end{aligned}$$

which ends the proof of Lemma 1.

3. Comparison of subharmonic functions

The proof of Lemma 2 relies on the following more general result.

Lemma 3. *Let $K \subset \mathbb{C}$ be a compact set such that $\mathbb{C} - K$ is connected. Let v be a subharmonic function on $\mathbb{C}$ such that Δv is supported on ∂K and does not charge the boundary of the connected components of the interior of K. Then, any subharmonic function u on $\mathbb{C}$ which coincides with v outside K coincides with v everywhere.*

Proof. According to [R, Theorem 3.8.3], for all $\zeta \in \partial K$,

$$u(\zeta) = \limsup_{\substack{z \in \mathbb{C} - K \\ z \to \zeta}} u(z) = \limsup_{\substack{z \in \mathbb{C} - K \\ z \to \zeta}} v(z) = v(\zeta). \tag{1}$$

So, $u = v$ on ∂K. The same theorem shows that, if ζ belongs to the boundary of a connected component U of the interior of K, then

$$\limsup_{\substack{z \in \overline{U} \\ z \to \zeta}} u(z) = u(\zeta) = v(\zeta) = \limsup_{\substack{z \in \overline{U} \\ z \to \zeta}} v(z). \tag{2}$$

First, consider the function w_1 defined by

$$w_1(z) = \begin{cases} \max(u, v) & \text{on } U \\ v & \text{on } \mathbb{C} - U. \end{cases}$$

According to Equations (1) and (2), w_1 is upper-semicontinuous. It is subharmonic on U and $\mathbb{C} - \overline{U}$. At any point $\zeta \in \partial U$, it satisfies the local submean inequality since $w_1(\zeta) = v(\zeta)$ and $v \leq w_1$ on $\mathbb{C}$. Thus, w_1 is globally subharmonic on $\mathbb{C}$ and coincides with v outside U. Consider a smooth test function $\chi : \mathbb{C} \to [0, 1]$ which vanishes near ∞ and is constant equal to 1 near K. On the one hand,

$$\Delta v(\mathbb{C}) = \int_{\mathbb{C}} \chi \cdot \Delta v = \int_{\mathbb{C}} \Delta \chi \cdot v = \int_{\mathbb{C}} \Delta \chi \cdot w_1 = \int_{\mathbb{C}} \chi \cdot \Delta w_1 = \Delta w_1(\mathbb{C}).$$

On the other hand, $\Delta v = \Delta w_1$ outside $\overline{U}$. Therefore,

$$\Delta w_1(\overline{U}) = \Delta w_1(\mathbb{C}) - \Delta w_1(\mathbb{C} - \overline{U}) = \Delta v(\mathbb{C}) - \Delta v(\mathbb{C} - \overline{U}) = \Delta v(\overline{U}) = 0.$$

So, $\Delta v = \Delta w_1$ on $\mathbb{C}$. The difference $v - w_1$ is harmonic and vanishes outside U. Therefore $w_1 = v$ on $\mathbb{C}$ and so, $\max(u, v) = v$ on U. It follows that $u \leq v$ on U. Since this holds for any connected component U of the interior of K, we have $u \leq v$ on $\mathbb{C}$.

Next, consider the function w_2 defined by

$$w_2(z) = \begin{cases} u & \text{on } U \\ v & \text{on } \mathbb{C} - U. \end{cases}$$

As previously, w_2 is upper-semicontinuous and subharmonic on U and $\mathbb{C} - \overline{U}$. At any point $\zeta \in \partial U$, it satisfies the local submean inequality since $w_2(\zeta) = u(\zeta)$ and $u \leq w_2$ on $\mathbb{C}$. As previously, $\Delta v = \Delta w_2$ outside $\overline{U}$ and vanishes on $\overline{U}$, so that $\Delta v = \Delta w_2$ on $\mathbb{C}$. The function $v - w_2$ is harmonic on $\mathbb{C}$ and vanishes outside $\overline{U}$. So, $w_2 = v$ on $\mathbb{C}$. In particular $u = v$ on U.

Since this is valid for any connected component U of the interior of K, we have $u = v$ on $\mathbb{C}$ as required. $\square$

Let us remark that the proof can be simplified if v is continuous, which is the case in our situation.

Proof of Lemma 2. According to Zakeri [Z], there is a set of parameters c which is of full measure for μ_{bif} such that

- J_c is locally connected and full, in particular c is not in the boundary of a hyperbolic component of M and
- the orbit of c is dense in J_c, in particular c is not renormalizable and so, not in the boundary of a queer component of M.

As a consequence, μ_{bif} does not charge the boundary of connected components of the interior of M. So, we may apply Lemma 3 with $K := M$ and $v := g_M + 2\log 2$. This yields Lemma 2. □

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