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The vertex test function for Hamilton-Jacobi equations on networks

C. Imbert* and R. Monneau†

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Abstract

A general method for proving comparison principles for Hamilton-Jacobi equations on networks is introduced. It consists in constructing a *vertex test function* to be used in the doubling variable technique. The first important consequence is that it provides very general existence and uniqueness results for Hamilton-Jacobi equations on networks with Hamiltonians that are not convex with respect to the gradient variable and can be discontinuous with respect to the space variable at vertices. It also opens many perspectives for the study of these equations in such a singular geometrical framework; to illustrate this fact, we show how to derive a homogenization result for networks from the comparison principle.

AMS Classification: 35F21, 49L25, 35B51.

Keywords: Vertex test function, Hamilton-Jacobi equations on networks, Comparison principle, discontinuous Hamiltonians.

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1 Introduction

This paper is concerned with Hamilton-Jacobi (HJ) equations on networks. In the Euclidian setting, the key result in the classical theory for HJ equations is the comparison principle

whose proof is based on the doubling variable technique. For Hamilton-Jacobi equations in a monodimensional space with piecewise constant Hamiltonians, such a method fails at the discontinuity of the Hamiltonians. Similarly, the classical doubling variable technique fails in the network setting at vertices. In this paper we show how we can modify the doubling variable technique by introducing a suitable *vertex test function* G at each vertex which will allow the edges of the network to exchange enough information. To be more specific, the usual penalization term, $\frac{(x-y)^2}{\varepsilon}$ with $\varepsilon > 0$, is replaced with $\varepsilon G(\varepsilon^{-1}x, \varepsilon^{-1}y)$. For a general HJ equation close to the node $x = 0$

$$u_t + H(x, u_x) = 0$$

the vertex test function will have to (“almost”) satisfy

$$H(y, -G_y(x, y)) - H(x, G_x(x, y)) \leq 0.$$

This inequality will be referred to as a *compatibility condition* on the gradients of the vertex test function since, if $x \in J_i$ and $y \in J_j$ with $i \neq j$, the Hamiltonian $H_i(p) = H(x, p)$ is not related to $H_j(p) = H(y, p)$. The construction of a (vertex) test function satisfying this condition allows us to circumvent the discontinuity of $H(x, p)$ at the junction point.

In order to explain the construction of the vertex test function and our main results, we focus in this introduction on the simplest network (which we will refer to as a *junction*) and on Hamiltonians which are constant with respect to the space variable on each edge. The case of a general network with (t, x) -dependent Hamiltonians will be presented in Section 5 below.

A *junction* is a network made of one node and a finite number of infinite edges and endowed with a flat metric on each edge. More precisely, it can be viewed as the set of N distinct copies ($N \geq 1$) of the half-line which are glued at the origin. For $i = 1, \dots, N$, each branch J_i is assumed to be isometric to $[0, +\infty)$ and

$$(1.1) \quad J = \bigcup_{i=1, \dots, N} J_i \quad \text{with} \quad J_i \cap J_j = \{0\} \quad \text{for} \quad i \neq j$$

where the origin 0 is called the *junction point*. For points $x, y \in J$, $d(x, y)$ denotes the geodesic distance on J defined as

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

For a real function u defined on J , $\partial_i u(x)$ denotes the (spatial) derivative of u at $x \in J_i$ and the “gradient” of u is defined as follows,

$$(1.2) \quad u_x(x) := \begin{cases} \partial_i u(x) & \text{if } x \in J_i^* := J_i \setminus \{0\}, \\ (\partial_1 u(0), \dots, \partial_N u(0)) & \text{if } x = 0. \end{cases}$$

We consider the following model Hamilton-Jacobi equation on the junction J

$$(1.3) \quad \begin{cases} u_t + H_i(u_x) = 0 & \text{for } t \in (0, +\infty) & \text{and } x \in J_i^*, \\ u_t + F_A(u_x) = 0 & \text{for } t \in (0, +\infty) & \text{and } x = 0 \end{cases}$$

submitted to the initial condition

$$(1.4) \quad u(0, x) = u^0(x) \quad \text{for } x \in J.$$

We will see below that the *junction function* F_A is constructed from the Hamiltonians H_i ($i = 1, \dots, N$) and a real parameter A . Equation (1.3) can be thought as a *system* of Hamilton-Jacobi equations associated with H_i coupled by the boundary condition involving F_A . This point of view can be useful, see Subsection 1.3.

More specifically, we consider the important case of Hamiltonians H_i satisfying the following structure condition: there exist numbers $p_i^0 \in \mathbb{R}$ such that for each $i = 1, \dots, N$,

$$(1.5) \quad \begin{cases} \text{(Continuity)} & H_i \in C(\mathbb{R}) \\ \text{(Bi-monotonicity)} & \begin{cases} H_i \text{ nonincreasing in } (-\infty, p_i^0] \\ H_i \text{ nondecreasing in } [p_i^0, +\infty) \end{cases} \\ \text{(Coercivity)} & \lim_{|q| \rightarrow +\infty} H_i(q) = +\infty \end{cases}$$

and define the A -limited flux through the junction point for $p = (p_1, \dots, p_N)$ as

$$(1.6) \quad F_A(p) = \max \left(A, \max_{i=1, \dots, N} H_i^-(p_i) \right)$$

for some given $A \in \mathbb{R} \cup \{-\infty\}$ where H_i^- is nonincreasing part of H_i defined by

$$H_i^-(q) = \begin{cases} H_i(q) & \text{if } q \leq p_i^0, \\ H_i(p_i^0) & \text{if } q > p_i^0. \end{cases}$$

It is important to treat the case of limited fluxes in (1.6) not only in view of the practical applications we have in mind (such as traffic flows), but also because we will show in a future work [16] that general classes of junction conditions reduce to F_A -conditions for a suitable parameter A . We also note that all the flux functions F_A are the same for $A \in [-\infty, A_0]$ where

$$(1.7) \quad A_0 = \max_{i=1, \dots, N} \min_{\mathbb{R}} H_i.$$

In the special case of convex Hamiltonians H_i with different minimum values, Problem (1.3) can be viewed as the Hamilton-Jacobi-Bellman equation satisfied by the value function of an optimal control problem (see for instance [17]). In this case, existence and uniqueness of viscosity solutions for (1.3)-(1.4) (with $A = -\infty$) have been established either with a very rigid method [17] based on an explicit Oleinik-Lax formula which does not extend easily to networks, or in cases reducing to $H_i = H_j$ for all i, j if Hamiltonians do not depend on the space variable [23, 1].

1.1 The main new idea: construction of the vertex test function

The goal of the present paper is to provide the reader with a general and flexible method to prove a comparison principle, allowing in particular to deal with Hamiltonians that are not convex with respect to the gradient variable and are possibly discontinuous with respect to the space variable at the vertices. As explained above, this method consists in combining the doubling variable technique with the construction of a vertex test function G . We took our inspiration for the construction of this function in papers like [14, 3] (where the case $N = 2$ is treated for corresponding scalar conservation laws). A natural family of explicit solutions is given by

$$u(t, x) = p_i x - \lambda t \quad \text{if } x \in J_i$$

for (p, λ) in the germ \mathcal{G}_A defined as follows,

$$(1.8) \quad \mathcal{G}_A = \begin{cases} \{(p, \lambda) \in \mathbb{R}^N \times \mathbb{R}, & H_i(p_i) = F_A(p) = \lambda \quad \text{for all } i = 1, \dots, N\} & \text{if } N \geq 2, \\ \{(p_1, \lambda) \in \mathbb{R} \times \mathbb{R}, & H_1(p_1) = \lambda \geq A\} & \text{if } N = 1. \end{cases}$$

In the special case of convex Hamiltonians satisfying $H_i'' > 0$ the vertex test function G is a regularized version of the function G^0 defined as follows: for $(x, y) \in J_i \times J_j$,

$$(1.9) \quad G^0(x, y) = \sup_{(p, \lambda) \in \mathcal{G}_A} (p_i x - p_j y - \lambda).$$

Such a function should indeed be regularized since it is not C^1 on the diagonal $\{x = y\}$ of J^2 .

1.2 Main results

The main result of this paper is the following comparison principle for Hamilton-Jacobi equations on a junction.

Theorem 1.1 (Comparison principle on a junction). *Let $A \in \mathbb{R} \cup \{-\infty\}$. Assume that the Hamiltonians satisfy (1.5) and that the initial datum u^0 is uniformly continuous. Then for all sub-solution u and super-solution w of (1.3), (1.4) satisfying for some $T > 0$ and $C_T > 0$:*

$$(1.10) \quad u(t, x) \leq C_T(1+d(0, x)), \quad w(t, x) \geq -C_T(1+d(0, x)), \quad \text{for all } (t, x) \in [0, T) \times J$$

we have

$$u \leq w \quad \text{on } [0, T) \times J.$$

As a consequence, we can prove the following result.

Theorem 1.2 (Existence and uniqueness on a junction). *Let $A \in \mathbb{R} \cup \{-\infty\}$. Assume that the Hamiltonians satisfy (1.5) and that the initial datum u^0 is uniformly continuous. Then there exists a unique viscosity solution u of (1.3), (1.4) such that for every $T > 0$, there exists a constant $C_T > 0$ such that*

$$|u(t, x) - u^0(x)| \leq C_T \quad \text{for all } (t, x) \in [0, T) \times J.$$

As explained above, the proof of the comparison principle on a junction relies on the construction of the vertex test function (Theorem 3.2). This construction is first achieved in the case of smooth strictly convex Hamiltonians (see Proposition 4.1).

The existence proof classically follows Ishii’s original idea [18] to use Perron’s method, reducing the construction of solutions to the construction of barriers.

We will extend easily our results to the case of networks and non-convex Hamiltonians depending on time and space and to limiting parameters A (appearing in the Hamiltonian at the junction point) depending on time and vertex, see Section 5. Noticeably, a localization procedure allows us to use the vertex test function constructed for a single junction.

In order to state the results in the network setting, we need to make precise the assumptions satisfied by the Hamiltonians associated with each edge and the limiting parameters associated with each vertex. This ends in a rather long list of assumptions. Still, when reading the proof, the reader may check that the main structure properties used in the proof of the comparison principle are gathered in the technical Lemma 5.2.

As an application of the comparison principle, we consider a model case for homogenization on a network. The network \mathcal{N}_ε whose vertices are $\varepsilon\mathbb{Z}^d$ is naturally embedded in \mathbb{R}^d . We consider for all edges a Hamiltonian only depending on the gradient variable but which is “repeated $\varepsilon\mathbb{Z}^d$ -periodically with respect to edges”. We prove that when $\varepsilon \rightarrow 0$, the solution of the “oscillating” Hamilton-Jacobi equation posed in \mathcal{N}_ε converges toward the unique solution of an “effective” Hamilton-Jacobi equation posed in \mathbb{R}^d .

Our proofs do not rely on optimal control interpretation (there is no representation formula of solutions for instance) but on PDE methods. We believe that the method consisting to construct a vertex test function is very flexible and opens many perspectives. To the best of our knowledge, it is also the first uniqueness results for a Hamilton-Jacobi equation posed on a network for Hamiltonians that are not convex with respect to the gradient variable and are possibly discontinuous at the vertices in the space variable.

1.3 Comparison with known results

Hamilton-Jacobi equations on networks. There is a growing interest in the study of Hamilton-Jacobi equations on networks. The first results were obtained in [23] for eikonal equations. Several years after this first contribution, the three papers [1, 17, 24] were published more or less simultaneously. In these three papers, the Hamiltonians are always *convex* with respect to the gradient variables and the optimal control interpretation of the equation is at the core of the proofs of comparison principles. Still, frameworks are significantly different.

First, the networks in [1] are embedded in \mathbb{R}^2 while in [23, 24, 17], the networks are understood as metric spaces and Hamilton-Jacobi equations are studied in such metric spaces. Recently, a general approach of eikonal equations in metric spaces has been proposed in [15].

In [1], the authors study an optimal control problem in \mathbb{R}^2 and impose a *state constraint*: the trajectories of the controlled system have to stay in the embedded network. From this

point of view, [1] is related to [12, 13] where trajectories in \mathbb{R}^N are constrained to stay in a closed set K which can have an empty interior. But as pointed out in [1], the framework from [12, 13] imply some conditions on the geometry of the embedded networks.

The main contribution of [17] in compare with [1, 24] comes from the dependence of the Hamiltonians with respect to the space variable. It is continuous in [1, 24] while [17] deals with Hamiltonians that possibly discontinuous at the junction point (but are independent of the space variable on each edge).

The reader is referred to [9] where the different notions of viscosity solutions used in [1, 17, 24] are compared; in the few cases where frameworks coincide, they are proved to be equivalent.

In [17], the comparison principle was a consequence of a super-optimality principle (in the spirit of [20] or [25, 26]) and the comparison of subsolutions with the value function of the optimal control problem. Still, the idea of using the “fundamental solution” \mathcal{D} to prove a comparison principle originates in the proof of the comparison of subsolutions and the value function. Moreover, as explained in Subsection 3.3, the comparison principle obtained in this paper could also be proved, for $A = -\infty$ and under more restrictive assumptions on the Hamiltonians, by using this fundamental solution.

The reader is referred to [1, 17, 24] for further references about Hamilton-Jacobi equations on networks.

Networks, regional optimal control and ramified spaces. We already pointed out that the Hamilton-Jacobi equation on a network can be regarded as a system of Hamilton-Jacobi equations coupled through vertices. In this perspective, our work can be compared with studies of Hamilton-Jacobi equations posed on say two domains separated by a frontier where some *transmission conditions* should be imposed. This can be even more general by considering equations in ramified spaces [8]. Contributions for such problems are [6, 7] and [22, 21]. Their approach differs from the one in papers like [1, 24, 17] and the present one since the idea is to write a Hamilton-Jacobi equation on the (lower-dimensional) frontier. Another difference is that techniques from dynamical systems play also an important role in these papers.

The contribution of the paper. In light of the review we made above, we can emphasize the main contributions of the paper: in compare with [23, 24], we deal not only with eikonal equations but with general Hamilton-Jacobi equations. In compare with [1], we are able to deal with networks with infinite number of edges, that are not embedded. In compare with [1, 17, 23, 24], we can deal with non-convex discontinuous Hamilton-Jacobi equations and we provide a flexible PDE method instead of a dynamical methodology.

Organization of the article. The paper is organized as follows. In Section 2, we introduce the notion of viscosity solution for Hamilton-Jacobi equations on junctions, we prove that they are stable (Proposition 2.2) and we give an existence result (Theorem 2.4). In Section 3, we prove the comparison principle in the junction case (Theorem 2.4). In

Section 4, we construct the vertex test function (Theorem 3.2). In Section 5, we explain how to generalize the previous results (viscosity solutions, HJ equations, existence, comparison principle) to the case of networks. In Section 6, we present a straightforward application of our results by proving a homogenization result passing from an “oscillating” Hamilton-Jacobi equation posed in a network embedded in an Euclidian space to a Hamilton-Jacobi equation in the whole space. Finally, we prove several technical results in Appendix A and we state results for stationary Hamilton-Jacobi equations in Appendix B.

Notation for a junction. A junction is denoted by J . It is made of a finite number of edges and a junction point. The N edges of a junction ($N \in \mathbb{N} \setminus \{0\}$) are isometric to $[0, +\infty)$. Given a final time $T > 0$, J_T denotes $(0, T) \times J$.

The Hamiltonians on the branches J_i of the junction are denoted by H_i ; they only depend on the gradient variable. The Hamiltonian at the junction point is denoted by F_A and is defined from all H_i and a constant A which “limits” the flux of information at the junction.

Given a function $u : J \rightarrow \mathbb{R}$, its gradient at x is denoted by u_x ; it is a real number if $x \neq 0$ but it is a vector of \mathbb{R}^N at $x = 0$. We let $|u_x|$ denote $|\partial_i u|$ outside the junction point and $\max_{i=1, \dots, N} |\partial_i u|$ at the junction point. If now $u(t, x)$ also depends on the time $t \in (0, +\infty)$, u_t denotes the time derivative.

Notation for networks. A network is denoted by \mathcal{N} . It is made of vertices $n \in \mathcal{V}$ and edges $e \in \mathcal{E}$. Each edge is either isometric to $[0, +\infty)$ or to a compact interval whose length is bounded from below; hence a network is naturally endowed with a metric. The associated open (resp. closed) balls are denoted by $B(x, r)$ (resp. $\bar{B}(x, r)$) for $x \in \mathcal{N}$ and $r > 0$.

In the network case, an Hamiltonian is associated with each edge e and is denoted by H_e . It depends on time and space; moreover, the limited flux functions A can depend on time and vertices: $A_n(t)$.

Further notation. Given a metric space E , $C(E)$ denotes the space of continuous real functions defined in E . A modulus of continuity is a function $\omega : [0, +\infty) \rightarrow [0, +\infty)$ which is non-increasing and $\omega(0+) = 0$.

2 Viscosity solutions on a junction

2.1 Definitions

Class of test functions. For $T > 0$, set $J_T = (0, T) \times J$. We define the class of test functions on $(0, T) \times J$ by

$$C^1(J_T) = \left\{ \varphi \in C(J_T), \text{ the restriction of } \varphi \text{ to } (0, T) \times J_i \text{ is } C^1 \text{ for } i = 1, \dots, N \right\}.$$

Viscosity solutions. In order to define viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a (locally bounded) function u defined on $[0, T) \times J$:

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

Definition 2.1 (Viscosity solutions). Assume the Hamiltonians satisfy (1.5) and let $u : [0, T) \times J \rightarrow \mathbb{R}$.

- i) We say that u is a *sub-solution* (resp. *super-solution*) of (1.3) in $(0, T) \times J$ if for all test function $\varphi \in C^1(J_T)$ such that

$$u^* \leq \varphi \quad (\text{resp.} \quad u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at (t_0, x_0) for some $t_0 > 0$, we have

$$(2.1) \quad \begin{array}{llll} \varphi_t + H_i(\varphi_x) \leq 0 & (\text{resp.} \quad \geq 0) & \text{at } (t_0, x_0) & \text{if } x_0 \in J_i^* \\ \varphi_t + F_A(\varphi_x) \leq 0 & (\text{resp.} \quad \geq 0) & \text{at } (t_0, x_0) & \text{if } x_0 = 0. \end{array}$$

- ii) We say that u is a *sub-solution* (resp. *super-solution*) of (1.3), (1.4) on $[0, T) \times J$ if additionally

$$u^*(0, x) \leq u_0(x) \quad (\text{resp.} \quad u_*(0, x) \geq u_0(x)) \quad \text{for all } x \in J.$$

- iii) We say that u is a (*viscosity*) *solution* if u is both a sub-solution and a super-solution.

2.2 Stability

An important property that we expect for viscosity solutions is their stability, either by passage to the limit, or for instance the stability of sub-solutions by passing to the supremum.

Proposition 2.2 (Stability by supremum/infimum). *Assume the Hamiltonians H_i satisfy (1.5). Let \mathcal{A} be non empty set and let $(u_a)_{a \in \mathcal{A}}$ be a family of sub-solutions (resp. super-solutions) of (1.3) on $(0, T) \times J$. Let us assume that*

$$u = \sup_{a \in \mathcal{A}} u_a \quad (\text{resp.} \quad u = \inf_{a \in \mathcal{A}} u_a)$$

is locally bounded on $(0, T) \times J$. Then u is a sub-solution (resp. super-solution) of (1.3) on $(0, T) \times J$.

Proving this result amounts to prove that the junction condition, which can be seen as a boundary condition, is always satisfied in the sense of Definition 2.1. Indeed, suprema (resp. infima) or upper (resp. lower) semi-limits of sub-solutions are known to satisfy boundary conditions *in a viscosity sense* [19, 5]: at the junction point either the junction condition or the equation is satisfied. The following proposition implies that, because of the junction functions F_A we are working with, we can prove that the junction condition is in fact always satisfied *in the classical sense*, that is to say in the sense of Definition 2.1.

Proposition 2.3 (The junction condition is always satisfied in the classical sense). *Assume the Hamiltonians satisfy (1.5) and let $u: (0, T) \times J \rightarrow \mathbb{R}$. If, for all test function $\varphi \in C^1(J_T)$ such that*

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at (t_0, x_0) for some $t_0 > 0$, we have

$$\left\{ \begin{array}{l} \varphi_t + F_A(\varphi_x) \leq 0 \quad (\text{resp. } \geq 0), \\ \text{OR} \\ \varphi_t + H_i(\partial_i \varphi) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{for some } i \in \{1, \dots, N\} \end{array} \right\} \quad \text{at } (t_0, x_0) \text{ if } x_0 = 0,$$

then u is a sub-solution (resp. super-solution) of (1.3) in $(0, T) \times J$.

Proof of Proposition 2.3. The proof was done in [17] for the case $A = -\infty$, using the monotonicities of the H_i . We follow the same proof and omit details.

Case 1: the super-solution case. Let u be a supersolution satisfying the junction condition in the viscosity sense and let us assume by contradiction that there exists a test function φ touching u from below at $P_0 = (t_0, 0)$ for some $t_0 \in (0, T)$, such that

$$(2.2) \quad \varphi_t + F_A(\varphi_x) < 0 \quad \text{at } P_0.$$

Then we can construct a test function $\tilde{\varphi}$ satisfying $\tilde{\varphi} \leq \varphi$ in a neighborhood of P_0 , with equality at P_0 such that

$$\tilde{\varphi}_t(P_0) = \varphi_t(P_0) \quad \text{and} \quad \partial_i \tilde{\varphi}(P_0) = \min(p_i^0, \partial_i \varphi(P_0)) \quad \text{for } i = 1, \dots, N.$$

Using the fact that $F_A(\varphi_x) = F_A(\tilde{\varphi}_x) \geq H_i^-(\partial_i \tilde{\varphi}) = H_i(\partial_i \tilde{\varphi})$ at P_0 , we deduce a contradiction with (2.2) using the viscosity inequality satisfied by φ for some $i \in \{1, \dots, N\}$.

Case 2: the sub-solution case. Let now u be a subsolution satisfying the junction condition in the viscosity sense and let us assume by contradiction that there exists a test function φ touching u from above at $P_0 = (t_0, 0)$ for some $t_0 \in (0, T)$, such that

$$(2.3) \quad \varphi_t + F_A(\varphi_x) > 0 \quad \text{at } P_0.$$

Let us define

$$I = \{i \in \{1, \dots, N\}, \quad H_i^-(\varphi) < F_A(\varphi_x) \quad \text{at } P_0\}$$

and for $i \in I$, let $q_i \geq p_i^0$ be such that

$$H_i(q_i) = F_A(\varphi_x(P_0))$$

where we have used the fact that $H_i(+\infty) = +\infty$. Then we can construct a test function $\tilde{\varphi}$ satisfying $\tilde{\varphi} \geq \varphi$ in a neighborhood of P_0 , with equality at P_0 , such that

$$\tilde{\varphi}_t(P_0) = \varphi_t(P_0) \quad \text{and} \quad \partial_i \tilde{\varphi}(P_0) = \begin{cases} \max(q_i, \partial_i \varphi(P_0)) & \text{if } i \in I, \\ \partial_i \varphi(P_0) & \text{if } i \notin I. \end{cases}$$

Using the fact that $F_A(\varphi_x) = F_A(\tilde{\varphi}_x) \leq H_i(\partial_i \tilde{\varphi})$ at P_0 , we deduce a contradiction with (2.3) using the viscosity inequality for φ for some $i \in \{1, \dots, N\}$. \square

2.3 Existence

Theorem 2.4 (Existence on the junction). *Let J be the junction defined in (1.1) and let $A \in \mathbb{R} \cup \{-\infty\}$. Assume condition (1.5) on the Hamiltonians and that the initial data u^0 is uniformly continuous. Let $T > 0$. Then there exists a viscosity solution u of (1.3),(1.4) on $[0, T) \times J$ and a constant $C_T > 0$ such that*

$$|u(t, x) - u^0(x)| \leq C_T \quad \text{for all } (t, x) \in [0, T) \times J.$$

Proof of Theorem 2.4. The proof follows classically along the lines of Perron's method (see [18, 10]), and then we omit details. We assume without loss of generality that $A > -\infty$ (and even $A > A_0 - 1$).

Step 1: Barriers. Because of the uniform continuity of u^0 , for any $\varepsilon \in (0, 1]$, it can be regularized by convolution to get a modified initial data u_ε^0 satisfying

$$(2.4) \quad |u_\varepsilon^0 - u^0| \leq \varepsilon \quad \text{and} \quad |(u_\varepsilon^0)_x| \leq L_\varepsilon$$

with $L_\varepsilon \geq \max_{i=1, \dots, N} |p_i^0|$. Let $C_\varepsilon = \max(|A|, \max_{i=1, \dots, N} \max_{|p_i| \leq L_\varepsilon} |H_i(p_i)|)$. Then the functions

$$(2.5) \quad u_\varepsilon^\pm(t, x) = u_\varepsilon^0(x) \pm C_\varepsilon t \pm \varepsilon$$

are global super and sub-solutions with respect to the initial data u^0 . We then define

$$u^+(t, x) = \inf_{\varepsilon \in (0, 1]} u_\varepsilon^+(t, x) \quad \text{and} \quad u^-(t, x) = \sup_{\varepsilon \in (0, 1]} u_\varepsilon^-(t, x).$$

Then we have $u^- \leq u^+$ with $u^-(0, x) = u^0(x) = u^+(0, x)$. Moreover, by stability of sub/super-solutions (see Proposition 2.2), we get that u^+ is a super-solution and u^- is a sub-solution of (1.3) on $(0, T) \times J$.

Step 2: Maximal sub-solution and preliminaries. Consider the set

$$S = \{w : [0, T) \times J \rightarrow \mathbb{R}, \quad w \text{ is a sub-solution of (1.3) on } (0, T) \times J, \quad u^- \leq w \leq u^+\}.$$

It contains u^- . Then the function

$$u(t, x) = \sup_{w \in S} w(t, x)$$

is a sub-solution of (1.3) on $(0, T) \times J$ and satisfies the initial condition. It remains to show that u is a super-solution of (1.3) on $(0, T) \times J$. This is classical for a Hamilton-Jacobi equation on an interval, so we only have to prove it at the junction point. We assume by contradiction that u is not a super-solution at $P_0 = (t_0, 0)$ for some $t_0 \in (0, T)$. Thanks

to Proposition 2.3, this implies that there exists a test function φ satisfying $u_* \geq \varphi$ in a neighborhood of P_0 with equality at P_0 , and such that

$$(2.6) \quad \left\{ \begin{array}{l} \varphi_t + F_A(\varphi_x) < 0, \\ \varphi_t + H_i(\partial_i \varphi) < 0, \quad \text{for } i = 1, \dots, N \end{array} \right\} \quad \text{at } P_0.$$

We also have $\varphi \leq u_* \leq u_*^+$. As usual, the fact that u^+ is a super-solution and condition (2.6) imply that we cannot have $\varphi = (u^+)_*$ at P_0 . Therefore we have for some $r > 0$ small enough

$$(2.7) \quad \varphi < (u^+)_* \quad \text{on } \overline{B_r(P_0)}$$

where we define the ball $B_r(P_0) = \{(t, x) \in (0, T) \times J, \quad |t - t_0|^2 + d^2(0, x) < r^2\}$. Subtracting $|(t, x) - P_0|^2$ to φ and reducing $r > 0$ if necessary, we can assume that

$$(2.8) \quad \varphi < u_* \quad \text{on } \overline{B_r(P_0)} \setminus \{P_0\}.$$

Further reducing $r > 0$, we can also assume that (2.6) still holds in $\overline{B_r(P_0)}$.

Step 3: Sub-solution property and contradiction. We claim that φ is a sub-solution of (1.3) in $B_r(P_0)$. Indeed, if ψ is a test function touching φ from above at $P_1 = (t_1, 0) \in B_r(P_0)$, then

$$\psi_t(P_1) = \varphi_t(P_1) \quad \text{and} \quad \partial_i \psi(P_1) \geq \partial_i \varphi(P_1) \quad \text{for } i = 1, \dots, N.$$

Using the fact that F_A is non-increasing with respect to all variables, we deduce that

$$\psi_t + F_A(\psi_x) < 0 \quad \text{at } P_1$$

as desired. Defining for $\delta > 0$,

$$u_\delta = \begin{cases} \max(\delta + \varphi, u) & \text{in } B_r(P_0), \\ u & \text{outside} \end{cases}$$

and using (2.8), we can check that $u_\delta = u > \delta + \varphi$ on $\partial B_r(P_0)$ for $\delta > 0$ small enough. This implies that u_δ is a sub-solution lying above u^- . Finally (2.7) implies that $u_\delta \leq u^+$ for $\delta > 0$ small enough. Therefore $u_\delta \in S$, but it is classical to check that u_δ is not below u for $\delta > 0$, which gives a contradiction with the maximality of u . \square

3 Comparison principle on a junction

This section is devoted to the proof of the comparison principle in the case of a junction (see Theorem 1.1).

It is convenient to introduce the following shorthand notation

$$(3.1) \quad H(x, p) = \begin{cases} H_i(p) & \text{for } p = p_i & \text{if } x \in J_i^*, \\ F_A(p) & \text{for } p = (p_1, \dots, p_N) & \text{if } x = 0. \end{cases}$$

In particular, keeping in mind the definition of u_x (see (1.2)), Problem (1.3) on the junction can be rewritten as follows

$$u_t + H(x, u_x) = 0 \quad \text{for all } (t, x) \in (0, +\infty) \times J.$$

We next make a trivial but useful observation.

Lemma 3.1. *It is enough to prove Theorem 1.1 further assuming that*

$$(3.2) \quad p_i^0 = 0 \quad \text{for } i = 1, \dots, N \quad \text{and} \quad 0 = H_1(0) \geq H_2(0) \geq \dots \geq H_N(0).$$

Proof of Lemma 3.1. We can assume without loss of generality that

$$H_1(p_1^0) \geq \dots \geq H_N(p_N^0).$$

Let us define

$$u(t, x) = \tilde{u}(t, x) + p_i^0 x - tH_1(p_1^0) \quad \text{for } x \in J_i.$$

Then u is a solution of (1.3) if and only if \tilde{u} is a solution of (1.3) with each H_i replaced with $\tilde{H}_i(p) = H_i(p + p_i^0) - H_1(p_1^0)$ and F_A replaced with $\tilde{F}_{\tilde{A}}$ constructed using the Hamiltonians \tilde{H}_i and the parameter $\tilde{A} = A - H_1(p_1^0)$. \square

3.1 The vertex test function

Then our key result is the following one.

Theorem 3.2 (The vertex test function – general case). *Let $A \in \mathbb{R} \cup \{-\infty\}$ and $\gamma > 0$. Assume the Hamiltonians satisfy (1.5) and (3.2). Then there exists a function $G : J^2 \rightarrow \mathbb{R}$ enjoying the following properties.*

i) (Regularity)

$$G \in C(J^2) \quad \text{and} \quad \begin{cases} G(x, \cdot) \in C^1(J) & \text{for all } x \in J, \\ G(\cdot, y) \in C^1(J) & \text{for all } y \in J. \end{cases}$$

ii) (Bound from below) $G \geq 0 = G(0, 0)$.

iii) (Compatibility condition on the diagonal) For all $x \in J$,

$$(3.3) \quad 0 \leq G(x, x) - G(0, 0) \leq \gamma.$$

iv) (Compatibility condition on the gradients) For all $(x, y) \in J^2$,

$$(3.4) \quad H(y, -G_y(x, y)) - H(x, G_x(x, y)) \leq \gamma$$

where notation introduced in (1.2) and (3.1) are used.

v) (Superlinearity) There exists $g : [0, +\infty) \rightarrow \mathbb{R}$ nondecreasing and s.t. for $(x, y) \in J^2$

$$(3.5) \quad g(d(x, y)) \leq G(x, y) \quad \text{and} \quad \lim_{a \rightarrow +\infty} \frac{g(a)}{a} = +\infty.$$

vi) (Gradient bounds) For all $K > 0$, there exists $C_K > 0$ such that for all $(x, y) \in J^2$,

$$(3.6) \quad d(x, y) \leq K \quad \implies \quad |G_x(x, y)| + |G_y(x, y)| \leq C_K.$$

3.2 Proof of the comparison principle

We will also need the following result whose classical proof is given in Appendix for the reader's convenience.

Lemma 3.3 (A priori control). *Let $T > 0$ and let u be a sub-solution and w be a super-solution as in Theorem 1.1. Then there exists a constant $C = C(T) > 0$ such that for all $(t, x), (s, y) \in [0, T] \times J$, we have*

$$(3.7) \quad u(t, x) \leq w(s, y) + C(1 + d(x, y)).$$

We are now ready to make the proof of comparison principle.

Proof of Theorem 1.1. The proof proceeds in several steps.

Step 1: the penalization procedure. We want to prove that

$$M = \sup_{(t,x) \in [0,T] \times J} (u(t, x) - w(t, x)) \leq 0.$$

Assume by contradiction that $M > 0$. Then for $\alpha, \eta > 0$ small enough, we have $M_{\varepsilon, \alpha} \geq M/2 > 0$ for all $\varepsilon, \nu > 0$ with

$$(3.8) \quad M_{\varepsilon, \alpha} = \sup_{(t,x), (s,y) \in [0,T] \times J} \left\{ u(t, x) - w(s, y) - \varepsilon G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha \frac{d^2(0, x)}{2} \right\}$$

where the vertex test function $G \geq 0$ is given by Theorem 3.2 for a parameter γ satisfying

$$0 < \gamma < \min\left(\frac{\eta}{2T^2}, \frac{M}{4\varepsilon}\right).$$

Thanks to Lemma 3.3 and (3.5), we deduce that

$$(3.9) \quad 0 < \frac{M}{2} \leq C(1 + d(x, y)) - \varepsilon g\left(\frac{d(x, y)}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha \frac{d^2(0, x)}{2}$$

which implies in particular that

$$(3.10) \quad \varepsilon g\left(\frac{d(x, y)}{\varepsilon}\right) \leq C(1 + d(x, y)).$$

Because of the superlinearity of g appearing in (3.5), we know that for any $K > 0$, there exists a constant $C_K > 0$ such that for all $a \geq 0$

$$Ka - C_K \leq g(a).$$

For $K \geq 2C$, we deduce from (3.10) that

$$(3.11) \quad d(x, y) \leq \inf_{K \geq 2C} \left\{ \frac{C}{K-C} + \frac{C_K}{C} \varepsilon \right\} =: \omega(\varepsilon)$$

where ω is a concave, nondecreasing function satisfying $\omega(0) = 0$. We deduce from (3.9) and (3.11) that the supremum in (3.8) is reached at some point $(t, x, s, y) = (t_\nu, x_\nu, s_\nu, y_\nu)$.

Step 2: use of the initial condition. We first treat the case where $t_\nu = 0$ or $s_\nu = 0$. If there exists a sequence $\nu \rightarrow 0$ such that $t_\nu = 0$ or $s_\nu = 0$, then calling (x_0, y_0) any limit of subsequences of (x_ν, y_ν) , we get from (3.8) and the fact that $M_{\varepsilon, \alpha} \geq M/2$ that

$$0 < \frac{M}{2} \leq u^0(x_0) - u^0(y_0) \leq \omega_0(d(x_0, y_0)) \leq \omega_0 \circ \omega(\varepsilon)$$

where ω_0 is the modulus of continuity of the initial data u^0 and ω is defined in (3.11). This is impossible for ε small enough.

Step 3: use of the equation. We now treat the case where $t_\nu > 0$ and $s_\nu > 0$. Then we can write the viscosity inequalities with $(t, x, s, y) = (t_\nu, x_\nu, s_\nu, y_\nu)$ using the shorthand notation (3.1) for the Hamiltonian,

$$\begin{aligned} \frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + H(x, G_x(\varepsilon^{-1}x, \varepsilon^{-1}y) + \alpha d(0, x)) &\leq 0, \\ \frac{t-s}{\nu} + H(y, -G_y(\varepsilon^{-1}x, \varepsilon^{-1}y)) &\geq 0. \end{aligned}$$

Substrating these two inequalities, we get

$$\frac{\eta}{T^2} \leq H(y, -G_y(\varepsilon^{-1}x, \varepsilon^{-1}y)) - H(x, G_x(\varepsilon^{-1}x, \varepsilon^{-1}y) + \alpha d(0, x)).$$

Using (3.4) with $\gamma \in (0, \frac{\eta}{2T^2})$, we deduce for $p = G_x(\varepsilon^{-1}x, \varepsilon^{-1}y)$

$$(3.12) \quad \frac{\eta}{2T^2} \leq H(x, p) - H(x, p + \alpha d(0, x)).$$

Because of (3.6) and (3.11), we see that p is bounded for ε fixed by $|p| \leq C_{\frac{\omega(\varepsilon)}{\varepsilon}}$. Finally, for $\varepsilon > 0$ fixed and $\alpha \rightarrow 0$, we have $\alpha d(0, x) \rightarrow 0$, and using the uniform continuity of $H(x, p)$ for $x \in J$ and p bounded, we get a contradiction in (3.12). The proof is now complete. \square

3.3 The function G^0 versus \mathcal{D}_0 from [17]

Recalling the definition of the germ \mathcal{G}_A (see (1.8)), let us associate with any $(p, \lambda) \in \mathcal{G}_A$ the following functions for $i, j = 1, \dots, N$,

$$u^{p, \lambda}(t, x, s, y) = p_i x - p_j y - \lambda(t - s) \quad \text{for } (x, y) \in J_i \times J_j, \quad t, s \in \mathbb{R}.$$

The reader can check that they solve the following system,

$$(3.13) \quad \begin{cases} u_t + H(x, u_x) = 0, \\ -u_s + H(y, -u_y) = 0. \end{cases}$$

Then, for $N \geq 2$, the function $\tilde{G}^0(t, x, s, y) = (t - s)G^0\left(\frac{x}{t-s}, \frac{y}{t-s}\right)$ can be rewritten as

$$(3.14) \quad \tilde{G}^0(t, x, s, y) = \sup_{(p, \lambda) \in \mathcal{G}_A} u^{p, \lambda}(t, x, s, y) \quad \text{for } (x, y) \in J \times J, \quad t - s \geq 0$$

which satisfies

$$(3.15) \quad \tilde{G}^0(s, x, s, y) = \begin{cases} 0 & \text{if } x = y, \\ +\infty & \text{otherwise.} \end{cases}$$

For $N \geq 2$ and $A > A_0$, it is possible to check (assuming (4.1)) that \tilde{G}^0 is a viscosity solution of (3.13) for $t - s > 0$, only outside the diagonal $\{x = y \neq 0\}$. Therefore, even if (3.14) appears as a kind of (second) Hopf formula (see for instance [4, 2]), this formula does not provide a true solution on the junction.

On the other hand, under more restrictive assumptions on the Hamiltonians and for $A = A_0$ and $N \geq 2$ (see [17]), there is a natural viscosity solution of (3.13) with the same initial conditions (3.15), which is $\mathcal{D}(t, x, s, y) = (t - s)\mathcal{D}_0\left(\frac{x}{t-s}, \frac{y}{t-s}\right)$ where \mathcal{D}_0 is a cost function defined in [17] following an optimal control interpretation. The function \mathcal{D}_0 is not C^1 in general (but it is semi-concave) and it is much more difficult to study it and to use it in comparison with G^0 . Nevertheless, under suitable restrictive assumptions on the Hamiltonians, it would be also possible to replace in our proof of the comparison principle the term $\varepsilon G(\varepsilon^{-1}x, \varepsilon^{-1}y)$ in (3.8) by $\varepsilon\mathcal{D}_0(\varepsilon^{-1}x, \varepsilon^{-1}y)$.

4 Construction of the vertex test function

This section is devoted to the proof of Theorem 3.2. Our construction of the vertex test function G is modelled on the particular subcase of normalized convex Hamiltonians H_i .

4.1 The case of smooth convex Hamiltonians

Assume that the Hamiltonians H_i satisfy the following assumptions for $i = 1, \dots, N$,

$$(4.1) \quad \begin{cases} H_i \in C^2(\mathbb{R}) & \text{with } H_i'' > 0 & \text{on } \mathbb{R}, \\ H_i' < 0 & \text{on } (-\infty, 0) & \text{and } H_i' > 0 & \text{on } (0, +\infty), \\ \lim_{|p| \rightarrow +\infty} \frac{H_i(p)}{|p|} = +\infty. \end{cases}$$

It is useful to associate with each H_i satisfying (4.1) its partial inverse functions π_i^\pm :

$$(4.2) \quad \text{for } \lambda \geq H_i(0), \quad H_i(\pi_i^\pm(\lambda)) = \lambda \quad \text{such that} \quad \pm \pi_i^\pm(\lambda) \geq 0.$$

Assumption (4.1) implies that $\pi_i^\pm \in C^2(\min H_i, +\infty) \cap C([\min H_i, +\infty))$ thanks to the inverse function theorem.

We recall that G^0 is defined, for $i, j = 1, \dots, N$, by

$$G^0(x, y) = \sup_{(p, \lambda) \in \mathcal{G}_A} (p_i x - p_j y - \lambda) \quad \text{if } (x, y) \in J_i \times J_j$$

where \mathcal{G}_A is defined in (1.8). Replacing A with $\max(A, A_0)$ if necessary, we can always assume that $A \geq A_0$ with A_0 given by (1.7).

Proposition 4.1 (The vertex test function – the smooth convex case). *Let $A \geq A_0$ with A_0 given by (1.7) and assume that the Hamiltonians satisfy (4.1). Then G^0 satisfies*

i) (Regularity)

$$G^0 \in C(J^2) \quad \text{and} \quad \begin{cases} G^0 \in C^1(\{(x, y) \in J \times J, x \neq y\}), \\ G^0(0, \cdot) \in C^1(J) \quad \text{and} \quad G^0(\cdot, 0) \in C^1(J); \end{cases}$$

ii) (Bound from below) $G^0 \geq G^0(0, 0) = -A$;

iii) (Compatibility conditions) (3.3) and (3.4) hold with $\gamma = 0$;

iv) (Superlinearity) (3.5) holds for some $g = g^0$;

v) (Gradient bounds) (3.6) holds only for $(x, y) \in J^2$ such that $x \neq y$ or $(x, y) = (0, 0)$;

vi) (Saturation close to the diagonal) For $i \in \{1, \dots, N\}$ and for $(x, y) \in J_i \times J_i$, we have $G^0(x, y) = \ell_i(x - y)$ with $\ell_i \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ and

$$\ell_i(a) = \begin{cases} +a\pi_i^+(A) - A & \text{if } 0 \leq a \leq z_i^+ \\ -a\pi_i^-(A) - A & \text{if } z_i^- \leq a \leq 0 \end{cases}$$

where $(z_i^-, z_i^+) := (H'_i(\pi_i^-(A)), H'_i(\pi_i^+(A)))$ and the functions π_i^\pm are defined in (4.2). Moreover $G^0 \in C^1(J_i \times J_i)$ if and only if $\pi_i^+(A) = 0 = \pi_i^-(A)$.

Remark 4.2. The compatibility condition (3.4) is in fact an equality with $\gamma = 0$ when $N \geq 2$.

The proof of this proposition is postponed until Subsection 4.4. With such a result in hand, we can now prove Theorem 3.2 in the case of smooth convex Hamiltonians.

Lemma 4.3 (The case of smooth convex Hamiltonians). *Assume that the Hamiltonians satisfy (4.1). Then the conclusion of Theorem 3.2 holds true.*

Proof. We note that the function G^0 satisfies all the properties required for G , except on the diagonal $\{(x, y) \in J \times J, x = y \neq 0\}$ where G^0 may not be C^1 . To this end, we first introduce the set I of indices such that $G^0 \notin C^1(J_i \times J_i)$. We know from Proposition 4.1 vi) that

$$I = \{i \in \{1, \dots, N\}, \pi_i^+(A) > \pi_i^-(A)\}.$$

For $i \in I$, we are going to construct a regularization $\tilde{G}^{0,i}$ of G^0 in a neighbourhood of the diagonal $\{(x, y) \in J_i \times J_i, x = y \neq 0\}$.

Step 1: Construction of $\tilde{G}^{0,i}$ for $i \in I$. Let us define

$$L_i(a) = \begin{cases} a\pi_i^+(A) & \text{if } a \geq 0, \\ -a\pi_i^-(A) & \text{if } a \leq 0. \end{cases}$$

We first consider a convex C^1 function $\tilde{L}_i : \mathbb{R} \rightarrow \mathbb{R}$ coinciding with L_i outside (z_i^-, z_i^+) , that we choose such that

$$(4.3) \quad 0 \leq \tilde{L}_i - L_i \leq 1.$$

Then for $\varepsilon \in (0, 1]$, we define

$$\ell_i^\varepsilon(a) := \begin{cases} \varepsilon \tilde{L}_i\left(\frac{a}{\varepsilon}\right) - A & \text{if } a \in [\varepsilon z_i^-, \varepsilon z_i^+], \\ \ell_i(a) & \text{otherwise.} \end{cases}$$

which is a $C^1(\mathbb{R})$ (and convex) function. We now consider a cut-off function ζ satisfying for some constant $B > 0$

$$(4.4) \quad \begin{cases} \zeta \in C^\infty(\mathbb{R}), \\ \zeta' \geq 0, \\ \zeta = 0 & \text{in } (-\infty, 0], \\ \zeta = 1 & \text{in } [B, +\infty), \\ \pm z_i^\pm \zeta' < 1 & \text{in } (0, +\infty) \end{cases}$$

and for $\varepsilon \in (0, 1]$, we define for $(x, y) \in J_i \times J_i$:

$$\tilde{G}^{0,i}(x, y) = \ell_i^{\varepsilon \zeta(x+y)}(x - y).$$

Step 2: First properties of $\tilde{G}^{0,i}$. By construction, we have $\tilde{G}^{0,i} \in C^1((J_i \times J_i) \setminus \{0\})$. Moreover we have

$$\tilde{G}^{0,i} = G^0 \quad \text{on } (J_i \times J_i) \setminus \delta_i^\varepsilon$$

where

$$\delta_i^\varepsilon = \{(x, y) \in J_i \times J_i, \quad \varepsilon z_i^- \zeta(x+y) < x - y < \varepsilon z_i^+ \zeta(x+y)\}$$

is a neighborhood of the diagonal

$$\{(x, y) \in J_i \times J_i, \quad x = y \neq 0\}.$$

Because of (4.3), we also have

$$(4.5) \quad G^0 \leq \tilde{G}^{0,i} \leq \varepsilon.$$

As a consequence of (4.4), we have in particular

$$(J_i \times J_i) \setminus \delta_i^\varepsilon \supset (J_i \times \{0\}) \cup (\{0\} \times J_i)$$

and moreover $\tilde{G}^{0,i}$ coincides with G^0 on a neighborhood of $(J_i^* \times \{0\}) \cup (\{0\} \times J_i^*)$, which implies that

$$(4.6) \quad \tilde{G}^{0,i} = G^0, \quad \tilde{G}_x^{0,i} = G_x^0 \quad \text{and} \quad \tilde{G}_y^{0,i} = G_y^0 \quad \text{on } (J_i \times \{0\}) \cup (\{0\} \times J_i).$$

Step 3: Computation of the gradients of $\tilde{G}^{0,i}$. For $(x, y) \in \delta_i^\varepsilon$, we have

$$\begin{cases} \tilde{G}_x^{0,i}(x, y) &= (\ell_i^{\varepsilon\zeta(x+y)})'(x-y) + \varepsilon\zeta'(x+y) \xi_i \left(\frac{x-y}{\varepsilon\zeta(x+y)} \right) \\ -\tilde{G}_y^{0,i}(x, y) &= (\ell_i^{\varepsilon\zeta(x+y)})'(x-y) - \varepsilon\zeta'(x+y) \xi_i \left(\frac{x-y}{\varepsilon\zeta(x+y)} \right) \end{cases}$$

with

$$\xi_i(b) = \tilde{L}_i(b) - b\tilde{L}'_i(b)$$

while if $(x, y) \in (J_i \times J_i) \setminus \delta_i^\varepsilon$ we have

$$\tilde{G}_x^{0,i}(x, y) = -\tilde{G}_y^{0,i}(x, y).$$

Given $\gamma > 0$, and using the local uniform continuity of H_i , we see that we have for ε small enough

$$H_i(\tilde{G}_x^{0,i}) \leq H_i(-\tilde{G}_y^{0,i}) + \gamma \quad \text{in } J_i^* \times J_i^*$$

and using (4.6), we get

$$(4.7) \quad H(x, \tilde{G}_x^{0,i}(x, y)) - H(y, -\tilde{G}_y^{0,i}(x, y)) \leq \gamma \quad \text{for all } (x, y) \in J_i \times J_i.$$

Step 4: Definition of G . We set for $(x, y) \in J_i \times J_j$:

$$G(x, y) = \begin{cases} G^0(x, y) - G^0(0, 0) & \text{if } i \neq j \text{ or } i = j \notin I, \\ \tilde{G}^{0,i}(x, y) - G^0(0, 0) & \text{if } i = j \in I. \end{cases}$$

From the properties of G^0 , we recover all the expected properties of G with $g(a) = g^0(a) - G^0(0, 0)$. In particular from (4.7) and (4.5), we respectively get the compatibility condition for the Hamiltonians (3.4) and the compatibility condition on the diagonal (3.3) for ε small enough. \square

4.2 The general case

Let us consider a slightly stronger assumption than (1.5), namely

$$(4.8) \quad \begin{cases} H_i \in C^2(\mathbb{R}) \quad \text{with } H_i''(p_i^0) > 0, \\ H_i' < 0 \quad \text{on } (-\infty, p_i^0) \quad \text{and } H_i' > 0 \quad \text{on } (p_i^0, +\infty), \\ \lim_{|q| \rightarrow +\infty} H_i(q) = +\infty. \end{cases}$$

We will also use the following technical result which allows us to reduce certain non convex Hamiltonians to convex Hamiltonians.

Lemma 4.4 (From non convex to convex Hamiltonians). *Given Hamiltonians H_i satisfying (4.8) and (3.2), there exists a function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ such that the functions $\beta \circ H_i$ satisfy (4.1) for $i = 1, \dots, N$. Moreover, we can choose β such that*

$$(4.9) \quad \beta \text{ is convex, } \beta \in C^2(\mathbb{R}), \quad \beta(0) = 0 \quad \text{and} \quad \beta' \geq \delta > 0.$$

Proof. Recalling (4.2), it is easy to check that $(\beta \circ H_i)'' > 0$ if and only if we have

$$(4.10) \quad (\ln \beta')'(\lambda) > -\frac{H_i''}{(H_i')^2} \circ \pi_i^\pm(\lambda) \quad \text{for } \lambda \geq H_i(0).$$

Because $H_i''(0) > 0$, we see that the right hand side is negative for λ close enough to $H_i(0)$. Then it is easy to choose a function β satisfying (4.10) and (4.9). Finally, composing β with another convex increasing function which is superlinear at $+\infty$ if necessary, we can ensure that $\beta \circ H_i$ superlinear. \square

Lemma 4.5 (The case of smooth Hamiltonians). *Theorem 3.2 holds true if the Hamiltonians satisfy (4.8).*

Proof. We assume that the Hamiltonians H_i satisfy (4.8). Thanks to Lemma 3.1, we can further assume that they satisfy (3.2). Let β be the function given by Lemma 4.4. If u solves (1.3) on $(0, T) \times J$, then u is also a viscosity solution of

$$(4.11) \quad \begin{cases} \bar{\beta}(u_t) + \hat{H}_i(u_x) = 0 & \text{for } t \in (0, T) & \text{and } x \in J_i^*, \\ \bar{\beta}(u_t) + \hat{F}_{\hat{A}}(u_x) = 0 & \text{for } t \in (0, T) & \text{and } x = 0 \end{cases}$$

with $\hat{F}_{\hat{A}}$ constructed as F_A where H_i and A are replaced with \hat{H}_i and \hat{A} defined as follows

$$\hat{H}_i = \beta \circ H_i, \quad \hat{A} = \beta(A)$$

and $\bar{\beta}(\lambda) = -\beta(-\lambda)$. We can then apply Theorem 3.2 in the case of smooth convex Hamiltonians to construct a vertex test function \hat{G} associated to problem (4.11) for every $\hat{\gamma} > 0$. This means that we have with $\hat{H}(x, p) = \beta(H(x, p))$,

$$\hat{H}(y, -G_y) \leq \hat{H}(x, G_x) + \hat{\gamma}.$$

This implies

$$H(y, -G_y) \leq \beta^{-1}(\beta(H(x, G_x)) + \hat{\gamma}) \leq H(x, G_x) + \hat{\gamma} |(\beta^{-1})'|_{L^\infty(\mathbb{R})}.$$

Because of the lower bound on β' given by Lemma 4.4, we get $|(\beta^{-1})'|_{L^\infty(\mathbb{R})} \leq 1/\delta$ which yields the compatibility condition (3.4) with $\gamma = \hat{\gamma}/\delta$ arbitrarily small. \square

We are now in position to prove Theorem 3.2 in the general case.

Proof of Theorem 3.2. Let us now assume that the Hamiltonians only satisfy (1.5). In this case, we simply approximate the Hamiltonians H_i by other Hamiltonians \tilde{H}_i satisfying (4.8) such that

$$|H_i - \tilde{H}_i| \leq \gamma.$$

We then apply Theorem 3.2 to the Hamiltonians \tilde{H}_i and construct an associated vertex test function \tilde{G} also for the parameter γ . We deduce that

$$H(y, -\tilde{G}_y) \leq H(x, \tilde{G}_x) + 3\gamma$$

with $\gamma > 0$ arbitrarily small, which shows again the compatibility condition on the Hamiltonians (3.4) for the Hamiltonians H_i 's. The proof is now complete in the general case. \square

Remark 4.6 (A variant in the proof of construction of G^0). When the Hamiltonians are not convex, it is also possible to use the function β from Lemma 4.4 in a different way by defining directly the function G^0 as follows

$$\tilde{G}^0(x, y) = \sup_{(p, \lambda) \in \mathcal{G}_A} (p_i x - p_j y - \beta(\lambda)).$$

4.3 A special function

In order to prove Proposition 4.1, we first need to study a special function \mathfrak{G} . Precisely, we define the following convex function for $z = (z_1, \dots, z_N) \in \mathbb{R}^N$,

$$\mathfrak{G}(z) = \sup_{(p, \lambda) \in \mathcal{G}_A} (p \cdot z - \lambda).$$

We then consider the following subsets of \mathbb{R}^N ,

$$Q_\sigma = \{z = (z_1, \dots, z_N) \in \mathbb{R}^N : \sigma_i z_i \geq 0, \quad i = 1, \dots, N\}$$

$$\Delta_\sigma = \{z = (z_1, \dots, z_N) \in Q_\sigma : \sum_{i=1}^N \frac{\sigma_i z_i}{\bar{z}_i^\sigma(A)} \leq 1\}$$

where $\bar{z}_i^\sigma(A) = \sigma_i H'_i(\pi_i^{\sigma_i}(A)) \geq 0$ and the functions π_i^\pm are defined in (4.2). We also make precise that we use the following convenient convention,

$$(4.12) \quad \frac{\bar{z}_i}{\bar{z}_i^\sigma(A)} = \begin{cases} 0 & \text{if } \bar{z}_i = 0, \\ +\infty & \text{if } \bar{z}_i > 0 \text{ and } \bar{z}_i^\sigma(A) = 0. \end{cases}$$

Lemma 4.7 (The function \mathfrak{G} in Q_σ). *Under the assumptions of Proposition 4.1, we have, for any $\sigma \in \{\pm\}^N$ with $\sigma \neq (+, \dots, +)$ if $N \geq 2$:*

- i) \mathfrak{G} is C^1 on Q_σ (up to the boundary).
- ii) For all $z \in Q_\sigma$, there exists a unique $\lambda = \mathfrak{L}(z) \geq A$ such that

$$\begin{aligned} \mathfrak{G}(z) &= p \cdot z - \lambda \\ \nabla \mathfrak{G}(z) &= p = (p_1, \dots, p_N) \\ p_i &= \pi_i^{\sigma_i}(\lambda) \end{aligned}$$

with $(p, \lambda) \in \mathcal{G}_A$.

- iii) For all $z \in Q_\sigma$, $\mathfrak{L}(z) = A$ if and only if $z \in \Delta_\sigma$. In particular, \mathfrak{G} is linear in Δ_σ .

Before giving global properties of \mathfrak{G} , we introduce the set

$$(4.13) \quad \bar{\Omega} = \begin{cases} \mathbb{R} & \text{if } N = 1, \\ \mathbb{R}^N \setminus (0, +\infty)^N & \text{if } N \geq 2. \end{cases}$$

Lemma 4.8 (Global properties of \mathfrak{G} and \mathfrak{L}). *Under the assumptions of Proposition 4.1, the function \mathfrak{G} is convex and finite in \mathbb{R}^N , reaches its minimum $-A$ at 0 and the function \mathfrak{L} is continuous in $\bar{\Omega}$.*

Proof of Lemmas 4.7 and 4.8. Let $\sigma \in \{\pm\}^N$ and $z \in Q_\sigma$. We set

$$\pi^\sigma(\lambda) = (\pi_1^{\sigma_1}(\lambda), \dots, \pi_N^{\sigma_N}(\lambda)).$$

Using the fact that $\pi^\sigma(A) \in \mathcal{G}_A$, we get $\mathfrak{G}(z) \geq \mathfrak{G}(0) = -A$.

Step 1: Explicit expression of \mathfrak{G} . For $\sigma \neq (+, \dots, +)$ if $N \geq 2$, we have

$$(4.14) \quad (p, \lambda) \in \mathcal{G}_A \cap (Q_\sigma \times \mathbb{R}) \iff \lambda \geq A \quad \text{and} \quad p = \pi^\sigma(\lambda).$$

This implies in particular that

$$(4.15) \quad \mathfrak{G}(z) = \sup_{\lambda \geq A} (z \cdot \pi^\sigma(\lambda) - \lambda).$$

Step 2: Optimization. Because of the superlinearity of the Hamiltonians H_i (see (4.1)), we have for $z \neq 0$,

$$\lim_{\lambda \rightarrow +\infty} f^\sigma(\lambda) = -\infty \quad \text{for} \quad f^\sigma(\lambda) := z \cdot \pi^\sigma(\lambda) - \lambda.$$

Therefore the supremum in (4.15) is reached for some $\lambda \in [A, +\infty)$, i.e.

$$\mathfrak{G}(z) = z \cdot \pi^\sigma(\lambda) - \lambda.$$

Then we have $\lambda = A$ or $\lambda > A$ and $(f^\sigma)'(\lambda) = 0$. Note that for $\lambda > A_0$, we can rewrite $(f^\sigma)'(\lambda) = 0$ as

$$\sum_{i=1, \dots, N} \frac{\bar{z}_i}{\bar{z}_i^\sigma} = 1 \quad \text{with} \quad \begin{cases} \bar{z}_i = \sigma_i z_i \geq 0, \\ \bar{z}_i^\sigma = \bar{z}_i^\sigma(\lambda) := \sigma_i H_i'(\pi_i^{\sigma_i}(\lambda)) > 0. \end{cases}$$

Moreover, we have

$$(\bar{z}_i^\sigma)'(\lambda) = \frac{H_i''(\pi_i^{\sigma_i}(\lambda))}{\sigma_i H_i'(\pi_i^{\sigma_i}(\lambda))} > 0$$

where the strict inequality follows from the strict convexity of Hamiltonians, see (4.1). Moreover, by definition of \bar{z}_i^σ , we have

$$\lim_{\lambda \rightarrow +\infty} \bar{z}_i^\sigma(\lambda) = +\infty$$

because H_i is convex and superlinear.

Step 3: Foliation and definition of \mathfrak{L} . Let us consider the sets

$$(4.16) \quad P^\sigma(\lambda) = \begin{cases} \left\{ \bar{z} \in [0, +\infty)^N, \quad \sum_{i=1, \dots, N} \frac{\bar{z}_i}{\bar{z}_i^\sigma(\lambda)} = 1 \right\} & \text{if } \lambda > A, \\ \left\{ \bar{z} \in [0, +\infty)^N, \quad \sum_{i=1, \dots, N} \frac{\bar{z}_i}{\bar{z}_i^\sigma(A)} \leq 1 \right\} & \text{if } \lambda = A \end{cases}$$

(keeping in mind convention (4.12)). Because for $\lambda > A$, the intersection points of the piece of hyperplane $P(\lambda)$ with each axis $\mathbb{R}e_i$ are $\bar{z}_i^0(\lambda)e_i$, we deduce that we can write the partition

$$[0, +\infty)^N = \bigcup_{\lambda \geq A} P^\sigma(\lambda)$$

where $P^\sigma(\lambda)$ gives a foliation by hyperplanes for $\lambda > A$. Then we can define for $z \in Q_\sigma$,

$$\mathfrak{L}^\sigma(z) = \{ \lambda \text{ such that } \bar{z} \in P^\sigma(\lambda) \text{ for } \bar{z}_i = \sigma_i z_i \text{ for } i = 1, \dots, N \}.$$

From our definition, we get that the function \mathfrak{L}^σ is continuous on Q_σ and satisfies $\mathfrak{L}^\sigma(0) = A$. For $z \in Q_\sigma$ such that $z_{i_0} = 0$, we see from the definition of P^σ given in (4.16) that the value of $\mathfrak{L}^\sigma(z)$ does not depend on the value of σ_{i_0} . Therefore we can glue up all the \mathfrak{L}^σ in a single continuous function \mathfrak{L} defined for $z \in \bar{\Omega}$ by

$$\mathfrak{L}(z) = \mathfrak{L}^\sigma(z) \quad \text{if } z \in Q_\sigma.$$

which satisfies $\mathfrak{L}(0) = A$.

Step 4: Regularity of \mathfrak{G} and computation of the gradients. For $z \in Q_\sigma \subset \bar{\Omega}$, we have

$$\mathfrak{G}(z) = \sup_{\lambda \geq A} (z \cdot \pi^\sigma(\lambda) - \lambda)$$

where the supremum is reached only for $\lambda = \mathfrak{L}(z)$. Moreover \mathfrak{G} is convex in \mathbb{R}^N . We just showed that the subdifferential of \mathfrak{G} on the interior of Q_σ is the singleton $\{\pi^\sigma(\lambda)\}$ with $\lambda = \mathfrak{L}(z)$. This implies that \mathfrak{G} is derivable on the interior of Q_σ and

$$\nabla \mathfrak{G}(z) = \pi^\sigma(\lambda) \quad \text{with } \lambda = \mathfrak{L}(z).$$

The fact that the maps π^σ and \mathfrak{L} are continuous implies that $\mathfrak{G}|_{Q_\sigma}$ is C^1 . □

4.4 Proof of Proposition 4.1

We now turn to the proof of Proposition 4.1.

Proof of Proposition 4.1. By definition of G^0 , we have

$$G^0(x, y) = \mathfrak{G}(Z(x, y)) \quad \text{with } Z(x, y) := xe_i - ye_j \in \bar{\Omega} \quad \text{if } (x, y) \in J_i \times J_j$$

where (e_1, \dots, e_N) is the canonical basis of \mathbb{R}^N and $\bar{\Omega}$ is defined in (4.13).

Step 1: Regularity. Then Lemmas 4.7 and 4.8 imply immediately that $G^0 \in C(J^2)$ and $G^0 \in C^1(R)$ for each region R given by

$$(4.17) \quad R = \begin{cases} J_i \times J_j & \text{if } i \neq j, \\ T_i^\pm = \{(x, y) \in J_i \times J_i, \pm(x - y) \geq 0\} & \text{if } i = j. \end{cases}$$

This regularity of \mathfrak{G} implies in particular the regularity of G^0 given in i).

Step 2: Computation of the gradients. We also deduce from Lemma 4.8 that

$$\Lambda(x, y) := \mathfrak{L}(Z(x, y))$$

defines a continuous map $\Lambda : J^2 \rightarrow [A, +\infty)$ which satisfies

$$(4.18) \quad \Lambda(x, x) = A$$

because of Lemma 4.7-iii) and $Z(x, x) = 0$. Moreover, for each R given by (4.17) and for all $(x, y) \in R \subset J_i \times J_j$ we have

$$G^0(x, y) = p_i x - p_j y - \lambda$$

and

$$(G^0|_R)_x(x, y) = p_i \quad \text{and} \quad (G^0|_R)_y(x, y) = -p_j$$

with $\lambda = \Lambda(x, y)$ and $(p, \lambda) \in \mathcal{G}_A$ and

$$(4.19) \quad (p_i, p_j) = \begin{cases} (\pi_i^+(\lambda), \pi_j^-(\lambda)) & \text{if } R = J_i \times J_j \quad \text{with } i \neq j, \\ (\pi_i^\pm(\lambda), \pi_i^\pm(\lambda)) & \text{if } R = T_i^\pm \quad \text{with } i = j. \end{cases}$$

Step 3: Checking the compatibility condition on the gradients. Let us consider $(x, y) \in J^2$ with $x = y = 0$ or $x \neq y$. We have

$$(\partial_i G^0(\cdot, y))(x) \in \{\pi_i^\pm(\lambda)\} \quad \text{and} \quad -(\partial_j G^0(x, \cdot))(y) \in \{\pi_j^\pm(\lambda)\} \quad \text{with } \lambda = \Lambda(x, y) \geq A.$$

We claim that

$$(4.20) \quad H(x, G_x^0(x, y)) = \lambda.$$

It is clear except in the special case where

$$(4.21) \quad x = 0 \quad \text{and} \quad (\partial_i G^0(\cdot, y))(0) = \pi_i^+(\lambda) \quad \text{for all } i = 1, \dots, N$$

If $0 \neq y \in J_j$, then $(x, y) = (0, y) \in T_j^-$ and $(\partial_j G^0(\cdot, y))(0) = \pi_j^-(\lambda)$. Therefore (4.21) only happens if $y = 0$ and then

$$H(0, G_x^0(0, 0)) = A$$

which still implies (4.20), because $\lambda = \Lambda(0, 0) = A$.

In view of (4.20), (3.4) with equality and $\gamma = 0$ is equivalent to

$$(4.22) \quad H(y, -G_y^0(x, y)) = \lambda.$$

This is clear except possibly in the special case where

$$(4.23) \quad y = 0 \quad \text{and} \quad -(\partial_j G^0(x, \cdot))(0) = \pi_j^\pm(\lambda) \quad \text{for all } j = 1, \dots, N.$$

If $x \in J_i$ and $N \geq 2$, then we can find $j \neq i$ such that $-(\partial_j G^0(x, \cdot))(0) = \pi_j^-(\lambda)$. Therefore (4.23) only happens if $N = 1$ and then

$$H(0, -G_y^0(x, 0)) = A \leq \lambda.$$

Step 4: Superlinearity. In view of the definition of G^0 , we deduce from (4.19) that

$$G^0(x, y) \geq \begin{cases} x\pi_i^+(\lambda) - y\pi_j^-(\lambda) - \lambda & \text{if } i \neq j, \\ (x - y)\pi_i^\pm(\lambda) - \lambda & \text{if } i = j \quad \text{and} \quad \pm(x - y) \geq 0 \end{cases}$$

Setting

$$\pi^0(\lambda) := \min_{\pm, i=1, \dots, N} \pm \pi_i^\pm(\lambda) \geq 0,$$

we get

$$G^0(x, y) \geq d(x, y)\pi^0(\lambda) - \lambda.$$

From the definition (4.2) of π_i^\pm and the assumption (4.1) on the Hamiltonians, we deduce that

$$\pi^0(\lambda) \rightarrow +\infty \quad \text{as } \lambda \rightarrow +\infty$$

which implies that for any $K \geq 0$, there exists a constant $C_K \geq 0$ such that

$$G^0(x, y) \geq Kd(x, y) - C_K.$$

Therefore we get (3.5) with

$$g^0(a) = \sup_{K \geq 0} (Ka - C_K).$$

Step 5: Gradient bounds. Note that

$$\sum_{i=1, \dots, N} |Z_i(x, y)| = d(x, y).$$

Because each component of the gradients of G^0 are equal to one of the $\{\pi_k^\pm(\lambda)\}_{\pm, k=1, \dots, N}$ with $\lambda = \mathfrak{L}(Z(x, y))$, we deduce (3.6) from the continuity of \mathfrak{L} and of the maps π_k^\pm .

Step 6: Saturation close to the diagonal. Point vi) in Proposition 4.1 follows from Lemma 4.7-iii), from the definition of \mathfrak{G} and from the regularity of G^0 . \square

5 Networks

5.1 Definition of a network

A general abstract network \mathcal{N} is characterized by the set (\mathcal{E} of its *edges* and the set \mathcal{V}) of its “nodes”. It is endowed with a distance.

Edges. \mathcal{E} is a finite or countable set of edges. Each edge $e \in \mathcal{E}$ is assumed to be either isometric to the half line $[0, +\infty)$ with $\partial e = \{e^0\}$ (where the endpoint e^0 can be identified to $\{0\}$), or to a compact interval $[0, l_e]$ with

$$(5.1) \quad \inf_{e \in \mathcal{E}} l_e > 0$$

and $\partial e = \{e^0, e^1\}$. Condition (5.1) implies in particular that the network is complete. The endpoints $\{e^0\}, \{e^1\}$ can respectively be identified to $\{0\}$ and $\{l_e\}$. The *interior* e^* of an edge e refers to $e \setminus (\partial e)$.

Vertices (or nodes). It is convenient to see vertices of the network as a partition of the sets of all edge endpoints,

$$\bigcup_{e \in \mathcal{E}} \partial e = \bigcup_{n \in \mathcal{V}} n;$$

we assume that each set n only contains a finite number of endpoints.

Here each $n \in \mathcal{V}$ can be identified as a vertex (or node) of the network as follows. For every $x, y \in \bigcup_{e \in \mathcal{E}} e$, we define the equivalence relation:

$$x \sim y \iff (x = y \text{ or } x, y \in n \in \mathcal{V})$$

and we define the network as the quotient

$$(5.2) \quad \mathcal{N} = \left(\bigcup_{e \in \mathcal{E}} e \right) / \sim = \left(\bigcup_{e \in \mathcal{E}} e^* \right) \cup \mathcal{V}.$$

We also define for $n \in \mathcal{V}$

$$\mathcal{E}_n = \{e \in \mathcal{E}, \quad n \in \partial e\}$$

and its partition $\mathcal{E}_n = \mathcal{E}_n^- \cup \mathcal{E}_n^+$ with

$$\mathcal{E}_n^- = \{e \in \mathcal{E}_n, n = e^0\}, \quad \mathcal{E}_n^+ = \{e \in \mathcal{E}_n, n = e^1\}.$$

Distance. We also define the distance function $d(x, y) = d(y, x)$ as the minimal length of a continuous path connecting x and y on the network, using the metric of each edge (either isometric to $[0, +\infty)$ or to a compact interval). Note that, because of our assumptions, if $d(x, y) < +\infty$, then there is only a finite number of minimal paths.

Remark 5.1. For any $\varepsilon > 0$, there is a bound (depending on ε) on the number of minimal paths connecting x to y for all $y \in B(\bar{y}, \varepsilon) = \{y \in \mathcal{N}, \quad d(\bar{y}, y) < \varepsilon\}$.

5.2 Hamilton-Jacobi equations on a network

Given a Hamiltonian H_e on each edge $e \in \mathcal{E}$, we consider the following HJ equation on the network \mathcal{N} ,

$$(5.3) \quad \begin{cases} u_t + H_e(t, x, u_x) = 0 & \text{for } t \in (0, +\infty) & \text{and } x \in e^*, \\ u_t + F_A(t, x, u_x) = 0 & \text{for } t \in (0, +\infty) & \text{and } x = n \in \mathcal{V} \end{cases}$$

submitted to an initial condition

$$(5.4) \quad u(0, x) = u^0(x) \quad \text{for } x \in \mathcal{N}.$$

The limited flux functions F_A associated with the Hamiltonians H_e are defined below. We first make precise the meaning of u_x in (5.3).

Gradients of real functions. For a real function u defined on the network \mathcal{N} , we denote by $\partial_e u(x)$ the (spatial) derivative of u at $x \in e$ and define the “gradient” of u by

$$u_x(x) := \begin{cases} \partial_e u(x) & \text{if } x \in e^* = e \setminus (\partial e), \\ ((\partial_e u(x))_{e \in \mathcal{E}_n^-}, (\partial_e u(x))_{e \in \mathcal{E}_n^+}) & \text{if } x = n \in \mathcal{V} \end{cases}.$$

The norm $|u_x|$ simply denotes $|\partial_e u|$ for $x \in e^*$ or $\max\{|\partial_e u| : e \in \mathcal{E}_n\}$ at the vertex $x = n$.

Limited flux functions. We also define for $(t, x) \in \mathbb{R} \times \partial e$,

$$H_e^-(t, x, q) = \begin{cases} H_e(t, x, q) & \text{if } q \leq p_e^0(t, x), \\ H_e(t, x, p_e^0(t, x)) & \text{if } q > p_e^0(t, x) \end{cases}$$

and

$$H_e^+(t, x, q) = \begin{cases} H_e(t, x, p_e^0(t, x)) & \text{if } q \leq p_e^0(t, x), \\ H_e(t, x, q) & \text{if } q > p_e^0(t, x). \end{cases}$$

Given limiting functions $(A_n)_{n \in \mathcal{V}}$, we define for $p = (p_e)_{e \in \mathcal{E}_n}$,

$$F_A(t, n, p) = \max \left(A_n(t), \max_{e \in \mathcal{E}_n^-} H_e^-(t, n, p_e), \max_{e \in \mathcal{E}_n^+} H_e^+(t, n, -p_e) \right).$$

In particular, for each $n \in \mathcal{V}$, the functions $F_A(t, n, \cdot)$ are the same for all $A_n(t) \in [-\infty, A_n^0(t)]$ with

$$(5.5) \quad A_n^0(t) := \max \left(\max_{e \in \mathcal{E}_n^-} H_e^-(t, n, p_e^0(t, n)), \max_{e \in \mathcal{E}_n^+} H_e^+(t, n, p_e^0(t, n)) \right).$$

A shorthand notation. As in the junction case, we introduce

$$(5.6) \quad H_{\mathcal{N}}(t, x, p) = \begin{cases} H_e(t, x, p) & \text{for } p \in \mathbb{R}, & t \in \mathbb{R}, & \text{if } x \in e^*, \\ F_A(t, x, p) & \text{for } p = (p_e)_{e \in \mathcal{E}_n} \in \mathbb{R}^{\text{Card } \mathcal{E}_n}, & t \in \mathbb{R}, & \text{if } x = n \in \mathcal{V} \end{cases}$$

in order to rewrite (5.3) as

$$(5.7) \quad u_t + H_{\mathcal{N}}(t, x, u_x) = 0 \quad \text{for all } (t, x) \in (0, +\infty) \times \mathcal{N}.$$

5.3 Assumptions on the Hamiltonians

For each $e \in \mathcal{E}$, we consider a Hamiltonian $H_e : [0, +\infty) \times e \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

- **(H0)** (Continuity) $H_e \in C([0, +\infty) \times e \times \mathbb{R})$.
- **(H1)** (Uniform coercivity) For all $T > 0$,

$$\lim_{|q| \rightarrow +\infty} H_e(t, x, q) = +\infty$$

uniformly with respect to $t \in [0, T]$ and $x \in e \subset \mathcal{N}$ and $e \subset \mathcal{N}$.

- **(H2)** (Uniform bound on the Hamiltonians for bounded gradients) For all $T, L > 0$, there exists $C_{T,L} > 0$ such that

$$\sup_{t \in [0, T], p \in [-L, L], x \in \mathcal{N} \setminus \mathcal{V}} |H_{\mathcal{N}}(t, x, p)| \leq C_{T,L}.$$

- **(H3)** (Uniform modulus of continuity for bounded gradients) For all $T, L > 0$, there exists a modulus of continuity $\omega_{T,L}$ such that for all $|p|, |q| \leq L$, $t \in [0, T]$ and $x \in e \in \mathcal{E}$,

$$|H_e(t, x, p) - H_e(t, x, q)| \leq \omega_{T,L}(|p - q|).$$

- **(H4)** (Bi-monotonicity) For all $n \in \mathcal{V}$, there exists a continuous function $t \mapsto p_e^0(t, n)$ such that

$$\begin{cases} H_e(t, n, \cdot) & \text{is nonincreasing on } (-\infty, p_e^0(t, n)], \\ H_e(t, n, \cdot) & \text{is nondecreasing on } [p_e^0(t, n), +\infty). \end{cases}$$

- **(H5)** (Uniform modulus of continuity in time) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0, T]$, $p \in \mathbb{R}$, $x \in e \in \mathcal{E}$,

$$H_e(t, x, p) - H_e(s, x, p) \leq \bar{\omega}_T(|t - s|(1 + \max(H_e(s, x, p), 0))).$$

- **(H6)** (Uniform continuity of A^0) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0, T]$ and $n \in \mathcal{V}$,

$$|A_n^0(t) - A_n^0(s)| \leq \bar{\omega}_T(|t - s|).$$

As far as flux limiters are concerned, the following assumptions will be used.

- **(A0)** (Continuity of A) For all $T > 0$ and $n \in \mathcal{V}$, $A_n \in C([0, T])$.
- **(A1)** (Uniform bound on A) For all $T > 0$, there exists a constant $C_T > 0$ such that for all $t \in [0, T]$ and $n \in \mathcal{V}$

$$|A_n(t)| \leq C_T.$$

- **(A2)** (Uniform continuity of A) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0, T]$ and $n \in \mathcal{V}$,

$$|A_n(t) - A_n(s)| \leq \bar{\omega}_T(|t - s|).$$

The proof of the following technical lemma is postponed until appendix.

Lemma 5.2 (Estimate on the difference of Hamiltonians). *Assume that the Hamiltonians satisfy (H0)-(H4) and (A0)-(A1). Then for all $T > 0$, there exists a constant $C_T > 0$ such that*

$$(5.8) \quad |p_e^0(t, x)| \leq C_T \quad \text{for all } t \in [0, T], \quad x \in \partial e, \quad e \in \mathcal{E},$$

$$(5.9) \quad |A_n^0(t)| \leq C_T \quad \text{for all } t \in [0, T], \quad n \in \mathcal{V}.$$

If we assume moreover (H5)-(H6) and (A2), then there exists a modulus of continuity $\tilde{\omega}_T$ such that for all $t, s \in [0, T]$, and x, p

$$(5.10) \quad H_{\mathcal{N}}(t, x, p) - H_{\mathcal{N}}(s, x, p) \leq \tilde{\omega}_T(|t - s|(1 + \max(0, H_{\mathcal{N}}(s, x, p)))).$$

Remark 5.3. The reader can check that Assumptions (H5)-(H6) and (A2) in the statement of Theorem 5.8 can in fact be replaced with (5.10).

Remark 5.4 (Example of Hamiltonians with uniform modulus of time continuity). Condition on the uniform modulus of continuity in time in (H5)-(H6) is for instance satisfied by Hamiltonians of the type for $q > 0$ and $\delta > 0$ such that for all $x \in e \in \mathcal{E}$ we have

$$H_e(t, x, p) = c_e(t, x)|p|^q \quad \text{with} \quad 0 < \delta \leq c_e(t, x) \leq 1/\delta$$

with c_e uniformly continuous in time and continuous in space.

5.4 Viscosity solutions on a network

Class of test functions. For $T > 0$, set $\mathcal{N}_T = (0, T) \times \mathcal{N}$. We define the class of test functions on $(0, T) \times \mathcal{N}$ by

$$C^1(\mathcal{N}_T) = \{\varphi \in C(\mathcal{N}_T), \text{ the restriction of } \varphi \text{ to } (0, T) \times e \text{ is } C^1, \text{ for all } e \in \mathcal{E}\}.$$

Definition 5.5 (Viscosity solutions). Assume the Hamiltonians satisfy (H0)-(H4) and (A0)-(A1) and let $u : [0, T) \times \mathcal{N} \rightarrow \mathbb{R}$.

- We say that u is a *sub-solution* (resp. *super-solution*) of (1.3) in $(0, T) \times \mathcal{N}$ if for all test function $\varphi \in C^1(\mathcal{N}_T)$ such that

$$u^* \leq \varphi \quad (\text{resp.} \quad u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in \mathcal{N}_T$$

with equality at (t_0, x_0) , we have

$$\varphi_t + H_{\mathcal{N}}(t, x, \varphi_x) \leq 0 \quad (\text{resp.} \quad \geq 0) \quad \text{at } (t_0, x_0).$$

ii) We say that u is a *sub-solution* (resp. *super-solution*) of (1.3), (1.4) in $[0, T) \times \mathcal{N}$ if additionally

$$u^*(0, x) \leq u_0(x) \quad (\text{resp.} \quad u_*(0, x) \geq u_0(x)) \quad \text{for all } x \in \mathcal{N}.$$

iii) We say that u is a (*viscosity*) *solution* if u is both a sub-solution and a super-solution.

Remark 5.6 (Touching sub-solutions with semi-concave functions). When proving the comparison principle in the network setting, sub-solutions (resp. super-solutions) will be touched from above (resp. from below) by functions that will not be C^1 , but only semi-concave (resp. semi-convex). We recall that a function is semi-concave if it is the sum of a concave function and a smooth (C^2 say) function. But it is a classical observation that, at a point where a semi-concave function is not C^1 , we can replace the semi-concave function by a C^1 test function touching it from above.

As in the case of a junction (see Proposition 2.2), viscosity solutions are stable through supremum/infimum. We also have the following existence result.

Theorem 5.7 (Existence on a network). *Assume (H0)-(H4) and (A0)-(A1) on the Hamiltonians and assume that the initial data u^0 is uniformly continuous on \mathcal{N} . Let $T > 0$. Then there exists a viscosity solution u of (5.7),(5.4) on $[0, T) \times \mathcal{N}$ and a constant $C_T > 0$ such that*

$$|u(t, x) - u^0(x)| \leq C_T \quad \text{for all } (t, x) \in [0, T) \times \mathcal{N}.$$

Proof. The proof follows along the lines of the ones of Theorem 1.1. The main difference lies in the construction of barriers. We proceed similarly and get a regularized initial data u_ε^0 satisfying

$$|u_\varepsilon^0 - u^0| \leq \varepsilon \quad \text{and} \quad |(u_\varepsilon^0)_x| \leq L_\varepsilon.$$

Then the functions

$$(5.11) \quad u_\varepsilon^\pm(t, x) = u_\varepsilon^0(x) \pm C_\varepsilon t \pm \varepsilon$$

are global super and sub-solutions with respect to the initial data u^0 if C_ε is chosen as follows,

$$(5.12) \quad C_\varepsilon = \max \left(\sup_{t \in [0, T]} \sup_{n \in \mathcal{V}} |\max(A_n(t), A_n^0(t))|, \sup_{t \in [0, T]} \sup_{e \in \mathcal{E}} \sup_{x \in e, |p_e| \leq L_\varepsilon} |H_e(t, x, p_e)| \right);$$

indeed, we use (5.9) in Lemma 5.2 to bound the first terms in (5.12). \square

5.5 Comparison principle on a network

Theorem 5.8 (Comparison principle on a network). *Assume the Hamiltonians satisfy (H0)-(H6) and (A0)-(A2) and assume that the initial data u^0 is uniformly continuous*

on \mathcal{N} . Let $T > 0$. Then for all sub-solution u and super-solution w of (5.7), (5.4) in $[0, T) \times \mathcal{N}$, satisfying for some $C_T > 0$ and some $x_0 \in \mathcal{N}$

$$(5.13) \quad u(t, x) \leq C_T(1 + d(x_0, x)), \quad w(t, x) \geq -C_T(1 + d(x_0, x)), \quad \text{for all } (t, x) \in [0, T) \times \mathcal{N},$$

we have

$$u \leq w \quad \text{on } [0, T) \times \mathcal{N}.$$

As a straightforward corollary of Theorems 5.8 and 5.7, we get

Corollary 5.9 (Existence and uniqueness). *Under the assumptions of Theorem 5.8, there exists a unique viscosity solution u of (5.7), (5.4) in $[0, T) \times \mathcal{N}$ such that there exists a constant $C > 0$ with*

$$|u(t, x) - u^0(x)| \leq C \quad \text{for all } (t, x) \in [0, T) \times \mathcal{N}.$$

In order to prove Theorem 5.8, we first need two technical lemmas that are proved in appendix.

Lemma 5.10 (A priori control – the network case). *Let $T > 0$ and let u be a sub-solution and w be a super-solution as in Theorem 5.8. Then there exists a constant $C = C(T) > 0$ such that for all $(t, x), (s, y) \in [0, T) \times \mathcal{N}$, we have*

$$(5.14) \quad u(t, x) \leq w(s, y) + C(1 + d(x, y)).$$

Lemma 5.11 (Uniform control by the initial data). *Under the assumptions of Theorem 5.8, for any $T > 0$ and $C_T > 0$, there exists a modulus of continuity $f : [0, T) \rightarrow [0, +\infty]$ satisfying $f(0^+) = 0$ such that for all sub-solution u (resp. super-solution w) of (5.7), (5.4) on $[0, T) \times \mathcal{N}$, satisfying (5.13) for some $x_0 \in \mathcal{N}$, we have for all $(t, x) \in [0, T) \times \mathcal{N}$,*

$$(5.15) \quad u(t, x) \leq u_0(x) + f(t) \quad (\text{resp. } w(t, x) \geq u_0(x) - f(t)).$$

We can now turn to the proof of Theorem 5.8. The proof is similar the comparison principle on a junction (Theorem 1.1). Still, a space localization procedure has to be performed in order to “reduce” to the junction case. From a technical point of view, a noticeable difference is that we will fix the time penalization (for some parameter ν small enough), and then will first take the limit $\varepsilon \rightarrow 0$ (ε being the parameter for the space penalization), and then take the limit $\alpha \rightarrow 0$ (α being the penalization parameter to keep the optimization points at a finite distance).

Proof of Theorem 5.8. Let $\eta > 0$ and $\theta > 0$ and consider

$$M(\theta) = \sup \left\{ u(t, x) - w(s, x) - \frac{\eta}{T-t}, \quad x \in \mathcal{N}, \quad t, s \in [0, T), \quad |t-s| \leq \theta \right\}.$$

We want to prove that

$$M = \lim_{\theta \rightarrow 0} M(\theta) \leq 0.$$

Assume by contradiction that $M > 0$. From Lemma 5.10 we know that M is finite.

Step 1: The localization procedure. Let ψ denote $\frac{d^2(x_0, \cdot)}{2}$.

Lemma 5.12 (Localization). *The supremum*

$$M_\alpha = \sup_{\substack{t, s \in [0, T], t < T \\ x \in \mathcal{N}}} \left\{ u(t, x) - w(s, x) - \alpha\psi(x) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} \right\}$$

is reached for some point $(t_\alpha, s_\alpha, x_\alpha)$. Moreover, for α and ν small enough, we have the following localization estimates

$$(5.16) \quad M_\alpha \geq 3M/4 > 0$$

$$(5.17) \quad d(x_0, x_\alpha) \leq \frac{C}{\sqrt{\alpha}}$$

$$(5.18) \quad 0 < \tau_\nu \leq t_\alpha, s_\alpha \leq T - \frac{\eta}{2C}$$

$$(5.19) \quad \lim_{\nu \rightarrow 0} \left(\limsup_{\alpha \rightarrow 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu} \right) = 0$$

where C is a constant which does not depend on α , ε , ν and η .

Proof of Lemma 5.12. Choosing α small enough, we have (5.16) for all $\nu > 0$. Because the network is complete for its metric, the supremum in the definition of M_α is reached at some point $(t_\alpha, s_\alpha, x_\alpha)$. From Lemma 5.10, we deduce that

$$0 < \frac{3M}{4} \leq M_\alpha \leq C - \alpha\psi(x_\alpha) - \frac{\eta}{T-t_\alpha} - \frac{(t_\alpha - s_\alpha)^2}{2\nu}$$

and then

$$(5.20) \quad \alpha\psi(x_\alpha) + \frac{\eta}{T-t_\alpha} + \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq C.$$

This implies (5.17) changing C if necessary.

On the one hand, we get from (5.20) the second inequality in (5.18) by choosing ν such that $\sqrt{2\nu C} \leq \eta/2C$. On the other hand, we get from Lemma 5.11

$$0 < M_\alpha \leq f(t_\alpha) + f(s_\alpha) - \frac{\eta}{T}.$$

In particular,

$$\frac{\eta}{T} \leq 2f(\tau + \sqrt{2\nu C})$$

where $\tau = \min(t_\alpha, s_\alpha)$. If both τ and ν are too small, we get a contradiction. Hence the first inequality in (5.18) holds for some constant τ_ν depending on ν but not on α , ε and η .

We now turn to the proof of (5.19). We know that for any $\delta > 0$, there exists $\theta(\delta) > 0$ (with $\theta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$) and $(t^\delta, s^\delta, x^\delta) \in [0, T] \times [0, T] \times \mathcal{N}$ such that

$$u(t^\delta, x^\delta) - w(s^\delta, x^\delta) - \frac{\eta}{T-t^\delta} \geq M - \delta \quad \text{and} \quad |t^\delta - s^\delta| \leq \theta(\delta).$$

Then from (5.20) we deduce that

$$M(\sqrt{2\nu C}) - \frac{(t_\alpha - s_\alpha)^2}{2\nu} \geq M_\alpha \geq M - \delta - \alpha\psi(x^\delta) - \frac{|\theta(\delta)|^2}{2\nu}$$

and then

$$\limsup_{\alpha \rightarrow 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq M(\sqrt{2\nu C}) - M + \delta + \frac{|\theta(\delta)|^2}{2\nu}.$$

Taking the limit $\delta \rightarrow 0$, we get

$$\limsup_{\alpha \rightarrow 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq M(\sqrt{2\nu C}) - M$$

which yields the desired result. \square

Step 2: Reduction when x_α is a vertex. We adapt here Lemma 3.1.

Lemma 5.13 (Reduction). *Assume that $x_\alpha = n \in \mathcal{V}$. Without loss of generality, we can assume that $\mathcal{E}_n^+ = \emptyset$ and $p_e^0(t_\alpha, x_\alpha) = 0$ for each $e \in \mathcal{E}_n$ with $n = x_\alpha$.*

Proof of Lemma 5.13. The orientation of the edges $e \in \mathcal{E}_n$ can be changed in order to reduce to the case $\mathcal{E}_n^+ = \emptyset$. In particular, for $p = (p_e)_{e \in \mathcal{E}_n}$,

$$F_A(t, n, p) = \max \left(A_n(t), \max_{e \in \mathcal{E}_n^-} H_e^-(t, n, p_e) \right).$$

We can then argue as in Lemma 3.1. This means that we redefine the Hamiltonians (and the flux limiter A_n) only locally for $e \in \mathcal{E}_n$. Therefore we can assume that Using (5.8), we can check that the new Hamiltonians (locally for $e \in \mathcal{E}_n$) and A_n still satisfy (H0)-(H6) and (A0)-(A2) (with the same modulus of continuity, and with some different controlled constants $C_{T,L}$). We also have (5.13) with some controlled different constants. \square

Step 3: The penalization procedure. We now consider for $\varepsilon > 0$ and $\gamma \in (0, 1)$

$$M_{\alpha, \varepsilon} = \sup_{\substack{(t, x), (s, y) \in [0, T] \times B(x_\alpha, r) \\ t < T}} \left\{ u(t, x) - w(s, y) - \alpha\psi(x) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} - G_\varepsilon^{\alpha, \gamma}(x, y) - \varphi^\alpha(t, s, x) \right\}$$

where the function φ^α

$$\varphi^\alpha(t, s, x) = \frac{1}{2} (|t - t_\alpha|^2 + |s - s_\alpha|^2 + d^2(x, x_\alpha))$$

will help us to localize the problem around $(t_\alpha, s_\alpha, x_\alpha)$, and $B(x_\alpha, r)$ is the open ball of radius $r = r(\alpha) > 0$ centered at x_α ; besides, we choose $r \in (0, 1)$ small enough such that

$B(x_\alpha, r) \subset e$ if $x_\alpha \in e \setminus \mathcal{V}$. Lemma A.2 ensures that ψ and φ^α are semi-concave and therefore can be used as test functions, see Remark 5.6.

We choose

$$G_\varepsilon^{\alpha, \gamma}(x, y) = \varepsilon G^{\alpha, \gamma}(\varepsilon^{-1}x, \varepsilon^{-1}y)$$

with

$$G^{\alpha, \gamma}(x, y) = \begin{cases} \frac{(x-y)^2}{2} & \text{if } x_\alpha \in \mathcal{N} \setminus \mathcal{V}, \\ G^{x_\alpha, \gamma}(x, y) & \text{if } x_\alpha \in \mathcal{V}, \end{cases}$$

where $G^{x_\alpha, \gamma} \geq 0$ is the vertex test function of parameter $\gamma > 0$ given by Theorem 3.2, built on the junction problem associated to the vertex x_α at time t_α , *i.e.* associated to junction problem for the Hamiltonian $H_\mathcal{V}^{t_\alpha, x_\alpha}$ given by

$$(5.21) \quad H_\mathcal{V}^{t_\alpha, n}(x, p) := \begin{cases} H_e(t_\alpha, n, p) & \text{if } x \in e \setminus \{n\} \quad \text{with } e \in \mathcal{E}_n, \\ F_A(t_\alpha, n, p) & \text{if } x = n. \end{cases}$$

The supremum in the definition of $M_{\alpha, \varepsilon}$ is reached at some point $(t, x), (s, y) \in [0, T] \times \overline{B(x_\alpha, r)}$ with $t < T$. These maximizers satisfy the following penalization estimates.

Lemma 5.14 (Penalization). *For $\varepsilon \in (0, 1)$ and $\gamma \in (0, M/4)$, we have*

$$(5.22) \quad M_{\alpha, \varepsilon} \geq M_\alpha - \varepsilon\gamma \geq M/2 > 0$$

$$(5.23) \quad \begin{aligned} d(x, y) &\leq \omega(\varepsilon) \\ 0 &< \tau_\nu \leq s, t \leq T - \sigma_\eta \end{aligned}$$

for some modulus of continuity ω (depending on α and γ) and τ_ν and σ_η not depending on (ε, γ) . Moreover,

$$(t, s, x, y) \rightarrow (t_\alpha, s_\alpha, x_\alpha, x_\alpha) \quad \text{as } (\varepsilon, \gamma) \rightarrow (0, 0).$$

In particular, we have $x, y \in B(x_\alpha, r)$ for $\varepsilon, \gamma > 0$ small enough.

Proof of Lemma 5.14. For all $\varepsilon, \nu > 0$, the compatibility on the diagonal (3.3) of the vertex test function $G^{x_\alpha, \gamma}$ yields the first inequality in (5.22). Then for $\varepsilon \in (0, 1]$, with a choice of γ such that $0 < \gamma < M/4$, we have the second one.

Bound on $d(x, y)$. Remark that

$$\varepsilon g\left(\frac{d(x, y)}{\varepsilon}\right) \leq G_\varepsilon^{x_\alpha, \gamma}(x, y)$$

where

$$g(a) = \begin{cases} \frac{a^2}{2} & \text{if } x_\alpha \in \mathcal{N} \setminus \mathcal{V}, \\ g^{x_\alpha, \gamma}(a) & \text{if } x_\alpha \in \mathcal{V}, \end{cases}$$

and where $g^{x_\alpha, \gamma}$ is the superlinear function associated to $G^{x_\alpha, \gamma}$ and given by Theorem 3.2. Thanks to Lemma 5.10, we deduce that

$$(5.24) \quad \begin{aligned} 0 < M/2 &\leq C(1 + d(x, y)) - G_\varepsilon^{\alpha, \gamma}(x, y) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha\psi(x) \\ &\leq C(1 + d(x, y)) - \varepsilon g\left(\frac{d(x, y)}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha\psi(x) \end{aligned}$$

which implies in particular that

$$\varepsilon g\left(\frac{d(x, y)}{\varepsilon}\right) \leq C(1 + d(x, y)).$$

This gives (5.23) as in Step 1 of the proof of Theorem 1.1.

First time estimate. From (5.24) with $G_\varepsilon^{\alpha, \gamma} \geq 0$ and (5.23), we deduce in particular that for $\varepsilon \in (0, 1]$

$$0 < M/2 \leq C' - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t}.$$

This implies in particular that

$$(5.25) \quad T - t \geq \frac{\eta}{C'}, \quad T - s \geq \frac{\eta}{C'} - \sqrt{2\nu C'} \geq \frac{\eta}{2C'} =: \sigma_\eta > 0$$

for $\nu > 0$ small enough, and up to redefine σ_η for the new constant $C' \geq C$.

Second time estimate. From Lemma 5.11, we have with

$$\begin{aligned} 0 < M/2 &\leq f(t) + f(s) + u^0(x) - u^0(y) - \frac{\eta}{T} - \frac{(t-s)^2}{2\nu} \\ &\leq f(t) + f(s) + \omega_0 \circ \omega(\varepsilon) - \frac{\eta}{T} - \frac{(t-s)^2}{2\nu} \end{aligned}$$

where ω_0 is the modulus of continuity of u^0 . Let us choose $\varepsilon > 0$ small enough such that

$$(5.26) \quad \omega_0 \circ \omega(\varepsilon) \leq \frac{M}{2}.$$

As in the proof of Lemma 5.12, for $\tau = \min(t, s)$, we get

$$\frac{\eta}{T} \leq 2f(\tau + \sqrt{2\nu C'}).$$

For ν small enough (with η fixed), we then get a contradiction if τ does not converge to 0 as ν does.

Convergence of maximizers. Because of (5.22) and using the fact that $G_\varepsilon^{\alpha,\gamma} \geq 0$, we get for $\varepsilon \in (0, 1]$

$$M_\alpha - \gamma \leq M_{\alpha,\varepsilon} \leq u(t, x) - w(s, y) - \alpha\psi(x) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} - \varphi^\alpha(t, s, x).$$

Extracting a subsequence if needed, we can assume

$$(t, x, s, y) \rightarrow (\bar{t}, \bar{x}, \bar{s}, \bar{y}) \quad \text{as } (\varepsilon, \gamma) \rightarrow (0, 0)$$

for some $\bar{t}, \bar{s} \in [\tau_\nu, T - \sigma_\eta]$, $\bar{x} \in \overline{B(x_\alpha, r)}$. We get

$$M_\alpha \leq u(\bar{t}, \bar{x}) - w(\bar{s}, \bar{y}) - \alpha\psi(\bar{x}) - \frac{\eta}{T-\bar{t}} - \frac{(\bar{t}-\bar{s})^2}{2\nu} - \varphi^\alpha(\bar{t}, \bar{s}, \bar{x}) \leq M_\alpha - \varphi^\alpha(\bar{t}, \bar{s}, \bar{x})$$

which implies that $(\bar{t}, \bar{s}, \bar{x}) = (t_\alpha, s_\alpha, x_\alpha)$. \square

Step 4: Viscosity inequalities. Then we can write the viscosity inequalities at (t, x) and (s, y) using the shorthand notation (5.6),

$$(5.27) \quad \begin{aligned} \frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + (t-t_\alpha) + H_{\mathcal{N}}(t, x, p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t, s, x)) &\leq 0 \\ \frac{t-s}{\nu} - (s-s_\alpha) + H_{\mathcal{N}}(s, y, p_y^{\alpha,\gamma,\varepsilon}) &\geq 0 \end{aligned}$$

where

$$\begin{cases} p_x^{\alpha,\gamma,\varepsilon} = G_x^{\alpha,\gamma}(\varepsilon^{-1}x, \varepsilon^{-1}y), \\ p_y^{\alpha,\gamma,\varepsilon} = -G_y^{\alpha,\gamma}(\varepsilon^{-1}x, \varepsilon^{-1}y). \end{cases}$$

We choose ε, γ small enough such that (Lemma 5.14) we have

$$|t-t_\alpha|, \quad |s-s_\alpha| \leq \frac{\eta}{4T^2}.$$

Subtracting the two viscosity inequalities, we get

$$(5.28) \quad \frac{\eta}{2T^2} \leq H_{\mathcal{N}}(s, y, p_y^{\alpha,\gamma,\varepsilon}) - H_{\mathcal{N}}(t, x, p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t, s, x)).$$

Step 5: Gradient estimates. We deduce from (5.27) that

$$\tilde{p}_x^{\alpha,\gamma,\varepsilon} = p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t, s, x)$$

satisfies

$$(5.29) \quad H_{\mathcal{N}}(t, x, \tilde{p}_x^{\alpha,\gamma,\varepsilon}) \leq \frac{s-t}{\nu} + t_\alpha - t \leq \frac{T}{\nu} + T.$$

Hence (H1) implies that there exists a constant C'_ν (independent of $\alpha, \varepsilon, \gamma$, but depending on η, ν) such that

$$\begin{cases} |\tilde{p}_x^{\alpha, \gamma, \varepsilon}| \leq C'_\nu & \text{if } x \neq x_\alpha \text{ or } x_\alpha \notin \mathcal{V}, \\ \tilde{p}_x^{\alpha, \gamma, \varepsilon} \geq -C'_\nu & \text{if } x = x_\alpha \text{ and } x_\alpha \in \mathcal{V}. \end{cases}$$

From (5.17), we deduce that

$$(5.30) \quad |\alpha\psi_x(x) + \varphi_x^\alpha(t, s, x)| \leq C\sqrt{\alpha} + d(x, x_\alpha) \leq C$$

for $\alpha \leq 1$ (using (5.17)). Therefore, we have for some constant C_ν (independent of $\alpha, \varepsilon, \gamma$):

$$\begin{cases} |p_x^{\alpha, \gamma, \varepsilon}| \leq C_\nu & \text{if } x \neq x_\alpha \text{ or } x_\alpha \notin \mathcal{V}, \\ p_x^{\alpha, \gamma, \varepsilon} \geq -C_\nu & \text{if } x = x_\alpha \text{ and } x_\alpha \in \mathcal{V}. \end{cases}$$

From the compatibility condition of the Hamiltonians satisfied by $G^{\alpha, \gamma}$ if $x_\alpha \in \mathcal{V}$, or the definition of $G^{\alpha, \gamma}$ if $x_\alpha \notin \mathcal{V}$, we have in both cases,

$$(5.31) \quad H^{t_\alpha, x_\alpha}(y, p_y^{\alpha, \gamma, \varepsilon}) \leq H^{t_\alpha, x_\alpha}(x, p_x^{\alpha, \gamma, \varepsilon}) + \gamma$$

where

$$H^{t_\alpha, x_\alpha}(x, p) = \begin{cases} H_{\mathcal{V}}^{t_\alpha, n}(x, p) & \text{if } x_\alpha = n \in \mathcal{V}, \\ H_e(t_\alpha, x_\alpha, p) & \text{if } x_\alpha \notin \mathcal{V}, x_\alpha \in e^*. \end{cases}$$

We deduce that $p_y^{\alpha, \gamma, \varepsilon}$ satisfies (modifying C_ν if necessary)

$$\begin{cases} |p_y^{\alpha, \gamma, \varepsilon}| \leq C_\nu & \text{if } y \neq x_\alpha \text{ or } x_\alpha \notin \mathcal{V}, \\ p_y^{\alpha, \gamma, \varepsilon} \geq -C_\nu & \text{if } y = x_\alpha \text{ and } x_\alpha \in \mathcal{V}. \end{cases}$$

Defining for $z = x, y$,

$$\bar{p}_z^{\alpha, \gamma, \varepsilon} = \begin{cases} (\min(K, (p_z^{\alpha, \gamma, \varepsilon})_{\bar{z}}))_{\bar{z} \in x_\alpha} & \text{if } z = x_\alpha \text{ and } x_\alpha \in \mathcal{V} \\ p_z^{\alpha, \gamma, \varepsilon} & \text{if not.} \end{cases}$$

with, in the case where $x_\alpha \in \mathcal{V}$, the constant K given by

$$K = \max_{e \in \mathcal{E}_{x_\alpha}} (p_e^0(s, x_\alpha), p_e^0(t_\alpha, x_\alpha), p_e^0(t, x_\alpha) + C) \leq C_T + C$$

(C comes from (5.30) and C_T from (5.8)), we have

$$|\bar{p}_z^{\alpha, \gamma, \varepsilon}| \leq C_\nu + C_T + C =: C_{\nu, T}$$

and

$$(5.32) \quad \frac{\eta}{2T^2} \leq H_{\mathcal{N}}(s, y, \bar{p}_y^{\alpha, \gamma, \varepsilon}) - H_{\mathcal{N}}(t, x, \bar{p}_x^{\alpha, \gamma, \varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t, s, x)),$$

$$(5.33) \quad H_{\mathcal{N}}(t, x, \bar{p}_x^{\alpha, \gamma, \varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t, s, x)) \leq \frac{s-t}{\nu} + t_\alpha - t \leq \frac{T}{\nu} + T,$$

$$(5.34) \quad H^{t_\alpha, x_\alpha}(y, \bar{p}_y^{\alpha, \gamma, \varepsilon}) \leq H^{t_\alpha, x_\alpha}(x, \bar{p}_x^{\alpha, \gamma, \varepsilon}) + \gamma.$$

Step 6: The limit $(\varepsilon, \gamma) \rightarrow (0, 0)$ and conclusion as $\alpha \rightarrow 0$. Up to a subsequence, we get in the limit $(\varepsilon, \gamma) \rightarrow (0, 0)$ for $z = x, y$:

$$\bar{p}_z^{\alpha, \gamma, \varepsilon} \rightarrow \bar{p}_z^\alpha \quad \text{with} \quad |\bar{p}_z^\alpha| \leq C_{\nu, T}.$$

Moreover, passing to the limit in (5.32) and (5.33), we get respectively

$$\frac{\eta}{2T^2} \leq H_{\mathcal{N}}(s_\alpha, x_\alpha, \bar{p}_y^\alpha) - H_{\mathcal{N}}(t_\alpha, x_\alpha, \bar{p}_x^\alpha + \alpha\psi_x(x_\alpha))$$

and

$$H_{\mathcal{N}}(t_\alpha, x_\alpha, \bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)) \leq \frac{s_\alpha - t_\alpha}{\nu} \leq \frac{T}{\nu}.$$

On the other hand, passing to the limit in (5.34) gives

$$H^{t_\alpha, x_\alpha}(x_\alpha, \bar{p}_y^\alpha) \leq H^{t_\alpha, x_\alpha}(x_\alpha, \bar{p}_x^\alpha).$$

Because

$$H_{\mathcal{N}}(t_\alpha, x_\alpha, p) = H^{t_\alpha, x_\alpha}(x_\alpha, p)$$

we get for any p ,

$$\frac{\eta}{2T^2} \leq I_1 + I_2$$

with

$$\begin{aligned} I_1 &= H_{\mathcal{N}}(s_\alpha, x_\alpha, \bar{p}_x^\alpha) - H_{\mathcal{N}}(s_\alpha, x_\alpha, \bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)), \\ I_2 &= H_{\mathcal{N}}(s_\alpha, x_\alpha, \bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)) - H_{\mathcal{N}}(t_\alpha, x_\alpha, \bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)). \end{aligned}$$

Thanks to (H3) and (5.17), we have $|\alpha\psi_x(x_\alpha)| \leq C_{\nu, T}$ and we thus get

$$(5.35) \quad I_1 \leq \omega_{T, 2C_{\nu, T}}(\alpha\psi_x(x_\alpha)) \leq \omega_{T, 2C_{\nu, T}}(C\sqrt{\alpha}).$$

Now thanks to Lemma 5.2, we also have

$$\begin{aligned} I_2 &\leq \tilde{\omega}_T(|t_\alpha - s_\alpha|(1 + \max(H_{\mathcal{N}}(t_\alpha, x_\alpha, \bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)), 0))) \\ &\leq \tilde{\omega}_T(|t_\alpha - s_\alpha|(1 + \max(\frac{s_\alpha - t_\alpha}{\nu}, 0))). \end{aligned}$$

Then taking first the limit $\alpha \rightarrow 0$ and then taking the limit $\nu \rightarrow 0$, we use (5.19) to get the desired contradiction. This achieves the proof of Theorem 5.8. \square

6 An application: a homogenization result for a network

In this section, we present an application of the comparison principle of viscosity sub- and super-solutions on networks.

6.1 A homogenization problem

We consider the simplest periodic network generated by $\varepsilon\mathbb{Z}^d$. Hence, the network is naturally embedded in \mathbb{R}^d . Let us be more precise now. At scale $\varepsilon = 1$, the edges are the following subsets of \mathbb{R}^d : for $k, l \in \mathbb{Z}^d$, $|k - l| = 1$,

$$e_{k,l} = \{\theta k + (1 - \theta)l : \theta \in [0, 1]\}.$$

If (e_1, \dots, e_d) denotes the canonical basis of \mathbb{R}^d , then for $l = k + e_i$, $e_{k,l}$ is oriented in the direction of e_i . The network \mathcal{N}_ε at scale $\varepsilon > 0$ is the one corresponding to

$$\begin{cases} \mathcal{E}_\varepsilon = \{\varepsilon e_{k,l}, k, l \in \mathbb{Z}^d, |k - l| = 1\} \\ \mathcal{V}_\varepsilon = \varepsilon\mathbb{Z}^d \end{cases}$$

endowed with the metric induced by the Euclidian norm. We next consider the following ‘‘oscillating’’ Hamilton-Jacobi equation on this network

$$(6.1) \quad \begin{cases} u_t^\varepsilon + H_\varepsilon(u_x^\varepsilon) = 0, & t > 0, x \in e^*, e \in \mathcal{E}_\varepsilon, \\ u_t^\varepsilon + F_A(\frac{x}{\varepsilon}, u_x^\varepsilon) = 0, & t > 0, x \in \mathcal{V}_\varepsilon \end{cases}$$

(for some $A \in \mathbb{R}$) submitted to the initial condition

$$(6.2) \quad u^\varepsilon(0, x) = u_0(x), \quad x \in \mathcal{N}_\varepsilon.$$

For $m \in \mathbb{Z}^d$, it is convenient to define

$$\varepsilon e_{k,l} + \varepsilon m = \varepsilon e_{k+m, l+m}.$$

Assumptions on H for the homogenization problem For each $e \in \mathcal{N}_1$, we associate a Hamiltonian H_e and we assume

• **(H’0)** (Continuity) For all $e \in \mathcal{E}_1$, $H_e \in C(\mathbb{R})$.

• **(H’1)** (Coercivity) $e \in \mathcal{E}_1$,

$$\liminf_{|q| \rightarrow +\infty} H_e(q) = +\infty.$$

• **(H’2)** (Bi-monotonicity) For all $e \in \mathcal{E}_1$, there exists a $p_e^0 \in \mathbb{R}$ such that

$$\begin{cases} H_e & \text{is nonincreasing on } (-\infty, p_e^0], \\ H_e & \text{is nondecreasing on } [p_e^0, +\infty). \end{cases}$$

• **(H’3)** (Periodicity) For all $m \in \mathbb{Z}^d$, $H_{e+m}(p) = H_e(p)$.

A homogenization result The goal of this section is to prove the following convergence result for the oscillating Hamilton-Jacobi equation.

Theorem 6.1. *Assume (H'0)-(H'3). Let u_0 be Lipschitz continuous and u^ε be the solution of (6.1)-(6.2). There exists a continuous function $\bar{H} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that u^ε converges locally uniformly towards the unique solution u^0 of*

$$(6.3) \quad u_t^0 + \bar{H}(\nabla_x u^0) = 0, \quad t > 0, x \in \mathbb{R}^d$$

$$(6.4) \quad \bar{u}^0(0, x) = u_0(x), \quad x \in \mathbb{R}^d.$$

Remark 6.2. The meaning of the convergence u^ε towards u^0 is

$$\lim_{\substack{(s,y) \rightarrow (t,x) \\ y \in \mathcal{N}_\varepsilon}} u^\varepsilon(s, y) = u^0(t, x).$$

6.2 The cell problem

Keeping in mind the definitions of networks and derivatives of functions defined on networks, solving the cell problem consists in finding specific global solutions of (6.1) for $\varepsilon = 1$, *i.e.*

$$(6.5) \quad \begin{cases} w_t + H_e(w_y) = 0, & t \in \mathbb{R}, y \in e^*, e \in \mathcal{E}_1, \\ w_t + F_A(y, w_y) = 0, & t \in \mathbb{R}, y \in \mathcal{V}_1. \end{cases}$$

Precisely, for some $P \in \mathbb{R}^d$, we look for solutions $w(t, y) = \lambda t + P \cdot y + v(y)$ with a \mathbb{Z}^d -periodic function v ; in other words, we look for (λ, v) such that

$$(6.6) \quad \begin{cases} \lambda + H_e((P \cdot y + v)_y) = 0, & y \in e^*, e \in \mathcal{E}_1, \\ \lambda + F_A(y, (P \cdot y + v)_y) = 0, & y \in \mathcal{V}_1. \end{cases}$$

Theorem 6.3. *For all $P \in \mathbb{R}^d$ there exists $\lambda \in \mathbb{R}$ and a \mathbb{Z}^d -periodic solution v of (6.6). Moreover, the function \bar{H} which maps P to $-\lambda$ is continuous.*

Proof. We consider the following \mathbb{Z}^d -periodic stationary problem

$$(6.7) \quad \begin{cases} \alpha v^\alpha + H_e((P \cdot y + v^\alpha)_y) = 0, & y \in e^*, e \in \mathcal{E}_1, \\ \alpha v^\alpha + F_A(y, (P \cdot y + v^\alpha)_y) = 0, & y \in \mathcal{V}_1. \end{cases}$$

We consider

$$C = \max_{e \in \mathcal{E}_1} |H_e((P \cdot y)_y)|.$$

Then the existence result and the comparison principle for the stationary equation (see Appendix B) imply that there exists a (unique) \mathbb{Z}^d -periodic solution v^α of (6.7) such that

$$|\alpha v^\alpha| \leq C.$$

Since H_ε is coercive, this implies that there exists a constant \tilde{C} such that for all $\alpha > 0$, v_α is Lipschitz-continuous and

$$|v_y^\alpha| \leq \tilde{C};$$

in other words, the family $(v^\alpha)_{\alpha>0}$ is equi-Lipschitz continuous. We then consider

$$\tilde{v}_\alpha = v_\alpha - v_\alpha(0).$$

By Arzelà-Ascoli theorem, there exists $\alpha_n \rightarrow 0$ such that $\tilde{v}^n := \tilde{v}_{\alpha_n}$ converges uniformly towards v . Moreover, we can also assume that

$$\alpha_n v_{\alpha_n}(0) \rightarrow \lambda.$$

Passing to the limit into the equation yields that (λ, v) solves the cell problem (6.6).

The continuity of λ is completely classical too. Consider $P_n \rightarrow P_\infty$ as $n \rightarrow \infty$ and consider (λ_n, v_n) solving (6.6). We proved above that

$$|\lambda_n| \leq C.$$

Hence, arguing as above, we can extract a subsequence from (λ_n, v_n) converging towards $(\lambda_\infty, v_\infty)$. Passing to the limit into the equation implies that $(\lambda_\infty, v_\infty)$ solves the cell problem (6.6). The uniqueness of λ yields the continuity of \bar{H} . The proof is now complete. \square

6.3 Proof of convergence

Before proving the convergence, we state without proof the following elementary lemma.

Lemma 6.4 (Barriers). *There exists $C > 0$ such that for all $\varepsilon > 0$,*

$$|u^\varepsilon(t, x) - u_0(x)| \leq Ct.$$

We can now turn to the proof of convergence.

Proof of Theorem 6.1. We classically consider the relaxed semi-limits

$$\begin{cases} \bar{u}(t, x) = \limsup_{\substack{\varepsilon \rightarrow 0, (s, y) \rightarrow (t, x) \\ y \in \mathcal{N}_\varepsilon}} u^\varepsilon(s, y), \\ \underline{u}(t, x) = \liminf_{\substack{\varepsilon \rightarrow 0, (s, y) \rightarrow (t, x) \\ y \in \mathcal{N}_\varepsilon}} u^\varepsilon(s, y). \end{cases}$$

In order to prove convergence of u^ε towards u^0 , it is enough to prove that \bar{u} is a subsolution of (6.3) and \underline{u} is a supersolution of (6.3). We only prove that \bar{u} is a subsolution since the proof for \underline{u} is very similar.

We consider a test function φ touching (strictly) \bar{u} from above at (t_0, x_0) : there exists $r_0 > 0$ such that for all $(t, x) \in B_{r_0}(t_0, x_0)$, $(t, x) \neq (t_0, x_0)$,

$$\varphi(t, x) > \bar{u}(t, x)$$

and $\varphi(t_0, x_0) = \bar{u}(t_0, x_0)$. We argue by contradiction by assuming that there exists $\theta > 0$ such that

$$(6.8) \quad \partial_t \varphi(t_0, x_0) - \lambda = \partial_t \varphi(t_0, x_0) + \bar{H}(\nabla_x \varphi(t_0, x_0)) = \theta > 0.$$

We then consider the following ‘‘perturbed test’’ function $\varphi^\varepsilon: \mathbb{R}^+ \times \mathcal{N}_\varepsilon \rightarrow \mathbb{R}$ [11],

$$\varphi^\varepsilon(t, x) = \varphi(t, x) + \varepsilon v(\varepsilon^{-1}x)$$

where (λ, v) solves the cell problem (6.6) for $P = \nabla_x \varphi(t_0, x_0)$.

Lemma 6.5. *For $r \leq r_0$ small enough, the function φ^ε is a supersolution of (6.1) in $B((t_0, x_0), r) \subset \mathcal{N}_\varepsilon$ and $\varphi^\varepsilon \geq u^\varepsilon + \eta_r$ in $\partial B((t_0, x_0), r)$ for some $\eta_r > 0$.*

Proof. Consider a test function ψ touching φ^ε from below at $(t, x) \in]0, +\infty[\times \mathcal{N}_\varepsilon$. Then the function

$$\psi_\varepsilon(s, y) = \varepsilon^{-1}(\psi(s, \varepsilon y) - \varphi(s, \varepsilon y))$$

touches v from below at $y = \frac{x}{\varepsilon} \in e$. In particular,

$$(6.9) \quad \partial_t \psi(t, x) = \partial_t \varphi(t, x),$$

$$(6.10) \quad \lambda + H_{\mathcal{N}_1}(y, \varphi_x(t_0, x_0) + \psi_x(t, x) - \varphi_x(t, x)) \geq 0.$$

Combine now (6.8), (6.9) and (6.10) and get

$$\partial_t \psi(t, x) + H_{\mathcal{N}_1}(y, \psi_x(t, x)) \geq \theta + E$$

where

$$E = (\varphi_t(t, x) - \varphi_t(t_0, x_0)) + (H_{\mathcal{N}_1}(y, \psi_x(t, x)) - H_{\mathcal{N}_1}(y, \psi_x(t, x) + \varphi_x(t_0, x_0) - \varphi_x(t, x))).$$

The fact that φ is C^1 implies that we can choose $r > 0$ small enough so that for all $(t, x) \in B((t_0, x_0), r)$,

$$E \geq -\theta.$$

Moreover, since φ is strictly above \bar{u} , we conclude that $\varphi^\varepsilon \geq u^\varepsilon + \eta_r$ on $\partial B((t_0, x_0), r)$ for some $\eta_r > 0$. This achieves the proof of the lemma. \square

From the lemma, we deduce thanks to the (localized) comparison principle that

$$\varphi^\varepsilon(t, x) \geq u^\varepsilon(t, x) + \eta_r.$$

In particular, this implies

$$u(t_0, x_0) = \varphi(t_0, x_0) \geq u(t_0, x_0) + \eta_r > u(t_0, x_0)$$

which is the desired contradiction. \square

6.4 Qualitative properties of the effective Hamiltonian

Proposition 6.6. *If for all $e \in \mathcal{E}_1$, $p \mapsto H_e(p)$ is convex, then so is $\bar{H}: \mathbb{R}^d \rightarrow \mathbb{R}$.*

This proposition is a simple consequence of (6.11) which is a classical characterization of λ if (λ, v) solves the cell problem.

Proposition 6.7. *Let (λ, v) be a solution of the cell problem (6.6). Then*

$$(6.11) \quad \begin{aligned} \lambda &= \inf\{\bar{\lambda} : \exists \text{ a } \mathbb{Z}^d\text{-periodic supersolution } \bar{v} \text{ of (6.6) with } \lambda = \bar{\lambda}\}, \\ \lambda &= \sup\{\underline{\lambda} : \exists \text{ a } \mathbb{Z}^d\text{-periodic subsolution } \underline{v} \text{ of (6.6) with } \lambda = \underline{\lambda}\}. \end{aligned}$$

A Appendix: proofs of some technical results

A.1 Technical results on a junction

In order to prove Lemma 3.3, we need the following one.

Lemma A.1 (A priori control at the same time). *Let $T > 0$ and let u be a sub-solution and w be a super-solution as in Theorem 1.1. Then there exists a constant $C_T > 0$ such that for all $t \in [0, T)$, $x, y \in J$, we have*

$$(A.1) \quad u(t, x) \leq w(t, y) + C_T(1 + d(x, y)).$$

We first derive Lemma 3.3 from Lemma A.1.

Proof of Lemma 3.3. Let us fix some $\varepsilon > 0$ and let us consider the sub-solution u_ε^- and super-solutions u_ε^+ defined in (2.5). Using (2.4), we see that we have for all $(t, x), (s, y) \in [0, T) \times J$

$$(A.2) \quad u_\varepsilon^+(t, x) - u_\varepsilon^-(s, y) \leq 2C_\varepsilon T + 2\varepsilon + L_\varepsilon d(x, y)$$

We first apply Lemma A.1 to control $u(t, x) - u_\varepsilon^+(t, x)$, and then apply Lemma A.1 to control $u_\varepsilon^-(s, y) - w(s, y)$. Finally we get the control on $u(t, x) - w(s, y)$, using (A.2). \square

We now turn to the proof of Lemma A.1.

Proof of Lemma A.1. Let us define

$$\varphi(x, y) = \sqrt{1 + d^2(x, y)}.$$

Then $\varphi \in C^1(J^2)$ and satisfies

$$(A.3) \quad |\varphi_x(x, y)|, |\varphi_y(x, y)| \leq 1.$$

For constants $C_1, C_2 > 0$ to be chosen, let us consider

$$M = \sup_{t \in [0, T), x, y \in J} (u(t, x) - w(t, y) - C_2 t - C_1 \varphi(x, y)).$$

The result follows if we show that M is non-positive for C_1 and C_2 large enough. Assume by contradiction that $M > 0$ for any C_1 and C_2 . Then for $\eta, \alpha > 0$ small enough, we have $M_{\alpha, \eta} \geq M/2 > 0$ with

$$(A.4) \quad M_{\eta, \alpha} = \sup_{t \in [0, T], x, y \in J} \left(u(t, x) - w(t, y) - C_2 t - C_1 \varphi(x, y) - \frac{\eta}{T-t} - \alpha \frac{d^2(x_0, x)}{2} \right).$$

From (1.10), we have

$$0 < \frac{M}{2} \leq C_T(2 + d(0, x) + d(0, y)) - C_2 t - C_1 \varphi(x, y) - \frac{\eta}{T-t} - \alpha \frac{d^2(x_0, x)}{2}$$

which shows that the supremum in (A.4) is reached at a point (t, x, y) , assuming $C_1 > C_T$. Moreover, we have (for $0 < \alpha \leq 1$)

$$(A.5) \quad \alpha d(0, x) \leq C = C(C_T).$$

From the uniform continuity of the initial data u^0 , there exists a constant $C_0 > 0$ such that

$$u^0(x) - u^0(y) \leq C_0 \varphi(x, y)$$

and therefore $t > 0$, assuming $C_1 > C_0$. Then the classical time penalization (or doubling variable technique) implies the existence of $a, b \in \mathbb{R}$ (that play the role of u_t and v_t) such that we have the following viscosity inequalities

$$\begin{cases} a + H(x, C_1 \varphi_x(x, y) + \alpha d(x_0, x)) \leq 0, \\ b + H(y, -C_1 \varphi_y(x, y)) \geq 0 \end{cases}$$

(using the shorthand notation (3.1) and writing $\alpha d(x_0, x)$ for $\alpha (d^2(x_0, x)/2)_x$ for the purposes of notation) with $a - b = C_2 + \eta(T-t)^{-2}$. Subtracting these inequalities yields

$$(A.6) \quad C_2 + \frac{\eta}{(T-t)^2} \leq H(y, -C_1 \varphi_y(x, y)) - H(x, C_1 \varphi_x(x, y) + \alpha d(0, x)).$$

Using bounds (A.3) and (A.5), this yields a contradiction in (A.6) for C_2 large enough. \square

A.2 Technical results on a network

A.2.1 Proof of Lemma 5.2

Proof of Lemma 5.2. (H1) and (H2) imply the uniform boundedness of the $p_e^0(t, x)$, i.e. (5.8). We also notice that because of (5.8), there exists a constant $C_0 > 0$ such that for all $t \in [0, T]$, $e \in \mathcal{E}$ and $n \in \partial e$,

$$(A.7) \quad |H_e(t, n, p_e^0(t, n))| \leq C_0$$

from which (5.9) is easily derived.

We now turn to the proof of (5.10). In view of the definition of F_A and (A2), (H5), we see that it is enough to prove that for all for $n \in \mathcal{V}$, $t, s \in [0, T]$, $p = (p_e)_{e \in \mathcal{E}_n} \in \mathbb{R}^{\text{Card } \mathcal{E}_n}$, $x \in \mathcal{V}$,

$$(A.8) \quad A_n^0(t, p) - A_n^0(s, p) \leq \tilde{\omega}_T \left(|t - s| (1 + \max(0, A_n^0(s, p))) \right).$$

where

$$A_n^0(t, p) = \max_{e \in \mathcal{E}_n^-} H_e^-(t, n, p_e) \geq A_n^0(t)$$

or

$$A_n^0(t, p) = \max_{e \in \mathcal{E}_n^+} H_e^+(t, n, p_e) \geq A_n^0(t).$$

We only treat the first case, since the second case reduces to the first one by a simple change of orientation of the network.

We have

$$A_n^0(a, p) = H_{e_a}^-(a, x, p_{e_a}) \quad \text{for } a = t, s.$$

Let us assume that we have (otherwise there is nothing to prove)

$$0 \leq I(t, s) := A_n^0(t, p) - A_n^0(s, p).$$

We also have

$$H_{e_s}^-(t, n, p_{e_s}) \leq A_n^0(t, p) = H_{e_t}^-(t, n, p_{e_t})$$

and

$$H_{e_t}^-(s, n, p_{e_t}) \leq A_n^0(s, p) = H_{e_s}^-(s, n, p_{e_s}).$$

We now distinguish three cases.

Case 1: $H_{e_t}^-(s, n, p_{e_t}) < H_{e_t}(s, n, p_{e_t})$. We first note that

$$(A.9) \quad 0 \leq I(t, s) \leq A_n^0(t, p) - A_n^0(s).$$

Let us define

$$\tau = \begin{cases} \inf \{ \sigma \in [t, s], & H_{e_t}^-(\sigma, n, p_{e_t}) < H_{e_t}(\sigma, n, p_{e_t}) \} & \text{if } t < s, \\ \sup \{ \sigma \in [s, t], & H_{e_t}^-(\sigma, n, p_{e_t}) < H_{e_t}(\sigma, n, p_{e_t}) \} & \text{if } t \geq s. \end{cases}$$

Let us consider a optimizing sequence $\sigma_k \rightarrow \tau$ such that

$$H_{e_t}^-(\sigma_k, n, p_{e_t}) < H_{e_t}(\sigma_k, n, p_{e_t}).$$

Then we have

$$H_{e_t}^-(\sigma_k, n, p_{e_t}) = H_{e_t}(\sigma_k, n, p_{e_t}^0(\sigma_k, n)) \leq A_n^0(\sigma_k) \leq A_n^0(\sigma_k, p).$$

Then passing to the limit $k \rightarrow +\infty$, we get

$$(A.10) \quad H_{e_t}^-(\tau, n, p_{e_t}) = H_{e_t}(\tau, n, p_{e_t}^0(\tau, n)) \leq A_n^0(\tau) \leq A_n^0(\tau, p).$$

If $\tau = t$, then (A.10) implies that $A_n^0(t, p) = A_n^0(t)$ (keeping in mind the definition of p_{e_t}).

SUBCASE 1.1: $\tau \neq t$. This shows that

$$H_{e_t}(\tau, n, p_{e_t}) \leq A_n^0(\tau) \quad \text{and} \quad H_{e_t}(t, n, p_{e_t}) \geq A_n^0(t).$$

We now choose some $\bar{\tau}$ in between t and τ such that

$$H_{e_t}(\bar{\tau}, n, p_{e_t}) = A_n^0(\bar{\tau})$$

and estimate, using (A.9) and (A.7) and (H5)-(H6),

$$\begin{aligned} 0 \leq I(t, s) &\leq \{A_n^0(t, p) - H_{e_t}(\bar{\tau}, n, p_{e_t})\} + \{A_n^0(\bar{\tau}) - A_n^0(s)\} \\ &\leq \{H_{e_t}(t, n, p_{e_t}) - H_{e_t}(\bar{\tau}, n, p_{e_t})\} + \{A_n^0(\bar{\tau}) - A_n^0(s)\} \\ &\leq \bar{\omega}_T(|t - \bar{\tau}|(1 + \max(A_n^0(\bar{\tau}), 0))) + \bar{\omega}_T(|\bar{\tau} - s|) \\ &\leq \bar{\omega}_T(|t - s|(1 + C_0)) + \bar{\omega}_T(|t - s|). \end{aligned}$$

SUBCASE 1.2: $\tau = t$. Then $A_n^0(t, p) = A_n^0(t)$. Using (A.9), this gives directly

$$0 \leq I(t, s) \leq A_n^0(t) - A_n^0(s) \leq \bar{\omega}_T(|t - s|).$$

Case 2: $H_{e_t}^-(s, n, p_{e_t}) = H_{e_t}(s, n, p_{e_t})$ **and** $H_{e_t}^-(t, n, p_{e_t}) = H_{e_t}(t, n, p_{e_t})$. We have

$$\begin{aligned} 0 \leq I(t, s) &= H_{e_t}^-(t, n, p_{e_t}) - A_n^0(s, p) \\ &\leq H_{e_t}^-(t, n, p_{e_t}) - H_{e_t}^-(s, n, p_{e_t}) \\ &= H_{e_t}(t, n, p_{e_t}) - H_{e_t}(s, n, p_{e_t}) \\ &\leq \bar{\omega}_T(|t - s|(1 + \max(H_{e_t}(s, n, p_{e_t}), 0))) \\ &\leq \bar{\omega}_T(|t - s|(1 + \max(H_{e_t}^-(s, n, p_{e_t}), 0))) \\ &\leq \bar{\omega}_T(|t - s|(1 + \max(A_n^0(s, p), 0))). \end{aligned}$$

Case 3: $H_{e_t}^-(s, n, p_{e_t}) = H_{e_t}(s, n, p_{e_t})$ **and** $H_{e_t}^-(t, n, p_{e_t}) < H_{e_t}(t, n, p_{e_t})$. Then

$$p_{e_t}^0(t, n) < p_{e_t} \leq p_{e_t}^0(s, n).$$

Because of (A.7) and the uniform bound on the Hamiltonians for bounded gradients, (H2), we deduce that

$$H_{e_t}(s, n, p_{e_t}) \leq C_1$$

for some constant $C_1 > 0$ only depending on our assumptions. Therefore, we have

$$\begin{aligned} 0 \leq I(t, s) &= H_{e_t}^-(t, n, p_{e_t}) - A_n^0(s, p) \\ &\leq H_{e_t}^-(t, n, p_{e_t}) - H_{e_t}^-(s, n, p_{e_t}) \\ &< H_{e_t}(t, n, p_{e_t}) - H_{e_t}(s, n, p_{e_t}) \\ &\leq \bar{\omega}_T(|t - s|(1 + C_1)). \end{aligned}$$

The proof is now complete. □

A.2.2 Semi-concavity of the distance

In order to prove Lemmas 5.10 and 5.11, we need the following one.

Lemma A.2 (Semi-concavity of φ and d^2). *Let \mathcal{N} be a network defined in (5.2) with edges \mathcal{E} and vertices \mathcal{V} . Let*

$$\varphi(x, y) = \sqrt{1 + d^2(x, y)}$$

where d is the distance function on the network \mathcal{N} . Then $\varphi(x, \cdot)$ and $\varphi(\cdot, y)$ are 1-Lipschitz for all $x, y \in \mathcal{N}$. Moreover φ and d^2 are semi-concave on $e_a \times e_b$ for all $e_a, e_b \in \mathcal{E}$.

Proof of Lemma A.2. The Lipschitz properties of φ are trivial. Since $r \mapsto r^2$ and $r \mapsto \sqrt{1 + r^2}$ are smooth increasing functions in \mathbb{R}^+ , the result follows from the fact that the distance function d itself is semi-concave; it is even the minimum of a finite number of smooth functions.

If $e_a = e_b$, then $d^2(x, y) = (x - y)^2$ which implies that $\varphi \in C^1(e_a \times e_a)$. Then we only consider the cases where $e_a \neq e_b$.

Case 1: e_a and e_b isometric to $[0, +\infty)$. Then for $(x, y) \in e_a \times e_b$, we have

$$d(x, y) = x + y + d(e_a^0, e_b^0)$$

which implies that $\varphi \in C^1(e_a \times e_b)$.

Case 2: e_a isometric to $[0, +\infty)$ and e_b isometric to $[0, l_b]$. Reversing the orientation of e_b if necessary, we can assume that

$$d_0 := d(e_a^0, e_b^0) \leq d(e_a^0, e_b^1) =: d_1$$

and then for $(x, y) \in e_a \times e_b$, we have

$$d(x, y) = x + \min(d_0 + y, d_1 + (l_b - y)) = \min(d_0 + x + y, d_1 + x + (l_b - y)).$$

Then φ is the minimum of two C^1 functions, it is semi-concave.

Case 3: e_a and e_b isometric to $[0, l_a]$ and $[0, l_b]$. Changing the orientations of both e_a and e_b if necessary, we can assume that

$$d(e_a^0, e_b^0) = \min_{i,j=0,1} d_{ij} \quad \text{with} \quad d_{ij} = d(e_a^i, e_b^j).$$

Therefore

$$d(x, y) = \min(d_{00} + x + y, d_{01} + x + (l_b - y), d_{10} + (l_a - x) + y, d_{11} + (l_a - x) + (l_b - y))$$

and again φ is the minimum of four C^1 functions, it is therefore semi-concave. \square

A.2.3 Proof of Lemma 5.10

Proof of Lemma 5.10. We first prove (5.14) for $t = s$ by adapting in a straightforward way the proof of Lemma A.1. The only difference is that for any $e_a, e_b \in \mathcal{E}$, the function

$$\varphi(x, y) = \sqrt{1 + d^2(x, y)}$$

may not be $C^1(e_a \times e_b)$. But Lemma A.2 and Remark 5.6 ensure that this is harmless. The remaining of the proof of Lemma A.1 is unchanged. In particular the uniform bound on the Hamiltonians for bounded gradients is used, see (H2).

Now (5.14) is obtained for $t \neq s$ by following the proof of Lemma 3.3 and using the barriers given in the proof of Theorem 5.7. \square

A.2.4 Proof of Lemma 5.11

Proof of Lemma 5.11. We do the proof for sub-solutions (the proof for super-solutions being similar). We consider the following barrier (similar to the ones in the proof of Theorem 5.7)

$$u_\varepsilon^+(t, x) = u_\varepsilon^0(x) + K_\varepsilon t + \varepsilon$$

with

$$|u_\varepsilon^0 - u^0| \leq \varepsilon \quad \text{and} \quad |(u_\varepsilon^0)_x| \leq L_\varepsilon$$

and $K_\varepsilon \geq C_\varepsilon$ with C_ε given in (5.12). It is enough to prove that for all $(t, x) \in [0, T] \times \mathcal{N}$,

$$u(t, x) \leq u_\varepsilon^+(t, x)$$

for a suitable choice of $K_\varepsilon \geq C_\varepsilon$ in order to conclude. Indeed, this implies

$$u(t, x) \leq u_0(x) + f(t)$$

with

$$f(t) = \min_{\varepsilon > 0} (K_\varepsilon t + \varepsilon)$$

which is non-negative, non-decreasing, concave and $f(0) = 0$.

We consider for $0 < \tau \leq T$,

$$M = \sup_{(t, x) \in [0, \tau] \times \mathcal{N}} (u - u_\varepsilon^+)(t, x)$$

and assume by contradiction that $M > 0$. We know by Lemma 5.10 that M is finite. Then for any $\alpha, \eta > 0$ small enough, we have $M_\alpha \geq M/2 > 0$ with

$$M_\alpha = \sup_{(t, x) \in [0, \tau] \times \mathcal{N}} \left\{ u(t, x) - u_\varepsilon^+(t, x) - \frac{\eta}{\tau - t} - \alpha \psi(x) \right\}.$$

(we recall that $\psi = d^2(x_0, \cdot)/2$). By the sublinearity of u and u_ε^+ , we know that this supremum is reached at some point (t, x) . Moreover $t > 0$ since $u(0, x) \leq u_0(x) \leq u_\varepsilon^+(0, x)$.

This implies in particular that

$$\begin{aligned}
0 < M/2 \leq M_\alpha &= u(t, x) - u_\varepsilon^+(t, x) - \frac{\eta}{\tau - t} - \alpha \frac{d^2(x_0, x)}{2} \\
&\leq C_T(1 + d(x_0, x)) - u_\varepsilon^0(x_0) + L_\varepsilon d(x, x_0) - \alpha \frac{d^2(x_0, x)}{2} \\
&\leq C_T(1 + d(x_0, x)) + |u_0(x_0)| + \varepsilon + L_\varepsilon d(x, x_0) - \alpha \frac{d^2(x_0, x)}{2} \\
&\leq R_\varepsilon(1 + d(x_0, x)) - \alpha \frac{d^2(x_0, x)}{2}
\end{aligned}$$

with

$$R_\varepsilon = C_T + \max(L_\varepsilon, |u_0(x_0)| + \varepsilon).$$

Then $z = \alpha d(x_0, x)$ satisfies

$$\frac{z^2}{2} \leq R_\varepsilon \alpha + R_\varepsilon z \leq R_\varepsilon \alpha + R_\varepsilon^2 + \frac{z^2}{4}$$

which implies that for $\alpha \leq 1$,

$$(A.11) \quad \alpha d(x_0, x) \leq 2\sqrt{R_\varepsilon + R_\varepsilon^2}.$$

Writing the sub-solution viscosity inequality, we get

$$K_\varepsilon + H_{\mathcal{N}}(t, x, (u_\varepsilon^0)_x(x) + \alpha \psi_x(x)) \leq 0$$

We get a contradiction for the choice

$$K_\varepsilon = 1 + \max \left(\sup_{t \in [0, T]} \sup_{n \in \mathcal{V}} |\max(A_n(t), A_n^0(t))|, \sup_{t \in [0, T]} \sup_{e \in \mathcal{E}} \sup_{x \in e} \sup_{|p_e| \leq L_\varepsilon + 2\sqrt{R_\varepsilon + R_\varepsilon^2}} |H_e(t, x, p_e)| \right).$$

□

B Stationary results

This short section is devoted to the statement of an existence and uniqueness result for the following stationary HJ equation posed on a network \mathcal{N} satisfying (5.1),

$$(B.1) \quad u + H_{\mathcal{N}}(x, u_x) = 0 \quad \text{for all } x \in \mathcal{N}.$$

For each $e \in \mathcal{E}$, we consider a Hamiltonian $H_e : e \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

- **(H0-s)** (Continuity) $H_e \in C(e \times \mathbb{R})$.
- **(H1-s)** (Uniform coercivity)

$$\liminf_{|q| \rightarrow +\infty} H_e(x, q) = +\infty$$

uniformly with respect to $x \in e$, $e \in \mathcal{E}$.

- **(H2-s)** (Uniform bound on the Hamiltonians for bounded gradients) For all $L > 0$, there exists $C_L > 0$ such that

$$\sup_{p \in [-L, L], x \in \mathcal{N} \setminus \mathcal{V}} |H_{\mathcal{N}}(x, p)| \leq C_L.$$

- **(H3-s)** (Uniform modulus of continuity for bounded gradients) For all $L > 0$, there exists a modulus of continuity ω_L such that for all $|p|, |q| \leq L$ and $x \in e \in \mathcal{E}$,

$$|H_e(x, p) - H_e(x, q)| \leq \omega_L(|p - q|).$$

- **(H4-s)** (Bi-monotonicity) For all $n \in \mathcal{V}$, there exists a $p_e^0(n)$ such that

$$\begin{cases} H_e(n, \cdot) & \text{is nonincreasing on } (-\infty, p_e^0(n)], \\ H_e(n, \cdot) & \text{is nondecreasing on } [p_e^0(n), +\infty). \end{cases}$$

As far as flux limiters are concerned, the following assumptions will be used.

- **(A1-s)** (Uniform bound on A) There exists a constant $C > 0$ such that for all $n \in \mathcal{V}$,

$$|A_n| \leq C.$$

The following result is a straightforward adaptation of Corollary 5.9. Proofs are even simpler since the time dependance was an issue when proving the comparison principle in the general case.

Theorem B.1 (Existence and uniqueness – stationary case). *Assume (H0-s)-(H4-s) and (A1-s). Then there exists a unique sublinear viscosity solution u of (B.1) in \mathcal{N} .*

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