HAL open science

Adaptive observers-based synchronization of a class of lur'e systems under transmission delays

Habib Dimassi, Antonio Loria, Safya Belghith

To cite this version:

Habib Dimassi, Antonio Loria, Safya Belghith. Adaptive observers-based synchronization of a class of lur'e systems under transmission delays. 2013. hal-00831476

HAL Id: hal-00831476

https://hal.science/hal-00831476v1

Preprint submitted on 7 Jun 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Adaptive observers-based synchronization of a class of Lur'e systems under transmission delays

Habib Dimassi†, Antonio Loría⋆, Safya Belghith•

†Namur Center for Complex Systems (NAXYS), Namur, Belgique, dimassihabib2007@yahoo.fr;
⋆CNRS, LSS-Supelec, 3 Rue Joliot-Curie, 91192, Gif/Yvette, France, loria@lss.supelec.fr;
•Sys'com-Ecole Nationale d'Ingénieurs de Tunis (ENIT), Tunisia, safya.belghith@enit.rnu.tn.

Abstract

We propose an adaptive observers-based synchronization approach for a class of chaotic Lur'e systems with slope-restricted nonlinearities and uncertain parameters, under transmission time-delays. The delay is assumed to be bounded and time varying and the uncertain parameters are assumed to be piece-wise constant. Based on the Lyapunov-Krasovskii approach, we show that for sufficiently short time-delays, master-slave synchronization is achieved and therefore, the uncertain parameters may be recovered. Then, the proposed approach is extended to the case of long constant time-delays by proposing a synchronization scheme based on cascade observers. Theoretical results are illustrated via two numerical examples.

Index Terms— Synchronization, Adaptive control, delays, chaos, nonlinear dynamical systems, observers, state estimation.

I. Introduction

Master-slave synchronization of chaotic systems is an active area of research with increasing attention from different research communities. This problem has been successfully solved in ideal conditions and many synchronization approaches for different classes of chaotic systems have been developed. However, in real applications, the "communication channel" linking the master to the slave systems is often subject to different kinds of communication constraints such as data packet dropouts, limited channel capacity, channel noise, transmission delays, *etc.*In particular, time delays occur frequently in master-slave synchronization configurations and may degrade the synchronization performance. Therefore, specific attention has been paid to the problem of synchronization of chaotic systems with transmission delays and, particularly, in the case where the master system belongs to the special family of chaotic Lur'e systems such as Chua's circuits [1] which possess interesting characteristics such as sector and slope-restriction properties.

To solve this problem, a common approach in the literature is to use error state or/and output feedback control –see [1], [2], [3], [4], [5]. In [1], a delay-dependent synchronization criterion was given based on a Lyapunov-Krasovskii functional. This result has motivated many studies on synchronization of chaotic Lur'e systems using feedback controllers. In [2], both delay dependent and delay independent criteria for synchronization are given. In [3], the proposed feedback controller includes both the current error state feedback and the delayed static error output feedback; and based on a more general Lur'e-Postnikov Lyapunov functional, a new delay dependent criteria are presented in the form of linear matrix inequalities (LMIs). This problem was extended in [4] to the case of continuous uniformly bounded time varying delays and differentiable uniformly bounded time varying delays with bounded derivatives. A less conservative delay dependent synchronization criterion was obtained in [5] using a new Lyapunov-Krasovskii functional which employs redundant state of differential equations shifted in time by a fraction of the time delay. More recently, the authors of [6] investigate the synchronization of a network of coupled systems and the stability analysis in [6] is based on passivity to achieve synchronization in a network of nonlinear systems subject to *constant* delays. It is also to be pointed out that the synchronization problem of Lur'e systems based on the Lyapunov-Krasovskii theory was investigated in the classical case where the delays appear in the states contrary to the more sophisticated case of this paper where the delays act in the transmitted signals (or outputs) – see for instance, the article [7] where the authors highlight that Lyapunov-Krasovskii theory still a powerful tool for the stability analysis of synchronization in coupled time-delay systems.

In the pioneering work [8], it was shown that from a control viewpoint, the master-slave synchronization problem may be considered as a paradigm of observer design where the slave system is an observer of the master system. Observer design for nonlinear systems with delayed outputs has been particularly investigated in the recent literature of *control and systems theory*.

For instance, based on the design of state observers from a drift-observability property, the authors of [9] present a solution for the case of delayed outputs by proposing a chain of observation algorithms which ensure global exponential convergence of the estimation error. A similar design method was adopted in [10] however, the proposed nonlinear observer relies on a state-dependent gain which is computed from the solution of a first-order singular partial differential equation. In [11] the authors propose two cascaded observers to reconstruct the system states of a linear time invariant plant, from measurements with time-delay which is defined by a known piecewise function of time. In [12] the authors propose a nonlinear observer for a class of drift observable nonlinear systems with a bounded time varying observation delay; asymptotic and exponential convergence of the estimation error is established relying on Lyapunov-Razumikhiin theory.

The problem of estimating simultaneously the state and the unknown parameters using adaptive observers has been extensively investigated in the literature for linear and nonlinear systems. For instance, in [13] the author introduces a unifying adaptive

observer form for nonlinear systems under an additional "passivity-like" condition on the observation error. In [14] the authors study a class of many-input-many-output linear time-varying systems. The approach therein was combined in [15], with high-gain observers to construct a new adaptive observer for a single-output uniformly observable systems. More recently, in [16] the authors address the problem of the adaptive observer design for a class of nonlinear time-varying systems with parametric uncertainties in the context of synchronization of chaotic systems, by exploiting the persistency-of-excitation property of chaotic systems.

The observer-based synchronization approach that we propose in this paper is tailored for a class of Lur'e systems with slope restricted nonlinearities, uncertain parameters (the regressor functions are also assumed to be slope-restricted) and delayed outputs. Our synchronization method is based on adaptive observer theory. The uncertain parameters are assumed to be piece-wise constant. The transmission delay is assumed to be known and to be defined by a bounded time function. To the best of our knowledge, adaptive observer-based synchronization in such complete scenario is not solved yet. Indeed, the synchronization problem was only solved in the case of the existence of either uncertain parameters [16] or transmission time delays [4], [6] but not simultaneously. Based on Lyapunov-Krasovskii's approach, we show that for sufficiently small values of the upper-bound on time-delay, the reconstruction of both, states and uncertain parameters, is ensured under a condition of persistency of excitation and after solving a convex optimization problem.

Assuming a small upper-bound on the time-delay is a classical technical limitation encountered in the literature, however in many control systems applications time-delays may take large values. To overcome this restriction, we extend the proposed synchronization approach to the case of long constant time delays by introducing a cascade observers-based synchronization scheme. For illustration, theoretical results are evaluated solving a problem of master-slave synchronization of uncertain Duffing systems subject to transmission time-delays.

The paper is organized as follows. In the following section, we motivate and formulate the problem. In section II, we introduce some notations and recall some definitions and results on the stability of time-delay systems. In Section III, we motivate and formulate the problem. In Section IV, we present the observer-based synchronization approach in the case of short transmission time-delays accompanied with illustrative numerical example, we present the stability analysis and prove both the master-slave synchronization and the parametric convergence. In Section V, we investigate the case of long constant time-delays and present a cascade observers-based synchronization scheme followed by illustrative numerical simulations. Finally, some concluding remarks are given in Section VI.

II. PRELIMINARIES

Notation. $|\cdot|$ denotes the absolute value for scalars, the Euclidean norm for vectors and the induced norm for matrices. $\mathcal{C} = C([-r,0],\mathbb{R}^n)$ denotes the set of continuous functions in the interval $[-r,0]$. For $\psi \in \mathcal{C}$, $|\psi|_c := \max_{-r\leq\theta\leq 0}|\psi(\theta)|$. I represents the identity matrix. The smallest and largest eigenvalues of P are denoted by p_m and p_M respectively.

Stability of Time-delay systems.

Consider the retarded functional differential equation

$$\dot{x}(t) = f(t, x_t), \qquad x_{t_0} = \phi \in \mathcal{C}; \qquad x_t \in \mathcal{C} \tag{1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ is continuous, $f(t,0) = 0$ for all $t \in \mathbb{R}$ and

$$x_t(\vartheta) = x(t+\vartheta), \qquad -r \leq \vartheta \leq 0.$$

Definition 1: (Asymptotic Stability) For a time-delay system described by (1), the trivial solution $x(t) \equiv 0$ is said to be stable if for any given $\tau \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_\tau|_c \leq \delta$ implies $|x_\tau|_c \leq \varepsilon$ for all $t \geq \tau$. It is said to be asymptotically stable if it is stable, and for any given $\tau \in \mathbb{R}$ and $\varepsilon > 0$, there exists, in addition, $\delta_a > 0$ such that $|x_\tau|_c \leq \delta_a$ implies $\lim_{t\to\infty} x(t) = 0$. It is globally asymptotically stable if it is asymptotically stable and δ_a can be made arbitrarily large.

Definition 2: (Exponential Stability)

The trivial solution of System (1) is exponentially stable if there exist $\alpha > 0$ and $\gamma > 0$ such that for any solution $x(\cdot, t_0, \phi)$, $\phi \in \mathcal{C}$:

$$|x(t, t_0, \phi)| \leq \gamma\,|\phi|_c \exp(-\alpha(t - t_0)), \forall t \geq t_0.$$

Theorem 1: (*Lyapunov-Krasovskii Theorem*)

Suppose $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ in (1) maps $\mathbb{R}\times$ (*bounded sets in* $\mathcal{C}$) into bounded sets in $\mathbb{R}^n$, and that u, v, $w : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous nondecreasing functions. In addition, $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$. If there exists a continuously differentiable functional $V : \mathbb{R}\times\mathcal{C} \to \mathbb{R}$ such that

$$u(|x(t)|) \leq V(t, x_t) \leq v(|x_t|_c)$$

and

$$\dot{V}(t, x_t) \leq -w(|x(t)|),$$

then the null solution of System (1) is stable. If $w(s) > 0$ for $s > 0$, then it is asymptotically stable. If, in addition, $\lim_{s\to\infty} u(s) = +\infty$, then it is globally asymptotically stable.

For Definition 1 and Theorem 1, see [17] and for Definition 2, see [18].

III. Context and problem statement

We consider a master-slave synchronization scheme for systems with time-varying transmission delay $h(t)$. The master system is a (chaotic) Lur'e system given by

$$\dot{x} = Ax + Ff(Hx,u) + B\sum_{k=1}^{q}\Psi^k(R_kx,u)m_k \tag{2a}$$

$$y = Cx(t-h(t)) \tag{2b}$$

where f is continuous and satisfies a "sector condition" –see below. The state is denoted by $x \in \mathbb{R}^n$, the outputs by $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^l$ denotes an exciting time-varying input signal. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times s}$, $F \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $H = \begin{bmatrix} H_1\, H_2 \cdots H_m \end{bmatrix}^\top \in \mathbb{R}^{m\times n}$ and $R_k = \begin{bmatrix} R_{k1} \cdots R_{ks} \end{bmatrix}^\top \in \mathbb{R}^{s\times n}$ are constant matrices; where H_i is the ith row of H; for $i \in \{1\dots m\}$ and R_{k_i} is the ith row of R_k; for $i \in \{1,\dots,s\}$. The function $f : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m$ is defined by $f(Hx,u) = \begin{bmatrix} f_1(H_1x,u) \cdots f_m(H_mx,u) \end{bmatrix}^\top$ and for each $k \in \{1\dots q\}$ the function $\Psi^k : \mathbb{R}^s \times \mathbb{R}^l \to \mathbb{R}^s$ is defined by $\Psi^k(R_kx,u) = \begin{bmatrix} \Psi_1^k(R_{k1}x,u) \cdots \Psi_s^k(R_{ks}x,u) \end{bmatrix}^\top$. The vector $m := [m_1 \cdots m_q]^\top$ denotes the uncertain parameters vector, where each $m_k : \mathbb{R}_+ \to \mathbb{R}$ is a piece-wise-constant function hence, there exists $\mu_m > 0$ such that

$$\sup_{t\geq 0}|m_k(t)| \leq \mu_m$$

and $\dot{m}(t) = 0$ a.e. The transmission delay $h(t)$ is assumed to be known and it satisfies $0 \leq h(t) \leq h_m$.

The objective is to design a slave dynamical system

$$\begin{aligned} \dot{z} &= \Phi(t,y,z,z(t-h(t)),\hat{m}) \\ \dot{\hat{m}} &= \Psi(t,y,z,z(t-h(t))) \end{aligned}$$

which synchronizes with the master system (2) and reconstructs the uncertain parameters $m(t)$ despite the transmission delay $h(t)$ that is,

$$\lim_{t\to\infty}|x(t)-z(t)| = 0, \qquad \lim_{t\to\infty}|m(t)-\hat{m}(t)| = 0.$$

Remark 1: In the literature of master-slave synchronization with transmission delays it is commonly assumed that the transmission time-delay is known, see *e.g.*, [2], [4], [5]. The same hypothesis is often employed as well in the literature of observer design for systems with delayed outputs, see *e.g.*, [12].

We also make the following hypotheses on the system's dynamics.

Assumption 1: There exists $b > 0$ such that for all $u \in \mathbb{R}^l$, all $\xi_1, \xi_2 \in \mathbb{R}^m$ such that $\xi_1 \neq \xi_2$, and all $i \in \{1\dots m\}$,

$$0 \leq \frac{f_i(\xi_1,u) - f_i(\xi_2,u)}{\xi_1 - \xi_2} \leq b. \tag{3}$$

Assumption 2: For each $k \in \{1\dots q\}$ there exists $\bar{b}_k > 0$ such that for all $u \in \mathbb{R}^l$, all $\zeta_1, \zeta_2 \in \mathbb{R}^s$ such that $\zeta_1 \neq \zeta_2$ and all $i \in \{1\dots s\}$,

$$0 \leq \frac{\Psi_i^k(\zeta_1,u) - \Psi_i^k(\zeta_2,u)}{\zeta_1 - \zeta_2} \leq \bar{b}_k.$$

Furthermore, there exists $\mu_\psi > 0$ such that for any bounded u, any $\zeta \in \mathbb{R}^s$ and $k \in \{1\dots q\}$

$$\left|\Psi^k(\zeta,u)\right| \leq \mu_\psi.$$

In the context of Lur'e chaotic systems, there is little loss of generality in the previous Assumptions 1 and 2 since the solutions of chaotic systems are uniformly bounded they remain in a compact, say $\mathcal{X}$. Then, if f_i is continuously differentiable which is the case of Lur'e systems, one may apply the mean value theorem to see that f_i may be replaced by $\tilde{f}_i : \mathcal{X}\times\mathbb{R}^l \to \mathbb{R}^m$ where $\tilde{f}_i(\xi,u) := F_i(\xi,u)\xi + f_i(0)$ and, for each u,

$$F_i(\xi,u) := \int_0^1 \frac{\partial f_i}{\partial \xi}(s\xi,u)ds$$

is bounded for all $\xi \in \mathcal{X}$. Note that the solutions of the system with $\tilde{f}_i$ in place of f_i are the same in either case. A similar argument applies to Ψ^k.

Remark 2: If the boundedness assumption on Ψ^k does not hold but the solutions $x(t)$ of the master system (2) lay in a compact set X, one can use a smooth saturation function $\sigma : \mathbb{R}^s \to X$, such that $\sigma(x) = x$ for all $x \in X$ and $\sigma(x) = \mathrm{sgn}(x)$ for all $x \notin X$, to construct a new function $\tilde{\Psi}^k(x,u) = \Psi^k(\sigma(x),u)$ which maps any $x \in \mathbb{R}^s$ into a bounded set. Then, the solutions of (2) with the unbounded Ψ^k coincide with the solutions of the same equations with the bounded function $\tilde{\Psi}^k$.

It may be argued that it is generally conservative to assume boundedness of solutions, even though it is common practice in the literature of observer design. However, boundedness of solutions is characteristic of chaotic $oscillators$ which are the subject of study in this paper, notably *nominal* chaotic Lur'e systems with dynamics given by

$$\dot{x}(t) = Ax(t) + Ff(Hx(t),u(t)). \tag{4}$$

Such systems cover in particular, autonomous systems where chaos is induced via an external signal, as in the case of the *Duffing* oscillator

$$\dot{x}_1 = x_2 \tag{5a}$$
$$\dot{x}_2 = -a_2 x_1 + \theta x_1^3 + b_2 cos(\omega_2 t) \tag{5b}$$

which has the form (4). The nonlinear term $f_0(x) = \theta x_1^3$ is continuously differentiable and the trajectories of System (5) are globally bounded in chaotic regime hence, f_0 may be replaced by $\tilde{f}_0(t,x) = \theta x_1(t)^2 x_1$ which satisfies the slope restriction (3) uniformly in t with b depending on the (uniform) bound on $|x_1(t)|$. It is clear that the solutions of (5) are the same as those of the system with θx_1^3 replaced by $\theta x_1(t)^2 x_1$. In this case, the input signal is $u(t) = cos(\omega_2 t)$. Other examples of chaotic systems of Lur'e type covered by the model (2) include the *generalized n-scroll Chua system*, the *coupled Chua circuits*, the *van der Pol oscillator*, *etc.*

IV. Observer-based synchronization under transmission time-delays

A. Design of the slave system

The problem of observer design for nonlinear systems with slope restricted nonlinearities in the case of non delayed outputs was investigated in the last decade and particularly by Arcak and Kokotović [19], for systems with slope-restricted nonlinearities. Their method was extended in [20] where the authors present a unified H_∞ adaptive observer for a class of systems with both Lipschitz and monotone nonlinearities. On the other hand, the problem of joint state and unknown parameters estimation using adaptive observers in the case of non delayed outputs was investigated in the literature –see [13], [14], [15]. The recent reference [16] investigates this problem in the context of synchronization of chaotic systems with parametric uncertainties, but without delay propagation.

Motivated by nonlinear observer design for systems with slope restricted nonlinearities and adaptive-observers design, the slave system that we propose for System (2) ensures both the objectives of master-slave synchronization and parametric convergence (despite the presence of transmission time-delays). It has the following structure:

$$\begin{aligned}
\dot{z} &= Az + Ff(Hz,u) + B\sum_{k=1}^{q}\Psi^k(R_k z,u)\hat{m}_k + K(y - Cz(t-h(t))) \\
\dot{\hat{m}}_k &= \rho\Upsilon_k^{-1}\Psi^k(R_k z,u)^T M(y - Cz(t-h(t))) \\
\dot{\Upsilon}_k &= -\alpha\Upsilon_k + \Psi^k(R_k z,u)^\top B^\top B\Psi^k(R_k z,u) \qquad \Upsilon_k(0) > 0, \qquad k = 1\dots q
\end{aligned} \tag{6a}$$

where z and $\hat{m}_k$ denote respectively the estimated states and the estimated the uncertain parameters vectors; ρ and α are positive constants; K and M are two design matrices to be determined later.

We define the synchronization error $e := x - z$ and the adaptation error $\tilde{m} := (\tilde{m}_1, \dots, \tilde{m}_q)$, where $\tilde{m}_k := m_k - \hat{m}_k$, for $k = 1\dots q$. Since $\dot{m}_k(t) = 0$ for almost all t, the error dynamics is described by the following equations

$$\begin{aligned}
\dot{e} &= Ae - KCe(t-h(t)) + F\eta(He,x,u) + B\sum_{k=1}^{s}(\bar{\eta}^k(R_k e,x,u)m_k + \Psi^k(R_k z,u)\tilde{m}_k) \\
\dot{\tilde{m}}_k &= -\rho\Upsilon_k^{-1}\Psi^k(R_k z,u)^T MCe(t-h(t)), \text{ a.e.}
\end{aligned} \tag{7a}$$

where $\eta(He,x,u) = \left[\eta_1(H_1 e,x,u) \cdots \eta_m(H_m e,x,u)\right]^\top$ where $\eta_i(H_i e,x,u) = f_i(H_i x,u) - f_i(H_i x - H_i e,u)$. Similarly, $\bar{\eta}^k(R_k e,x,u) = \left[\bar{\eta}_1^k(R_1 e,x,u) \cdots \bar{\eta}_s^k(R_s e,x,u)\right]^\top$ where $\bar{\eta}_i^k(R_{ki} e,x,u) = \Psi^k(R_{ki} x,u) - \Psi^k(R_{ki} x - R_{ki} e,u)$. The error system (7) is a set of functional differential equations with initial states

$$e(\vartheta) = \phi_s(\vartheta), \tag{8}$$
$$\tilde{m}(\vartheta) = \phi_a(\vartheta), \qquad \forall\vartheta \in [-h_m, 0]. \tag{9}$$

where $\phi_s(\cdot)$ and $\phi_s(\cdot)$ are continuous functions defined on the interval $[-h_m, 0]$.

To establish master-slave synchronization, the gain matrix K is designed so that $A - KC$ is Hurwitz and the design matrix M should be selected to cancel some residual terms generated by the nonlinear regressor Ψ^k –*cf.* Section IV-C. To that end, we introduce the following hypothesis

Assumption 3: There exist a symmetric positive definite matrix P, diagonal positive matrices $D, \bar{D}_1, \dots, \bar{D}_q$; regular matrices M and K of appropriate dimensions and a positive constant ε such that

$$\mathcal{S} = \begin{bmatrix} -Q + \varepsilon I & PF + bH^\top D & \Lambda \\ F^\top P + bDH & -2D & 0 \\ \Lambda^\top & 0 & -2\bar{D} \end{bmatrix} \leq 0 \tag{10}$$

$$B^\top P = MC \tag{11}$$

where $Q = -[(A-KC)^T P + P(A-KC)]$, $\Lambda = (PB + \bar{b}_1 R_1^\top \bar{D}_1, \cdots, PB + \bar{b}_q R_q^\top \bar{D}_q)$ and

$$\bar{D} = \begin{bmatrix} \bar{D}_1 & 0 & \dots & 0 \\ 0 & \bar{D}_2 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & \dots & \bar{D}_q \end{bmatrix}$$

Similar forms to the matrix inequality (10) may be found in the literature of master-slave synchronization of Lur'e systems under transmission delays, and particularly in feedback-control based synchronization methods exploiting the sector and slope-restricted nonlinearities of Lur'e systems – see for instance [1], [2], [3], [4]. On the other hand, it is to be noticed that a necessary condition for the linear matrix Equation (11) to be solved is the *relative degree one assumption i.e.*, $\text{Rank}(CB) = \text{Rank}(B)$, which is often used for the design of unknown input observers (UIO) [21] and sliding mode observers (SMO) [22] in the case of nonlinear systems with unknown inputs and also in the literature of adaptive observers for systems with unknown parameters –see [23].

In order to find the matrices P, D, $\bar{D}_1, \dots, \bar{D}_q$, M and K in Assumption 3, one may consider the following convex optimization problem: to minimize ς subject to

$$P > 0, \quad D > 0, \quad \varepsilon > 0,$$

$$\begin{bmatrix} \Xi & PF + bH^\top D & \Lambda \\ F^\top P + bDH & -2D & 0 \\ \Lambda^\top & 0 & -2\bar{D} \end{bmatrix} \leq 0 \qquad \begin{bmatrix} \varsigma I & B^\top P - MC \\ PB - C^\top M^\top & \varsigma I \end{bmatrix} \leq 0,$$

where $\Xi = PA + A^\top P + WC_1 + C^\top W^\top + \varepsilon I$. The solution to this problem yields the minimum $\varsigma = 0$, ε, P, D, $\bar{D}_1, \dots, \bar{D}_q$, M and W such that $K = -P^{-1}W$ satisfies (10). The problem may be solved via convex optimization algorithms such as the LMI solver `feasp`, which is well developed in Matlab LMI Toolbox. Another alternative is to use the Matlab-based package `cvx` for convex optimization that in particular handles SPDs and LMIs in a convenient way.

Convergence of the estimation errors relies upon the so-called persistency of excitation property, on the nonlinear regressor Ψ^k *i.e.*,

Assumption 4: We assume that the input signal u is such that for any trajectory $z(t)$ of System (6), $\forall k = 1 \dots q$, there exist $\mu_k, T_k > 0$, such that $\forall\, t \geq 0$

$$\int_t^{t+T_k} \Psi^k(R_k z(s), u(s))^\top B^\top B \Psi^k(R_k z(s), u(s)) ds \ \geq \mu_k. \tag{12}$$

In words, it is required that the input signal $u(t)$ is sufficiently "rich". Persistency of excitation is commonly used in several forms accordingly to the considered class of systems –see for instance [14], [13], [16]. It is used to ensure the positivity of the solution $\Upsilon_k(t)$ of Equation (6a) – *cf.* Lemma 1 in Section IV-C. In turn, the positivity of $\Upsilon_k(t)$ ensures the parametric convergence as it will be shown in section IV-C. The condition is usually verified numerically, indeed, establishing conditions on the inputs u such that the persistency of excitation (12) is satisfied is still an open problem. A similar assumption is used in [24] for the design of adaptive observers for a class of uniformly observable nonlinear systems with nonlinear parameterizations.

The following statement provides conditions for master-slave synchronization and parametric convergence. The detailed proof is given in Section IV-C.

Theorem 2: Consider the master system (2) and the slave system (6) under Assumptions 1–4. Then, there exist h_m, K, α and ρ such that the null solution $(e, \tilde{m}) = (0, 0)$ of the error system (7) is globally asymptotically stable for all $h(t) \in (0, h_m]$.

B. Example 1: Master-slave synchronization of Duffing oscillators subject to parametric uncertainties and time-varying transmission delays

We illustrate the performance of our synchronization approach through a master-slave synchronization case-study. The master system is a chaotic Duffing system subject to parametric uncertainties. The output y is corrupted by a known and time-varying transmission delay $h(t) \leq h_m = 0.04s$. Then, the dynamics of the master system is given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -0.4x_2 - 1.1x_1 - \Theta x_1^3 + \cos(1.8t) \\ y(t) &= x_1(t-h(t)) + x_2(t-h(t)). \end{aligned} \tag{13}$$

which is of the form (2) with $f(Hx, u) = u$, $u(t) = \cos(1.8t)$, $k = 1$, $R_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\Psi^1(R_1 x, u) = x_1^3$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 \\ 0.4 & -1.1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, F = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

We assume that the uncertain parameter to be estimated is the unknown parameter $\Theta = 1$. The initial state of the system (13) is set to $x(s) = [0, 0]^\top$, $\forall s \in [-0.04, 0]$. Under these conditions, the attractor of the master system (13) is depicted in Figure 1. Figure 2 illustrates the evolution of the time delay.

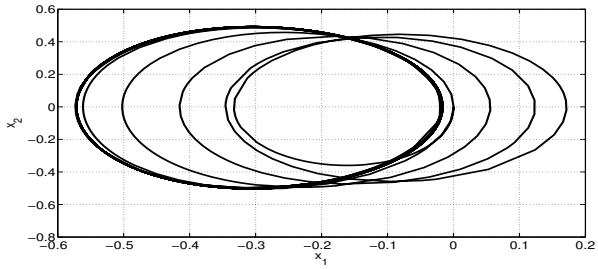

Fig. 1. The attractor of Duffing system in the (x_1, x_2) plane

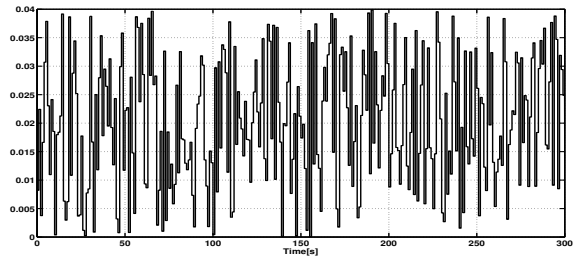

Fig. 2. Evolution of the time delay function $h(t)$

We remark that the nonlinear functions f and Ψ^1 are continuously differentiable therefore, as it is discussed in Section II, we may exploit the boundedness of the solutions of the master system to show that f and Ψ^1 satisfy Assumptions 1 and 2. More precisely, we can apply saturation functions to obtain a globally bounded counter-part function of Ψ^1. To that end, we consider the compact set $\Omega = \{x \in \mathbb{R}^2 : |x_i| \leq \omega_i,\ 1 \leq i \leq 2\}$ with $\omega_1 = \omega_2 = 0.6$. We deduce from Figure 1, that Ω contains strictly the attractor of the system (13). Then let $\sigma(\cdot)$ be a saturation function defined as follows:

$$\sigma(x_1) = \begin{cases} \omega_1 & \text{if } x_1 > \omega_1 \\ x_1 & \text{if } -\omega_1 \leq x_1 \leq \omega_1 \\ -\omega_1 & \text{if } x_1 < -\omega_1. \end{cases}$$

In this case, for each $x \in \Omega$, $\tilde{\Psi}^1(x) = \Psi^1(\sigma(x_1)) = \sigma(x_1)^3$, hence for the design of the observer, we can use $\sigma(z_1)^3$ which is globally bounded, instead of z_1^3.

The slave system is given by (6). We solve the optimization problem described in Section IV using the Matlab-based package `cvx` to obtain:

$$D = \bar{D}_1 = 2.6464,\ P = \begin{bmatrix} 12.1437 & 1.5679 \\ 1.5679 & 1.5679 \end{bmatrix},\ \varepsilon = 5.9542.\ K = \begin{bmatrix} 0.0596 \\ 3.4896 \end{bmatrix},\ M = -1.5679,\ \rho = 10,\ \alpha = 0.5.$$

The state x of System (13) is initialized at $x_0 = [0, 0]^\top$, $\forall s \in [-0.04, 0]$. The observer is started with initial conditions $z(s) = [1, 2]^\top$, $\forall s \in [-0.04, 0]$, $\hat{m} = 0.6$, $\Upsilon = 0.2$.

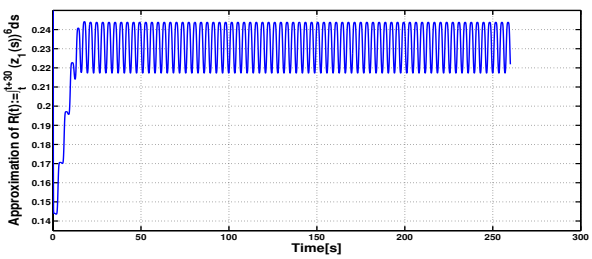

Fig. 3. Numerical approximation of $R(t) := \int_t^{t+30} (z_1(s))^6 ds$ for $T_1 = 30s$, $\mu_1 := \min_{t\in[0,300-T_1]}\{R(t)\} = 0.1435$ for the verification of the persistency of excitation condition.

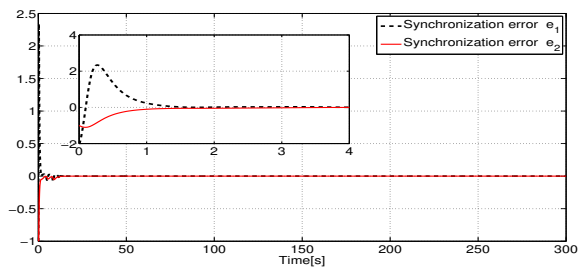

Fig. 4. Synchronization errors $e_1 = x_1 - \hat{x}_1$ and $e_2 = x_2 - \hat{x}_2$ in the presence of a time-varying transmission delay of upper-bound $h_m = 0.04s$

It is to be noticed that the persistency of excitation condition 4 on $\Psi^1(z(t), u(t)) = z_1(t)^3$ may be verified numerically. First, we fix arbitrarily $T_k = T_1 = 30s$, then we compute the integral term $R(t) := \int_t^{t+30} (z_1(s))^6 ds$, for $t \in [0, 300 - T_1]$ with a step of $0.1s$, by using the Matlab function `trapz` which allows to compute an approximation of the integral via the trapezoidal method. Hence, an approximation of the integral term $R(t)$ is obtained as illustrated in Figure 3 from which we deduce an estimate of the persistency of excitation bound $\mu_1 := \min_{t\in[0,300-T_1]}\{R(t)\} = 0.1435$.

The simulation results are as follows. Figure 4 illustrates the synchronization between the master and the slave systems and figure 5 illustrates the parametric convergence based on the proposed observer.

We have also performed additional tests to verify the behavior of the proposed synchronization scheme in the presence of noise in the communication channel. That is, we assume that the output signal $y(t)$ is affected by an additive zero mean Gaussian noise $\bar{\omega}(t)$ of variance equal to one. In order to quantify the rate between the amplitude of the output signal and the channel noise, we measure the *Signal to Noise Ratio* $SNR(y, \bar{\omega}) = 20 \log \frac{y(t)}{\bar{\omega}(t)}$, expressed in dB. Figure 6 illustrates the effect of the channel noise on the synchronization errors in the case of two different values of *SNR* equal to 20dB and 6dB, respectively.

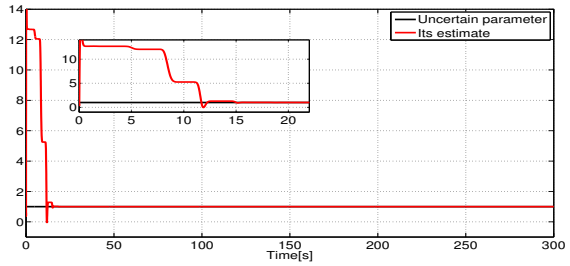

Fig. 5. Parametric convergence in presence of a time-varying transmission delay of upper-bound $h_m = 0.04s$

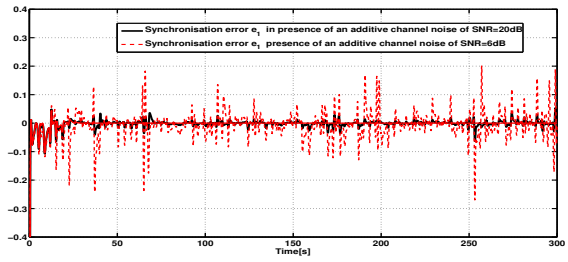

Fig. 6. Effect of the additive channel noise on the synchronization error e_1

C. Proof of Theorem 2

The demonstration of Theorem 2 relies on Lyapunov-Krasovskii's method. First, we apply the Leibniz-Newton formula to the synchronization error $e(t)$ to obtain

$$e(t) - e(t-h(t)) \quad = \quad \int_{-h(t)}^{0} \dot{e}(t+\vartheta)d\vartheta. \tag{14}$$

Hence, The dynamics of $e(t)$ can be rewritten as

$$\dot{e} = (A-KC)e + F\eta(He,x,u) + KC\int_{-h(t)}^{0}\dot{e}(t+\vartheta)d\vartheta + B(\sum_{k=1}^{q}\bar{\eta}^k(R_k e,x,u)m_k + \sum_{k=1}^{q}\Psi^k(R_k z,u)\tilde{m}_k). \tag{15}$$

Let $\omega(t) = (e(t), \tilde{m}(t))^\top$ and

$$\omega_t(\vartheta) = \omega(t+\vartheta), \qquad -h_m \le \vartheta \le 0.$$

and consider the Lyapunov-Krasovskii functional

$$\begin{aligned} V(t,\omega_t) &= V_1(e(t)) + V_2(\omega_t) + V_3(t,\tilde{m}(t)) \\ V_1(e) &= e^\top Pe, \\ V_2(\omega_t) &= \int_{-h_m}^{0}(\vartheta+h_m)|\dot{e}(t+\vartheta)|^2 d\vartheta, \\ V_3(t,\tilde{m}) &= \sum_{k=1}^{q}\rho^{-1}\Upsilon_k\tilde{m}_k^2, \end{aligned} \tag{16}$$

where $\Upsilon_k(t)$ is defined by Equation (6a) and satisfies the following.

Lemma 1: Let $\upsilon_m := \min_{k\in\{1\dots q\}}\{\mu_k e^{-\alpha T_k}\}$. If $B\Psi^k$ satisfies Assumption 4 then

$$\Upsilon_k(t) \ge \upsilon_m \quad \forall t \ge \max_{k\in\{1\dots q\}}\{T_k\}. \tag{17}$$

The proof of Lemma 1 is omitted due to space constraints; it follows by integrating 0 to $t+T_k$,

$$\frac{d}{dt}(e^{\alpha t}\Upsilon_k(t)) = e^{\alpha t}(\dot{\Upsilon}_k(t) + \alpha\Upsilon_k(t))$$

then, multiplying on both sides by $e^{-\alpha(t+T_k)}$ and straightforward computations which lead to $\Upsilon_k(t) \ge \upsilon_m$ for all $t \ge \max\{T_k, k=1\dots q\}$, where $\upsilon_m = \min\{\mu_k e^{-\alpha T_k}, k=1\dots q\} > 0$.

Next, we introduce the following symbols to avoid a cumbersome notation,

$$\begin{array}{lllllll} \Gamma(t) & := & \int_{-h(t)}^{0}\dot{e}(t+\vartheta)d\vartheta & ; & \beta_a & := & |A|\,; \\ \beta_b & := & |B| & ; & \beta_c & := & |KC|\,. \end{array}$$

Then, the total derivative of V_1 along the trajectories of (7) yields

$$\begin{aligned} \dot{V}_1 = {} & e^\top[(A-KC)^\top P + P(A-KC)]e + 2e^\top PF\eta(He,x,u) + 2e^\top PB\sum_{k=1}^{q}\bar{\eta}^k(R_k e,x,u)m_k \\ & + 2e^\top PKC\int_{-h(t)}^{0}\dot{e}(t+\vartheta)d\vartheta + 2e^\top PB\sum_{k=1}^{q}\Psi^k(R_k z,u)\tilde{m}_k. \end{aligned} \tag{18a}$$

In view of Assumption 1, we have

$$0 \leq \frac{\eta_i(H_i e, x, u)}{H_i e} \leq b$$

and multiplying on both sides of the latter by $\eta_i(H_i e, x, u) H_i e$, we obtain for $i = 1 \dots m$, $\forall t \geq 0$, $\forall e, x, u$,

$$\eta_i(H_i e, x, u)[\eta_i(H_i e, x, u) - b H_i e] \leq 0$$

which means that $\eta(He, x, u)$ belongs to the sector [0,b]. Next, let $D = diag(d_1, d_2, \dots, d_m)$ with $d_i > 0$, $\forall i = 1 \dots m$; after some manipulations one obtains

$$-\eta^\top(He, x, u) D \eta(He, x, u) + b\eta(He, x, u) DHe \geq 0. \tag{19}$$

Proceeding with the same reasoning for $\Psi^k(R_k x, u)$, we obtain

$$-\bar{\eta}^{k\top}(R_k e, x, u) \bar{D}_k \bar{\eta}^k(R_k e, x, u) + \bar{b}^k \bar{\eta}^k(R_k e, x, u) \bar{D}_k R_k e \geq 0. \tag{20}$$

for all $k \in \{1 \dots q\}$ and $\bar{D}_k = \text{diag}[\bar{d}_{k1} \cdots \bar{d}_{ks}]$ with $\bar{d}_{ki} > 0$, for all $i \in \{1 \dots s\}$. Then, we add the left hand side terms of Inequalities (19) and (20) to the right hand side of Inequality (18a) to obtain

$$\begin{aligned}
\dot{V}_1 \quad \leq \quad & e^\top[(A - KC)^\top P + P(A - KC)]e + 2b\eta(He, x, u) DHe + 2e^\top PKC \int_{-h(t)}^{0} \dot{e}(t + \vartheta) d\vartheta \\
& -2 \sum_{k=1}^{q} \bar{\eta}^{k\top}(R_k e, x, u) m_k \bar{D}_k \bar{\eta}^k(R_k e, x, u) \\
& + \sum_{k=1}^{q} \bar{b}_k \bar{\eta}^k(R_k e, x, u) m_k \bar{D}_k R_k e + 2e^\top PB \sum_{k=1}^{q} \Psi^k(R_k z, u) \tilde{m}_k + 2e^\top PF\eta(He, x, u) \\
& -2\eta^\top(He, x, u) D\eta(He, x, u) + 2e^\top PB \sum_{k=1}^{q} \bar{\eta}^k(R_k e, x, u) m_k.
\end{aligned}$$

Let us define

$$\zeta := [e, \eta(He, x, u), \bar{\eta}^1(R_1 e, x, u) m_1, \dots, \bar{\eta}^q(R_q e, x, u) m_q]^\top$$

then,

$$\dot{V}_1 \leq \zeta^\top \mathcal{S} \zeta + 2e^\top PKC \textstyle\int_{-h(t)}^{0} \dot{e}(t + \vartheta) d\vartheta - \varepsilon |e|^2 + 2e^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z, u) \tilde{m}_k,$$

where the matrix $\mathcal{S} \leq 0$ is defined in Assumption 3 and where we have also used (11). Applying Young's inequality $\left|2c^\top d\right| \leq \gamma c^\top c + \frac{1}{\gamma} d^\top d$ to the term $2e(t)^\top PKC\dot{e}(t+\vartheta)$ with $c^\top = e(t)^\top PKC$, $d = \dot{e}(t+\vartheta)$ and $\gamma = 2$, and integrating from $\vartheta = -h(t)$ to $\vartheta = 0$, we obtain

$$\begin{aligned}
2e(t)^\top PKC \int_{-h(t)}^{0} \dot{e}(t + \vartheta) d\vartheta \leq \; & 2he(t)^\top C^\top K^\top P^\top PKCe(t) + \frac{1}{2} \int_{-h(t)}^{0} |\dot{e}(t + \vartheta)|^2 d\vartheta \\
\leq \; & 2h_m \beta_c^2 p_M^2 |e(t)|^2 + \frac{1}{2} \int_{-h_m}^{0} |\dot{e}(t + \vartheta)|^2 d\vartheta
\end{aligned}$$

hence,

$$\dot{V}_1 \quad \leq \quad -(\frac{\varepsilon}{p_M} + 2h_m \beta_c^2 p_M^2 p_m^{-1}) V_1 + \frac{1}{2} \int_{-h_m}^{0} |\dot{e}(t + \vartheta)|^2 d\vartheta + 2e^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z, u) \tilde{m}_k.$$

Next, the time derivative of V_3 along the trajectories of (7) yields

$$\dot{V}_3 = \rho^{-1} \sum_{k=1}^{q} (\dot{\tilde{m}}_k^\top \Upsilon_k \tilde{m}_k + \tilde{m}_k^\top \Upsilon_k \dot{\tilde{m}}_k + \tilde{m}_k^\top \dot{\Upsilon}_k \tilde{m}_k).$$

and after (6a) and (7a), one obtains

$$\dot{V}_3 \quad = \quad -2e(t - h(t))^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z, u) \tilde{m}_k - \alpha V_3 + \rho^{-1} \sum_{k=1}^{q} \Psi^k(R_k z, u)^\top B^\top B \Psi^k(R_k z, u) \tilde{m}_k^2.$$

Now, we apply Young's inequality and Equation (11) to obtain

$$2\Gamma^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z, u) \tilde{m}_k \leq q |\Gamma|^2 + (\mu_\psi \beta_b p_M)^2 \sum_{k=1}^{q} \tilde{m}_k^2,$$

from which together with Equation (14), we get

$$\dot{V}_3 \quad \leq \quad \upsilon_m^{-1}((\rho^{-1}+p_M^2)(\beta_b\mu_\psi)^2-\alpha)V_3 - 2e^\top C^\top M^\top \sum_{k=1}^{q}\Psi^k(R_k z,u)\tilde{m}_k + q\left|\Gamma\right|^2$$

where we have also used positivity of $\Upsilon_k(t)$.

The time derivative of V_2 is given by

$$\dot{V}_2 \quad = \quad h_m\left|\dot{e}\right|^2 - \int_{-h_m}^{0}\left|\dot{e}(t+\vartheta)\right|^2 d\vartheta. \tag{21}$$

From (15) and using the sector condition on $\eta(\cdot)$ and $\bar{\eta}^k(\cdot)$, one obtains

$$h_m\left|\dot{e}\right|^2 \quad \leq \quad h_m(\beta_a+\beta_c+b+\sum_{k=1}^{q}\bar{b}_k\mu_m)^2 p_m^{-1}V_1 + h_m\upsilon_m^{-1}(\mu_\psi\beta_b)^2V_3 + h_m\beta_c^2\left|\Gamma\right|^2,$$

so, using Jensen's inequality it follows that

$$h_m\int_{-h_m}^{0}\left|\dot{e}(t+\vartheta)\right|^2 d\vartheta \geq \left|\int_{-h(t)}^{0}\dot{e}(t+\vartheta)d\vartheta\right|^2 = \left|\Gamma\right|^2. \tag{22}$$

Using Inequalities (21), (21)–(22) and re-arranging terms, we see that the total time derivative of $V(t,\omega_t)$ along the trajectories of (7) satisfies

$$\dot{V} \quad \leq \quad (-\frac{\varepsilon}{p_M}+h_mC_e)V_1 + (q-\frac{1}{2h_m}+h_m\beta_c^2)\left|\Gamma\right|^2 + (C_m+h_m\upsilon_m^{-1}(\mu_\psi\beta_b)^2-\alpha)V_3$$

where

$$C_e \quad = \quad p_m^{-1}((\beta_a+\beta_c+b+\sum_{k=1}^{q}\bar{b}_k\mu_m)^2+2\beta_c^2p_M^2),$$

$$C_m \quad = \quad \upsilon_m{}^{-1}(\beta_b\mu_\psi)^2(\rho^{-1}+p_M^2).$$

Therefore, if h_m satisfies the following system of in-equations

$$h_mC_e - \frac{\varepsilon}{2p_M} \leq 0 \tag{23a}$$

$$C_m + h_m\upsilon_m^{-1}(\mu_\psi\beta_b)^2 - \frac{\alpha}{2} \leq 0 \tag{23b}$$

$$q - \frac{1}{2h_m} + h_m\beta_c^2 \leq 0, \tag{23c}$$

it holds that

$$\dot{V}(t,\omega_t) \leq -\frac{\varepsilon}{2p_M}V_1(e(t)) - \frac{\alpha}{2}V_3(t,\tilde{m}(t)) \qquad a.e. \tag{24}$$

Solving for h_m the system of in-equations (23a)–(23c), we deduce that (24) holds for

$$h_m \leq \min\{\pi_a,\pi_b,\pi_c\}$$

where $\pi_a = C_e^{-1}(\frac{\varepsilon}{2p_M})$, $\pi_c = \frac{-q+\sqrt{q^2+2\beta_c^2}}{2\beta_c^2}$ and $\pi_b = \upsilon_m(\mu_\psi\beta_b)^{-2}(\frac{\alpha}{2}-C_m)$.

Therefore, applying Lyapunov-Krasovskii Stability Theorem 1 we conclude that the null solution $(e,\tilde{m}) = (0,0)$ (i.e $\omega = 0$) of the error system (7) is globally asymptotically stable. ■

The following statement follows as a direct corollary.

Corollary 1: The origin of the error synchronization system under the conditions of Theorem 2 is exponentially stable.

Proof: Let θ be a positive constant which depends on $\beta_c = |KC|$ and α. Proceeding as in the proof of Theorem 2 and using the fact that

$$V_2 \leq h_m\int_{-h_m}^{0}\left|\dot{e}(t+\vartheta)\right|^2 d\vartheta,$$

we obtain

$$\begin{aligned}\dot{V}+\theta V \quad \leq \quad & (\theta-\frac{\varepsilon}{p_M}+h_mC_e)V_1 + (q+h_m\beta_c^2)\left|\Gamma\right|^2 - (\frac{1}{2}-\theta h_m)\int_{-h_m}^{0}\left|\dot{e}(t+\vartheta)\right|^2 d\vartheta \\ & +(\theta+C_m+h_m\upsilon_m^{-1}(\mu_\psi\beta_b)^2-\alpha)V_3 \quad a.e.\end{aligned}$$

Let h_m satisfy the following system of in-equations:

$$\theta + h_m C_e - \frac{\varepsilon}{2p_M} \leq 0, \tag{25a}$$

$$\theta + C_m + h_m \upsilon_m^{-1}(\mu_\psi \beta_b)^2 - \frac{\alpha}{2} \leq 0, \tag{25b}$$

$$\theta + q - \frac{1}{2h_m} + h_m \beta_c^2 \leq 0, \tag{25c}$$

$$h_m \leq \frac{1}{2\theta}, \tag{25d}$$

which holds for

$$h_m \leq \min\{\bar{\pi}_a, \bar{\pi}_b, \bar{\pi}_c, \bar{\pi}_d\} \tag{26}$$

where $\bar{\pi}_a = C_e^{-1}(\frac{\varepsilon}{2p_M} - \theta)$, $\bar{\pi}_c = \frac{-q-\theta+\sqrt{(q+\theta)^2+2\beta_c^2}}{2\beta_c^2}$, $\bar{\pi}_b = \upsilon_m(\mu_\psi\beta_b)^{-2}(\frac{\alpha}{2} - C_m - \theta)$ and $\pi_d = \frac{1}{2\theta}$. Hence, using again Inequalities (22), (25a), (25b) and (25d), it follows that

$$\dot{V} + \theta V \quad \leq \quad -\frac{\varepsilon}{2p_M} V_1 - \frac{\alpha}{2} V_3 - (\frac{1}{2h_m} - \theta - q - h_m \beta_c^2) \left|\Gamma\right|^2 \quad a.e,$$

and after Equation (25c), we obtain

$$\dot{V}(t, \omega_t) \leq -\theta V(t, \omega_t) \quad a.e.$$

Multiplying on both sides of the latter by $1/V(t, \omega_t)$ and integrating from t_0 to t, we obtain

$$V(t, \omega_t) \leq V(t_0, \omega_{t_0}) \exp(-\theta(t - t_0))$$

where $\omega_{t_0} = (\phi_s, \phi_a)^\top$ is the initial condition of the error system – *cf.* Equations (8) and (9). From Equations (16) and (17), it follows that

$$\min\{p_m, \rho^{-1}\upsilon_m\} \left|\omega(t)\right|^2 \leq V(t, \omega_t).$$

hence

$$\left|(e(t), \tilde{m}(t))^\top\right| \leq \sqrt{\frac{V(t_0, \omega_{t_0})}{\min\{p_m, \rho^{-1}\upsilon_m\}}} \exp\Big(-\frac{\theta}{2}(t - t_0)\Big).$$

We conclude that the null solution $(e, \tilde{m}) = (0, 0)$ of the error system (7) is exponentially stable. ■

V. Cascade observers-based synchronization in the case of long and constant transmission-delays

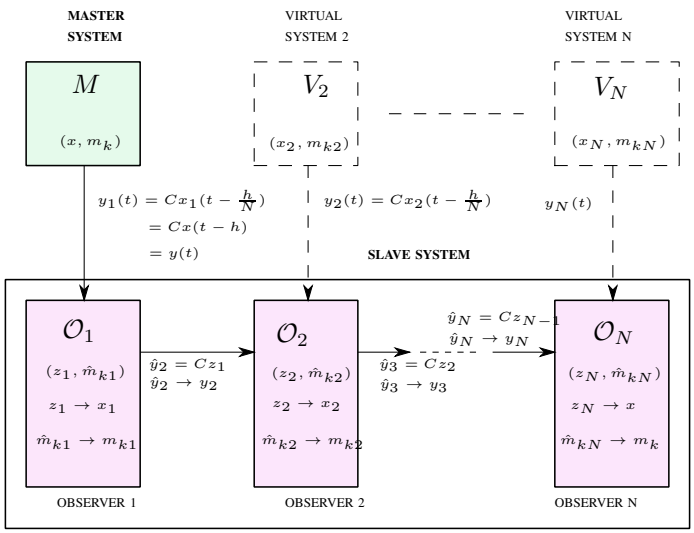

Fig. 7. Cascade observers-based synchronization scheme

Inequality (26), which imposes a limitation on the upper bound of the transmission time-delay, is a classical technical restriction encountered in the literature of master-slave synchronization with transmission delays and in the literature of state estimation under delayed outputs. To overcome this restriction, cascade observer design for systems with long time-delays

was adopted in some recent papers –see for instance [10]. Roughly speaking, the idea is to divide the long time-delay into sufficiently short time-delays which are allowed by each observer of the cascade configuration, hence the state estimation is established for longer time-delays. In this section, we extend the results obtained in the previous section IV to the particular case where the transmission delay is constant and takes large values. The synchronization scheme based on cascade observers that we propose is represented in Figure 7.

A. Design of the slave system with cascaded structure

Consider Lur'e systems (2) with constant time-delay $\bar{h}$. Following [10] we consider the delayed states $x_j = x(t-\bar{h}+j\frac{\bar{h}}{N})$ corresponding to the time instants $t_j = t-\bar{h}+j\frac{\bar{h}}{N}$, with $j = 1,\dots,N$. Then, the dynamics of the delayed states x_j are

$$\begin{aligned} \dot{x_j} &= Ax_j + Ff(Hx_j,u_j) + B\sum_{k=1}^{q}\Psi^k(R_kx_j,u_j)m_{kj} \\ y_1 &= Cx_1(t-\frac{\bar{h}}{N}) = y(t) = Cx(t-\bar{h}) \\ y_j &= Cx_{j-1} = Cx_j(t-\frac{\bar{h}}{N}), \quad j = 2\dots N \end{aligned}$$

where $u_j := u(t-\bar{h}+j\frac{\bar{h}}{N})$, $y_j := y(t-\bar{h}+j\frac{\bar{h}}{N})$, $m_{kj} := m_k(t-\bar{h}+j\frac{\bar{h}}{N})$ are respectively the delayed inputs, delayed outputs and delayed uncertain parameters at the instants t_j, for $j = 1\dots N$. Note that the dynamics of the delayed state x_N of the last virtual system V_N corresponds exactly to the dynamics of the actual state $x(t)$ of the master system (2) ($x(t) = x_N(t)$) and that the output $y_1(t)$ of the first virtual system is equal to the output $y(t)$ of the master system (2) ($y_1(t) = y(t)$).

In the chain of N estimators, the observer $\mathcal{O}_j$ estimates the delayed state x_j of the virtual system V_j in the presence of a delay $\frac{\bar{h}}{N}$ which may be reduced at will by manipulating N. Hence, the Nth observer ensures the estimation of the actual state $x(t)$ and reconstructs the uncertain parameters despite the presence of long time-delays. The cascade observer is

$$\begin{aligned} \dot{z}_j &= Az_j + Ff(Hz_j,u_j) + B\sum_{k=1}^{q}\Psi^k(R_kz_j,u_j)\hat{m}_{kj} + K(\hat{y}_j - Cz_j(t-\frac{\bar{h}}{N})) \\ \dot{\hat{m}}_{kj} &= \rho\Upsilon_{kj}^{-1}\Psi^k(R_kz_j,u_j)^TM(\hat{y}_j - Cz_j(t-\frac{\bar{h}}{N})) \qquad (27a) \\ \dot{\Upsilon}_{kj} &= -\alpha\Upsilon_{kj} + \Psi^k(R_kz_j,u_j)^\top B^\top B\Psi^k(R_kz_j,u_j) \\ & \Upsilon_{kj}(0) > 0, \quad k\in\{1\dots q\}, \quad j\in\{1\dots N\} \qquad (27b) \end{aligned}$$

where

$$\begin{aligned} \hat{y}_1(t) &= y_1(t) = y(t) = Cx(t-\bar{h}) \qquad (28a) \\ \hat{y}_j(t) &= Cz_{j-1}(t), \quad j = 2\dots N, \qquad (28b) \end{aligned}$$

$z_j(t)$ and $\hat{m}_{kj}(t)$ denote respectively the estimates of the delayed state x_j and the estimated uncertain parameters m_{kj}, at the instants t_j, for $j = 1\dots N$. We define the observation error $e_j(t) := x_j(t) - z_j(t)$ and the adaptation errors $\tilde{m}_{kj}(t) := m_{kj}(t) - \hat{m}_{kj}(t)$. Then, since $\dot{m}_{kj}(t) = 0$ almost every where, the error system is

$$\begin{aligned} \dot{e}_j &= Ae_j - KCe_j(t-\frac{\bar{h}}{N}) + F\eta(He_j,x_j,u_j) - K\bar{e}_{j-1} + B\sum_{k=1}^{s}(\bar{\eta}^k(R_ke_j,x_j,u_j)m_{kj} + \Psi^k(R_kz_j,u_j)\tilde{m}_{kj}) \\ \dot{\tilde{m}}_{kj} &= -\rho\Upsilon_{kj}^{-1}\Psi^k(R_kz_j,u_j)^TMC(e_j(t-\frac{\bar{h}}{N}) - \bar{e}_{j-1}), \end{aligned} \qquad (29a)$$

where $\bar{e}_0 = y_1 - \hat{y}_1 = 0$; $\bar{e}_{j-1} = Ce_{j-1} = y_j - \hat{y}_j = Cx_{j-1} - Cz_{j-1}$, $j = 2\dots N$, $\eta(He_j,x_j,u_j) = (\eta_1(H_1e_j,x_j,u_j),\dots,\eta_m(H_me_j,$ such that for $i = 1\dots m$, $\eta_i(H_ie_j,x_j,u_j) = f_i(H_ix_j,u_j) - f_i(H_ix - H_ie_j,u_j)$ and $\bar{\eta}^k(R_ke_j,x_j,u_j) = (\bar{\eta}_1^k(R_1e_j,x_j,u_j),\dots,\bar{\eta}_s^k(R_se$ such that for $i = 1\dots s$, $\bar{\eta}_i^k(R_{ki}e_j,x_j,u_j) = \Psi^k(R_{ki}x_j,u_j) - \Psi^k(R_{ki}x_j - R_{ki}e_j,u_j)$. Now, in view of the slope restriction (3) with $\xi_1 = H_ix_j$ and $\xi_2 = H_ix_j - H_ie_j$, we have

$$0 \le \frac{\eta_i(H_ie_j,x_j,u_j)}{H_ie_j} \le b. \qquad (30)$$

Multiplying on both sides of (30) by $\eta_i(H_ie_j,x_j,u_j)H_ie_j$, we get for $i = 1\dots m$, $\forall t\ge 0$, $\forall e_j, \forall x_j$

$$\eta_i(H_ie_j,x_j,u_j)[\eta_i(H_ie_j,x_j,u_j) - bH_ie_j] \le 0$$

which means that $\eta(H_ie_j,x_j,u_j)$ belongs to the sector $[0,\ b]$. Let $D = diag(d_1,d_2,\dots,d_m)$ where $\forall i = 1\dots m$, $d_i > 0$. After some manipulations, one can easily obtain the following relation

$$b\eta(He_j,x_j,u_j)DHe_j \ge \eta^\top(He_j,x_j,u_j)D\eta(He_j,x_j,u_j) \qquad (31)$$

Proceeding with the same reasoning for $\Psi^k(R_k x_j, u_j)$, we obtain also the relations: for $k = 1 \dots q$ and $\bar{D}_k = diag(\bar{d}_{k1}, \dots, \bar{d}_{ks})$ where $\forall i = 1 \dots s$, $\bar{d}_{ki} > 0$.

$$-\bar{\eta}^{kT}(R_k e_j, x_j, u_j)\bar{D}_k \bar{\eta}^k(R_k e_j, x_j, u_j) + \bar{b}^k \bar{\eta}^k(R_k e_j, x_j, u_j)\bar{D}_k R_k e_j \geq 0.$$

We make now the following persistency of excitation condition:

Assumption 5: We assume that the delayed inputs u_j are such that for any trajectory of the delayed state $z_j(t)$ of the cascaded observer (27), $\forall j = 1..N$, $\forall k = 1..q$, there exist $\mu_k, T_k > 0$, such that $\forall\, t \geq 0$

$$\int_t^{t+T_k} \Psi^k(R_k z_j(s), u_j(s))^\top B^\top B \Psi^k(R_k z_j(s), u_j(s)) ds \ \geq \mu_k. \tag{32}$$

Theorem 3: Consider the system (2) and the cascade observer (27)–(28). Let Assumptions 1–3 and 5 hold. Then, for any constant time-delay $\bar{h}$, there exists an integer N, the estimated state $\hat{x}_N(t)$ and the parameters $\hat{m}_{kN}(t)$ $(k = 1 \dots s)$ of the Nth observer (27) converge exponentially respectively towards the actual state $x(t)$ and the uncertain parameters $m_k(t)$ of System (2).

Proof: The Leibniz-Newton formula applied to the observation error $e_j(t)$ yields

$$e_j(t) - e_j(t - \frac{\bar{h}}{N}) = \int_{-\frac{\bar{h}}{N}}^0 \dot{e}_j(t+\vartheta) d\vartheta. \tag{33}$$

In the sequel, $k = 1 \dots s$ and $j = 1 \dots N$. The dynamics of the observation error $e(t)$ can be rewritten as follows

$$\begin{aligned} \dot{e}_j &= (A - KC)e_j + F\eta(He_j, x_j, u_j) + B(\sum_{k=1}^q \bar{\eta}^k(R_k e_j, x_j, u_j) m_{kj} + \sum_{k=1}^q \Psi^k(R_k z_j, u_j)\tilde{m}_{kj}) \\ &+ KC \int_{-\frac{\bar{h}}{N}}^0 \dot{e}_j(t+\vartheta) d\vartheta - K\bar{e}_{j-1}. \end{aligned} \tag{34}$$

Next, consider the following Lyapunov-Krasovskii functional

$$\begin{aligned} V_j(t, e_j, \tilde{m}_{kj}) &= V_{1j}(e_j) + V_{2j}(t) + V_{3j}(t, \tilde{m}_{kj}) \\ V_{1j}(e_j) &= e_j^\top P e_j \\ V_{2j}(t) &= \int_{-\frac{\bar{h}}{N}}^0 (\vartheta + \frac{\bar{h}}{N}) |\dot{e}_j(t+\vartheta)|^2 d\vartheta \\ V_{3j}(t, \tilde{m}_{kj}) &= \sum_{k=1}^q \rho^{-1} \Upsilon_{kj}(t) \tilde{m}_{kj}^2, \end{aligned}$$

where $\Upsilon_{kj}(t)$ is governed by Equation (27b). Let us define

$$\begin{aligned} \Gamma_j(t) &:= \textstyle\int_{-\frac{\bar{h}}{N}}^0 \dot{e}_j(t+\vartheta) d\vartheta \ ; \ \beta_a := |A|; \\ \beta_b &:= |B| \ ; \ \beta_c := |KC|. \end{aligned}$$

The time derivative of V_{1j} is given by

$$\begin{aligned} \dot{V}_{1j}(e_j) &= e_j^\top[(A-KC)^\top P + P(A-KC)]e_j 2e_j^\top PF\eta(He_j, x_j, u_j) - 2e_j^\top PK\bar{e}_{j-1} \\ &+ 2e_j^\top PKC \int_{-\frac{\bar{h}}{N}}^t \dot{e}_j(t+\vartheta) d\vartheta + 2e_j^\top PB \sum_{k=1}^q \Psi^k(R_k z_j, u_j)\tilde{m}_{kj} + 2e_j^\top PB \sum_{k=1}^q \bar{\eta}^k(R_k e_j, x_j, u_j) m_{kj}. \end{aligned}$$

Using (31) and (32), we obtain

$$\begin{aligned} \dot{V}_{1j}(e_j) &\leq e_j^\top[(A-KC)^\top P + P(A-KC)]e_j + 2b\eta(He_j, x_j, u_j)DHe_j \\ &+ 2e_j^\top PKC \int_{-\frac{\bar{h}}{N}}^0 \dot{e}_j(t+\vartheta) d\vartheta - 2\sum_{k=1}^q \bar{\eta}^{kT}(R_k e_j, x_j, u_j) m_{kj} \bar{D}_k \bar{\eta}^k(R_k e_j, x_j, u_j) \\ &+ \sum_{k=1}^q \bar{b}_k \bar{\eta}^k(R_k e_j, x_j, u_j) m_{kj} \bar{D}_k R_k e_j + 2e_j^\top PB \sum_{k=1}^q \Psi^k(R_k z_j, u_j)\tilde{m}_{kj} - 2e_j^\top PK\bar{e}_{j-1} \\ &- 2\eta^\top(He_j, x_j, u_j)D\eta(He_j, x_j, u_j) + 2e_j^\top PF\eta(He_j, x_j, u_j) + 2e_j^\top PB \sum_{k=1}^q \bar{\eta}^k(R_k e_j, x_j, u_j) m_k. \end{aligned}$$

Let

$$\zeta_j := [e_j,\ \eta(He_j, x_j, u_j), \bar{\eta}^1(R_1 e_j, x_j, u_j) m_{1j},\ \bar{\eta}^2(R_2 e_j, x_j, u_j) m_{2j}, \dots, \bar{\eta}^q(R_q e_j, x_j, u_j) m_{qj}]^\top$$

then using (11), one has

$$\dot{V}_{1j}(e) \leq 2e_j^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z_j, u_j)\tilde{m}_{kj} - 2e_j^\top PK\bar{e}_{j-1}, + \zeta^\top \mathcal{S}\zeta + 2e_j^\top PKC \int_{-\frac{\bar{h}}{N}}^{0} \dot{e}_j(t+\vartheta)d\vartheta - \varepsilon \left|e_j\right|^2$$

where $\mathcal{S} \leq 0$ is in Assumption 3. Applying Young's inequality $\left|2c^\top d\right| \leq \gamma c^\top c + \frac{1}{\gamma} d^\top d$ to the term "$2e_j(t)^\top PKC\dot{e}_j(s)$" with $c^\top = e_j(t)^\top PKC$, $d = \dot{e}_j(s)$ and $\gamma = 2$, and integrating from $s = t - \frac{\bar{h}}{N}$ to $s = t$, we obtain

$$\begin{aligned} 2e_j(t)^\top PKC \int_{-\frac{\bar{h}}{N}}^{0} \dot{e}_j(t+\vartheta)d\vartheta &\leq 2\frac{\bar{h}}{N} e_j(t)^\top C^\top K^\top P^2 KCe_j(t) + \frac{1}{2}\int_{-\frac{\bar{h}}{N}}^{0} \left|\dot{e}_j(t+\vartheta)\right|^2 d\vartheta \\ &\leq 2\frac{\bar{h}}{N}\beta_c^2 p_M^2 \left|e_j(t)\right|^2 + \int_{-\frac{\bar{h}}{N}}^{0} \left|\dot{e}_j(t+\vartheta)\right|^2 d\vartheta. \end{aligned}$$

In the same manner, by applying the Young's inequality to the term "$2e_j^\top PK\bar{e}_{j-1}$", one obtains

$$2e_j^\top PK\bar{e}_{j-1} \leq \bar{e}_{j-1}^\top K^\top P^2 K\bar{e}_{j-1} + e_j^\top e_j \leq (\left|K\right| p_M)^2 \left|\bar{e}_{j-1}\right|^2 + P_m^{-1} V_{1j}.$$

hence,

$$\begin{aligned} \dot{V}_{1j}(e_j) \leq & -\frac{\varepsilon}{p_M} V_{1j} + (\left|K\right| p_M)^2 \left|\bar{e}_{j-1}\right|^2 + (1 + 2\bar{h}\beta_c^2 p_M^2) P_m^{-1} V_{1j} + \frac{1}{2}\int_{-\frac{\bar{h}}{N}}^{0} \left|\dot{e}_j(t+\vartheta)\right|^2 d\vartheta \\ & + 2e_j^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z_j, u_j)\tilde{m}_{kj}. \end{aligned}$$

The time derivative of V_{3j} is given by

$$\dot{V}_{3j}(t, \tilde{m}_{kj}) = \rho^{-1} \sum_{k=1}^{q} (\dot{\tilde{m}}_{kj}^\top \Upsilon_{kj} \tilde{m}_{kj} + \tilde{m}_{kj}^\top \Upsilon_{kj} \dot{\tilde{m}}_{kj} + \tilde{m}_{kj}^\top \dot{\Upsilon}_{kj} \tilde{m}_{kj}).$$

After (27b) and (29a) and (33), one obtains

$$\begin{aligned} \dot{V}_{3j}(t, \tilde{m}_{kj}) = & -2e_j(t - \frac{\bar{h}}{N})^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z_j, u_j)\tilde{m}_{kj} + \rho^{-1} \sum_{k=1}^{q} \Psi^k(R_k z_j, u_j)^\top B^\top B \Psi^k(R_k z_j, u_j)\tilde{m}_{kj}^2 \\ & - 2\bar{e}_{j-1}^T M^\top \sum_{k=1}^{q} \Psi^k(R_k z_j, u_j)\tilde{m}_{kj} - \alpha V_{3j}. \end{aligned}$$

Applying Young's inequality and Equation (11), we obtain

$$\begin{aligned} 2\Gamma_j^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z_j, u_j)\tilde{m}_{kj} &\leq q\left|\Gamma_j\right|^2 + (\mu_\psi \beta_b p_M)^2 \sum_{k=1}^{q} \tilde{m}_{kj}^2, \\ 2\bar{e}_{j-1}^T M^\top \sum_{k=1}^{q} \Psi^k(R_k z_j, u_j)\tilde{m}_{kj} &\leq q(\mu_\psi \left|M\right|)^2 \left|\bar{e}_{j-1}\right|^2 + \sum_{k=1}^{q} \tilde{m}_{kj}^2. \end{aligned}$$

So, it follows that

$$\begin{aligned} \dot{V}_{3j}(t, \tilde{m}_{kj}) \leq & \upsilon_m^{-1}(1 + (\rho^{-1} + p_M^2)(\beta_b \mu_\psi)^2 - \alpha) V_{3j} - 2e^\top C^\top M^\top \sum_{k=1}^{q} \Psi^k(R_k z_j, u_j)\tilde{m}_{kj} \\ & + q(\mu_\psi \left|M\right|)^2 \left|\bar{e}_{j-1}\right|^2 + q\left|\Gamma_j\right|^2 \quad a.e, \end{aligned}$$

where we have also used positivity of $\Upsilon_j(t)$.
Now, the time derivative of V_{2j} is given by

$$\dot{V}_{2j} = \frac{\bar{h}}{N}\left|\dot{e}_j\right|^2 - \int_{-\frac{\bar{h}}{N}}^{0} \left|\dot{e}_j(t+\vartheta)\right|^2 d\vartheta.$$

From (34) and using the sector condition on $\eta(\cdot)$ and $\bar{\eta}^k(\cdot)$, one easily obtains

$$\frac{\bar{h}}{N}\left|\dot{e}_j\right|^2 \leq \frac{\bar{h}}{N}(\beta_a + \beta_c + b + \sum_{k=1}^{q} \bar{b}_k \mu_m)^2 P_m^{-1} V_{1j} + \frac{\bar{h}}{N}(\left|K\right|^2 \left|\bar{e}_{j-1}\right|^2 + \upsilon_m^{-1}(\mu_\psi \beta_b)^2 V_{3j} + \beta_c^2 \left|\Gamma_j\right|^2),$$

and using Jensen's inequality, it follows that

$$\frac{\bar{h}}{N}\int_{-\frac{\bar{h}}{N}}^{0} \left|\dot{e}_j(t+\vartheta)\right|^2 d\vartheta \geq \left|\int_{-\bar{h}}^{0} \dot{e}_j(t+\vartheta)d\vartheta\right|^2 = \left|\Gamma_j\right|^2.$$

Let γ be a positive constant which depends on $\beta_c = |KC|$ and α. Hence, after re-arranging terms and using the fact that

$$V_{2j} \leq \frac{\bar{h}}{N}\int_{-\frac{\bar{h}}{N}}^{0} |\dot{e}_j(t+\vartheta)|^2 \, d\vartheta,$$

we deduce that the total time derivative of V_j along the trajectories of (29) satisfies for any γ,

$$\begin{aligned}\dot{V}_j + \gamma V_j \quad \leq \quad & (\gamma - \frac{\varepsilon}{p_M} + p_m^{-1} + \frac{\bar{h}}{N}C'_e)V_{1j} + (\gamma + C'_m + \frac{\bar{h}}{N}\upsilon_m^{-1}(\mu_\psi\beta_b)^2 - \alpha)V_{3j} + (q + \frac{\bar{h}}{N}(1+\beta_c^2))\,|\Gamma_j|^2 \\ & -(\frac{1}{2} - \gamma\frac{\bar{h}}{N})\int_{-\bar{h}}^{0} |\dot{e}_j(t+\vartheta)|^2 \, d\vartheta + \left[(|K|\,p_M)^2 + \frac{\bar{h}}{N}\,|K|^2 + q(\mu_\psi\,|M|)^2\right]|\bar{e}_{j-1}|^2\end{aligned}$$

where $C'_m = \upsilon_m{}^{-1}(1 + (\beta_b\mu_\psi)^2(\rho^{-1} + p_M^2))$ and $C'_e = p_m^{-1}((\beta_a + \beta_c + b + \sum_{k=1}^{q}\bar{b}_k\mu_m)^2 + 2\beta_c^2 p_M^2)$. Therefore, if $\bar{h}$ satisfies the following system of in-equations

$$\gamma + \frac{\bar{h}}{N}C'_e + p_m^{-1} \leq \frac{\varepsilon}{2p_M}, \tag{35a}$$

$$\gamma + C'_m + \frac{\bar{h}}{N}\upsilon_m^{-1}(\mu_\psi\beta_b)^2 \leq \frac{\alpha}{2}, \tag{35b}$$

$$\gamma + q + \frac{\bar{h}}{N}(1+\beta_c^2) \leq \frac{N}{2\bar{h}}, \tag{35c}$$

$$\frac{\bar{h}}{N} \leq \frac{1}{2\gamma} \tag{35d}$$

and applying again the Jensen's Inequality, it follows that

$$\dot{V}_j(t,\omega_{jt}) \quad \leq \quad ((|K|\,p_M)^2 + \frac{\bar{h}}{N}\,|K|^2 + q(\mu_\psi\,|M|)^2)\,|\bar{e}_{j-1}|^2 - \gamma V_j(t,\omega_{jt})$$

and solving for $\bar{h}$ the system of in-equations (35), we deduce that (36) holds for

$$\bar{h} \leq N\min\{\pi'_a, \pi'_b, \pi'_c, \pi'_d\} \tag{36}$$

where $\pi'_a = C_e'^{-1}(\frac{\varepsilon}{2p_M} - \gamma - p_m^{-1})$,

$$\pi'_c = \frac{-(q+\gamma) + \sqrt{(q+\gamma)^2 + 2(1+\beta_c^2)}}{2(1+\beta_c^2)}$$

, $\pi'_b = \upsilon_m(\mu_\psi\beta_b)^{-2}(\frac{\alpha}{2} - C'_m - \gamma)$ and $\pi'_d = \frac{1}{2\gamma}$. One observes that for any constant delay $\bar{h}$, there exists an integer N such that the relation (36) is satisfied. For $j = 1$, we have $\bar{e}_{j-1} = \bar{e}_0 = y_1 - \hat{y}_1 = 0$, since $y_1(t) = \hat{y}_1(t) = y(t) = Cx(t-\bar{h})$. It follows that $\dot{V}_1(t,\omega_{1t}) \leq -\gamma V_1(t,\omega_{1t})$. which implies that $\hat{x}_1(t)$ and $\hat{m}_{k1}(t)$ converge exponentially to $x_1(t)$ and $m_{k1}(t)$ respectively. Using the comparison Lemma [25], we deduce from (36), that for $j = 2,\ldots,N$, if $\bar{e}_{j-1} \to 0$ exponentially, then $e_j \to 0$ and $\tilde{m}_{kj} \to 0$ exponentially. Therefore, based on a mathematical induction, we conclude that $\forall j = 1,\ldots,N$, $e_j \to 0$ and $\tilde{m}_{kj} \to 0$ exponentially. We recall that $e_N(t) = x_N(t) - \hat{x}_N(t) = x(t) - \hat{x}_N(t)$ and $\tilde{m}_{kN}(t) = m_{kN}(t) - \hat{m}_{kN}(t) = m_k(t) - \hat{m}_{kN}(t)$ to conclude that $\hat{x}_N$ and $\hat{m}_{kN}$ converge exponentially to the actual state $x(t)$ and uncertain parameters $m_k(t)$, where $k = 1\ldots s$. ■

B. Synchronization of Duffing oscillators via cascaded observers

Let us reconsider the synchronization of the Duffing equation given by Equation (13) under the assumption that the output y is corrupted by a known and constant transmission delay of value ($\bar{h} = 0.08s$) and the parameter is $\Theta = 1$ is corrupted by an unknown piece-wise constant perturbation $\Delta_\Theta(t) \in [0, 0.5]$ –see Figure 9. We aim at estimating the state of the master system and the parametric perturbation $\Delta_\Theta(t)$.

In preliminary simulations with one observer, we observe that the estimated state fails to converge to the actual state. Now we use two cascaded observers given by (27) with $N = 2$. The state $x(t)$ of (13) is initialized at $x_0 = [0, 0]^\top$, $\forall s \in [-0.08, 0]$. The two cascaded observers are started with initial conditions $z_1(s) = z_2(s) = [1, 2]^\top$, $\forall s \in [-0.04, 0]$, $\hat{m}_{10} = \hat{m}_{20} = 0.6$, $\Upsilon_1(0) = \Upsilon_2(0) = 0.2$. Figure 8 illustrates the synchronization between the master and the slave and Figure 9 depicts the parametric convergence based on the proposed cascade observer.

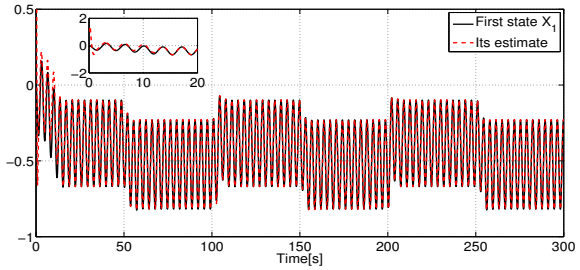

Fig. 8. The first state X_1 and its estimate in the presence of a constant transmission delay with $\bar{h} = 0.08s$ using two cascaded observers

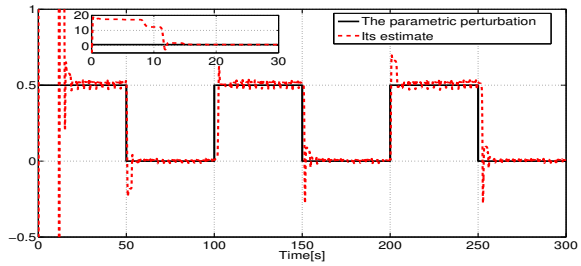

Fig. 9. The parametric convergence in presence of a constant transmission delay $\bar{h} = 0.08s$ using two cascaded observers

VI. Conclusion

We proposed an observers-based synchronization method for a class of Lur'e systems with slope restricted nonlinearities and uncertain parameters, such that the transmission channel is subject to a bounded time-varying delay. Both master-slave synchronization and parametric convergence are ensured for sufficiently small delays and if a condition of persistency of excitation is satisfied. The design matrices of the observer are obtained via convex optimization. The case of constant time-delays of large upper-bounds is also investigated and a cascade observers-based synchronization scheme was proposed. For illustration, theoretical results evaluated through master-slave synchronization of uncertain "Duffing" oscillators subject to transmission time-delays.

References

[1] M. E. Yalcin, J. A. K. Suykens, and J. Vandewalle, "Master-slave synchronization of lur'e systems with time-delay," *Int. J. of Bifurcat. and Chaos*, vol. 11 (6), pp. 1707–1722, 2001.

[2] J. Cao, H. X. Li, and D. W. C. Ho, "Synchronization criteria of lur'e systems with time-delay feedback control," *Chaos, Solitons and Fractals*, vol. 23, pp. 1285–1298, 2005.

[3] Q. L. Han, "New delay-dependent synchronization criteria for lur'e systems using time delay feedback control," *Phys. Lett. A*, vol. 360, pp. 563–569, 2007.

[4] Q. L. Han, "On designing time-varying delay feedback controllers for master-slave synchronization of lur'e systems," *IEEE Trans. on Circ. Syst. I: Regular Papers*, vol. 54, pp. 1573–1583, 2007.

[5] S. M. Lee, S. J. Choi, D. H. Ji, J. H. Park, and S. C.Won, "Synchronization for chaotic lure systems with sector-restricted nonlinearities via delayed feedback control," *Nonlinear Dyn*, vol. 59, pp. 277–288, 2010.

[6] E. Steur and H. Nijmeijer, "Synchronization in networks of diffusively time-delay coupled (semi-) passive systems," *IEEE Trans. on Circ. Syst. I: Regular Papers*, vol. 58, no. 6, pp. 1358–1371, 2011.

[7] D. V. Senthilkumar, J. Kurths, and M. Lakshmanan, "Stability of synchronization in coupled time-delay systems using krasovskii-lyapunov theory," *Phys. Rev. E*, vol. 79, 2009.

[8] H. Nijmeijer and I. Mareels, "An observer looks at synchronization," *IEEE Trans. on Circ. Syst. I: Fundamental Theory and Applications*, vol. 44 (10), pp. 882–890, 1997.

[9] A. Germani, C. Manes, and P. Pepe, "A new approach to state observation of nonlinear systems with delayed output," *IEEE Trans. on Automat. Contr.*, vol. 47 (1), pp. 96–101, 2000.

[10] N. Kazantzis and R. A. Wright, "Nonlinear observer design in the presence of delayed output measurements," *Sys.& Control Letters*, vol. 54, pp. 877–886, 2005.

[11] K. Subbarao and P. C. Muralidhar, "A state observer for lti systems with delayed outputs: Time-varying delay," in *American Control Conference*, (Washington, USA), pp. 3029–3033, 2008.

[12] F. Cacace, A. Germani, and C. Manes, "An observer for a class of nonlinear systems with time varying observation delay," *Sys.& Control Letters*, vol. 59, pp. 305–312, 2010.

[13] G. Besançon, "Remarks on nonlinear adaptive observer design," *Systems and Control Letters*, vol. 41, pp. 271–280, 2000.

[14] Q. Zhang, "Adaptive observers for mimo linear time-varying systems," *IEEE Trans. on Automat. Contr.*, vol. 47, pp. 525–529, 2002.

[15] A. Xu and Q. Zhang, "Nonlinear system fault diagnosis based on adaptive estimation," *Automatica*, vol. 40, pp. 1183–1193, 2004.

[16] A. Loria, E. Panteley, and A. Zavala, "Adaptive observers with persistency of excitation for synchronization of chaotic systems," *IEEE Trans. on Circ. Syst. I: Regular Papers*, vol. 56, no. 12, pp. 2703–2716, 2009.

[17] N. N. Krasovskii, *Stabiliy of motion*. California: Stanford universiry press, 1963.

[18] V. L. Kharithonov, "Lyapunov functionals and matrices," *Annual reviews in control*, vol. 34, pp. 13–20, 2010.

[19] M. Arcak and P. Kokotović, "Observer-based control of systems with slope-restricted nonlinearities," *IEEE Trans. on Automat. Contr.*, vol. 46, no. 7, pp. 1146–1150, 2001.

[20] M. B. A. Zemouche, "A unified adaptive observer synthesis method for a class of systems with both lipschitz and monotone nonlinearities," *IEEE Trans. on Automat. Contr.*, vol. 5, no. 4, pp. 282–288, 2009.

[21] M. Darouach, M. Zasadzinski, and S. J. Xu, "Full-order observers for lienar systems with unknown inputs," *IEEE Trans. on Automat. Contr.*, vol. 39, no. 3, pp. 606–609, 1994.

[22] X. Yan and C. Edwards, "Nonlinear robust fault reconstruction and estimation using a sliding mode observer," *Automatica*, vol. 43, pp. 1605–1614, 2007.

[23] M. Feki, "An adaptive chaos synchronization scheme applied to secure communication," *Chaos, Solitons and Fractals*, vol. 18, pp. 141–148, 2003.

[24] M. Farza, M. Msaad, T. Maatoug, and M. Kamounb, "Adaptive observers for nonlinearly parameterized class of nonlinear systems," *Automatica*, vol. 45, pp. 2292–2299, 2009.

[25] H. Khalil, *Nonlinear systems*. New York: Prentice Hall, 3rd ed., 2002.