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# Exponential stabilization of switched-reluctance motors via speed-sensorless feedback\*

Antonio Loria Gerardo Espinosa–Pérez Erik Chumacero

## Abstract

We solve the control problem of switched-reluctance motors without velocity measurements. Our controller is composed of a loop in the mechanical dynamics which consists of a PI<sup>2</sup>D controller and a “tracking” controller closing an inner loop with the stator currents dynamics. The PI<sup>2</sup>D controller consists in a linear proportional derivative controller in which the measurement of velocities is replaced by approximate derivatives of angular position. Then a double integrator is added, composed of an integral of the angular position errors and a second integral correction term in function of the approximate derivative. We show global exponential stability and illustrate the performance of our controller in numerical simulations.

**Keywords:** Switched-reluctance motor, PID control, nonlinear control.

## I. INTRODUCTION

Switched-reluctance drives are typically very reliable and produce high torque at low speeds. This eliminates the use of gear boxes and makes them a suitable candidate for direct-drive applications. Nonetheless, the price paid for high performance is a technical difficulty –they are highly nonlinear electro-mechanical machines since the generated torque is a nonlinear function of stator currents and rotor position, and the magnetic saturation is required for its basic operation in order to maximize torque/mass ratio.

In technological systems relying on electrical machines the necessity of eliminating the use of sensors for the mechanical variables (position and/or speed) is of special interest. This problem which is well known as *sensorless* control, is beyond the theoretical interest entailed by its difficulty –the requirement of not using mechanical sensors has its roots both in practical limitations and economic factors. On one hand, mechanical sensors exhibit undesirable behaviors in several scenarios such as high-noise sensitivity and reduced reliability. On the other, economic factors cannot be overestimated –the operation and implementation of this type of measurement devices may drastically increase the cost of a given setup.

There exist a large number of efficient heuristically-based and experimentally-validated control approaches to reduce the number of mechanical sensors in the loop –see *e.g.*, [1], [2], [3], [4], [5], [6], [7], [8]. However, to the best of the authors knowledge, articles on control of switched-reluctance drives that include a rigorous stability analysis, especially in a sensorless context, are rare. The main result in [9] establishes global asymptotic stability for a passivity-based controller in the case of unknown load however, it uses both mechanical variables, angular velocity and position measurements. A proportional-derivative-based controller is proposed in [10] but relying on

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the knowledge of the torque load. Other works on sensorless control and containing a theoretical analysis concern different electrical machines such as the induction motor –see *e.g.*, [11].

This paper is the outgrowth of [12], we exploit the physical characteristics of the nonlinear model of the switched-reluctance machine. The control design relies on the ability to separate the machine model into its electrical and mechanical components. Torque generation is achieved by following the “torque-sharing” approach of [10] with the aim of reducing the ripple in the mechanical variables that appears due to the electric commutation. A first control loop is designed to steer the stator currents to a desired “reference” that may be regarded as a virtual control input for the mechanical dynamics. Then, an “outer” control-loop is designed including a controller of proportional-integral-derivative type, probably the most often used in control practice. Specifically, we use the so-called PI<sup>2</sup>D controller originally proposed in [13] for robot manipulators. The control law consists in a term proportional to the angular position error, a term which is proportional to an approximate derivative of the angular position, and two integrators: of angular position error and of the approximate derivative. We establish *global exponential* stability of the origin of the closed-loop via Lyapunov’s direct method.

The rest of the paper is organized as follows. In the next section we present the motor model and explain the sharing-functions implementation approach. In Section III we present our main result and in Section IV we present a case-study in simulation, which reproduces a practically reasonable scenario. We close with some concluding remarks in Section V.

## II. THE MOTOR MODEL

It is well accepted that the three stator phases of a switched-reluctance motor may be assumed to be magnetically decoupled *i.e.*, the mutual inductance among stator phases is negligible. Under such hypothesis, a general three-phase dynamic model is given by

$$\dot{\psi}_j(\theta, i) + Ri_j = u_j, \quad j \in \{1, 2, 3\} \quad (1a)$$

$$J\dot{\omega} = T_e(\theta, i) - T_L(\theta, \omega) \quad (1b)$$

where for each phase  $j$ ,  $u_j$  is the voltage applied to the stator terminals,  $i_j$  is the stator current and  $\psi_j$  is the flux linkage.  $R$  and  $J$  are physical parameters; the former accounts for the stator winding resistance and the latter denotes the total rotor inertia. The state variables are the angular velocity  $\omega$ , and the angular rotor position  $\theta$ , hence  $\omega = \dot{\theta}$ . The rotor dynamics is driven by the two inputs  $T_L$  which is the load torque and  $T_e$  which accounts for the mechanical torque of electrical origin.  $T_e$  depends both on the angular rotor position and on all the stator currents  $i = [i_1, i_2, i_3]^\top$ .

The flux linkage may be modeled in a number of ways as for instance the experimentally obtained and well-assimilated structure proposed in [14],

$$\psi_j(\theta, i) = \psi_s \left( 1 - e^{-i_j f_j(\theta)} \right), \quad i_j \geq 0 \quad (2)$$

where  $\psi_s$  is the saturated flux linkage and  $f_j$  which is known as the winding inductance, is strictly positive and periodic. Although Equation (2) accounts for magnetic saturation it leads to the definition of a non-invertible map for the generated torque  $i \mapsto T_e$ . Therefore, the inductance of each phase may be expressed as a strictly positive

Fourier series truncated at the first harmonic,  $L_j(\theta)$  –see [14], [9]; this implies that  $\psi_j(\theta, i) = L_j(\theta)i_j$  where

$$L_j(\theta) = l_0 - l_1 \cos\left(N_r\theta - (j-1)\frac{2\pi}{3}\right),$$

$l_0 > l_1 > 0$  and  $N_r$  is the number of rotor poles. Then, the dynamic model of the motor may be expressed as

$$u_j = L_j(\theta)\frac{di_j}{dt} + K_j(\theta)\omega i_j + Ri_j \quad (3a)$$

$$J\dot{\omega} = T_e(\theta, i) - T_L(\theta, \omega) \quad (3b)$$

where

$$K_j(\theta) = \frac{\partial L_j}{\partial \theta} = N_r l_1 \sin\left(N_r\theta - (j-1)\frac{2\pi}{3}\right)$$

corresponds to the phase-inductance variation relative to the rotor angular position. It is clear that

$$0 < l_m \leq |L_j(\theta)| \leq l_M, \quad |K_j(\theta)| \leq k_M. \quad (4)$$

for some positive constants  $l_m$ ,  $l_M$  and  $k_M$ .

Considering the decoupled behavior of stator windings, the mechanical torque  $T_e$  corresponds to the sum of torques  $T_j$  produced by each phase, that is

$$T_e = \sum_{j=1}^3 T_j, \quad T_j = \frac{1}{2}K_j(\theta)i_j^2. \quad (5)$$

This model is adopted in both the electrical-machines and the control research communities –*cf.* [14].

The **speed-less control problem** for (3) is to design a dynamic controller with output  $u = [u_1 \ u_2 \ u_3]^T$  depending on the stator currents and rotor angular positions, such that  $\omega(t)$  tracks any bounded smooth desired trajectory  $\omega^*$ . Hence, the purpose is to avoid the use of speed sensors in the control scheme. This problem cannot be overestimated; from a control design viewpoint it is a crucial step towards full-sensorless control while from an implementation perspective, it eliminates the use of noisy velocity sensors. As mentioned in the introduction, speed-sensorless and full-sensorless have been addressed before however, never from an automatic-control perspective that is, without theoretical foundation.

### III. SPEED-LESS MOTOR CONTROL

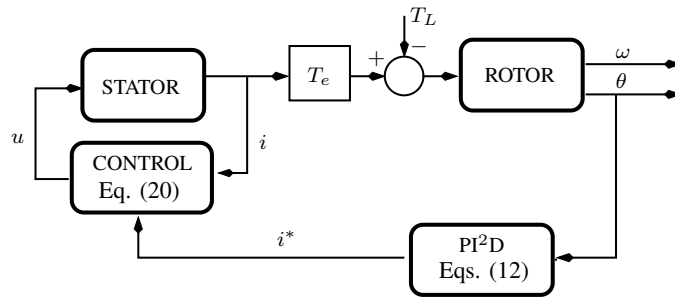


Fig. 1: Implementation of control scheme, composed of an outer rotor control loop and an inner stator control loop

The control approach consists in designing two controllers separately: the first closes an inner loop with the stator dynamics and is implemented using the input voltages that is, the actual control inputs. An outer control law, to

drive the rotor dynamics, is implemented using the mechanical torque of electrical origin as a virtual control input. The control implementation relies on the so-called sharing-torque technique –see Section III-A for a description and see Figure 1 for an illustration.

#### A. Torque delivery

Generally speaking, the control design starts with a given desired reference  $\omega^*$ . Then, a *desired* control input  $T_d$  is designed for the mechanical equation (3b) to steer  $\omega \rightarrow \omega^*$ . The control  $T_d$  must be implemented “through” the mechanical torque  $T_e$  that is, we define the reference mechanical torque  $T_e^*$  which satisfies

$$T_e^*(\theta, i^*) = \frac{1}{2} \left( K_1(\theta) i_1^{*2} + K_2(\theta) i_2^{*2} + K_3(\theta) i_3^{*2} \right) \quad (6)$$

where  $i_j^*$  is a current reference trajectory for each phase, which must be defined as a solution to

$$\frac{T_e^*}{J} = T_d \quad (7)$$

for any given  $T_d$ . That is, provided that  $T_e = T_e^*$  the desired control torque  $JT_d$  acts upon the mechanical equation to drive  $\omega \rightarrow \omega^*$ . By ensuring an accurate current tracking that is,  $i \rightarrow i^*$  it is guaranteed that  $T_e \rightarrow T_e^*$  and consequently, that  $\omega \rightarrow \omega^*$ .

In order to solve (6) for  $i_j^*$ , we exploit the physics of the *switched* reluctance motor. Note that the torque sign is only determined by the variation of the inductance that is,  $T_j > 0$  if and only if  $K_j > 0$  and  $K_j = 0$  implies that no torque is produced ‘through’ the phase  $j$  –see Equation (5). Then, using ideas reported in [14] and [10] we introduce the following smooth current-switching policy. Define the sets

$$\begin{aligned} \Theta_j^+ &= \{ \theta \in [-\pi, \pi] : K_j(\theta) \geq 0 \} \\ \Theta_j^- &= \{ \theta \in [-\pi, \pi] : K_j(\theta) < 0 \} \end{aligned}$$

where the superscripts  $+$  and  $-$  stand for required positive and negative torques respectively. Accordingly, let us introduce

$$\begin{aligned} m_j^+(\theta) &> 0, \quad \sum_{j=1}^3 m_j^+(\theta) = 1 \quad \forall \theta \in \Theta^+, \\ m_j^-(\theta) &> 0, \quad \sum_{j=1}^3 m_j^-(\theta) = 1 \quad \forall \theta \in \Theta^- \end{aligned}$$

and given  $T_d$ , define

$$m_j(\theta) = \begin{cases} m_j^+(\theta) & \text{if } T_d \geq 0, \\ m_j^-(\theta) & \text{if } T_d < 0. \end{cases} \quad (8)$$

Then, we define the reference currents for  $j \in \{1, 2, 3\}$  as

$$i_j^* = \begin{cases} \left[ \frac{2Jm_j(\theta)T_d}{K_j(\theta)} \right]^{1/2} & \text{if } |K_j(\theta)| \neq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

The definition of  $m_j(\theta)$  ensures that  $i_j^*$  exists for any  $\theta$  and  $T_d$ . That is, depending on the current phase of the motor, the function  $m_j(\theta)$  ensures that the respective signs of the numerator and of the denominator in the previous expression are equal for *at least one*  $j \in \{1, 2, 3\}$  and the denominator  $\sin\left(\theta - (j-1)\frac{2\pi}{3}\right) \neq 0$ . Furthermore, for

implementation purposes hysteresis is typically introduced in the definition of  $i_j^*$  –see Eq. (9) to smoothly avoid the singularity at the points  $\theta$  in the neighbourhood of  $\{K_j(\theta) = 0\}$ . In addition, it is imposed that  $m_1 + m_2 + m_3 \equiv 1$  so by construction,

$$T_d = m_1(\theta)T_d + m_2(\theta)T_d + m_3(\theta)T_d.$$

Thus, roughly speaking the virtual control  $T_d$  is induced into the mechanical dynamics through a different reference, depending on the current phase. The transition is ensured by a proper definition of the functions  $m_j$  –cf. [9]. See also Section IV for a case-study.

### B. Control of the rotor dynamics

The rotor controller that produces  $T_d$  is of the proportional-integral-derivative type, possibly the most popular controller in practice. It is reminiscent of the Proportional-Derivative-plus-load-compensation controller presented in [14] where the sharing-functions approach was proposed. However, we relax the assumption that the torque-load is known and that velocity is measured. Our main result extends previous work using both mechanical velocity and position measurements as in [9] as well as [10]. In contrast with available sensorless *ad hoc* controllers proposed without theoretical foundation, we establish global exponential stability hence robustness with respect to external disturbances, neglected dynamics, *etc.*

For control purposes, we view the rotor dynamics equation (1b), as a drift-less system perturbed by the input  $T_L$ . Its complexity comes from the fact that it is non-affine in the (virtual control) inputs  $i$ ; such difficulty is overcome by using the torque-sharing technique described previously. Then, for a given reference  $\omega^*$  and defining  $e_\omega = \omega - \omega^*$ , Equation (3b) may be equivalently written as

$$\dot{e}_\omega = T_d - \frac{T_L}{J} + \frac{T_e}{J} - \frac{T_e^*}{J} - \dot{\omega}^* \quad (10)$$

and the control problem comes to designing a control law  $T_d$  to stabilize the origin  $\{e_\omega = 0\}$  of (10). Such controller must compensate for the constant perturbation  $T_L/J$  and be robust to the “disturbance”  $T_e - T_e^*$  which vanishes by design. The former is compensated for by integral action; the latter, which accounts for the difference between the ideal mechanical torque  $T_e^*/J$  that stabilizes the system and the actual mechanical torque which depends on the stator currents, vanishes provided that the external control loop achieves current tracking. Indeed, this difference satisfies

$$|T_e - T_e^*| \leq \frac{k_M}{2} \left[ |e_i|^2 + 2|i^*||e_i| \right], \quad (11)$$

since  $T_e$  and  $T_e^*$  are quadratic functions uniformly bounded in  $\theta$ .

For the rotor control law  $T_d$ , we chose to use the PI<sup>2</sup>D controller introduced in [13] for robot manipulators. Its name comes from the fact that it corresponds to a modified PID controller; it consists in a correction term proportional to the tracking errors  $e_\theta$ , a ‘derivative’ term proportional to filtered velocities  $\vartheta$  and a double integral action, both on  $e_\theta$  and  $\vartheta$ . The choice of this controller is motivated by its mathematical simplicity and its physical properties: it is a linear controller which conserves the passivity properties of Lagrangian systems and ensures asymptotic stability, provided a property of detectability holds –see [15].

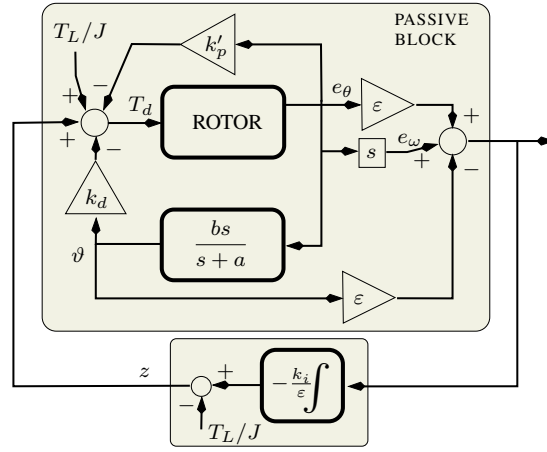


Fig. 2: For analysis purpose note that the rotor dynamics in closed loop forms a passive systems interconnection. The control implementation is illustrated in Figure 1.

The PI<sup>2</sup>D controller for the rotor dynamics is defined by

$$T_d = -k_p e_\theta - k_d \vartheta + \nu + \dot{\omega}^* \quad (12a)$$

$$\dot{\nu} = -k_i (e_\theta - \vartheta) \quad (12b)$$

$$\dot{q}_c = -a(q_c + b e_\theta) \quad (12c)$$

$$\vartheta = q_c + b e_\theta \quad (12d)$$

where  $k_p$ ,  $k_i$ ,  $k_d$ ,  $a$ ,  $b$  are positive constants and we use the tracking errors  $e_\omega = \omega - \omega^*$  and  $e_\theta = \theta - \theta^*$  where

$$\theta^*(t) = \int_0^t \omega^*(s) ds, \quad \theta^*(0) = \theta_0^* \in [-\pi, \pi].$$

Note that since the variable to be controlled is  $\omega$ , the initial value of  $\theta^*(t)$  is innocuous. As usual, the integral gain  $k_i$  is chosen ‘small’ consequently, the estimation error of the load torque  $\tilde{\nu}$  converges slowly to zero; this is not a drawback since the variable of interest is the angular velocity  $\omega$ .

The last two equations in (12) correspond to the well-known and widely used ‘dirty derivatives’ or ‘approximate differentiation’. The nicknames come from the observation that they are equivalent to

$$\vartheta = \frac{b}{s+a} e_\omega$$

where  $s$  is the Laplace variable. That is,  $\vartheta$  is not an estimate of the velocities  $e_\omega$  but a filtered version of it. In the limit case, when the pole is at  $-\infty$ ,  $\vartheta = e_\omega$  modulo the DC gain  $b/a$ .

One of the most attractive features of the controller (12) is that as for robot manipulators –see [15], the system (10) in closed loop with (12) may be regarded as the feedback interconnection of two passive systems; see Figure 2. The first passive block (shaded in the figure) is defined by the map  $z \mapsto \varepsilon[e_\theta - \vartheta] + e_\omega$  where

$$z = \nu - \frac{T_L}{J} - \frac{k_i}{\varepsilon} e_\theta \quad (13)$$

and  $\varepsilon > k_i$  is a small positive parameter. The second passive block is a simple integrator.

The rotor closed-loop dynamics, which corresponds to the feedback interconnection showed in Figure 2 is obtained by substituting (12a) and (13) into (10) and by differentiating on both sides of (12d) and (13), to obtain

$$\dot{e}_\theta = e_\omega \quad (14a)$$

$$\dot{e}_\omega = -k_d \vartheta - k'_p e_\theta + z + \frac{1}{J} (T_e - T_e^*) \quad (14b)$$

$$\dot{\vartheta} = -a \vartheta + b e_\omega \quad (14c)$$

$$\dot{z} = -k_i (e_\theta - \vartheta) - \frac{k_i}{\varepsilon} e_\omega \quad (14d)$$

where  $k'_p := k_p - k_i/\varepsilon$ ,  $k_i \leq \varepsilon$ . Note that we have added the input  $(T_e - T_e^*)$  that comes from the stator dynamics although this is not represented in Figure 2. This input vanishes provided that the stator controller performs a perfect current tracking; this is better seen from the structure of (14) which may be rewritten in the clearer form

$$\begin{bmatrix} \dot{e}_\theta \\ \dot{e}_\omega \\ \dot{\vartheta} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -k'_p & 0 & -k_d & 1 \\ 0 & b & -a & 0 \\ -k_i & -k_i/\varepsilon & k_i & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} e_\theta \\ e_\omega \\ \vartheta \\ z \end{bmatrix}}_{\zeta_1} + \underbrace{\begin{bmatrix} 0 \\ 1/J \\ 0 \\ 0 \end{bmatrix}}_B (T_e - T_e^*).$$

Note that the only nonlinearity corresponds to the additive input term  $(T_e - T_e^*)$  and on the other hand, for appropriate values of the control gains, one can render the matrix  $A$  Hurwitz. Therefore, the system may be made input to state stable with respect to the input  $(T_e - T_e^*)$  and asymptotic stability follows if  $T_e - T_e^*$  tends to zero; this is the job of the stator controller, which we present next.

### C. Control of the stator dynamics

The controller (12) stabilizes (the origin of) (10) exponentially, provided that perfect current tracking is achieved. Therefore, an inner control loop is added to steer  $e_i = i - i^*$  to zero. This task may be achieved with the model-based nonlinear tracking controller

$$u^* = L(\theta) \frac{di^*}{dt} + K(\theta) \omega^* i + R i^* - k_{px} e_i, \quad k_{px} > 0 \quad (15)$$

where for convenience, we introduced  $L(\theta) = \text{diag}\{L_j(\theta)\}$  and  $K(\theta) = \text{diag}\{K_j(\theta)\}$ .

The motivation to use this controller relies on the fact that it stabilizes the stator dynamics, robustly with respect to additive disturbances. This may be clear from the expression of the closed-loop dynamics (3), (15) to which we add an external input  $v$ , that is

$$\dot{e}_i = L(\theta)^{-1} [-(R + k_{px}) e_i - K(\theta) i e_\omega] + v. \quad (16)$$

To see that the origin of the ‘nominal’ system  $\dot{e}_i = -L(\theta)^{-1} (R + k_{px}) e_i$  is exponentially stable, it suffices to invoke Lyapunov’s direct method with the Lyapunov function

$$V_2(e_i) = \frac{1}{2} |e_i|^2,$$

whose total time derivative along the trajectories of the nominal system yields

$$\dot{V}_2 = -(R + k_{px}) e_i^\top L^{-1} e_i \leq -\frac{1}{l_M} (R + k_{px}) |e_i|^2$$



where for the last inequality, we used (4).

Exponential stability implies robustness with respect to disturbances, more precisely, local input to state stability. To make this property global, we chose the control gain  $k_{px}$  as a function of the measured states. To that end, note that

$$|L(\theta)^{-1}[-K(\theta)ie_\omega]| \leq \frac{k_M}{l_m} \left( |e_\omega| |i| \right),$$

so the total derivative of  $V_2$  along the trajectories of the ‘perturbed’ system (16), yields

$$\dot{V}_2 \leq -\left( \frac{R + k_{px}}{l_M} - \frac{k_M}{2l_m} |i|^2 \right) |e_i|^2 + v^\top e_i + \frac{k_M}{2l_m} e_\omega^2. \quad (17)$$

Hence, it suffices to make  $k_{px}$  dependent on the measured stator currents  $i$ , to render the factor of  $|e_i|^2$  negative. Under this condition and in the case of perfect rotor velocity tracking ( $e_\omega = 0$ ), the map  $v \mapsto e_i$  is output strictly passive. Note also that the system is input to state stable with respect to the input  $v + e_\omega$ .

Although the control law (15) renders the stator dynamics robustly stable, note that the column vector  $\frac{di^*}{dt} = \frac{d}{dt} [i_1^* \ i_2^* \ i_3^*]^\top$  depends on  $\dot{T}_d$  where  $T_d$  is defined in (12a) hence, also on the unmeasurable error  $e_\omega$ . That is,

$$\frac{di_j^*}{dt} = \begin{cases} \alpha_j [\rho_j + \delta_j e_\omega] & \text{if } |K_j(\theta)| \neq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

where

$$\alpha_j = \left( \frac{2Jm_j T_d}{K_j} \right)^{-\frac{1}{2}}, \quad (19a)$$

$$\rho_j = \frac{J}{K_j} (\beta_j + m'_j \omega^* T_d), \quad (19b)$$

$$\beta_j = m_j \left( k_{id} \vartheta - k_i e_\theta + \ddot{\omega}^* - \frac{T_d K'_j \omega^*}{K_j} \right), \quad (19c)$$

$$\delta_j = \frac{J}{K_j} \left( m'_j T_d - m_j \left( k_{pd} + \frac{T_d K'_j}{K_j} \right) \right), \quad (19d)$$

$k_{id} = k_i + ak_d$ ,  $k_{pd} = k_p + bk_d$ ,  $m'_j = \frac{\partial m_j}{\partial \theta}$  and  $K'_j = \frac{\partial K_j}{\partial \theta}$ . Since it is assumed that  $e_\omega$  is not measured, only the terms  $\alpha = \text{diag}\{\alpha_j\}$  and  $\rho = [\rho_1 \ \rho_2 \ \rho_3]^\top$  may be implemented in the control law. Therefore, we use the control law

$$u = L(\theta)\alpha\rho + K(\theta)\omega^*i + Ri^* - k_{px}e_i \quad (20)$$

which satisfies  $u = u^* - L(\theta)\alpha\delta e_\omega$  where  $\delta = [\delta_1 \ \delta_2 \ \delta_3]^\top$ . With this modification, the closed-loop dynamics takes the form (16) with  $v = \alpha\delta e_\omega$ . Therefore, after (17) and the triangle inequality, the total derivative of  $V_2$  along the closed-loop trajectories of (3a), (20) yields

$$\dot{V}_2 \leq -\left( \frac{R + k_{px}}{l_M} - \frac{k_M}{2l_m} |i|^2 - \frac{1}{2} |\alpha\delta|^2 \right) |e_i|^2 + \frac{1}{2} \left( \frac{k_M}{l_m} + 1 \right) |e_\omega|^2 \quad (21)$$

that is, for an adequate choice of  $k_{px}$  the factor of  $|e_i|^2$  is negative and the system is input to state stable with respect to the input  $e_\omega$ .

### D. Main result

We are ready to present our main result: global exponential stabilization of the switched-reluctance motor without velocity measurements. The result relies on the fact that the rotor dynamics in closed-loop with the PI<sup>2</sup>D controller is input to state stable with respect to the input  $e_i$  while the stator dynamics in closed loop with the control law (20) is input to state stable with respect to the input  $e_\omega$ .

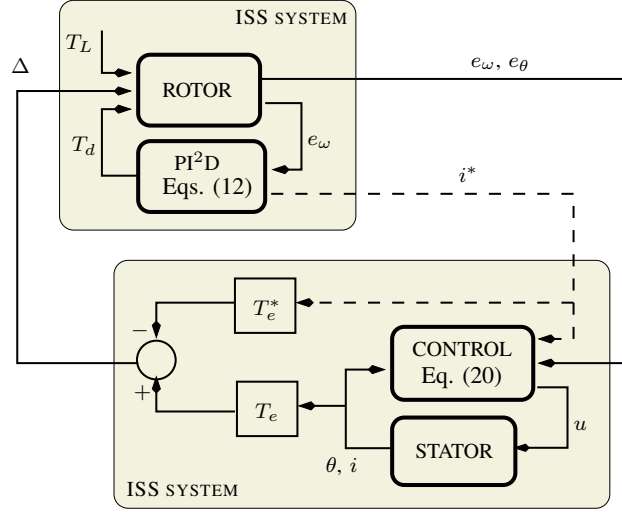


Fig. 3: Analysis: the closed-loop system consists in two ISS controlled sub-systems

*Proposition 1:* Consider the system (3) in closed loop with the controller (12), (19), (20). Let  $k'_{px}$ ,  $\gamma_1 > 0$ ,  $\lambda \in (0, 1)$ ,

$$k_d > \frac{3b}{b-1} \left[ \frac{k_M}{l_m} + 1 \right], \quad k_d < \frac{3a}{2} \frac{\lambda}{\lambda+1} \quad (22)$$

$$b \geq 2a + 1, \quad k'_p > k_d \quad (23)$$

$$k_{px} := k'_{px} + \frac{l_M}{2} \left[ \frac{k_M}{l_m} |i|^2 + |\alpha\delta|^2 + \frac{\gamma_1 k_M l_m}{2} \left( \frac{|e_i|^2}{\mu_2} + \frac{4|i^*|^2}{\mu_1} \right) \right]. \quad (24)$$

Then, the origin of the closed-loop system is globally exponentially stable.

The conditions of the proposition hold if the control gains are chosen as follows: 1) define  $c = k_M/l_m + 1$  and let  $k_d \approx 3c$  but satisfying  $k_d > 3c$ ; 2) pick  $\lambda \in (0, 1)$  and define  $k'_p := k_d/\lambda$ ; 3) choose  $a$  such that the second inequality in (22) holds; 4) define  $b$  such that  $b \geq 2a + 1$  and verify that  $k_d > 3bc/(b-1)$ ; if necessary, redefine  $k_d$  so as to satisfy the latter.

The proof of Proposition 1 is based on Lyapunov's direct method and the previous developments. Define  $\zeta_1 = [e_\theta \ e_\omega \ \vartheta \ z]^\top$  then we write (14) as  $\dot{\zeta}_1 = A\zeta_1 + B(T_e - T_e^*)$ . Let  $\varepsilon_1 = \varepsilon > k_i$  and  $\varepsilon_2$  be two "small" positive numbers and consider the Lyapunov function  $V_1(\zeta_1) = \frac{1}{2}\zeta_1^\top P\zeta_1$  with

$$P = \begin{bmatrix} k'_p & \varepsilon_1 & 0 & 0 \\ \varepsilon_1 & 1 & -\varepsilon_1 & -\varepsilon_2 \\ 0 & -\varepsilon_1 & k_d/b & 0 \\ 0 & -\varepsilon_2 & 0 & \varepsilon_1/k_i \end{bmatrix}.$$

This matrix is diagonal dominant, hence positive definite, if

$$\begin{aligned} k'_p &> \varepsilon_1, \quad 1 > 2\varepsilon_1 + \varepsilon_2 \\ \frac{k_d}{b} &> \varepsilon_1 \quad \frac{\varepsilon_1}{k_i} > \varepsilon_2 \end{aligned}$$

which hold for sufficiently small  $k_i$ ,  $\varepsilon_1$  and  $\varepsilon_2$ . Also, we have

$$\begin{aligned} -\frac{1}{2}(A^\top P + PA) = & \\ & \underbrace{\frac{1}{2} \begin{bmatrix} \varepsilon_1 k'_p & -\varepsilon_2 k_i & \varepsilon_1(k_d - k'_p) & 0 \\ -\varepsilon_2 k_i & \varepsilon_1(b-1) - 2\frac{\varepsilon_2 k_i}{\varepsilon_1} & \varepsilon_2 k_i - \varepsilon_1 a & 0 \\ \varepsilon_1(k_d - k'_p) & \varepsilon_2 k_i - \varepsilon_1 a & k_d\left(\frac{a}{b} - \varepsilon_1\right) & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{bmatrix}}_{Q_1} + \\ & \underbrace{\frac{1}{2} \begin{bmatrix} \varepsilon_1 k'_p & 0 & 0 & -\varepsilon_2 k'_p \\ 0 & \varepsilon_1(b-1) & 0 & 0 \\ 0 & 0 & k_d\left(\frac{a}{b} - \varepsilon_1\right) & -k_d \varepsilon_2 \\ -\varepsilon_2 k'_p & 0 & -k_d \varepsilon_2 & \varepsilon_2 \end{bmatrix}}_{Q_2} \end{aligned} \quad (25)$$

The matrix  $Q_1$  in (25) is positive definite if each element in its main diagonal is positive and the matrix is diagonal dominant. This holds for small values of  $\varepsilon_1$  and  $\varepsilon_2$  and control gains satisfying (23). The matrix  $Q_2$  is positive semidefinite for sufficiently small values of  $\varepsilon_1$  and  $\varepsilon_2$ ; indeed,  $-\zeta_1^\top Q_2 \zeta_1 \leq -(1/2)\varepsilon_1(b-1)e_\omega^2$ . See Appendix VI-A for details. We conclude that

$$-\frac{1}{2}(A^\top P + PA) = Q, \quad Q = Q^\top > 0$$

and the total time derivative of  $V_1(\zeta_1)$  yields

$$\dot{V}_1 = -\zeta_1^\top Q \zeta_1 + \zeta_1^\top PB(T_e - T_e^*).$$

To prove input-to-state stability with input  $e_i$  let  $q_m$  be the smallest eigenvalue of  $Q_1$  and let  $\gamma_1 \geq |PB|$  then,

$$\dot{V}_1 \leq -q_m |\zeta_1|^2 + \gamma_1 |\zeta_1| |T_e - T_e^*|.$$

Next, we use (11) to obtain

$$\begin{aligned} \dot{V}_1 &\leq -q_m |\zeta_1|^2 + \frac{\gamma_1 k_M}{2} |\zeta_1| \left[ |e_i|^2 + 2|i^*||e_i| \right] + \frac{\gamma_1 k_M}{4} \left[ \frac{4|i^*|^2|e_i|^2}{\mu_1} + \mu_1 |\zeta_1|^2 \right] \\ &\leq -\left[ q_m - \frac{\gamma_1 k_M}{4}(\mu_1 + \mu_2) \right] |\zeta_1|^2 + \frac{\gamma_1 k_M}{4} \left[ \frac{|e_i|^2}{\mu_2} + \frac{4|i^*|^2}{\mu_1} \right] |e_i|^2 \end{aligned}$$

and we see that for sufficiently small values of  $\mu_1$  and  $\mu_2$  the factor of  $|\zeta_1|^2$  is positive, say actually not smaller than  $q_m/2$ . Thus, in view of (24) and  $-\zeta_1^\top Q_2 \zeta_1 \leq -(1/2)\varepsilon_1(b-1)e_\omega^2$ ,

$$\dot{V}_1 + \dot{V}_2 \leq -\frac{q_m}{2} |\zeta_1|^2 - \frac{[R + k'_{px}]}{l_M} |e_i|^2 - \frac{1}{2} \left[ \varepsilon_1(b-1) - \frac{k_M}{l_m} - 1 \right] e_\omega^2 \quad (26)$$

in which the factor of  $e_\omega^2$  is positive for sufficiently large  $b$ . Moreover, in view of the positivity of  $P$ , for appropriate (large) values of the control gains  $a, b, k_p, k_d, k_{px}$  and small values of  $k_i, \varepsilon_1$  and  $\varepsilon_2$  there exist positive constants  $q_1, q_2, q_3$  such that (see the appendix for details)

$$\begin{aligned} q_1 \left( |e_i|^2 + |\zeta_1|^2 \right) &\leq V_1(\zeta_1) + V_2(e_i) \leq q_2 \left( |e_i|^2 + |\zeta_1|^2 \right) \\ \dot{V}_1(\zeta_1) + \dot{V}_2(e_i) &\leq -q_3 \left( |e_i|^2 + |\zeta_1|^2 \right). \end{aligned}$$

Global exponential stability of the origin follows.

#### IV. SIMULATION RESULTS

In the previous section we establish global exponential stability for the system in closed loop with the controller (20). This is a strong stability property since it implies robustness with respect to perturbations with arbitrarily large initial tracking errors. However, the theoretical contribution comes at the expense of high-order nonlinear gains –see (24) which in turn, is likely to cause large control values  $u$ . Other causes of large voltage inputs are clear from formula (20); notice that  $\dot{\omega}^*$  is used in  $T_d$  and  $\ddot{\omega}^*$  enters in  $u$  through the expression  $\beta_j$  –see (19c). In addition to this, the nature of the switched-reluctance motor which imposes the use of the sharing-torque technique, induces discontinuous switches in the expression of  $i^*$  –note that  $\alpha_j$  may be considerably large in the neighborhood of  $\{K_j(\theta) = 0\}$ .

Thus, in order to implement the controller (20) with reasonable control inputs some precautions must be taken. Firstly, a smooth reference velocity is to be preferred even if stability is guaranteed for piecewise differentiable reference trajectories. Hence, we mostly use a sequence of smoothed steps generated by

$$\omega^*(t) = \omega_0^* + \frac{(\omega_f^* - \omega_0^*)}{2} (1 + \tanh(t - T)) \quad (27)$$

–see Figure 4 for an illustration and the definition of the different design parameters. Note that different reference

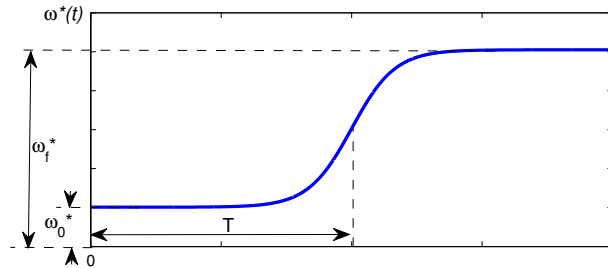


Fig. 4: Generic velocity reference profile constructed using hyperbolic tangent function

signals can be constructed by changing the numerical values of  $\omega_0^*, \omega_f^*, T$  and  $\gamma$ .

The functions  $m_j$ , which ensure the proper commutation of the motor, are defined via (8) where

$$m_j^+(\theta) = \begin{cases} f_j(q_{aj}) & 0 < q_{aj} \leq \frac{\pi}{3N_r} \\ 1 & \frac{\pi}{3N_r} < q_{aj} \leq \frac{2\pi}{3N_r} \\ 1 - f_j(q_{aj}) & \frac{2\pi}{3N_r} < q_{aj} \leq \frac{\pi}{N_r} \\ 0 & \text{otherwise} \end{cases} \quad (28)$$

$$m_j^-(\theta) = \begin{cases} f_j(q_{aj}) & \frac{\pi}{N_r} < q_{aj} \leq \frac{4\pi}{3N_r} \\ 1 & \frac{4\pi}{3N_r} < q_{aj} \leq \frac{5\pi}{3N_r} \\ 1 - f_j(q_{aj}) & \frac{5\pi}{3N_r} < q_{aj} \leq \frac{2\pi}{N_r} \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

and

$$q_{aj} = \text{mod} \left( q, \frac{2\pi}{N_r} \right) + (j-1) \frac{2\pi}{N_r}. \quad (30)$$

Thirdly, it is customary to protect the motor to introduce input voltage saturation that is, using  $\text{sat}_\delta(u_i)$  in place of  $u_i$  where  $\text{sat}_\delta(u_i) = u_i$  if  $|u_i| < \delta$  and  $\text{sat}_\delta(u_i) = \delta \text{sgn} u_i$  otherwise.

Under the previous conditions we used SIMULINK<sup>TM</sup> of MATLAB<sup>TM</sup> to test the controller (15) on the *nonlinear* model (1), (2) with parameters borrowed from [9] and [14]:  $R = 5$ ,  $l_0 = 0.030H$ ,  $l_1 = 0.020H$ ,  $J = 0.001kg-m^2$ ,  $\psi_s = 0.25Wb$  and  $N_r = 8$ .

The controller gains are set to  $k_{px} = 2000$ ,  $a = 2580$ ,  $b = 1900$ ,  $k_d = 5050$ ,  $k_i = 5e-4$  and  $k_p = 900$ . Based on these values and the definitions of  $K_j(\theta)$ ,  $L_j(\theta)$ ,  $P$  and  $Q$  we have  $k_M = 0.085$ ,  $l_m = 0.01$ ,  $l_M = 0.05$ ,  $\gamma_1 = 105$  and  $q_m = 1e-5$  respectively then, the conditions for positive definiteness of  $P$  and  $Q$  are satisfied if  $\mu_1 = 3e-6$ ,  $\mu_2 = 1e-6$ ,  $\varepsilon_1 = 0.04$  and  $\varepsilon_2 = 1e-5$ .

For the sake of comparison, we present the results corresponding to two different runs of simulations: in the first case the control inputs are implemented without saturation, in the second case we set the saturation level to  $\delta = 100$ . The reference velocity consists in a smooth step as described above.

The simulation results are presented in Figs. 5–10. In Figs. 5, 6 and 9 are depicted, respectively, the current, the input voltages and the velocity tracking responses for the simulation test without input saturation. In Figs. 7, 8 and 10 we present the simulation results for the controller with an input saturation level of 100V. In both cases one may appreciate the fairly good performance; note the almost perfect velocity tracking albeit the slight increase of the response time, in the case that the inputs are saturated. Furthermore, the smaller plots in Figs. 5 and 7 show a zoom on the currents to appreciate the switching mechanism with hysteresis, employed to implement the virtual control input  $T_d$ .

## V. CONCLUDING REMARKS

We have presented the first controller guaranteeing global exponential stability for switched-reluctance motors without velocity measurements. Our main result is a preliminary step towards the open problem of theoretically-validated global sensorless control for switched-reluctance motors. Current research is being carried out in this direction, in particular it focuses on the design of an angular-position observer to be implemented with a certainty-equivalence controller. Other significant problems under study include control under parametric uncertainty and control for the fully-nonlinear motor model (with inductances depending both on the angular position and the currents).

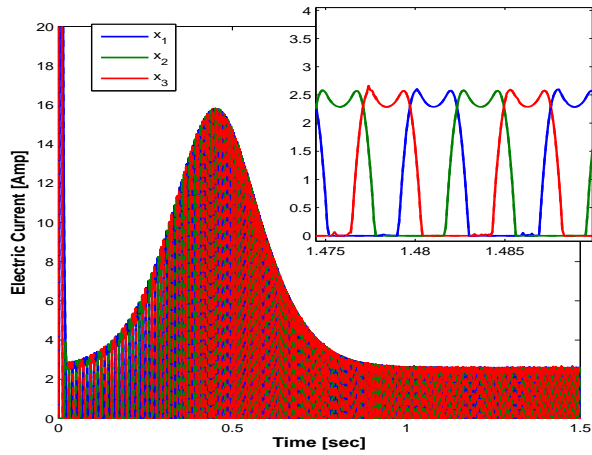


Fig. 5: Unsaturated controls: stator currents for the three phases and references

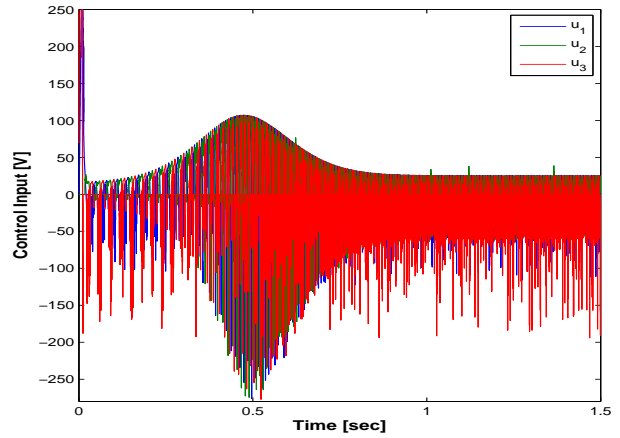


Fig. 6: Unsaturated controls: Input voltages

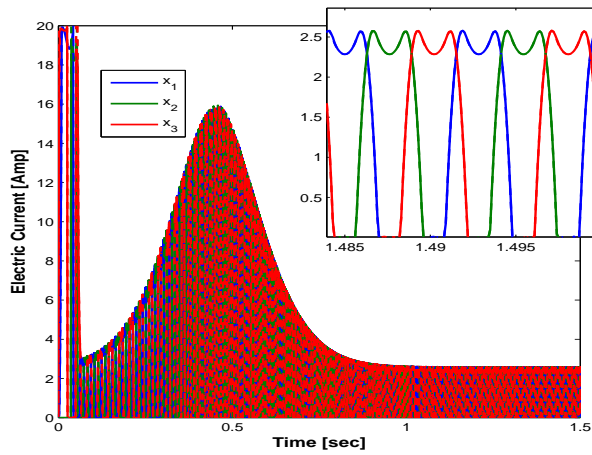


Fig. 7: Saturated controls (at 100V): stator currents for the three phases and references

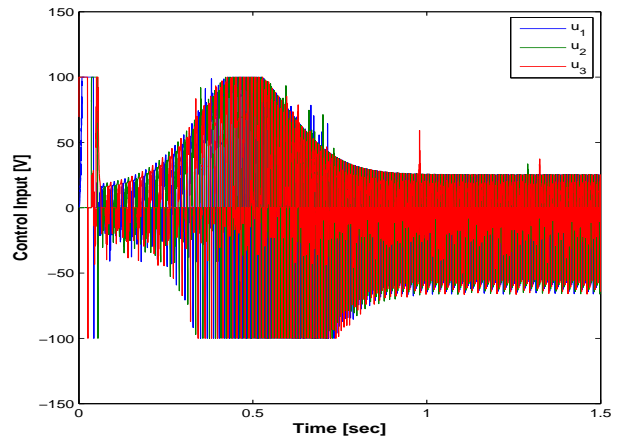


Fig. 8: Saturated controls (at 100V): Input voltages

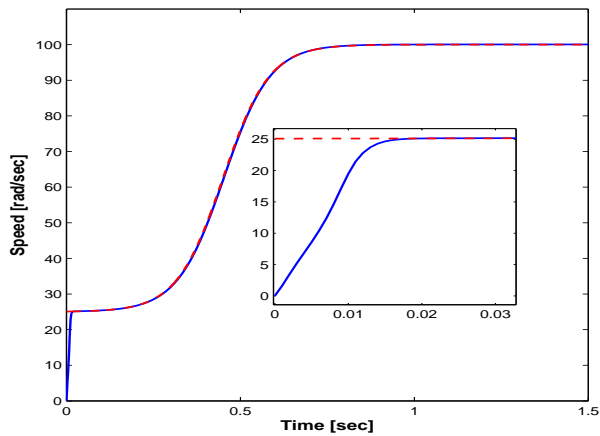


Fig. 9: Unsaturated controls: velocity tracking response

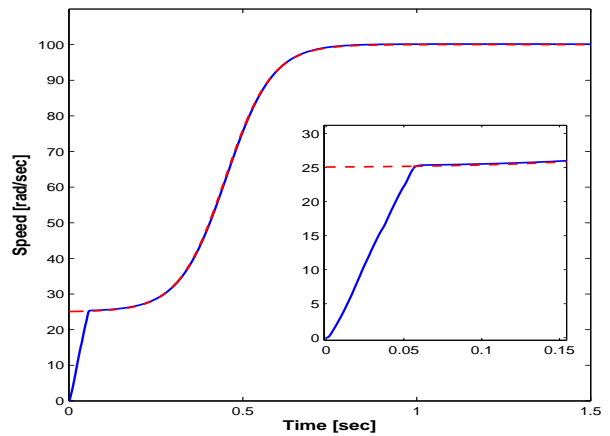


Fig. 10: Saturated controls (at 100V): velocity tracking response

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## VI. APPENDIX

### A. Positivity of $Q$

#### Positivity of $Q_1$

*Claim 1:* Given  $k'_p, k_d, k_i$  and  $a, b$  such that:

$$b \geq 2a + 1 \quad k'_p > k_d \quad (31)$$

then  $\exists \varepsilon_1, \varepsilon_2 > 0$  s.t.  $Q_1 \geq 0$

**Proof of Claim 1.** Let

$$\lambda = \frac{k_d}{k'_p} < 1, \quad (32)$$

$$\varepsilon_1 \leq \min \left\{ 1, \frac{k_d}{3b}, \frac{a\lambda}{2b(\lambda+1)} \right\}, \quad (33)$$

$$\lambda_2 := \frac{a}{\varepsilon_1 b} \quad (34)$$

and  $\varepsilon_2$  be such that

$$\frac{\varepsilon_2}{\varepsilon_1} \leq \frac{1}{k_i} \min \left\{ \alpha_1, \alpha_2, \alpha, \alpha_4 \right\} \quad (35)$$

where  $\alpha_1 = k_d$ ,  $\alpha_2 = \frac{(b-1-2a)}{2}$ ,  $\alpha_3 = \frac{\varepsilon_1 a}{2}$  and  $\alpha_4 = \frac{k_i}{k'_p} \left[ 1 + \frac{\lambda}{\lambda_2 - 1} \right]^{-1}$ .

Note that given  $a, b, k'_p, k_i$  and  $k_d$  satisfying (31), the conditions (33) and (35) hold for sufficiently small values of  $\varepsilon_1$  and  $\varepsilon_2$ . Also note that  $\lambda_2 > 1$  since, in view of (33) we have  $a > b\varepsilon_1$ . We show that if (32)–(35) hold then  $Q_1$  is diagonal dominant. It is so if the following hold

$$\varepsilon_1 |\lambda - 1| k'_p + \varepsilon_2 k_i \leq \varepsilon_1 k'_p \quad (36)$$

$$\varepsilon_2 k_i + |\varepsilon_2 k_i - \varepsilon_1 a| \leq \varepsilon_1 (b-1) - \frac{2\varepsilon_2 k_i}{\varepsilon_1} \quad (37)$$

$$\varepsilon_1 k'_p |\lambda - 1| + \varepsilon_2 k_i + \varepsilon_1 a \leq \lambda k'_p \left( \frac{a}{b} - \varepsilon_1 \right) \quad (38)$$

Since  $\lambda < 1$  (36) holds if

$$\varepsilon_1 k'_p \geq \varepsilon_2 k_i + \varepsilon_1 (1 - \lambda) k'_p \quad \Leftrightarrow \quad \varepsilon_1 \lambda k'_p \geq \varepsilon_2 k_i$$

which is equivalent to

$$\frac{\varepsilon_2}{\varepsilon_1} \leq \frac{\lambda k'_p}{k_i} \quad (39)$$

which holds in view of (35) and (32).

Now, we show that (37) holds. Again, from (35) we have

$$\frac{\varepsilon_2}{\varepsilon_1} \leq \frac{a\varepsilon_1}{2k_i} \quad \Leftrightarrow \quad 2\varepsilon_2 k_i \leq \varepsilon_1^2 a \quad (40)$$

hence, (37) holds if

$$\varepsilon_1 (b-1) - \varepsilon_1 a \geq 2\varepsilon_2 k_i + \varepsilon_1 a$$

which is satisfied if

$$\varepsilon_1 (b-1) \geq 2(\varepsilon_2 k_i + \varepsilon_1 a)$$

or equivalently if

$$b-1-2a \geq \frac{2\varepsilon_2 k_i}{\varepsilon_1}$$



which in turn, holds in view of (35).

Finally, we show that inequality (38) holds. Indeed since  $\lambda < 1$ , (38) is implied by

$$\frac{\lambda k'_p a}{b} \geq \varepsilon_1 k'_p + \varepsilon_2 k_i + \varepsilon_1 a + \lambda k'_p \varepsilon_1$$

which holds if

$$\frac{\lambda a}{2b} \geq \varepsilon_1(1 + \lambda) \quad (41)$$

and

$$\frac{\lambda k'_p a}{2b} \geq \varepsilon_2 k_i + \varepsilon_1 a. \quad (42)$$

Inequality (41) holds in view of (33) and since  $2\varepsilon_2 k_i \leq \varepsilon_1^2 a$ . Inequality (42) holds if

$$\frac{\lambda k'_p a}{2b} \geq \varepsilon_1 \left( \frac{a\varepsilon_1}{2} + a \right) \Leftrightarrow \frac{\lambda k'_p}{2b} \geq \left( \frac{\varepsilon_1^2}{2} + \varepsilon_1 \right).$$

Since  $\varepsilon_1 < 1$ , the latter holds if

$$\frac{\lambda k'_p}{2b} \geq \frac{3\varepsilon_1}{2} \Leftrightarrow \frac{k_d}{3b} \geq \varepsilon_1$$

which holds in view of (33).

#### Positivity of $Q_2$

Define  $\eta := (1/2)\varepsilon_1(b-1)$  and  $Q'_2 := \text{diag}\{[0 \ \eta \ 0 \ 0]\}$  then,  $\zeta_1^\top Q_2 \zeta_1 = \zeta_1^\top [Q_2 - Q'_2] \zeta_1 + (1/2)\varepsilon_1(b-1)e_\omega^2$ .

The Schur Complement of  $[Q_2 - Q'_2]$  is non-negative if

$$\varepsilon_2 \geq \begin{bmatrix} -\varepsilon_2 k'_p \\ -k_d \varepsilon_2 \end{bmatrix}^\top \begin{bmatrix} \varepsilon_1 k'_p & 0 \\ 0 & k_d \left( \frac{a}{b} - \varepsilon_1 \right) \end{bmatrix}^{-1} \begin{bmatrix} -\varepsilon_2 k'_p \\ -k_d \varepsilon_2 \end{bmatrix}$$

the latter is equivalent to

$$\varepsilon_2 \geq \frac{\varepsilon_2^2 k'_p}{\varepsilon_1} + \frac{\varepsilon_2^2 k'_p \lambda}{\left( \frac{a}{b} - \varepsilon_1 \right)} \quad (43)$$

using  $\lambda_2 := \frac{a}{b\varepsilon_1}$  we see that (43) holds if

$$\varepsilon_2 \geq \frac{\varepsilon_2^2 k'_p}{\varepsilon_1} + \frac{\varepsilon_2^2 k'_p \lambda}{(\lambda_2 - 1)\varepsilon_1} = \frac{\varepsilon_2^2 k'_p}{\varepsilon_1} \left[ 1 + \frac{\lambda}{\lambda_2 - 1} \right]$$

which holds in view of (35).

#### B. Negativity of $\dot{V}$

We establish that the factor of  $e_\omega^2$  in (26) may be rendered positive. From (22) and (23) we see that the argument of  $\min\{\cdot\}$  in (33) equals  $k_d/3b$ . Therefore,  $\varepsilon_1$  must belong to the interval

$$\left[ \frac{c}{b-1}, \frac{k_d}{3b} \right]$$

which is non-empty in view of (22).