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# Cascaded-based Stabilization of Time-varying Systems Using 2nd-order Sliding Modes

Antonio Estrada\*    Antonio Loria<sup>†</sup>    Raúl Santiesteban<sup>‡</sup>    Leonid Fridman<sup>§¶</sup>

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## Abstract

We present a result on stabilization of strict feedback systems with unknown disturbances based on the so-called twisting algorithm, a second-order sliding mode controller for the double integrator. The novelty of the note relies in the stability analysis of the closed-loop system and the relaxation of stability conditions (we do not assume boundedness of trajectories). The proof of the main result is constructed along similar lines as for well-established theorems for nonlinear time-varying systems in *cascade*, with continuous right-hand sides. Although we restrict our analysis to second-order systems, the purpose of this note is to settle the basis for a methodological stability analysis approach for higher-order systems in strict feedback form, under the influence of uncertain perturbations. An illustrative example is provided.

## 1 Introduction

We study strict-feedback systems affected by additive and possibly unbounded disturbances. The control goal is to stabilize the origin of the closed-loop system to zero in finite time. Roughly, the control method consists in a two-loop design: an outer loop in which the control law is designed by defining a virtual control input as in backstepping control, and an inner loop in which the twisting algorithm of [1] is used to exactly compensate for the disturbances. First-order sliding-mode controllers have been applied combined with different robust techniques in order to reduce the effect of such perturbations [2]–[3].

In [4] the authors explore the combination of backstepping and sliding mode control is for systems in strict-feedback form with parameter uncertainties; this is extended to the multi input case in [5]. The procedure proposed in [4],[5] reduces the computational load as compared to standard backstepping, because it only retains  $n-2$  steps of the original backstepping technique, coupling them with an auxiliary second-order subsystem to which a second-order sliding mode control is applied. A differentiation-based coordinate transformation is applied to the auxiliary second-order subsystem in order to obtain a system with the perturbations appearing in the input channel.

The controllers proposed in [2],[3] do not ensure the exact tracking of output unmatched variables. The contributions in [4],[5] rely on an explicit representation of the second-order auxiliary subsystem in which perturbations are *matched*. The controller proposed in [6, 7] which utilizes the so-called quasi-continuous high-order sliding mode algorithm of [8], ensures exact tracking of a smooth signal despite the presence of unmatched perturbations. It uses the idea of virtual control instead of transforming the system as one with matched perturbations. Nevertheless, the scheme in [6] only guarantees *local* stability hence, nothing is ensured about the transient phase. In [7], the transient stage is handled by using integral high-order sliding modes [9], which increases the control complexity.

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Our main contribution strives in the method of stability analysis which particularly, leads to relaxed assumptions. Indeed, instead of constructing a (control) Lyapunov function for the whole system, as it is done in backstepping, we propose regard the system as a cascade that is, in which one of the systems is decoupled from the first. As is well-understood now, at the core of strict-feedback forms one finds *cascaded* systems –[10]. These have been thoroughly studied in the literature of (continuous) nonlinear systems for the last 20 years or so. Cascaded systems consist in two subsystems which independently, are stable and are interconnected by a nonlinearity. Under such setting, a necessary and sufficient condition for stability is that the trajectories of the cascaded system remain bounded –see [11]. As we show in this paper for certain nonlinear systems, the boundedness condition may be relaxed to forward completeness provided that the two cascaded systems separately, are finite-time stable.

A significant difficulty in the analysis of cascaded systems in the context of sliding-modes is the discontinuities of the control laws. In particular, for the twisting algorithm backstepping control leads to a complex cascaded system described by integral-differential equations and equations with discontinuous right-hand sides. One way to analyze the stability of the integral-differential equation is to differentiate however, this leads to ever more complex equations and eventually, to restrictive conditions of boundedness of trajectories –cf. [8]. Besides, backstepping naturally leads to the analysis of a *cascaded* system. Following that train of thought, in the recent note [12] we relaxed the restrictive hypotheses from [8].

Nonetheless, the results available in the literature of cascaded systems are inapplicable as such, in the present setting. A fundamental, yet commonly-made assumption in the analysis of cascaded systems is that one disposes of a Lyapunov function for the perturbed system, taken independently. In the present setting, this is a stumbling block as it comes to asking for a (converse) Lyapunov function for an integral-differential equation, having specific growth-order properties.

This note continues and improves the main results in [12]. Firstly, we use the twisting controller for which a Lyapunov function has been recently proposed in [13]. Then, a theorem for stability of cascades of systems with discontinuous right-hand sides is established. Although the scope of the present paper is limited to second order stems our main result constitutes a first step towards a recursive design method for nonlinear systems in strict feedback form with unmatched uncertainties, reminiscent of backstepping and based on high-order sliding mode control. The rest of the paper is organized as follows. In the following section we present the problem statement and our main result; in Section 4 we present an illustrative example and we conclude with some remarks in Section 5.

## 2 Problem statement and its solution

Consider second-order nonlinear systems of the form

$$\dot{\xi}_1 = f_1(t, \xi_1) + g_1(t, \xi)\xi_2 + \omega_1(t, \xi) \quad (1a)$$

$$\dot{\xi}_2 = f_2(t, \xi) + g_2(t, \xi)u + \omega_2(t, \xi) \quad (1b)$$

where  $\xi = [\xi_1, \xi_2]^T$ ,  $\xi_1, \xi_2 \in \mathbb{R}$  is the state vector and is assumed to be known;  $u \in \mathbb{R}$  is the control input. For simplicity, we assume that  $f_i$  and  $g_i$  are smooth functions and the unknown perturbations  $\omega_1, \omega_2$  are assumed to be bounded on their domain and  $\omega_1$  is assumed to be once continuously differentiable with bounded derivative. For the application of backstepping control, we also assume that  $g_i(t, \xi) \neq 0$  for all  $(t, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$  and all functions  $f_1, f_2, g_1$  and  $g_2$  are available for feedback.

The problem of interest is to design a controller such that the state  $\xi_1$  tracks a desired smooth reference  $t \mapsto \xi_d$  in spite of the presence of the unknown bounded perturbations  $\omega_1, \omega_2$ . For further development, we introduce the tracking error  $\sigma_1 = \xi_1 - \xi_d$ .

We propose a dynamic controller which consists in two nested control loops. For the inner-loop, let  $\xi_2$  correspond to a virtual control input acting on the dynamics equation (1a) with the aim at steering  $\sigma_1 \rightarrow 0$  in finite time. We define the second-order sliding mode controller<sup>1</sup>

$$\phi_1(t, \xi_1) = g_1^{-1}(t, \xi)[-f_1(t, \xi_1) + u_{11} + \dot{\xi}_d] \quad (2a)$$

$$\dot{u}_{11} = -\alpha_1 \text{sgn}(\sigma_1) - \beta_1 \text{sgn}(\dot{\sigma}_1) \quad (2b)$$

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<sup>1</sup>The right hand side of (2b) is referred to as the twisting controller –see [1].

as the ideal control input to (1a) that is, we claim and we shall prove that if  $\xi_2 = \phi_1$ ,  $\sigma_1$  converges to zero in finite time.

With this in mind, we design the outer control loop in order to steer the error  $\sigma_2 = \xi_2 - \phi_1$  to zero in finite time. Let

$$u = g_2^{-1}(t, \xi)[-f_2 - \alpha_2 \text{sgn}(\sigma_2) + v] \quad (3)$$

then, using  $\xi_2 = \sigma_2 + \phi_1$  and the expressions (2a) and (3) into the system's equations (1) we obtain

$$\dot{\sigma}_1 = u_{11} + \omega_1 + g_1'(t, \sigma_1, \sigma_2)\sigma_2 \quad (4a)$$

$$\dot{\sigma}_2 = -\alpha_2 \text{sgn}(\sigma_2) + \omega_2 - \dot{\phi}_1 + v \quad (4b)$$

where

$$g_1'(t, \sigma_1, \sigma_2) = g_1 \left( t, \begin{bmatrix} \sigma_1 + \xi_d(t) \\ \sigma_2 + \phi_1(t, \sigma_1 + \xi_d(t)) \end{bmatrix} \right)$$

and the additional control input  $v$  is left to be defined. For instance if  $\dot{\phi}_1$  is bounded, one can set  $v \equiv 0$  and redefine  $\omega_2$  to incorporate  $\dot{\phi}_1$  as a perturbation in the second equation. Otherwise it is convenient to set  $v = \dot{\phi}_1$  at the expense of further complexity in the controller. Thus, the control problem is solved if we show that the origin  $(\sigma_1, \sigma_2) = (0, 0)$  is finite-time stable for (4).

What is remarkable about the system (4) in closed loop with the dynamic controller (2) is that it is in cascaded form. To see this, we introduce the new variables  $z_1 = \sigma_1$ ,  $z_2 = \omega_1 + u_{11}$  and  $\Phi = -\beta_1 \text{sgn}(\dot{z}_1) + \beta_1 \text{sgn}(z_2)$ . Then, (4) is equivalent to

$$\underbrace{\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} z_2 \\ \dot{\omega}_1 - \alpha_1 \text{sgn}(z_1) - \beta_1 \text{sgn}(z_2) \end{bmatrix}}_{F_1(z)} + \underbrace{\begin{bmatrix} g_1'(t, z_1, \sigma_2)\sigma_2 \\ \Phi \end{bmatrix}}_{G(t, z, \sigma_2)} \quad (5a)$$

$$\dot{\sigma}_2 = -\alpha_2 \text{sgn}(\sigma_2) + \omega_2. \quad (5b)$$

Note that  $\Phi$  is a set-valued map of  $z_1$ ,  $z_2$  and  $\sigma_2$  but it is uniquely defined at  $\sigma_2 = 0$ , in this case  $\Phi = -\beta_1 \text{sgn}(u_{11} + \omega_1) + \beta_1 \text{sgn}(z_2) = 0$  hence,  $\sigma_2 = 0 \Rightarrow G = 0$ .

In summary, the closed-loop dynamics of (1) with (2) and (3) is given by the cascaded system (5) for which we have  $G(t, z, 0) \equiv 0$  if  $\sigma_2 = 0$ . In this case the dynamics is simply  $\dot{z} = F_1(z)$ , which corresponds to the double integrator, known to be finite-time stable under the twisting algorithm –cf. [13]. Based on these observations the following theorem establishes global finite-time stability of the origin of the closed-loop system following arguments as for cascaded systems with continuous right-hand sides.

**Theorem 1** *The origin of the system (1) in closed loop with (2) and (3) is globally finite-time stable provided that  $\alpha_1 - |\dot{\omega}_1| > \beta_1 > |\omega_1|$ ,  $\alpha_2 > |\omega_2|$  and that there exist continuous non-decreasing functions  $\theta_1, \theta_2$  such that*

$$|g_1'(t, z_1, \sigma_2)| \leq \theta_1(|\sigma_2|)|z_1| + \theta_2(|\sigma_2|). \quad (6)$$

**Remark 1** *The functions  $\theta_i$  exist if  $g_1'$  is uniformly bounded in  $t$  for each fixed  $\sigma$  and  $z_1$ . The main restriction imposed on  $g_1'$  is that for each  $\sigma$  and  $t$  the growth order of  $g_1'(t, \cdot, \sigma_2)$  must be linear. There exists no restriction locally, in the neighbourhood of  $(z_1, z_2) = (0, 0)$ .*

### 3 Proof of Main Result

The proof of Theorem 1 is inspired from the main results in [14]. It is composed of three major steps: 1) to show that the origin of (5b) is finite-time stable; 2) to prove that so is the origin of  $\dot{z} = F_1(z)$  and 3) to establish that the trajectories of (5a) do not explode in a finite time smaller than the settling time for (5b).

#### 3.1 Finite-time stability of (5b)

Let  $V_2 = \sigma_2^2$ , its time derivative along the trajectories of (5b) yields  $\dot{V}_2 \leq -2(\alpha_2 - |\omega_2|)|\sigma_2|$  i.e.,

$$\dot{V}_2 \leq -2(\alpha_2 - |\omega_2|)V_2^{1/2}$$

choosing  $\alpha_2$  such that  $\alpha_2 - |\omega_2| > 0$  finite-time stability follows integrating on both sides of the previous expression, along trajectories.

### 3.2 Finite-time stability of $\dot{z} = F_1(z)$

The system (5a) with zero input  $\sigma_2$  has the form of a perturbed double integrator,

$$\dot{z}_1 = z_2 \quad (7a)$$

$$\dot{z}_2 = -\alpha_1 \text{sgn}(z_1) - \beta_1 \text{sgn}(z_2) + \dot{\omega}_1(t, z) \quad (7b)$$

where  $z_1$  and  $z_2 \in \mathbb{R}$  and  $\dot{\omega}_1$  is a bounded perturbation. Such type of systems under the so-called twisting controller have been studied for instance in [13], where a proof of global finite-time stability via Lyapunov's direct method is established. The following proposition establishes global finite stability of (7) with a similar Lyapunov function as in [13] however, the proof provided here has the merit of fitting the cascaded-based design method described previously. We also provide a simple rule to tune the gains in order to obtain a faster convergence and most significantly, our estimate of the settling time is tighter than that in [13].

**Proposition 1** *Let  $M > 0$  be such that  $|\dot{\omega}_1(t, z)| \leq M$  for all  $(t, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ ,  $\gamma_1 > 0$ ,  $\gamma_2 > 0$  and consider the function*

$$V(z_1, z_2) = \alpha_1^2 \gamma_1 z_1^2 + \gamma_2 |z_1|^{3/2} \text{sgn}(z_1) z_2 + \alpha_1 \gamma_1 |z_1| z_2^2 + \frac{1}{4} \gamma_1 z_2^4. \quad (8)$$

Then, we have the following:

- Given any  $\alpha_1 > 0$  and  $\beta_1$  such that  $\alpha_1 > 2\beta_1 > 2M$  there always exist parameters  $\gamma_1, \gamma_2$  such that  $V$  is a strict Lyapunov function for the system (7) that is, it is positive definite, proper and its derivative along the trajectories of (7) is negative definite;
- provided that  $\beta_1 > M$  there always exist  $c_1 > 0$  and  $c_2 \in (0, 1)$  such that

$$\frac{dV}{dz_1} \dot{z}_1 + \frac{dV}{dz_2} \dot{z}_2 \leq -c_1 V^{c_2}; \quad (9)$$

- (hence) for any pair  $(z_{1o}, z_{2o}) \in \mathbb{R}^2$  all generated solutions satisfying  $(z_1(t_0), z_2(t_0)) = (z_{1o}, z_{2o})$  converge to the origin  $(z_1, z_2) = (0, 0)$  in finite time  $t_f$  where

$$t_f \leq \frac{4}{c_1} V(z_{1o}, z_{2o})^{1/4} \quad (10)$$

and  $c_1$  is proportional to  $\beta_1 - M$ .

#### Proof of Proposition 1

Firstly, we show that  $V$  is positive definite and proper without any restriction on the control gains, other than  $\alpha_1 > 0$ . Let  $\mu > 0$  and observe that

$$V(z_1, z_2) = \mu \left( |z_1|^{1/2} + |z_2| \right)^4 + W \quad (11)$$

$$\begin{aligned} W(z_1, z_2) &= (\alpha_1^2 \gamma_1 - \mu) |z_1|^2 - 4\mu |z_1|^{3/2} |z_2| + \gamma_2 |z_1|^{3/2} \text{sgn}(z_1) z_2 \\ &\quad + (\alpha_1 \gamma_1 - 6\mu) |z_1| |z_2|^2 - 4\mu |z_1|^{1/2} |z_2|^3 \\ &\quad + \left( \frac{1}{4} \gamma_1 - \mu \right) |z_2|^4. \end{aligned} \quad (12)$$

Let us show that for any given control gain  $\alpha_1 > 0$  and an appropriate choice of the parameters  $\gamma_1, \gamma_2$  we have  $W \geq 0$ , thereby implying that  $V$  is positive definite and radially unbounded. To that end, let

$$\eta_m = \min \left\{ (\alpha_1^2 \gamma_1 - \mu), \frac{1}{6} (\alpha_1 \gamma_1 - 6\mu), \left( \frac{1}{4} \gamma_1 - \mu \right) \right\}.$$

Note that for any given control gain  $\alpha_1$  one can find parameters  $\gamma_1$  and  $\mu$  such that  $\eta_m > 0$ . Hence

$$W(z_1, z_2) \geq -(4\mu + \gamma_2) \left[ |z_1|^{3/2}|z_2| + |z_1|^{1/2}|z_2|^3 \right] + \eta_m \left[ |z_1|^2 + 6|z_1||z_2|^2 + |z_2|^4 \right]. \quad (13)$$

Furthermore, for any given parameters  $\alpha_1, \gamma_1, \mu > 0$  such that  $\eta_m > 0$ , pick  $\gamma_2 > 0$  such that  $\gamma_2 \leq 4(\eta_m - \mu)$ . Under such conditions Inequality (13) implies that

$$W(z_1, z_2) \geq \eta_m \left[ |z_1|^{1/2} - |z_2| \right]^4 \geq 0.$$

We emphasize that there always exists  $\gamma_2 > 0$  satisfying  $\gamma_2 \leq 4(\eta_m - \mu)$ . Indeed, if for a particular choice of  $\gamma_1$  and  $\mu$  we have  $\eta_m < \mu$  we can redefine  $\gamma_1$  and  $\mu$  such that for any given  $n > 4$

$$\min \left\{ \alpha_1^2 \gamma_1, \frac{\alpha_1 \gamma_1}{6}, \frac{\gamma_1}{4} \right\} \geq \frac{(n+4)\mu}{4}$$

which implies that  $4\eta_m \geq n\mu > 4\mu$ . We conclude that  $V$  is positive definite and radially unbounded for large values of  $\mu, \alpha_1, \gamma_1$  and small values of  $\gamma_2$ .

Now we proceed to compute an upper-bound for  $V$ . To that end we observe from (12), that

$$W(z_1, z_2) \leq (\alpha_1^2 \gamma_1 - \mu)|z_1|^2 + (\gamma_2 - 4\mu)|z_1|^{3/2}|z_2| + (\alpha_1 \gamma_1 - 6\mu)|z_1||z_2|^2 + (\gamma_2 - 4\mu)|z_1|^{1/2}|z_2|^3 + \left(\frac{1}{4}\gamma_1 - 4\mu\right)|z_2|^4 \quad (14)$$

which together with (11), implies that

$$V(z_1, z_2) \leq (\mu + \eta_M) \left[ |z_1|^{1/2} + |z_2| \right]^4 \quad (15)$$

with

$$\eta_M = \max \left\{ (\alpha_1^2 \gamma_1 - \mu), \frac{(\alpha_1 \gamma_1 - 6\mu)}{6}, \frac{(\gamma_2 - 4\mu)}{4}, \left(\frac{1}{4}\gamma_1 - 4\mu\right) \right\}.$$

Next, we compute the total time derivative of  $V$  along the trajectories of (7). We have

$$\begin{aligned} \dot{V}(z_1, z_2) &= 2\alpha_1^2 \gamma_1 z_1 z_2 \\ &+ \gamma_2 |z_1|^{3/2} \text{sgn}(z_1) \left( -\alpha_1 \text{sgn}(z_1) - \beta_1 \text{sgn}(z_2) + \dot{\omega}_1 \right) \\ &+ \frac{3}{2} \gamma_2 |z_1|^{1/2} z_2^2 + \alpha_1 \gamma_1 z_2^2 \text{sgn}(z_1) z_2 \\ &+ 2\alpha_1 \gamma_1 |z_1| z_2 \left( -\alpha_1 \text{sgn}(z_1) - \beta_1 \text{sgn}(z_2) + \dot{\omega}_1 \right) \\ &+ \gamma_1 z_2^3 \left( -\alpha_1 \text{sgn}(z_1) - \beta_1 \text{sgn}(z_2) + \dot{\omega}_1 \right). \end{aligned} \quad (16)$$

which implies that

$$\begin{aligned} \dot{V}(z_1, z_2) &\leq -2\alpha_1 \gamma_1 (\beta_1 - M) |z_1||z_2| - \gamma_1 (\beta_1 - M) |z_2|^3 \\ &- \gamma_2 (\alpha_1 - \beta_1 - M) |z_1|^{3/2} + \frac{3}{2} \gamma_2 z_2^2 |z_1|^{1/2} \end{aligned} \quad (17)$$

Let  $\kappa > 0$  and add  $\kappa$  times

$$-(|z_1|^{1/2} + |z_2|)^3 + \left[ |z_1|^{3/2} + 3|z_1||z_2| + 3|z_1|^{1/2}|z_2|^2 + |z_2|^3 \right] = 0$$

to the right-hand side of (17). We obtain

$$\begin{aligned} \dot{V}(z_1, z_2) &\leq -[\gamma_2 (\alpha_1 - \beta_1 - M) - \kappa] |z_1|^{3/2} \\ &+ \frac{3}{2} [\gamma_2 + 2\kappa] z_2^2 |z_1|^{1/2} \\ &+ [3\kappa - 2\alpha_1 \gamma_1 (\beta_1 - M)] |z_1||z_2| \\ &- [\gamma_1 (\beta_1 - M) - \kappa] |z_2|^3 - \kappa (|z_1|^{1/2} + |z_2|)^3. \end{aligned}$$

which implies that

$$\begin{aligned}\dot{V}(z_1, z_2) &\leq -[\gamma_2(\alpha_1 - \beta_1 - M) - \kappa]|z_1|^{3/2} \\ &\quad - |z_2| \begin{bmatrix} |z_1|^{1/2} \\ |z_2| \end{bmatrix}^\top N \begin{bmatrix} |z_1|^{1/2} \\ |z_2| \end{bmatrix} \\ &\quad - \kappa(|z_1|^{1/2} + |z_2|)^3.\end{aligned}\tag{18}$$

where

$$N = \begin{bmatrix} 2\alpha_1\gamma_1(\beta_1 - M) - 3\kappa & -\frac{3}{4}[\gamma_2 + 2\kappa] \\ -\frac{3}{4}[\gamma_2 + 2\kappa] & \gamma_1(\beta_1 - M) - \kappa \end{bmatrix},$$

which is positive semidefinite for sufficiently large values of  $\gamma_1$  and  $\beta_1 > M$ . Additionally, the first term on the right-hand side of (18) is negative if  $\alpha_1 - M > \beta_1 > M$  and  $\gamma_2(\alpha_1 - \beta_1 - M) - \kappa > 0$ . In particular, for any given gain  $\beta_1 > M$  one can find parameters  $\gamma_1 \gg 1$  and  $\kappa' \in (0, 1)$  such that  $N \geq 0$  with  $\kappa = \kappa'\gamma_1(\beta_1 - M)$ . To see this observe that

$$N = \gamma_1(\beta_1 - M) \begin{bmatrix} 2\alpha_1 - 3\kappa' & -\frac{3}{4}\left(\frac{\gamma_2}{(\beta_1 - M)\gamma_1} + 2\kappa'\right) \\ -\frac{3}{4}\left(\frac{\gamma_2}{(\beta_1 - M)\gamma_1} + 2\kappa'\right) & 1 - \kappa' \end{bmatrix}.$$

Let  $k > 1$  be such that  $\beta_1 - M = k$  and  $\gamma_1 = k\gamma_2$  and let  $\kappa' = 1/k$ . Then, the matrix  $N$  is positive semidefinite if  $2\alpha_1 \geq 3/k$  and

$$\left(2\alpha_1 - \frac{3}{k}\right) \left(1 - \frac{1}{k}\right) \geq \frac{9}{16} \left(\frac{1}{k^2} + \frac{3}{k}\right)^2$$

which is easily fulfilled since the number on the right-hand side is smaller than 9. On the other hand, the condition  $\gamma_2(\alpha_1 - \beta_1 - M) - \kappa > 0$  is equivalent to  $\gamma_2[\alpha_1 - (\beta_1 - M) - \kappa'k(\beta_1 - M)] > 0$  which holds if  $\alpha_1 > 2\beta_1$  which is satisfied by assumption.

Finally, we proceed to find  $c_1$  and  $c_2$  such that (9) holds. From the above we have

$$\dot{V}(z_1, z_2) \leq -\kappa(|z_1|^{1/2} + |z_2|)^3$$

and on the other hand, from (15) we have

$$-V(z_1, z_2)^{3/4} \geq -(\mu + \eta_M)^{3/4} \left[|z_1|^{1/2} + |z_2|\right]^3$$

hence,

$$\dot{V}(z_1, z_2) \leq -\frac{\kappa}{(\mu + \eta_M)^{3/4}} V(z_1, z_2)^{3/4}.\tag{19}$$

In particular, (9) holds for any given  $\alpha_1 > 0$  and  $\beta_1 > M$ , with  $\kappa' \in (0, 1)$ ,  $\gamma_1 \gg 1$  and

$$c_1 = \frac{\kappa'\gamma_1(\beta_1 - M)}{(\mu + \eta_M)^{c_2}}, \quad c_2 = \frac{3}{4}.$$

Therefore, an upper bound for time convergence of the trajectories to zero, for the perturbed case, may be computed by integrating  $\dot{V} \leq -c_1 V^{c_2}$  along the trajectories generated by (7) from any pair of initial conditions  $(z_{1o}, z_{2o}) \in \mathbb{R}^2$  that is,

$$t_f \leq \frac{4}{c_1} V(z_{1o}, z_{2o})^{1/4}.\tag{20}$$

We show that  $c_1$  may be taken proportional to  $\beta_1 - M$ . For  $\alpha_1 > 1$  and  $\gamma_2 \leq \alpha_1^2\gamma_1 - 5\mu$  we have  $\eta_M = \alpha_1^2\gamma_1 - \mu$  hence

$$c_1 = \kappa' \frac{\gamma_1^{1/4}}{\alpha_1^{3/2}} (\beta_1 - M)$$

so  $c_1 \propto (\beta_1 - M)$  provided that  $\gamma_1 \propto \alpha_1^6$ . Note that  $\gamma_2 \leq \alpha_1^2\gamma_1 - 5\mu$  holds if  $\gamma_2 = 5\mu$  and  $\alpha_1 \geq \sqrt{2/k}$  which in turn holds if  $\alpha_1 > \sqrt{2}$ .  $\blacksquare$

### 3.3 Proof of finite-time stability for (5a)

This is the last part of the proof of Theorem 1. It relies on the observation that the last two terms on the right-hand side of the total time-derivative of  $V$  along the trajectories of (5a),

$$\frac{dV}{dz}[F_1(z) + G(t, z, \sigma_2)] \leq -c_1 V^{c_2} + \frac{\partial V}{\partial z_1} g'_1(t, z_1, \sigma_2) \sigma_2 + \frac{\partial V}{\partial z_2} \Phi, \quad (21)$$

become zero in finite time. However, to use this argument we first need to show first that the solutions  $z(t)$  of (5a) exist for all  $t$  and in particular, that they do not explode before the settling time of  $\sigma_2(t)$ . To that end, we observe that since  $\sigma_2(t)$  converges to zero in finite time it is globally uniformly bounded hence, for  $\sigma_2 = \sigma_2(t)$  we have from (6),

$$\begin{aligned} |g'_1(t, z_1, \sigma_2(t))| |\sigma_2(t)| &\leq \left[ \theta_1(|\sigma_2(t)|) |z_1| + \theta_2(|\sigma_2(t)|) \right] |\sigma_2(t)| \\ &\leq c_3 |z_1| + c'_3 \quad \forall t \geq 0, z_1 \in \mathbb{R} \end{aligned} \quad (22)$$

where  $c_3$  and  $c'_3$  depend only on the size of  $\sigma_2(t_0)$ . Next, observe that  $|z_1|^{3/2} \text{sgn}(z_1) = z_1 |z_1|^{1/2}$ , hence

$$\begin{aligned} \frac{\partial V}{\partial z_1} &= 2\alpha_1^2 \gamma_1 z_1 + \frac{3}{2} \gamma_2 |z_1|^{1/2} z_2 + \alpha_1 \gamma_1 \text{sgn}(z_1) z_2^2 \\ \frac{\partial V}{\partial z_2} &= \gamma_2 |z_1|^{3/2} \text{sgn}(z_1) + 2\alpha_1 \gamma_1 |z_1| z_2 + \gamma_1 z_2^3 \end{aligned}$$

and defining  $c_4 := \max \{ 2\alpha_1^2 \gamma_1, \frac{3}{4} \gamma_2, \alpha_1 \gamma_1 \}$  we see that

$$\left| \frac{\partial V}{\partial z_1} \right| \leq c_4 \left( |z_1|^{1/2} + |z_2| \right)^2$$

which implies that

$$\left| \frac{\partial V}{\partial z_1} \right| |z_1| \leq c_4 \left( |z_1|^{1/2} + |z_2| \right)^4 \quad \forall z_1 \in \mathbb{R}, z_2 \in \mathbb{R}.$$

From this, (11) and the fact that  $W \geq 0$  we see that

$$\left| \frac{\partial V}{\partial z_1} \right| |z_1| \leq \frac{c_4}{\mu} V(z_1, z_2). \quad (23)$$

Similarly, define  $c_9 := \max \{ \gamma_2, \frac{2\alpha_1}{3} \gamma_1, \gamma_1 \}$  then,

$$\left| \frac{\partial V}{\partial z_2} \right| \leq c_9 \left( |z_1|^{1/2} + |z_2| \right)^3 \quad (24)$$

which implies that

$$\left| \frac{\partial V}{\partial z_2} \right| |z_2| \leq \frac{c_9}{\mu} V(z_1, z_2) \quad \forall z_1 \in \mathbb{R}, z_2 \in \mathbb{R}. \quad (25)$$

To conclude, consider (21) and (22) together with (23) and (25). Then, since  $|\Phi| \leq 2\beta_1$ ,

$$\{t \in \mathbb{R}_{\geq 0} : |z_1(t)|, |z_2(t)| \geq 1\} \implies \frac{dV}{dz}[F_1(z(t)) + G(t, z(t), \sigma_2(t))] \leq \frac{(c_3 + c'_3)c_4 + 2\beta_1 c_9}{\mu} V(z_1(t), z_2(t)).$$

Similarly, using (24) we deduce that

$$\{t \in \mathbb{R}_{\geq 0} : |z_1(t)|, |z_2(t)| \leq 1\} \implies \frac{dV}{dz}[F_1(z(t)) + G(t, z(t), \sigma_2(t))] \leq \frac{c_3 c_4}{\mu} V(z_1(t), z_2(t)) + 16c_9 \beta_1 + 4c'_3 c_4.$$

The last two implications lead to the inequality

$$\dot{V}(z_1(t), z_2(t)) \leq c_{10} V(z_1(t), z_2(t)) + c_{11} \quad \forall t \in \mathbb{R}_{\geq 0}$$

with  $\mu c_{10} = [(2c_3 + c'_3)c_4 + \beta_1 c_9]$  and  $c_{11} = 16c_9 \beta_1 + 4c'_3 c_4$ . Integrating on both sides of the latter inequality from any  $t_0 \in \mathbb{R}_{\geq 0}$  to  $\infty$  we conclude that the trajectories exist for all  $t$ .

Next, let  $t_f < \infty$  be the settling time for  $\sigma_2(t)$ . From forward completeness, for all  $t$  such that  $t > t_f$  and observing that  $G(t, z, 0) = 0$  we obtain, invoking (21),

$$\frac{dV}{dz}[F_1(z(t)) + G(t, z(t), \sigma_2(t))] \leq -c_1 V(z_1(t), z_2(t))^{c_2}$$

for all  $t \geq t_f$ . Finite-time stability follows integrating on both sides of the latter inequality.

## 4 Example

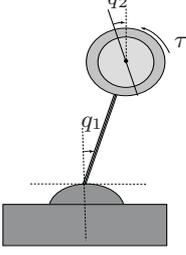


Fig. 1. Inertial wheel pendulum

Consider the inertia-wheel pendulum illustrated in Figure 4. The control goal is to stabilize the system to  $q_1 = q_d$  constant. The dynamics in Lagrangian coordinates is given by

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ J_2 & J_2 \end{bmatrix}^{-1} \begin{bmatrix} h \sin(q_1) \\ 0 \end{bmatrix} + \begin{bmatrix} J_1 & J_2 \\ J_2 & J_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \quad (26)$$

where  $J_1 > J_2$  are constant inertia parameters,  $\tau$  is the control torque and  $\delta = [\delta_1, \delta_2]^\top$  is a bounded perturbation with continuous and uniformly bounded derivatives<sup>2</sup>.

To apply our main result, we must express the system's dynamics in the form (1). To that end, we start by applying a global coordinate transformation reported in [15] to transform the Lagrangian equations into a strict-feedback form. We introduce the new state variables

$$z_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = J_1 \dot{q}_1 + J_2 \dot{q}_2 \quad (27a)$$

$$z_2 = \dot{q}_2 \quad (27b)$$

where  $\mathcal{L} = \mathcal{T} - V$  corresponds to the Lagrangian that is, the difference between kinetic and potential energy. Then, we differentiate on both sides of (27) and use (26) to obtain

$$\dot{z}_1 = h \sin(q_1) \quad (28a)$$

$$\dot{z}_2 = \frac{h \sin(q_1)}{J_2 - J_1} - \frac{J_1}{J_2(J_2 - J_1)} [\tau + \delta_2] \quad (28b)$$

and, from (27),

$$\begin{aligned} \dot{q}_1 &= \frac{1}{J_1} [z_1 - J_2 z_2] + J_1 \delta_1 + J_2 \delta_2 \\ \dot{q}_2 &= z_2. \end{aligned} \quad (29)$$

Equations (28) may be written in the form (1) with  $\xi_1 = q_1$  and  $\xi_2 = z_2$ , and replacing  $z_1$  with  $z_1(t)$  on the interval of definition that is,

$$\begin{aligned} f_1 &= \frac{1}{J_1} z_1(t), \\ g_1 &= -\frac{J_2}{J_1}, \\ \omega_1 &= J_1 \delta_1 + J_2 \delta_2, \\ f_2 &= \frac{1}{J_2 - J_1} h \sin(\xi_1), \\ g_2 &= -\frac{J_1}{J_2(J_2 - J_1)}, \\ \omega_2 &= \delta_2 \end{aligned}$$

so condition (6) holds with  $\theta_1 \equiv 0$  and a constant  $\theta_2 \geq |J_2/J_1|$ .

The controller is constructed following the method explained in Section 2. The first sliding surface  $\sigma_1 = \xi_1 - \xi_d$  in this case corresponds to  $\sigma_1 = q_1 - q_d$  and the virtual control input  $\phi_1$  is

$$\begin{aligned} \phi_1(t, \xi_1) &= -\frac{J_1}{J_2} \left[ -\frac{1}{J_1} z_1(t) + u_{1,1} \right] \\ \dot{u}_{1,1} &= -\alpha_1 \text{sgn}(\sigma_1) - \beta_1 \text{sgn}(\dot{\sigma}_1). \end{aligned}$$

<sup>2</sup>The function  $\delta$  which may depend on time and state, is assumed once continuously differentiable; and such that  $\delta$  and  $\dot{\delta}$  map their domain (not specified on purpose) into compact sets.

in which expression the derivative  $\dot{\sigma}_1$  is computed by means of the robust differentiator —see [16],

$$\begin{aligned}\dot{s}_0 &= -\lambda_2 L^{1/2} |s_0 - \sigma_1|^{1/2} \text{sgn}(s_0 - \sigma_1) + s_1 \\ \dot{s}_1 &= -\lambda_1 L \text{sgn}(s_0 - \dot{s}_0).\end{aligned}$$

Next, we define  $\sigma_2 = z_2 - \phi_1(t, \xi_1)$  and the control input as

$$u = -\frac{J_2(J_2 - J_1)}{J_1} \left[ \frac{h \sin(\xi_1)}{(J_2 - J_1)} - \alpha_2 \text{sgn}(\sigma_2) \right].$$

Note that  $\dot{\phi}_1(t, \xi_1)$  is not cancelled in the control input since, as a simple inspection shows,  $\dot{\phi}_1(t, \xi_1)$  is bounded. To compensate for this disturbance we use the dynamically defined gain

$$\alpha_2(\sigma_2) = \alpha_{2a} e^{\alpha_{2b} |\sigma_2|}.$$

The choice for this gain as opposed to a constant is motivated by performance improvement; note that the gain is large when the sliding error is large and small in the vicinity of the sliding surface, in order to reduce overshoot while reducing convergence time.

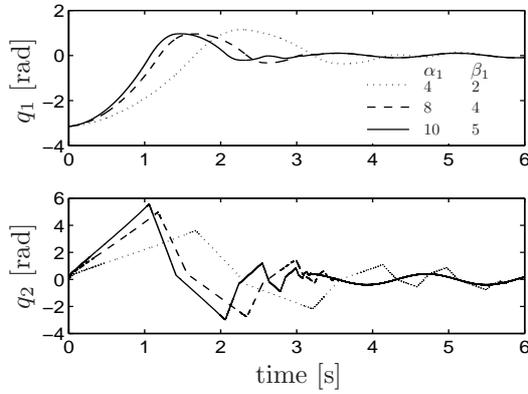


Figure 2: Positions  $q_1$  and  $q_2$ .

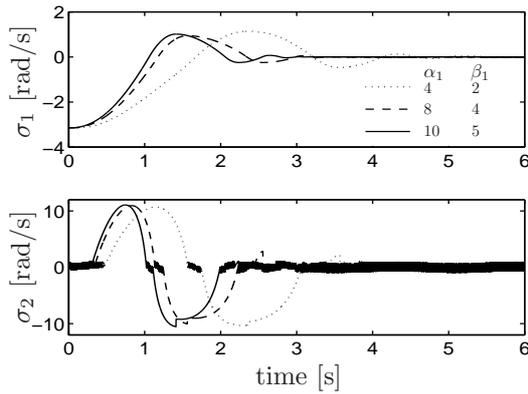


Figure 3: Errors  $\sigma_1$  and  $\sigma_2$ .

We have performed some simulations to test the efficiency of the controller above. The parameters  $J_i$  and  $h$  are computed from an experimental benchmark manufactured by Quanser Inc; we have  $J_1 = 4.572 \times 10^{-3}$ ,  $J_2 = 2.495 \times 10^{-5}$ , and  $h = 0.3544$ . The desired position is set to  $0.1 \sin(4t)$ ; the initial conditions for the inertia wheel pendulum are all set to zero except for  $q_1(0) = -\pi$ , *i.e.*, the pendulum is assumed to start off from the downward position. The disturbances are  $\omega_1 = 0.1 \cos(40t)$  and

$\omega_2 = 0.1 \sin(40t)$ . The controller parameters  $\alpha_1 = 5$ ,  $\beta_1 = 3$ ,  $\alpha_{2a} = 5000$  and  $\alpha_{2b} = 0.1$  and, for the differentiator,  $\lambda_1 = 1.1$ ,  $\lambda_2 = 1.5$ ,  $L = 10$  were used. The graphs of the system's responses for three different set of values of the parameters  $\alpha_1, \beta_1$  are depicted in Figures 2-3. The Figure 4, is the control signal corresponding to the simulation for  $\alpha_1 = 4, \beta_1 = 2$ . The simulations confirm that the finite-time convergence  $T_f$  is inversely proportional to  $\beta_1$ . Note also that increasing  $\beta_1 - M$ , eventually implies to increase  $\alpha_1$  in order to fulfill the conditions of Proposition 1.

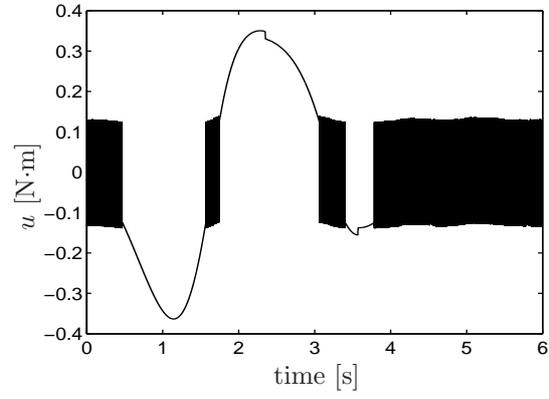


Figure 4: Control signal  $u$  ( $\alpha_1 = 4, \beta_1 = 2$ ).

## 5 Conclusion

We believe that the analysis based on arguments for cascaded systems shall be useful in extending the control method to systems in strict feedback form, of order higher than two. So far, there remains a fundamental obstacle: beyond second-order systems, the design procedure is systematic (reminiscent of backstepping) however, it is considerably intricate since it involves high-order sliding modes. The lack of Lyapunov functions for the case of order larger than two, fundamental in the stability analysis of the cascade, hampers further extensions at this point.

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