A family of functions with two different spectra of singularities

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Study of a "sparse" multifractal signal

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Abstract

Our goal is to study the multifractal properties of functions of a given family which have few non vanishing wavelet coefficients. They are indeed somehow "sparse" signals. We compute at each point the pointwise Hölder exponent of these functions and also their local L^p regularity, computing the so-called p-exponent. We prove that in the general case the Hölder and p exponent are different at each point. We also compute the dimension of the sets where the functions have a given pointwise regularity and prove that these functions are multifractal both from the point of view of Hölder and L^p local regularity with different spectra of singularities. Furthermore, we check that multifractal formalism type formulas hold for the functions in that family.

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1 Introduction

Multifractal analysis for signal analysis was developped in the context of fully developped turbulence in order to study the signal of velocity of turbulent fluid, whose regularity is changing from point to point.

Indeed one criterium for estimating pointwise regularity of a signal at a point x_0 is to compute the pointwise Hölder exponent . Recall its definition.

Definition 1. *Let $x_0 \in \mathbb{R}^d$ and $\alpha \geq 0$.*
A locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $C^\alpha(x_0)$ if there exists $C > 0$ and a polynomial P_{x_0} with $deg(P) \leq [\alpha]$ and such that on a neighborhood of x_0,

$$|f(x) - P_{x_0}(x)| \leq C|x - x_0|^\alpha. \tag{1}$$

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The pointwise Hölder exponent of f at x_0 is $h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$.

Under these assumptions, let $\alpha \in \mathbb{R}$ be fixed and set $E_f(\alpha) = \{x_0 : h_f(x_0) = \alpha\}$. Performing the multifractal analysis of the signal f is computing for every α the Hausdorff dimension $d_f(\alpha)$ of the set $E_f(\alpha)$.

If $E_f(\alpha)$ is non empty for more than two values of α the function is called a multifractal function (if it isn't empty for one single value of α it will be called a monofractal function). The map $\alpha \mapsto d_f(\alpha)$ is called the spectrum of Hölder singularities of the function f.

For example one can see that the classical Weierstrass function

$$f : x \mapsto \sum_{n\in\mathbb{N}} A^n \cos(B^n x) \qquad \text{with } AB > 1 \text{ and } A < 1$$

is a monofractal function, indeed $d_f(\alpha) = -\infty$ for $\alpha \neq -\frac{\ln(A)}{\ln(B)}$ and 1 for $\alpha = -\frac{\ln(A)}{\ln(B)}$. Other examples can be found in numerous works and studies of multifractal functions (see [12] for references).

Other types of pointwise singularities can be studied. Calderon and Zygmund in [3] introduced a local exponent based on L^p norms, the "p exponent".

Definition 2. *Let $p \in [1, \infty]$ and u such that $u \geq -\frac{d}{p}$. Let f be a function in L^p_{loc}. f belongs to $T^p_u(x_0)$ if there exists $R > 0$, P a polynomial such that $deg(P) \leq u$, and $C > 0$ such that*

$$\forall \rho \leq R : \quad \left(\frac{1}{\rho^d} \int_{|x-x_0|\leq\rho} |f(x) - P(x)|^p dx \right)^{\frac{1}{p}} \leq C\rho^u. \tag{2}$$

The p-exponent of f at x_0 is $u^p_f(x_0) = \sup\{u : f \in T^p_u(x_0)\}$

Under these assumptions, let $\alpha \in \mathbb{R}$ be fixed. We denote $E_{f,p}(\alpha) = \{x_0 : u^p_f(x_0) = \alpha\}$ and $d_{f,p}(\alpha)$ the Hausdorff dimension of $E_{f,p}(\alpha)$. The map $\alpha \mapsto d_{f,p}(\alpha)$ is called the spectrum of p-singularities of the function f.

The p exponent was studied in the context of multifractal analysis in [9], [8] for instance, but as far as we know there weren't many contributions where examples of multifractal functions were studied from this point of view. The work of [6] proves that generically (in the sense of prevalence) in a given functional space like a Sobolev or Besov space, the spectrum of Hölder singularities and the one of p singularities coincide. Indeed the author proves the existence of a prevalent set of functions in a given Sobolev or Besov space (i.e a set whose complement is a Haar-null set) which have the same spectra. Our family of functions don't belong to this prevalent set, especially because the structure of

the wavelet coefficients for our family is not at all the same as the one of the functions in this set. Indeed in our case we have "few" non vanishing coefficients (there is actually a lot of scales where they all vanish) which is not the case in the prevalent set. One of the consequences of many vanishing coefficients is that the p exponent and Hölder exponent may differ as we will see in the following. Let us mention that this could be the goal of a further study to check if there are other prevalent sets based on such families of sparse signals where the spectrum of Hölder singularities and the spectrum of p-singularities don't coincide.

One can easily build functions were the two pointwise exponents are different at least at one given point.

Let f such that at x_0 $u_p^f(x_0)$ and $h_f(x_0)$ are defined. Then remark that $u_p^f(x_0) \geq h_f(x_0)$. One can check that the p exponent doesn't provide the same local information as the Hölder exponent. For example let $\alpha > 0$ and the function $x \mapsto g(x) = |\, x \,|^\alpha \sum_{j=1}^{\infty} I_{D_j}(x)$ with $D_j = [\, 1/2^j - 1/2^{3j},\, 1/2^j \,]$ for $j \geq 0$. The function g satisfies $h_g(0) = \alpha < u_g^p(0) = \alpha + 1/p$ for any $p \geq 1$.

Remark that the signal g is in the same time a sparse signal (with few non vanishing entries) and oscillating since as far as we go to zero it has oscillations. Actually one can check using the properties of the p exponent, that if at a point it is different from the pointwise Hölder exponent there should be oscillations of the function at this point (see for example [9] for details).

In this work we study a slight modification of the family of functions introduced by S. Jaffard [11], and prove that for each member of this family the spectrum of Hölder singularities and the one of p singularities are different. Actually we will compute at each point of $\mathbb{R}$ the pointwise Hölder exponent, as well as the p-exponent. One will see with the description of this kind of signals that it can be described as "sparse" since many of its wavelet coefficients vanish. On the other hand, it is multifractal since the computation of its spectra of singularities proved that they are non trivial. Our results seem to confirm the idea that the p exponent could be a good indicator of sparsity of multifractal signals whenever the computation proves that it is different from the Hölder exponent.

Furthermore we will also check that the spectra of singularities satisfy multifractal formalism type formulas. These formulas are generally heuristic formulas whose goal is to compute the spectrum of singularities with the help of global quantities. The first one was introduced by Frish and Parisi in the context of fully developped turbulence [14]. It was then rewritten using wavelet analysis by A. Arnéodo and al. [1]. The domain of validity, counter examples and generic properties of this kind of formulas were the subjects of studies and they are still active fields [15]. We will focus on the multifractal type formulas described in [10] since they fit with the kind of singularities we study. We will prove that these formulas are satisfied by our signals.

2 Notations and definitions

- In all the following Λ denotes the set of all dyadic intervals $\lambda = [k2^{-j}, (k+1)2^{-j}]$, $j \in \mathbb{Z}, k \in \mathbb{Z}$ and Λ_j with $j \in \mathbb{Z}$ the subset of dyadic cubes λ of the type $\lambda = [k2^{-j}, (k+1)2^{-j}]$ with $k \in \mathbb{Z}$. We will sometimes write $\lambda = (j,k)$ if no confusion is possible. The notation $[x]$ significates that $[x]$ is the integer part of x, and $\lceil x \rceil$ denotes the smallest integer not less than x.
  Recall that a wavelet basis is a set of functions such that ϕ and ψ are functions in $L^2(\mathbb{R})$ and such that they satisfy $\Psi = \{\phi(.-k), k \in \mathbb{Z}\} \bigcup \{2^{j/2}\psi(2^j.-k), j \geq 0; k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. We will use a wavelet basis regular enough, i.e ϕ and ψ are functions in $C^{r+1}(\mathbb{R})$ and with compact support. We will call r the regularity of the basis Ψ. This is always possible (see for example [4] for such constructions).

  In order to simplify the notations we will write $\psi_\lambda(x) = 2^{j/2}\psi(2^j x - k)$.

  To sum up we have the following equality in $L^2(\mathbb{R})$

$$\forall f \in L^2(\mathbb{R}),\ f(x) = \sum_{k\in\mathbb{Z}} c_k \phi(x-k) + \sum_{j\geq 0}\sum_{\lambda\in\Lambda_j} c_\lambda \psi_\lambda(x) \tag{3}$$

  with

$$\begin{aligned} c_k &= \int f(x)\phi(x-k)dx, \forall k \in \mathbb{Z} \\ c_{j,k} = c_\lambda &= \int f(x)\psi_\lambda(x)dx. \end{aligned} \tag{4}$$

- Let $\alpha \geq 1$ and $\mathcal{S}_\alpha$ the set of dyadic points such that $\frac{k'}{2^{j'}} \in \mathcal{S}_\alpha$ if one can find (j,k) such that $\frac{k'}{2^{j'}} = \frac{k}{2^{j-1}} \pm \frac{1}{2^{\alpha j}}$ with $\frac{k}{2^{j-1}}$ an irreducible fraction of order $j-1$.

  We will need the rate of approximation of x_0 by dyadics in a given set $\mathcal{S}_\alpha$ given by

$$r_\alpha(x_0) = \limsup_{j'\to\infty} \frac{\log(|K_{j'}(x_0)2^{-j'} - x_0|)}{\log(2^{-j})}, \tag{5}$$

  where $K_{j'}(x_0) = \text{argmin}_{k,k2^{-j'}\in\mathcal{S}_\alpha}(|x_0 - k2^{-j'}|)$ and $(j, k_{j-1}(x_0))$ the integers such that $\frac{K_{j'}(x_0)}{2^{j'}} = \frac{k_{j-1}(x_0)}{2^{j-1}} \pm \frac{1}{2^{\alpha j}}$ with $\frac{k_{j-1}(x_0)}{2^{j-1}}$ an irreducible fraction.

  Since we always have $|\frac{K_{j'}(x_0)}{2^{j'}} - x_0| \leq 2^{-j}$, then $r_\alpha(x_0) \geq 1$. Given the definition of $r_\alpha(x_0)$, for every $\delta > 0$ there exists a subsequence m'_n

$(m'_n \to \infty$ when $n \to \infty)$ and m_n,
with $\frac{K_{m'_n}(x_0)}{2^{m'_n}} = \frac{k_{m_n-1}}{2^{m_n-1}} \pm \frac{1}{2^{\alpha m_n}}$ such that,

$$|K_{m'_n}(x_0)2^{-m'_n} - x_0| < 2^{-m_n(r_\alpha(x_0)-\delta)}. \tag{6}$$

Furthermore, still using the definition of $r_\alpha(x_0)$, for every $\varepsilon > 0$, there exists a constant $M > 0$ such that for all $m' \geq M$,there is $m \geq M/\alpha$

$$|K_{m'}(x_0)2^{-m'} - x_0| > 2^{-m(r_\alpha(x_0)+\varepsilon)}. \tag{7}$$

- We need to recall the definition of the *Hausdorff dimension*.

  Definition 3. *Let $A \subset \mathbb{R}^d$; if $\varepsilon > 0$, an ε-covering of A is a countable collection $R = \{A_i\}_{i\in\mathbb{N}}$ such that each diameter $|A_i|$ is less than ε, and $R \subset \bigcup_{i=1}^{\infty} A_i$. If $\delta \in [0, d]$, set*

$$M_\varepsilon^\delta = \inf_R \left(\sum_i |A_i|^\delta \right),$$

where the infimum is taken on all ϵ-coverings R.

For any $\delta \in [0, d]$, the δ-dimensional Hausdorff measure of A is $mes_\delta(A) = \lim_{\varepsilon\to 0} M_\varepsilon^\delta$. There exists $\delta_0 \in [0, d]$ such that

$$\forall \delta < \delta_0, \quad mes_\delta(A) = +\infty \quad \text{and} \quad \forall \delta > \delta_0, \quad mes_\delta(A) = 0;$$

this critical δ_0 is the Hausdorff dimension of A, and will be denoted by $dim(A)$.

Suppose that A is a subset of $\mathbb{R}^d$, and that a numerical quantity $\mathcal{H}(x)$ taking values in $\mathbb{R}^m$ is attached to each point of A. If $\mathcal{H}(x)$ has no regularity, then the level sets of $\mathcal{H}$

$$E_H = \{x : \ \mathcal{H}(x) = H\}$$

may be fractal sets. We will consider here the special case where $\mathcal{H}(x)$ is the pointwise Hölder exponent at point x of function f, i.e $h_f(x)$, and for $p \geq 1$ be fixed the case where $\mathcal{H}(x)$ is the p-exponent at point x of function f, i.e $u_p^f(x)$.

We will denote the Hölder spectrum of f $d_f : h \mapsto dim(E_h)$ with $E_h = \{x : \ h_f(x) = h\}$ and the p spectrum of f $d_f^p : u \mapsto dim(E_u^p)$ with $E_u^p = \{x : \ u_p^f(x) = u\}$.

3 Description and results on Jaffard's model

3.1 Notations and definitions

Let $\alpha \geq 1, \beta \geq 1$ be two integers.
In [11], S. Jaffard considers the following function. Let G_m the set of dyadic numbers such that if $\frac{k}{2^j} \in G_m$ then $j = [\alpha\beta m]$ and one can find $(\varepsilon_1, ..., \varepsilon_m) \in \{-1, 1\}^m$ such that $\frac{k}{2^j} = \sum_{l=1}^{m} \frac{\varepsilon_l}{2^{\alpha l}}$.

The set of dyadic cubes $\lambda = (j, k)$ with $\frac{k}{2^j} \in G_m$ is denoted $L_m^{(\alpha,\beta)}$, and $L(\alpha, \beta) = \bigcup_{m\geq 1} L_m^{(\alpha,\beta)}$.

Let $\gamma > 0$ and g be

$$g(x) = \sum_{\lambda \in L(\alpha,\beta)} 2^{-(\gamma+1/2)j} \psi_\lambda(x),$$

Since the set $L(\alpha, \beta)$ is symetric we will restrict our study to the interval $[0, 1]$.

We consider here a slight modification of Jaffard's model which coincides with it in the special case $\alpha = 1$.

3.2 Modification of Jaffard's Model

We define a function f as a modification of Jaffard's Model. This function has three parameters α, β and γ, with $\alpha \geq 1$ and $\beta \geq 1$ integers and $\gamma > 0$ a non integer. We set

$$f(x) = \sum_{\lambda \in \Lambda(\alpha,\beta)} 2^{-(\gamma+1/2)j} \psi_\lambda(x). \tag{8}$$

In (8), $\Lambda(\alpha, \beta) = \bigcup_{m\geq 1} \Lambda_m^{(\alpha,\beta)}$, where $\Lambda_m^{(\alpha,\beta)}$ is the set of $\lambda = (j, k)$ such that

- $j = \alpha\beta m$
- $2^{-j}k = \varepsilon_1\ell_1 + \ldots + \varepsilon_{m-1}\ell_{m-1} + \varepsilon'_m\ell'_m$, where $\varepsilon_i = \pm 1$, $\varepsilon'_i = \pm 1$, $\ell_i = 2^{-i}$ and $\ell'_i = 2^{-\alpha i}$, for each $i \geq 1$ and $\varepsilon_1 = 1$.

Remark that this definition implies that each $j > 1$ creates 2^{m-1} coefficients located at $\frac{k}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$, k odd, at the scale $\alpha\beta m$, and its values are all equal to $2^{-(\gamma+1/2)\alpha\beta m}$.

We can verify that for $\alpha = 1$, the function is the same as the one of Jaffard.
Following the characterization of $C^\gamma(\mathbb{R})$ (see for example [12]) with the help of wavelet coefficients we have $f \in C^\gamma(\mathbb{R})$ since its wavelet coefficients satisfy: for all $j \geq 0$ and all dyadic cube $\lambda \in \Lambda_j$, $|c_\lambda| \leq 2^{-j(\gamma+1/2)}$.

Remark also that f is in fact compactly supported, thus it is in all L^p spaces for $p \geq 1$. Following [7] (see Chapter 5 Section 5.3), and the fact that f is bounded and in $C^\gamma(\mathbb{R})$, we get that the serie in (8) converges in all L^p for $1 \leq p \leq \infty$.

3.3 Results

We will prove the following results.

Theorem 1. *Let α, β and γ, with $\alpha \geq 1$ and $\beta \geq 1$ two integers and $\gamma > 0$ a non integer. Let $p \geq 1$.*

- *Suppose $x_0 \in [-1,1]$ and $r_\alpha(x_0) \leq \alpha\beta$ then $h_f(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)}$ and $u_f^p(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)} + \left(\frac{\alpha\beta}{r_\alpha(x_0)} - 1\right)\frac{1}{p}$*
- *Suppose $x_0 \in [-1,1]$ and $r_\alpha(x_0) > \alpha\beta$ then $h_f(x_0) = \alpha\beta\gamma$ and $u_f^p(x_0) = \alpha\beta\gamma + \frac{\alpha\beta-1}{p}$.*
- *$x_0 \notin [-1,1]$ then $h_f(x_0) = u_f^p(x_0) = +\infty$*

As a corollary we get

Corollary 2. *The Hölder spectrum of f is the function d_f defined on the interval $[\gamma, \alpha\beta\gamma]$ such that $d_f(h) = \frac{h}{\alpha\beta\gamma}$. The p spectrum is the function $d_{f,p}$ defined on the interval $[\gamma, \alpha\beta\gamma + \frac{\alpha\beta-1}{p}]$ and such that $d_{f,p}(u) = \frac{u+\frac{1}{p}}{\alpha\beta\left(\gamma+\frac{1}{p}\right)}$.*

The proof of Theorem 1 is given in Section 4 and the one of Corollary 2 in Section 5.

The results on the multifractal formalisms can be found in Section 6.

4 Local regularity

4.1 Hölder regularity

We first want to study the pointwise Hölder regularity of the function f at each point $x_0 \in \mathbb{R}$. This amounts to compute the pointwise Hölder exponent at $x_0 \in \mathbb{R}$. Recall that we can apply Theorem 1 of [12] which relates the so called wavelet leaders, which depend on the wavelet coefficients of f, with the pointwise Hölder exponent at x_0. We need to start with a definition.

Definition 4. *[12] Two dyadic cubes λ_1 and λ_2 are called adjacent if they are at the same scale and if $dist(\lambda_1, \lambda_2) = 0$ (note that a dyadic cube is adjacent to himself). We denote by $\lambda_j(x_0)$ the dyadic cube of size 2^{-j} containing x_0 and $3\lambda_j(x_0)$ the set of 3 dyadic cubes adjacent to $\lambda_j(x_0)$.*

More precisely if $\lambda = (j,k)$ then we denote $\lambda^l = (j, k-1)$ and $\lambda^r = (j, k+1)$.

Then

$$d_j(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda \tag{9}$$

with

$$d_\lambda = \sup_{\lambda' \subset \lambda} |2^{j'/2} c_{\lambda'}| \tag{10}$$

d_λ is called a "wavelet leader".

Theorem 3. *[12] Let $\alpha > 0$, $x_0 \in \mathbb{R}$ and f be a function in $L^\infty(\mathbb{R})$. Suppose Ψ is a wavelet basis of regularity $r > [\alpha] + 1$.*

- *Suppose f is in $C^\alpha(x_0)$. Then there exists $C > 0$ such that*

$$\forall j \geq 0,\ d_j(x_0) \leq C 2^{-\alpha j} \tag{11}$$

- *Conversely suppose that (11) holds and furthermore there exists $\varepsilon > 0$ such that $f \in C^\varepsilon(\mathbb{R})$. Thus f belongs to $C^{\alpha'}(x_0)$ for all $\alpha' < \alpha$. In particular this means that $h_f(x_0) \geq \alpha$.*

- *Suppose $f \in C^\varepsilon(\mathbb{R})$. Then $h_f(x_0) = \liminf_{j \to \infty} \frac{\ln(d_j(x_0))}{\ln(2^{-j})}$*

Since f belongs to $C^\gamma(\mathbb{R})$ we only need to compute, at each point x_0, $d_j(x_0)$ at each scale $j \geq 0$. This is what we do in Section 4.3.

4.2 L^p pointwise regularity

To study and compute the p exponent at each point x_0 in $\mathbb{R}$, we also compute some quantities related to wavelet coefficients.
Define the so-called p leader

$$D_{\lambda,p} = \left(\sum_{\lambda' \subset \lambda} |c'_\lambda|^p 2^{j'(\frac{p}{2}-1)} \right)^{1/p}, \tag{12}$$

We set

$$D_{j,p}(x_0) = \left(\sum_{\lambda' \subset 3\lambda_j(x_0)} |c'_\lambda|^p 2^{j'(\frac{p}{2}-1)} \right)^{1/p}, \tag{13}$$

with the notation $\lambda' = (j', k')$.

It is easy to see that actually $D_{j,p}(x_0) = \left(\sum_{\lambda \in 3\lambda_j(x_0)} D^p_{\lambda,p} \right)^{1/p}$
Before stating the characterization theorem of [10], we need to recall the characterization of Besov spaces $B^{s,p}_p$ [13].

Theorem 4. *Let $s \in \mathbb{R}$ and $\infty > p > 0$, $q > 0$ and r an integer such that $r > [s] + 1$. Let Ψ be a r regular wavelet basis.*

Suppose f is a tempered distribution with $c_k, k \in \mathbb{Z}$, $(c_{jk})_{j \geq 0; k \in \mathbb{Z}}$ its wavelet coefficients defined by (4).

A tempered distribution f belongs to $B_p^{s,p}$ if (c_k) belongs to l^p and if

$$\sum_{j\geq 0}\sum_{k}\left|c_{j,k}2^{(s+1/2-1/p)j}\right|^p < +\infty \tag{14}$$

Remark that a compactly supported function in $C^\varepsilon(\mathbb{R})$ belongs to any Besov space $B_p^{s,p}$ for $s < \varepsilon$.

We have the following theorem of [10] in a slightly modified version in comparison to the original one

Theorem 5. *Let $p \geq 1$ and $u > \frac{-1}{p}$. Let Ψ a r regular wavelet basis with $r \geq [u]+1$.*

- *Suppose f belongs to $T_u^p(x_0)$ then there exists a constant $C > 0$ such that for all $j \geq 0$*
$$D_{j,p}(x_0) \leq C2^{-j(u+1/p)} \tag{15}$$
- *Suppose f belongs to $B_p^{\delta,p}$ for some $\delta > 0$. If there exists a constant $C > 0$ such that (15) holds for all $j \geq 0$ then $f \in T_{u'}^p(x_0)$ for all $u' < u$.*
- *Suppose $f \in B_p^{\delta,p}$ for some $\delta > 0$. Then $u_p^f(x_0) = \liminf\limits_{j\to\infty} \frac{\ln(D_{j,p}(x_0))}{\ln(2^{-j})} - \frac{1}{p}$*

Remark that we always have $d_j(x_0) \leq D_{j,p}(x_0)$. This means that whenever the Hölder exponent at a point x_0 of a function f is defined, and if this function satisfies the hypothesis of Theorems 3 and 5 we immediately have

$$h_f(x_0) \leq u_f^p(x_0) \tag{16}$$

4.3 Study of the pointwise regularity of the function f

4.3.1 Wavelet and p leaders

Let λ be a dyadic cube indexed by (j,k). Let m_0 such that $\alpha\beta(m_0-1) < j \leq \alpha\beta m_0$ and m_1 such that $\alpha(m_1-1) \leq j < \alpha m_1$. Obviously we have $m_0 \leq m_1$ since α and β are some positive integers.

Let $p \geq 1$ and denote $ap = \gamma p + 1$. Remark that $-ap\alpha\beta + 1 < 0$.

1. Suppose $\alpha = 1$.

   Remark that in this case the dyadic cubes $\lambda = (j,k)$ of $\Lambda(\alpha,\beta)$ satisfy that $\frac{k}{2^j}$ is an irreducible fraction of order j in $]-1,1[$, i.e k is an odd number. Actually it is easy to prove that the set of irreducible fractions of order j in $]-1,1[$ $\frac{k}{2^j}$ with k odd, matches exactly the set of dyadic cubes in $\Lambda(\alpha,\beta)$.

(a) If $\beta = 1$ remark that the computation of d_λ is trivial since in this case j has to be exactly m_0. Since every two dyadic number at scale j is an irreducible fraction we have $d_\lambda = 2^{j/2}2^{-j(\gamma+1/2)} = 2^{-j\gamma}$ or $d_\lambda = 2^{(j+1)/2}2^{-(j+1)(\gamma+1/2)} = 2^{-(j+1)\gamma}$. The same kind of computation yields $D_{\lambda,p} = C2^{-j(\gamma+1/p)}$.

(b) If $\beta \geq 2$, remark that the first scale $j' \geq j$ at which non vanishing coefficients appear is the scale βm_0. These non vanishing coefficients take place at all locations $\frac{K}{2^{m_0}}$ with $\frac{K}{2^{m_0}}$ an irreducible fraction such that K is an odd number.

Remark that we have the following obvious partition.

$$\begin{aligned}\Lambda_j &= \{(j,k), k = -2^j + 1, \dots 2^j - 1\} \\ &= \bigcup_{m=m_0}^{j} \{(m,K)K = -2^m + 1, -2^m + 3, \dots, -3, -1, 1, 3, \dots 2^m - 1\}\end{aligned}$$

i. Thus, we have for $\lambda = (j,k)$ with $\frac{k}{2^j} = \frac{K}{2^m}$ and $j \geq m \geq m_0$.

$$d_\lambda = \sup_{\lambda' \subset \lambda} 2^{j'/2}|c_{\lambda'}| = 2^{-\gamma\beta m} \tag{17}$$

For what concerns the p leader we get, summing all the coefficients located in the dyadic cube λ

$$\begin{aligned}D^p_{\lambda,p} &= 2^{-(\gamma+\frac{1}{2})\beta mp+\beta m(\frac{p}{2}-1)} + 2^{-(\gamma+\frac{1}{2})\beta(j+1)p+\beta(j+1)(\frac{p}{2}-1)} \\ &\quad + \sum_{\ell=j+2}^{\infty} 2^{\ell-j}2^{-(\gamma+\frac{1}{2})\beta\ell p+\beta\ell(\frac{p}{2}-1)} \\ &= 2^{-ap\beta m} + 2^{-ap\beta(j+1)} + \sum_{\ell=j+2}^{\infty} 2^{\ell-j}2^{-ap\beta\ell} \\ &= C\Big[2^{-ap\beta m} + 2^{-ap\beta j}\Big].\end{aligned}$$

with $C > 0$ independent of j and k.

ii. If $m < m_0$ we have

$$d_\lambda = \sup_{\lambda' \subset \lambda} 2^{j'/2}|c_{\lambda'}| = 2^{-\gamma\beta(j+1)} \tag{18}$$

For what concerns the p leader the only difference with the previous case will be on the first terms.

$$\begin{aligned}D^p_{\lambda,p} &= 2^{-ap\beta(j+1)} + 2^{-ap\beta(j+2)} + \sum_{\ell=j+3}^{\infty} 2^{\ell-j}2^{-ap\beta\ell} \\ &= C2^{-ap\beta(j+1)}\end{aligned} \tag{19}$$

with $C > 0$ independent of j and k.

2. If $\alpha > 1$, we can do the same kind of computation for the wavelet leader, we just need to have more cases.

   (a) suppose first that $\frac{k}{2^j} = \frac{K}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$ with $m \geq m_0$. This yields $m_0 \leq m \leq m_1 - 1$ since we need to have $m\alpha \leq j$. Thus $d_\lambda = 2^{-\beta\gamma(\alpha m)}$.

   For what concerns the p leader since $m \leq m_1 - 1$ we can have a non vanishing coefficient located at $\lambda' \subset \lambda$ with $\lambda' = (j', k')$ such that $\frac{k'}{2^{j'}} = \frac{k}{2^j} + \frac{1}{2^{\alpha(\alpha m+1)}}$ if $\alpha(\alpha m+1) \geq j$. Otherwise the first scale $j' > j$ at which a non vanishing coefficient can appear is $\alpha\beta(j+1)$. This yields in all the cases

$$\begin{aligned} 2^{-ap\alpha\beta m} + \sum_{\ell=j+1}^{\infty} 2^{\ell-j} 2^{-ap\alpha\beta\ell} &\leq D^p_{\lambda,p} \leq 2 \times 2^{-ap\alpha\beta m} + \sum_{\ell=j+1}^{\infty} 2^{\ell-j} 2^{-ap\alpha\beta\ell} \\ C' 2^{-ap\alpha\beta m} &\leq D^p_{\lambda,p} \leq C 2^{-ap\alpha\beta m} \end{aligned} \tag{20}$$

   (b) Suppose $\frac{k}{2^j} = \frac{K}{2^{m'}}$ is an irreducible fraction which is not of the previous type, with $m' < m_1$. Thus the first non vanishing coefficient appears at location $\frac{K}{2^{m'-1}} + \frac{1}{2^j} - \frac{1}{2^{\alpha(j+1)}}$. This yields $d_\lambda = 2^{-\alpha\beta\gamma(j+1)}$.

   The p leader is given by the same kind of computation as previously and we get

$$D^p_{\lambda,p} = 2^{-ap\alpha\beta(j+1)} + \sum_{\ell=j+2}^{\infty} 2^{\ell-j} 2^{-ap\alpha\beta\ell} = C 2^{-ap\alpha\beta(j+1)} \tag{21}$$

   If $\frac{k}{2^j} = \frac{K}{2^{m'}} - \frac{1}{2^j}$ with $\frac{K}{2^{m'}}$ irreducible, and $m' < m_1$, we have $d_\lambda = 2^{-\alpha\beta\gamma(j+1)}$. The p leader is given by the computation of (21)

   (c) Suppose then that $\frac{k}{2^j} = \frac{K}{2^{m'}}$ is an irreducible fraction, and $m' \geq m_1$. Then $d_\lambda = 2^{-\alpha\beta\gamma m'}$.

   Again the p leader is simply given by

$$D^p_{\lambda,p} = 2^{-ap\alpha\beta m'} + \sum_{\ell=j+2}^{\infty} 2^{\ell-j} 2^{-ap\alpha\beta\ell} = 2^{-ap\alpha\beta m'} + C 2^{-ap\alpha\beta j} \tag{22}$$

   Suppose $\frac{k}{2^j}$ can be written $\frac{k}{2^j} = \frac{K}{2^{m'-1}} - \frac{1}{2^j}$ with $m' \geq m_1$. Thus $d_\lambda = 2^{-\alpha\beta\gamma m'}$. Formula (22) holds for the p leader.

4.3.2 Computation of the local regularity of f

Let us now prove Theorem 1. Let $x_0 \in \mathbb{R}$ and $p \geq 1$.

1. Let us first remark that for $x_0 \notin [-1,1]$, we have for j large enough $d_j(x_0) = 0 = D_{j,p}(x_0)$ since for j large enough all the wavelet coefficients adjacent to $\lambda_j(x_0)$ are vanishing. Thus $h_f(x_0) = u_p^f(x_0) = +\infty$.

2. Let $x_0 \in [-1,1]$ and $r_\alpha(x_0)$ defined as in (5). Let $\delta > 0$. As it is explained in Section 2, one can find sequences $m'_n \to +\infty$ and $m_n \to +\infty$ which satisfy (6).

   Let $j_n = [m_n(r_\alpha(x_0) - \delta)]$. Let $m_0^{(n)}$ defined by

   $$\alpha\beta(m_0^{(n)}-1) < j_n \leq \alpha\beta m_0^{(n)} \quad \text{i.e.} \quad \alpha\beta(m_0^{(n)}-1) < [m_n(r_\alpha(x_0)-\delta)] \leq \alpha\beta m_0^{(n)}. \tag{23}$$

   and $m_1^{(n)}$ by

   $$\alpha(m_1^{(n)}-1) \leq j_n < \alpha m_1^{(n)} \quad \text{i.e.} \quad \alpha(m_1^{(n)}-1) \leq m_n(r_\alpha(x_0)-\delta) < \alpha m_1^{(n)}. \tag{24}$$

   Let k_n such that $\lambda_{j_n}(x_0) = (j_n, k_n) = \lambda_n$.

   Recall that $\lambda_n^l = (j_n, k_n - 1)$, and $\lambda_n^r = (j_n, k_n + 1)$.

   As we already mentioned it, we have $d_{j_n}(x_0) = \sup\{d_{\lambda_n^l}, d_{\lambda_n^r}, d_{\lambda_n}\}$

   On the other hand for $\varepsilon > 0$ one can find M such that for $m' \geq M$ (7) is satisfied.

   Let us consider $3\lambda_j(x_0) = [(k_j - 1)2^{-j}, (k_j + 2)2^{-j}]$. Choose m' such that it is the smallest as possible such that $\frac{K_{m'}}{2^{m'}} = \frac{k_{m-1}}{2^{m-1}} \pm \frac{1}{2^{\alpha m}} \in 3\lambda_j(x_0)$. We have clearly $\alpha m \leq j - 1$.

   Remark also that it is always possible to choose j large enough such that $m' \geq M$.

   Thus

   $$\left| \frac{K_{m'}}{2^{m'}} - x_0 \right| > 2^{-m(r(x_0)+\varepsilon)}$$

   Since $K_{m'} 2^{-m'} \in 3\lambda_j(x_0)$ we have

   $$\begin{aligned} \frac{3}{2^{j+1}} &> 2^{-m(r(x_0)+\varepsilon)} \\ \frac{\ln(3)}{\ln(2)} - 1 + m(r(x_0)+\varepsilon) &> j \end{aligned} \tag{25}$$

Thus $j \leq m(r_\alpha(x_0) + \varepsilon)$.

Again define m_0 and m_1 such that

$$\alpha\beta(m_0 - 1) < j \leq \alpha\beta m_0 \tag{26}$$

and m_1 by

$$\alpha(m_1 - 1) \leq j < \alpha m_1 \tag{27}$$

We consider the following cases

(a) $r_\alpha(x_0) \leq \alpha$.
Thus we have immediately $m_n \geq m_1^{(n)}$, which yields $m_n \geq m_0^{(n)}$. Since (6) is satisfied, $\frac{K_{m'_n}}{2^{m'_n}} \in 3\lambda_{j_n}(x_0)$.
And it is related to Case 2c and yields

$$d_{j_n}(x_0) \geq C2^{-\alpha\beta\gamma m_n}. \tag{28}$$

Thus for any $\delta > 0$

$$\begin{aligned} h_f(x_0) &= \liminf_{j\to\infty} \frac{\log d_{\lambda_j(x_0)}}{\log 2^{-j}} \\ &\leq \liminf_{n\to\infty} \frac{\log d_{\lambda_{j_n}(x_0)}}{\log 2^{-j_n}} \\ &\leq \frac{\alpha\gamma\beta m_n}{(r_\alpha(x_0) - \delta)m_n} = \frac{\alpha\beta\gamma}{r_\alpha(x_0) - \delta}. \end{aligned} \tag{29}$$

On the other hand we choose $\varepsilon > 0$ small enough so that $r_\alpha(x_0)+\varepsilon < \alpha$ if $r_\alpha(x_0) < \alpha$, and we will have $r_\alpha(x_0) + \varepsilon > \alpha$ if $r_\alpha(x_0) = \alpha$.

i. If $r_\alpha(x_0) < \alpha$ then $\alpha(m_1 - 1) \leq m(r(x_0)+\varepsilon) < \alpha m$ and $m_1 \leq m$. This is again related to Case 2c and yields

$$d_j(x_0) \leq 2^{-\alpha\beta\gamma m} \tag{30}$$

Thus we get

$$h_f(x_0) \geq \frac{\alpha\beta\gamma}{r_\alpha(x_0)} \tag{31}$$

Thus together with (29) we have $h_f(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)}$.

Furthermore following the same proof and using the p leader computed in Case 2c we have $u_f^p(x_0) = \frac{\alpha\beta\left(\gamma+\frac{1}{p}\right)}{r_\alpha(x_0)} - \frac{1}{p}$

ii. If $r(x_0) = \alpha$, then we still have $m_0 \leq m$ and may have $m \leq m_1$, with $\frac{K_{m'}}{2^{m'}} = \frac{k}{2^{m-1}} + \frac{1}{2^{\alpha m}}$ inside $3\lambda_j(x_0)$.

This yields in all cases again

$$d_j(x_0) \leq 2^{-\alpha\beta\gamma m} \tag{32}$$

Thus we get

$$\begin{aligned} \frac{\ln(d_j(x_0))}{\ln(2^{-j})} &\geq \frac{\alpha\beta\gamma(m+1)}{j} \\ &\geq \frac{\alpha\beta\gamma m}{m(\alpha+\varepsilon)} \\ &\geq \frac{\alpha\beta\gamma}{\alpha+\varepsilon} \end{aligned} \tag{33}$$

Thus together with (29) we have $h_f(x_0) = \frac{\alpha\beta\gamma}{\alpha} = \frac{\alpha\beta\gamma}{r_\alpha(x_0)}$.

The same computation yields $u_f^p(x_0) = \frac{\alpha\beta\left(\gamma+\frac{1}{p}\right)}{r_\alpha(x_0)} - \frac{1}{p}$.

(b) Suppose now $\alpha < r(x_0) \leq \alpha\beta$. This yields $m_0^{(n)} \leq m_n < m_1^{(n)}$. It is related to Case 2a and yields

$$d_{j_n}(x_0) \geq C2^{-\alpha\beta\gamma m_n} \tag{34}$$

Thus we get

$$h_f(x_0) \leq \frac{\alpha\beta\gamma}{r_\alpha(x_0)} \tag{35}$$

The same computation yields in the case of the p exponent

$$u_f^p(x_0) \leq \frac{\alpha\beta\left(\gamma+\frac{1}{p}\right)}{r_\alpha(x_0)} - \frac{1}{p} \tag{36}$$

On the other hand we have to consider the following cases.

i. If $\alpha < r_\alpha(x_0) < \alpha\beta$, choosing ε small enough we have $\alpha < r_\alpha(x_0) + \varepsilon$ and $r_\alpha(x_0) + \varepsilon < \alpha\beta$. Using the same notations as in the general setting of this section we get $m_0 \leq m$ but may have $m < m_1$. This is again related to Case 2a and yields

$$d_j(x_0) \leq C2^{-\alpha\beta\gamma m} \tag{37}$$

Thus the following upper-bound holds

$$h_f(x_0) \geq \frac{\alpha\beta\gamma}{r_\alpha(x_0)} \tag{38}$$

Together with (35) this yields

$$h_f(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)} \tag{39}$$

The same computation yields $u_f^p(x_0) = \frac{\alpha\beta(\gamma+\frac{1}{p})}{r_\alpha(x_0)} - \frac{1}{p}$

ii. If $r_\alpha(x_0) = \alpha\beta$ remark that the upperbound (35) yields

$$h_f(x_0) \leq \gamma$$

Since we already know that $h_f(x_0) \geq \gamma$ because $f \in C^\gamma(\mathbb{R})$ this yields

$$h_f(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)} = \gamma \tag{40}$$

For what concerns the p exponent the bound (36) yields

$$u_f^p(x_0) \leq \gamma$$

Since we know already that $u_f^p(x_0) \geq h_f(x_0) = \gamma$ (see (16)) we get

$$u_f^p(x_0) = \frac{\alpha\beta(\gamma+\frac{1}{p})}{r_\alpha(x_0)} - \frac{1}{p} = \gamma \tag{41}$$

(c) Suppose $r(x_0) > \alpha\beta$. Thus $m_0^{(n)} > m_n$. It is related to Case 2b and yields

$$d_{j_n}(x_0) \geq C2^{-\alpha\beta\gamma j_n} \tag{42}$$

Thus we have the upper-bound

$$\begin{aligned} h_f(x_0) &\leq \liminf_{n\to\infty} \frac{\log d_{\lambda_{j_n}(x_0)}}{\log 2^{-j_n}} \\ &\leq \frac{\alpha\gamma\beta j_n}{j_n} = \alpha\beta\gamma. \end{aligned} \tag{43}$$

For the lower bound we pick up ε small enough and have $r_\alpha(x_0)+\varepsilon > \alpha\beta$. This yields that we may have $m \leq m_0$ and get

$$d_j(x_0) \leq C2^{-\alpha\beta\gamma j} \tag{44}$$

This yields

$$h_f(x_0) \leq \alpha\beta\gamma. \tag{45}$$

Together with (43) we have

$$h_f(x_0) = \alpha\beta\gamma \tag{46}$$

The same computation yields $u_f^p(x_0) = \alpha\beta(\gamma+\frac{1}{p}) - \frac{1}{p}$.

This proves Theorem 1.

5 Spectra of singularities

Let F_r^α the set of points x_0 such that $r_\alpha(x_0) = r$ with $r < \infty$.

Remark first that we can describe F_r^α as $F_r^\alpha = \limsup_{j \to +\infty} B(x_j, r_j^r)$ with $x_j \in S_\alpha, x_j = \frac{1}{2^{j-1}} \pm \frac{1}{2^{\alpha j}}$ and $r_j = 2^{-j}$.

We can thus cover F_r^α with $B(x_j, r_j^r), j \in \mathbb{N}$ and this yields that the Hausdorff dimension of F_r^α is less or equal than r.

Several ways are possible in order to get a lower bound of the Hausdorff dimensions. We will make use of recent results by Durand [5] in the version proposed by A. Amou and Y. Bugeaud [2] since this result can be applied directly in our case.

Definition 5. *Let U be a real open interval. Let $(x_i)_{i\geq 1}$ be points in U and let $(r_i)_{i\geq 1}$ be a sequence of positive real numbers such that $\lim_{i\to\infty} r_i = 0$. The family $(x_i, r_i)_{i\geq 1}$ is a homogeneous ubiquitous system in U if the set $\limsup_i B(x_i, r_i)$ is of full Lebesgue measure in U.*

Theorem D of [2] proved in [5] yields the following.

Theorem 6. *Let τ be a real number with $\tau \geq 1$. Let the family $(x_i, r_i)_{i\geq 1}$ be a homogeneous ubiquitous system in U, then the Hausdorff dimension of the set $\limsup B(x_i, r_i^\tau)$ is at least equal to $\frac{1}{\tau}$.*

We have clearly that $\limsup_{j\to+\infty} B(x_j, r_j)$ is of full Lebesgue measure since all points in $[-1, 1]$ can be approximated by dyadics. Thus the dimension of F_r^α is exactly r.

This yields the spectra and Corollary 2.

6 Multifractal formalism

Let us now check if the function f satisfies a formula of multifractal formalism type.

6.1 Multifractal formalism with Oscillation spaces

Jaffard in [12] (Definition 15) gives a multifractal type formula to compute the Hölder spectrum of singularities, the so called multifractal formalism for Hölder

spectrum. This formula, unlike previous formulas which were stated before, is stable under oscillating behaviors and is easy to compute once we have the wavelet leaders d_λ. We will check that it is satisfied in our case.
Recall the definition with the help of wavelet leaders. Indeed we want to compute the following function of q $\omega_f(q) = \sup\{s : \forall j \geq 0, 2^{j(s-1)} \sum_{\lambda \in \lambda_j} d_\lambda^q < +\infty\}$.
Then the multifractal formalism claims $d(h) = d_O(h)$ with

$$d_O(h) = \inf_q (hq - \omega_f(q) + 1)$$

Let us check if this formula is true for our function f.
As usual define $m_0 = \lceil \frac{j}{\alpha\beta} \rceil$ and $m_1 = \lfloor \frac{j}{\alpha} \rfloor$. We have 2^{j+1} dyadics cubes at scale j inside $[-1, 1]$ with 2^m irreducible fractions of type $\frac{k}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$ for $m_0 \leq m \leq m_1$ with a general count of $2^{\alpha m+1}$ of irreducible fractions at scale αm.
Let

$$2^{j(s-1)} \left(\sum_{m=0}^{m_1-1} 2^{m+1} 2^{-q\alpha\gamma\beta(j+1)} + \sum_{m=m_1}^{j} 2^{m+1} 2^{-\alpha\beta\gamma q m} + \sum_{m=m_0}^{m_1-1} 2^m 2^{-\alpha\beta\gamma q m} \right) = \Omega_f(j, q) \tag{47}$$

Some dyadic cubes are counted twice in $\Omega_f(j, q)$. But we can say that we have anyway

$$\frac{1}{2}\Omega_f(j, q) \leq 2^{j(s-1)} \sum_{\lambda \in \lambda_j} d_\lambda^q \leq \Omega_f(j, q) \tag{48}$$

we have the following cases

1. Suppose $1 - \alpha\gamma\beta q < 0$, thus $\frac{1}{\alpha\gamma\beta} < q$ then we have
   hence $\omega_f(q) = -\frac{1}{\alpha\beta} + q\gamma$
2. Suppose $1 - \alpha\gamma\beta q \geq 0$, thus $\frac{1}{\alpha\gamma\beta} \geq q$ then we have,
   $\omega_f(q) = -1 + q\alpha\beta\gamma$

Let h be fixed and $f(q) = hq - \omega_f(q) + 1$. We have

$$f(q) = \begin{cases} hq - \gamma\beta q + \frac{1}{\alpha\beta} & \text{if } \frac{1}{\gamma\beta\alpha} < q \\ hq - \alpha\beta\gamma q + 1 & \text{otherwise} \end{cases} \tag{49}$$

- Suppose $h > \alpha\gamma\beta$ or $h < \gamma$. Thus f is unbounded by below and $d_O(h) = -\infty$.
- Suppose $\gamma \leq h \leq \alpha\gamma\beta$. The minimum of f is at $p_0 = \frac{1}{\alpha\gamma\beta}$ and we have $f(p_0) = \frac{h}{\alpha\gamma\beta}$

Since $d_f(h) = d_O(h)$ for all h the multifractal formalism with oscillation spaces is satisfied.

6.2 Multifractal formalism with p-Oscillation spaces

The same kind of formula as in the Hölder case exists in order to compute the p spectrum. The claim is the following (see [10] for details): compute $\omega_f(p,q) = \sup\{s : \forall j \geq 0, 2^{j(s-1)} \sum_{\lambda \in \lambda_j} D^q_{\lambda,p} < +\infty\}$.

Then

$$d_p(h) = \inf_q (hq - \omega_f(p,q) + 1)$$

should give $d_p(h) = d_{f,p}(h)$.

We will check that this is actually true for our function f. Indeed remark that it is enough to replace γ by a in the previous computation of Section 6.1 to compute exactly the formula for the p spectrum and get it exactly. The multifractal formalism for the p exponent is satisfied.

References

[1] Arneodo, A. Bacry, E. and Muzy J-F.: The thermodynamics of fractals revisited with wavelets. Physica A **213**, 232-275 (1995)

[2] Amou, M. and Bugeaud, Y.: Exponents of Diophantine approximation and expansions in integer bases. J. London Math. Soc. **81** (2) 297-316 (2010)

[3] Calderón, A. P. and Zygmund A.: Local properties of solutions of elliptic partial differential equations. Studia Matematica **20** 171–227 (1961)

[4] Daubechies, I: Ten lectures on wavelets. SIAM: Society for Industrial and Applied Mathematics. (1992)

[5] Durand, A.: Sets with large intersection and ubiquity. Math. Proc. Cambridge Philos. Soc. **144** 119-144 (2008)

[6] Fraysse, A.: Regularity criteria for almost every function in Sobolev spaces. J. Funct. Anal. **258**, no. 6, 1806-1821. (2010)

[7] Hernández, E. and Weiss, G.: A First Course on Wavelets. CRC Press, New York, (1996).

[8] Heurteaux, Y. and Jaffard, S.: Multifractal analysis of images: new connexions between analysis and geometry. Proceedings of the NATO-ASI Conference on Imaging for Detection and Identification, Springer (2006)

[9] Jaffard, S. and Melot, C.: Wavelet Analysis of Fractal Boundaries. Part 1: Local Exponents. Commun. Math. Phys. **258** 513-539 (2005)

[10] Jaffard, S. and Melot, C.: Wavelet Analysis of Fractal Boundaries. Part 2: Multifractal analysis. Commun. Math. Phys. **258** 541-565 (2005)

[11] Jaffard, S.: Construction de fonctions multifractales ayant un spectre de singularités prescrit. C.R. Acad. Sci. Série I. **315**, 19-24. (1992)

[12] Jaffard, S.: Wavelet techniques in multifractal analysis, in Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot, M. Lapidus et M. van Frankenhuijsen Eds., Proceedings of Symposia in Pure Mathematics (AMS) **72** Part 2,91-151 (2004)

[13] Meyer, Y.: Ondelettes et opérateurs, Herman, Paris (1990)

[14] Parisi G. and Frisch U.: On the singularity structure of fully developed turbulence; appendix to Fully developed turbulence and intermittency, by U. Frisch; Proc. Int. Summer school Phys. Enrico Fermi, 84-88 North Holland. (1985)

[15] Wendt H., Abry P., Roux S.G, Jaffard S. and Vedel B.: The contribution of wavelets in multifractal analysis; In Series in contemporary applied mathematics, A. Damlamian, S. Jaffard, L.T. Tsien, Eds., Higher education press and World scientific publishing, (2009)