



HAL
open science

Investment under uncertainty, competition and regulation

Adrien Nguyen Huu

► **To cite this version:**

Adrien Nguyen Huu. Investment under uncertainty, competition and regulation. 2013. hal-00831263v2

HAL Id: hal-00831263

<https://hal.science/hal-00831263v2>

Preprint submitted on 6 Sep 2013 (v2), last revised 3 Feb 2014 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Investment under uncertainty, competition and regulation

Adrien Nguyen Huu¹

McMaster University, Hamilton, Canada

*Fields Institute for Research in Mathematical Sciences, 222 College Street, Toronto,
Canada*

Abstract

We investigate a randomization procedure undertaken in real option games which can serve as a raw model of regulation in a duopoly model of preemptive investment. We recall the rigorous framework of Grasselli and al. [7] and extend it to a random regulator. This model generalizes and unifies the different competitive frameworks proposed in the literature, and creates a new one similar to a Stackelberg leadership. We fully characterize strategic interactions in the several situations following from the parametrization of the regulator. Finally, we study the effect of the coordination game and uncertainty of outcome when agents are risk-averse, providing new intuitions for the standard case.

Keywords: real option game, preemption game, timing game, regulation, asymmetry, Stackelberg competition, Cournot competition

1. Introduction

A significant progress on the net present valuation method has been made by real option theory for new investment valuation. The latter uses recent methods from stochastic finance to price uncertainty and competition in complex investment situations. Real option games are the part of this theory dedicated to the competitive dimension. They correspond to an extensively studied situation, where two or more economical agents face with time a common project to invest into, and where there might be an advantage of being a leader (*preemptive* game) or a follower (*attrition* game). On these problems and the related literature, Azevedo and Paxson [1] or Chevalier-Roignant and al. [5] provide comprehensive broad scopes. Real option

¹corresponding email: adrien.nguyenhuu@gmail.com

games are especially dedicated to new investment opportunities following R&D developments, like technological products, a new drug or even real estate projects. The regulatory dimension is an important part of such investment projects, and the present paper attempts to contribute in this direction.

The economical situation can be the following. We consider an investment opportunity on a new market for an homogeneous good, available for two agents labeled one and two. This investment opportunity is not constrained in time, but the corresponding market is regulated by a supreme instance. Both agents have the same model for the project's incomes, and both have also access to a financial market with one risky asset and a risk-free asset, i.e., a bank account.

A simple approach to regulation is to consider that when an agent wants to take advantage of such an opportunity, he must validate some non-financial criteria which are scrutinized by the regulator. Even though a model of regulatory decision is at the heart of this matter, we will take the most simple and naive approach. We will just assume that the regulator can accept or refuse an agent to proceed into the investment project, and that this decision is taken randomly. This extremely simple model finds its root in a widespread convenient assumption. In the standard real option game representing a Stackelberg duopoly, the settlement of leadership and followership is decided via the flipping of an even coin, independent of the model. The procedure is used for example in Weeds [15], Tsekeros [12] or Paxson and Pinto [10] who all refer to Grenadier [8] for the justification, which appears in a footnote for a real estate investment project:

A potential rationale for this assumption is that development requires approval from the local government. Approval may depend on who is first in line or arbitrary considerations.

This justification opens the door to many alterations of the assumption inspired from other similar economical situations.

Let us take a brief moment to describe one of those we have in mind. Assume that two economical agents are running for the investment in a project with possibility of simultaneous investment, as in Grasselli and *al.* [7]. In practice, even if they are accurately described as symmetrical, they would never act at the exact same time, providing that instantaneous action in a continuous time model is just an idealized situation. Instead, they show their intention to invest to a third party (a regulator, a public institution) at approximatively the same time. An answer to a call for tenders is a typical example of such interactions. Assume now that this arbitrator decides at last

on how to proceed and has some flexibility in his judgment. For example, he can evaluate whose agent is the most suitable to be granted the project as a leader regarding qualitative criteria. This situation might be illustrated in particular where numerous environmental or health exigences are in line. When simultaneous investment is impossible, the real estate market example of Grenadier [8] can also be cited again with in mind that safety constraints, but also aesthetic or confidence dimension, can intervene in the decision of a market regulator. We emphasize here that the arbitrator is not explicitly unfair, but that exogenous criteria might influence his decision. There is an asymmetry in the chance to be elected as a leader and perfectly informed agents should take into account this fact into their decision to invest or to differ. Those are some of the situations the introduction of a random regulator can shed some light on.

The naive approach we follow is also inspired from Fudenberg and Tirole [6], Thijssen and *al.* [14] and Grasselli and *al.* [7] for providing rigorous mathematical grounds to real option game models. Our present model appears to encompass in a continuous manner the Cournot competition setting of Grasselli and *al.* [11] and the Stackelberg competition of Grenadier [8]. Other competitive situations are of course possible. As yet another study on the standard real option game setting, we additionally provide new insights on the strategic interactions of players when they are risk-averse. Against the existing literature, the present contribution situates itself as a mathematically rigorous framework for the general duopoly game, and can thus be seen as an addendum to Grasselli and *al.* [7].

Let us present how the remaining of the paper proceeds. Section 2 introduces the standard model and its extension to the random regulatory framework. Section 3 provides the study of the model and optimal strategies in the general case. In Section 4, we present how the proposed model encompasses the usual types of competition, and propose a new asymmetrical situation related to the Stackelberg competition framework. Section 5 introduces risk-aversion in agents evaluation to study the effect of a random regulator and the uncertain outcome of a coordination game on equilibrium strategies of two opponents.

2. The model

2.1. *The investment opportunity*

We assume for the moment that the regulator does not intervene. The framework is thus the standard one that one shall find in Grasselli and *al.* [7]. Notations and results of this section follow from the latter.

We consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is the filtration of a one-dimensional Brownian motion $(W_t)_{t \geq 0}$, i.e., $\mathcal{F}_t := \sigma\{W_s, 0 \leq s \leq t\}$. The project delivers for each involved agent a random continuous stream of cash-flows $(D_{Q(t)}Y_t)_{t \geq 0}$. Here, Y_t is the stochastic profit per unit sold and $D_{Q(t)}$ is the quantity of sold units per agent actively involved on the market, when $Q(t)$ agents are actively participating at time t . The increments of $(Q(t))_{t \geq 0}$ inform on the timing decision of agents, and considering a perfect information setting, we assume that $(Q(t))_{t \geq 0}$ is a \mathbb{F} -adapted right-continuous process. It is natural to assume that being alone in the market is better than sharing it with a competitor:

$$0 =: D_0 < D_2 < D_1, \quad (1)$$

but we also assume these quantities to be known and constant. The process $(Y_t)_{t \geq 0}$ is \mathbb{F} -adapted, non negative and continuous with dynamics given by

$$dY_t = Y_t(\nu dt + \eta dW_t), \quad t \geq 0, \quad (2)$$

with $(\nu, \eta) \in \mathbb{R} \times (0, +\infty)$. We assume that cash-flows are perfectly correlated to a liquid traded asset whose price dynamics is given by

$$dP_t = P_t(\mu dt + \sigma dW_t) = P_t(r dt + \sigma dW_t^{\mathbb{Q}}) \quad (3)$$

where $(\mu, \sigma) \in \mathbb{R} \times (0, +\infty)$, r is the constant interest rate of the risk-free bank account available for both agents, and $W_t^{\mathbb{Q}} = W_t + \lambda t$ is a Brownian motion under the unique risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ of the arbitrage-free market. The variable $\lambda := (\mu - r)/\sigma$ in (3) is the Sharpe ratio. The present financial setting is thus the standard Black-Scholes-Merton model [2] of a complete perfect financial market.

2.2. The follower's problem

In this setting, since $D_{Q(t)}$ takes known values, the future stream of cash-flows can be evaluated under the risk-neutral measure under which

$$dY_t = Y_t((\nu - \eta\lambda)dt + \eta dW_t^{\mathbb{Q}}). \quad (4)$$

Assume that one of the two agents, say agent one, desires to invest at time t when $Y_t = y$. If $Q(t) = 1$, then the available market for agent one is D_2 . The risk-neutral expectation of the project's discounted cash flows are then given by

$$V^F(t, y) := \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\infty} e^{-r(s-t)} D_2 Y_s ds \right] = \frac{D_2 y}{\eta\lambda - (\nu - r)} = \frac{D_2 y}{\delta} \quad (5)$$

with $\delta := \eta\lambda - (\nu - r)$. We assume from now on $\delta > 0$. Now to price the investment option of a follower, we recall that agent one can wait to invest as long as he wants. We also recall the sunk cost K at time τ he invests. In the financial literature, this is interpreted as a Russian call option of payoff $(D_2 Y_\tau / \delta - K)^+$. The value function of this option is given by

$$F(t, y) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau-t)} \left(\frac{D_2 Y_\tau}{\delta} - K \right)^+ \mathbb{1}_{\{\tau < +\infty\}} | \mathcal{F}_t \right] \quad (6)$$

where \mathcal{T}_t denotes the collection of all \mathbb{F} -stopping times with values in $[t, \infty]$. The solution to (6) is well-known in the literature. A formal proof can be found in Grasselli and *al.* [7].

Proposition 2.1 (Prop.1, [7]). *The solution to (6) is given by*

$$F(y) = \begin{cases} \frac{K}{\beta-1} \left(\frac{y}{Y_F} \right)^\beta & \text{if } y \leq Y_F, \\ \frac{D_2 y}{\delta} - K & \text{if } y > Y_F, \end{cases} \quad (7)$$

with a threshold Y_F given by

$$Y_F := \frac{\delta K \beta}{D_2 (\beta - 1)} \quad (8)$$

and

$$\beta := \left(\frac{1}{2} - \frac{r - \delta}{\eta^2} \right) + \sqrt{\left(\frac{1}{2} - \frac{r - \delta}{\eta^2} \right)^2 + \frac{2r}{\eta^2}} > 1. \quad (9)$$

The behavior of the follower is thus quite explicit. He will differ investment until the demand reaches at least the level $Y_F = \beta / (\beta - 1) K > K$ which depends on the profitability of the investment opportunity, the latter being conditioned by $\delta > 0$. We thus introduce

$$\tau_F := \tau(Y_F) = \inf\{t \geq 0 : Y_t \geq Y_F\}. \quad (10)$$

2.3. The leader's problem

Assume now that instead of having $Q(t) = 1$ we have $Q(t) = 0$. Agent one investing at time t will receive a stream of cash-flows associated to the level D_1 for some time, but he expects agent two to enter the market when the threshold Y_F is triggered. After the moment τ_F , both agents share the market and agent one receives cash-flows determined by level D_2 . The project value is thus

$$\begin{aligned}
V^L(t, y) &:= \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\infty} e^{-r(s-t)} (D_1 \mathbf{1}_{\{s < \tau_F\}} + D_2 \mathbf{1}_{\{s \geq \tau_F\}}) Y_s^{t,y} ds \right] \\
&= \frac{D_1 y}{\delta} - \frac{(D_1 - D_2) Y_F}{\delta} \left(\frac{y}{Y_F} \right)^{\beta}
\end{aligned}$$

where detailed computation can be found in Grasselli and *al.* [7]. This allows to characterize the leader's value function $L(t, y)$, i.e., the option to invest at time t for a demand y , as well as the value of the project $S(t, y)$ in the situation of simultaneous investment.

Proposition 2.2 (Prop. 2, [7]). *The value function of a leader is given by*

$$L(y) = \begin{cases} \frac{D_1 y}{\delta} - \frac{(D_1 - D_2) K \beta}{D_2^{\beta-1}} \left(\frac{y}{Y_F} \right)^{\beta} & \text{if } y < Y_F, \\ \frac{D_2 y}{\delta} - K & \text{if } y \geq Y_F, \end{cases} \quad (11)$$

If both agents invest simultaneously, we have

$$S(y) := \frac{D_2 y}{\delta} - K. \quad (12)$$

Remark 2.1. *Notice that no exercise time is involved as we consider the interest of exercising immediately, Y being non-negative. Notice also that L , F and S do not depend on t , since the problem is stationary.*

The payoff of the investment opportunity is then fully characterized for any situation in the case of no regulatory intervention.

2.4. The regulator

Let us define τ_1 and τ_2 the time at which agents one and two respectively express their firm desire to invest. In full generality, τ_i for $i = 1, 2$ can depend on a probability of acting in a game, see the next subsection. We assume that agents cannot predict the decision of the regulator, so that τ_1, τ_2 are \mathbb{F} -adapted stopping times. The regulator only intervenes at such times. If at time τ_i for $i = 1, 2$, $Q(\tau_i^-) = 0$, but $Q(\tau_i) = 1$ then his decision affects only agent i who expresses his desire to be a leader. If however $Q(\tau_i) = 2$, then $\tau_1 = \tau_2$ and the regulator shall decide if one or the other agent is accepted, none is or both are. Finally, if $Q(\tau_i^-) = 1$, then the regulator takes his decision upon the follower's fate. The regulator decision thus depends on \mathbb{F} .

We introduce a probability space $(\Lambda, \mathcal{P}(\Lambda), \mathbb{A})$ where $\Lambda = \{\alpha_0, \alpha_1, \alpha_2, \alpha_S\}$. We then introduce the product space $(\Omega \times \Lambda, \mathcal{F} \times \mathcal{P}(\Lambda), \mathbb{P}^+)$ and the augmented filtration $\mathbb{F}^+ := (\mathcal{F}_t^+)_{t \geq 0}$ with $\mathcal{F}_t^+ := \sigma\{\mathcal{F}_t, \mathcal{P}(\Lambda)\}$. The regulator is modeled by a \mathbb{F}^+ -adapted process.

Definition 2.1. Fix t and $Y_t = y$. For $i = 1, 2$, if $j = 3 - i$ is the index of the opponent and τ_j his time of investment, then agent i desiring to invest at time t receives

$$R_i(t, y) := \begin{cases} 0 & \text{if } \alpha = \alpha_0 \\ L(y)\mathbb{1}_{\{t \leq \tau_j\}} + F(y)\mathbb{1}_{\{t > \tau_j\}} & \text{if } \alpha = \alpha_i \\ F(y)\mathbb{1}_{\{t = \tau_j\}} & \text{if } \alpha = \alpha_j \\ L(y)\mathbb{1}_{\{t < \tau_j\}} + S(y)\mathbb{1}_{\{t = \tau_j\}} + F(y)\mathbb{1}_{\{t > \tau_j\}} & \text{if } \alpha = \alpha_S \end{cases} . \quad (13)$$

Let us discuss briefly this representation. According to (13), agent i is accepted if alternative α_i or α_S is picked up, and denied if α_0 or α_j is picked up. It is therefore implicit in the model that time does not affect the regulator's decision upon acceptability. However $Q(t)$ affects the position of leader or follower. Probability \mathbb{P}^+ can thus be given by $\mathbb{P} \times \mathbb{A}$, and probability \mathbb{A} given by the quartet $\{q_0, q_1, q_2, q_S\}$. However, as we will see shortly, the alternative α_0 is irrelevant in continuous time. We assume $q_0 < 1$. Since for the unregulated model we assumed that agents are symmetrical, the general study of the regulator's parameters given by \mathbb{A} will be chosen without loss of generality such that $q_1 \geq q_2$.

An additional major change comes into the evaluation of payoffs. Agents information is reduced to \mathbb{F} , and therefore the final settlement is not evaluable as in the complete market setting. We therefore make the assumption that agents are risk-neutral, i.e., they evaluate the payoffs by taking the expectation of previously computed values under \mathbb{A} .

Remark 2.2. *Expectations of (13) are made under the minimal entropy martingale measure $\mathbb{Q} \times \mathbb{A}$ for this problem, meaning that uncertainty of the model follows a semi-complete market hypothesis: if we reduce uncertainty to the market information \mathbb{F} , then the market is complete. See Becherer [4] for details.*

Once the outcome of Λ is settled, the payoff then depends on $Q(t)$, i.e., τ_j . We thus focus now on strategic interactions that determine $Q(t)$.

2.5. Timing strategies

It has been observed since Fudenberg & Tirole [6] that real time $t \geq 0$ is not sufficient to describe strategic possibilities of opponents in a coordination game. If $Q(t) = 0$ and agents coordinate by deciding to invest with probabilities $p_i \in (0, 1)$ for $i = 1, 2$, then a one-round game at time t implies a probability $(1 - p_1)(1 - p_2)$ to exit without at least one investor. However, another coordination situation appears for the instant just after and the game shall be repeated with the same parameters.

The problem has been settled by Fudenberg & Tirole [6] in the deterministic case, and recently by Thijssen and *al.* [14] for the stochastic setting. We extend it to the existence of the regulator who adds a step into the game. It consists in extending time $t \in \mathbb{R}_+$ to $(t, k, l) \in \mathbb{R}_+ \times \mathbb{N}^* \times \mathbb{N}^*$. The filtration \mathbb{F} is augmented via $\mathcal{F}_{t,i,j} = \mathcal{F}_{t,k,l} \subseteq \mathcal{F}_{t',i,j}$ for any $t < t'$ and any $(i, j) \neq (k, l)$, and the state process is extended to $Y_{(t,k,l)} := Y_t$. Therefore, when both agents desire to invest at the same time t , we extend the time line by freezing real time t and indefinitely repeating a game on natural time k . Once the issue of the game is settled, the regulator intervenes on natural time $l = 1$. If no participant is accepted, the game is replayed for $l = 2$, and so on.

Definition 2.2. *A strategy for agent $i \in \{1, 2\}$ is defined as a pair of \mathbb{F} -adapted processes $(G_{(t,k,l)}^i, p_{(t,k,l)}^i)$ taking values in $[0, 1]^2$ such that*

- (i) *The process $G_{(t,k,l)}^i$ is of the type $G_{(t,k,l)}^i(Y_t) = G_t^i(Y_t) = \mathbb{1}_{\{t \geq \tau(\hat{y})\}}$ with $\tau(\hat{y}) := \inf\{t \geq 0 : Y_t \geq \hat{y}\}$.*
- (ii) *The process $p_i(t, k, l)$ is of the type $p_i(t, k, l) = p_i(t) = p^i(Y_t)$.*

The reduced set of strategies is motivated by several facts. Since the process Y is Markov, we can focus without loss of generality on Markov subgame perfect equilibrium strategies. The process $G_{(t,k)}^i$ is a non-decreasing càdlàg process, and refers to the cumulative probability of agent i exercising before t . Its use is kept when agent i does not exercise immediately the option to invest, and when exercise depends on a specific stopping time of the form $\tau(\hat{Y})$, such as the follower strategy. The process $p_{(t,k,l)}^i$ denotes the probability of exercising in a coordinate game at round k , after $l - 1$ denials of the regulator, when $\alpha = \alpha_0$. It should be stationary and not depend on the previous rounds of the game since no additional information is given. For both processes, the information is given by \mathbb{F} and thus reduces to Y_t at time t . Additional details can be found in Thijssen and *al.* [14].

3. Optimal behavior and Nash equilibria

3.1. Conditions for a coordination game

A first statement about payoffs can be immediately provided. A formal proof can be found in Grasselli and *al.* [7].

Proposition 3.1 (Prop. 1, [9]). *There exists a unique point $Y_L \in (0, Y_F)$ such that*

$$\begin{cases} S(y) < L(y) < F(y) & \text{for } y < Y_L, \\ S(y) < L(y) = F(y) & \text{for } y = Y_L, \\ S(y) < F(y) < L(y) & \text{for } Y_L < y < Y_F, \\ S(y) = F(y) = L(y) & \text{for } y \geq Y_F. \end{cases} \quad (14)$$

Fix t and $Y_t = y$. In the deregulated situation, three different cases are thus possible, depending on the three intervals given by $0 < Y_L < Y_F < +\infty$. It appears that in our framework, the discrimination also applies.

Consider the following situation. Assume $Q(t) = 0$ and $t = \tau_1 < \tau_2$: agent one wants to start investing in the project as a leader and agent two allows it, i.e., $(p_1(t), p_2(t)) = (1, 0)$. By looking at (13), agent one receives $L(y)$ with probability $q_1 + q_S$ and 0 with probability $q_2 + q_0$. However, as noticed for the coordination game, if agent one is denied investment at $(t, 1, 1)$, he can try at $(t, 1, 2)$ and so on until he obtains $L(y)$.

The situation is identical if $\tau_2 < \tau_1$. Our setting is then limited by the continuous time approach and Proposition 3.1 applies as well as in the standard case.

Corollary 3.1. *Let $t \geq 0$ and $y > 0$. Then for $i = 1, 2$*

- (d) *if $y < Y_L$, agent i differs action until $\tau(Y_L)$;*
- (e) *if $y > Y_F$, agent i exercises immediately, i.e., $\tau_i = t$.*

Proof. (d) According to (14), if $\tau_1 \neq \tau_2$, expected payoffs given to (13) verify $(q_S + q_i)L(y) < (q_S + q_i)F(y)$ and there is no incentive to act for agent i , $i = 1, 2$.

(e) Since $S(y) = F(y) = L(y)$ and $\tau_F = t$, agents act with probability $(p_1(t), p_2(t)) = (1, 1)$. Since $q_0 < 1$, they submit their request as much as needed and receive $S(y)$ with probability $(1 - q_0) \sum_{l \in \mathbb{N}} q_0^l = 1$. \square

Corollary 3.1 does not describe which action is undertaken at $\tau(Y_L)$ or if $y = Y_F$. We are thus left to study the case where $Y_L \leq y \leq Y_F$. This is done by considering a coordination game. Following the reasoning of the above proof, definition (13) allows to give the payoffs of the game. Let us define

$$(S_1, S_2) := \left(\frac{1}{1 - q_0} (q_1 L + q_2 S + q_S S), \frac{1}{1 - q_0} (q_2 L + q_1 F + q_S S) \right). \quad (15)$$

Proposition 3.2. *For any $q_0 < 1$, agents play the coordination game given by Table 1 if $Y_L < y < Y_F$. Consequently, we can assume $q_0 = 0$ without loss of generality.*

	Exercise	Differ
Exercise	$(S_1(y), S_2(y))$	$(L(y), F(y))$
Differ	$(F(y), L(y))$	Repeat

Table 1: Coordination game at t for $\alpha \neq \alpha_0$.

Proof. If $\alpha = \alpha_0$ after settlement at round k of the game, time goes from (t, k, l) to $(t, 1, l + 1)$ and the game is repeated. Therefore, according to (13), the game takes the form of Table 1 for a fixed l and is settled with probability $1 - q_0$, or canceled and repeated with probability q_0 . If (p_1, p_2) is a given strategy for the game at t and $(E_1(p_1, p_2), E_2(p_1, p_2))$ the consequent expected payoff for agents one and two in the game of Table 1, then the total expected payoff for agent i at time t is

$$E_i(p_1, p_2)(1 - q_0) \sum_{l \in \mathbb{N}} q_0^l = E_i(p_1, p_2). \quad (16)$$

The game is thus not affected by $q_0 < 1$, which reduces the game to one intervention of the regulator, i.e. $l \leq 1$.

When $\tau_1 \neq \tau_2$, the probability that the regulator accepts agent i demand for investment goes from $q_i + q_S$ to $(q_i + q_S)/(q_i + q_j + q_S)$ with $j = 3 - i$. We can thus settle $q_0 = 0$ in full generality. \square

As for a single investor, continuous time implies strong limitations to the issues of the game. The confrontation has three outcomes and the exit of the repeated game without the investment of at least one player is not permitted. It also implies that we can reduce the regulator intervention to the only case of simultaneous investment. From now on $q_0 = 0$. Let's turn to the solution to Table 1.

3.2. Solution in the regular case

We reduce the analysis here to the case where $0 < q_2 \leq q_1 < 1 - q_2$. We can now assume that

$$\max(p_1, p_2) > 0. \quad (17)$$

We now introduce $p_0(y) := (L(y) - F(y))/(L(y) - S(y))$ and two functions

$$P_i(y) := \frac{p_0(y)}{q_i p_0(y) + q_S} = \frac{L(y) - F(y)}{L(y) - S_j(y)}, \quad i \neq j \in \{1, 2\}^2. \quad (18)$$

The values of P_i strongly discriminates the issue of the game. Since $q_1 \geq q_2$, if $Y_L \leq y \leq Y_F$, then $S_1(y) \geq S_2(y)$ according to (14) on that interval and $P_2(y) \geq P_1(y)$.

Lemma 3.1. *The functions P_2 and P_1 are increasing on $[Y_L, Y_F]$.*

Proof. By taking $d_1(y) := L(y) - F(y)$ and $d_2(y) := S(y) - F(y)$, we get

$$P_i(y) = \frac{1}{q_i} \left[\frac{d_1(y)}{d_1(y) + \gamma_i(d_1(y) - d_2(y))} \right]$$

and

$$P'_i(y) = \frac{1}{q_i} \left[\frac{\gamma_i(d_1(y)d'_2(y) - d_2(y)d'_1(y))}{(d_1(y) + \gamma_i(d_1(y) - d_2(y)))^2} \right]$$

where $\gamma_i := q_S/q_i \leq 1$ with $i \in \{1, 2\}$. We are thus interested in the sign of the quantity $g(y) := d_1(y)d'_2(y) - d_2(y)d'_1(y)$ which quickly leads to

$$\frac{g(y)\delta}{yD_2} = \left(\frac{y}{Y_F} \right)^{\beta-1} \left[(D_1 - D_2) \left(\beta + \frac{1}{\beta} - 2 - \frac{\delta K}{D_2} \right) \right] + \frac{\delta K}{D_2} (D_1 - D_2) .$$

Since $\beta > 1$, $(y/Y_F)^{\beta-1}$ is increasing in y . Since $0 < y \leq Y_F$, it suffices to verify that $(\delta K)/D_2 \geq (\beta + 1/\beta - 2 - (\delta K)/D_2)$, which is naturally the case for any β , to obtain that g is non-negative on the interval. \square

We will omit for now the symmetric case and assume $P_1(y) < P_2(y)$ for $Y_L < y < Y_F$. Recall now that $q_i > 0$ for $i = 1, 2$. Then $P_i(Y_F) = 1/(q_i + q_S) > 1$ for $i = 1, 2$. Accordingly and by Lemma 3.1, there exists $Y_L < Y_1 < Y_2 < Y_F$ such that

$$\begin{aligned} F(Y_1) &= q_1 L(Y_1) + q_2 F(Y_1) + q_S S(Y_1) = S_1(Y_1) , \\ F(Y_2) &= q_2 L(Y_2) + q_1 F(Y_2) + q_S S(Y_2) = S_2(Y_2) . \end{aligned} \quad (19)$$

Proposition 3.3. *Assume $Y_L < y < Y_F$. Then solutions of Table 1 are of three types:*

- (a) *If $Y_L < y < Y_1$ the game has three Nash Equilibria given by two pure strategies $(1, 0)$ and $(1, 0)$, and one mixed strategy $(P_1(y), P_2(y))$.*
- (b) *If $Y_1 \leq 1 < Y_2$, the game has one Nash Equilibrium given by strategies $(1, 0)$.*
- (c) *If $Y_2 \leq y < Y_F$, the game has one Nash Equilibrium given by strategies $(1, 1)$.*

Proof. For $Y_L < y < Y_1$, we have $P_1 < P_2 < 1$. Fix an arbitrary constant strategy $(p_1, p_2) \in [0, 1]^2$. Considering the \mathbb{A} -expected payoffs, agent one receives $L(y)$ at the end of the game with a probability

$$a_1 := p_1 \sum_{k \in \mathbb{N}^*} (1 - p_1)^{k-1} (1 - p_2)^k = \frac{p_1(1 - p_2)}{p_1 + p_2 - p_1 p_2} . \quad (20)$$

Symmetrically, he receives $F(y)$, and agent two receives $L(y)$ with probability

$$a_2 := \frac{p_2(1-p_1)}{p_1+p_2-p_1p_2}, \quad (21)$$

and they receives $(S_1(y), S_2(y))$ with probability

$$a_S := \frac{p_1p_2}{p_1+p_2-p_1p_2}. \quad (22)$$

The expected payoff of the game for agent one is given by

$$\begin{aligned} E_1(p_1, p_2) &:= a_1L(y) + a_2F(y) + a_S S_1(y) \\ &= (a_1 + a_S q_1)L(y) + (a_2 + a_S q_2)F(y) + a_S q_S S(y). \end{aligned} \quad (23)$$

A similar expression E_2 is given for agent two. Now fix p_2 . Since E_1 is a continuous function of both variables, maximum of (23) depends on

$$\frac{\partial E_1}{\partial p_1}(p_1, p_2) = \frac{p_2(L(y) - F(y) + p_2^2(S_1(y) - L(y)))}{(p_1(1-p_2) + p_2)^2}. \quad (24)$$

One can then see that the sign of (24) is the sign of $(p_2 - P_2)$. A similar discrimination for agent two implies P_1 .

(a) If $Y_L < y < Y_1$, then according to (18) and (19), $P_i < 1$ for $i = 1, 2$. Three situations emerge:

- (i) If $p_2 > P_2$, the optimal p_1 is 0. Then by (24), E_2 should not depend on p_2 , and the situation is stable for any pair $(0, p_2)$ with p_2 in $(P_2, 1]$.
- (ii) If $p_2 = P_2$, E_1 is constant and p_1 can take any value. If $p_1 < P_1$, then by symmetry p_2 should take value 1, leading to case (i). If $p_1 = P_1$, E_2 is constant and either $p_2 = P_2$, or we fall in case (i) or (iii). The only possible equilibrium is thus (P_1, P_2) .
- (iii) If $p_2 < P_2$, E_1 is increasing with p_1 and agent one shall play with probability $p_1 = 1 > P_1$. Therefore p_2 optimizes E_2 when being 0, and E_1 becomes independent of p_1 . Altogether, situation stays unchanged if $p_1 \in (P_1, 1]$ or if $p_1 = 0$. Otherwise, if $p_1 \leq P_1$, we fall back into cases (i) or (ii). The equilibria here are $(p_1, 0)$ with $p_1 \in (P_1, 1]$, and the trivial case $(0, 0)$.

Recalling constraint (17), we get rid of case $(0, 0)$. Coming back to the issue of the game when k goes to infinity in $(t, k, 1)$, three situations emerge from the above calculation. Two of them are pure coordinated equilibriums, of the type $(a_1, a_2) = (1, 0)$ or $(0, 1)$, which can be produced by pure coordinated

strategies $(p_1, p_2) = (a_1, a_2)$, settling the game in only one round. The third one is a mixed equilibrium given by $(p_1, p_2) := (P_1, P_2)$.

(b) According to (19), $S_1(Y_1) = F(Y_1)$. Following Lemma 3.1, agent one prefers being a leader for $y \geq Y_1$ and prefers a regulator intervention rather than the follower position, i.e. $S_1(y) \geq F(y)$. Thus $p_1 = 1$. For agent two, differing means receiving $F(y) > F(Y_1)$ and exercising implies a regulation intervention. Since $y < Y_2$, $q_1F(y) + q_2L(y) + q_S S(y) < F(y)$ and differing is his best option: $p_2 = 0$. That means that on (Y_1, Y_2) , the equilibrium strategy is $(p_1, p_2) = (1, 0)$.

(c) On the interval $[Y_2, Y_F)$, the reasoning of (b) still applies for agent one by monotony, and $p_1 = 1$. The second agent can finally bear the same uncertainty if $y \geq Y_2$, and $p_2 = 1$. Here, $1 \leq P_1(y) \leq P_2(y)$ and both agents have greater expected payoff by letting the regulator intervene rather than being follower. Equilibrium exists when both agents exercise. \square

Two reasons force us to prefer the strategy (P_1, P_2) on $(1, 0)$ and $(0, 1)$ in case (a). First, it is the only one which extends naturally the symmetric case. Second, it is the only trembling-hand equilibrium. Considering (P_1, P_2) in the interval (a), (a_1, a_2, a_S) follows according to (20), (21) and (22),

$$(a_1, a_2, a_S) = \left(\frac{1 - p_0}{2 - p_0}, \frac{1 - p_0}{2 - p_0}, \frac{p_0}{2 - p_0} \right), \quad (25)$$

If we plug (25) into (23), we obtain that the payoff of respective agents do not depend on (q_1, q_2, q_S) :

$$E_1(P_1, P_2) = E_2(P_1, P_2) = \frac{1 - p_0}{2 - p_0} (L(y) + F(y)) + \frac{a_0}{2 - p_0} S(y). \quad (26)$$

In the case $q_S > 0$, they are equal to F . This recalls the rent equalization principle of Fudenberg and Tirole [6]: in the preemption situation, agents are indifferent between playing the game and being the follower. In addition, asymmetry $q_1 \geq q_2$ does not affect the payoffs and the final outcome of the game after decision of the regulator has the same probability as in the deregulated situation, see Grasselli and *al.* [7].

3.2.1. Endpoints and overall strategy

We have to study junctures of areas (d) and (a), and (c) and (e). The technical issue has been settled in Thijssen and *al.* [14].

Lemma 3.2. *Assume $y = Y_L$. Then both agents have a probability 1/2 to be leader or follower, and receive $L(Y_L) = F(Y_L)$.*

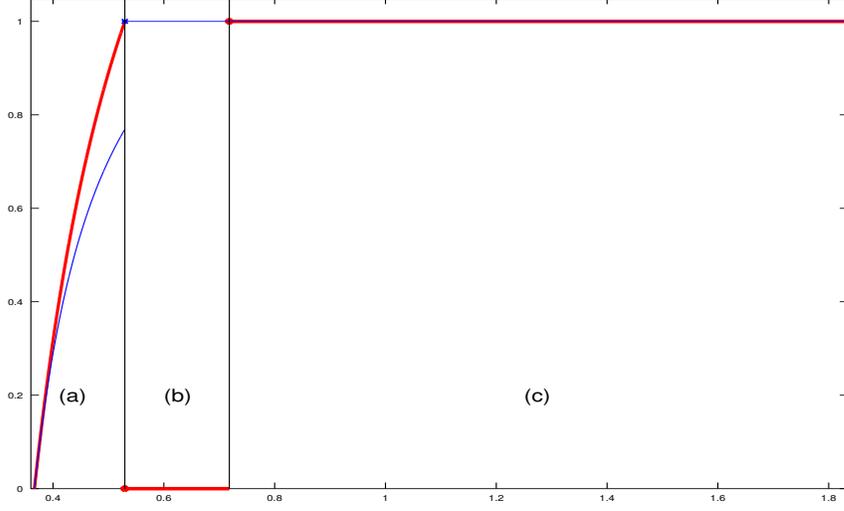


Figure 1: Values of equilibrium mixed strategies $p_1(y)$ (blue) and $p_2(y)$ (red) in the asymmetrical case $(q_1, q_2, q_S) = (0.5, 0.2, 0.3)$. Areas (a), (b) and (c) are separated by vertical lines at $Y_1 = 0.53$ and $Y_2 = 0.72$ on $[Y_L, Y_F] = [0.37, 1.83]$. Area (d) is then at the left of the graph and (e) at the right of it. Note that p_1 and p_2 are right-continuous. Parameters set at $(K, \nu, \eta, \mu, \sigma, r, D_1, D_2) = (10, 0.01, 0.2, 0.04, 0.3, 0.03, 1, 0.35)$.

Proof. The juncture of $[0, Y_L)$ with $[Y_L, Y_1]$ is a delicate point. At the left of point Y_L , no agent wants to invest. We thus shall use the strategy $G^i(Y_L)$ for both agents. By right-continuity of this process, both agents shall exercise with probability 1 at point Y_L . Each agent receives $R_i(y)$ which takes value $L(y) = F(y)$ with probability $q_1 + q_2$ and $S(y)$ with probability q_S :

$$E_1(y) = E_2(y) = (q_1 + q_2)F(y) + q_S S(y) . \quad (27)$$

At the right side of point Y_L however, P_i converges to 0 when y converges to Y_L , for $i = 1, 2$. Therefore, so do (p_1, p_2) toward $(0, 0)$. We cannot reconcile $G_{\tau(Y_L)}^i(Y_L)$ with $(p_1(Y_L), p_2(Y_L)) = (0, 0)$ and shall compare the payoffs. A short calculation provides

$$\lim_{y \downarrow Y_L} \frac{a_1(y)}{a_2(y)} = 1 \quad \text{and} \quad \lim_{y \downarrow Y_L} a_s(y) = 0 . \quad (28)$$

Therefore at point Y_L , $(a_1(Y_L), a_2(Y_L), a_S(Y_L)) = (1/2, 1/2, 0)$. It is clear at point Y_L that the second option is better for both agents. \square

Remark 3.1. *There is a continuity of behavior between (d) and (a) from the fact that regulatory intervention does not impact the outcome of the game, from the point of view of discrimination between the two agents: simultaneous investment is improbable at point Y_L , and probability $a_S(y)$ is continuous and null at this point. Consequently, the local behavior of agents around Y_L is similar to the one given in Grasselli and al. [7].*

Let us complete the strategic interaction of the two agents by summarizing the previous results in the following Theorem. As we will see in the next section for singular cases, it extends Theorem 4 in [7].

Theorem 3.1. *Consider the strategic economical situation presented in Section 2, with*

$$\min\{q_1, q_2, q_S\} > 0 \text{ and } q_0 = 0. \quad (29)$$

Then there exists a Markov sub-game perfect equilibrium with strategies depending on the level of profits as follows:

- (i) If $y < Y_L$, both agents wait for the profit level to rise and reach Y_L .*
- (ii) At $y = Y_L$, there is no simultaneous exercise and each agent has an equal probability of emerging as a leader while the other becomes a follower and waits until the profit level reaches Y_F .*
- (iii) If $Y_L < y < Y_1$, each agent choses a mixed strategy consisting of exercising the option to invest with probability $P_i(y)$. Their expected payoffs are equal, and the regulator intervenes on the settlement of positions.*
- (iv) If $Y_1 \leq y < Y_2$, agent one exercises his option and agent two becomes a follower and waits until y reaches Y_F .*
- (v) If $Y_2 \leq y < Y_F$, both agents express their desire to invest immediately, and the regulator is called. If one agent is elected leader, the other one becomes follower and waits until y reaches Y_F .*
- (vi) If $Y_F \geq y$, both agents act as in (v), but if a follower emerges, he invest immediately after the other agent.*

Few comments are in order. First, we shall emphasize that the regulator theoretically intervenes for any situation where $y \geq Y_L$. However, as explained at the beginning of this section, its intervention becomes mostly irrelevant in continuous time when agents don't act simultaneously. Its impact is unavoidable to settle the final payoff after the game, but its influence on agents' strategies boils down to the interval $(Y_L, Y_2]$.

At point Y_1 , there is a strong discontinuity in the optimal behavior of both agents. For $y < Y_1$, the mixed strategy used by agent two tends toward

a pure strategy with systematic investment. However at the point itself, the second agent differs investment and becomes the follower. It follows from the fact that agent one is indifferent between being follower or letting the regulator decide the outcome at this point. He suddenly seeks, without hesitation, for the leader's position, creating a discontinuity in his behavior. The same happens for agent two at Y_2 , creating another discontinuity in the strategy of the latter.

4. Singular cases

The proposed framework encompasses in a natural way the two main competition situations encountered in the literature, namely the Cournot competition and the Stackelberg competition. By introducing minor changes in the regulatory intervention, it is also possible to represent the situation of Stackelberg leadership advantage. Finally, our setting allows to study a new and weaker type of advantage we call *weak Stackelberg leadership*.

4.1. The Cournot and Stackelberg competitions

The Cournot duopoly refers to a situation when both players adjust quantities simultaneously. This is the framework of Grasseli and *al.* [7], involving the payoff $S(y)$ if agents invest at the same time. Recalling Definition 2.1, this framework corresponds to

$$(q_1, q_2, q_S) = (0, 0, 1) . \quad (30)$$

We notice that agents then become symmetrical. This appears from (18) which implies $P_1(y) = P_2(y) = p_0(y)$. Additionally, $p_0(y) \in (0, 1)$ if $y \in (Y_L, Y_F)$, and is increasing on this interval according to Lemma 3.1.

The Stackelberg competition refers to a situation where competitors adjust the level $D_{Q(t)}$ sequentially. In a pre-emptive game, this implies that in the case of simultaneous investment, one agent is elected leader and the other one becomes follower. This setting implies an exogenous randomization procedure, which by symmetry is given by a flipping of a fair coin. The procedure is described as such in Grenadier [8], Weeds [15], Tsekeros [12] or Paxson and Pinto [10]. Recalling Definition 2.1, the setting is retrieved by fixing

$$(q_1, q_2, q_S) = (1/2, 1/2, 0) . \quad (31)$$

The implication in our context is the following: by symmetry, $P_1(y) = P_2(y) = 2$. Therefore, the interval (Y_L, Y_2) reduces to nothing, i.e., $Y_L = Y_1 = Y_2$, and the strategical behavior boils down to (i), (v) and (vi) in Theorem 3.1.

Remark 4.1. Notice that any combination $q_2 = 1 - q_1 \in (0, 1)$ provides the same result as in (31), as $P_i(y) = 1/q_i > 1$ for $i = 1, 2$. The strategic behavior is unchanged with an unfair coin flipping. This is foreseeable as $q_i L(y) + (1 - q_i)F(y) > F(y)$ on (Y_L, Y_F) . This will also hold for a convex combination, i.e., for risk-averse agents.

Remark 4.2. Assume now symmetry in the initial framework, i.e., $q := q_1 = q_2 \in (0, 1/2)$. We have $Y_S := Y_1 = Y_2$, and the region (b) reduces to nothing. By recalling (19), we straightly obtain

$$\lim_{q \uparrow 1/2} Y_S = Y_L \quad \text{and} \quad \lim_{q \downarrow 0} Y_S = Y_F. \quad (32)$$

Therefore, the regulation intervention encompasses in a continuous manner the two usual types of games described above.

4.2. Stackelberg leadership

This economical situation represents an asymmetrical competition where the roles of leader and follower are predetermined exogenously. It can be justified as in Bensoussan and *al.* [3] by regulatory or competitive advantage. However Definition 2.1 does not allow to retrieve this case directly. Instead, we can extend it by conditioning the probability quartet $\{q_0, q_1, q_2, q_S\}$. In this situation, the probability \mathbb{P}^+ depends on \mathbb{F} -adapted events in the following manner:

$$\mathbb{P}^+(\alpha_0 | t < \tau_1) = 1 \quad \text{and} \quad \mathbb{P}^+(\alpha_1 | t \geq \tau_1) = 1. \quad (33)$$

This means that no investment is allowed until agent one decides to invest, which leads automatically to the leader position. The strategical interaction is then pretty different from the endogenous attribution of roles. See Grasselli and *al.*[7] for a comparison of this situation with the Cournot game.

4.3. The weak Stackelberg advantage

We propose here a different type of competitive situation. Consider the investment timing problem in a duopoly where agent one has, as in the Stackelberg leadership, a significant advantage due to exogenous qualities. We assume that for particular reasons, the regulator only allows one agent at a time to invest in the shared opportunity. The advantage of agent one thus translates into a preference of the regulator, but only in the case of simultaneous move of the two agents. That means that agent two can still,

in theory, preempt agent one. This situation can be covered by simply setting

$$(q_1, q_2, q_S) = (1, 0, 0) . \quad (34)$$

In this setting, the results of Section 3 apply without loss of generality.

Proposition 4.1. *Assume (34). Then the optimal strategy for agents one and two is given by $(G^1(Y_L), G^2(Y_F))$.*

Proof. It suffices to consider the game of Table 1 on the interval $[Y_L, Y_F]$. The Nash equilibrium is given by $(p_1(y), p_2(y)) = (1, 0)$ for any $y \in [Y_L, Y_F]$. Corollary 3.1 applies for other values. \square

Remark 4.3. *As in Remark 4.2, we can observe that this situation corresponds to strategy (iv) of Theorem 3.1 expanded to the interval $[Y_L, Y_F]$. This situation can be obtained continuously with q_1 converging to one.*

Following Grasselli and *al.* [7], we compare the advantage given by such an asymmetry to the usual Cournot game. In Grasselli and *al.* [7], the positive difference between the Stackelberg leadership and the Cournot game provides a financial advantage which is called a priority option. By similarity, we call the marginal advantage of the weak Stackelberg advantage on the Cournot game a *preference option*.

Corollary 4.1. *Let us assume $q_2 = q_0 = 0$. Let us denote $E_1^{(q_1, q_S)}(y)$ the expected payoff of agent one following from Theorem 3.1 when (q_1, q_S) is given, for a level $y \in \mathbb{R}_+$. Then the preference option value is given by $\pi^0(y) := E_1^{(1,0)}(y) - E_1^{(0,1)}(y)$ for all $y \in \mathbb{R}_+$. Its value is equal to*

$$\pi^0(y) = (L(y) - F(y))^+ \quad \forall y \in \mathbb{R}_+ . \quad (35)$$

Proof. The proof is straightforward and follows from (26), where $E_1^{(0,1)}(y) = F(y)$ for $y \in [Y_L, Y_F]$. Following Proposition 4.1, agent one shall invest at $\tau(Y_L)$ if $y \leq Y_L$ at first. In this condition, his payoff is $L(Y_L) = F(Y_L)$, which provides (35). \square

This option gives an advantage to its owner, agent one, without penalizing the other agent, who can always expect the payoff F . A comparison with the priority option is given in Figure 2.

A very similar situation to the weak Stackelberg advantage can be observed if we take $q_2 = 0$ but $q_S > 0$. The economical situation however loses a part of its meaning. It would convey the situation where agent two

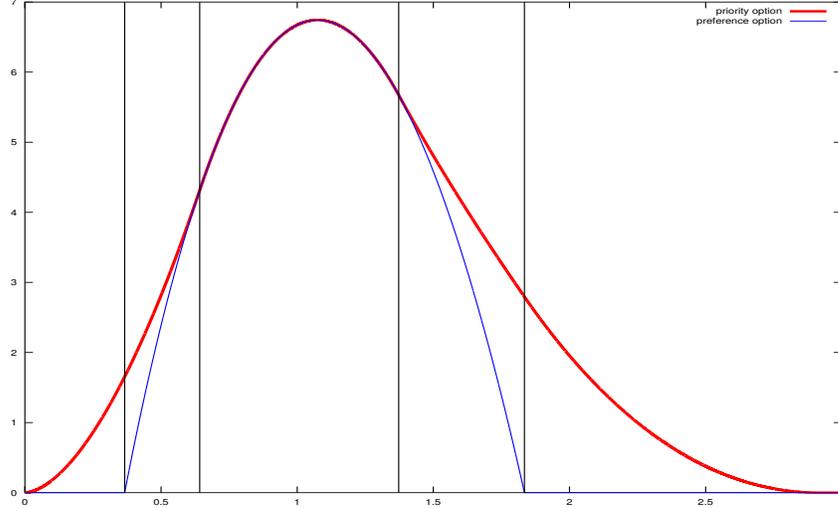


Figure 2: Priority option value (red) and Preference option value (blue) in function of y . Vertical lines at $Y_L = 0.37$, $Y_1 = 0.64$, $Y_2 = 1.37$ and $Y_F = 1.83$. Option values are equal on $[Y_1, Y_2]$. Same parameters as in Figure 1.

is accepted as investor only if he shares the market with agent one or become a follower. In this setting we can apply also the results of Section 3. We observe from definition (19) that in this case, $Y_2 = Y_F$ and Y_1 verifies $F(Y_1) = q_1 L(Y_1) + (1 - q_1) S(Y_1)$. The consequence is straightforward: the interval $[Y_1, Y_2]$ expands to $[Y_1, Y_F]$. The fact that $Y_1 > Y_L$ for $q_1 < 1$ has also a specific implication.

In that case, the equilibrium $(p_1(y), p_2(y)) = (1, 0)$ is more relevant than $(p_1(y), p_2(y)) = (P_1(y), P_2(y))$ on $[Y_L, Y_1]$. Indeed, if agent one invests systematically on that interval then agent two has no chance of being the leader since $q_2 = 0$. Thus $p_2 = 0$ and the payoffs become

$$(E_1(y), E_2(y)) = (L(y), F(y)) \text{ for } y \in [Y_L, Y_1]. \quad (36)$$

In opposition of the trembling-hand equilibrium, this pure strategy can be well figured by being called *steady-hand*. Comparing (26) to (36), agent one shall use the pure strategy, whereas agent two is indifferent between the mixed and the pure strategy. Setting $q_2 = 0$ thus provides a preference option to agent one.

5. Aversion for confrontation

Although its action is purely random in the presented model, the regulator represents a third actor in the game. We thus study an original dimension we denote *aversion for confrontation*, which conveys the impact of risk aversion on the coordination game only.

5.1. Risk aversion in complete market

We keep the market model of Section 2. However, we now endow each agent with the same CARA utility function

$$U(x) = -\exp(-\gamma x) \quad (37)$$

where $\gamma > 0$ is the risk-aversion of agents. The function U is strictly concave and is chosen to avoid initial wealth dependency. Since the market is complete and free of arbitrage, both agents still price the leader, the follower and sharing positions with the unique risk-neutral probability \mathbb{Q} . Thus, agents compare the utility of the different investment option market prices, denoted $l(y) := U(L(y))$, $f(y) := U(F(y))$ and $s(y) := U(S(y))$. The s_i , $i = 1, 2$ are defined the same way. When needed, variables will be indexed with γ to make the dependence explicit. The definition of regulation is updated.

Definition 5.1. Fix t and $Y_t = y$. For $i = 1, 2$, if $j = 3 - i$ is the index of the opponent and τ_j his time of investment, then agent i receives utility

$$r_i(t, y) := \begin{cases} 0 & \text{if } \alpha = \alpha_0 \\ l(y)\mathbb{1}_{\{t \leq \tau_j\}} + f(y)\mathbb{1}_{\{t > \tau_j\}} & \text{if } \alpha = \alpha_i \\ f(y)\mathbb{1}_{\{t = \tau_j\}} & \text{if } \alpha = \alpha_j \\ l(y)\mathbb{1}_{\{t < \tau_j\}} + s(y)\mathbb{1}_{\{t = \tau_j\}} + f(y)\mathbb{1}_{\{t > \tau_j\}} & \text{if } \alpha = \alpha_S \end{cases} \quad (38)$$

From Definition 5.1, it appears that by monotonicity of U , the game is strictly the same as in Section 2, apart from payoffs. Both agents differ for $y < Y_L$ and both act immediately for $y \geq Y_F$. Each agent uses now a mixed strategy p_i^γ , in order to maximize a expected utility slightly different from (23) on (Y_L, Y_F) :

$$E_1^\gamma(y) = (a_1^\gamma + a_s^\gamma q_1)l(y) + (a_2^\gamma + a_s^\gamma q_2)f(y) + a_s^\gamma q_S s(y) \quad (39)$$

Results of Section 3 thus hold by changing (L, F, S) for (l, f, s) . It follows that in that case

$$P_{i,\gamma}(y) = \frac{l(y) - f(y)}{q_i(l(y) - f(y)) - q_S(l(y) - s(y))} \text{ with } i \in \{1, 2\} \quad (40)$$

play the central role and that optimal strategic interaction of agents can be characterized by the value of y , or equivalently on the interval $[Y_L, Y_L]$ by the values of $P_{1,\gamma}(y)$ and $P_{2,\gamma}(y)$. The question we address in this section is how risk-aversion influences the different strategic interactions.

5.2. Influence of γ on strategies

First, aversion for confrontation is expressed through diminishing probability of intervention with γ .

Proposition 5.1. *Assume $q_S = 1$, and $y \in (Y_L, Y_S)$. Denote $p_\gamma(y) := P_{1,\gamma}(y) = P_{2,\gamma}(y)$. Then $p_\gamma \leq p_0$ and furthermore,*

$$\lim_{\gamma \downarrow 0} p_\gamma = p_0 \quad \text{and} \quad \lim_{\gamma \uparrow \infty} p_\gamma = 0. \quad (41)$$

Proof. According to (40) with $q_S = 1$, $p_\gamma(y) \in (0, 1)$ on (Y_L, Y_S) . From (40) we get

$$p_\gamma(y) = \frac{l(y) - f(y)}{l(y) - s(y)} = \frac{-e^{\gamma L(y)} + e^{\gamma F(y)}}{-e^{\gamma L(y)} + e^{\gamma S(y)}} = \frac{e^{\gamma(L-F(y))} - 1}{e^{\gamma(L-S(y))} - 1}. \quad (42)$$

Since $u(x) := -1 - U(-x) = e^{\gamma x} - 1$ is a positive strictly convex function on $[0, \infty)$ with $u(0) = 0$, we have that

$$p_\gamma(y) = \frac{u(L(y) - F(y))}{u(L(y) - S(y))} < \frac{L(y) - F(y)}{L(y) - S(y)} =: p_0(y). \quad (43)$$

For γ going to zero, we apply l'Hôpital's rule to obtain that $\lim_{\gamma \downarrow 0} p_\gamma = p_0$. The other limit follows from expliciting the utility function:

$$\lim_{\gamma \uparrow \infty} p_\gamma(y) = \lim_{\gamma \uparrow \infty} e^{-\gamma(F(y)-S(y))} = 0. \quad (44)$$

□

Remark 5.1. *Notice that there is no uniform convergence since p_0 is continuous and $p_0(Y_F) = 1$. The above convergence holds for all $y \in [Y_L, Y_F)$. It is clear from (43) that p_γ is monotonous in γ on $(0, \infty)$. Then according to (44), it is convex decreasing with γ .*

For the general case $q_S < 1$, the above result still holds, but on a reduced interval. Nevertheless, this interval depends on γ .

Proposition 5.2. *Assume $\min\{q_1, q_2, q_S\} > 0$. Let $Y_{1,\gamma} \in [Y_L, Y_F]$ be such that $P_{2,\gamma}(Y_{1,\gamma}) = 1$, and $Y_{2,\gamma} \in [Y_L, Y_F]$ such that $P_{1,\gamma}(Y_{2,\gamma}) = 1$. Then for $i = 1, 2$, $Y_{i,\gamma}$ is increasing in γ and*

$$\lim_{\gamma \uparrow \infty} Y_{i,\gamma} = Y_F. \quad (45)$$

Proof. Consider $i \in \{1, 2\}$. First notice that along Lemma 3.1, $Y_{i,\gamma}$ is uniquely defined on the designed interval. Following Remark 5.1, $P_{i,\gamma}$ is a concave non-decreasing functions of p_γ . It is then a decreasing function of γ . Since $P_{i,\gamma}(y)$ is decreasing with γ , $Y_{i,\gamma}$ is an increasing function of γ : the region $(Y_L, Y_{1,\gamma})$ spreads on the right with γ . Adapting (19) to the present values, $Y_{i,\gamma}$ shall verify:

$$q_i(1 - e^{\gamma(L(Y_{i,\gamma}) - F(Y_{i,\gamma}))}) + q_S(1 - e^{\gamma(S(Y_{i,\gamma}) - F(Y_{i,\gamma}))}) = 0$$

and when γ goes to ∞ , following (44), we need $L(Y_{i,\gamma}) - F(Y_{i,\gamma})$ to go to 0, so that $Y_{i,\gamma}$ tends toward Y_F . \square

With risk aversion, the region (a) takes more importance and the competitive advantage of agent one decreases with γ . Figure 3 resumes the evolution.

The interval $(Y_L, Y_{1,\gamma})$ describing the case (a) is the more relevant in terms of strategies, as the only one to involve mixed strategies. Propositions 5.1 and 5.2 imply the following result on this interval.

Corollary 5.1. *Assume $y \in [Y_L, Y_{1,\gamma}]$. Let $(a_1^\gamma, a_2^\gamma, a_S^\gamma)$ be defined by (20), (21) and (22) where (p_1, p_2) is replaced by $(P_{1,\gamma}, P_{2,\gamma})$. Then*

$$\lim_{\gamma \uparrow \infty} a_S = 0 \quad \text{and} \quad \lim_{\gamma \uparrow \infty} \frac{a_1^\gamma}{a_2^\gamma} = 1. \quad (46)$$

Proof. Denote $p_i^\gamma := P_{i,\gamma}$ and $p_\gamma := (l(y) - f(y))/(l(y) - s(y))$ the equivalent of p_0 in the risk averse setting. From (22), a_S^γ is a decreasing function of both p_1^γ and p_2^γ , and the first limit is a straightforward consequence of (41).

Plugging (40) into a_i^γ , we obtain

$$a_i^\gamma = \frac{p_i^\gamma(1 - p_j^\gamma)}{p_1^\gamma + p_2^\gamma - p_1^\gamma p_2^\gamma}, \quad i \neq j \in \{1, 2\}^2, \quad (47)$$

and differentiating in p_γ , we obtain that a_i^γ is increasing in γ . It also follows that

$$\frac{a_1^\gamma}{a_2^\gamma} = \frac{p_1^\gamma - p_1^\gamma p_2^\gamma}{p_2^\gamma - p_1^\gamma p_2^\gamma} = \frac{q_S - (1 - q_2)p_\gamma}{q_S - (1 - q_1)p_\gamma} \leq 1. \quad (48)$$

The second limit of (46) follows immediately from Proposition 5.1. \square

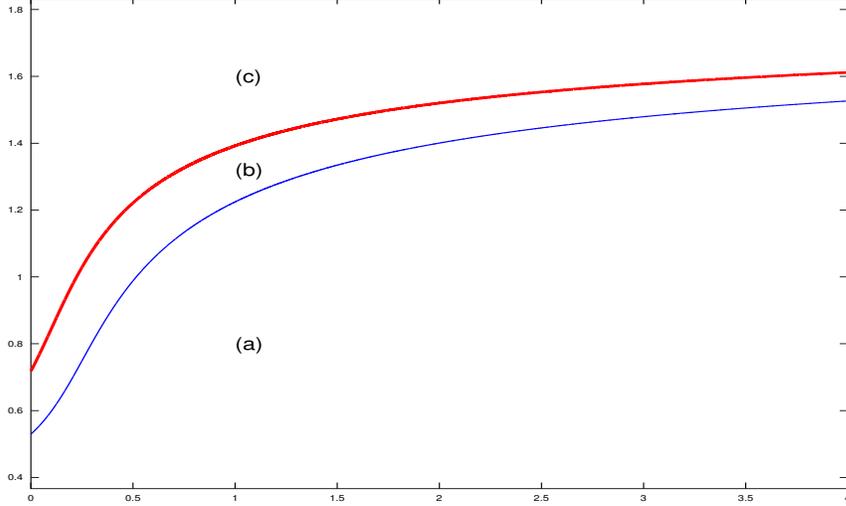


Figure 3: Values of Y_1 (blue) and Y_2 (red) as a function of risk aversion γ . Y-axis limited to $[Y_L, Y_F] = [0.37, 1.83]$. Limit values of (Y_1, Y_2) for γ going to 0 corresponds to $(0.53, 0.72)$ of Figure 1. Same parameters as previous figures.

Remark 5.2. We can easily assert as in Proposition 5.1 that $a_S^\gamma < a_S$ for any $\gamma > 0$, where a_S is the probability of simultaneous action in the game for risk-neutral agents. Equivalently,

$$\frac{a_1}{a_2} = \frac{F(y) - S_1(y)}{F(y) - S_2(y)} < \frac{a_1^\gamma}{a_2^\gamma}, \quad \forall \gamma > 0. \quad (49)$$

The above results can be interpreted as follows. The parameter γ is an aversion for the uncertainty following from the coordination game and the regulatory intervention. As expected, the higher this parameter, the lower the probability to act $P_{i,\gamma}$ in the game. However the game being infinitely repeated until action of at least one agent, for values of $P_{i,\gamma}$ lower than one, the game is extended to a bigger interval $[Y_L, Y_{1,\gamma})$ with γ . Then, it naturally reduces simultaneity of investment, but tends also to an even chance of becoming a leader for both agents. There is thus a consequence to a high risk-aversion γ : agents synchronize to avoid playing the game and the regulator decision. In some sense, the behavior in a Stackelberg competition is the limit with risk-aversion of the behavior of competitors in a Cournot game.

How does γ impact the outcome of the game? As above since we are in complete market, the risk aversion makes no relevant modification to the cases where the coordination game is not played. On the interval $(Y_L, Y_{1,\gamma})$ then, we need to compare the expected values of options L, F and S to homogeneously quantities.

Proposition 5.3. *For $y \in [Y_L, Y_{1,\gamma})$, let $E_1^\gamma(y)$ be the expected utility of agent one in the coordination game when strategies $(P_{1,\gamma}(y), P_{2,\gamma}(y))$ are played. Then the indifference value of the game for agent one is given by*

$$e_{1,\gamma}(y) := U^{-1}(E_1^\gamma(y)) = -\frac{1}{\gamma} \log(-a_1^\gamma l(y) - a_2^\gamma f(y) - a_S^\gamma s_i(y)) . \quad (50)$$

We define $e_{2,\gamma}$ similarly. Assume now $q_S > 0$. For $y \in [Y_L, Y_{1,\gamma})$, we have for $i = 1, 2$ that

$$e_{i,\gamma}(y) = E_i(y) = F(y) \quad \forall \gamma > 0 , \quad (51)$$

with $E_i(y)$ defined in (26).

Proof. Using (47), we can proceed as in the end of Section 3 to retrieve

$$a_{1,\gamma} l(y) + a_{2,\gamma} f(y) + a_{S,\gamma} s_i(y) = f(y), \quad i = 1, 2 , \quad (52)$$

and (51) follows from the definition of $e_{i,\gamma}$. \square

Risk-aversion in complete market has thus the interesting property to keep the rent equalization principle of Fudenberg and Tirole [6]: agents adapt their strategies to be indifferent between playing the game or obtaining the follower's position.

6. Critics and extensions

If real option games model are to be applied, the role of a regulator shall be introduced for many applications. Indeed regulators often intervene for important projects in energy, territorial acquisition and highly sensitive products such as drugs. Despite its simplicity and its idealization, the presented model has still something to say. One can see it as an archetype model, unifying mathematically the Stackelberg and the Cournot competition frameworks and leading to simple formulas. The model we proposed is a first attempt, and could be improved on several grounds. Three main objections can be raised.

As observed in Section 3, the continuous-time setting with perfect information as extended by Fudenberg and Tirole [6] has unrealistic and strongly

constrained implications. This is emphasized here where the regulator can only intervene when agents simultaneously invest. A prioritize objective would be to propose a new setting for the standard game conveying a dynamical dimension to strategies.

A realistic but involved complication is the influence of explicit parameters on the law \mathbb{P}^+ , parameters on which agents have some control. The value of being preferred, introduced as a financial option in subsection 4.3, would then provide a price to a competitive advantage and to side efforts to satisfy non-financial criteria. Eventually, a game with the regulator as a third player appears as a natural track of inquiry.

Finally, the introduction of risk-aversion should not be restrained to the coordination game. This issue is already tackled in Bensoussan and al. [3], and Grasselli and al. [7] for the incomplete market setting. But as a fundamentally different source of risk, the coordination game and the regulator's decision shall be evaluated with a different risk-aversion parameter, as we proposed in Section 5. An analysis of asymmetrical risk aversion as in Appendix C of Grasselli and al. [7] shall naturally be undertaken.

Acknowledgement

This work was performed as at the Fields Institute for Research in Mathematical Sciences in Toronto. The author thanks Matheus Grasselli for fruitful comments on a very first draft of the article. Remaining errors are the author's responsibility alone.

- [1] Azevedo, A. F., Paxson, D. A., 2010, in: Real options game models: A review. Real Options 2010.
- [2] Black, F. and Scholes, M. 1973. The pricing of options and corporate liabilities. In: The journal of political economy, 637-654.
- [3] Bensoussan, A., Diltz, J. D. and Hoe, S. .2010. Real options games in complete and incomplete markets with several decision makers. In: SIAM Journal on Financial Mathematics, 1(1), 666-728.
- [4] Becherer, D., 2003. Rational hedging and valuation of integrated risks under constant absolute risk aversion. In: Insurance: Mathematics and economics, 33(1), pp.1-28.
- [5] Chevalier-Roignant, B., Flath, C. M., Huchzermeier, A., Trigeorgis, L. , 2011. Strategic investment under uncertainty: a synthesis. In: European Journal of Operational Research, 215(3), pp.639-650.

- [6] Fudenberg, D., & Tirole, J., 1985. Preemption and rent equalization in the adoption of new technology. In: *The Review of Economic Studies*, 52(3), pp.383-401.
- [7] Grasselli, M., Leclere, V., & Ludkovski, M., 2013. Priority Option: The Value of Being a Leader. In press: *International Journal of Theoretical and Applied Finance*.
- [8] Grenadier, S. R., 1996. The strategic exercise of options: Development cascades and overbuilding in real estate markets. In: *The Journal of Finance*, 51(5), pp.1653-1679.
- [9] Grenadier, S. R. .2000. Option exercise games: the intersection of real options and game theory. In: *Journal of Applied Corporate Finance*, 13(2), 99-107
- [10] Paxson, D., & Pinto, H., 2005. Rivalry under price and quantity uncertainty. In: *Review of Financial Economics*, 14(3), pp.209-224.
- [11] Smets, F., 1993. *Essays on Foreign Direct Investment*, Ph. D. Dissertation, Yale University.
- [12] Tsekrekos, A., 2003. The effect of first-mover's advantages on the strategic exercise of real options. In: *Real R&D Options* D. Paxson (Ed.), pp.185-207.
- [13] Thijssen, J. J., 2010. Preemption in a real option game with a first mover advantage and player-specific uncertainty. In: *Journal of Economic Theory*, 145(6), pp.2448-2462.
- [14] Thijssen, J. J., Huisman, K. J., & Kort, P. M., 2002. Symmetric equilibrium strategies in game theoretic real option models. *CentER DP*, 81.
- [15] Weeds, H., 2002. Strategic delay in a real options model of R&D competition. In: *The Review of Economic Studies*, 69(3), pp.729-747.