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Real option game with intervention of a random partial arbitrator

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Abstract

We formalize the randomization procedure undertaken to discriminate players in real option games with Stackelberg leadership advantage. This is done by introducing a random arbitrator with specific parameterizations, and allows to propose a unified treatment of both Cournot and Stackelberg competition in real option games. We extend the study to a partial arbitrator, which leads to competitive advantages in various asymmetrical situations. We fully characterize strategic interactions. We then apply the procedure to risk-neutral and risk-averse firms in a stochastic preemptive real option game in complete market under regulation. The risk-averse case gives us the opportunity to study a new phenomenon we call aversion for confrontation, and its impact on the asymmetrical game.

Key words : real option game, timing game, regulator, asymmetry, Stackelberg competition, Cournot competition

1 Introduction

Real option games are commonly known as a collection of games with payoffs provided by the value function of some optimal control problems. This corresponds to an extensively studied situation, where two or more economical agents face with time a common project to invest into, and where there might be an advantage of being a leader (*pre-preemptive* game) or a follower (*attrition* game). On these problems and the related literature, [1, 3] provide comprehensive broad scopes.

From Azavedo and Paxson [1] emerges a *standard real option game*: a symmetric duopoly preemption situation depending on a continuous Markov state variable with risk-neutral agents. However, two approaches stand out on the final outcome of the game (the mechanism of settlement) depending on the economical situation. It shall be pointed out if simultaneous action of the two agents results in a fair division of the total demand, a Cournot competition as in the recent [5], or the election of one agent as the leader, as the Stackelberg competition of [8]. If the project can be shared, e.g., an open market for a technological product, there is a certain configuration when players do play a coordination game (when players want to be leader but prefer being follower rather than share the project incomes). Otherwise, and [6] gives the example of a real estate project, there is a trigger

state at which both players invest simultaneously, but the role of leader is decided via the flipping of an even coin. This procedure turns out to be perfectly consistent for risk-neutral and risk-averse players, i.e., when independence axiom is valid. But this procedure also seems to come out of nowhere, simplifying the calculus and terminating the litigate between players in a rather direct and eluded way. The procedure is used for example in [12, 9, 7] who refer to Grenadier [6] for the justification, which appears in a footnote:

A potential rationale for this assumption is that development requires approval from the local government. Approval may depend on who is first in line or arbitrary considerations.

This justification opens the door to many alterations of the assumption inspired from other similar economical situations.

We concern ourselves in this paper with properly formalizing this ambiguity, using the formal mathematical background of [11]. We do that by simply introducing a random arbitrator who decides what is happening in the case where both players act simultaneously, i.e., who is the leader and who is the follower. Starting from this minor idea, we actually reach several interesting key points of the real option game framework. Allowing the arbitrator to allow simultaneous investment also, we provide a unification of both endogenous attribution of roles with and without Stackelberg advantage. And then, allowing any weights on the arbitrator's decision, we make appearing new economical interactions.

Let us take a brief moment to describe one of those we have in mind. Assume that two economical agents are running for the investment in a project in time with possibility of simultaneous investment, as in [5]. In practice, even if they are accurately described as symmetrical, they would never act at the exact same time, providing that instantaneous action in a continuous time model is just an idealized situation. Instead, they show their intention to invest to a third party (a regulator, a public institution) at approximatively the same time. An answer to a call for tenders is a typical example of such interactions. Assume now that this arbitrator decides at last on how to proceed and has some flexibility in his judgment. For example, he can evaluate whose agent is the most suitable to be granted the project as a leader regarding qualitative criteria. This situation might be illustrated in particular where numerous environmental or health exigences are in line. When simultaneous investment is impossible, the real estate market example of [6] can also be cited again with in mind that safety constraints, but also aesthetic or confidence dimension, can intervene in the decision of a market regulator. We emphasize here that the arbitrator is not explicitly unfair, but that exogenous criteria might be taken into account in his decision. In those cases, there is an asymmetry in the chance to be elected as a leader and perfectly informed players should take into account this fact into their decision to invest or to differ. Those are some of the situations the introduction of a partial arbitrator can shed some light on.

Let us present how the remaining of the paper proceeds. In the next section, we introduce the theoretical framework of real option games with an arbitrator. This abstract setting allows to introduce very easily the arbitrator as a specific asymmetry and already provides Nash equilibriums in the different possible situations we encounter hereafter. Moreover, it gives a road-map for alternative models, such as incomplete market and war of attrition games.

In Section 3, we apply the above result to the real option framework. Therefore, we will speak of

firms to designate players, and the arbitrator of the game will be referred as the *regulator* on the project market. This is not an intention to create confusion, but to suit the economical situation it is supposed to describe. We rely on the ideal situation where the project cash flows are perfectly correlated to an asset price. We could then deduce that we fall in the setting of a complete market, but provided that firms cannot hedge or predict the decision of the regulator, this situation falls actually in the incomplete market framework. We then assume that firms are *risk-neutral*, as in the standard real option model, see [1]. This section is the opportunity to make case studies of different parameterizations of the regulator in order to see the effect of the market advantage provided by the latter. Singular parameterizations allow to retrieve standard cases or purely asymmetrical situations in the regulator preferences. We even provide a real option price for the advantage of being completely preferred by the regulator, in the manner and spirit of the *priority option* of [5]. We end the section discussing pertinent choices over multiple Nash equilibriums.

We continue and end the study in Section 4 by assuming that firms are *risk-averse*, and have a constant absolute risk aversion. This allows us to focus on the effect of the additional randomness induced by the regulator decision and the coordination game on firms decision to invest: since payoffs are already computed in a complete market setting, the risk aversion is reduced to an aversion for confrontation. We prove that in the complete market setting, the risk-averse case is a continuous extension of the risk-neutral case. We also emphasize the effect of aversion for confrontation in the asymmetrical case. We conclude by a discussion in Section 5.

One last thing shall be said before getting to the heart of the matter. The real option framework implies the derivation of value functions corresponding to expected project values. For this matter, we follow and rely heavily on the recent [5], where the complete market model is fully explored. We thus avoid technical well-known mathematical developments and focus on the interactions between firms. We prefer to ease reading fluidity by developing as little as possible mathematics along the text, which are actually straightforward computations. We acknowledge [5] as a starting point for our study, and refer the reader to this article for many details, as also a very good first incursion in the real option game treated with formality. We apologize for such a subjective suggestion and the omission of the impressive amount of literature on the topic since the seminal paper of Fudenberg and Tirole [4] or the PhD thesis of Smets [8]. Again, we refer to [1, 3] for a fairer and more exhaustive homage to the many contributions to real option games.

2 The continuous Markov preemption game

2.1 Framework

Time is represented by the positive half-line $\{t \geq 0\}$. We consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying usual nice properties. We consider a continuous and Markov state process Y , \mathbb{F} -adapted, which takes values in \mathbb{R} . The real option game with a random arbitrator is the following. Two symmetrical players (endowed with a utility function U) start at time 0 with an option to exercise at an arbitrary time $t \geq 0$, provided the information given by the σ -algebra \mathcal{F}_t . The first one to exercise the option at time t receives the payoff $L(t, Y_t)$ (as *leader*). The other player then receives the payoff $F(t, Y_t)$ (as *follower*). If both players act at the same time, an *arbitrator* intervenes instantaneously. In our setting, he can choose one alternative among three: he can give

$L(t, Y_t)$ to first player and $F(t, Y_t)$ to second player, or the converse, or give an equal payoff $S(t, Y_t)$ to both players. More explicitly L , F and S are the *expected utility* of some future cash-flows depending on the evolution of the state process. In full generality, they are measurable functions of t and Y_t . The latter being Markov, the dependence is reduced to the actual value Y_t at time t , as we will see in application. If nothing happens at some time t , the system evolves and is infinitely repeated as time goes by, see Fudenberg & Tirole [4] for the seminal formalization of the continuous-time game.

We assume that agents cannot predict the decision of the arbitrator, but the arbitrator decision can depend on \mathbb{F} . We introduce a probability space $(\Lambda, \mathcal{P}(\Lambda), \mathbb{A})$ where $\Lambda = \{\alpha_1, \alpha_2, \alpha_S\}$. We then introduce the product space $(\Omega \times \Lambda, \mathcal{F} \times \mathcal{P}(\Lambda), \mathbb{P} \times \mathbb{A})$ and the augmented filtration $\mathbb{F}^+ := (\mathcal{F}_t^+)_{t \geq 0}$ with $\mathcal{F}_t^+ := \sigma\{\mathcal{F}_t, \mathcal{P}(\Lambda)\}$. The arbitrator is represented by stochastic processes (A_1, A_2) which is \mathbb{F}^+ -adapted. For $t \geq 0$ and $(\omega, \alpha) \in \Omega \times \Lambda$,

$$(A_1, A_2)(t, (\omega, \alpha)) := \begin{cases} (L, F)(t, Y_t(\omega)) & \text{if } \alpha = \alpha_1 \\ (F, L)(t, Y_t(\omega)) & \text{if } \alpha = \alpha_2 \\ (S, S)(t, Y_t(\omega)) & \text{if } \alpha = \alpha_S \end{cases} .$$

The probability \mathbb{A} is given by the triplet $\{q_1, q_2, q_S\}$. We will assume without loss of generality that $q_1 \geq q_2$ in all the paper. The model can be easily extended by refining the arbitrator's representation: one can introduce time dependence, additional outcomes or asymmetrical payoffs, for example as in [10]. We omit these complications and let them to practitioners, bearing in mind that there is no additional difficulty.

2.2 Condition for the coordination game

Let us fix t and Y_t and denote $L := L(t, Y_t)$, $F := F(t, Y_t)$ and $S := (t, Y_t)$. Payoffs L , F and S cannot take arbitrary values for the preemptive game and the taxonomy of situations produces three standard cases. We will illustrate this reduction in the next section. We now consider two among them, which are the non ambiguous cases for the regular framework:

- (d) When $F > L > S$, both players desire to acquire the payoff F . Neither player wants to act first, so that the situation is standard: players *differ* action.
- (e) When $L = F = S$, each firm is indifferent to receiving F or S , which is also equal to L . Both players act, whatever the decision of the arbitrator. We then retrieve the standard case: players *exercise* their option together.

The attentive reader might wonder what happens at end-points of the above cases, when $F = L > S$ for example. This is postponed to the specific cases of Sections 3. One can easily extend the above procedure to the attrition game also to obtain another taxonomy of cases.

The coordination game The third case, when $L > F > S$, presents the situation where both players want to act, but might prefer to receive F rather than S . It is the occasion to introduce a coordination game. Players are in a coordination situation. To overcome the specificity of the latter in the continuous time setting, Fudenberg and Tirole [4] developed a timing game equilibrium framework. A recent contribution of Thijssen & al. [11] extended the approach to the stochastic setting. It consists in extending the time domain to index pairs $(t, k) \in \mathbb{R}_+ \times \mathbb{N}^*$, with the lexical

order. That means that being at time t , with the payoffs defined above, we *extend* the time line by freezing real time t and indefinitely repeating the following game on natural time k :

	Exercise	Differ
Exercise	(S_1, S_2)	(L, F)
Differ	(F, L)	Repeat

Table 1: Coordination game at a fixed time t .

The game has three different outcomes and the exit of the repeated game without the action of at least one player is not permitted. For the continuous time game, this implies that in the present situation of confrontation there must be instantaneous emergence of at least one player, and the game ends here. The situation when players exercise simultaneously and the arbitrator is called upon is actually a particular example of asymmetric game. Indeed in that situation, the outcome of the game is random for both players. Each player then has its own expected utility S_i for an outcome, given by the expectation of A_i according to its law \mathbb{A} .

$$(S_1, S_2) := (q_1L + q_2S + q_S S, q_2L + q_1F + q_S S). \quad (2.1)$$

2.3 Sub-game perfect Nash equilibria of the coordination game

Following [11] and [5], we introduce the formal framework of strategies for the coordination game. We raise the reader's awareness on the assumption that t does not intervene in any payoff function due to the Markov structure of Y . This is why we focus on *Markov sub-game perfect equilibrium* strategies. To obtain the latter, we proceed to the time extension $(t, k) \in \mathbb{R}_+ \times \mathbb{N}^*$ of the model. The filtration is defined via $\mathcal{F}_{t,i} = \mathcal{F}_{t,j} \subseteq \mathcal{F}_{t',i}$ for any $t < t'$ and any $i \neq j$, and the state process is extended to $Y_{(t,k)} := Y_t$. A *simple strategy* for player $i \in \{1, 2\}$ is defined as a pair of \mathbb{F} -adapted processes $(G_{(t,k)}^i, p_{(t,k)}^i)$ taking values in $[0, 1]^2$. The process $G_{(t,k)}^i$ must be a non-decreasing càdlàg process, and refers to the cumulative probability of firm i exercising before (t, k) , whereas $p_{(t,k)}^i$ denotes the probability of exercising in the coordinate game we introduced with Table 1. As we noticed, Y is a continuous Markov process and that allows us to focus without loss of generality on Markov hitting-time strategies of the type $G_{(t,k)}^i(\hat{y}) \equiv G_t^i(\hat{y}) \equiv \mathbf{1}_{\{t \geq \tau(\hat{Y})\}}$ with $\tau(\hat{Y}) := \inf\{t \geq 0 : Y_t \geq \hat{Y}\}$. Explicitly, $p_{(t,k)}^i$ is a strategy for the discretely repeated game at time t , so that it should be stationary and not depend on the previous rounds of the game: $p_{(t,k)}^i \equiv p_t^i \equiv p^i(Y_t)$ for $i = 1, 2$. We openly address the conclusion of [5] on the use of strategies: the repeated discrete game is played only with $(p_{(t,k)}^i)_{i=1,2}$ if $L(t, Y_t) \geq F(t, Y_t) > S(t, Y_t)$. The processes $(G_{(t,k)}^i)_{i=1,2}$ are kept when players do not exercise immediately the option, and when exercise depends on a specific stopping time of the form $\tau(\hat{Y})$. Details can be found in [11] and application is reported to further section. Let us now assume that

$$\max(p_1, p_2) > 0, \quad (2.2)$$

to remind us that $L \geq F > S$ and that we are in situation where players want to act. In this infinite 2-by-2 game, there are three possible outcomes:

- Player one receives payoff L if he exercises alone at any round $k > 0$, provided that nobody exercised before. The probability is given by

$$\begin{aligned}
a_1 &:= p_1(1 - p_2) + (1 - p_1)p_1(1 - p_2)^2 + \dots \\
&= p_1 \sum_{k \in \mathbb{N}^*} (1 - p_1)^{k-1} (1 - p_2)^k \\
&= \frac{p_1(1 - p_2)}{p_1 + p_2 - p_1 p_2}.
\end{aligned}$$

Player two receives payoff F .

- Symmetrically, Player two receives payoff L (and Player one, F) with probability

$$a_2 := \frac{p_2(1 - p_1)}{p_1 + p_2 - p_1 p_2}.$$

- Players play simultaneously with probability

$$a_S := \frac{p_1 p_2}{p_1 + p_2 - p_1 p_2}$$

and the arbitrator is invoked.

Nash equilibria Each player regards the expected utility provided by each possible decision, and then chooses a mixed strategy in consequence. The confrontation being instantaneous, players know that just after it they will receive a specific expected utility of the form L, F or S_i . Therefore, they are taking into account the uncertainty of their own strategy and the uncertainty of the arbitrator's decision. Focus on that particular phenomenon is the purpose of Section 4. Players will to maximize the quantity

$$E_1 := a_1 L + a_2 F + a_S S_1 \tag{2.3}$$

for player one, $E_2 := a_2 L + a_1 F + a_S S_2$ for player two. As explained in Appendix C of [5], we look for Nash equilibriums by first fixing p_2 and differentiate E_1 with respect to p_1 to obtain a first order condition for the optimal value of p_1 for player one:

$$\frac{\partial E_1}{\partial p_1}(p_1, p_2) = \frac{p_2(L - F) + p_2^2(S_1 - L)}{(p_1(1 - p_2) + p_2)^2} \tag{2.4}$$

Paying attention to this expression, one can then see that by denoting

$$P_2 := \frac{L - F}{L - S_1} = \frac{L - F}{q_2(L - F) + q_S(L - S)} , \tag{2.5}$$

the sign of (2.4) is the sign of $p_2 - P_2$. We have a similar discrimination for second player by symmetry of expressions, necessitating to introduces the value $P_1 := (L - F)/(L - S_2)$. We then look for naive equilibriums in the following manner:

- (i) If $p_2 > P_2$, the optimal p_1 is 0. Then by (2.4) E_2 should not depend on p_2 , and the situation is stable for any pair $(0, p_2)$ with p_2 in $(P_2, 1]$.
- (ii) If $p_2 = P_2$, E_1 is constant and p_1 can take any value. If $p_1 < P_1$, then by symmetry p_2 should take value 1, leading to case (i). If $p_1 = P_1$, E_2 is constant and either $p_2 = P_2$, or we fall in case (i) or (iii). The only possible equilibrium is thus (P_1, P_2) .

- (iii) If $p_2 < P_2$, E_1 is increasing with p_1 and player one shall play with probability $p_1 = 1 > P_1$. Therefore p_2 optimizes E_2 when being 0, and E_1 becomes independent of p_1 . Altogether, situation stays unchanged if $p_1 \in (P_1, 1]$ or if $p_1 = 0$. Otherwise, if $p_1 \leq P_1$, we fall back into cases (i) or (ii). The equilibria here are $(p_1, 0)$ with $p_1 \in (P_1, 1]$, and the trivial case $(0, 0)$.

Recalling constraint (2.2), we get rid of case $(0, 0)$. Coming back to the issue of the game when k goes to infinity in (t, k) , three situations emerge from the above calculation. Two of them are pure coordinated equilibriums, of the type $(a_1, a_2) = (1, 0)$ or $(0, 1)$, which can be produced by pure coordinated strategies $(p_1, p_2) = (a_1, a_2)$, settling the game in only one round. The third one is a mixed equilibrium given by $(p_1, p_2) := (P_1, P_2)$, for which (a_1, a_2, a_S) follows according to its definition,

$$(a_1, a_2, a_S) = \left(\frac{1-p_0}{2-p_0}, \frac{1-p_0}{2-p_0}, \frac{p_0}{2-p_0} \right), \quad (2.6)$$

with $p_0 := (L-F)/(L-S)$. If we plug (2.6) into (2.3), we obtain that the payoff of respective players do not depend on (q_1, q_2, q_S) :

$$E_1 = E_2 = \frac{1-p_0}{2-p_0}(L+F) + \frac{a_0}{2-p_0}S. \quad (2.7)$$

In the case $q_S > 0$, they are equal to F . This recalls the rent equalization principle of [4]: players are indifferent between playing the game and being the follower. More than that, the asymmetry does not affect the payoffs and the final outcome of the game after decision of the arbitrator has the same probability as in [5]. Comparing the payoffs for pure and mixed strategies, we find no unique Pareto-optimal strategy.

Remark 2.1. Take $q_S = 1$. Note that by assuming that the symmetry of firms imposes $p_1 = p_2 =: p_0$ and $a_1 = a_2$ in (2.5), we retrieve the case of [5], where we actually obtain $a_1 = a_2$. Conversely, optimal probabilities P_i in the general case can be computed from p_0 defined for (2.6):

$$P_i = \frac{p_0}{q_i p_0 + q_S}, \quad i \in \{1, 2\}. \quad (2.8)$$

Influence of arbitrator's preferences Prior to the selection among multiple Nash equilibria, one shall ask whether value P_i is greater or equal to 1 for $i = 1, 2$, so that the previous equilibriums may not be reachable. If we recall that $q_1 \geq q_2$, we directly obtain from (2.1) that

$$S_1 \geq S_2.$$

According to that, we have $L > F \geq S_1 \geq S_2$ so that $P_2 \geq P_1 \geq 0$. Therefore, using the same procedure to find Nash equilibria as above, we find the following possibilities, which we detail in the next section:

- (a) $P_1 < P_2 < 1$: the three equilibria from above calculation are $(0, 1)$, (P_1, P_2) and $(1, 0)$.
- (b) $P_1 < 1 \leq P_2$: we find only one Nash equilibrium which is $(1, 0)$.
- (c) $1 \leq P_1 < P_2$: we find only one Nash equilibrium which is $(1, 1)$.

Recall that (P_1, P_2) is the only mixed strategy in case (a), and thus the only trembling-hand sub-game perfect equilibrium. It will correspond to the more natural strategy in the sequel, not only because of the robustness of this additional criterion and its pertinence in real world situations, but also because it is also the natural extension of the symmetrical case where both players have the same strategy. We will however fall in a special situation in the next section where one of the three others is more relevant. In case (b), we observe the asymmetry between players. On this frame, there is a strategic advantage of player one, and expected payoffs follow: $(E_1, E_2) = (L, F)$. In case (c), both players exercise the option as in the case (e): the arbitrator decides and the advantage of player one resides in $q_1 \geq q_2$: $(E_1, E_2) = (S_1, S_2)$.

Formalization of the coin toss procedure Assume a Stackelberg competition where simultaneous investment is impossible, i.e., $q_S = 0$. Following definitions (2.1) and (2.5), and assuming $q_i \in (0, 1)$, we have $P_i = 1/q_i > 1$ for $i = 1, 2$. Therefore, only case (c) from above is possible, which is the situation commonly found in the framework using a coin toss procedure. Here $(p_1, p_2) = (1, 1)$, giving the instantaneous situation $(a_1, a_2, a_S) = (1/2, 1/2, 0)$. To our knowledge, papers invoking a coin toss, i.e., an arbitrator, assume that both players have the same chance to be elected as a leader. There is however no need for the regulator to be *fair*. This is a foreseeable phenomenon, since for $L > F$, it is always more interesting to obtain $q_i L + (1 - q_i)F$ rather than F . Note that if players behave symmetrically here, the expected payoff is not the same for each one and depends on q_1 . This will also be true for any convex combination instead of a linear one, meaning that the behavior is expected to be the same with risk-averse players. However, for risk-affine players, this does not hold any more. We do not investigate this peculiar setting reserved to particular economical situations, although it might suit the real option framework, and leave this for further research. The case where $q_1 = 1$ is left to next section.

3 Risk neutral agents in a semi-complete market

We now apply this framework to the investment decision when the project value is driven by a Brownian motion and is perfectly correlated with a traded asset in a complete market. This is the most common studied economical situation, see [8, 6, 7, 1, 5]. We focus on the different laws \mathbb{A} to study the effect of a regulator on firms decision. The introduction of the regulator however makes the payoffs S_i not duplicable, so that we need a pricing criterion for firms. We assume here they are risk-neutral, i.e., their utility function is given by $U(x) = x$.

3.1 The model and value functions

We consider two firms that can invest in a similar project with random cash flows and a fixed initial sunk cost K . The cash-flows are the product of a demand variable process Y and an inverse demand curve $(D_{Q(t)})_{t \geq 0}$, where $Q(t)$ is the number of firms which have invested in the project by time t . The process D_Q can only take the following three values $D_0 = 0 < D_2 < D_1$ and the stochastic process Y is precisely denoted $Y^{t,y}$ if it verifies

$$Y_t^{t,y} = y \text{ and } dY_s = Y_s(\nu ds + \eta dW_s), \quad s \geq t,$$

with $(W_t)_{t \geq 0}$ a standard Brownian motion under the historical measure \mathbb{P} . We assume that Y is perfectly correlated with a liquid traded asset P with dynamics

$$dP_t = P_t(\mu dt + \sigma dW_t) = P_t(r dt + \sigma dW_t^{\mathbb{Q}})$$

where r is the constant interest rate of a risk-less bank account at the disposal of each firm, and $W_t^{\mathbb{Q}} = W_t + \lambda t$ is a Brownian motion under the unique risk-neutral measure \mathbb{Q} of the arbitrage-free market. The variable λ in its expression is the Sharpe ratio equal to $(\mu - r)/\sigma$.

The follower's problem Assume that one of the two firms, say firm one, desires to invest at time t . If $Q(t) = 1$, then the available market for firm one is D_2 . Having a complete financial market at her disposal to replicate the uncertain incomes of the project, the value of the latter at time t if $Y_t^{t,y} = y$ is given by the risk-neutral expectation of the project's discounted cash flows

$$V^F(t, y) = \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\infty} e^{-r(s-t)} D_2 Y_s^{t,y} ds \right] = \frac{D_2 y}{\eta \lambda - (\nu - r)} = \frac{D_2 y}{\delta}$$

with $\delta := \eta \lambda - (\nu - r)$. We assume from now on $\delta > 0$. But firm one can wait to invest, and we suppose she can wait as long as she wants. She has to pay the cost K at time τ she invests. In the financial literature, this is interpreted as a Russian call option of payoff $(D_2 Y_{\tau} / \delta - K)^+$. The value function of this option is given by

$$F(t, y) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau-t)} \left(\frac{D_2 Y_{\tau}^{t,y}}{\delta} - K \right)^+ \mathbb{1}_{\{\tau < +\infty\}} | \mathcal{F}_t \right] \quad (3.1)$$

where \mathcal{T}_t denotes the collection of all \mathbb{F} -stopping times with values in $[t, \infty]$. We now recall Proposition 1 of [5] which gives the explicit solution to (3.1).

Proposition 3.1. *The solution to (3.1) is given by*

$$F(t, y) = \begin{cases} \frac{K}{\beta-1} \left(\frac{y}{Y_F} \right)^{\beta} & \text{if } y \leq Y_F, \\ \frac{D_2 y}{\delta} - K & \text{if } y > Y_F, \end{cases} \quad (3.2)$$

with a threshold Y_F given by

$$Y_F := \frac{\delta K \beta}{D_2 (\beta - 1)} \quad (3.3)$$

and

$$\beta := \left(\frac{1}{2} - \frac{r - \delta}{\eta^2} \right) + \sqrt{\left(\frac{1}{2} - \frac{r - \delta}{\eta^2} \right)^2 + \frac{2r}{\eta^2}} > 1. \quad (3.4)$$

The behavior of the follower firm is thus quite explicit. She will differ investment until the demand reaches at least the level $Y_F = \beta / (\beta - 1) K > K$ which depends on the profitability of the project.

The leader's problem Assume now that instead of having $Q(t) = 1$, $Q(t) = 0$. Firm one investing at time t will receive cash-flows associated to the level D_1 for some time, but she expects firm two to enter the market when the threshold Y_F is triggered. After the moment $\tau(Y_F)$, both firms share the market and firm one receives cash flows determined by level D_2 . The project value is thus

$$\begin{aligned}
V^L(t, y) &:= \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\infty} e^{-r(s-t)} (D_1 \mathbb{1}_{\{s < \tau_s\}} + D_2 \mathbb{1}_{\{s \geq \tau_s\}}) Y_s^{t,y} ds \right] \\
&= \frac{D_1 y}{\delta} - \frac{(D_1 - D_2) Y_F}{\delta} \left(\frac{y}{Y_F} \right)^{\beta}
\end{aligned}$$

where detailed computation can be found in [5]. This allows to characterize the leader's value function $L(t, y)$, i.e., the option to invest at time t for a demand y , as well as the value of the project $S(t, y)$ in the situation of simultaneous investment (Proposition 2 in [5]). Note that here no exercise time is involved as we consider the interest of exercising immediately, Y being non-negative.

Proposition 3.2. *The value of a leader's option is given by*

$$L(t, y) = \begin{cases} \frac{D_1 y}{\delta} - \frac{(D_1 - D_2) K \beta}{D_2^{\beta-1}} \left(\frac{y}{Y_F} \right)^{\beta} & \text{if } y < Y_F, \\ \frac{D_2 y}{\delta} - K & \text{if } y \geq Y_F, \end{cases} \quad (3.5)$$

If both firms invest simultaneously, we have

$$S(t, y) := \frac{D_2 y}{\delta} - K. \quad (3.6)$$

The last case applies if the follower invests immediately after the leader or if the regulator picks up the third outcome: $(A_1, A_2) = (S, S)$.

3.2 Equilibrium study with a partial regulator

It is now appropriate to recall the preemptive game we stated in Section 2. To this end, we need in this setting the following proposition, again formally proved in [5].

Proposition 3.3. *There exists a unique point $Y_L \in (0, Y_F)$ such that*

$$\begin{cases} S(t, y) < L(t, y) < F(t, y) & \text{for } y < Y_L, \\ S(t, y) < L(t, y) = F(t, y) & \text{for } y = Y_L, \\ S(t, y) < F(t, y) < L(t, y) & \text{for } Y_L < y < Y_F, \\ S(t, y) = F(t, y) = L(t, y) & \text{for } y \geq Y_F. \end{cases} \quad (3.7)$$

We retrieve two explicit cases: when $y < Y_L$, where the two firms wait to obtain a better return of the project, and when $y \geq Y_F$, leading to immediate commitment of firms, whatever the decision of the regulator. These correspond respectively to cases (d) and (e) of previous section. We now look more deeply into the case $Y_L \leq y \leq Y_F$.

Risk-neutral agents and semi-complete market According to the Markovian framework we will fix $t = 0$ and consider the process $Y^{0,y}$ without loss of generality. Henceforth, we omit the t dependency in value functions and strategies. Coming back to the situation where $y \in (Y_L, Y_F)$, firms are facing a coordination game similar to the one of subsection 2.3. Notice that the regulator intervenes in the coordination game only once and if both firms want to invest at the same time. Since we assumed that the regulator decision is not \mathbb{F} -adapted but \mathbb{F}^+ -adapted, firms face a market

incompleteness. We thus introduce now the fact that firms are *risk-neutral*, i.e., they will estimate the value of the outcome A_i defined above by taking their expectations under \mathbb{A} :

$$\begin{cases} S_1(y) = q_1 L(y) + q_2 F(Y) + q_S S(y) & \text{for firm 1,} \\ S_2(y) = q_2 L(y) + q_1 F(Y) + q_S S(y) & \text{for firm 2.} \end{cases} \quad (3.8)$$

Values $L(y)$, $F(y)$ and $S(y)$ are still computed under the risk-neutral measure \mathbb{Q} on Ω . That means that expectations of (3.8) are made under the *minimal entropy martingale measure* $\mathbb{Q} \times \mathbb{A}$ for this problem, and that uncertainty of the model follows a *semi-complete* market hypothesis: if we reduce uncertainty to the market information, i.e., the filtration \mathbb{F} , then the market is considered as complete. It is a very convenient assumption when central limit theorem can be invoked for the orthogonal random variable, as in diversification for insurance purpose. See [2] for the formal definition and corresponding situations. Here it is not the case, and we include the risk-neutral assumption to obtain simple solutions. Firms value the arbitrator's intervention by just weighting each arbitrage-free price by its *historical* probability. The risk-averse case is postponed to the next section.

The case study Since the cases $q_S = 1$ and $q_S = 0$ were studied in section 2, we now study the general case $0 < q_2 \leq q_1 < 1 - q_2$ of subsection 2.3 with respect to the value of y . In reference to [5], we provide a proposition allowing to define the regions (a), (b) and (c).

Proposition 3.4. *The functions P_2 and P_1 are increasing on $[Y_L, Y_F]$.*

Proof By taking $d_1(y) := L(y) - F(y)$ and $d_2(y) := S(y) - F(y)$, we get

$$P_i(y) = \frac{1}{q_i} \left[\frac{d_1(y)}{d_1(y) + \gamma_i(d_1(y) - d_2(y))} \right]$$

and

$$P'_i(y) = \frac{1}{q_i} \left[\frac{\gamma_i(d_1(y)d'_2(y) - d_2(y)d'_1(y))}{(d_1(y) + \gamma_i(d_1(y) - d_2(y)))^2} \right]$$

where $\gamma_i := q_S/q_i \leq 1$ with $i \in \{1, 2\}$. We are thus interested in the sign of the quantity $g(y) := d_1(y)d'_2(y) - d_2(y)d'_1(y)$ which quickly leads to

$$\frac{g(y)\delta}{yD_2} = \left(\frac{y}{Y_F} \right)^{\beta-1} \left[(D_1 - D_2)(\beta + \frac{1}{\beta} - 2 - \frac{\delta K}{D_2}) \right] + \frac{\delta K}{D_2} (D_1 - D_2) .$$

Since $\beta > 1$, $(y/Y_F)^{\beta-1}$ is an increasing function. Since $0 < y \leq Y_F$, it suffices to verify that $(\delta K)/D_2 \geq (\beta + 1/\beta - 2 - (\delta K)/D_2)$, which is naturally the case for any β , to obtain that g is non-negative on the interval.

As a by-product, this proves that $p_0 := p_1 = p_2$ in the case $q_S = 1$ is increasing with y . This fact was already observed in [5].

Subregions Note that the quantities q_1 and q_2 do not affect the monotonicity of P_2 and P_1 . We already know that $P_2(y) \geq P_1(y)$. Looking back at (2.5), we also have that $P_2(Y_L) = P_1(Y_L) = 0$ and $P_i(Y_F) = 1/(q_i + q_S) > 1$ for $i = 1, 2$. This implies that we retrieve the three subregions (a), (b) and (c) of previous section:

- (a) the interval (Y_L, Y_1) for some $Y_1 \in (Y_L, Y_F)$ on which $P_1(y) \leq P_2(y) < 1$ and such that $P_2(Y_1) = 1$. The point Y_1 verifies

$$F(Y_1) = q_1L(Y_1) + q_2F(Y_1) + q_S S(Y_1) = S_1(Y_1) . \quad (3.9)$$

For this interval, we are in the situation described in subsection 2.3 where three Nash equilibria are possible, and only $(p_1(y), p_2(y)) = (P_1(y), P_2(y))$ is kept, as it is the only trembling-hand perfect equilibrium. We also keep that one because it is the one involved in the fair situation of [5]. For this strategy, the probabilities of outcomes of the game are given by the triplet (2.6) and the payoffs by (2.7).

- (b) The next case is given by interval (Y_1, Y_2) for some $Y_2 \in (Y_1, Y_F)$, on which $P_1(y) < 1 \leq P_2(y)$. The reader will guess that Y_2 is chosen such that $P_1(Y_2) = 1$, and verifies

$$F(Y_2) = q_2L(Y_2) + q_1F(Y_2) + q_S S(Y_2) = S_2(Y_2) . \quad (3.10)$$

We just said that $S_1(Y_1) = F(Y_1)$. Being risk-neutral, the first firm prefers being a leader after $\tau(Y_1)$ and is indifferent between being follower or letting the regulator chose. For the other firm, differing means receiving $F(Y_1)$ and exercising means letting the regulator give an expected payoff $q_1F(Y_1) + q_2L(Y_1) + q_S S(Y_1) < F$. Differing is her best option. That means that on (Y_1, Y_2) , the equilibrium strategies are $(p_1, p_2) = (1, 0)$.

- (c) On the interval (Y_2, Y_F) , the second firm can finally bear the regulator preference for first firm and be indifferent between *followership* and a decision of the regulator. Here, $1 \leq P_1(y) \leq P_2(y)$ and firms has a greater expected payoff by letting the regulator choose rather than being follower. According to the facts stated in subsection 2.3, equilibrium exists when both firms exercise.

Endpoints We separate the study of junctures of areas (d),(a),(b),(c) and (e). The technical issue has been settled in [11] and we recall the strategy G when needed.

- (a) The juncture of $[0, Y_L)$ with $[Y_L, Y_1]$ is a delicate point. At the left of point Y_L , no firm wants to invest. We thus shall use the strategy $G^i(Y_L)$ for both players. By right-continuity of this process, both players shall exercise with probability 1 at point Y_L . At the right side of point Y_L however, P_i converges to 0 when y converges to Y_L , for $i = 1, 2$. Therefore, so do (p_1, p_2) toward $(0, 0)$. We cannot reconcile $G_{\tau(Y_L)}^i(Y_L)$ with $(p_1(Y_L), p_2(Y_L)) = (0, 0)$ and shall compare the payoffs. It is clear at point Y_L that the second option is better for both firms. There is also a continuity of behavior between (d) and (a) from the fact that simultaneous investment is improbable at point Y_L , and probability $a_S(y)$ is continuous at this point: a short calculation provides

$$\lim_{y \downarrow Y_L} \frac{a_1(y)}{a_2(y)} = 1 \quad \text{and} \quad \lim_{y \downarrow Y_L} a_s(y) = 0 . \quad (3.11)$$

Therefore at point Y_L ,

$$(a_1(Y_L), a_2(Y_L), a_S(Y_L)) = (1/2, 1/2, 0) .$$

- (b) On the left of Y_1 , we find that $P_2(y)$ goes to 1 and (a_1, a_2, a_S) tends toward $(0, 1 - P_1(Y_1), P_1(Y_1))$. As long as $y < Y_1$ the equilibrium is given by $(p_1(y), p_2(y)) = (P_1(y), P_2(y))$. However, following (3.9), $S_1(y)$ goes to $F(Y_1)$ and at point Y_1 , firm one has interest to exercise the option with probability one. Nevertheless, she does not make use of strategy $G^1(Y_1)$ since the coordination game implies the settlement of the game on $[Y_L, Y_1)$. There is still a discontinuity, and behavior at Y_1 is the same as on (Y_1, Y_2) , given by $(p_1, p_2) = (1, 0)$.
- (c) It is clear that for $y < Y_2$, $F(y) > S_2(y)$, and since first firm has a definite advantage not to hesitate exercising her option, the second firm will differ exercise until $\tau(Y_2)$. For point Y_2 , we shall then introduce the strategy $G^2(Y_2)$ for firm two. By right-continuity of this process, the strategy at point Y_2 is given by $(p_1, p_2) = (1, 1)$. There is no conflict in doing such, but another discontinuity.
- (e) The continuity of strategies at point Y_F spares us from detailing.

Few comments are in order. On the right of point Y_L , the asymptotic probabilities of outcomes given by equation (3.11) tend to the fair distribution of [5]. This is a direct consequence of a_S vanishing to 0, and thus less intervention from the regulator. At point Y_1 , there is a strong discontinuity in the optimal behavior of the second firm. For $y < Y_1$, the mixed strategy used by the latter tends toward a pure strategy with systematic investment. However at the point itself, the second firm differs investment and becomes the follower. We propose the following interpretation. As y tends to Y_1 from left, firm one is less refrained from investing, although her probability to act is still lower than the one of firm two. For firm two, this is the signal that she should invest and try to preempt firm one, otherwise she will clearly lose interest in the preemption situation when Y_1 is reach. For firm one, being a follower is better than letting the regulator decide before Y_1 . At the point however, she is indifferent between these two positions so that she will suddenly seek for the leader's position, creating another discontinuity in her behavior.

3.3 Extreme cases and the value of being preferred

Continuous unification of classical frameworks Starting from the framework where sharing the market is the only outcome of a simultaneous investment, we can see that the progressive introduction of alternatives in the regulator's range of decisions reduces the synchronization game to $[Y_L, Y_1]$, and intervention of the regulator comes at point Y_2 , lower than Y_F . In the case where $q_1 = q_2 =: q$, and $Y_1 = Y_2 =: Y_S$, the intermediate advantage given to the first firm reduces to nothing. Assuming this is the case, we obtain two limits

$$\lim_{q \uparrow 1/2} Y_S = Y_L \quad \text{and} \quad \lim_{q \downarrow 0} Y_S = Y_F .$$

Therefore, our setting encompasses in a continuous manner the two usual types of games described in the previous section.

The value of being preferred We continue the case study with $q_1 = 1 - q_2 = 1$. Firm one is completely preferred to firm two. We intend to draw the reader's attention on the fact that the leader's role is not decided exogenously. The advantage of firm one is conditioned to the simultaneity

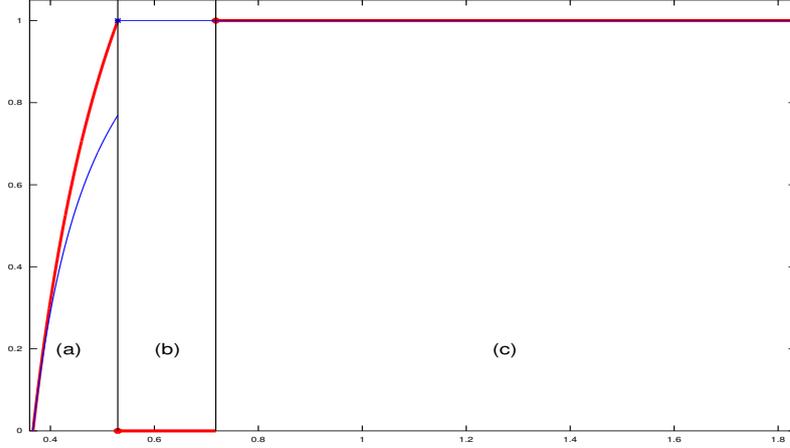


Figure 1: Values of equilibrium mixed strategies $p_1(y)$ (blue) and $p_2(y)$ (red) in the asymmetrical case $(q_1, q_2, q_S) = (0.5, 0.2, 0.3)$. Areas (a), (b) and (c) are separated by vertical lines at $Y_1 = 0.53$ and $Y_2 = 0.72$ on $[Y_L, Y_F] = [0.37, 1.83]$. Area (d) is then at the left of the graph and (e) at the right of it. Note that p_1 and p_2 are right-continuous. Parameters set at $(K, \nu, \eta, \mu, \sigma, r, D_1, D_2) = (10, 0.01, 0.2, 0.04, 0.3, 0.03, 1, 0.35)$.

of investment decision of firms. If firm two preempts firm one, this advantage is useless. For firm two also, this is a privilege in the case of simultaneous investment since it gives her the choice to postpone investment. In that case $P_1 = 1$, and firm one has always interest to exercise the option for $y \geq Y_L$. Knowing that, firm two shall wait the trigger Y_F to exercise her option. To sum up, the equilibrium behavior of firms is $(p_1, p_2) = (1, 0)$ on $[Y_L, Y_F)$, which corresponds to case (b) exclusively.

For curiosity, we compare here this situation to the case $q_S = 1$. We do that because [5] already compared the situation $q_S = 1$ to the *ex-ante* attribution of roles, where firm two is the follower and must wait for firm one to exercise her option. The difference of value for the leader is designed to be a *priority option*, i.e., the value of being designated leader. Here, we introduce a weaker option, which only gives the value of being preferred by the regulator. The option allows to shift (q_1, q_S) from $(0, 1)$ to $(1, 0)$, and is more straightforward to compute. We compare in Figure 2 the two option values. Following definition (2.3), we denote $E_1^{(q_1, q_S)}(y)$ the expected payoff of firm one when $Q = 0$, with q_1 and q_S explicitly given. Let us define now the value of being preferred by $\pi^0(y) := E_1^{(1,0)}(y) - E_1^{(0,1)}(y)$. For $y \geq Y_L$, $E_1^{(1,0)}(y) = L(y)$. By equation (2.7), we retrieve the rent equalization principle which says that $E_1^{(0,1)}(y) = F(y)$. If $y < Y_L$, by continuity of the demand process, firm one should exercise at time $\tau(Y_L)$ where $L(Y_L) = F(Y_L)$ to avoid preemption of her opponent. Her expected additional value is thus 0. Altogether, π^0 is given by

$$\pi^0(y) = (L(y) - F(y))^+ \quad \text{for all } y \geq 0.$$

Remark 3.1. *This option has three interesting properties.*

- *Firstly, it gives an advantage to its owner without penalizing the other firm, who can still expect the payoff F .*
- *Consider the incomplete information assumption where firm one knows that $(q_1, q_S) = (1, 0)$*

and firm two assumes wrongly that $(q_1, q_S) = (0, 1)$. Then the behavior of firm two will change to a mixed strategy $P_2(y)$ on $[Y_L, Y_1)$, differs on $[Y_1, Y_2)$ and exercises on $[Y_2, \infty)$. Firm one's best behavior is thus to invest with probability one on interval $[Y_L, Y_F]$, which is exactly what is already recommended with the option in hand in the perfect information setting. Therefore, no advantage is given by hiding the regulator's decision to the penalized firm.

- There is no advantage of being preferred if the follower's option value is greater or equal to the leader's value function, even with the consideration of evolution of the latter, see Figure 2. This is due to the continuity of the state variable Y . A question that naturally arises is whether the introduction of jumps in the state variable dynamics has a pertinent effect on the value of being preferred.

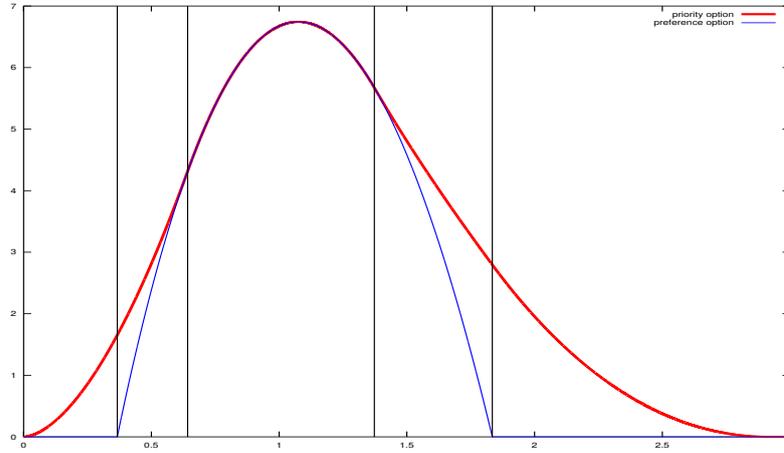


Figure 2: Priority option value (red) and Preference option value (blue) in function of y . Vertical lines at $Y_L = 0.37$, $Y_1 = 0.64$, $Y_2 = 1.37$ and $Y_F = 1.83$. Option values are equal on $[Y_1, Y_2]$. Same parameters as in Figure 1.

Pure strategy as the best strategy One mathematical situation is left to explore, i.e. $q_2 = 0$ and $q_1 \in (0, 1)$, but reminding the introduction of the present subsection, we think that it is also economically relevant to study the exclusion by the regulator of second firm. Imagine for example that firm one has a definitive advantage, like safety and health standards in the case of a new drug product, but that simultaneous investment is not literally forbidden and the regulator shall publicly prove his fairness if simultaneity is acknowledged. We simply observe from equations (3.9) and (3.10) that if $q_2 = 0$, then $Y_2 = Y_F$ and Y_1 verifies $F(Y_1) = q_1 L(Y_1) + (1 - q_1) S(Y_1)$. The consequence is straightforward: the interval $[Y_1, Y_2)$ expands to $[Y_1, Y_F)$. The fact that $Y_1 > Y_L$ for $q_1 < 1$ implies also a specific situation. We intuit the equilibrium $(p_1, p_2) = (1, 0)$ to be more relevant than $(p_1, p_2) = (P_1, P_2)$ on $[Y_L, Y_1)$. Indeed, if firm one invests systematically on that interval then firm two has no chance of being the leader (by drastic choice of the regulator), and therefore finds satisfying the follower's position. In opposition of the trembling-hand equilibrium, this pure strategy can be

well figured by being called *steady-hand*. In that case

$$(E_1(y), E_2(y)) = (L(y), F(y)) \text{ for } y \in [Y_L, Y_1]. \quad (3.12)$$

Comparing (2.7) to (3.12), it appears more interesting for firm one to push firm two to this strategy with indifference from the latter. There is thus a definitive advantage to play this strategy rather than the former. Notice also the connexion with the preference option we described in the last paragraph: $E_1(y) = E_1^{(1,0)}(y)$.

4 Aversion for confrontation

This section can be considered as an addendum because most ideas regarding asymmetry in front of a regulator's decision has been presented in the previous section. We will essentially focus on the effect of the random decision of the regulator and the mixed strategy by means of risk-aversion of the firms. We study the combination of both effects in what we call *aversion for confrontation*. We emphasize the intricate relation between asymmetry and risk-aversion.

Risk profile in complete market In this section, we do not change the setting of the market, nor the project's value: the market is complete. However, we now endow each firm with the same utility function U designed to be strictly concave. To avoid initial wealth dependency, we focus on the CARA utility function given by

$$U(x) = -\exp(-\gamma x)$$

where $\gamma > 0$ is the risk-aversion of firms. We then assume that firms start with a null initial wealth. Since the market is complete and free of arbitrage, both firms still price the leader, the follower and sharing positions with the unique risk-neutral probability \mathbb{Q} . For each of them, a firm receives the corresponding utility. We denote $l(y) := U(L(y))$, $f(y) := U(F(y))$ and $s(y) := U(S(y))$, and index variables with γ to make the dependence explicit. In the case where both firms simultaneously declare their desire to exercise the option, the regulator comes into play. In that case, he randomly attributes a payoff to each player, which receives a utility from it. The regulator is then defined via

$$(A_1, A_2)(y) := \begin{cases} (l, f)(y) & \text{with probability } q_1 \\ (f, l)(y) & \text{with probability } q_2 \\ (s, s)(y) & \text{with probability } q_S \end{cases}.$$

As the reader is now aware of, the regulator only intervenes in one outcome of the coordination game of Table 1. Prior to that, each firm uses a mixed strategy p_i^γ , in order to maximize the expected utility (2.3):

$$E_1^\gamma(y) = (a_1^\gamma + a_s^\gamma q_1)l(y) + (a_2^\gamma + a_s^\gamma q_2)f(y) + a_s^\gamma q_S s(y) \quad (4.1)$$

We then understand that all the calculus of Section 3 hold in this setting by just changing (L, F, S) for (l, f, s) . It follows that in that case

$$P_{i,\gamma}(y) = \frac{l(y) - f(y)}{q_i(l(y) - f(y)) - q_S(l(y) - s(y))} \text{ with } i \in \{1, 2\}, \quad (4.2)$$

and that, as in section 3, the behavior of firms can be characterized by four strategic interaction types depending on the value of y on four intervals (d), (a), (b) and (c)+(e).

Influence of aversion on regions (a),(b) and (c) The question we address in this section is how risk-aversion influences the different intervals. Notice that the definition of regions (d) and (e) do not change since they are independent from risk-profile of players: prices are sufficiently explicit under the complete market hypothesis. Let's come to the analysis of regions (a), (b) and (c), and focus on (a) where (4.2) is used. To that purpose, we first consider the case $q_S = 1$ where $P_{1,\gamma}(y) = P_{2,\gamma}(y) =: p_\gamma(y)$, and (b) and (c) reduce to nothing. Consider a fixed value $y \in (Y_L, Y_F)$ to avoid putting it in the notations. From (4.2) we get

$$p_\gamma = \frac{l-f}{l-s} = \frac{-e^{\gamma L} + e^{\gamma F}}{-e^{\gamma L} + e^{\gamma S}} = \frac{e^{\gamma(L-F)} - 1}{e^{\gamma(L-S)} - 1}.$$

Since $u(x) := -1 - U(-x) = e^{\gamma x} - 1$ is a positive strictly convex function on $[0, \infty)$ with $u(0) = 0$, we have that

$$p_\gamma = \frac{u(L-F)}{u(L-S)} < \frac{L-F}{L-S} =: p_0 \quad (4.3)$$

Aversion for confrontation is expressed through diminishing probability of intervention with γ . For γ going to zero, we apply l'Hôpital's rule to obtain that $\lim_{\gamma \downarrow 0} p_\gamma = p_0$. By augmenting the risk aversion γ , the probability to act in the infinite game of section 2 reduces to an asymptotic limit:

$$\lim_{\gamma \uparrow \infty} p_\gamma = \lim_{\gamma \uparrow \infty} e^{-\gamma(F-S)} = 0.$$

Note that there is no uniform convergence since p_0 is continuous and $p_0(Y_F) = 1$. But the above convergence holds for all $y \in [Y_L, Y_F)$. It is clear from (4.3) that p_γ is monotonous for $\gamma \in (0, \infty)$, and then according to the above limit, it is convex decreasing with γ .

We now take $q_2 \leq q_1 < 1 - q_2$. Following Remark 2.1 and relation (2.8), we get that $P_{i,\gamma}$ is a concave non-decreasing functions of p_γ . It is then a decreasing function of γ , but we cannot easily state convexity or concavity of the function. Now recalling that the regions are separated via conditions $P_{2,\gamma} = 1$ and $P_{1,\gamma} = 1$, we can state the following:

- (a) Since $P_{2,\gamma}(y)$ is decreasing with γ , Y_1 is an increasing function of γ : the region (a) spreads on the right with γ . Adapting (3.9) to the present values, Y_1 shall verify:

$$q_1(1 - e^{\gamma(L(Y_1) - F(Y_1))}) + q_S(1 - e^{\gamma(S(Y_1) - F(Y_1))}) = 0$$

and when γ goes to ∞ , we need $L(Y_1) - F(Y_1)$ to go to 0, so that Y_1 tends toward Y_F .

- (b) The width of region (b) is not monotonous in γ . From numerical simulation, we observe that the region is increasing and then decreasing, converging to zero according to (a).
- (c) Since $P_{1,\gamma}(y)$ is decreasing with γ , Y_2 is an increasing function of γ : the regions (c) reduces from the left with γ , until disappearance.

With risk aversion, the region (a) takes more importance and the competitive advantage of firm one decreases with γ . Figure 3 resumes the evolution.

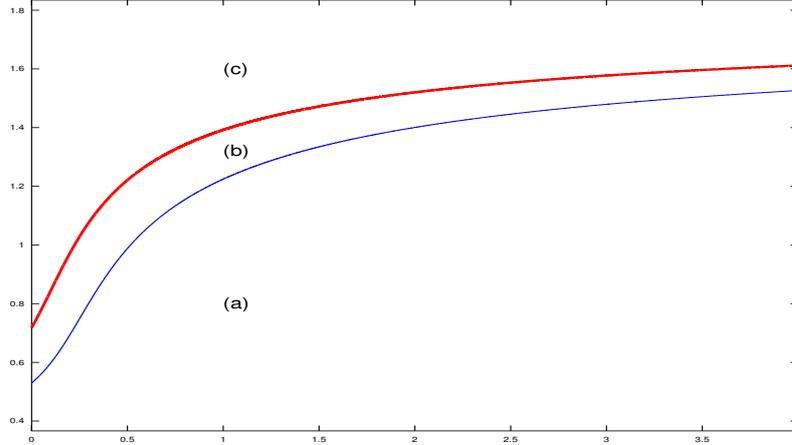


Figure 3: Values of Y_1 (blue) and Y_2 (red) as a function of risk aversion γ . Y-axis limited to $[Y_L, Y_F] = [0.37, 1.83]$. Limit values of (Y_1, Y_2) for γ going to 0 corresponds to $(0.53, 0.72)$ of Figure 1. Same parameters as previous figures.

Highlight on region (a) Let us denote $(a_1^\gamma, a_2^\gamma, a_S^\gamma)$ the probabilities of outcomes with risk averse firms who uses the mixed strategy $p_i^\gamma := P_{i,\gamma}$, and (a_1, a_2, a_S) the probabilities of outcomes by using probability p_i of Section 3. From above $p_i^\gamma < p_i$ and since a_S^γ is increasing in both variables (p_1^γ, p_2^γ) ,

$$a_S^\gamma < a_S . \quad (4.4)$$

Putting (2.8) into a_i^γ , we get

$$a_i^\gamma = \frac{1 - p_\gamma}{2 - p_\gamma} - \frac{q_i}{q_S} \frac{p_\gamma}{2 - p_\gamma} \quad (4.5)$$

and differentiating in p_γ , we obtain that a_i^γ is increasing in γ . Then we can see that

$$\frac{a_1^\gamma}{a_2^\gamma} = \frac{p_1^\gamma - p_1^\gamma p_2^\gamma}{p_2^\gamma - p_1^\gamma p_2^\gamma} = \frac{q_S - (1 - q_2)p_\gamma}{q_S - (1 - q_1)p_\gamma} \quad (4.6)$$

and at the limit or using (2.6),

$$\frac{a_1}{a_2} = \frac{p_1 - p_1 p_2}{p_2 - p_1 p_2} = \frac{q_S - (1 - q_2)p_0}{q_S - (1 - q_1)p_0} = \frac{F - S_1}{F - S_2} .$$

Notice that this last term is lower than 1 on region (a). Differentiating equation (4.6) in p_γ and recalling that $q_1 \geq q_2$ we obtain that a_1^γ/a_2^γ is decreasing in p_γ , and therefore increasing in γ . As a corollary,

$$\frac{a_1}{a_2} < \frac{a_1^\gamma}{a_2^\gamma} < \lim_{\gamma \uparrow \infty} \frac{a_1^\gamma}{a_2^\gamma} = 1 . \quad (4.7)$$

From inequality (4.4) we shall conclude the following. Aversion for confrontation instinctively leads to a lower probability of simultaneity. This is due to the case that both firms increase their utility by reducing the risk of their decision. Consider the two reasonable directions : both firms act more steadily by increasing p_1^γ and p_2^γ , or on the contrary are more hesitant and reduce these quantities. By augmenting their probability to act, a_S^γ grows up. However, the intervention of the regulator

makes leader and follower positions remaining alternatives. Therefore the risk keeps high. If both firms reduce their instantaneous probability to exercise, a_S^γ reduces and the risk reduces to the two positions of leader and follower by preemption. When the risk aversion augments, this is the most desirable alternative. Equation (4.7) confirms that conclusion. The lower the probability of a regulator intervention, the lower is the impact of the asymmetry. When the regulator is fair, only the simultaneous exercise probability a_S^γ is affected.

Indifference prices How does γ impact the real outcome of the game? To compare homogeneously the expected values of options L, F and S to the expected utility provided by (4.1), we inverse the utility and compute indifference prices e_i :

$$e_{1,\gamma}(y) := U^{-1}(E_1^\gamma(y)) = -\frac{1}{\gamma} \log(-a_1^\gamma l(y) - a_2^\gamma f(y) - a_S^\gamma s_i(y)) .$$

It is clear that this computation is relevant only in the region (a). Applying Jensen's inequality to U^{-1} and using (4.5),

$$\begin{aligned} e_{1,\gamma}(y) &\leq (a_1^\gamma + a_s^\gamma q_1)L(y) + (a_2^\gamma + a_S^\gamma q_2)F(y) + a_s^\gamma q_S S(y) \\ &\leq \frac{1-p_\gamma}{2-p_\gamma}(L(y) + F(y)) + \frac{p_\gamma}{2-p_\gamma}S(y). \end{aligned}$$

Since p_γ is decreasing with γ , comparing $e_{1,\gamma}(y)$ with $E_1(y)$ is not straightforward. We can however recall that a_S goes to 0 when γ increases, and using the above inequality, we get that

$$\lim_{\gamma \uparrow \infty} e_{1,\gamma}(y) < \frac{1}{2}(L(y) + F(y)) \quad \text{for } y \in (Y_L, Y_F).$$

where right-hand term is precisely the expected payoff in the real option game with $q_S = 0$. The same calculus apply for firm two.

5 Discussion

If real option games model are to be applied, the role of a regulator shall be introduced for many applications. Indeed regulators often intervene for important projects in energy, territorial acquisition and highly sensitive products such as drugs. The model we proposed is a first attempt, and could be improved on several grounds. A realistic but involved complication is the influence of explicit parameters on the law \mathbb{A} , parameters on which agents have some control. The value of being preferred, introduced as a financial option, would then provide a price to a competitive advantage and to side efforts to satisfy non-financial criteria. The abstract approach undertaken in Section 2 could be used with little modifications

Despite its simplicity and its idealization, the real option game with the three outcomes regulator has still something to say. One can see it as an archetype model, unifying mathematically the Stackelberg and the Cournot competition frameworks and leading to simple formulae. Beside the original motivation to rigorously formulate the Stackelberg competition problem, we obtain a complex and intriguing form of asymmetrical competition, which can settle the ground for more complex situations. One can see that a partial regulator influences outcomes of the game only partially. If the

expected value of the project is sufficiently high (when $y \geq Y_2$), payoffs and strategies are impacted, and if it is low (when $y \leq Y_1$), only strategies are. In between lies the real advantage for the preferred agent. Altogether, it is a complex strategic behavior that emerges from the simplest model.

Finally, the introduction of an additional source of uncertainty has been the opportunity to propose another dimension of analysis. By focusing risk-aversion only on the uncertainty of the game, we can extend the model in a completely different direction, where agents are averse to the confrontation itself. The short analysis provides analytical and numerical foundations to many economical intuitions. The most relevant here is that aversion for confrontation leads to an implicit fair agreement among participants to avoid confrontation and the regulator's intervention. The simplicity and abstraction of the model forbid us to develop more precisely. However, this method to analyse risk-aversion implication appears, to our knowledge, a new treatment of real-option games. If it suits particularly well to our initial setting, we are confident that many symmetrical or asymmetrical situations can be studied and benefit from such an approach.

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