UTILITY MAXIMISATION AND UTILITY INDIFFERENCE PRICE FOR EXPONENTIAL SEMI-MARTINGALE MODELS WITH RANDOM FACTOR

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UTILITY MAXIMISATION AND UTILITY INDIFFERENCE PRICE FOR EXPONENTIAL SEMI-MARTINGALE MODELS WITH RANDOM FACTOR

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Abstract. We consider utility maximization problem for semi-martingale models depending on a random factor ξ. We reduce initial maximization problem to the conditional one, given $\xi = u$, which we solve using dual approach. For HARA utilities we consider information quantities like Kullback-Leibler information and Hellinger integrals, and corresponding information processes. As a particular case we study exponential Levy models depending on random factor. In that case the information processes are deterministic and this fact simplify very much indifference price calculus. Then we give the equations for indifference prices. We show that indifference price for seller and minus indifference price for buyer are risk measures. Finally, we apply the results to Geometric Brownian motion case. Using identity in law technique we give the explicit expression for information quantities. Then, the previous formulas for indifference price can be applied.

Key words and phrases: utility maximisation, utility indifference price, semi-martingale, f-divergence minimal martingale measure, exponential Levy model

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1. Introduction

In the real financial market investors can held traded risky assets of maturity time T and receive some particular derivatives such as contingent claims offering some pay-off at maturity time $T' > T > 0$. It

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can happen that the assets related with contingent claims can not be traded since the trading is difficult or impossible for investor because of lack of liquidity or legal restrictions. In this situation the investor would like maximize expected utility of total wealth and at the same time reduce the risk due to the uncertainty of pay-off of the contingent claim. In such situations the utility indifference pricing become to be a main tool for option pricing.

To be more precise, let us suppose that our market consists on non-risky asset $B_t = B_0 \exp(rt)$, where r is interest rate, and two risky assets

$$S_t = S_0\, \mathcal{E}(X)_t, \;\; \tilde{S}_t = \tilde{S}_0\, \mathcal{E}(\tilde{X})_t$$

where X and $\tilde{X}$ are semi-martingales with jumps $\Delta X > -1$, $\Delta \tilde{X} > -1$, and $\mathcal{E}$ is Dolean-Dade exponential. The investor can trade S and at the same time he has a European type claim on $\tilde{S}$ given by $g(\tilde{S}_{T'})$ where g is some real-valued Borel function. Let us denote by Π the set of self-financing admissible strategies. Then, for utility function U and initial capital x, the optimal expected utility related with S will be

$$V_T(x) = \sup_{\phi \in \Pi} E[U(x + \int_0^T \phi_s \, dS_s)]$$

and if we add an option, then the optimal utility will be equal to

$$V_T(x, g) = \sup_{\phi \in \Pi} E[U(x + \int_0^T \phi_s \, dS_s \; + \; g(\tilde{S}_{T'})]$$

As known, the indifference price p_T^b for buyer of the option $g(\tilde{S}_{T'})$ is a solution to the equation

$$V_T(x - p_T^b, g) = V_T(x)$$

and it is an amount of money which the investor would be willing to pay today for the right to receive the claim and such that he is no worse off in expected utility terms then he would have been without the claim. The indifference price for the seller p_T^s of the option is a solution to the equation

$$V_T(x + p_T^s, -g) = V_T(x)$$

and it is an amount of money which the seller of the option would be willing to receive in counterpart of the option in order to preserve his own optimal utility.

The optimal utility of assets containing the options highly depends on the level of information of the investor about $\tilde{S}$. More precisely, the investor can be non-informed, partially informed or perfectly informed

agent and the level of information changes the class Π mentioned in previous formulas. Namely, a non-informed agent can maximize his expected utility taking the strategies only from the set of self-financing admissible strategies with respect to the natural filtration $\mathbf{F}$ of X. At the same time, a partially informed agent can built his optimal strategy using the set of self-financing admissible strategies with respect to the progressively enlarged filtration $\tilde{\mathbf{F}}$ with the process $\tilde{X}$. Finally, a perfectly informed agent can use the self-financing admissible strategies with respect to initially enlarged filtration $\mathbf{G}$ with $\tilde{S}_{T'}$.

Utility maximisation and utility indifference pricing was considered in a number of books and papers, see for instance [3], [4],[6], [12], [18], [31], [28], [29], [32],[33]. Some explicit formulas for the indifference prices were obtained for Brownian motion models, where the incompleteness on the market comes from the non-traded asset (see [18], [28], [29]). Close to our setting case, but for complete markets, was considered in [2] and one can find there nice explicit formulas for indifference price.

In this note we concentrate ourselves on non-complete market case, and we establish some explicit formulas for the indifference prices for semi-martingale models when the traded and non-traded asset are dependent. This dependence is modelled by including the non-traded asset into the structure of the traded asset as a factor influencing its price dynamics. We will concentrate ourselves on the problem of utility maximisation and utility indifference pricing for perfectly informed agents. Our aim is to obtain explicit and numerically tractable solutions for these questions, especially for exponential Levy models and diffusions.

It should be noticed that the indifference price for partially informed agents and non-informed agents will be the same in considered case since the σ-algebras at time T in all three cases coincide(to do calculus, one has to ensure that $g(\xi)$ is measurable!), and the last fact implies that the minimal equivalent martingale measure, when exist, will be the same in three cases, too. Contrarily to this, the optimal strategies will depend on the used filtration. It should be noticed that in the case of exponential Levy models and HARA utilities the optimal strategy for initial enlargement, when it exists, is always progressively adapted. The same is true for the processes with independent increments. This fact can be explained by preservation of Levy property and "independent increments" property by minimal equivalent martingale

measure when such measure exists, and explicit formulas for optimal strategies(cf.[7],[8]).

From point of view of modelling our approach consists to introduce semi-martingales depending on a random factor ξ. Namely, the considered risky asset S will be of the form $S(\xi) = \mathcal{E}(X(\xi))$ with the semi-martingale $X(\xi) = (X_t(\xi))_{t\geq 0}$ leaving on a canonical probability space and depending on a supplementary random factor ξ. The random variable ξ is given on Polish space $(\Xi, \mathcal{H})$. We denote by α the law of this variable ξ. The details concerning such mathematical framework is given in section 2 and they are close to the approach in [17].

The section 3 is devoted to the general results about the maximisation of utility for semi-martingale models depending on a random factor. As previously let us introduce the total utility with the option $g(\xi)$:

$$V(x,g) = \sup_{\varphi\in\Pi(\mathbf{G})} E_{\mathbb{P}}\left[U\left(x + \int_0^T \varphi_s dS_s(\xi) + g(\xi)\right)\right]$$

Here $\Pi(\mathbf{G})$ is the set of all self-financing and admissible trading strategies related with the initially enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,T]}$, where $\mathcal{G}_t = \bigcap_{s>t}(\mathcal{F}_s \otimes \sigma(\xi))$. To solve the utility maximisation problem in the initially enlarged filtration we make an assumption about the absolute continuity of the conditional laws $\alpha^t = \mathbb{P}(\xi \,|\, \mathcal{F}_t)$ of the random variable ξ given $\mathcal{F}_t$ with respect to α, namely

$$\alpha^t \ll \alpha$$

for $t \in]0,T]$. Then we define the conditional laws $(P^u)_{u\in\Xi}$ of our semi-martingale $S(\xi)$ given $\{\xi = u\}$ and we reduce the initial utility maximisation problem to the conditional utility maximisation problem on the asset prices filtration $\mathbf{F}$ (see Proposition 1). Proposition 1 says that to solve the utility maximisation problem on the enlarged filtration it is enough to solve the conditional utility maximisation problem on the asset prices filtration $\mathbf{F}$

$$V^u(x,g) = \sup_{\varphi\in\Pi^u(\mathbf{F})} E_{P^u}\left[U\left(x + \int_0^T \varphi_s(u) dS_s(u) + g(u)\right)\right]$$

and then integrate the solution with respect to α. To solve conditional utility maximisation problem we use dual approach. Let us denote by f a convex conjugate of U. Under the assumption about the existence of an equivalent f-divergence minimal measure for the conditional semi-martingale model, we give the expression for conditional maximal utility (cf. Proposition 2). The main result of this section is

Theorem 1 which gives the final result for general utility maximisation problem.

In section 3.1 we study HARA utilities. For HARA utilities we introduce corresponding information quantities and we give the expression for the maximal expected utility in terms of these quantities (cf. Theorem 2). Finally, we introduce the information processes and we give the expression of the maximal expected utility involving these information processes (see Propositions 3, 4, 5 and Theorem 3).

In section 5 we give the formulas for indifference price of buyers and sellers of the option for HARA utilities. Then we discuss risk measure properties of the mentioned indifference prices. We show that $-p_T^b(g)$ and $p_T^s(g)$ are risk measures.

In the section 6 we study utility maximisation and utility indifference pricing of exponential Levy models. It should be noticed that in Levy models case the information processes are deterministic processes containing the constants which are the solutions of relatively simple integral equations. It gives us the possibility to calculate the indifference prices relatively easy.

The section 7 in devoted to the explicit calculus of information quantities for Geometric Brownian motion case and use identity in law technique.

2. Mathematical Framework

We consider a semi-martingale $X(\xi) = (X_t(\xi))_{t\geq 0}$ of the law P, depending on a supplementary factor ξ which can be a random process or a random variable. The semi-martingale $X(\xi)$ is given on a canonical probability space $(\Omega, \mathcal{F}, P)$, equipped with the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions: $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$, $\mathcal{F}_t = \bigcap_{u>t} \sigma\{X_v(\xi), v \leq u\}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

We suppose that the law P of $X(\xi)$ is uniquely defined by its semi-martingale characteristics (B, C, ν). We recall here the notion of the characteristics for the convenience of the readers. Let μ be a jump measure of the process $X = X(\xi)$ and $l : \mathbb{R} \to \mathbb{R}$ be a truncation function: $l(x) = x$ in the neighbourhood of 0 and l has a compact support. Then one can write the semi-martingale X as

$$X = (X - X(l)) + X(l),$$

where $X(l)$ is a 'big' jumps process, defined as

$$X(l)_t = \sum_{s \leq t} (\Delta X_s - l(\Delta X_s))$$

with $\Delta X_s = X_s - X_{s-}$. The process $\tilde{X} = (X - X(l))$ is a special semi-martingale with the bounded jumps and allows the representation

$$\tilde{X}_t = X_0 + X_t^c + \int_0^t \int_{\mathbb{R}} l(x) \left(\mu(ds, dx) - \nu(ds, dx)\right) + B_t(l),$$

where X^c is the continuous local martingale part of X, ν is the $(P, \mathbf{F})$ compensator of μ, $B = B(l)$ is the unique $(P, \mathbf{F})$-predictable locally integrable process such that the process $\tilde{X} - B(l)$ is a $(P, \mathbf{F})$-local martingale. Let C be a continuous process such that the process $(X^c)^2 - C$ is a $(P, \mathbf{F})$-local martingale. We have defined the triplet of predictable characteristics of the $(P, \mathbf{F})$-semi-martingale $X = X(\xi)$ as $T^{\mathbf{F}} = (B, C, \nu)$ (see also [23]).

We suppose that the supplementary random factor ξ is given on the probability space $(\Xi, \mathcal{H}, \alpha)$ with α being the law of ξ.

We assume that our market contains a single traded risky asset with the price process $S = S(\xi)$ and without any loss of generality we will assume that the riskless interest rate is 0, and then the riskless bond process is identically equal to 1. Our risky asset $S = S(\xi)$ which we consider will be simply of the form

$$S(\xi) = \mathcal{E}\left(X(\xi)\right), \tag{1}$$

where $\mathcal{E}(\cdot)$ is a stochastic exponential,

$$\mathcal{E}(X)_t = \exp\left\{X_t - \frac{1}{2} < X^c >_t \right\} \prod_{0 \leq s \leq t} \exp\{-\Delta X_s\}(1 + \Delta X_s).$$

We prefer the representation (1) of the risky asset more than the representation with the usual exponent by the simple reason that if the process X is a local $(P, \mathbf{F})$-martingale then the process S inherits this property, i.e. is a local $(P, \mathbf{F})$-martingale. To ensure that $S_t > 0$ for all $t \geq 0$ we assume that $\Delta X_t > -1$.

The process $X(\xi)$ can be defined in a different way. One of the possibilities is to give, when they exist, a family of the regular conditional laws of $X(\xi)$ given $\xi = u$, denoted by $(P^u)_{u \in \Xi}$. Such family of conditional laws should verify: for all $t \geq 0$, for all $A \in \mathcal{F}$

$$P(A) = \int_{\Xi} P^u(A)\, d\alpha(u). \tag{2}$$

Now, on the product space $(\Omega \times \Xi, \mathcal{F} \otimes \mathcal{H})$ we can also define a probability $\mathbb{P}$ as it follows: for all $A \in \mathcal{F}$ and $B \in \mathcal{H}$

$$\mathbb{P}(A \times B) = \int_B P^u(A) d\alpha(u), \tag{3}$$

such that $\mathbb{P}(A \times \Xi) = P(A)$ and $\mathbb{P}(\Omega \times B) = \alpha(B)$. In such situation for all $A \in \mathcal{F}$

$$P^u(A) = \mathbb{P}(A \,|\, \xi = u)$$

Now we define the initially enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{t \geq 0}$ by

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \otimes \sigma(\xi)) \,. \tag{4}$$

Let $t \in \mathbb{R}_+$ and α^t be a regular conditional distribution of the random variable ξ given the information $\mathcal{F}_t$, i.e.

$$\alpha^t(\omega, du) = \mathbb{P}(\xi \in du | \mathcal{F}_t)(\omega).$$

We make the following assumption

Assumption 1. *The regular conditional distribution of random variable ξ is absolutely continuous with respect to its law, i.e.*

$$\alpha^t \ll \alpha, \quad \forall t \in]0, T] \,.$$

Lemma 1. *(see*[21]*) Under Assumption 1 there exists a positive $\mathcal{O}(\mathbb{G})$ measurable function $(\omega, t, u) \to p_t^u(\omega)$ such that*

1. *For each $u \in supp(\alpha)$, p^u is $(P, \mathbf{F})$-martingale.*
2. *For each $t \in [0, T]$, the measure $p_t^u \alpha(du)$ is a version of the regular conditional distribution $\alpha^t(du)$ so that $P_t \times \alpha$-a.s.*

$$\frac{d\alpha^t}{d\alpha}(u) = p_t^u. \tag{5}$$

To avoid unnecessary complications, we introduce also

Assumption 2. *For each $u \in \Xi$ the probability P^u is locally absolutely continuous with respect to P, i.e*

$$P^u \overset{loc}{\ll} P.$$

The Assumptions 1, 2 and Lemma 1 imply that for all $t \in [0, T]$ and $P_t \times \alpha$-a.s.

$$\frac{dP^u|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} = p_t^u \tag{6}$$

The process X is also $(P^u, \mathbf{F})$-semi-martingale, $u \in \Xi$. If we know the density p^u, then using Ito formula we can write the semi-martingale decomposition of it and restore the $(P^u, \mathbf{F})$-characteristic triplet $T^{\mathbf{F}}(u) = (B^u, C^u, \nu^u)$. This triplet is related to the triplet $T^{\mathbf{F}} = (B, C, \nu)$ as follows

$$\begin{aligned} B^u &= B + \int_0^{\cdot} \beta_s^u \, dC_s + \int_0^{\cdot} \int_{\mathbb{R}} l(x) \, (Y_s^u(x) - 1) \nu(ds, dx), \\ C^u &= C, \\ \nu^u &= Y^u \cdot \nu, \end{aligned} \tag{7}$$

with certain $(P^u, \mathbf{F})$-predictable process $\beta^u = (\beta_t^u)_{t \in [0,T]}$ and $Y^u = (Y_t^u)_{t \in [0,T]}$ such that $P - a.s$ for all $t \in [0, T]$

$$\int_0^t (\beta_s^u)^2 \, dC_s + \int_0^t \int_{\mathbb{R}} |\, l(x) \, (Y_s^u(x) - 1)| \, \nu(ds, dx) < \infty.$$

For the details about the integration with respect to the random measures and its compensators , stochastic integration with respect to a local martingales and Riemann-Stieltjes integral see [23].

Since the density process p^u is a $(P, \mathbf{F})$-martingale, we define the stochastic logarithm m^u of p^u by:

$$dm_t^u = \frac{dp_t^u}{p_{t-}^u}.$$

Then m^u is a $(P, \mathbf{F})$-local martingale and p^u is a stochastic exponential of m^u

$$p^u = \mathcal{E}(m^u).$$

By the predictable representation property we have that the local martingale m^u has the following semi-martingale representation

$$m^u = \int_0^{\cdot} \beta_s^u \, dX_s^c + \int_0^{\cdot} \int_{\mathbb{R}} \left(Y_s^u - 1 + \frac{\hat{Y}_s^u - \hat{1}}{1 - \hat{1}} \right) (\mu - \nu)(ds, dx),$$

where the process β^u and Y^u are the same as in (7) and the processes $\hat{Y}^u$ and $\hat{1}$ are related to the compensator ν, namely

$$\hat{1}_t(\omega) = \nu(\omega, \{t\} \times \mathbb{R}_0)$$

and

$$\hat{Y}_t^u(\omega) = \int_{\mathbb{R}_0} Y_t^u(\omega, x) \nu(\omega, \{t\}, dx).$$

For more information see again [23].

3. Utility maximisation problem

In this section we introduce the sets of the self-financing admissible trading strategies and the sets of the equivalent martingale measures for the initially enlarged filtration and we establish the connection between them and the analogous sets on the $(\Omega, \mathcal{F}, \mathbf{F}, P^u)$ filtered space. Then we show that the solution of the utility maximisation problem in the enlarged filtration can be reduced to the solution of the conditional utility maximisation problem (cf. Proposition 1) which in turn, we solve using the dual approach (cf. Proposition 2). The final result on utility maximisation is given in Theorem 1 at the end of this section.

3.1. **Utility maximisation problem in enlarged filtration.**

We consider a utility function $U : \mathbb{R} \to \mathbb{R} \bigcup \{-\infty\}$, which is assumed to be strictly increasing, strictly concave, continuously differentiable in $dom(U) = \{x \in \mathbb{R} | U(x) > -\infty\}$ and is supposed to satisfy the Inada conditions

$$U^{'}(\infty) = \lim_{x \to +\infty} U^{'}(x) = 0,$$
$$U^{'}(\overline{x}) = \lim_{x \downarrow \underline{x}} U^{'}(x) = \infty,$$

where $\underline{x} = \inf\{x \in \mathbb{R} | U(x) > -\infty\}$. We require that the utility function is the increasing function of the wealth because with the growth of wealth the investor's usefulness also grows. The concavity of the function reflects a phenomenon of risk-aversion for the investor.

Suppose that the investor carries out the trading on the finite time interval $[0, T]$ and holds a European type option with the pay-off function $G_T = g(\xi)$ in his portfolio, where g is an $\mathcal{H}$-measurable function. We define by $\Pi(\mathbf{G})$ the set of admissible and self-financing strategies $\varphi(\xi)$, such that $\varphi(\xi)$ is $\mathbf{G}$-predictable and $S(\xi)$-integrable on $[0, T]$ $P - a.s.$, with the integrals bounded from below. To describe this set we recall the known result about $\mathbf{G}$-predictable processes, denoted by $\mathcal{P}(\mathbf{G})$.

Lemma 2. (*cf.* [4]) *A random process $\varphi(\xi)$ is $\mathbf{G}$-predictable if and only if the application $(t, \omega, \xi) \to \varphi_t(\xi)$ is a $\mathcal{P}(\mathbf{F}) \otimes \mathcal{H}$-measurable random process where $\mathcal{P}(\mathbf{F})$ is the set of $\mathbf{F}$-predictable processes.*

Thus, the set of the admissible and self-financing strategies $\Pi(\mathbf{G})$ on $(P, \mathbf{G})$ is of the form

$$\Pi(\mathbf{G}) = \bigcup_{c>0} \left\{ \varphi(\xi) \in \mathcal{P}(\mathbf{F}) \otimes \mathcal{H} \,|\, \int_0^t \varphi_s(\xi) dS_s(\xi) \geq -c,\ \forall t \in [0,T]\ (\mathbb{P}\text{-a.s.}) \right\}$$

The classical utility maximisation problem consists to find the optimal investment portfolio over set of all self-financing and admissible strategies in order to maximise the given expected utility, namely

$$V(x,g) = \sup_{\varphi \in \Pi(\mathbf{G})} E_{\mathbb{P}} \left[U\left(x + \int_0^T \varphi_s(\xi) dS_s(\xi) + g(\xi) \right) \right], \tag{8}$$

where U is the given utility function and x is the initial endowment.

We define also the set $\Pi^u(\mathbf{F})$ of the admissible and self-financing strategies related with the filtration $\mathbf{F}$:

$$\Pi^u(\mathbf{F}) = \bigcup_{c>0} \left\{ \varphi \in S_u(\mathcal{P}(\mathbf{F}) \otimes \mathcal{H}) \,|\, \int_0^t \varphi_s(\xi) dS_s(\xi) \geq -c,\ \forall t \in [0,T]\ (\mathbb{P}\text{-a.s.}) \right\}$$

where $S_u(\mathcal{P}(\mathbf{F}) \otimes \mathcal{H})$ is a section of $\mathcal{P}(\mathbf{F}) \otimes \mathcal{H}$ in u. For any $u \in \Xi$ we denote

$$V^u(x,g) = \sup_{\varphi \in \Pi^u(\mathbf{F})} E_{P^u} \left[U\left(x + \int_0^T \varphi_s(u) dS_s(u) + g(u) \right) \right] \tag{9}$$

The next result establishes that the value of the maximal utility in enlarged filtration $\mathbf{G}$ can be obtained from the solutions of the conditional utility maximisation problem.

Proposition 1. *Let us suppose that Assumptions 1 and 2 hold. Then we can reduce classical utility maximisation problem to the corresponding conditional utility maximisation problem in the sense that*

$$V(x,g) = \int_\Xi V^u(x,g) d\alpha(u). \tag{10}$$

To prove this proposition we prove first one lemma.

Lemma 3. *Let $\varphi(\xi) \in \Pi(\mathbf{G})$. Then for $t \in [0,T]$ and $u \in \Xi$*

$$\mathcal{L}_{\mathbb{P}} \left(\left(\int_0^t \varphi_s(\xi) dS_s(\xi), \xi \right) \Big| \xi = u \right) = \mathcal{L}_{P^u} \left(\left(\int_0^t \varphi_s(u) dS_s(u), u \right) \right). \tag{11}$$

As consequence, we get that

$$E_{\mathbb{P}} \left[U\left(x + \int_0^t \varphi_s(\xi) dS_s(\xi) + g(\xi) \right) \Big| \xi = u \right] = E_{P^u} \left[U\left(x + \int_0^t \varphi_s(u) dS_s(u) + g(u) \right) \right].$$

Proof: It is known that $\Pi(\mathbf{G})$ can be generated by the simple functions of the type $\varphi(\xi) = 1_A(\xi)\varphi_{t_1}1_{]t_1,t_2]}$, where $t_1, t_2 \in \mathbb{R}_+$, $t_1 \leq t_2$, $A \in \mathcal{H}$ and φ_{t_1} is $\mathcal{F}_{t_1}$-measurable random variable. For such $\varphi(\xi)$ we have:

$$\int_0^T \varphi_s(\xi)dS_s(\xi) = 1_A(\xi)\varphi_{t_1}(S_{t_2}(\xi) - S_{t_1}(\xi)).$$

Since the filtration $\mathbf{F}$ is natural, $\varphi_{t_1} = F(X_v(\xi), 0 \leq v \leq T)$ where F is a measurable functional. Since

$$\mathcal{L}_{\mathbb{P}}\left((\xi, X(\xi))|\xi = u\right) = \mathcal{L}_{P^u}\left(u, X\right), \tag{12}$$

the same identity in law is true for measurable functional of $(\xi, X(\xi))$ such as $(S(\xi), \varphi_{t_1}, 1_A(\xi))$ and it gives (11) for a special type of $\varphi(\xi)$.

For general $\varphi(\xi) \in \Pi(\mathbf{G})$ there exists a sequence of linear combination of simple functions, $(\varphi^n(\xi))_{n\in\mathbb{N}_+}$ such that for $s \in [0, T]$ and $P \times \alpha$-a.s.

$$\varphi_s^n(\xi) \to \varphi_s(\xi),$$

and $|\varphi_s^n(\xi)| \leq |\varphi_s(\xi)|$. Since $\varphi(\xi)$ is locally bounded predictable function, then, according to Theorem I.4.31 in [23], we have the convergence in P-law:

$$\int_0^T \varphi_s^n(\xi)dS_s(\xi) \underset{n\to\infty}{\longrightarrow} \int_0^T \varphi_s(\xi)dS_s(\xi). \tag{13}$$

For the same reason and since for $s \in [0, T]$ ($P^u \times \alpha$-a.s)

$$\varphi_s^n(u) \longrightarrow \varphi_s(u),$$

we have the convergence in P^u-law:

$$\int_0^T \varphi_s^n(u)dS_s(u) \underset{n\to\infty}{\longrightarrow} \int_0^T \varphi_s(u)dS_s(u). \tag{14}$$

From (12), (13) and (14) we obtain (11). If we denote $\Phi(v, r) = U(x + v + g(r))$ then it is a $\mathcal{B}(\mathbb{R}^2)$-measurable function of (v, r) for all $x \in \mathbb{R}_+$, and it gives the second claim. Then lemma is proved. □

Proof of Proposition 1: If in Lemma 3 we take regular versions of stochastic integrals and conditional expectations (cf. [35]), then we

have:

$$\begin{aligned} & E_{\mathbb{P}}\left[U\left(x+\int_0^T \varphi_s(\xi)dS_s(\xi)+g\left(\xi\right)\right)\right] = \\ = & \int_{\Xi} E_{P^u}\left[U\left(x+\int_0^T \varphi_s(u)dS_s(u)+g(u)\right)\right]d\alpha(u) \\ \leq & \int_{\Xi} sup_{\varphi\in\Pi^u(\mathbf{F})}E_{P^u}\left[U\left(x+\int_0^T \varphi_s(u)dS_s(u)+g(u)\right)\right]d\alpha(u), \end{aligned}$$

and hence,

$$V(x,g) \leq \int_{\Xi} V^u(x,g)d\alpha(u). \tag{15}$$

For each $\epsilon > 0$ there exists $\varphi^{(\epsilon)} \in \Pi^u(\mathbf{F})$ such that

$$sup_{\varphi\in\Pi^u(\mathbf{F})}E_{P^u}\left[U\left(x+\int_0^T \varphi_s(u)dS_s(u)+g(u)\right)\right] \leq$$
$$E_{P^u}\left[U\left(x+\int_0^T \varphi_s^{(\epsilon)}(u)dS_s(u)+g(u)\right)\right]+\epsilon$$

Integration with respect to α gives:

$$\begin{aligned} & \int_{\Xi} sup_{\varphi\in\Pi^u(\mathbf{F})}E_{P^u}\left[U\left(x+\int_0^T \varphi_s(u)dS_s(u)+g(u)\right)\right]d\alpha(u) \leq \\ \leq & \int_{\Xi} E_{P^u}\left[U\left(x+\int_0^T \varphi_s^{(\epsilon)}(u)dS_s(u)+g(u)\right)\right]d\alpha(u)+\epsilon \\ = & E_{\mathbb{P}}\left[U\left(x+\int_0^T \varphi^{(\epsilon)}(\xi)_s dS_s(\xi)+g(\xi)\right)\right]+\epsilon \\ \leq & V(x,g)+\epsilon \end{aligned}$$

Combining the two previous inequalities we have (10). □

3.2. **The solution to conditional utility maximisation problem.** To solve the conditional utility maximisation problem $V^u(x,g)$ we use the dual approach. For that we consider the equivalent martingale measures in the enlarged filtration $\mathbf{G}$ and then we provide the link between them and the equivalent martingale measures related to $(P^u,\mathbf{F})$.

Let $\mathcal{M}(\mathbf{G})$ be a set of $\mathbb{P}$-equivalent martingale measures on product space $(\Omega\times\Xi,\mathcal{F}\otimes\mathcal{H})$ defined as

$$\mathcal{M}(\mathbf{G}) = \left\{\mathbb{Q}:\ \mathbb{Q}\overset{loc}{\sim}\mathbb{P} \text{ and such that } S(\xi) \text{ is an } (\mathbb{Q},\mathbf{G})\text{-martingale}\right\}.$$

Let T be a finite time horizon. Then the restrictions of the measures $\mathbb{Q}$ on the σ-algebra $\mathcal{G}_T$ can be given by density process $Z(\xi)$:

$$\frac{d\mathbb{Q}|_{\mathcal{G}_T}}{d\mathbb{P}|_{\mathcal{G}_T}} = Z_T(\xi) \tag{16}$$

The density process $Z(\xi) = (Z_t(\xi))_{t\in[0,T]}$ is a uniformly integrable positive $(\mathbb{P}, \mathbf{G})$-martingale with $E_{\mathbb{P}}[Z_T(\xi)] = 1$. We recall the following known result about $\mathbf{G}$-martingales. Let us fix $u \in supp(\alpha)$ and let the process $Z(u)$ be obtained from the process $Z(\xi)$ by replacing of ξ by u.

Lemma 4. (*cf.* [4]) *Under Assumptions 1 and 2 there exists a version of density process $Z(\xi)$ such that the following two statements are equivalent:*

(*i*) *The process $Z = Z(\xi)$ is a $(\mathbb{P}, \mathbf{G})$-martingale*

(*ii*) *The process $Z(u) = (Z_t(u))_{t\in[0,T]}$ is a $(P^u, \mathbf{F})$-martingale, for all $u \in supp(\alpha)$.*

As it was mentioned, $Z(u)$ is a positive $(P^u, \mathbf{F})$-martingale. However $Z(u)$ is not a density process because of the fact that

$$E_P[Z_T(u)] = Z_0(u)$$

with $Z_0(u)$ which is not necessarily equal to 1. But the modified density process process $\tilde{Z}(u) = \frac{Z(u)}{Z_0(u)}$ describes the equivalent martingale measures Q^u such that

$$\frac{dQ^u|_{\mathcal{F}_t}}{dP^u|_{\mathcal{F}_t}} = \tilde{Z}_t(u). \tag{17}$$

We denote by $\mathcal{M}^u(\mathbf{F})$ the set of such measures, namely

$$\mathcal{M}^u(\mathbf{F}) = \left\{Q^u : \quad Q^u \overset{loc}{\sim} P^u, \quad S \text{ is an } (Q^u, \mathbf{F})\text{-martingale}\right\} \tag{18}$$

Let us denote by f the convex conjugate of U obtained by Frenchel-Legendre transform of U:

$$f(y) = \sup_{x>0} (U(x) - yx).$$

Let us denote by $I(y) = -f'(y), \quad y \in \mathbb{R}_+$, then

$$f(y) = U(I(y)) - yI(y), \tag{19}$$

Now we consider the dual problem of finding

$$\inf_{Q^u\in\mathcal{M}^u(\mathbf{F})} E_{P^u}\left[f\left(\frac{dQ^u_T}{dP_T}\right)\right].$$

If minimum is reached on the set $\mathcal{M}^u(\mathbf{F})$, then the corresponding measure $Q^{u,*}$ is called f-divergence minimal martingale measure.

Let also $u \in \Xi$ to be fixed, and the set $\mathcal{K}^u$ be defined as follows:

$$\mathcal{K}^u = \Big\{ Q^u \in \mathcal{M}^u(\mathbf{F}) : \; E_{P^u} \left| f\left(\lambda \frac{dQ_T^u}{dP_T}\right)\right| < \infty, \; E_{Q^u} \left| f'\left(\lambda \frac{dQ_T^u}{dP_T}\right)\right| < \infty, \forall \lambda > 0 \Big\}.$$

We introduce two additional Assumptions.

Assumption 3. *For each $u \in \Xi$, there exists f-divergence minimal equivalent martingale measure $Q^{u,*}$, it belongs to the set $\mathcal{K}^u$ and verify scaling property: for each $\lambda > 0$*

$$\inf_{Q^u \in \mathcal{M}^u(\mathbf{F})} E_{P^u}\left[f\left(\lambda \frac{dQ_T^u}{dP_T}\right)\right] = E_{P^u}\left[f\left(\lambda \frac{dQ_T^{u,*}}{dP_T}\right)\right].$$

Remark 1. *For HARA utilities the scaling property is automatically verified and the definition of the set $\mathcal{K}^u$ can be simplified:*

$$\mathcal{K}^u = \Big\{ Q^u \in \mathcal{M}^u(\mathbf{F}) : \; E_{P^u} | f\left(\frac{dQ_T^u}{dP_T}\right) | < \infty \Big\}.$$

The next assumption is related with the properties of the density $\tilde{Z}_T^*(u)$ of f-divergence minimal equivalent martingale measure $Q_T^{u,*}$ with respect to P_T^u.

Assumption 4. *There exists $\mathcal{H}$- measurable function λ_g, which verifies:*

$$\int_\Xi E_{P^u} |f(\lambda_g(u)\, \tilde{Z}_T^*(u)| d\alpha(u) < \infty, \; \int_\Xi E_{Q^u} |f'(\lambda_g(u)\, \tilde{Z}_T^*(u)| d\alpha(u) < \infty$$

and such that for each $u \in \Sigma$ and $x > \underline{x}$

$$E_{P^u}\left[\tilde{Z}_T^*(u)\, I(\, \lambda_g(u) \tilde{Z}_T^*(u)\,)\right] = x + g(u). \tag{20}$$

Remark 2. *For HARA utilities the integrability conditions of the Assumption 4 is reduced to the first one.*

Proposition 2. *Let the Assumptions 3 and 4 hold. Then there exists an optimal strategy $\varphi \in \Pi^u(\mathbf{F})$ such that*

$$V^u(x,g) = E_{P^u}\left[U\left(x + \int_0^T \varphi_s^*(u) dS_s(u) + g(u)\right)\right]$$

Moreover, we have

$$V^u(x,g) = E_{P^u}\left[U\left(I(\, \lambda(u) \tilde{Z}_T^*(u)\,)\right)\right] \tag{21}$$

Proof: For any martingale measure $\mathbb{Q}_T$ equivalent to $\mathbb{P}_T$, and $Z_T(\xi)$ its Radon-Nikodym derivative which is $\mathcal{F}\otimes\mathcal{H}$-measurable, we write:

$$E_{\mathbb{P}}\, f(Z_T(\xi)) = \int_\Xi E_{P^u}\, f(Z_T(u))\, d\alpha(u) =$$

$$\int_\Xi E_{P^u}\, f(Z_0(u)\tilde{Z}_T(u))\, d\alpha(u)$$

where $\tilde{Z}_T(u)) = Z_T(u)/Z_0(u)$. Now, we consider conditional f-divergence minimisation problem, i.e. find $\inf E_{P^u}\, f(\tilde{Z}_T(u))$ over all martingale measures Q^u equivalent to P^u with $\tilde{Z}_T(u) = \frac{dQ^u_T}{dP^u_T}$, under initial capital equal to $x+g(u)$. Let $Q^{u,*}_T$ be f-minimal equivalent martingale measure and $\tilde{Z}^*_T(u) = \frac{dQ^{u,*}_T}{dP^u_T}$. According to Assumption 4, there exists $\lambda_g(u)$ such that

$$E_{Q^{u,*}}(\, I(\lambda_g(u)\, \tilde{Z}^*_T(u)) = x + g(u)$$

One can show that λ_g is unique $\alpha - a.s.$.

Since standard f-divergences verify scaling property, we get for any $\mathbb{Q}_T$:

$$E_{\mathbb{P}}\, f(Z_T(\xi)) \geq \int_\Xi E_{P^u}\, f(Z_0(u)\, \tilde{Z}^*_T(u))\, d\alpha(u)$$

But $Z_0(u)$ is entirely defined by the restriction on conditional initial capital, so the minimum over all equivalent martingale measures $\mathbb{Q}_T$ with this restriction is $\int_\Xi E_{P^u}\, f(\lambda_g(u)\tilde{Z}^*_T(u))\, d\alpha(u)$.

Then we can use the result of [16] and write ($\mathbb{P}$ - a.s.):

$$I(\lambda_g(\xi)\, \tilde{Z}^*_T(\xi)) = x + g(\xi) + \int_0^T \varphi^*_s(\xi)\, dS(\xi)$$

where $\varphi^* \in \mathcal{P}(\mathbf{G})$, it is self-financing and admissible, and such that $\int_0^\cdot \varphi^*_s(\xi)\, dS(\xi)$ is $\mathbb{Q}^*$-martingale. The previous expression conditioned in $\xi = u$ gives (P^u-a.s.):

$$I(\lambda_g(u)\, \tilde{Z}^*_T(u)) = x + g(u) + \int_0^T \varphi^*_s(u)\, dS(u)$$

We see that $\varphi^*(u) \in \Pi^u(\mathbf{F})$, it is self-financing, admissible and such that $\int_0^\cdot \varphi^*_s(u)\, dS(u)$ is $Q^{u,*}$-martingale.

Now we show that $\varphi^*(u)$ is optimal strategy. Let us put $x + g(u) = \tilde{x}$. Then, since (19) and $I(y) = -f'(y)$,

$$U(\tilde{x} + \int_0^T \varphi^*_s(u)\, dS(u)) = f(\lambda_g(u)\tilde{Z}^*_T(u)) + \tilde{Z}^*_T(u)\, f'(\lambda_g(u)\, \tilde{Z}^*_T(u))$$

We show easily that the left-hand side of the previous equality is integrable:

$$E_{P^u}|\,U(\tilde{x}+\textstyle\int_0^T \varphi_s^*(u)\,dS(u))\,| \leq$$

$$E_{P^u}|f(\lambda_g(u)\tilde{Z}_T^*(u))| + E_{P^u}|\tilde{Z}_T^*(u)\,f'(\lambda_g(u)\,\tilde{Z}_T^*(u))| < \infty$$

We write for any $\varphi \in \Pi^u(\mathbf{F})$ using the definition of Fenchel-Legendre transform:

$$U(\tilde{x}+\int_0^T \varphi_s(u)\,dS(u)) \leq \left[\tilde{x}+\int_0^T \varphi_s(u)\,dS(u)\right]\lambda_g(u)\tilde{Z}_T^*(u)$$

$$+f(\lambda_g(u)\tilde{Z}_T^*(u)) \leq \left[\tilde{x}+\int_0^T \varphi_s(u)\,dS(u)\right]\lambda_g(u)\tilde{Z}_T^*(u)+U(I(\lambda_g(u)\,\tilde{Z}_T^*(u))-$$

$$\lambda_g(u)\,\tilde{Z}_T^*(u)\,I(\lambda_g(u)\,\tilde{Z}_T^*(u))$$

We take an expectation with respect to P^u, then we use the fact that $\int_0^{\cdot} \varphi_s(u)\,dS(u)$ is a super-martingale started from zero and that $\int_0^{\cdot} \varphi_s^*(u)\,dS(u)$ is a martingale with respect to $Q^{u,*}$. Finally we get that

$$E_{P^u}[\,U(\tilde{x}+\int_0^T \varphi_s(u)\,dS(u))] \leq E_{P^u}[\,U(\tilde{x}+\int_0^T \varphi_s^*(u)\,dS(u))] \qquad \square$$

3.3. **Final result on utility maximisation problem.** We combine the results of Proposition 1 and Proposition 2 to get the following final result on utility maximisation.

Theorem 1. *We suppose that The Assumptions 1, 2, 3, 4 hold. Then, the maximal expected utility verify:*

$$V(x,g) = \int_{\Xi} E_{P^u}\left[U\left(I\left(\lambda_g(u)\tilde{Z}_T^*(u)\right)\right)\right] d\alpha(u), \tag{22}$$

and

$$V(x,0) = \int_{\Xi} E_{P^u}\left[U\left(I\left(\lambda_0(u)\tilde{Z}_T^*(u)\right)\right)\right] d\alpha(u), \tag{23}$$

where $\lambda_g(u)$ is a solution of (20) and λ_0 is a solution of (20) with replacing $g(u)$ by 0.

4. Utility maximisation for HARA utilities

In the overwhelming part of the literature, the utility maximisation analysis is carried out under the hyperbolic absolute risk utilities (HARA), which are logarithmic, power and exponential utilities represented below:

$$\begin{aligned} U(x) &= \ln x, \text{ then } f(x) = -\ln x - 1, \\ U(x) &= \frac{x^p}{p},\ p \in (-\infty, 0) \cup (0, 1), \text{ then } f(x) = -\frac{p-1}{p} x^{\frac{p}{p-1}}, \\ U(x) &= 1 - e^{-\gamma x},\ \gamma > 0, \text{ then } f(x) = 1 - \frac{x}{\gamma} + \frac{1}{\gamma} x \ln x - \frac{1}{\gamma} x \ln \gamma \end{aligned}$$

where $x > 0$. This choice can be explained by the good properties of these functions such as scaling property, time horizon invariance property, preservation of Levy property and so on (see for instance [8]).

We introduce the information quantities related with HARA utilities. The corresponding maximal utilities are given in Theorem 2. Then, we express these information quantities via information processes (cf. Propositions 3, 4, 5). The final result on utility maximisation is given in Theorem 3.

4.1. **Maximal utilities and information quantities.**

As before, we assume the existence of f-divergence minimal martingale measure $Q^{u,*} \in \mathcal{K}^u$. We introduce three important quantities related with P^u_T and $Q^{u,*}_T$ namely the entropy of P^u with respect to $Q^{u,*}_T$,

$$\mathbf{I}(P^u_T | Q^{u,*}_T) = -E_{P^u}\left[\ln \tilde{Z}^*_T(u)\right],$$

the entropy of $Q^{u,*}_T$ with respect to P^u_T,

$$\mathbf{I}(Q^{u,*}_T | P^u_T) = E_{P^u}\left[\tilde{Z}^*_T(u) \ln \tilde{Z}^*_T(u)\right],$$

and Hellinger type integrals

$$\mathbf{H}^{(q),*}_T(u) = E_{P^u}\left[(\tilde{Z}^*_T(u))^q\right],$$

where $q = \frac{p}{p-1}$ and $p < 1$.

Now we give the expressions of the value function $V(x, g)$ involving relative entropies and Hellinger type integrals.

Theorem 2. *Under The Assumptions 1, 2, 3, 4 the information quantities are $\mathcal{H}$-measurable and we have the following expressions for $V_T(x,0)$:*

(i) If $U(x) = \ln x$ then

$$V_T(x,0) = \int_\Xi [\,\ln x + \mathbf{I}(P_T^u|Q_T^{u,*})\,]d\alpha(u) \tag{24}$$

(ii) If $U(x) = \frac{x^p}{p}$ with $p < 1, p \neq 0$ then

$$V_T(x,0) = \frac{1}{p}\int_\Xi x^p \left(\mathbf{H}_T^{(q),*}(u)\right)^{1-p} d\alpha(u) \tag{25}$$

(iii) If $U(x) = 1 - e^{-\gamma x}$ with $\gamma > 0$ then

$$V_T(x,0) = 1 - \int_\Xi \exp\{-[\,\gamma x + \mathbf{I}(Q_T^{u,*}|P_T^u)\,]\}\, d\alpha(u) \tag{26}$$

The expressions for $V_T^u(x,g)$ can be obtained from previous expressions replacing in right-hand side x by $x + g(u)$.

Proof: First of all we remark that $\tilde{Z}_T^*$ is $\mathcal{F}_T \otimes \mathcal{H}$ measurable and $\frac{dP_T^u}{dP_T} = p_T$ is also $\mathcal{F}_T \otimes \mathcal{H}$ measurable. Hence, the information quantities are $\mathcal{H}$-measurable.

(i) The Theorem 1 states that

$$V_T(x,0) = \int_\Xi E_{P^u}\left[U\left(I\left(\lambda_0(u)\,\tilde{Z}_T^*(u)\right)\right)\right] d\alpha(u), \tag{27}$$

where $\lambda_0(u)$ is defined from the equation

$$E_{P^u}\left[\tilde{Z}_T^*(u) I\left(\lambda_0(u)\tilde{Z}_T^*(u)\right)\right] = x. \tag{28}$$

The corresponding inverse function of the derivative of the logarithmic utility is $I(y) = \frac{1}{y}$, then $\lambda_0(u) = \frac{1}{x}$. Putting this result into (27) we have that

$$\begin{aligned} V_T(x,0) &= \int_\Xi E_{P^u}\left[\ln\left[\frac{x}{\tilde{Z}_T^*(u)}\right]\right] d\alpha(u) \\ &= \int_\Xi E_{P^u}\left[\ln x - \ln \tilde{Z}_T^*(u)\right] d\alpha(u) \\ &= \int_\Xi [\,\ln x + \mathbf{I}(P_T^u|Q_T^{u,*})\,]d\alpha(u). \end{aligned} \tag{29}$$

The formula (53) has been proved.

(*ii*) The corresponding inverse function of the derivative of the power utility is $I(y) = y^{\frac{1}{p-1}}$. Then, using (28) we deduce that

$$\lambda_0(u) = \frac{x^{p-1}}{\left(E_{P^u}\left[\left(\tilde{Z}^*_T(u)\right)^q\right]\right)^{p-1}}.$$

with $q = \frac{p}{p-1}$, and we get finally that

$$\begin{aligned} E_{P^u}\left[U\left(I\left(\lambda_0(u)\,\tilde{Z}^*_T(u)\right)\right)\right] &= E_{P^u}\left[\frac{1}{p}\left(\lambda_0(u)\,\tilde{Z}^*_T(u)\right)^q\right] \\ &= \frac{x^p}{p}\left(E_{P^u}\left[\left(\tilde{Z}^*_T(u)\right)^q\right]\right)^{1-p}. \end{aligned} \tag{30}$$

Then, we integrate over Ξ with respect to α and we obtain (25).

(*iii*) The corresponding inverse function of the derivative of the exponential utility is $I(y) = -\frac{1}{\gamma}(\ln y - \ln \gamma)$. Then, value function in the case of the exponential utility can be simplified to the form

$$\begin{aligned} V_T(x,0) &= \int_\Xi E_{P^u}\left[U\left(I\left(\lambda_0(u)\tilde{Z}^*_T(u)\right)\right)\right] d\alpha(u) \\ &= 1 - \frac{1}{\gamma}\int_\Xi \lambda_0(u)\, d\alpha(u). \end{aligned} \tag{31}$$

The corresponding λ_0 is given by

$$\lambda_0(u) = \gamma \exp\left\{ -\gamma x - E_{P^u}\left[\tilde{Z}^*_T(u)\ln \tilde{Z}^*_T(u)\right]\right\}.$$

Taking into account that

$$E_{P^u}\left[\tilde{Z}^*_T(u)\ln \tilde{Z}^*_T(u)\right] = \mathbf{I}(Q^{u,*}_T|P^u_T)$$

we get that

$$\lambda_0(u) = \gamma \exp\left\{ -\gamma x - \mathbf{I}(Q^{u,*}_T|P^u_T)\right\}.$$

and it gives us (26). □

4.2. **Information quantities and information processes.** In this subsection we express the information quantities via corresponding information processes. As previously, we assume the existence of an equivalent f-divergence minimal martingale measure $Q^{u,*}$. We recall that a semi-martingale $X(\xi)$ under P^u is also a semi-martingale with the triplet $T^{\mathbf{F}}(u) = (B^u, C, \nu^u)$ defined by (7). To avoid non-necessary complications we introduce the following additional assumption.

Assumption 5. *For each $u \in \Xi$, the $(P^u, \mathbf{F})$-semi-martingale X is a quasi-left continuous, i.e. for any predictable stopping time τ, the jump $\Delta X_\tau = 0$ on the set $\{\tau < \infty\}$.*

Let us denote by $\beta^{u,*}$ and $Y^{u,*}(x)$ two $(P^u, \mathbf{F})$-predictable processes known as Girsanov parameters for the changing of measure from P^u into $Q^{u,*}$ such that: $\forall t \geq 0$ and P^u-a.s.

$$\int_0^t \int_{\mathbb{R}} |\, l(x)\,(Y_s^{u,*}(x) - 1)|\nu^u(ds, dx) < \infty, \quad \int_0^t (\beta_s^{u,*})^2 dC_s < \infty.$$

In the case of logarithmic utility we consider the entropy $\mathbf{I}(P_T^u \,|\, Q_t^{u,*})$ and we introduce the corresponding predictable process $\mathcal{I}^*(u) = (\mathcal{I}_t^*(u))_{t\in[0,T]}$

(32)
$$\mathcal{I}_t^*(u) = \frac{1}{2}\int_0^t (\beta_s^{u,*})^2 dC_s - \int_0^t \int_{\mathbb{R}} \left(\ln(Y_s^{u,*}(x)) - Y_s^{u,*}(x) + 1\right) \nu^u(ds, dx).$$

Proposition 3. *We suppose that $E_{P^u}|\ln \tilde{Z}_T^*(u)| < \infty$ and Assumption 5 holds. Then*

$$\mathbf{I}(P_T^u \,|\, Q_T^{u,*}) = E_{P^u}\mathcal{I}_T^*(u). \tag{33}$$

Proof: To avoid the complicated notations we omit for the proof of this Proposition the indexes u, $*$, and replace the notation of $\tilde{Z}$ by Z. Let Q and P be two equivalent probability measures on canonical space and let $(Z_t)_{t\in[0,T]}$ be the Radon-Nikodym density, $Z_t = \frac{dQ_t}{dP_t}$, where Q_t and P_t are the restrictions of Q and P on σ-algebra $\mathcal{F}_t$. Let X be P-semi-martingale with the characteristics (B, C, ν).

For $\epsilon > 0$ we put

$$\tau_\epsilon = \inf\{0 \leq t \leq T | Z_t \leq \epsilon\} \tag{34}$$

with $\inf\{\emptyset\} = +\infty$. We remark that τ_ϵ is a stopping time and that the sequence of stopping times (τ_ϵ) is increasing to infinity as $\epsilon \to 0$, and hence, it is localising sequence. Then, by Ito formula we have:

$$\begin{aligned}\ln Z_{T\wedge\tau_\epsilon} = \ln Z_0 + \int_0^{T\wedge\tau_\epsilon} \frac{1}{Z_{s-}} dZ_s - \frac{1}{2}\int_0^{T\wedge\tau_\epsilon} \frac{1}{(Z_{s-})^2} d < Z^c >_s \\ + \sum_{0<s\leq T\wedge\tau_\epsilon} \left(\ln Z_s - \ln Z_{s-} - \frac{1}{Z_{s-}}\Delta Z_s\right),\end{aligned} \tag{35}$$

where $\Delta Z_s = Z_s - Z_{s-}$ and $< Z^c >_s$ is a predictable variation of the continuous martingale part of Z. We remark that $\left(\int_0^{t\wedge\tau_\epsilon} \frac{1}{Z_{s-}} dZ_s\right)_{t\in[0,T]}$ is a $(P, \mathbf{F})$-martingale started from zero, since it is stochastic integral

with respect to $(P,\mathbf{F})$-martingale Z and since $Z_{s-} \geq \epsilon > 0$ on the stochastic interval $[0, T \wedge \tau_\epsilon]$. By Theorem 1.8, p.66 in [23], we get

$$E_P \int_0^{t\wedge\tau_\epsilon} \int_{\mathbb{R}} \left(\ln\left(1 + \frac{x}{Z_{s-}}\right) - \frac{x}{Z_{s-}} \right) \mu_Z(ds,dx) = E_P \int_0^{t\wedge\tau_\epsilon} \int_{\mathbb{R}} \left(\ln\left(1 + \frac{x}{Z_{s-}}\right) - \frac{x}{Z_{s-}} \right) \nu_Z(ds,dx),$$

where μ_Z and ν_Z are measure of jumps of Z and its compensator. Finally, from (35) and the fact that $Z_0 = 1$, we have:

$$E_P \ln Z_{T\wedge\tau_\epsilon} = E_P \left[-\frac{1}{2} \int_0^{T\wedge\tau_\epsilon} \frac{1}{(Z_{s-})^2} d < Z^c >_s + \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} \left(\ln\left(1 + \frac{x}{Z_{s-}}\right) - \frac{x}{Z_{s-}} \right) \nu_Z(ds,dx) \right]. \tag{36}$$

Now, since $Z = \mathcal{E}(M)$ where $\mathcal{E}(\cdot)$ is a Dolean-Dade exponential, for all $t \in [0,T]$ we get:

$$M_t = \int_0^t \beta_s dX_s^c + \int_0^t \int_{\mathbb{R}} (Y_s - 1)(\mu - \nu)(ds,dx) \tag{37}$$

Then, $dZ_t = Z_{t-} dM_t$, and in particular, $dZ_t^c = Z_t dM_t^c$ and $\Delta Z_t = Z_{t-} \Delta M_t$. In addition, (37) implies that for $t \in [0,T]$

$$M_t^c = \int_0^t \beta_s dX_s^c \text{ and } \Delta M_t = \left(Y_t(\Delta X_t) - 1\right), \tag{38}$$

and, hence,

$$d < Z^c >_t = (Z_{t-})^2 \beta_t^2 d < X^c >_t$$

and

$$\Delta Z_t = Z_{t-} \left(Y_t(\Delta X_t) - 1\right).$$

Using mentioned above relations we obtain:

$$E_P \ln Z_{T\wedge\tau_\epsilon} = E_P \left[-\frac{1}{2} \int_0^{T\wedge\tau_\epsilon} \beta_s^2 dC_s + \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} \left(\ln Y_s(x) - Y_s(x) + 1\right) \nu(ds,dx) \right]. \tag{39}$$

Since $\ln(1+x) \leq x$, both integrands in the right hand side are negatives. So, using the Lebesgue monotone convergence theorem, we can pass to the limit on the right-hand side.

It remains to pass to the limit on the left hand side in (39), i.e. to prove

$$\lim_{\epsilon \to 0} E_P \ln Z_{T\wedge\tau_\epsilon} = E_P \ln Z_T. \tag{40}$$

We can write

$$E_P \ln Z_{T\wedge\tau_\epsilon} - E_P \ln Z_T = E_P\left[\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right] - E_P\left[\ln Z_T 1_{\{\tau_\epsilon<T\}}\right]. \tag{41}$$

We show that two last terms in (41) tend to zero as $\epsilon \to 0$.

Let $\hat{Z}_t = \frac{1}{Z_t}$ for $t \in [0,T]$. We remark that $(\hat{Z}_t)_{[0,T]}$ is a Q-martingale. Then, by maximal inequality for positive martingales

$$Q\left(\tau_\epsilon < T\right) \le Q\left(\sup_{0\le t\le T} \hat{Z}_t \ge \frac{1}{\epsilon}\right) \le E_Q \hat{Z}_T \cdot \epsilon = \epsilon \tag{42}$$

Finally,

$$P\left(\tau_\epsilon < T\right) \le E_Q(\hat{Z}_T 1_{\{\sup\limits_{0\le t\le T} \hat{Z}_t \ge \frac{1}{\epsilon}\}}) \to 0$$

as $\epsilon \to 0$ since $E_Q \hat{Z}_T = 1$. Since $\ln Z_T$ is P-integrable, the relation (42) implies that

$$\lim_{\epsilon\to 0} E_P\left(\ln Z_T 1_{\{\tau_\epsilon<T\}}\right) = 0. \tag{43}$$

Since $Z_{\tau_\epsilon} \le \epsilon$, for $\epsilon < 1$ we get $E_P(\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}) \le 0$. From concavity of $\ln x, x > 0$

$$E_P(\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}) \ge E_P\left(\ln Z_T 1_{\{\tau_\epsilon<T\}}\right)$$

The relation (43) implies that $E_P\left(\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right) \to 0$ when $\epsilon \to 0$. Finally, we proved (40). □

In the case of exponential utility we consider the entropy $\mathbf{I}(\, Q_T^{u,*} \mid P_T^u\,)$, which is known also as Kullback-Leiber information. We introduce the corresponding Kullback-Leiber process $I^*(u) = (I_t^*(u))_{t\in[0,T]}$ with

$$I_t^*(u) = \frac{1}{2}\int_0^t (\beta_s^{u,*})^2 dC_s + \int_0^t \int_{\mathbb{R}} \left[Y_s^{u,*}(x)\ln(Y_s^{u,*}(x)) - Y_s^{u,*}(x) + 1\right] \nu^u(ds,dx). \tag{44}$$

We remark that, Kullback-Leibler process was first introduced in [25] and it was studied in [27], [13], [20]. We give here some properties of this process needed for our final results.

Proposition 4. *We suppose that $E_{P^u}|\tilde{Z}_T^*(u)\ln\tilde{Z}_T^*(u)| < \infty$ and that the Assumption 5 holds. Then,*

$$\mathbf{I}(\, Q_T^{u,*} \mid P_T^u\,) = E_{P^u}\left[\int_0^T \tilde{Z}_{s-}^*(u)\, dI_s^*(u)\right] = E_{Q^{u,*}}\left(\, I_T^*(u)\,\right) \tag{45}$$

Proof: We continue in the framework of Proposition 3 to prove the first equality. The second one is a consequence of integration by part

formula. Let $\epsilon > 0$ and the localising sequence (τ_ϵ) : for $\epsilon > 0$

$$\tau_\epsilon = \inf\{0 \leq t \leq T | Z_t \leq \epsilon \text{ or } Z_t \geq \frac{1}{\epsilon}\}$$

with $\inf\{\emptyset\} = +\infty$. Then, by Ito formula we have

(46)

$$Z_{T\wedge\tau_\epsilon} \ln Z_{T\wedge\tau_\epsilon} = Z_0 \ln Z_0 + \int_0^{T\wedge\tau_\epsilon} (\ln Z_{s-}+1) dZ_s + \frac{1}{2}\int_0^{T\wedge\tau_\epsilon} \frac{1}{Z_{s-}} d < Z^c >_s$$
$$+ \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} \left[(Z_{s-}+x)\ln(Z_{s-}+x) - Z_{s-}\ln Z_{s-} - (\ln Z_{s-}+1)x\right] \mu_Z(ds,dx).$$

We remark that $\left(\int_0^{T\wedge\tau_\epsilon} (\ln Z_{s-}+1) dZ_s\right)_{t\in[0,T]}$ is a $(P,\mathbf{F})$-martingale started from zero, since it is stochastic integral with respect to $(P,\mathbf{F})$-martingale Z such that $Z_{s-} \geq \epsilon$ and $Z_{s-} \leq \frac{1}{\epsilon}$ on the stochastic interval $[0, T\wedge\tau_\epsilon]$. Using Theorem 1.8, p.66, in [23], we get

$$E_P \int_0^{t\wedge\tau_\epsilon} \int_{\mathbb{R}} \left[(Z_{s-}+x)\ln(Z_{s-}+x) - Z_{s-}\ln Z_{s-} - (\ln Z_{s-}+1)x\right] \mu_Z(ds,dx) =$$
$$E_P \int_0^{t\wedge\tau_\epsilon} \int_{\mathbb{R}} \left[(Z_{s-}+x)\ln(Z_{s-}+x) - Z_{s-}\ln Z_{s-} - (\ln Z_{s-}+1)x\right] \nu_Z(ds,dx),$$

where μ_Z and ν_Z are the measure of jumps of Z and its compensator.

Finally, from (46) and the fact that $Z_0 = 1$, we have:

$$E_P\left[Z_{T\wedge\tau_\epsilon} \ln Z_{T\wedge\tau_\epsilon}\right] = E_P\left[\frac{1}{2}\int_0^{T\wedge\tau_\epsilon} \frac{1}{Z_{s-}} d < Z^c >_s \right. \tag{47}$$
$$\left. + \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} \left[(Z_{s-}+x)\ln(Z_{s-}+x) - Z_{s-}\ln Z_{s-} - (\ln Z_{s-}+1)x\right] \nu_Z(ds,dx)\right].$$

Using the relations between $Z = \mathcal{E}(M)$, M and the process X given by (37) and (38), we get

$$E_P\left[Z_{T\wedge\tau_\epsilon} \ln Z_{T\wedge\tau_\epsilon}\right] = E_P\left[\frac{1}{2}\int_0^{T\wedge\tau_\epsilon} Z_{s-}\beta_s^2 dC_s \right. \tag{48}$$
$$\left. + \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} Z_{s-}\left(Y_s(x)\ln Y_s(x) - Y_s(x) + 1\right)\nu(ds,dx)\right].$$

Since $\tau_\epsilon \to +\infty$ as $\epsilon \to 0$ and $x\ln x - x + 1 \geq 0$ for all $x > 0$, by Lebesgue monotone convergence theorem we can pass to the limit in

the right-hand side of (48). It remains to show that the left-hand side of (48) converges to $E_P[Z_T \ln Z_T]$. We can write

$$E_P[Z_{T\wedge\tau_\epsilon} \ln Z_{T\wedge\tau_\epsilon}] - E_P[Z_T \ln Z_T] = \tag{49}$$
$$E_P\left[Z_{\tau_\epsilon} \ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right] - E_P\left[Z_T \ln Z_T 1_{\{\tau_\epsilon<T\}}\right].$$

We show that the last two terms in (49) tends to zero as $\epsilon \to 0$. Since $Z_T \ln Z_T$ is P-integrable and $P(\tau_\epsilon < T) \to 0$ as $\epsilon \to 0$, we get that

$$\lim_{\epsilon\to 0} E_P\left[Z_T \ln Z_T 1_{\{\tau_\epsilon<T\}}\right] = 0.$$

Using the inequality $x \ln x \geq \frac{1}{e}$ for all $x \in \mathbb{R}_+$, we have for $0 \leq \epsilon \leq \frac{1}{e}$

$$-\frac{1}{e} \cdot P(\tau_\epsilon < T) \leq E_P\left[Z_{\tau_\epsilon} \ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right] \leq 0,$$

and $E_P\left[Z_{\tau_\epsilon} \ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right] \to 0$ when $\epsilon \to 0$. Finally,

$$\lim_{\epsilon\to 0} E_P[Z_{T\wedge\tau_\epsilon} \ln Z_{T\wedge\tau_\epsilon}] = E_P[Z_T \ln Z_T].$$

and the proposition is proved. □

For the case of power utility we consider Hellinger types integrals

$$\mathbf{H}_T^{(q),*}(u) = E_{P^u}\left[(\tilde{Z}_T^*(u))^q\right],$$

where $q = \frac{p}{p-1}$, $p < 1$. We notice that if $p < 0$ then $0 < q < 1$ and if $0 < p < 1$ then $q < 0$, so, in any cases $q < 1$.

We introduce the corresponding predictable process called Hellinger type process $h^{(q),*}(u) = (h_t^{(q),*}(u))_{t\in[0,T]}$

$$h_t^{(q),*}(u) = \frac{1}{2}q(q-1)\int_0^t (\beta_s^{u,*})^2 dC_s + \tag{50}$$
$$\int_0^t \int_{\mathbb{R}} \left[(Y_s^{u,*}(x))^q - q(Y_s^{u,*}(x) - 1) - 1\right] \nu(ds, dx),$$

In the case when $0 < q < 1$ the Hellinger processes was studied in [36], [23], [10],[11]. We show that the result can be extended for $q < 1$.

Proposition 5. *Suppose that $\mathbf{H}_T^{(q),*}(u) < \infty$ and that the Assumption 5 holds. Then*

$$\mathbf{H}_T^{(q),*}(u) = 1 + E_{P^u}\left[\int_0^T (\tilde{Z}_{s-}^*)^q \, dh_s^{(q),*}(u)\right]$$

or, in the terms of the stochastic exponential:

$$\mathbf{H}_T^{(q),*}(u) = E_{P^u}\left[\mathcal{E}\left(h^{(q),*}\right)_T\right]. \tag{51}$$

Proof: We continue in the framework of Proposition 3. Let $\epsilon > 0$ and the localising sequence (τ_ϵ) defined by (34). Applying Ito's formula we have:

$$\begin{aligned} Z^q_{T\wedge\tau_\epsilon} &= 1 + q\int_0^{T\wedge\tau_\epsilon} Z^{q-1}_{s-}dZ_s + \frac{1}{2}q(q-1)\int_0^{T\wedge\tau_\epsilon} Z^{q-2}_{s-}d<Z^c>_s \\ &+ \int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}}\left[Z^q_{s-}\left(\left(1+\frac{x}{Z_{s-}}\right)^q - 1\right) - qZ^{q-1}_{s-}x\right]\mu_Z(ds,dx). \end{aligned}$$

Since $\left(\int_0^{t\wedge\tau_\epsilon} Z^{q-1}_{s-}dZ_s\right)_{t\in[0,T]}$ is a $(P,\mathbf{F})$-martingale starting from 0 and due to the projection theorem we get:

$$\begin{aligned} E_P Z^q_{T\wedge\tau_\epsilon} &= 1 + E_P\left[\frac{1}{2}q(q-1)\int_0^{T\wedge\tau_\epsilon} Z^{q-2}_{s-}d<Z^c>_s \right. \\ &\left. + \int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}}\left[Z^q_{s-}\left(\left(1+\frac{x}{Z_{s-}}\right)^q - 1\right) - qZ^{q-1}_{s-}x\right]\nu_Z(ds,dx)\right]. \end{aligned}$$

Using the relation between $Z = \mathcal{E}(M)$, M and the initial process X, we get

$$\begin{aligned} E_P Z^q_{T\wedge\tau_\epsilon} &= 1 + E_P\left\{\frac{1}{2}q(q-1)\int_0^{T\wedge\tau_\epsilon} Z^q_{s-}\beta^2_s dC_s \right. \\ &\left. + \int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}} Z^q_{s-}\left[Y^q_s(x) - q(Y_s(x)-1) - 1\right]\nu(ds,dx)\right\} \end{aligned}$$

We remark that $\lim_{\epsilon\to 0}\tau_\epsilon = +\infty$. Since for $0 < q < 1$, $q(q-1) < 0$ and $x^q - qx - 1 \leq 0$ and for $q < 0$, $q(q-1) > 0$ and $x^q - qx - 1 \geq 0$, the right hand side of above expression contains the integral of some negative function. Then, by Lebesgue monotone convergence theorem we can pass to the limit on the right hand side.

It remains to show that

$$\lim_{\epsilon\to 0} E_P Z^q_{T\wedge\tau_\epsilon} = E_P Z^q_T.$$

We have:

$$E_P Z^q_{T\wedge\tau_\epsilon} - E_P Z^q_T = E_P\left(Z^q_{\tau_\epsilon}1_{\{\tau_\epsilon<T\}}\right) - E_P\left(Z^q_T 1_{\{\tau_\epsilon<T\}}\right). \tag{52}$$

Since $P(\tau_\epsilon < T) \to 0$ as $\epsilon \to 0$ and Z^q_T is P-integrable, the first term in the right-hand side of (52) tends to zero. For the second term we distinguish two cases: $0 < q < 1$ and $q < 0$. In the first case,

$$E_P\left(Z^q_{\tau_\epsilon}1_{\{\tau_\epsilon<T\}}\right) \leq \epsilon^q P(\tau_\epsilon < T) \to 0$$

as $\epsilon \to 0$. In the second case we have:

$$E_P\left(Z^q_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right) = E_Q\left(\hat{Z}^{1-q}_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right),$$

where $\hat{Z}_{\tau_\epsilon} = \dfrac{1}{Z_{\tau_\epsilon}}$. From maximal inequalities for the martingales we have:

$$E_Q\left[\sup_{0\le t\le T} \hat{Z}_t\right]^{1-q} \le c(q) E_Q(\hat{Z}^{1-q}_T) = c(q) E_P(Z^q_T) < \infty$$

where $c(q)$ is a constant. In addition, $Q(\tau_\epsilon < T) \to 0$ as $\epsilon \to 0$, and, then,

$$\lim_{\epsilon\to 0} E_Q\left(\hat{Z}^{1-q}_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right) = 0.$$

We prove now (51). From Ito formula we also get that

$$Z^q_{T\wedge\tau_\epsilon} = 1 + \int_0^{T\wedge\tau_\epsilon} Z^q_s \, dK_s$$

where $K = N + h^{(q)}$ is a sum of a martingale N and predictable process $h^{(q)}$ with

$$N_t = q\int_0^t \beta_{s-}\, dX^c_s + \int_0^t \int_{\mathbb{R}} (Y^q_s(x) - 1)(\mu - \nu)(ds, dx)$$

Then, we have:

$$Z^q_{T\wedge\tau_\epsilon} = \mathcal{E}(N + h^{(q)})_{T\wedge\tau_\epsilon} = \mathcal{E}(N)_{T\wedge\tau_\epsilon}\, \mathcal{E}(h^{(q)})_{T\wedge\tau_\epsilon}$$

We take the expectation with respect to P and we show that $(\mathcal{E}(N)_{t\wedge\tau_\epsilon})_{0\le t\le T}$ is uniformly integrable martingale with expectation 1. Since $h^{(q)}$ is monotone and continuous process, we can pass to the limit and it gives (51). □

4.3. **Maximal utility and information processes.** The final result for maximal utility in terms of information processes follows directly from Theorem 2 and Propositions 3, 4 and 5 and is given in the following Theorem 3.

Theorem 3. *Under The Assumptions 1, 2, 3, 4, 5 and for HARA utilities we have the following expressions for* $V_T(x, 0)$:

(*i*) *If* $U(x) = \ln x$, *then*

$$V_T(x, 0) = \int_\Xi E_{P^u}[\ln x + \mathcal{I}^*_T(u)]\, d\alpha(u) \tag{53}$$

(ii) If $U(x) = \frac{x^p}{p}$ *with* $p < 1, p \neq 0$, *then*

$$V_T(x,0) = \frac{1}{p}\int_\Xi x^p \left(E_{P^u}\left[\mathcal{E}\left(h^{(q),*}(u)\right)_T\right]\right)^{1-p} d\alpha(u) \tag{54}$$

(iii) If $U(x) = 1 - e^{-\gamma x}$ *with* $\gamma > 0$, *then*

$$V_T(x,0) = 1 - \int_\Xi \exp\{-(\gamma x + E_{Q^{u,*}}(\, I_T^*(u)\,)\}\, d\alpha(u) \tag{55}$$

The expressions for $V_T(x,g)$ *can be obtained from previous expressions replacing* x *by* $x + g(u)$ *in right-hand side.*

5. Indifference pricing on the initially enlarged filtration

We consider the situation when the investor carries out the trading of risky asset $S(\xi)$ on the finite time interval $[0,T]$ and has a European type option with the pay-off function $G_T = g(\xi)$, g is an $\mathcal{H}$-measurable real-valued function. Then, as it was already mentioned a buyer's indifference price p_T^b is the solution to the equation

$$V_T(x,0) = V_T(x - p_T^b, g). \tag{56}$$

and a seller's indifference price p_T^s is defined from

$$V_T(x,0) = V_T(x + p_T^s, -g). \tag{57}$$

We notice that the indifference prices p_T^b and p_T^s are related, namely

$$p_T^b(g) = -p_T^s(-g). \tag{58}$$

5.1. Indifference price formulas.

Now we apply the results of Theorem 1 and Theorem 2 to give the formulas for the indifference prices in the cases of the exponential, power and logarithmic utilities.

Proposition 6. *In the case of logarithmic utility* $U(x) = \ln x$, $x > 0$, *and under the Assumptions 1, 2, 3, 4, and* $g(\xi) \in]0, x[$ *(α-a.s.), the buyer's and seller's indifference price satisfy:*

$$\int_\Xi \ln\left[1 - \frac{p_T^b}{x} + \frac{g(u)}{x}\right] d\alpha(u) = 0 \tag{59}$$

and

$$\int_\Xi \ln\left[1 + \frac{p_T^s}{x} - \frac{g(u)}{x}\right] d\alpha(u) = 0. \tag{60}$$

Moreover, if $\ln(g(\xi))$, $\ln(x-g(\xi))$ *are integrable functions then the solutions of the equations* (59) *and* (60) *exist, they are unique and* $p_T^b, p_T^s \in [0,x]$.

Remark 3. *It should be noticed that in logarithmic utility case, the formulas for indifference price do not reflect the dependence between* $X(\xi)$ *and* ξ*: exactly the same equations will hold when* $X(\xi)$ *and* ξ *are independent.*

Proof: From (24) and (56) we have (59). Formula (60) is obtained from the relation (58). We see that

$$F(y) = \int_\Xi \ln\left[1 - \frac{y}{x} + \frac{g(u)}{x}\right] d\alpha(u), \quad y \in [0,x]\,,$$

is well-defined strictly decreasing function and $F(0) \geq 0$. If $\ln g(\xi)$ is integrable with respect to α, then F is a continuous by Lebesgue dominated theorem. Under the condition that $g(\xi) \in]0,x[$ (α-a.s.), $F(x) \leq 0$. Then, solution exists by the mean-value theorem and it is unique.

In the case of the seller's indifference price, the function

$$F(y) = \int_\Xi \ln\left[1 + \frac{y}{x} - \frac{g(u)}{x}\right] d\alpha(u), \quad y \in [0,x]\,,$$

is a strictly increasing continuous function with $F(0) \leq 0$ and $F(x) \geq 0$ and then, there exists a unique solution of (60). □

Proposition 7. *In the case of the power utility* $U(x) = \frac{x^p}{p}$, $x > 0$, *with* $p < 1$, $p \neq 0$, *we suppose that the Assumptions 1, 2, 3, 4 hold,* $g(\xi) \in]0,x[$*(*α *-a.s.) and*

$$\int_\Xi \left(\mathbf{H}_T^{(q),*}(u)\right)^{1-p} d\alpha(u) < \infty.$$

Then, the buyer's and seller's indifference prices are defined respectively from the equations:

$$\int_\Xi [(1 - \frac{p_T^b}{x} + \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u)\right)^{1-p} d\alpha(u) = 0 \tag{61}$$

and

$$\int_\Xi [(1 + \frac{p_T^s}{x} - \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u)\right)^{1-p} d\alpha(u) = 0 \tag{62}$$

Moreover, the equations (61) *and* (62) *have unique solutions belonging to the interval* $[0,x]$.

Remark 4. *In the case when $X(\xi)$ and ξ are independent, the information quantity $\mathbf{H}_T^{(q),*}(u)$ does not depend on u and we get from Proposition 7 the following equations for indifference price:*

$$\int_\Xi [(1 - \frac{p_T^b}{x} + \frac{g(u)}{x})^p - 1] d\alpha(u) = 0,$$

$$\int_\Xi [(1 + \frac{p_T^s}{x} - \frac{g(u)}{x})^p - 1] d\alpha(u) = 0.$$

Proof: The formula (61) follows from (25) and (56), then, the formula (62) can be obtained from the relation (58).

We denote for $y \in [0, x]$

$$F(y) = \int_\Xi \left[(1 - \frac{y}{x} + \frac{g(u)}{x})^p - 1 \right] \left(\mathbf{H}_T^{(q),*}(u) \right)^{1-p} d\alpha(u)$$

We see that F is continuous strictly decreasing function for $p \in (0, 1)$ on $[0, x]$ and that $F(0) \geq 0$ and $F(x) \leq 0$. Then the solution of the equation exists by mean value theorem and it is unique. For $p < 0$, F is a strictly increasing continuous function with $F(0) \leq 0$ and $F(x) \geq 0$, then (61) has a unique solution. The case of seller's indifference price can be considered in a similar way. □

Proposition 8. *In the case of the exponential utility $U(x) = 1 - e^{-\gamma x}$, $x > 0$, with $\gamma > 0$ and under the Assumptions 1, 2, 3, 4 , the buyer's and seller's indifference prices verify:*

$$p_T^b = \frac{1}{\gamma} \ln \left[\frac{\int_\Xi \exp \left\{ - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)}{\int_\Xi \exp \left\{ - \gamma g(u) - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)} \right] \tag{63}$$

and

$$p_T^s = -\frac{1}{\gamma} \ln \left[\frac{\int_\Xi \exp \left\{ - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)}{\int_\Xi \exp \left\{ \gamma g(u) - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)} \right] \tag{64}$$

Remark 5. *In the case when $X(\xi)$ and ξ are independent, the information quantity $\mathbf{I}(Q_T^{u,*} | P_T^u)$ does not depend on u and we get from Proposition 8 the following equations for indifference price:*

$$p_T^b = -\frac{1}{\gamma} \ln \left[\int_\Xi \exp \left\{ - \gamma g(u) \right\} d\alpha(u) \right],$$

$$p_T^s = \frac{1}{\gamma} \ln \left[\int_\Xi \exp \Big\{ \gamma g(u) \Big\} d\alpha(u) \right].$$

Proof: In the case of the exponential utility $I(y) = -\frac{1}{\gamma}(\ln y - \ln \gamma)$ and from (56) and (26) we get the equation for buyer's indifference price:

$$\int_\Xi \left(\lambda_g(u) - \lambda_0(u) \right) d\alpha(u) = 0,$$

where λ_g is given by

$$\lambda_g(u) = \gamma \exp \Big\{ -\gamma \left(x - p_T^b + g(u) \right) - \mathbf{I}(\, Q_T^{u,*} \, |, P_T^{u*} \,) \Big\},$$

and

$$\lambda_o(u) = \gamma \exp \Big\{ -\gamma \, x - \mathbf{I}(\, Q_T^{u,*} \, |, P_T^{u*} \,) \Big\}$$

These formulas give us (63). The seller's indifference price (64) can be obtained from the relation (58). □

5.2. **Indifference prices and risk measure properties.** In this subsection we will show that indifference prices p_T^s and $-p_T^b$ are risk measures. First we recall here the definition of risk measure.

The application $\rho : \mathcal{F}_T \to \mathbb{R}^+$ is convex risk measure if for all contingent claims $C_T^{(1)}, C_T^{(2)} \in \mathcal{F}_T$ and all $0 < \gamma < 1$ we have:

(1) convexity of ρ with respect to the claims:

$$\rho(\gamma \, C_T^{(1)} + (1 - \gamma) \, C_T^{(2)}) \leq \gamma \rho(C_T^{(1)}) + (1 - \gamma) \rho(C_T^{(2)})$$

(2) it is increasing function with respect to the claim:

$$\text{for } C_T^{(1)} \leq C_T^{(2)}, \text{ we have } \rho(C_T^{(1)}) \leq \rho(C_T^{(2)})$$

(3) it is invariant with respect to the translation: for $m > 0$

$$\rho(C_T^{(1)} + m) = \rho(C_T^{(1)}) + m$$

Proposition 9. *We suppose that The Assumptions 1, 2, 3, 4 hold. Then for HARA utilities the indifference prices for sellers $p_T^s(g)$ and $(-p_T^b)$ for buyers obtained in the Propositions 6, 7, 8 are risk measures.*

Proof: We prove the claim for seller's indifference price since the corresponding properties for $-p_T^b$ will follow from (58).

(i) Let $U(x) = \ln(x), x > 0$. From the Proposition 6 the indifference price for seller $p_T^s = p_T^s(g)$ is defined from the equation:

$$\int_\Xi \ln\left[1 + \frac{p_T^s(g)}{x} - \frac{g(u)}{x}\right] d\alpha(u) = 0.$$

Since for each $m \in \mathbb{R}^+$, $p_T^s(g+m)$ verify

$$\int_\Xi \ln\left[1 + \frac{p_T^s(g+m)}{x} - \frac{g(u)+m}{x}\right] d\alpha(u) = 0.$$

and the solution of this equation is unique,

$$p_T^s(g+m) = p_T^s(g) + m$$

and, hence, the property (3) holds.

Let $g_1(u) \le g_2(u)$ for $u \in \Xi$. Then we have:

$$0 = \int_\Xi \ln\left[1 + \frac{p_T^s(g_1)}{x} - \frac{g_1(u)}{x}\right] d\alpha(u) \ge \int_\Xi \ln\left[1 + \frac{p_T^s(g_1)}{x} - \frac{g_2(u)}{x}\right] d\alpha(u)$$

and it gives (2).

We put $g(u) = \gamma\, g_1(u) + (1-\gamma)\, g_2(u)$. Then from concavity of ln we get:

$$\int_\Xi \ln\left[1 + \frac{\gamma\, p_T^s(g_1) + (1-\gamma)\, p_T^s(g_2)}{x} - \frac{g(u)}{x}\right] d\alpha(u) \ge$$

$$\gamma \int_\Xi \ln\left[1 + \frac{p_T^s(g_1)}{x} - \frac{g_1(u)}{x}\right] d\alpha(u) + (1-\gamma) \int_\Xi \ln\left[1 + \frac{p_T^s(g_2)}{x} - \frac{g_2(u)}{x}\right] d\alpha(u)$$

Then, since the right-hand side of previous expression is equal to zero,

$$p_T^s(\gamma\, g_1(u) + (1-\gamma)\, g_2(u)) \le \gamma\, p_T^s(g_1) + (1-\gamma)\, p_T^s(g_2)$$

and we proved the relation (1).

(ii) Let $U(x) = \frac{x^p}{p}, p < 1, p \ne 0$. From the Proposition 7 the indifference price for seller is defined from the equation:

$$\int_\Xi [(1 + \frac{p_T^s(g)}{x} - \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u)\right)^{1-p} d\alpha(u) = 0$$

The properties (2) and (3) can be proved in the same way as in (i). Let us denote by $g(u) = \gamma\, g_1(u) + (1-\gamma)\, g_2(u)$ and let us suppose that $0 < p < 1$. Then using the concavity of the function $(1+x)^p - 1$ we have:

$$\int_\Xi [(1 + \frac{\gamma\, p_T^s(g_1) + (1-\gamma)\, p_T^s(g_2)}{x} - \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u)\right)^{1-p} d\alpha(u) \ge$$

$$\gamma \int_\Xi [(1+\frac{p^s_T(g_1)}{x}-\frac{g_1(u)}{x})^p-1]\left(\mathbf{H}^{(q),*}_T(u)\right)^{1-p} d\alpha(u)+$$
$$(1-\gamma)\int_\Xi [(1+\frac{p^s_T(g_2)}{x}-\frac{g_2(u)}{x})^p-1]\left(\mathbf{H}^{(q),*}_T(u)\right)^{1-p} d\alpha(u)$$

Since the right-hand side of above expression is equal to zero, we get the property (1). The case $p<0$ can be considered in similar way.

(iii) Let $U(x)=1-e^{-\gamma_0 x}$, $x>0$, with $\gamma_0>0$. From the Proposition 8 the indifference price for the seller is defined by the formula:

$$p^s_T(g)=-\frac{1}{\gamma_0}\ln\left[\frac{\int_\Xi \exp\left\{-\mathbf{I}(Q^{u,*}_T|P^u_T)\right\}d\alpha(u)}{\int_\Xi \exp\left\{\gamma_0 g(u)-\mathbf{I}(Q^{u,*}_T|P^u_T)\right\}d\alpha(u)}\right]$$

We see directly from this formula that the properties (2) and (3) are verified. Let us take $g(u)=\gamma\, g_1(u)+(1-\gamma)\, g_2(u)$. Then, by Holder inequality with $p=\frac{1}{\gamma}$ and $q=\frac{1}{1-\gamma}$ we get:

$$\int_\Xi \exp\left\{\gamma_0 g(u)-\mathbf{I}(Q^{u,*}_T|P^u_T)\right\}d\alpha(u)\le\left(\int_\Xi \exp\left\{\gamma_0 g_1(u)-\mathbf{I}(Q^{u,*}_T|P^u_T)\right\}d\alpha(u)\right)^{\gamma}$$
$$\left(\int_\Xi \exp\left\{\gamma_0 g_2(u)-\mathbf{I}(Q^{u,*}_T|P^u_T)\right\}d\alpha(u)\right)^{1-\gamma}$$

and it gives the property (1). □

6. Conditionally exponential Levy models

We continue in the framework of Section 2. In addition, and it will be specific for this part, we assume that conditionally to $\xi=u$, $X(\xi)$ is a Levy process. It means that P^u is the law of Levy process with the parameters $(b^u,(\sigma^u)^2,\nu^u)$, where b^u is a drift parameter, $(\sigma^u)^2$ is a parameter related with a continuous martingale part and ν^u is a Levy measure, such that

$$\int_{\mathbb{R}}(x^2\wedge 1)\,\nu^u(dx)<\infty$$

Since $P^u\ll P$ we deduce that $(\sigma^u)^2$ does not depend on u and then $(\sigma^u)^2=\sigma^2$. We recall that according to the Levy-Ito decomposition theorem, conditional Levy process X with the generating triplet (b^u,σ^2,ν^u) can be represented in the following form

$$X_t=\sigma W^u_t+b^u t+\int_0^t\int_{|x|>1} xN^u(ds,dx)+\int_0^t\int_{|x|\le 1} x\tilde{N}^u(ds,dx),$$

where W^u is a $(P^u, \mathbf{F})$ standard Wiener process, $N^u(ds, dx)$ is a $(P^u, \mathbf{F})$ Poisson random measure and $\tilde{N}^u(ds, dx)$ is a $(P^u, \mathbf{F})$ compensated Poisson random measure with a compensator $\nu^u(dx)ds$.

The parameters of Levy process define entirely the law of the process via its one-dimensional distributions : for all $\lambda \in \mathbb{R}$

$$E_{P^u} e^{i\lambda X_t} = e^{t\psi^u(\lambda)},$$

where the characteristic exponent $\psi^u(\lambda)$ is given by Levy-Kinchin formula:

$$\psi^u(\lambda) = i\lambda b^u - \frac{1}{2}\lambda^2\sigma^2 + \int_{\mathbb{R}} \left(e^{itx} - 1 - x1_{\{|x|\leq 1\}}\right) \nu^u(dx).$$

Exponential Levy models was very well studied (see for instance [1], [34]). In particular, the notion of minimal entropy martingale measure was introduced first in [27], the question of existence of f-divergences minimal martingale measures, for the first time was studied in [16], and for classical f-divergences for the exponential Levy models it was done in its generalised version in [7]. Again, we denote by $\beta^{u,*}$ and $Y^{u,*}(x)$ two $(P^u, \mathbf{F})$-predictable processes known as Girsanov parameters for the changing of measure from P^u to $Q^{u,*}$ such that: $\forall t \geq 0$ and P^u-a.s.

$$\int_0^t \int_{\mathbb{R}} |l(x)\,(Y_s^{u,*}(x) - 1)|\,\nu^u(dx)\,ds < \infty, \quad \int_0^t (\beta_s^{u,*})^2\,ds < \infty.$$

We recall that for HARA utilities, the equivalent f-divergence minimal martingale measures when they exist, have Levy preservation property, i.e. being Levy process under P^u, the process remains Levy process under $Q^{u,*}$. More about preservation of Levy property see [8]. The preservation of Levy property implies that all information processes introduced in section 5 are deterministic and this fact simplifies very much the expression for maximal utility of Theorem 3.

Proposition 10. *Let $U(x) = \ln x$, $x > 0$, and Assumption 1 holds. We suppose that there exists a solution β^u to the equation*

$$b^u + \beta^u\sigma^2 + \int_{\mathbb{R}} (Y^u(x) - 1)\,1_{\{|x|\leq 1\}}\nu^u(dx) = 0, \tag{65}$$

with

$$Y^u(x) = (1 - \beta^u\,x\,1_{\{|x|\leq 1\}})^{-1} \tag{66}$$

and such that $Y^u(x) > 0 \quad \nu^u-a.s.$ *Then, there exists* f*-divergence minimal equivalent martingale measure* Q^u *and the corresponding information process is equal to:*

$$\mathcal{I}_T(u) = T\left\{\frac{1}{2}\left(\beta^u\sigma\right)^2 + \int_{\mathbb{R}}\left(-\ln\left(Y^u(x)\right) + Y^u(x) - 1\right)\nu^u(dx)\right\} \tag{67}$$

If we assume, in addition, that $g(\xi) \in]0, x[$ *(*α*-a.s.) and that* $\ln g(\xi)$, $\ln(x - g(\xi))$ *and* $\mathcal{I}_T(\xi)$ *are* α*-integrable random variables, then*

$$V(x, 0) = \int_{\Xi}\mathcal{I}_T(u)d\alpha(u) - \ln x \tag{68}$$

and for the buyer of option

$$V(x - p_T^b, g) = V(x, 0) + \int_{\Xi}\ln\left(1 - \frac{p_T^b}{x} + \frac{g(u)}{x}\right)d\alpha(u), \tag{69}$$

for the seller of option

$$V(x + p_T^s, g) = V(x, 0) + \int_{\Xi}\ln\left(1 + \frac{p_T^s}{x} - \frac{g(u)}{x}\right)d\alpha(u). \tag{70}$$

Moreover, the indifference prices p_T^b, p_T^s *are defined by the formulas* (59) *and* (60) *respectively.*

Proof: The stochastic exponent of X will be a $(Q^u, \mathbf{F})$ local martingale if the process X is a $(Q^u, \mathbf{F})$ local martingale. The process X will be a local martingale under the measure Q^u if and only if the corresponding drift parameter B^{Q^u} is equal to 0 for each t. It was shown in [26] that Girsanov parameters of the minimal martingale measure does not depend on (ω, t) (see also [8]). Since β^u and $Y^u(x)$ denote the Girsanov parameters for the changing of measure from P^u to Q^u, according to [23], Theorem 3.24, p. 159, we have $\forall t \in [0, T]$

$$B_t^{Q^u} = b^u t + \beta^u\sigma^2 t + t\int_{\mathbb{R}}(Y^u(x) - 1)1_{\{|x|\leq 1\}}(x)d\nu(x).$$

Equating B^{Q^u} to 0, one gets the relation (65).

It was shown in [26] (see also [8]) that if $S_t = \exp(\tilde{X}_t)$ then

$$Y^u(\Delta\tilde{X}) = (1 - \beta^u(e^{\Delta\tilde{X}} - 1))^{-1}$$

But at the same time $S_t = \mathcal{E}(X)_t$ and writing the relation for the jumps we get

$$\exp(\Delta\tilde{X}) - 1 = \Delta X_t 1_{\{|\Delta X_t|\leq 1\}} \tag{71}$$

and it gives the formula (66). From (32) we get the expression of the corresponding information process (67). From Theorem 3 we obtain

the expressions for maximal expected utility, and Proposition 6 gives us the formulas for indifference prices. □

Proposition 11. *Let* $U(x) = \frac{x^p}{x}, \quad x > 0$, *with* $p < 1, p \neq 0$ *and Assumption 1 holds. We suppose that there exists a solution to the equation* (65) *with*

$$Y^u(x) = \left(1 - \frac{|p|}{(p-1)^2}\beta^u\, x\, 1_{\{|x|\leq 1\}}\right)^{p-1} \tag{72}$$

such that $1 - \frac{|p|}{(p-1)^2}\beta^u\, x\, 1_{\{|x|\leq 1\}} > 0$ *(ν^u-a.s.) Then, there exists f-divergence minimal equivalent martingale measure Q^u and the corresponding Hellinger type process of order $q = \frac{p}{p-1}$ is given by:*

(73)
$$h_T^{(q)}(u) = T\left\{\frac{1}{2}q(q-1)\,(\beta^u\sigma)^2 + \int_{\mathbb{R}}\left[(Y^u(x))^q\ - q(Y^u(x)-1) - 1\right]\nu^u(dx)\right\}$$

If we assume, in addition, that $g(\xi) \in]0, x[$ *(α-a.s.) and that* $e^{h_T^{(q)}(\xi)}$ *is α-integrable random variable, then*

$$V(x,0) = x^p \int_\Xi e^{h_T^{(q)}(u)} d\alpha(u),$$

for the buyer of the option

$$V(x - p_T^b, g) = \int_\Xi (x - p_T^b + g(u))^p\, e^{h_T^{(q)}(u)} d\alpha(u),$$

for the seller of the option

$$V(x + p_T^s, g) = \int_\Xi (x + p_T^b - g(u))^p\, e^{h_T^{(q)}(u)} d\alpha(u),$$

Moreover, the buyer's and seller's indifference prices are defined by the formulas (61) *and* (62) *respectively with* $\mathbf{H}_T^{(q),*}(u) = e^{h_T^{(q)}(u)}$.

Proof: The same reasons as in the proof of the Proposition 10 gives us (65). It was shown in [24] that Girsanov parameters of the minimal martingale measure does not depend on (ω, t) (see also [8]) and that if $S_t = \exp(\tilde{X}_t)$ then

$$Y^u(\Delta\tilde{X}) = \left(1 - \frac{|p|}{(p-1)^2}\beta^u(e^{\Delta\tilde{X}} - 1)\right)^{p-1}$$

But then $S_t = \mathcal{E}(X)_t$ and (71) gives us the formula (72). Then from (50) we deduce the expression (73). From Theorem 3 we obtain the

expressions for maximal expected utility, and Proposition 7 gives us the formulas for indifference prices. □

Proposition 12. *Let $U(x) = 1 - e^{-\gamma x}$, $x > 0$, with $\gamma > 0$ and Assumption 1 holds. We suppose that there exists a solution to the equation* (65) *with*

$$Y^u(x) = \exp\{\beta^u x \, 1_{\{|x| \leq 1\}}\} \tag{74}$$

Then, there exists f-divergence minimal equivalent martingale measure Q^u and the corresponding information process is given by:

$$I_T(u) = T\left\{\frac{1}{2}\left(\beta^u \sigma\right)^2 + \int_{\mathbb{R}} \left[Y^u(x) \ln Y^u(x) - Y^u(x) + 1\right] \nu^u(dx)\right\}, \tag{75}$$

If we suppose that $I_T(u)$ is finite (α-a.s.) then we have:

$$V(x,0) = 1 - \int_{\Xi} \exp\{-\gamma x - I_T(u)\} \, d\alpha(u),$$

for the buyer,

$$V(x - p_T^b, g) = 1 - \int_{\Xi} \exp\{-\gamma \left(x - p_T^b + g(u)\right) - I_T(u)\} \, d\alpha(u),$$

for the seller,

$$V(x + p_T^s, g) = 1 - \int_{\Xi} \exp\{-\gamma \left(x + p_T^s - g(u)\right) - I_T(u)\} \, d\alpha(u).$$

Moreover, the buyer's and seller's indifference prices are defined by the formulas (63) *and* (64) *respectively with $\mathbf{I}(Q_T^{u,*} \mid P_T^u) = I_T(u)$.*

Proof: It was shown in [13] that Girsanov parameters of the minimal martingale measure does not depend on (ω, t) (see also [8]). Since β^u and $Y^u(x)$ denote the Girsanov parameters for the changing of measure from P^u to Q^u, according to [23], Theorem 3.24, p. 159, we have (65).

It was shown in [20] (see also [8])that if $S_t = \exp(\tilde{X}_t)$ then

$$Y^u(\Delta \tilde{X}) = e^{\beta^u \left(e^{\Delta \tilde{X}} - 1\right)}$$

But since $S_t = \mathcal{E}(X)_t$ and (71) we get the formula (74). From (44) we get the expression of the corresponding information process (75). From Theorem 3 we obtain the expressions for maximal expected utility, and Proposition 8 gives us the formulas for indifference prices. □

7. Applications to Geometric Brownian motion case

Let $(W^{(1)}, W^{(2)})$ bi-dimensional standard Brownian motions with correlation ρ, $|\rho| < 1$ on $[0,T]$. Let $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1 > 0, \sigma_2 > 0$. We put

$$S_t^{(1)} = \exp\{(\mu_1 - \frac{\sigma_1^2}{2})t + \sigma_1 W_t^{(1)}\}$$
$$S_t^{(2)} = \exp\{(\mu_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_t^{(2)}\}$$

for two risky assets.

The first asset will play the role of $S(\xi)$ and $X_t(\xi) = \mu_1 t + \sigma_1 W_t^{(1)}$ in this case. We take $\xi = W_{T'}^{(2)}$ instead of $S_{T'}^{(2)}$ since they generate the same σ-algebras. In this case $\alpha = \mathcal{N}(0, T')$.

We know that for all $t \in [0,T]$

$$W_t^{(1)} = \rho W_t^{(2)} + \sqrt{1-\rho^2}\gamma_t$$

where γ is independent from $W^{(2)}$ standard Brownian motion. Then, the conditional law of X given $W_{T'}^{(2)} = u$ coincide with the law of

$$X_t(u) = \mu_1 t + \sigma_1 \rho V_t(u) + \sigma_1 \sqrt{1-\rho^2}\gamma_t \tag{76}$$

where $V(u)$ is a Brownian bridge starting from 0 at $t = 0$ and ending in u at $t = T'$ which is independent from the process γ. As known,

$$V_t(u) = \int_0^T \frac{u - V_s(u)}{T' - s} ds + \eta_t$$

where η is standard Brownian motion independent from γ. Finally, since $\hat{\gamma} = \rho\eta + \sqrt{1-\rho^2}\gamma$ is again standard Brownian motion, we get:

$$X_t(u) = \mu_1 t + \sigma_1 \rho \int_0^t \frac{u - V_s(u)}{T' - s} ds + \sigma_1 \hat{\gamma}_t \tag{77}$$

Hence, $P_t^u \ll P_t$ for all $u \in \mathbb{R}$ and $t \in [0,T]$, and the Assumptions 1 and 2 are satisfied.

Let au calculate the conditional law $\alpha^t = P(\xi \,|\, \mathcal{F}_t)$ given $\mathcal{F}_t = \sigma(W_s^{(1)}, s \leq t)$. By Markov property we get: for $A \in \mathcal{B}(\mathbb{R})$

$$\alpha^t(A) = P(W_{T'}^{(2)} \in A \,|\, \mathcal{F}_t) = P(W_{T'}^{(2)} \in A \,|\, W_t^{(1)}) = P(W_{T'}^{(2)} - W_t^{(2)} + W_t^{(2)} \in A \,|\, W_t^{(1)})$$

Since $W_{T'}^{(2)} - W_t^{(2)}$ is independent from $(W_t^{(1)}, W_t^{(2)})$, the law of ξ given $W_t^{(1)} = x$ is $\mathcal{N}(\rho\, x, T' - \rho^2 t)$. So, since $T' - \rho^2 t \neq 0$ for $t \in [0,T]$, it is equivalent to the law of $W_{T'}^{(2)}$ being $\mathcal{N}(0, T')$.

To give the formulas for indifference price it is convenient to remark that $Q^{u,*}$ is a unique martingale measure which annulate the drift of $X(u)$ given by

$$B_t(u) = \mu_1 t + \sigma_1 \rho \int_0^t \frac{u - V_s(u)}{T' - s} ds$$

If we denote

$$\beta_s^u = \mu_1 + \sigma_1 \rho \frac{u - V_s(u)}{T' - s}$$

then

$$\frac{dQ_T^{u,*}}{dP_T^u} = \exp\{\sigma_1 \int_0^T \beta_s^u d\hat{\gamma}_s + \frac{\sigma_1^2}{2} \int_0^T (\beta_s^u)^2 ds\}$$

Let us write the information processes corresponding to $(P^u, Q^{u,*})$. For Hellinger process we have:

$$h_t^{(q)} = \frac{q(1-q)\,\sigma_1^2}{2} \int_0^T (\beta_s^u)^2 ds$$

and

$$\mathbf{H}_T^{(q)}(u) = E_{P^u}\left[\exp\{\frac{q(1-q)\,\sigma_1^2}{2} \int_0^T (\beta_s^u)^2 ds\}\right].$$

For Kullback-Leibler process we get:

$$I_T^*(u) = \frac{\sigma_1^2}{2} \int_0^T (\beta_s^u)^2 ds$$

and Kullback-Leibler information

$$\mathbf{I}(Q^{u,*} \,|\, P^u) = E_{Q^{u,*}}(\, I_T^*(u)\,).$$

For the entropy of P_T^u with respect to $Q^{u,*}$ we deduce that:

$$\mathcal{I}_T^*(u) = \frac{\sigma_1^2}{2} \int_0^T (\beta_s^u)^2 ds$$

and

$$\mathbf{I}(P^u \,|\, Q^{u,*}) = E_{P^u}(\mathcal{I}_T^*(u)).$$

Proposition 13. *For mentioned three information quantities we have the following result:*

$$\mathbf{I}(P^u \,|\, Q^{u,*}) = \frac{\sigma_1^2}{2}\left[\left(\mu_1 - \frac{\sigma_1 \rho u}{T'}\right)^2 T + \frac{\sigma_1^2 \rho^2}{T'}\left(T' \ln(\frac{T'}{T'-T}) - T\right)\right],$$

$$\mathbf{I}(Q^{u,*} \,|\, P^u) = \frac{\sigma_1^2}{2}\left\{\mu_1^2\, T + 2\sigma_1\, \mu_1\, \rho\, u\, \ln(\frac{T'}{T'-T}) + \sigma_1^2 \rho^2\, u^2 \frac{T}{T'(T'-T)}\right.$$
$$\left. + \sigma_1^2 \rho^2 \left[\frac{T}{T'-T} - \ln(\frac{T'}{T'-T})\right]\right\},$$

$$\mathbf{H}_T^{(q)}(u) = \left(\frac{T'}{T'-T+qT}\right)^{1/2} \exp\left\{-\frac{(1-q)}{2}\left[\frac{u^2}{T'} - \frac{(u+cT)^2}{T'-T+qT}\right]\right\}$$

where $q > -(\frac{T'}{T} - 1)$ *and* $c = \dfrac{\mu_1}{\sigma_1\sqrt{1-\rho^2}}$

Proof: We begin with the calculus of $E_{P^u}(\beta_t^u)^2$. We have:

$$(\beta_t^u)^2 = \mu_1^2 + 2\mu_1\sigma_1\rho\frac{u - V_t(u)}{T'-t} + \sigma_1^2\rho^2\frac{(u-V_t(u))^2}{(T'-t)^2}$$

From second representation for Brownian bridge we know that

$$(V_t(u))_{0\le t\le T'} \overset{\mathcal{L}}{=} (W_t - \frac{t}{T'}(W_{T'} - u))_{0\le t\le T'}$$

Then,

$$E_{P^u}(u - V_t(u)) = \frac{u(T'-t)}{T'}$$

and

$$E_{P^u}(u - V_t(u))^2 = \frac{t(T'-t)}{T'} + u^2\frac{(T'-t)^2}{(T')^2}$$

Hence,

$$E_{P^u}(\beta_t^u)^2 = (\mu_1 + \frac{\sigma_1\rho u}{T'})^2 + \sigma_1^2\rho^2\frac{t}{T'(T'-t)},$$

and using Fubini theorem and the expression for $\mathcal{I}_T^*(u)$ we get the first equality.

The semi-martingale characteristics $X(u)$ under P^u are: $(B(u), I, 0)$ where $I(t) = t$ and

$$B_t(u) = \mu_1 t + \sigma_1\rho\int_0^t \frac{u - V_s(u)}{T'-s}ds.$$

From another side, the change of the measure P^u into $Q^{u,*}$ annulate the drift of $X(u)$, i. e. the semi-martingale characteristics of $X(u)$ will be $(0, I, 0)$. One of the possibilities to do this is annulate the drift of $V(u)$ transforming this process into Brownian motion, then to annulate the drift $\mu_1 I$ using independent Brownian motion γ. All successive equivalent change of the measures will give the same final result in terms of information quantities. Hence,

$$E_{Q^{u,*}}(\beta_t^u)^2 = \mu_1^2 + 2\mu_1\sigma_1\rho\frac{u}{T'-t} + \sigma_1^2\rho^2\frac{u^2+t}{(T'-t)^2}$$

Using Fubini theorem and the expression for $I_T(u)$ given previously, we get the second result.

Now, we calculate $\mathbf{H}_T^{(q),*}$ applying the definition of Hellinger integral, namely,

$$\mathbf{H}_T^{(q),*} = E_{P^u}(Z_T^*(u))^q = E_{Q^{u,*}}(Z_T^*(u))^{q-1} \tag{78}$$

We will find $Z_T^*(u)$ first. For that we remark that P^u is the law of the process $X(u) = (X_t(u))_{0\le t\le T}$ with

$$X_t(u) = \mu_1 t + \sigma_1\rho V_t(u) + \sigma_1\sqrt{1-\rho^2}\gamma_t$$

where $V(u)$ is Brownian bridge independent from standard Brownian motion γ. As it was mentioned, the change of the measure P^u into $Q^{u,*}$ annulate the drift of $X(u)$. This annulation can be made in two steps: annulate the drift of $V(u)$ transforming $V(u)$ into standard Brownian motion, and then, annulate the drift $\mu_1 I$ using the change of the measure related with the process γ. More precisely, we do the following transformations:

$$(V(u), \sigma_1\sqrt{1-\rho^2}\gamma+\mu_1 I) \to (W, \sigma_1\sqrt{1-\rho^2}\gamma+\mu_1 I) \to (W, \sigma_1\sqrt{1-\rho^2}\gamma)$$

where W and γ are standard independent Brownian motions.

Let us denote $\hat{P}^u$ the law of $(V_t(u))_{0\le t\le T}$ and by P_T the law of $(W_t)_{0\le t\le T}$. We show that

$$Z_T^{(1)} = \frac{dP_T}{d\hat{P}_T} = \sqrt{\frac{T'}{T'-T}}\exp\left\{-\frac{(u-W_T)^2}{2(T'-T)} + \frac{u^2}{2T'}\right\} \tag{79}$$

In fact, for any measurable bounded functionals F and G we have:

$$E[F(W_s, s\le T)G(W_{T'})] = E\left\{E(F(W_s, s\le T)\,|\,W_T)\int_{\mathbb{R}}\frac{\exp(\frac{(x-W_T)^2}{2(T'-T)})}{\sqrt{2\pi(T'-T)}}G(x)dx\right\} \tag{80}$$

since $W_{T'} = W_{T'} - W_T + W_T \stackrel{\mathcal{L}}{=} \tilde{W}_{T'-T} + W_T$ where $\tilde{W}$ is independent from W standard Brownian motion. Hence,

$$E[F(W_s, s\le T)G(W_{T'})] = E\left[F(W_s, s\le T)\int_{\mathbb{R}}\frac{\exp(\frac{(x-W_T)^2}{2(T'-T)})}{\sqrt{2\pi(T'-T)}}G(x)dx\right]$$

In addition,

$$E[F(W_s, s\le T)G(W_{T'})] = E\left[E(F(W_s, s\le T)\,|\,W_{T'})\int_{\mathbb{R}}\frac{\exp(\frac{x^2}{2T'})}{\sqrt{2\pi T'}}G(x)dx\right] \tag{81}$$

Since (80) and (81) are verified for any bounded G we get:

$$E(F(W_s, s \leq T) \,|\, W_{T'}) = E\left[F(W_s, s \leq T) \frac{\exp(\frac{(x-W_T)^2}{2(T'-T)} + \frac{x^2}{2T'}}{\sqrt{(T'-T)/T'}} \right]$$

The last equality proves that (79) holds.

To annulate the drift $\mu_1 I$ we use the process γ and the change of the measure with the density:

$$Z_T^{(2)} = \exp\left\{ \frac{\mu_1 \gamma_T}{\sigma_1\sqrt{1-\rho^2}} - \frac{\mu_1^2 T}{2\sigma_1^2(1-\rho^2)} \right\}$$

So, the measure which transform $(V(u), \sigma_1\sqrt{1-\rho^2}\gamma + \mu_1 I)$ into $(W, \sigma_1\sqrt{1-\rho^2}\gamma]$ has a density $Z_T^{(1)} Z_T^{(2)}$. Using the theorem about of change of variables we find equivalent to $Z_T^*(u)$ expression in law with respect to $Q^{u,*}$:

$$Z_T^*(u) \stackrel{\mathcal{L}}{=} \int Z_T^{(1)}(W_T - y) Z_T^{(2)}(y) dP_{\gamma_T}(y)$$

Simple calculations gives us that

$$Z_T^*(u) \stackrel{\mathcal{L}}{=} \exp\left(\frac{u^2 - (u - W_T + cT)^2}{2T'} \right)$$

with $c = \frac{\mu_1}{\sigma_1\sqrt{1-\rho^2}}$. Then, using (78) we get after simple calculus the third result.□

Now, from Propositions 6, 7, 8 and 13 we can find indifference prices taking $\alpha = \mathcal{N}(0, T')$.

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