B-spline techniques for volatility modeling

Sylvain Corlay

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Abstract

This paper is devoted to the application of B-splines to volatility modeling, specifically the calibration of the leverage function in stochastic local volatility models and the parameterization of an arbitrage-free implied volatility surface calibrated to sparse option data. We use an extension of classical B-splines obtained by including basis functions with infinite support.

We first come back to the application of shape-constrained B-splines to the estimation of conditional expectations, not merely from a scatter plot but also from the given marginal distributions. An application is the Monte Carlo calibration of stochastic local volatility models by Markov projection.

Then we present a new technique for the calibration of an implied volatility surface to sparse option data. We use a B-spline parameterization of the Radon-Nikodym derivative of the underlying's risk-neutral probability density with respect to a roughly calibrated base model. We show that this method provides smooth arbitrage-free implied volatility surfaces.

Finally, we propose a Galerkin method with B-spline finite elements to the solution of the differential equation satisfied by the Radon Nikodym derivative.

Keywords: B-splines, Tikhonov regularization, Radon-Nikodym, local volatility, finite elements

Introduction

This article is concerned with the calibration of volatility models to market option prices. We address the problem of fitting a smooth and arbitrage-free implied volatility surface to sparse option data, the Monte Carlo calibration of the leverage function in stochastic local volatility models and the numerical treatment of the Kolmogorov forward equation.

We advocate a B-spline parameterization of the Radon-Nikodym derivative of the underlying's risk-neutral distribution with respect to a base model. This approach has numerous advantages over previous methods for fitting a vanilla option price surface to market data. First, it is a linear transformation of the price space, so that we can formulate the conditions for absence of static arbitrage with linear constraints. Moreover, it allows for beliefs on the asymptotics of the volatility surface to be accounted for through a choice of the base model. Finally, the problem of calibrating an arbitrage-free surface to market option prices can be formulated as a second-order cone program, which is solved efficiently using off-the-shelf software like CVXOPT [3]. An advantage of this method over a direct price space interpolation is that if the base model has a smooth implied volatility surface, it will be the case for the calibrated surface as well. Another benefit is that it allows one to avoid the numerical approximation of a Dirac mass in the finite-element approximation of the Kolmogorov forward equation.

We dedicate the first section to giving background on B-splines. The classical B-spline basis functions have compact support and practitioners usually handle extrapolation by adding external "ghost knots" with a certain multiplicity. We favor an alternative extrapolation scheme which consists of supplementing the basis with functions of infinite support as proposed in [19].

Section 2 is devoted to the problem of estimating a conditional expectation from given bivariate data and marginal distributions. We review the Bayesian interpretation of Tikhonov regularization and we address the problem of compatibility between the marginal distributions and the conditional expectation. We show that shape-constrained B-splines are well suited to the problem and apply the technique with

*Bloomberg Quant Research, 731 Lexington Avenue, New York, NY 10022, USA.

the extended infinite-support basis to a recently devised particle method for calibrating the leverage function in stochastic local volatility models. Section 3 gives some general background on second-order cone programming and its application to B-splines.

Section 4 is devoted to the problem of calibrating of a smooth arbitrage-free implied volatility surface from sparse option data. Most approaches proposed in the literature rely on some kind of general-purpose nonlinear optimizer such as the Levenberg-Marquardt algorithm. Our method is based on a B-spline parameterization of the Radon-Nikodym derivative with respect to a prior density. The calibration amounts to a second order cone program.

Finally, in Section 5 we look at discretization methods of the Kolmogorov forward P.D.E.. We propose using a B-spline-based finite element space discretization of the forward P.D.E. satisfied by the Radon-Nikodym derivative of the underlying's risk-neutral distribution with respect to a base model. This leads to the kind of surface parameterization that we have considered earlier.

Notation: We use the following conventions: $\inf(\emptyset) = +\infty$ and $\sup(\emptyset) = -\infty$. $\mathcal{P}_n(\mathbb{R})$ is the set of real polynomials of order n. If X is a random variable on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\mathbb{P}_X$ denotes its pushforward measure. Indexing of knots and B-spline basis functions start at 0.

1 Univariate B-splines and extrapolation

1.1 B-splines of infinite support

In this section, we present the extension of classical B-splines devised in [19] by Schumaker to include basis functions with infinite support.

Definition 1.1 (B-splines of infinite support, [19])**.** *Let k be a nonnegative integer and $\Gamma := \{\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{k-1}\}$ be a sorted collection of k knots. (If $k = 0$, $\Gamma := \emptyset$.) Let C_0 and C_1 be two positive constants. For a nonnegative integer $n \leq k$, a B-spline of order n associated with the knots Γ is a function of the form $\sum_{j=0}^{k+n} w_j b_{j,n}^{\Gamma}$, where the weights $(w_j)_{0\leq j\leq k+n}$ are real numbers and where the functions $\left(b_{j,n}^{\Gamma}\right)_{n\geq 0,\ 0\leq j\leq k+n}$ are defined by*

$$\begin{cases} b_{0,0}^{\Gamma}(x) := \mathbf{1}_{(-\infty,\inf(\Gamma))}(x), \qquad b_{k,0}^{\Gamma} := \mathbf{1}_{[\sup(\Gamma),+\infty)}(x), \\ \text{and} \quad b_{j,0}^{\Gamma}(x) = \mathbf{1}_{[\gamma_{j-1},\gamma_j)}(x), \quad 1 \leq j \leq k-1, \end{cases} \tag{1}$$

and for $1 \leq n \leq k$, with the induction formula

$$\begin{cases} b_{j,n}^{\Gamma}(x) := \frac{(\gamma_j - x)}{C_0} b_{j,n-1}^{\Gamma}(x), & j = 0, \\ b_{j,n}^{\Gamma}(x) := b_{j-1,n-1}^{\Gamma}(x) + \frac{(\gamma_j - x)}{C_0} b_{j,n-1}^{\Gamma}(x), & 1 \leq j < \min(n,k), \\ b_{j,n}^{\Gamma}(x) := \begin{cases} b_{j-1,n-1}^{\Gamma}(x) + \frac{\gamma_j - x}{\gamma_j - \gamma_{j-n}} b_{j,n-1}^{\Gamma}(x), & \text{if } k > n, \\ b_{j-1,n-1}^{\Gamma}(x) + b_{j,n-1}^{\Gamma}(x), & \text{if } k = n, \end{cases} & j = \min(n,k), \\ b_{j,n}^{\Gamma}(x) := \frac{x - \gamma_{j-n-1}}{\gamma_{j-1} - \gamma_{j-n-1}} b_{j-1,n-1}^{\Gamma}(x) + \frac{\gamma_j - x}{\gamma_j - \gamma_{j-n}} b_{j,n-1}^{\Gamma}(x), & \min(n,k)+1 \leq j < \max(n,k), \\ b_{j,n}^{\Gamma}(x) := \begin{cases} \frac{x - \gamma_{j-n-1}}{\gamma_{j-1} - \gamma_{j-n-1}} b_{j-1,n-1}^{\Gamma}(x) + b_{j,n-1}^{\Gamma}(x), & \text{if } k > n, \\ b_{j-1,n-1}^{\Gamma}(x) + b_{j,n-1}^{\Gamma}(x), & \text{if } k = n, \end{cases} & j = \max(n,k), \\ b_{j,n}^{\Gamma}(x) := \frac{(x - \gamma_{j-n-1})}{C_1} b_{j-1,n-1}^{\Gamma}(x) + b_{j,n-1}^{\Gamma}(x), & \max(n,k)+1 \leq j < k+n, \\ b_{j,n}^{\Gamma}(x) := \frac{(x - \gamma_{j-n-1})}{C_1} b_{j-1,n-1}^{\Gamma}(x), & j = k+n. \end{cases} \tag{2}$$

Regarding the terms $\frac{x - \gamma_{j-n-1}}{\gamma_{j-1} - \gamma_{j-n-1}}$ and $\frac{\gamma_j - x}{\gamma_j - \gamma_{j-n}}$, the convention when the denominator is equal to zero is to replace it by 1 and 0 respectively.
The so-defined collection of functions $\left(b_{i,n}^{\Gamma}\right)_{0\leq n\leq k, 0\leq i<k+n+1}$ are called the B-spline basis functions of order n. The B-splines of order n form a vector space of dimension $k+n+1$.

Definition 1.2 (B-splines of higher order)**.** *With the same notations as in Definition 1.1, for a nonnegative integer $n > k$, a B-spline of order n associated with the knots Γ is a function of the form $\sum_{j=0}^{k+n} w_j b_{j,n}^{\Gamma}$,*

where the weights $(w_j)_{0\leq j\leq k+n}$ *are real numbers and where the functions* $\left(b^\Gamma_{j,n}\right)_{n>k,0\leq j\leq k+n}$ *are defined by the induction formula*

$$\begin{cases} b^\Gamma_{j,n}(x) := \frac{(\gamma_j - x)}{C_0} b^\Gamma_{j,n-1}(x), & j = 0, \\ b^\Gamma_{j,n}(x) := b^\Gamma_{j-1,n-1}(x) + \frac{(\gamma_j - x)}{C_0} b^\Gamma_{j,n-1}(x), & 1 \leq j < \min(n,k), \\ \left(b^\Gamma_{j,n}(x)\right)_{\min(n,k)\leq j<\max(n,k)+1} & \text{any basis of } \mathcal{P}_{\max(n,k)-\min(n,k)}(\mathbb{R}) = \mathcal{P}_{n-k}(\mathbb{R}) \\ b^\Gamma_{j,n}(x) := \frac{(x - \gamma_{j-n-1})}{C_1} b^\Gamma_{j-1,n-1}(x) + b^\Gamma_{j,n-1}(x), & \max(n,k)+1 \leq j < k+n, \\ b^\Gamma_{j,n}(x) := \frac{(x - \gamma_{j-n-1})}{C_1} b^\Gamma_{j-1,n-1}(x), & j = k+n. \end{cases} \tag{3}$$

The positive constants C_0 *and* C_1 *are the same as in Definition 1.1*

Remark (On the choice of the constants C_0 and C_1)**.** *A desirable property for the B-spline basis functions is that if the set of knots* Γ *is affinely transformed, the corresponding B-splines are affinely transformed as well. In other words,* C_0 *and* C_1 *should scale with* Γ*. In our implementation, we used* $C_0 = C_1 = \frac{\gamma_{k-1} - \gamma_0}{k-1}$ *if* $\gamma_{k-1} > \gamma_0$ *and* $C_0 = C_1 = 1$ *otherwise.*

We have defined B-spline basis functions of arbitrary order associated with an arbitrary finite collection of knots. In Figure 1, we display the B-spline basis functions of order 0, 1, 2 and 3, for the same set of 8 knots, where we have taken $C_0 = C_1 = \frac{\gamma_{k-1} - \gamma_0}{k-1}$.

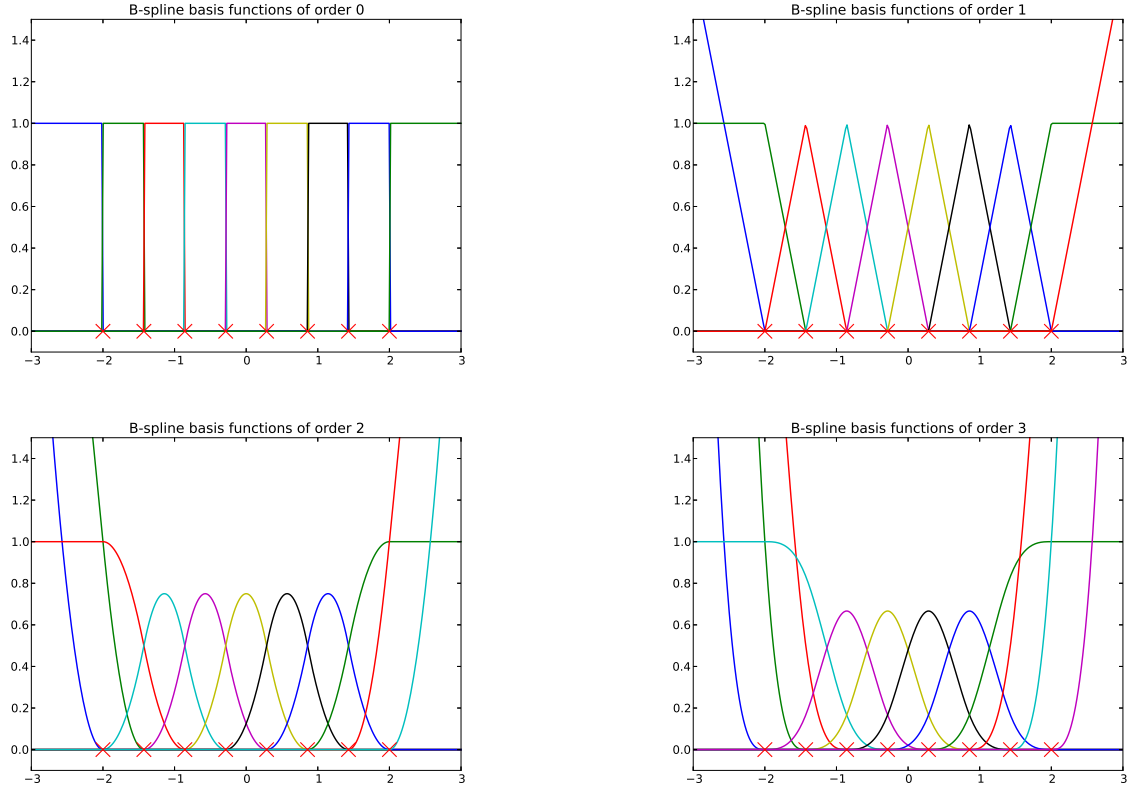


Figure 1: The B-spline basis functions of order 0, 1, 2 and 3, corresponding to the same set of 8 knots.

Proposition 1.1 (Properties of B-splines)**.** *With the same notation as in Definition 1.1:*

- *If* $\gamma_0 < \gamma_1 < \cdots < \gamma_{k-1}$ *(with strict inequalities), then the vector space spanned by* $\left(b^\Gamma_{j,n}\right)_{0\leq j\leq k+n}$ *is the set of* C^{n-1} *piecewise polynomial functions of order* n *over* $\mathbb{R}$ *with breakpoints* Γ*.*
- *If* $\gamma_{l-1} < \gamma_l = \cdots = \gamma_{l+m-1} < \gamma_{l+m}$ *for some* $0 \leq l \leq k-1$ *and* $m \geq 1$*, then the functions contained in* $\operatorname{span}\left(b^\Gamma_{j,n}\right)_{0\leq j\leq k+n}$ *are piecewise polynomial functions of order* n *and are only* C^{n-m} *at* γ_l*.*

In other words, the multiplicity of a knot diminishes the regularity of the spanned set of piecewise polynomial functions at the corresponding breakpoint.

Remark (Basis truncation)**.** *When using B-splines for regression, a good way to avoid explosion of the extrapolation is to remove the basis functions of unbounded support that have a polynomial order strictly higher than t,* $b^\Gamma_{0,n}, b^\Gamma_{1,n}, \cdots, b^\Gamma_{n-t-1,n}$ *and* $b^\Gamma_{k+t+1,n}, \cdots, b^\Gamma_{k+n,n}$*. The resulting vector space has dimension* $k - 2t - n + 1$*. For* $t = -1$*, this reduces to the usual B-splines of compact support.*

The derivative of these B-spline basis functions can be decomposed onto a B-spline basis of lower order.

Proposition 1.2 (Differentiation of B-splines)**.** *With the same notation, if* $0 < n \leq k$*,*

$$\left(b^\Gamma_{j,n}\right)' = \begin{vmatrix} -\frac{n}{C_0} b^\Gamma_{j,n-1} & 0 \leq j < \min(n,k), \\ -\frac{n}{\gamma_n - \gamma_0} b^\Gamma_{n,n-1} & \text{for } j = n, \\ \frac{n}{\gamma_{j-1}-\gamma_{j-n-1}} b^\Gamma_{j-1,n-1} - \frac{n}{\gamma_j - \gamma_{j-n}} b^\Gamma_{j,n-1} & n+1 \leq j < k, \\ \frac{n}{\gamma_{k-1}-\gamma_{k-n-1}} b^\Gamma_{k-1,n-1} & \text{for } j = k, \\ \frac{n}{C_1} b^\Gamma_{j-1,n-1} & \max(n,k)+1 \leq j < k+n+1. \end{vmatrix} \tag{4}$$

For $0 \leq p \leq n$, the decomposition of the pth derivative of B-spline basis functions of order n onto the basis of order $n - p$ is obtained by iterating over this decomposition.

Proposition 1.3 (Higher-order derivatives)**.** *We can define the B-splines of order* -1 *by* $b^\Gamma_{j,-1} := \delta_{x_j}$ *for* $j = 0, \cdots, k-1$*, where* δ_x *denotes the Dirac mass centered at* x*. If* $k > 0$ *and* $n = 0$*, we have*

$$\left(b^\Gamma_{j,n}\right)' = \begin{vmatrix} 0 & 0 \leq j < \min(n,k), \\ -b^\Gamma_{n,n-1} & \text{for } j = n, \\ b^\Gamma_{j-1,n-1} - b^\Gamma_{j,n-1} & n+1 \leq j < k, \\ b^\Gamma_{k-1,n-1} & \text{for } j = k, \\ 0 & \max(n,k)+1 \leq j < k+n+1, \end{vmatrix} \tag{5}$$

which can be seen as a limiting version of Equation (4)*.*

Remark (Integration and inner products of B-splines)**.** *Primitives and integrals of B-splines, as well as inner products of B-splines have closed form expressions. An exact quadrature method is to use Gauss-Legendre points on each interval defined by the knots. A comprehensive study of methods to compute inner products of B-splines is carried out in [20].*

1.2 Evaluation and representation of B-splines

The forward evaluation scheme for basis functions

We can reformulate (2) in a simpler way. Starting from $b^\Gamma_{j,n}(x) = 0$, we write for $0 < n \leq k$

$$\begin{array}{ll} \text{For } 0 \leq j < \min(n,k) & b^\Gamma_{j,n}(x) \mathrel{+}= \frac{\gamma_j - x}{C_0} b^\Gamma_{j,n-1}(x), \\ & b^\Gamma_{j+1,n}(x) \mathrel{+}= b^\Gamma_{j,n-1}(x), \\ \text{For } \min(n,k) \leq j < \max(n,k) & b^\Gamma_{j,n}(x) \mathrel{+}= \frac{\gamma_j - x}{\gamma_j - \gamma_{j-n}} b^\Gamma_{j,n-1}(x), \\ & b^\Gamma_{j+1,n}(x) \mathrel{+}= \frac{x - \gamma_{j-n}}{\gamma_j - \gamma_{j-n}} b^\Gamma_{j,n-1}(x), \\ \text{For } \max(n,k) \leq j < k+n & b^\Gamma_{j,n}(x) \mathrel{+}= b^\Gamma_{j,n-1}(x), \\ & b^\Gamma_{j+1,n}(x) \mathrel{+}= \frac{x - \gamma_{j-n}}{C_1} b^\Gamma_{j,n-1}(x). \end{array} \tag{6}$$

This formulation is used to evaluate B-spline basis function in two ways:

1. The first and most natural approach is to use Formula (6) at each query point.

2. The second method is to implement Formula (6) in terms of operations in the *polynomial algebra*. With this pre-processing stage, we end up with a representation of the B-spline basis as the collection of their polynomial coefficients on each interval.

   The evaluation from the piecewise polynomial representation can be carried out using Horner's method, which is more efficient than recomputing the basis functions at new query points.

   Therefore the use of the second method, which involves a pre-processing stage, is beneficial if we evaluate the B-spline on a large number of points. The threshold for the number of evaluations is approximately equal to n evaluations by interval. Its actual value depends on the implementation.

In every case, one can use knowledge of the support of B-spline basis functions for their representation in memory and their evaluation. For a fixed $x \in \mathbb{R}$, if $0 \leq n \leq k$ there are $n+1$ B-spline basis functions of order n that can be non-zero at x. More precisely, if $\gamma_i \leq x \leq \gamma_{i+1}$ (with the conventions that $\gamma_{-1} = -\infty$ and $\gamma_k = +\infty$) the only B-spline basis functions that are not equal to zero are $b^\Gamma_{j,n}$ for $i \leq j \leq i+n$.

The backward evaluation scheme

Regarding the evaluation of a B-spline function $f = \sum \alpha_j b^\Gamma_{j,n}$, the natural and naive approach would be to use the evaluation method already presented for the basis function and to sum over the basis. This is efficient if the B-spline basis functions have already been evaluated. However, if this is not the case, there is a more direct algorithm.

Indeed, using that the basis functions $b^\Gamma_{j,n}$ are decomposed onto the basis functions $b^\Gamma_{j-1,n-1}$ and $b^\Gamma_{j,n-1}$ (Equation (2)), we can show that $f = \sum \alpha^{(1)}_j b^\Gamma_{j,n-1}$ where the loadings $\alpha^{(1)}_j$ are piecewise polynomial of order 1 and carry on with the decomposition of $b^\Gamma_{j,n-1}$ onto a lower order basis. We find $f = \sum \alpha^{(i)}_j b^\Gamma_{j,n-i}$ where the loadings $\alpha^{(i)}_j$ are piecewise polynomial of order i. The algorithm stops when $i = n$ with the decomposition of f onto the trivial basis $\left(b^\Gamma_{j,0}\right)_{0 \leq j \leq k}$. To get the loadings $\alpha^{(i+1)}_j$ from $\alpha^{(i)}_j$, we start from $\alpha^{(i+1)}_j(x) = 0$ and write

$$
\begin{array}{ll}
\text{For } 0 \leq j < \min(n-(i+1), k) & \alpha^{(i+1)}_j(x) \mathrel{+}= \alpha^{(i)}_j(x) \frac{\gamma_j - x}{C_0} + \alpha^{(i)}_{j+1}(x), \\
\text{For } \min(n-(i+1), k) \leq j < \max(n-(i+1), k) & \alpha^{(i+1)}_j(x) \mathrel{+}= \alpha^{(i)}_j(x) \frac{\gamma_j - x}{\gamma_j - \gamma_{j-n}} + \alpha^{(i)}_{j+1}(x) \frac{x - \gamma_{j-n}}{\gamma_j - \gamma_{j-n}}, \\
\text{For } \max(n-(i+1), k) \leq j < k+n-(i+1) & \alpha^{(i+1)}_j(x) \mathrel{+}= \alpha^{(i)}_j(x) + \alpha^{(i)}_{j+1}(x) \frac{x - \gamma_{j-n}}{C_1}.
\end{array}
\tag{7}
$$

This method is called backward evaluation. The scheme was proposed in [19, Chapter 5] for the case of bounded splines. It can be carried out in the polynomial algebra as well, to obtain a piecewise polynomial representation of f.

Remark. *Backward and forward evaluation schemes can be used for the evaluation of derivatives of B-splines using Equation* (4).

The case of equally spaced knots

A critical stage of all evaluation schemes is the localization of the query points in the knot vector. In the general case, this is done by bisection with $O(\log(k))$ complexity. However, in the case where the knots are evenly spaced, this is reduced to an integer part computation. The case of equally spaced knots leads to further simplifications: all bounded spline basis functions have the same polynomial representation up to a parallel shift, and unbounded basis functions are symmetric. We can exploit these properties to save a significant amount of memory and computing.

2 Multiple regression and Bayesian considerations

In this section, we address the estimation of conditional expectations by multiple regression, with special attention paid to the case of the B-splines. We also recall the Bayesian theoretical foundation of Tikhonov regularization.

Then we tackle the problem of estimating a conditional expectation, not merely from a scatter plot but also given the marginal distributions $\mathbb{P}_X$ and $\mathbb{P}_Y$. The regression problem must be constrained

to account for these compatibility conditions. The problem can be formulated as a second-order cone program.

Eventually, we show that this technique can be used as a time-stepping scheme in the particle method proposed in [11] for the calibration of stochastic local volatility models.

Multiple regression as an approximation of conditional expectation

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X, Y be two real random variables such as $Y \in L^2(\mathbb{P})$.

- $\mathbb{E}[Y|X]$ is the projection of Y onto the vector space $\big\{f(X), f \in L^2(\mathbb{P}_X)\big\}$, *i.e.* it is the solution of

$$\min_{f \in L^2(\mathbb{P}_X)} \|Y - f(X)\|_2 \,,$$

- while the multiple regression of Y with respect to a finite collection $(f_i)_{i\in I} \in \big(L^2(\mathbb{P}_X)\big)^I$ is the projection of Y onto the subspace $\mathrm{span}_{i\in I}\{f_i(X)\}$, *i.e.* it is the solution of

$$\min_{w \in \mathbb{R}^I} \left\| Y - \sum_{i \in I} w_i f_i(X) \right\|_2 .$$

Hence, the larger the vector space $\mathrm{span}_{i\in I}(f_i)$, the better the approximation of $\mathbb{E}[Y|X]$ by the multiple regression of Y with respect to $(f_i(X))_{i\in I}$.

Regression of empirical distributions

In practice, we usually only have a finite sample $(x_j, y_j)_{1\le j\le N}$ of (X, Y). A common approach is then to approximate the multiple regression of Y with respect to $(f_i(X))_{i\in I}$ by the regression of the corresponding empirical distributions. When doing so, enlarging the vector space onto which we project can be detrimental rather than beneficial. Indeed, performing a better regression of the empirical distribution does not mean that we get a better regression of the actual distribution of X with respect to Y. This phenomenon, also called "over-fitting", occurs for example when using a very fine grid for piecewise linear regression. Certain practitioners refrain from using a fine grid because of it. This means that they do not believe in wiggly results, that is, they have a prior belief on the smoothness of the conditional expectation.

Rather than refraining from refining the grid, another approach to the problem of over-fitting is the Bayesian approach, that is, to determine the most likely conditional expectation of Y with respect to X given the observed sample $(x_j, y_j)_{1\le j\le N}$ and the prior distribution.

Bayesian foundations of Tikhonov regularization

We now assume that X and $\mathcal{E}$ are L^2 real random variables and F is a random variable valued in $L^2(\mathbb{P}_X)$. We also assume that X, $\mathcal{E}$ and F are independent, and that $\mathcal{E} \overset{\mathcal{L}}{\sim} \mathcal{N}(0, \sigma_{\mathcal{E}}^2)$. We define $Y := F(X) + \mathcal{E}$. If we assume that our prior distribution for F is proportional to $e^{-\frac{\psi(F)}{2\sigma_F^2}}$ for some functional $\psi : L^2(\mathbb{P}_X) \to \mathbb{R}_+$, using Bayes' lemma and the independence of X, $\mathcal{E}$ and F, we find

$$L(F|(X,Y)) \propto L(F)L((X,Y)|F) = L(F)L((X,\mathcal{E})|F) \propto L(F)L(\mathcal{E}) \propto e^{-\frac{1}{2\sigma_{\mathcal{E}}^2}(Y-F(X))^2 - \frac{\psi(F)}{2\sigma_F^2}} .$$

The functional ψ is usually a measure of irregularity such as $\psi(f) := C\int_R \big(f^{(p)}(x)\big)^2\,dx$ for a nonnegative integer p. If $(X_i, Y_i)_{1\le i\le N}$ are N independent copies of (X, Y), the likelihood of $F \in L^2(\mathbb{P}_X)$ given this sample satisfies

$$L(F|(X_i,Y_i)_{1\le i\le N}) \propto e^{-\frac{1}{2\sigma_{\mathcal{E}}^2}\sum_{i=1}^{N}(Y_i-F(X_i))^2 - \frac{\psi(F)}{2\sigma_F^2}} .$$

Hence, maximizing the likelihood of f amounts to solving the minimization problem

$$\min_{f \in \text{span}_{i\in I}(f_i)} \underbrace{\frac{1}{N}\sum_{i=1}^{N}(y_i - f(x_i))^2}_{\text{estimator of } \mathbb{E}[(Y-f(X))^2]} + \underbrace{\frac{\sigma_{\mathcal{E}}^2}{\sigma_F^2}\frac{1}{N}}_{\text{Tikhonov factor}} \psi(f). \tag{8}$$

This shows us that the Tikhonov factor should be proportional to $\frac{1}{N}$ where N is the sample size, which is consistent with the intuition that the larger the sample is, the less we need to regularize to avoid over-fitting.

The quadratic case

In the case where the functional ψ is such that $\psi\left(\sum_{i\in I} w_i f_i\right)$ is a quadratic form in the loadings $(w_i)_{i\in I}$, the minimization problem (8) simply amounts to the minimization of a quadratic form. It is the case, for example when $\psi(f) = \int_R (f^{(p)}(x) - g(x))^2 dx$ for some $g \in L^2(\mathbb{R})$. We solve the Tikhonov-regularized regression problem by solving the corresponding set of normal equations.

Let V be defined by $V_{ij} := \frac{1}{N}\sum_{l=0}^{N} f_i(x_l)f_j(x_l)$, $i,j \in I$ and c be the vector defined by $c_i := \frac{1}{N}\sum_{l=0}^{N} f_i(x_l)y_l$, $i \in I$. We assume that the quadratic form ψ is defined by $\psi(w) = \frac{1}{2}wPw + qw$, $w \in \mathbb{R}^I$. After algebra, the minimization problem (8) amounts to

$$\min_{w\in\mathbb{R}^I} \frac{1}{2}wVw + cw + \lambda\left(\frac{1}{2}wPw + qw\right). \tag{9}$$

We obtain the following system of normal equations by differentiating (9)

$$(V + \lambda P)w + (c + \lambda q) = 0. \tag{10}$$

Quadratic forms of interest and measure of smoothness

In the case where the basis functions $(f_i)_{i\in I}$ are B-spline basis functions, measures of smoothness of the form $\psi(f) = \int_{\mathbb{R}} \left(f^{(p)}\right)^2 (x)dx$ for some p can be explicitly derived in terms of the loadings w.

To begin with, if $f = \sum_{j=0}^{k+n} w_j b_{j,n}^{\Gamma}$ has non-zero weights on basis functions that have unbounded support and of extrapolating order higher or equal to p, we get $\psi(f) = +\infty$. Hence, the basis truncation order t should always satisfy $t < p$. In other words, $w_i = 0$ for $0 \le j \le n-p$ and $k+p \le j \le k+n+1$. For example, with a penalization order $p = 2$, the maximum extrapolation order t should be strictly lower than 2. We obtain

$$\int_{\mathbb{R}} \left(\sum_{j=0}^{n+k} w_j \left(b_{j,n}^{\Gamma}\right)^{(p)}(x)\right)^2 dx = \sum_{i=0}^{n+k}\sum_{j=0}^{n+k} w_i w_j \int_{\mathbb{R}} \left(b_{i,n}^{\Gamma}\right)^{(p)}(x)\left(b_{j,n}^{\Gamma}\right)^{(p)}(x)dx.$$

Using the explicit decomposition of $\left(b_{j,n}^{\Gamma}\right)^{(p)}$ onto the B-spline basis of order $n-p$, $\left(b_{j,n-p}^{\Gamma}\right)_{0\le i<k+n-p+1}$, the coefficients of the quadratic form depend on inner products of basis functions of order $n-p$, $P_{ij} := C\int_{\mathbb{R}} b_{i,n-p}^{\Gamma}(x)b_{j,n-p}^{\Gamma}(x)dx$, which can be computed exactly using Gauss-Legendre quadrature or one of the other methods to compute inner products of B-splines presented in [20].

Remark (Penalization of order $n+1$)**.** *We are restricted to a penalization order satisfying $p \le n$. Using the Dirac comb $\left(b_{j,-1}^{\Gamma}\right)_{0\le j<k}$ introduced in Proposition 1.3, we can penalize the derivative of order $n+1$ in the same fashion. Regarding the inner product of B-splines of order -1, we use the convention*

- $\int_{\mathbb{R}}(b_{j,-1}^{\Gamma}(x))^2dx := \frac{1}{2}\left(\frac{1}{\gamma_j - \gamma_{j-1}} + \frac{1}{\gamma_{j+1}-\gamma_j}\right)$, $1 \le j < k-1$,

- $\int_{\mathbb{R}}(b^{\Gamma}_{0,-1}(x))^2dx := \frac{1}{2}\frac{1}{\gamma_1-\gamma_0}$ *and* $\int_{\mathbb{R}}(b^{\Gamma}_{k-1,-1}(x))^2dx := \frac{1}{2}\frac{1}{\gamma_{k-1}-\gamma_{k-2}}$,

which corresponds to the trapezoidal rule.

Remark (Invariance of the penalization by re-scaling)**.** *A desirable property is that if the sample* $(x_j, y_j)_{1\leq j\leq N}$ *and the knots* Γ *are simultaneously affinely transformed, the result of the penalized regression remains the same.*

On the one hand, an affine transformation of the y-axis affects the regression error term and the penalization term in the same fashion and will not change the shape of the penalized regression. In the other hand, an affine transformation of the x-axis only affects the smoothness penalization term and thus its relative importance w.r.t. the regression error. In general, if $g(x/\lambda) = f(x)$ *for some* $\lambda > 0$*, then* $g^{(p)}(x/\lambda)/\lambda^p = f^{(p)}(x)$ *and* $\int_{\mathbb{R}}\left(f^{(p)}(x)\right)^2 dx = \frac{1}{\lambda^{2p-1}}\int_{\mathbb{R}}\left(g^{(p)}(u)\right)^2 du$.

Hence, for a penalization order $p > 0$*, we should use a penalization of the form* $\sigma_X^{2p-1}\int_{\mathbb{R}}\left(f^{(p)}(x)\right)^2 dx$ *where the quantity* σ_X *scales proportionally with* X*, like the mean absolute deviation or the standard deviation.*

Numerical experiments with penalized regression

In Figure 2, we present the penalized regression of the same sample of (X, Y) with B-splines of various orders, various numbers of knots and penalization order $p = 2$. In every case, we used a Tikhonov regularization factor of $\frac{\sigma_X^{2p-1}}{N}$. We observe that the results are not very dependent on the spline order, or the number of knots once it is large enough. No additional tuning has been done. For these experiments, the random variables X and Y are defined by

$$\begin{aligned} &X \overset{\mathcal{L}}{\sim} \mathcal{N}\left(0, \sigma_X^2\right) \\ &Y := \tanh(2X/\sigma_X) + Z^2, \qquad \text{where} \quad \mathbb{Z} \overset{\mathcal{L}}{\sim} \mathcal{N}(0, \sigma_Z^2) \ \text{ is independent of } X, \end{aligned} \tag{11}$$

with $\sigma_X = \sigma_Z = 1$. This test case is nonlinear and presents changes of convexity.

Penalized B-spline regression has proven to be a rather robust regression method in comparison with classical non-parametric approaches. Non-parametric regression methods are very sensitive to parameters such as the regression order, bandwidth selection, and give poor control on extrapolation. Moreover, the piecewise polynomial form of B-splines allows for a natural representation of the regression result in memory and a natural way to evaluate it at new values. It has a solid theoretical foundation as a maximum likelihood estimator of the conditional expectation. Another advantage of B-splines is their possibility to account for shape constraints in the optimization.

Compatibility with the marginal distributions

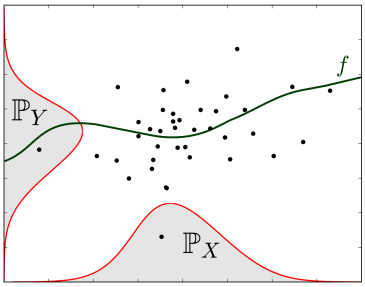
A common application of multiple regression is the estimation of the conditional expectation of a random variable Y given another random variable X from a scatter plot of the joint distribution.

However, very often it happens that we have additional information. For example, it is common that we completely know the marginal distributions of X and Y. If X and Y are two real (Borel) L^1 random variables and $f : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\mathbb{E}[Y|X] = f(X)$ then we have:

(a) $\mathbb{E}[Y] = \int_{\mathbb{R}} f(x)d\mathbb{P}_X(x)$.

(b) $f(x)$ is $\mathbb{P}_X$-almost surely in the convex hull of $\text{supp}(\mathbb{P}_Y)$.

(c) For every nonnegative convex function ϕ, $\mathbb{E}[\phi(f(X))] \leq \mathbb{E}[\phi(Y)]$.



Thus, from a Bayesian point of view, it does not make sense to consider an estimate of the conditional expectation that does not satisfy these properties.

- The first property amounts to a linear integral equality constraint.

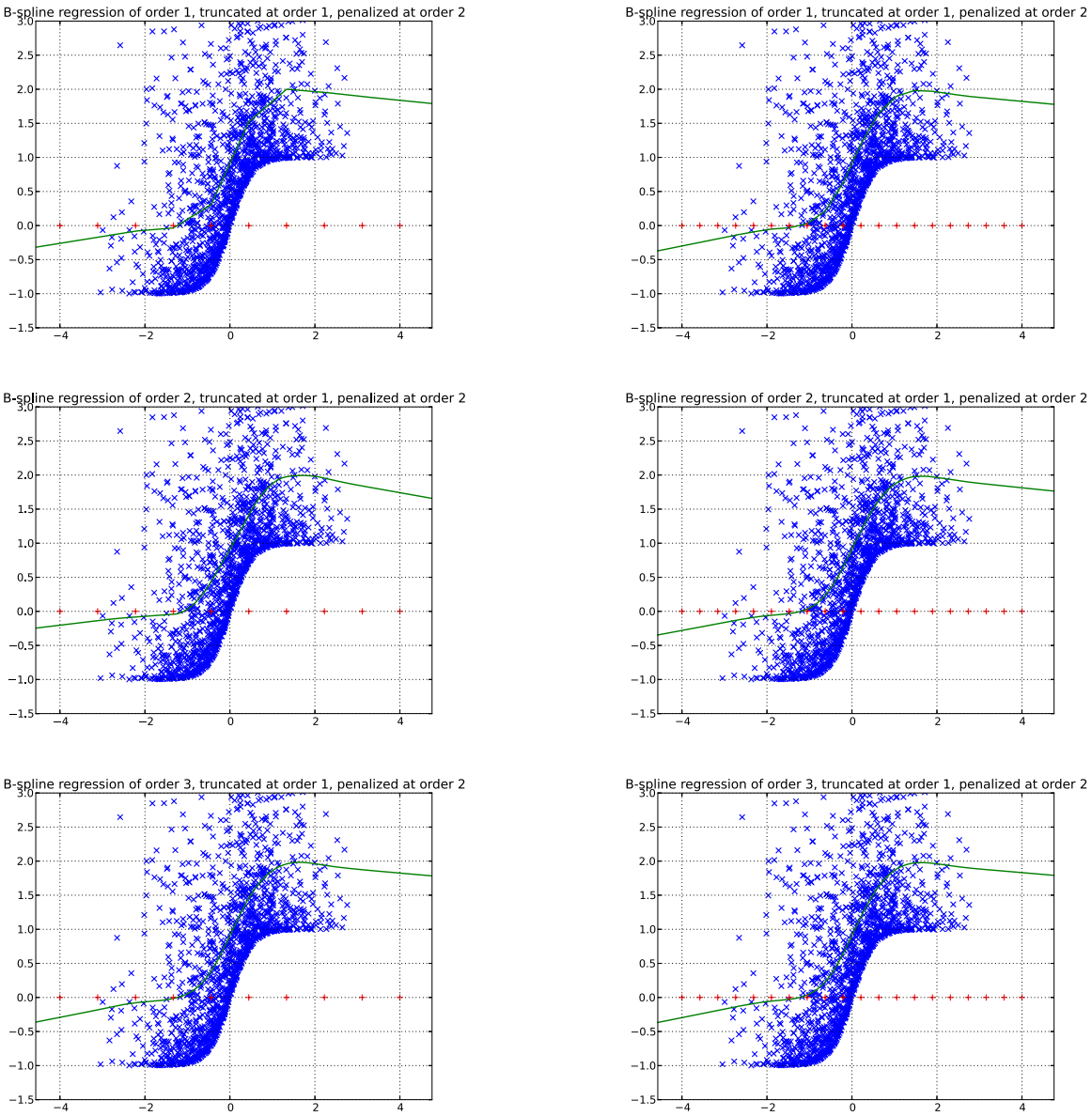


Figure 2: Penalized B-spline regression of a scatter plot sampled from Distribution (11). In every case, the penalization order is $p = 2$ and the truncation order is $t = 1$. In every case, the Tikhonov regularization factor is $\frac{\sigma_X^{2p-1}}{N}$ where $N = 1600$ is the sample size and σ_X the standard deviation of the X sample. We notice that the result does not depend significantly on the spline order. When using a finer discretization grid, the results obtained with different spline orders get closer to each other.

- The second one consists of a set of linear inequality constraints. (In practice, it often amounts to a nonnegativity constraint in the regression.)
- A consequence of the third property is that $\mathbb{E}\left[f(X)^2\right] \leq \mathbb{E}\left[Y^2\right]$, which is a quadratic constraint.

As we will see in Section 3, the resulting constrained optimization problem can be formulated as a second-order cone program. This special class of optimization problems can be solved efficiently using specialized software.

Remark (Estimation of the conditional median rather than the conditional expectation)**.** *In [14], He and Ng proposed a constrained L^1 regression technique based on B-splines. More precisely, the quantity*

of interest that is parameterized with a spline is a conditional quantile distribution $\mathbb{P}[Y \leq g_\tau(x)|X = x]$, *and* $\tau = 1/2$ *corresponds to the conditional median.*

Application to Guyon and Henry-Labordère's particle method [11]

Let S_t be the price of some tradeable asset at time t . Let us assume for the sake of simplicity that the asset does not pay any dividend and that interest and repo rates are zero. Then, arbitrage pricing theory tells us that under any risk-neutral probability, S_t is a martingale.

Knowledge of the call and put option prices of all strikes and maturities is equivalent to the knowledge of the risk-neutral densities of S_t for every maturity t. The celebrated Local Volatility Model [7] is the only Markov diffusion to match the corresponding continuum of marginal distributions. The local volatility function is given by Dupire's stripping formula. (We use the Bachelier convention for instantaneous volatilities, that is $dS = \sigma dW$, rather than the lognormal convention $dS = S\sigma dW$.)

$$\sigma^2_{\text{Dup}}(T, x) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}\frac{\partial^2 C}{\partial K^2}}. \tag{12}$$

However, it is an arbitrary choice for the modeling of transition probabilities. It may not be a good model to price and hedge products that depend on these transition probabilities. Pure stochastic volatility models, such as SABR [13], the Heston model [15] or Bergomi's model [5] are a first attempt of the modeling of these transition probabilities. A potential problem is that they do not have enough degrees of freedom to match all quoted vanilla option prices. A widespread approach is the embedding of an additional level-dependent function $l(t, x)$, the leverage function, into the diffusion equation:

$$dS_t = a_t l(t, S_t) dW_t.$$

The process a_t is a pure stochastic volatility process. It depends on S_t only through the correlation of its driving process with the Brownian motion W_t. We assume that a_0 is deterministic. Gyöngy's Markov projection theorem [12] tells us that the so-defined process S_t will have same marginals as Dupire's Local Volatility Model $d\widetilde{S}_t = \sigma_{\text{Dup}}\left(t, \widetilde{S}_t\right) d\widetilde{W}_t$ if and only if

$$\sigma^2_{\text{Dup}}(t, x) = l(t, x)^2 \mathbb{E}\left[a_t^2 \middle| S_t = x\right]. \tag{13}$$

In [11], Guyon and Henry-Labordère devised a purely forward Monte Carlo method to solve the resulting nonlinear stochastic differential equation satisfied by (S_t, a_t), and calibrate the leverage function $l(t, x)$:

$$dS_t = \frac{\sigma_{\text{Dup}}(t, S_t)}{\sqrt{\mathbb{E}\left[a_t^2 | S_t\right]}} a_t dW_t.$$

Let T be the horizon maturity for the calibration and $t_0 = 0 < \cdots < t_n = T$ a subdivision of $[0, T]$. We assume that the Dupire local volatility function σ_{Dup} is already calibrated and that the model parameters for a_t are already fixed. The calibration algorithm proceeds as follow.

Monte Carlo calibration of the leverage function

For every $0 \leq k < n$, simulate N independent draws of $S_{t_{k+1}}, a_{t_{k+1}}$ with a single Euler step (or another stepping scheme) from

- the N draws of the previous time step of S_{t_k}, a_{t_k}
- the calibrated leverage function $x \mapsto l(t_k, x)$.

We ensure the calibration condition (13) for the next time step by setting

$$l(t_{k+1}, x) := \frac{\sigma_{\text{Dup}}(t_{k+1}, x)}{\sqrt{\mathbb{E}\left[a^2_{t_{k+1}} \middle| S_{t_{k+1}} = x\right]}}.$$

The first step can be carried out by setting $l(t_0, x) := \frac{\sigma_{\text{Dup}}(t_0, x)}{a_0}$.

Each step in the calibration procedure involves an estimation of the conditional distribution a_t^2 with respect to S_t. In addition to the scatter plot, the risk-neutral probability distribution of S_t is known. Furthermore, the distribution of a_t^2 in most stochastic volatility models if also a given. For example, in the case of the Heston model, it is a noncentral chi-squared distribution. In the cases of the SABR model and Bergomi's model, a_t^2 is lognormal. Hence, we are in the situation studied in the previous section, where the marginal distributions are known.

(a) We have the integral equality constraint for the instantaneous forward variance. $\mathbb{E}\left[a_t^2\right] = \int_{\mathbb{R}} f_w(x) d\mathbb{Q}_{S_t}(x)$.

(b) The regression should be constrained to be nonnegative.

(c) The inequality constraint $\mathbb{E}\left[a_t^4\right] \geq \mathbb{E}\left[f_w(S_t)^2\right] = \sum_{i=0}^{k+n} \sum_{j=0}^{k+n} w_i w_j \int b_{i,n}^{\Gamma}(x) b_{j,n}^{\Gamma}(x) d\mathbb{Q}_{S_t}(x)$ is a quadratic inequality constraint on the loadings.

We have experienced that using this shape-constrained B-spline regression rather than non-parametric methods allowed us to dramatically reduce the number of Monte Carlo runs necessary to reach a desired level of accuracy.

Remark (Evaluation and representation of B-splines for the particle method)**.**

- *In the case of the particle method, the resulting B-splines are evaluated at the very same points where the basis functions where evaluated in the first place. Hence, we should keep the corresponding values and simply sum over the already evaluated basis functions rather than relying on backward evaluation schemes.*
- *The basis functions are evaluated on a Monte Carlo sample. It is beneficial to pre-compute the piecewise polynomial representation of the basis for the evaluation on this large number of points.*
- *Another advantage for the piecewise polynomial representation is that it is the most convenient memory representation of the calibrated leverage function for future use, that does not rely on the definition of the B-splines.*

3 Shape constraints and second-order cone programming

In this section, we first give some background on second-order cone programming and quadratic programming. Then, we review the shape constraints on B-splines that qualify as second-order cone constraints.

Second-order cone programming

A second-order cone program is a minimization problem of the form

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \cdot, N, \end{array} \tag{14}$$

where $A_i \in M_{n_i - 1, n}(\mathbb{R})$, $b_i \in \mathbb{R}^{n_i - 1}$, $c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$. Second-order cone constraints, of the form $\|A_i x + b_i\|_2 \leq c_i^T x + d_i$ reduce to

- linear inequality constraints if $n_i = 1$, $(0 \leq c_i^T x + d_i)$,
- quadratic constraints if $c_i = 0$, $(\|A_i x + b_i\|_2^2 \leq d_i^2)$.

Moreover, in the case where the objective function itself is a positive definite quadratic form, we can recast it as a second-order cone program by appending an additional scalar t to the optimization variable. The optimization problem

$$\begin{array}{ll} \text{minimize} & x^T P^T P x + 2 q_0^T x \\ \text{subject to} & \text{a collection of second-order cone constraints} \end{array}$$

where P is an invertible matrix, amounts to the minimization problem

$$\begin{array}{rl} \text{minimize} & t \\ \text{subject to} & \text{the same collection of second-order cone constraints on } x \\ \text{and} & \left\|Px + P^{-1} q_0\right\|_2 \leq t \end{array}$$

where the new optimization variable is $(x_1, \cdots, x_n, t) \in \mathbb{R}^{n+1}$. Highly efficient software packages to solve second-order cone program are available, such as primal-dual interior point methods [3, 2].

Shape constraints on B-splines

As we have seen in Section 1, if k is a nonnegative integer, $\Gamma := \{\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{k-1}\}$ is a sorted collection of k knots and if $n \leq k$, the B-spline basis functions of order n are nonnegative functions. Hence, the nonnegativity of each one of the the loadings is a sufficient condition for nonnegativity. It is also a finite set of linear constraints. This condition happens to be necessary for B-splines of order 0 and 1 as well as for the Dirac comb $\left(b^{\Gamma}_{i,-1}\right)_{0\leq i<k}$ introduced in Proposition 1.3.

As we have seen in Proposition 1.2, derivatives of B-spline basis functions are explicitly decomposed onto the basis of lower order. Thus nonnegativity constraints on the first and second derivatives of B-splines translate into monotonicity and convexity constraints.

Remark. *There is no simple sufficient and necessary condition for spline nonnegativity of order $n \geq 2$. However, in [18] Papp and Alizadeh devised a method to handle the global nonnegativity constraints on B-splines without restraining to the case of nonnegative coefficients on a nonnegative basis, while remaining within the scope of second-order cone programs. In this article, we settle for the sufficient condition mentioned already.*

Other linear constraints

Equality and inequality constraints on the value of a B-spline or one of its derivatives at a certain point obviously qualify as linear constraints. It is also the case for inequality and equality constraints on limits of a B-spline or its derivatives at $-\infty$ or $+\infty$.

Regarding integral constraints, if $f_w = \sum_{j=0}^{k+n} w_j b^{\Gamma}_{j,n}$ is a B-spline of order n and ϕ a given locally integrable function, we have $I(w) := \int_{\mathbb{R}} f_w(x)\phi(x)dx = \sum_{j=0}^{k+n} w_j \int_{\mathbb{R}} b^{\Gamma}_{j,n}(x)\phi(x)dx$. Thus, (if the quantities $\int_{\mathbb{R}} b^{\Gamma}_{j,n}(x)\phi(x)dx$ are known for $0 \leq j < k+n+1$), equality and inequality constraints on $I(w)$ qualify as linear constraints.

Hierarchy of equality constraints: a modified Moore-Penrose pseudoinverse

Shape-constrained B-splines can be used as an interpolation method rather than a multiple regression method. In this case, there is temptation to consider the interpolation condition as firm equality constraints, and to use a measure of smoothness for the objective function in the resulting second-order cone program. However, we can encounter feasibility issues when using this approach. The input data could be unreachable with the given knots and spline order.

A more robust approach is to use all the degrees of freedom to achieve a least-square fit of the data points, and among the solutions of this problem, maximize smoothness. If interpolation is feasible, it will be achieved and the most regular interpolator will be returned.

The singular value decomposition of a real matrix $A \in M_{l,p}(\mathbb{R})$ is the decomposition $A = UDV^*$ where U and V are (complex) unit matrices and D is a $l \times p$ diagonal matrix. We denote by D^+ the $p \times l$ diagonal matrix obtained by inverting non-zero entries of D and transposing it. It satisfies the following properties:

- For $b \in \mathbb{R}^l$, $\widehat{x} := \overbrace{(VD^+U^*)}^{=:A^+} b$ is a solution to $\min_{x\in\mathbb{R}^p} \|Ax - b\|_2$.
- If the minimization problem $\min_{x\in\mathbb{R}^p} \|Ax-b\|_2$ has several solutions, $\widehat{x}$ is the one which has the minimal Euclidean norm. $A^+ := VD^+U^*$ is called the Moore-Penrose pseudoinverse of A.

One could prefer to minimize another quadratic form $x \mapsto x^TQx$, different from the Euclidean norm. If Q if positive definite and $Q = G^TG$ is its Cholesky decomposition, we define $\widehat{x} := G^{-1}\left(AG^{-1}\right)^+ b$. Using the properties of the Moore-Penrose pseudoinverse mentioned already, we find

- $\widehat{u} := G\widehat{x}$ minimizes $\left\|AG^{-1}u - b\right\|_2$, which implies that $\widehat{x}$ minimizes$\|Ax - b\|_2$,
- Gx has a minimal Euclidean norm, and thus x^TQx is minimal.

The matrix $A_Q^+ := G^{-1}\left(AG^{-1}\right)^+$ is the pseudoinverse of A that minimizes the quadratic form $Q = G^TG$. In general, we would always recommend to use this approach when all constraints are linear equality constraints, in order to give a best fit result in the case of infeasibility. However, the same analysis cannot be carried out in presence of inequality constraints. We refer to [8] for a thorough review of methods to handle hierarchies of constraints with more general quadratic programs.

4 Application to arbitrage-free completion of sparse option data

In this section, we address the problem of arbitrage-free completion of the vanilla option price surface from a sparse grid of vanilla option prices of various maturities and strikes. For the sake of simplicity, we first consider the case of a single maturity.

We propose to use a B-spline parameterization of the Radon-Nikodym derivative of the risk-neutral distribution with respect to a simple roughly calibrated base model. The base model can be normal (Bachelier), lognormal (Black and Scholes [6]) or more sophisticated like S.V.I. (Gatheral and Jacquier [10]). The resulting calibrated risk-neutral density inherits certain properties of the base model. For example, in the case of the S.V.I. prior, the calibrated density will also have fat tails.

Arbitrage conditions on vanilla option prices

As we have seen in Section 3, convexity, integral constraints, equalities and inequalities on values and derivatives of B-splines qualify as linear constraints. All the conditions of absence of arbitrage on call option prices fall into these categories. Namely, if $K \mapsto C_T(K)$ and $K \mapsto P_T(K)$ are the (undiscounted) call and put option prices of strike K and fixed maturity T on some given underlying, the conditions of absence of arbitrage for a given maturity are given by:

For call option prices:

- $K \mapsto C_T(K)$ is nonnegative, nonincreasing and convex on $\mathbb{R}$,
- $\lim_{K\to+\infty} C_T(K) = 0$ and $\lim_{K\to-\infty} C_T'(K) = -1$.

For put option prices:

- $K \mapsto P_T(K)$ is nonnegative, nondecreasing and convex on $\mathbb{R}$,
- $\lim_{K\to-\infty} P_T(K) = 0$ and $\lim_{K\to+\infty} P_T'(K) = 1$.

All these conditions put together mean that the second derivative of the (undiscounted) call or put option price as a function of the strike in the sense of distributions is a probability distribution. This probability distribution is the risk-neutral probability of the underlying asset at maturity[1]. Arbitrage-free interpolation of vanilla option prices amounts to a density estimation problem.

The forward price F_T, (the risk-neutral expectation of the underlying) must be equal to $\lim_{K\to-\infty} C_T(K)+K$ and to $\lim_{K\to+\infty} K - P_T(K)$. This amounts to $C_T(0) = F_T$ in the case where the underlying can only be nonnegative (such as a stock price). Bid-ask spreads on the forward price and call option prices, as well as firm equality constraints on these quantities qualify as linear constraints. Hence, there is a temptation to use a B-spline parameterization of the call option price as a function of strike (as in [16] and [9]). However, the resulting probability distribution would have compact support which is unrealistic. It has also been proposed to directly use a spline parameterization of the risk-neutral probability distribution [17], which has an identical flaw: the resulting density has compact support.

B-spline parameterization of the Radon-Nikodym derivative with respect to a base model

The method of our choice is to start from a prior (L^1) probability distribution, $\mathbb{Q}^0_{S_T}$, roughly calibrated to the data. For example, $\mathbb{Q}^0_{S_T}$ could correspond to a rough calibration of the Black-Scholes model. We then use a B-spline parameterization of the Radon-Nikodym derivative of the risk-neutral distribution of S_T, $\mathbb{Q}_{S_T}$ with respect to $\mathbb{Q}^0_{S_T}$. For the sake of brevity, we will use the shorter notations $\mathbb{Q}^0_T := \mathbb{Q}^0_{S_T}$ and $\mathbb{Q}_T := \mathbb{Q}_{S_T}$.

$$\mathbb{Q}_T = f_w \mathbb{Q}^0_T = \left(\sum_{i=0}^{k+n} w_i b^{\Gamma}_{i,n}\right) \mathbb{Q}^0_T.$$

[1] Actually, it is the so-called forward-probability of the underlying asset, which coincides with the risk-neutral density if interest rates are assumed to be deterministic. We will not make the distinction in the rest of the paper.

We truncate the B-spline to 0th order (constant) extrapolation so that the resulting density is also ensured to be L^1. Constraints on f_w are

$$f_w \geq 0 \qquad \text{and} \qquad \int_{\mathbb{R}} f_w(x) d\mathbb{Q}_T^0(x) = 1.$$

The forward price is given by

$$F_T = \int_{\mathbb{R}} x f_w(x) d\mathbb{Q}_T^0(x),$$

while the undiscounted call and put option prices of strike K are given by

$$C_T(K) = \int_K^{+\infty} (x-K) f_w(x) d\mathbb{Q}_T^0(x) \qquad \text{and} \qquad P_T(K) = \int_{-\infty}^K (K-x) f_w(x) d\mathbb{Q}_T^0(x).$$

All these quantities happen to be linear forms of the loadings w. Thus, problems such as

- the least-square fit of (mid) vanilla option prices under these constraints and a firm equality constraint on the forward,
- minimization of the mean-square second derivative of f_w under the firm constraint of yielding a price within bid-ask for each listed strike and for the forward price,

all qualify as second-order cone programs.

Choice of the base model

The advantage of lognormal base models is that integrals of the form $\int_a^b x^n d\mathbb{Q}_T^0(x)$ have closed-form expressions. However, this base model fails to account for implied volatility skew. The consequence is that the calibrated Radon-Nikodym density typically either explodes or almost vanishes in the wings. Therefore, it is not very well approximated by piecewise polynomials. The advantage of more sophisticated models such as the SVI parameterization is that it can account for general features of the implied risk-neutral density of S_T and thus more regular results for the Radon-Nikodym derivative. However, there is no closed-form expression for quantities of the form $\int_a^b x^n d\mathbb{Q}_T^0(x)$ if $n > 1$. And one needs to resort to approximate quadrature methods.

In Figure 3, we have calibrated a B-spline parameterization of the Radon-Nikodym derivative $\frac{d\mathbb{Q}_T}{d\mathbb{Q}_T^0}$ where $\mathbb{Q}_T^0$ is the risk-neutral probability distribution of the VIX Volatility Index on October 10th 2013 seen from May 6th 2013.

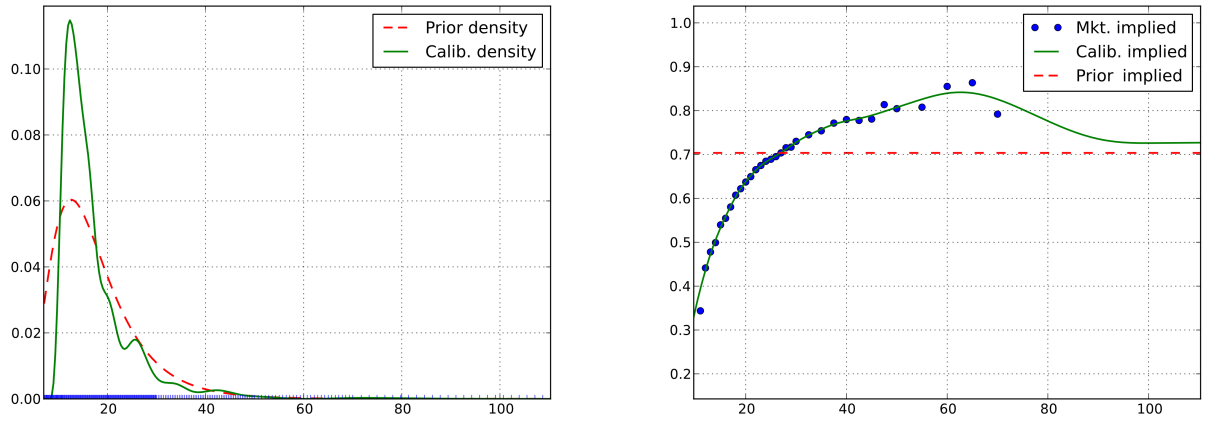


Figure 3: Risk-neutral density (left) and implied volatility smile (right) of the VIX index for expiry October 16th 2013 as seen on May 6th 2013. The red curve corresponds to the Black-Scholes base model, and the green dotted curve to the calibrated implied risk-neutral density.

Eventually, we managed to obtain a parameterization of the Radon-Nikodym density that also gives a smooth implied volatility smile.

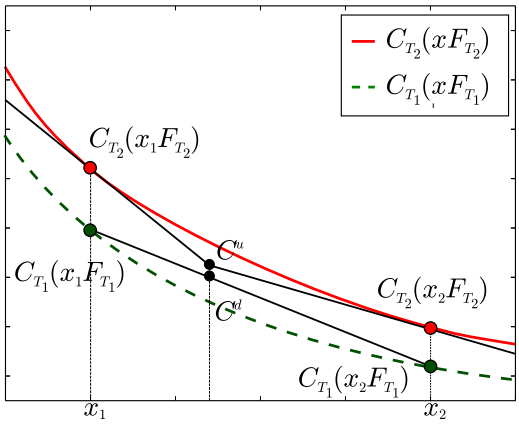


Figure 4: C^u is the ordinate of the interpolation between the tangents of the call option price of maturity T_2 as a function of the relative strike x. C^d is the value of the linear interpolation between $C_{T_1}(x_1F_{T_1})$ and $C_{T_1}(x_2F_{T_1})$ at the corresponding abscissa. $C^u \geq C^d$ is a sufficient condition for the absence of calendar arbitrage on $[x_1, x_2]$. The same equivalent argument holds for the Put option prices.

Calendar arbitrage conditions

So far, we have only addressed the case of a single maturity. If one assumes that rates are deterministic, the calendar arbitrage constraint is that undiscounted call and put option prices of constant relative strike (with respect to the forward price of the same maturity) is a non-decreasing function of maturity.

If we are given a sequence of listed maturities, we can enforce this condition on a fine grid of relative strikes, which amounts to a finite set of linear constraints. The arbitrage-free calibration of the implied volatility surface then amounts to a second-order cone program. However, this is not entirely satisfactory:

- On the one hand, we need the calendar arbitrage condition to be satisfied at any point lower than the first knot or higher than the last knot. If $0 \leq T_1 \leq T_2$ are two maturities and x is a fixed relative strike, we denote $K_1 := xF_{T_1}$ and $K_2 := xF_{T_2}$. Now, if x is such that K_1 and K_2 are both greater than the maximum knot, we have

$$C_{T_1}(K_1) = \int_{K_1}^{\infty} w_{T_1}(x - K_1) d\mathbb{Q}^0_{T_1}(x) \qquad \text{and} \qquad C_{T_2}(K_2) = \int_{K_2}^{\infty} w_{T_2}(x - K_2) d\mathbb{Q}^0_{T_2}(x),$$

  where w_{T_1} and w_{T_2} are the coefficients of the order-zero extrapolating B-spline basis function in the parameterizations of the Radon-Nikodym derivative for maturity T_1 and T_2 respectively. If the base model is assumed to be free of static arbitrage, $w_{T_2} \geq w_{T_1}$ is a sufficient condition to ensure that $C_{T_2}(K_2) \geq C_{T_1}(K_1)$. Regarding the low-strike extrapolation, the same argument holds when working with the put option prices.

- On the other hand, we need to ensure absence of calendar arbitrage in the interpolation region. In practice, we can settle for enforcing inequalities $C_{T_1}(xF_{T_1}) \leq C_{T_2}(xF_{T_2})$ only on a fine grid of values of x, for each couple of subsequent listed maturities ($T_1 < T_2$). However, it can still happen that the calendar arbitrage occurs between two points of the grid. A sufficient condition to ensure absence of calendar arbitrage in the interpolation is detailed in Figure 4. This second-order inequality condition is rather strong and it is the only nonlinear constraint. Thus, for the numerical experiments, we settled for the fine grid condition.

This discussion shows that the calibration of the Radon-Nikodym density with respect to a prior model, under the constraints of absence of arbitrage amounts to a global second-order cone program. Second order cone programs can be solved in minimum running time using available software such as [3]. If the base model is Black-Scholes, the pricing of call and put options in the calibrated model remains explicit. The risk-neutral density in the calibrated model also has a closed-form expression. Hence, the denominator in the stripping formula (12) for local volatility has a closed-form expression.

Smoothness in the time direction

In [4], Andreasen and Huge proposed an interpolation and extrapolation method which is guaranteed to yield arbitrage-free surfaces. Nonetheless, the resulting implied volatility surface fails to be differentiable in time at listed market maturities and thus, the corresponding local volatility has discontinuities in the time direction. Even though it is not formally an arbitrage, one could take advantage of a market maker using this method in markets where listed maturities are sliding with the current date, like foreign exchange markets. If the discontinuity is positive one can sell a "calendar butterfly" $C_{T+h} + C_{T-h} - 2C_T$ where T is a given market maturity and h is 1 day. One day later, all the tenors of the corresponding options will be lower than the new listed market maturity and the calendar butterfly will be valued at (almost) 0 by the market if he has recalibrated his model. One can then re-buy it for a much lower price. In the case where the discontinuity is negative, we can apply the opposite strategy.

Therefore, one should, wherever possible, produce implied volatility surfaces that are C^1 in the time direction. This can be ensured in our method by adding a term corresponding to the second time-derivative of the B-spline loadings in the regularity penalization.
In Figure 5, we display the results of the global calibration to a sparse grid of vanilla option prices. Some of the maturities do not have any listed option prices. We can add as many of those intermediate maturities as necessary to obtain a fine grid in the time direction. The base model is the Black & Scholes model with volatility equal to 20%.

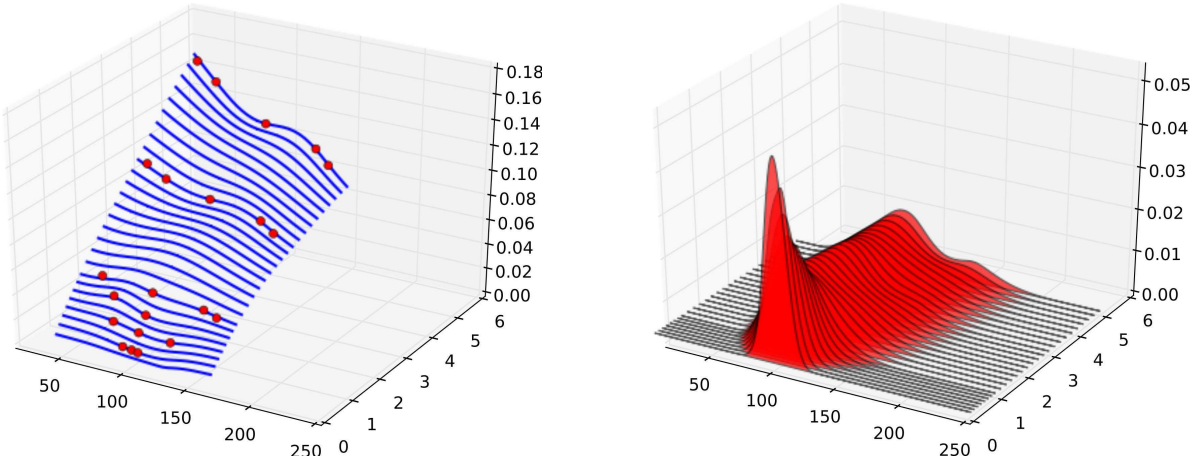


Figure 5: Calibration of Radon-Nikodym densities from sparse vanilla option data. On the left-hand side, we plot the sequence of implied total lognormal variances (input data and interpolation). On the right-hand side, we show the calibrated risk-neutral densities for the same maturities.

5 Kolmogorov forward P.D.E. for the Radon-Nikodym derivative

As we have seen earlier, parameterizing the Radon Nikodym derivative with respect to a simple base model allows to account for a prior belief on the risk-neutral distribution tails. If we believe in fat tails, we can use a fat-tailed prior model and the calibrated risk-neutral density will have the same rate of decay as the prior density. Moreover, as it is a linear transformation of the price space, conditions of absence of arbitrage remain linear conditions.

In this section, we show that the Radon-Nikodym derivative is also a convenient space for the numerical treatment of the Kolmogorov forward P.D.E. with the Galerkin method using B-spline finite elements.

We consider the case of a simple S.D.E. $dS_t = \sqrt{v(t, S_t)}dW_t$ for some instantaneous variance $v(t, x)$ and deterministic initial condition S_0. The base model is assumed to be driven by the S.D.E. $dS_t = \sqrt{v_0(t, S_t)}dW_t$ where $v_0(0, S_0) = v(0, S_0)$. The Kolmogorov forward Equation for the risk-neutral density $\phi(t, x)$ and the prior density ϕ_0 are

$$\partial_t \phi = \frac{1}{2}\partial^2_{xx}\left(v\phi\right) \qquad \text{and} \qquad \partial_t \phi_0 = \frac{1}{2}\partial^2_{xx}\left(v_0\phi_0\right).$$

If we assume that $f(t,x) = \frac{d\mathbb{Q}_t}{d\mathbb{Q}_t^0} = \frac{\phi(t,x)}{\phi_0(t,x)}$ is the Radon-Nikodym derivative of the risk-neutral probability distribution of S_t with respect to the base model, we obtain a linear P.D.E. for f,

$$\partial_t f = \frac{1}{2}\frac{\partial^2_{xx}(fv\phi_0)}{\phi_0} - f\frac{1}{2}\frac{\partial^2_{xx}(v_0\phi_0)}{\phi_0}, \qquad \text{and} \qquad f(0,x) \equiv 1. \tag{15}$$

The main benefit of this formulation with respect to the usual Kolmogorov forward P.D.E. is that the initial condition is constant rather than being a Dirac mass. Hence, we do not need to tackle the numerical approximation of the Dirac mass with the basis functions. The formulation in terms of Radon-Nikodym derivative allows us to work with far more regular functions.

In the simple case of a Bachelier base model (*i.e.* constant instantaneous variance v_0), $\phi_0(t,x) = \frac{1}{\sqrt{2\pi v_0 t}}\exp\left(-\frac{(x-S_0)^2}{2v_0 t}\right)$. In this case, the terms of the form $\frac{\partial_x(\phi_0)}{\phi_0}$ and $\frac{\partial_{xx}(\phi_0)}{\phi_0}$ involved in the P.D.E. (15) are polynomials of order 1 and 2 respectively.

Variational formulation

We first derive a spatially weak formulation of P.D.E. (15). Multiplying by a test function u of the space variable x and integrating over $\mathbb{R}$, we get

$$\partial_t \int fu = \frac{1}{2}\int v\partial^2_{xx}(f)u + \int\left(\frac{\partial_x(v\phi_0)}{\phi_0}\right)\partial_x(f)u + \int\frac{1}{2}\left(\frac{\partial^2_{xx}((v-v_0)\phi_0)}{\phi_0}\right)fu.$$

Using an integration by parts, we get the weak formulation of the P.D.E.

$$\partial_t \int fu = -\frac{1}{2}\int v\partial_x(f)\partial_x(u) + \int \underbrace{\left(\frac{\partial_x(\phi_0)}{\phi_0} + \frac{1}{2}\partial_x(v)\right)}_{=:c}\partial_x(f)u + \int \underbrace{\frac{1}{2}\left(\frac{\partial^2_{xx}((v-v_0)\phi_0)}{\phi_0}\right)}_{=:e}fu. \tag{16}$$

B-spline finite elements

We plug a B-spline or order n, $f(t,x) = \sum_{j=0}^{k+n} w_j(t)b^\Gamma_{j,n}(x)$ into Equation (16) and we obtain

$$\partial_t \sum_{j=0}^{k+n+1} w_j(t)\int b^\Gamma_{j,n}u = -\frac{1}{2}\sum_{j=0}^{k+n+1} w_j(t)\int v\partial_x\left(b^\Gamma_{j,n}\right)\partial_x(u) + \sum_{j=0}^{k+n+1} w_j(t)\int c\partial_x\left(b^\Gamma_{j,n}\right)u + \sum_{j=0}^{k+n+1} w_j(t)\int eb^\Gamma_{j,n}u,$$

that is $\partial_t \sum_{j=0}^{k+n+1} w_j(t)\int b^\Gamma_{j,n}u = \sum_{j=0}^{k+n+1} w_j(t)\left(-\frac{1}{2}\int v\partial_x\left(b^\Gamma_{j,n}\right)\partial_x(u) + \int c\partial_x\left(b^\Gamma_{j,n}\right)u + \int eb^\Gamma_{j,n}u\right)$. Following Galerkin, we plug $u = b^\Gamma_{j,n}$ for $n \le j < k+1$ (the basis functions of compact support) into (16), we get a matrix equation $A\partial_t w = B(t)w$, where A is defined by $A_{ij} = \int b^\Gamma_{j,n}b^\Gamma_{i,n}$, and is a banded matrix independent of t, and B is defined by

$$B(t)_{ij} = -\frac{1}{2}\int v\partial_x\left(b^\Gamma_{j,n}\right)\partial_x(b^\Gamma_{i,n}) + \int c\partial_x\left(b^\Gamma_{j,n}\right)b^\Gamma_{i,n} + \int eb^\Gamma_{j,n}b^\Gamma_{i,n},$$

which is time-dependent (because of the time dependence of v, c and e.) If p is the truncation order, we need to have as much as $2(p+1)$ additional linear equations or boundary conditions to obtain a full rank system. Hence, in the case of a flat extrapolation ($p = 0$), we need two boundary conditions. We can use the integral constraints

$$\int_{\mathbb{R}} f(x)\phi_0(x)dx = 1 \qquad \text{and} \qquad \int_{\mathbb{R}} xf(x)\phi_0(x)dx = S_0. \tag{17}$$

For higher order extrapolation, additional boundary conditions must be added to the system. In the end, we obtain a $(k+2p-n+1)$-dimensional ordinary differential equation, which can be solved with the Euler scheme or by using a more sophisticated stepping method.

The time-independent case

A P.D.E. of the form $\partial_t f = \mathcal{L}_x f$ where $\mathcal{L}_x$ is a linear differential operator independent of t can be seen as an infinite-dimensional O.D.E. whose solution is directly given by the exponential of the operator $\mathcal{L}_x$: $f(t,\cdot) \equiv \exp(\mathcal{L}_x) f_0(\cdot)$ where $f_0(\cdot) \equiv f(0,\cdot)$ is the initial condition. A space-discretized version of the P.D.E. can be solved in the same way with a matrix exponential. This has been used for option pricing with time-independent local volatility models by Albanese and Trovato in [1]. However, even if the local variance v is time-independent, the forward P.D.E. for the Radon-Nikodym derivative (15) is not of the required form, unless we use a time-independent base density ϕ_0.

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