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ON MEASURE SOLUTIONS OF THE BOLTZMANN EQUATION PART II: RATE OF CONVERGENCE TO EQUILIBRIUM

XUGUANG LU AND CLÉMENT MOUHOT

ABSTRACT. The paper considers the convergence to equilibrium for measure solutions of the spatially homogeneous Boltzmann equation for hard potentials with angular cutoff. We prove the exponential sharp rate of strong convergence to equilibrium for conservative measure solutions having finite mass and energy. The proof is based on the regularizing property of the iterated collision operators, exponential moment production estimates, and some previous results on the exponential rate of strong convergence to equilibrium for square integrable initial data. We also obtain a lower bound of the convergence rate and deduce that no eternal solutions exist apart from the trivial stationary solutions given by the Maxwellian equilibrium. The constants in these convergence rates depend only on the collision kernel and conserved quantities (mass, momentum, and energy). We finally use these convergence rates in order to deduce global-in-time strong stability of measure solutions.

Mathematics Subject Classification (2000): 35Q Equations of mathematical physics and other areas of application [See also 35J05, 35J10, 35K05, 35L05], 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05].

Keywords: Boltzmann equation; spatially homogeneous; hard potentials; measure solutions; equilibrium; exponential rate of convergence; eternal solution; global-in-time stability.

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1. INTRODUCTION

The Boltzmann equation describes evolution of a dilute gas. Investigations of the spatially homogeneous Boltzmann equation have made a lot of progresses in the last decades and it is hoped to provide useful clues for the understanding of the complete (spatially inhomogeneous) Boltzmann equation. The complete equation is more realistic and interesting to physics and mathematics but remains still largely out of reach mathematically and will most likely need long term preparations and efforts. For review and references of these areas, the reader may consult for instance [30, 13, 22].

The present paper is a follow-up to our previous work [22] on measure-valued solutions¹ to the spatially homogeneous Boltzmann equation for hard potentials. In this second part, we prove that, under some angular cutoff assumptions (which include the hard sphere model), solutions with measure-valued initial data having finite mass and energy converge strongly to equilibrium in the exponential rate $e^{-\lambda t}$, where $\lambda > 0$ is the spectral gap of the corresponding linearized collision operator. This sharp exponential rate was first proved in [25] for initial data with bounded energy, and belonging to L^1 (for the hard sphere model) or to $L^1 \cap L^2$ (for all hard potentials with cutoff). The core idea underlying our improvement of this result to measure solutions is that instead of considering a one-step iteration of the collision integral which produces the $L^1 \cap L^2$ integrability for the hard sphere model (as first observed by Abrahamsson [1], elaborating upon an idea in [23]), we consider a *multi-steps* iteration which produces the $L^1 \cap L^\infty$ integrability for all hard potentials with angular cutoff. This, together with approximation by L^1 solutions through the Mehler transform, and the property of the exponential moment production, enables us to apply the results of [25] and obtain the same convergence rate $e^{-\lambda t}$ for measure solutions. We also obtain a lower bound of the convergence rate and establish the global in time strong stability estimate. As a consequence we prove that, for any hard potentials with cutoff, there are no eternal measure solutions with finite and non-zero temperature, apart from the Maxwellians.

1.1. The spatially homogeneous Boltzmann equation. The spatially homogeneous Boltzmann equation takes the form

$$(1.1) \quad \frac{\partial}{\partial t} f_t(v) = Q(f_t, f_t)(v), \quad (v, t) \in \mathbb{R}^N \times (0, \infty), \quad N \geq 2$$

with some given initial data $f_t(v)|_{t=0} = f_0(v) \geq 0$, where Q is the *collision integral* defined by

$$(1.2) \quad Q(f, f)(v) = \iint_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(v - v_*, \sigma) \left(f(v') f(v'_*) - f(v) f(v_*) \right) d\sigma dv_*.$$

¹As in our previous work [22], the “measure-valued solutions” will be also called “measure solutions”.

In the latter expression, v, v_* and v', v'_* stand for velocities of two particles after and before their collision, and the microscopic conservation laws of an elastic collision

$$(1.3) \quad v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

induce the following relations:

$$(1.4) \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma$$

for some unit vector $\sigma \in \mathbb{S}^{N-1}$.

The collision kernel $B(z, \sigma)$ under consideration is assumed to have the following *product form*

$$(1.5) \quad B(z, \sigma) = |z|^\gamma b \left(\frac{z}{|z|} \cdot \sigma \right), \quad \gamma > 0$$

where b is a nonnegative Borel function on $[-1, 1]$. This corresponds to the so-called *inverse power-law interaction potentials* between particles, and the condition $\gamma > 0$ corresponds to the so-called *hard potentials*. Throughout this paper we assume that the function b satisfies *Grad's angular cutoff*:

$$(1.6) \quad A_0 := \int_{\mathbb{S}^{N-1}} b \left(\frac{z}{|z|} \cdot \sigma \right) d\sigma = |\mathbb{S}^{N-2}| \int_0^\pi b(\cos \theta) \sin^{N-2} \theta d\theta < \infty$$

and it is always assumed that $A_0 > 0$, where $|\mathbb{S}^{N-2}|$ denotes the Lebesgue measure of the $(N-2)$ -dimensional sphere \mathbb{S}^{N-2} (recall that in the case $N = 2$ we have $\mathbb{S}^0 = \{-1, 1\}$ and $|\mathbb{S}^0| = 2$). This enables us to split the collision integral as

$$Q(f, g) = Q^+(f, g) - Q^-(f, g)$$

with the two bilinear operators

$$(1.7) \quad Q^+(f, g)(v) = \iint_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(v - v_*, \sigma) f(v') g(v'_*) d\sigma dv_*,$$

$$(1.8) \quad Q^-(f, g)(v) = A_0 f(v) \int_{\mathbb{R}^N} |v - v_*|^\gamma g(v_*) dv_*.$$

which are nonnegative when applied to nonnegative functions.

The bilinear operators Q^\pm are bounded from $L_{s+\gamma}^1(\mathbb{R}^N) \times L_{s+\gamma}^1(\mathbb{R}^N)$ to $L_s^1(\mathbb{R}^N)$ for $s \geq 0$, where $L_s^1(\mathbb{R}^N)$ is a subspace of $L_0^1(\mathbb{R}^N) := L^1(\mathbb{R}^N)$ defined by

$$(1.9) \quad f \in L_s^1(\mathbb{R}^N) \iff \|f\|_{L_s^1} := \int_{\mathbb{R}^N} \langle v \rangle^s |f(v)| dv < \infty.$$

where we have used the standard notation

$$\forall v \in \mathbb{R}, \quad \langle v \rangle := \sqrt{1 + |v|^2}.$$

Since in the equation (1.1), $f = g = f_t$, by replacing

$$B(v - v_*, \sigma) \quad \text{with} \quad \frac{1}{2} [B(v - v_*, \sigma) + B(v - v_*, -\sigma)]$$

one can assume without loss of generality that the function b is even: $b(-t) = b(t)$ for all $t \in [-1, 1]$. This in turn implies that the polar form of Q^+ satisfies

$$(1.10) \quad Q^+(f, g) \equiv Q^+(g, f).$$

1.2. The definition of the solutions. The equation (1.1) is usually solved as an integral equation as follows. Given any $0 \leq f_0 \in L^1_2(\mathbb{R}^N)$, we say that a nonnegative Lebesgue measurable function $(v, t) \mapsto f_t(v)$ on $[0, \infty) \times \mathbb{R}^N$ is a mild solution to (1.1) if for every $t \geq 0$, $v \mapsto f_t(v)$ belongs to $L^1_2(\mathbb{R}^N)$, $\sup_{t \geq 0} \|f_t\|_{L^1_2} < \infty$, and there is a Lebesgue null set Z_0 (which is independent of t) such that

$$(1.11) \quad \begin{cases} \forall t \in [0, \infty), \quad \forall v \in \mathbb{R}^N \setminus Z_0, & \int_0^t Q^\pm(f_\tau, f_\tau)(v) d\tau < \infty, \\ \forall t \in [0, \infty), \quad \forall v \in \mathbb{R}^N \setminus Z_0, & f_t(v) = f_0(v) + \int_0^t Q(f_\tau, f_\tau)(v) d\tau. \end{cases}$$

The bilinear operators $(f, g) \mapsto Q^\pm(f, g)$ can now be extended to measures. For every $s \geq 0$, let $\mathcal{B}_s(\mathbb{R}^N)$ with the norm $\|\cdot\|_s$ be the Banach space of real Borel measures on \mathbb{R}^N defined by

$$(1.12) \quad F \in \mathcal{B}_s(\mathbb{R}^N) \iff \|F\|_s := \int_{\mathbb{R}^N} \langle v \rangle^s d|F|(v) < \infty,$$

where the positive Borel measure $|F|$ is the total variation of F . This norm $\|\cdot\|_s$ can also be defined by duality:

$$(1.13) \quad \|F\|_s = \sup_{\varphi \in C_c(\mathbb{R}^N), \|\varphi\|_{L^\infty} \leq 1} \left| \int_{\mathbb{R}^N} \varphi(v) \langle v \rangle^s dF(v) \right|.$$

The latter form is convenient when dealing with the difference of two positive measures.

The norms $\|\cdot\|_s$ and $\|\cdot\|_{L^1_s}$ are related by

$$(1.14) \quad \|F\|_s = \|f\|_{L^1_s} \quad \text{if} \quad dF(v) = f(v) dv.$$

For any $F, G \in \mathcal{B}_{s+\gamma}(\mathbb{R}^N)$ ($s \geq 0$), we define the Borel measures $Q^\pm(F, G)$ and

$$Q(F, G) = Q^+(F, G) - Q^-(F, G)$$

through Riesz's representation theorem by

$$(1.15) \quad \int_{\mathbb{R}^N} \psi(v) dQ^+(F, G)(v) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} L_B[\psi](v, v_*) dF(v) dG(v_*),$$

$$(1.16) \quad \int_{\mathbb{R}^N} \psi(v) dQ^-(F, G)(v) = A_0 \iint_{\mathbb{R}^N \times \mathbb{R}^N} |v - v_*|^\gamma \psi(v) dF(v) dG(v_*)$$

for all bounded Borel functions ψ , where

$$(1.17) \quad L_B[\psi](v, v_*) = |v - v_*|^\gamma \int_{\mathbb{S}^{N-1}} b(\mathbf{n} \cdot \sigma) \psi(v') d\sigma, \quad \mathbf{n} = \frac{v - v_*}{|v - v_*|}$$

and in case $v = v_*$ we define \mathbf{n} to be a fixed unit vector \mathbf{e}_1 . It is easily shown (see Proposition 2.3 of [22]) that the extended bilinear operators Q^\pm are also bounded from

$\mathcal{B}_{s+\gamma}(\mathbb{R}^N) \times \mathcal{B}_{s+\gamma}(\mathbb{R}^N)$ to $\mathcal{B}_s(\mathbb{R}^N)$ for $s \geq 0$: if $F, G \in \mathcal{B}_{s+\gamma}(\mathbb{R}^N)$ then $Q^\pm(F, G) \in \mathcal{B}_s(\mathbb{R}^N)$ and

$$(1.18) \quad \|Q^\pm(F, G)\|_s \leq 2^{(s+\gamma)/2} A_0 (\|F\|_{s+\gamma} \|G\|_0 + \|F\|_0 \|G\|_{s+\gamma}),$$

(1.19)

$$\|Q^\pm(F, F) - Q^\pm(G, G)\|_s \leq 2^{(s+\gamma)/2} A_0 (\|F + G\|_{s+\gamma} \|F - G\|_0 + \|F + G\|_0 \|F - G\|_{s+\gamma}).$$

Let us finally define the cone of positive distributions with s moments bounded:

$$\mathcal{B}_s^+(\mathbb{R}^N) := \{F \in \mathcal{B}_s(\mathbb{R}^N) \mid F \geq 0\}.$$

We can now define the notion of solutions that we shall use in this paper. We note that the condition $\gamma \in (0, 2]$ as assumed in the following definition is mainly used for ensuring the existence of solutions.

Definition 1.1 (Measure strong solutions). Let $B(z, \sigma)$ be given by (1.5) with $\gamma \in (0, 2]$ and with b satisfying the condition (1.6). Let $\{F_t\}_{t \geq 0} \subset \mathcal{B}_2^+(\mathbb{R}^N)$. We say that $\{F_t\}_{t \geq 0}$, or simply F_t , is a *measure strong solution* of equation (1.1) if it satisfies the following:

$$(1.20) \quad \begin{aligned} & \text{(i) } \sup_{t \geq 0} \|F_t\|_2 < \infty, \\ & \text{(ii) } t \mapsto F_t \in C([0, \infty); \mathcal{B}_2(\mathbb{R}^N)) \cap C^1([0, \infty); \mathcal{B}_0(\mathbb{R}^N)) \text{ and} \\ & \forall t \in [0, \infty), \quad \frac{d}{dt} F_t = Q(F_t, F_t). \end{aligned}$$

Furthermore F_t is called a *conservative solution* if F_t conserves the mass, momentum and energy, i.e.

$$\forall t \geq 0, \quad \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dF_t(v) = \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dF_0(v).$$

Observe that (1.18) and (1.19) imply the strong continuity of $t \mapsto F_t \in C([0, \infty); \mathcal{B}_2(\mathbb{R}^N))$ and therefore the strong continuity of $t \mapsto Q(F_t, F_t) \in C([0, \infty); \mathcal{B}_0(\mathbb{R}^N))$. Hence the differential equation (1.20) is equivalent to the integral equation

$$(1.21) \quad \forall t \geq 0, \quad F_t = F_0 + \int_0^t Q(F_\tau, F_\tau) d\tau,$$

where the integral is taken in the sense of the Riemann integration or more generally in the sense of the Bochner integration. Recall also that here the derivative $d\mu_t/dt$ and integral $\int_a^b \nu_t dt$ are defined by

$$\left(\frac{d}{dt} \mu_t \right) (E) = \frac{d}{dt} \mu_t(E), \quad \left(\int_a^b \nu_t dt \right) (E) = \int_a^b \nu_t(E) dt$$

for all Borel sets $E \subset \mathbb{R}^N$.

1.3. Recall of the main results of the first part. The following results concerning moment production and uniqueness of conservative solutions which will be used in the present paper are extracted from our previous paper [22]. The following properties (a) and (b) are a kind of “gain of decay” property of the flow stating and quantifying how moments of the solutions become bounded for any positive time even they are not bounded at initial time; the following properties (c)-(d)-(e) concern the stability of the flow.

Theorem 1.2 ([22]). *Let $B(z, \sigma)$ be defined in (1.5) with $\gamma \in (0, 2]$ and with the condition (1.6). Then for any $F_0 \in \mathcal{B}_2^+(\mathbb{R}^N)$ with $\|F_0\|_0 > 0$, there exists a unique conservative measure strong solution F_t of equation (1.1) satisfying $F_t|_{t=0} = F_0$. Moreover this solution satisfies:*

(a) F_t satisfies the moment production estimate:

$$(1.22) \quad \forall t > 0, \quad \forall s \geq 0, \quad \|F_t\|_s \leq \mathcal{K}_s \left(1 + \frac{1}{t}\right)^{\frac{(s-2)^+}{\gamma}}$$

where $(x - y)^+ = \max\{x - y, 0\}$,

$$(1.23) \quad \mathcal{K}_s := \mathcal{K}_s(\|F_0\|_0, \|F_0\|_2) = \|F_0\|_2 \left[2^{s+7} \frac{\|F_0\|_2}{\|F_0\|_0} \left(1 + \frac{1}{16\|F_0\|_2 A_2 \gamma}\right)\right]^{\frac{(s-2)^+}{\gamma}}$$

$$(1.24) \quad A_2 := |\mathbb{S}^{N-2}| \int_0^\pi b(\cos \theta) \sin^N \theta \, d\theta.$$

(b) If $\gamma \in (0, 2)$ or if

$$(1.25) \quad \gamma = 2 \quad \text{and} \quad \exists 1 < p < \infty \quad \text{s.t.} \quad \int_0^\pi [b(\cos \theta)]^p \sin^{N-2} \theta \, d\theta < \infty$$

then F_t satisfies the exponential moment production estimate:

$$(1.26) \quad \forall t > 0, \quad \int_{\mathbb{R}^N} e^{\alpha(t)\langle v \rangle^\gamma} \, dF_t(v) \leq 2\|F_0\|_0$$

where

$$(1.27) \quad \alpha(t) = 2^{-s_0} \frac{\|F_0\|_0}{\|F_0\|_2} \left(1 - e^{-\beta t}\right), \quad \beta = 16\|F_0\|_2 A_2 \gamma > 0,$$

and $1 < s_0 < \infty$ depends only on the function b and γ .

(c) Let G_t be a conservative measure strong solutions of equation (1.1) on the time-interval $[\tau, \infty)$ with an initial datum $G_t|_{t=\tau} = G_\tau \in \mathcal{B}_2^+(\mathbb{R}^N)$ for some $\tau \geq 0$.

Then:

– If $\tau = 0$, then

$$(1.28) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq \Psi_{F_0}(\|F_0 - G_0\|_2) e^{C(1+t)},$$

where

$$(1.29) \quad \Psi_{F_0}(r) = r + r^{1/3} + \int_{|v| > r^{-1/3}} |v|^2 dF_0(v), \quad r > 0, \quad \Psi_{F_0}(0) = 0,$$

and $C = \mathcal{R}(\gamma, A_0, A_2, \|F_0\|_0, \|F_0\|_2)$ is an explicit positive continuous function on $(\mathbb{R}_+^*)^5$.

– If $\tau > 0$, then

$$(1.30) \quad \forall t \in [\tau, \infty), \quad \|F_t - G_t\|_2 \leq \|F_\tau - G_\tau\|_2 e^{C_\tau(t-\tau)},$$

where

$$C_\tau := 4(\mathcal{K}_{2+\gamma} + \|F_0\|_2) \left(1 + \frac{1}{\tau}\right),$$

and $\mathcal{K}_{2+\gamma}$ is defined by (1.23) with $s = 2 + \gamma$.

(d) If F_0 is absolutely continuous with respect to the Lebesgue measure, i.e.

$$dF_0(v) = f_0(v) dv \quad \text{with} \quad 0 \leq f_0 \in L_2^1(\mathbb{R}^N),$$

then F_t is also absolutely continuous with respect to the Lebesgue measure: $dF_t(v) = f_t(v) dv$ for all $t \geq 0$, and f_t is the unique conservative mild solution of equation (1.1) with the initial datum f_0 .

(e) If F_0 is not a single Dirac distribution, then there is a sequence $f_{k,t}$, $k \geq 1$, of conservative mild solutions of equation (1.1) with initial data $0 \leq f_{k,0} \in L_2^1(\mathbb{R}^N)$ satisfying

$$(1.31) \quad \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_{k,0}(v) dv = \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dF_0(v), \quad k = 1, 2, \dots$$

such that

$$(1.32) \quad \forall \varphi \in C_b(\mathbb{R}^N), \quad \forall t \geq 0, \quad \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(v) f_{k,t}(v) dv = \int_{\mathbb{R}^N} \varphi(v) dF_t(v).$$

Besides, the initial data can be chosen of the form $f_{k,0} = I_{n_k}[F_0]$, $k = 1, 2, 3, \dots$ where $\{I_{n_k}[F_0]\}_{k=1}^\infty$ is a subsequence of the Mehler transforms $\{I_n[F_0]\}_{n=1}^\infty$ of F_0 .

Remarks 1.3. (1) In the physical case, $N = 3$ and $0 < \gamma \leq 1$, the moment estimates (1.22) and (1.26) also hold for conservative weak measure solutions of equation (1.1) without angular cutoff (see [22]).

(2) The Mehler transform

$$I_n[F](v) := e^{Nn} \int_{\mathbb{R}^N} M_{1,0,T} \left(e^n \left(v - u - \sqrt{1 - e^{-2n}} (v_* - u) \right) \right) dF(v_*) \in L_2^1(\mathbb{R}^N)$$

of a measure $F \in \mathcal{B}_2^+(\mathbb{R}^N)$ (which is not a single Dirac distribution) will be studied in Section 4 (after introducing other notations) where we shall show that $I_n[F]$ has

a further convenient property:

$$\lim_{n \rightarrow \infty} \|I_n[F] - M\|_2 = \|F - M\|_2$$

and thus it is a useful tool in order to reduce the study of properties of measure solutions to that of L^1 solutions. Here M is the Maxwellian (equilibrium) having the same mass, momentum, and energy as F , see (1.42)-(1.43) below.

1.4. Normalization. In most of the estimates in this paper, we shall try as much as possible to make explicit the dependence on the basic constants in the assumptions. But first let us study the reduction that can be obtained by scaling arguments.

Under the assumption (1.6), it is easily seen that F_t is a measure solution of equation (1.1) with the angular function b if and only if $t \mapsto F_{A_0^{-1}t}$ is a measure solution of equation (1.1) with the scaled angular function $A_0^{-1}b$. Therefore without loss of generality we can assume the normalization

$$(1.33) \quad A_0 = |\mathbb{S}^{N-2}| \int_0^\pi b(\cos \theta) \sin^{N-2} \theta \, d\theta = 1.$$

Next given any $\rho > 0$, $u \in \mathbb{R}^N$ and $T > 0$, we define the bounded positive linear operator $\mathcal{N}_{\rho,u,T}$ on $\mathcal{B}_2(\mathbb{R}^N)$ as follows: for any $F \in \mathcal{B}_2(\mathbb{R}^N)$, there is a unique $\mathcal{N}_{\rho,u,T}(F) \in \mathcal{B}_2(\mathbb{R}^N)$ such that (thanks to Riesz representation theorem),

$$\forall \psi \text{ Borel function s.t. } \sup_{v \in \mathbb{R}^N} |\psi(v)| \langle v \rangle^{-2} < +\infty,$$

$$\int_{\mathbb{R}^N} \psi(v) \, d\mathcal{N}_{\rho,u,T}(F)(v) = \frac{1}{\rho} \int_{\mathbb{R}^N} \psi\left(\frac{v-u}{\sqrt{T}}\right) \, dF(v).$$

We call $\mathcal{N}_{\rho,u,T}$ the *normalization operator* associated with ρ, u, T . The inverse $\mathcal{N}_{\rho,u,T}^{-1}$ of $\mathcal{N}_{\rho,u,T}$ is given by $\mathcal{N}_{\rho,u,T}^{-1} = \mathcal{N}_{1/\rho, -u/\sqrt{T}, 1/T}$, i.e.

$$\int_{\mathbb{R}^N} \psi(v) \, d\mathcal{N}_{\rho,u,T}^{-1}(F)(v) = \rho \int_{\mathbb{R}^N} \psi(\sqrt{T}v + u) \, dF(v).$$

It is easily seen that for every $F \in \mathcal{B}_2(\mathbb{R}^N)$

$$(1.34) \quad \|\mathcal{N}_{\rho,u,T}(F)\|_0 = \frac{1}{\rho} \|F\|_0,$$

$$(1.35) \quad \|\mathcal{N}_{\rho,u,T}(F)\|_2 \leq C_{\rho,|u|,T} \|F\|_2,$$

$$(1.36) \quad \|\mathcal{N}_{\rho,u,T}^{-1}(F)\|_2 \leq C_{1/\rho,|u|/\sqrt{T},1/T} \|F\|_2$$

where

$$(1.37) \quad C_{\rho,|u|,T} = \frac{1}{\rho} \max \left\{ 1 + \frac{|u|^2 + |u|}{T}; \frac{1 + |u|}{T} \right\},$$

$$(1.38) \quad C_{1/\rho,|u|/\sqrt{T},1/T} = \rho \max \left\{ 1 + |u|^2 + \sqrt{T}|u|; T + \sqrt{T}|u| \right\}.$$

We then introduce the subclass $\mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ of $\mathcal{B}_2^+(\mathbb{R}^N)$ by

$$(1.39) \quad \begin{cases} F \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N) \iff F \in \mathcal{B}_2^+(\mathbb{R}^N) \quad \text{and} \\ \int_{\mathbb{R}^N} dF(v) = \rho, \quad \frac{1}{\rho} \int_{\mathbb{R}^N} v dF(v) = u, \quad \frac{1}{N\rho} \int_{\mathbb{R}^N} |v-u|^2 dF(v) = T. \end{cases}$$

In other words, $F \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ means that F has the mass ρ , mean-velocity u , and the kinetic temperature T . It is obvious that F_t conserves mass, momentum, and energy is equivalent to that F_t conserves mass, mean-velocity, and kinetic temperature.

When restricting $\mathcal{N}_{\rho,u,T}$ on $\mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$, it is easily seen that

$$\mathcal{N}_{\rho,u,T} : \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N) \rightarrow \mathcal{B}_{1,0,1}^+(\mathbb{R}^N), \quad \mathcal{N}_{\rho,u,T}^{-1} : \mathcal{B}_{1,0,1}^+(\mathbb{R}^N) \rightarrow \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N).$$

Similarly we define $L_{\rho,u,T}^1(\mathbb{R}^N)$ by

$$(1.40) \quad f \in L_{\rho,u,T}^1(\mathbb{R}^N) \iff \begin{cases} 0 \leq f \in L_2^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} f(v) dv = \rho, \\ \frac{1}{\rho} \int_{\mathbb{R}^N} v f(v) dv = u, \quad \frac{1}{N\rho} \int_{\mathbb{R}^N} |v-u|^2 f(v) dv = T. \end{cases}$$

In this case, the normalization operator $\mathcal{N} = \mathcal{N}_{\rho,u,T} : L_{\rho,u,T}^1(\mathbb{R}^N) \rightarrow L_{1,0,1}^1(\mathbb{R}^N)$ is written directly as

$$(1.41) \quad \mathcal{N}(f)(v) = \frac{T^{N/2}}{\rho} f(\sqrt{T}v + u).$$

Recall that the Maxwellian $M \in L_{\rho,u,T}^1(\mathbb{R}^N)$ is given by

$$(1.42) \quad M(v) := \frac{\rho}{(2\pi T)^{N/2}} \exp\left(-\frac{|v-u|^2}{2T}\right).$$

For notational convenience we shall do not distinguish between a Maxwellian distribution $M \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ and its density function $M \in L_{\rho,u,T}^1(\mathbb{R}^N)$: we write without risk of confusion that

$$(1.43) \quad dM(v) = M(v) dv.$$

Due to the homogeneity of $z \mapsto B(z, \sigma) = |z|^\gamma b(\frac{z}{|z|} \cdot \sigma)$, we have

$$L_B \left[\psi \left(\frac{\cdot - u}{\sqrt{T}} \right) \right] (v, v_*) = T^{\gamma/2} L_B[\psi] \left(\frac{v-u}{\sqrt{T}}, \frac{v_*-u}{\sqrt{T}} \right)$$

and then by Fubini theorem we get (denoting simply $\mathcal{N} = \mathcal{N}_{\rho,u,T}$ when no ambiguity is possible)

$$\forall F \in \mathcal{B}_2^+(\mathbb{R}^N), \quad \mathcal{N}(Q^\pm(F, F)) = \rho T^{\gamma/2} Q^\pm(\mathcal{N}(F), \mathcal{N}(F)).$$

Since \mathcal{N} is linear and bounded, this implies that if F_t is a measure strong solution of equation (1.1) and $c = \rho T^{\gamma/2}$, then

$$\frac{d}{dt} \mathcal{N}(F_t/c) = Q(\mathcal{N}(F_t/c), \mathcal{N}(F_t/c)).$$

This together with (1.34)-(1.36) leads to the following statement:

Proposition 1.4 (Normalization). *Let $B(z, \sigma)$ be defined by (1.5) with $\gamma \in (0, 2]$ and with the condition (1.33). Let $F_0 \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$ with $\rho > 0$, $u \in \mathbb{R}^N$ and $T > 0$, and let F_t be the unique conservative measure strong solution of equation (1.1) with the initial datum F_0 . Let $M \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$ be the Maxwellian defined by (1.42), let $\mathcal{N} := \mathcal{N}_{\rho, u, T}$ be the normalization operator, and let $c = \rho T^{\gamma/2}$. Then:*

(I) *The normalization $t \mapsto \mathcal{N}(F_{t/c})$ is the unique conservative measure strong solution of equation (1.1) with the initial datum $\mathcal{N}(F_0) \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$.*

(II) *For all $t \geq 0$*

$$\begin{cases} \|F_t - M\|_0 = \rho \|\mathcal{N}(F_t) - \mathcal{N}(M)\|_0, \\ \|F_t - M\|_2 \leq C_{1/\rho, |u|/\sqrt{T}, 1/T} \|\mathcal{N}(F_t) - \mathcal{N}(M)\|_2, \\ \|\mathcal{N}(F_t) - \mathcal{N}(M)\|_2 \leq C_{\rho, |u|, T} \|F_t - M\|_2 \end{cases}$$

where $C_{\rho, |u|, T}$ and $C_{1/\rho, |u|/\sqrt{T}, 1/T}$ are given in (1.37)-(1.38).

1.5. Linearized collision operator and spectral gap. For any nonnegative Borel function W on \mathbb{R}^N we define the weighted Lebesgue space $L^p(\mathbb{R}^N, W)$ with $1 \leq p < \infty$ by

$$f \in L^p(\mathbb{R}^N, W) \iff \|f\|_{L^p(W)} := \left(\int_{\mathbb{R}^N} |f(v)|^p W(v) \, dv \right)^{1/p} < \infty.$$

Let $B(z, \sigma)$ as defined in (1.5) with $\gamma \in (0, 2]$ and with b satisfying (1.33). Let M be the Maxwellian with mass $\rho > 0$, mean velocity u and temperature $T > 0$ defined in (1.42), and let

$$L_M : L^2(\mathbb{R}^N, M^{-1}) \rightarrow L^2(\mathbb{R}^N, M^{-1})$$

be the linearized collision operator associated with $B(z, \sigma)$ and $M(v)$, i.e.

$$(1.44) \quad L_M(h)(v) = \iint_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(v - v_*, \sigma) M(v_*) (h' + h'_* - h - h_*) \, d\sigma \, dv_*.$$

It is well-known that the spectrum $\Sigma(L_M)$ of L_M is contained in $(-\infty, 0]$ and has a positive spectral gap $S_{b, \gamma}(\rho, \mu, T) > 0$, i.e.

$$S_{b, \gamma}(\rho, u, T) := \inf \{ \lambda > 0 \mid -\lambda \in \Sigma(L_M) \} > 0.$$

Moreover by simple calculations, one has the following scaling property on this spectral gap

$$S_{b, \gamma}(\rho, u, T) = \rho T^{\gamma/2} S_{b, \gamma}(1, 0, 1).$$

In the spatially homogeneous case, the study of the linearized collision operator goes back to Hilbert [17, 18] who computed the collisional invariant, the linearized operator and its kernel in the hard spheres case, and showed the boundedness and “complete continuity” of its non-local part. Carleman [6] then proved the existence of a spectral gap by using Weyl’s theorem and the compactness of the non-local part proved by Hilbert. Grad [14, 15] then extended these results to the case of hard potentials with cutoff. All these results

are based on non-constructive arguments. The first constructive estimates in the hard spheres case were obtained only recently in [2] (see also [24] for more general interactions, and [26] for a review). Let us also mention the works [32, 3, 4] for the different setting of *Maxwell molecules* where the eigenbasis and eigenvalues can be explicitly computed by Fourier transform methods. Although these techniques do not apply here, the explicit formula computed are an important source of inspiration for dealing with more general physical models.

1.6. Main results. In order to use the results obtained in [25] (see also [31, 27]) for L^1 solutions, we shall need the following additional assumptions for some of our main results:

$$(1.45) \quad \|b\|_{L^\infty} := \sup_{t \in [-1, 1]} b(t) < \infty,$$

$$(1.46) \quad \inf_{t \in [-1, 1]} b(t) > 0.$$

Recall that for the hard sphere model, i.e. $N = 3, \gamma = 1$, and $b \equiv \text{const.} > 0$, the conditions (1.45)-(1.46) are satisfied.

The first main result of this paper is concerned with the upper bound of the rate of convergence to equilibrium when the dimension N is greater or equal to 3.

Theorem 1.5 (Sharp exponential relaxation rate). *Suppose $N \geq 3$ and let $B(z, \sigma)$ be given by (1.5) with $\gamma \in (0, \min\{2, N - 2\}]$ and with b satisfying (1.33), (1.45), and (1.46). Let $\rho > 0, u \in \mathbb{R}^N$ and $T > 0$, and let*

$$\lambda = S_{b, \gamma}(\rho, \mu, T) = S_{b, \gamma}(1, 0, 1) \rho T^{\gamma/2} > 0$$

be the spectral gap for the linearized collision operator (1.44) associated with $B(z, \sigma)$ and the Maxwellian $M \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$. Then for any conservative measure strong solution F_t of the equation (1.1) with $F_0 \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$ we have:

$$\forall t \geq 0, \quad \|F_t - M\|_2 \leq C \|F_0 - M\|_2^{1/2} e^{-\lambda t}$$

where

$$C := C_0 C_{1/\rho, |u|/\sqrt{T}, 1/T} (C_{\rho, |u|, T})^{1/2}$$

with $C_{\rho, |u|, T}$ and $C_{1/\rho, |u|/\sqrt{T}, 1/T}$ given in (1.35) and (1.36), and with some constant $C_0 < \infty$ which depends only on N, γ , and the function b (through the bounds (1.45), (1.46)).

Remark 1.6. (1) It should be noted that, in addition to the exponential rate, Theorem 1.5 also shows that for the hard potentials considered here, the convergence to equilibrium is *grossly determined*, i.e. the speed of the convergence only depends only on the collision kernel and the conserved macroscopic quantities (mass, momentum, energy). This is essentially different from those for non-hard potentials (i.e. $\gamma \leq 0$), see for instance [11].

- (2) Applying Theorem 1.5 to the normal initial data and the Maxwellian $F_0, M \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$, and using $\|F_0 - M\|_2^{1/2} \leq (\|F_0\|_2 + \|M\|_2)^{1/2} = (2(1+N))^{1/2}$ we have

$$\forall t \geq 0, \quad \|F_t - M\|_2 \leq C_0 e^{-\lambda t}, \quad \lambda = S_{b,\gamma}(1, 0, 1)$$

where $C_0 < \infty$ depends only on N, γ , and the function b . Then by normalization (using Proposition 1.4) and the relation $S_{b,\gamma}(\rho, \mu, T) = S_{b,\gamma}(1, 0, 1)\rho T^{\gamma/2}$, we conclude that if $F_0, M \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$, then for the same constant C_0 we have

$$(1.47) \quad \|F_t - M\|_2 \leq C_0 C_{1/\rho, |u|/\sqrt{T}, 1/T} e^{-\lambda t}, \quad t \geq 0; \quad \lambda = S_{b,\gamma}(\rho, u, T).$$

This estimate will be used in proving our next results Corollary 1.9 and Theorem 1.10.

- (3) In general, in this paper we say that a constant C depends only on some parameters x_1, x_2, \dots, x_m , if $C = C(x_1, x_2, \dots, x_m)$ is an explicit continuous function of $(x_1, x_2, \dots, x_m) \in I$ where $I \subset \mathbb{R}^m$ is a possible value range of the parameters (x_1, x_2, \dots, x_m) . In particular this implies that if K is a compact subset of I , then C is bounded on K .

The second main result is concerned with the lower bound of the rate of convergence to equilibrium.

Theorem 1.7 (Lower bound on the relaxation rate). *Let $B(z, \sigma)$ be given by (1.5) with $\gamma \in (0, 2]$ and with the condition (1.33). Let $\rho > 0, u \in \mathbb{R}^N, T > 0$ and let $M \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ be the Maxwellian. Then for any conservative measure strong solution F_t of equation (1.1) with initial data $F_0 \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ we have:*

- (i) *If $0 < \gamma < 2$, then*

$$\forall t \geq 0, \quad \|F_t - M\|_0 \geq (4\rho)^{1-\alpha} \|F_0 - M\|_0^\alpha \exp\left(-\beta t^{\frac{2}{2-\gamma}}\right)$$

where

$$\alpha = \left(\frac{2}{\gamma}\right)^{\frac{\gamma}{2-\gamma}} \quad \text{and} \quad \beta = \left(1 - \frac{\gamma}{2}\right) \left(2^6(N+1)^2 \rho T^{\gamma/2}\right)^{\frac{2}{2-\gamma}}.$$

- (ii) *If $\gamma = 2$, then*

$$\forall t \geq 0, \quad \|F_t - M\|_0 \geq 4\rho \left(\frac{\|F_0 - M\|_0}{4\rho}\right)^{e^\kappa t}$$

with $\kappa = 2^6(N+1)^2 \rho T$.

Remarks 1.8. (1) The lower bounds established with the norm $\|\cdot\|_0$ imply certain lower bounds in terms of the norm $\|\cdot\|_2$. In fact, on one hand, it is obvious that

$\|F_t - M\|_2 \geq \|F_t - M\|_0$. On the other hand, for the standard case $F_0, M \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$, applying the inequalities (5.9) and $\log y \leq \sqrt{y}$ ($y \geq 1$) we have

$$\|F_0 - M\|_0 \geq \left(\frac{1}{4(N+1)} \|F_0 - M\|_2 \right)^2.$$

Then, for the general case $F_0, M \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$, we use part (II) of Proposition 1.4 (normalization) to deduce

$$\|F_0 - M\|_0 \geq \rho \left(\frac{1}{4(N+1)} \cdot \frac{1}{C_{1/\rho,|u|/\sqrt{T},1/T}} \|F_0 - M\|_2 \right)^2.$$

- (2) To our knowledge, Theorem 1.7 is perhaps the first result concerning the lower bounds on the relaxation rate for the hard potentials. Of course –and in spite of the fact that the assumptions of Theorem 1.7 are weaker than those of Theorem 1.5–, these lower bounds are very rough as compared with the corresponding upper bounds in Theorem 1.5. The particular formula in these lower bounds come from limitations of the method we adopted. We conjecture that under the same assumptions on the initial data (i.e. assuming only that F_0 have finite mass, momentum and energy), the lower bounds have the same form $\text{cst.}e^{-\text{cst.}t}$ as the upper bounds. This may be investigated in the future.

Now let us state an important corollary of Theorem 1.5 and Theorem 1.7, which gives a positive answer (see the part (iii) below), for hard potentials, to the question of *eternal solutions* raised in [30, Chapter 1, subsection 2.9] (see also [21]).

Corollary 1.9. *Under the same assumptions on N, γ and $B(z, \sigma)$ as in Theorem 1.5, let $F_0 \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ with $\rho > 0$, $u \in \mathbb{R}^N$ and $T > 0$, and let $M \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ be the Maxwellian. Then we have:*

- (i) *Let $F_t \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ be the unique conservative measure solution of equation (1.1) on $[0, \infty)$ with the initial datum F_0 . If $F_0 \neq M$, then $F_t \neq M$ for all $t \geq 0$. In other words, F_t can not arrive at equilibrium state in finite time unless F_0 is an equilibrium.*
- (ii) *Let $F_t \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ be a conservative backward measure strong solution of equation (1.1) on an interval $(-t_\infty, 0]$ for some $0 < t_\infty \leq \infty$, i.e.*

$$\frac{d}{dt} F_t = Q(F_t, F_t), \quad t \in (-t_\infty, 0].$$

Then if $F_0 \neq M$, then $(-t_\infty, 0]$ must be bounded, and if $F_0 = M$, then $F_t \equiv M$ on $(-t_\infty, 0]$. In particular we have

- (iii) *[Eternal measure solutions are stationary.] If a conservative measure strong solution F_t of equation (1.1) in $\mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ is eternal, i.e. defined for all $t \in \mathbb{R}$, then it has to be stationary and $F_t = M$ for all $t \in \mathbb{R}$.*

The proof of this Corollary is easy and we would like to present it here.

Proof of Corollary 1.9. Part (i) follows simply from the lower bound in Theorem 1.7. Part (iii) follows from part (ii). In fact let F_t be an eternal solution of equation (1.1) as defined in the part (iii) of the statement. Then F_t is also a backward measure strong solution of equation (1.1) on the unbounded time-interval $(-\infty, 0]$. By part (ii) we conclude that $F_0 = M$ and thus $F_t \equiv M$ on $(-\infty, 0]$. Then by the uniqueness of forward solutions we conclude that $F_t = M$ for all $t \in \mathbb{R}$.

To prove part (ii), we use the existence and the uniqueness theorem of conservative measure strong solutions (see Theorem 1.2) to extend the backward solution F_t to the whole interval $(-t_\infty, \infty)$. Fix any $\tau \in (-t_\infty, 0)$. Then $t \mapsto F_{\tau+t}$ is a conservative measure strong solution of equation (1.1) on $[0, \infty)$ with the initial datum F_τ . By using the upper bound of the convergence rate in Theorem 1.5 (see also (1.47)), together with the conservation of mass, momentum, and energy we have (with $\lambda = S_{b,\gamma}(\rho, u, T)$)

$$(1.48) \quad \forall t \geq 0, \quad \|F_{\tau+t} - M\|_0 \leq C e^{-\lambda t},$$

where $C > 0$ only depends on N, γ, b, ρ, u, T . Taking $t = -\tau$ gives

$$\|F_0 - M\|_0 \leq C e^{\lambda \tau}.$$

Thus if $\|F_0 - M\|_0 > 0$, then

$$-\tau \leq \frac{1}{\lambda} \log \left(\frac{C}{\|F_0 - M\|_0} \right) < \infty.$$

Letting $\tau \rightarrow -t_\infty$ leads to

$$t_\infty \leq \frac{1}{\lambda} \log \left(\frac{C}{\|F_0 - M\|_0} \right) < \infty.$$

Next, applying Theorem 1.7, we have for all $t \geq 0$

$$(1.49) \quad \|F_{\tau+t} - M\|_0 \geq (4\rho)^{1-\alpha} \|F_\tau - M\|_0^\alpha e^{-\beta t^{\frac{2}{2-\gamma}}} \quad (\text{if } 0 < \gamma < 2)$$

$$(1.50) \quad \|F_{\tau+t} - M\|_0 \geq 4\rho \left(\frac{\|F_\tau - M\|_0}{4\rho} \right)^{e^{\kappa t}} \quad (\text{if } \gamma = 2).$$

Now suppose $\|F_0 - M\|_0 = 0$. Then taking $t = -\tau$ so that $\|F_{\tau+t} - M\|_0 = 0$ we obtain from (1.49), (1.50) that $\|F_\tau - M\|_0 = 0$. Since $\tau \in (-t_\infty, 0)$ is arbitrary, this shows that $F_t \equiv M$ on $(-t_\infty, 0]$ and concludes the proof. \square

The third main result is concerned with the global-in-time stability of measure strong solutions.

Theorem 1.10. *Let N, γ and $B(z, \sigma)$ satisfy the same assumptions in Theorem 1.5. Let $\rho_0 > 0$, $u_0 \in \mathbb{R}^N$, $T_0 > 0$ and let $M \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$ be the Maxwellian.*

Then for any conservative measure strong solutions F_t, G_t of equation (1.1) with $F_0 \in \mathcal{B}_{\rho_0, u_0, T_0}^+(\mathbb{R}^N)$, there are explicit constants $\eta \in (0, 1)$, $C \in (0, \infty)$ only depending on $N, \gamma, b, \rho_0, u_0, T_0$, such that

$$\sup_{t \geq 0} \|F_t - G_t\|_2 \leq \tilde{\Psi}_{F_0}(\|F_0 - G_0\|_2)$$

where

$$\forall r \geq 0, \quad \tilde{\Psi}_{F_0}(r) := C \left(r + [\Psi_{F_0}(r)]^\eta \right)$$

with $\Psi_{F_0}(r)$ defined in (1.29).

1.7. Previous results and references. Apart from the paper [25] already mentioned concerning the sharp rate of relaxation for L^1 solutions in the case of hard spheres or hard potentials with cutoff, let us mention the many previous works that developed quantitative estimates on the rate of convergence [7, 8, 9, 10, 12, 11, 11, 11]. Let us also mention the recent work [16] obtaining sharp rates of relaxation for $L_v^1 L_x^\infty$ solutions in the spatially inhomogeneous case in the torus.

1.8. Strategy and plan of the paper. The rest of the paper is organized as follows: In Section 2 we give an integral representation for the one-step iterated collision operator $(f, g, h) \mapsto Q^+(f, Q^+(g, h))$ and prove an L^p gain of integrability for this operator. This is a generalization of Abrahamsson's result [1] which is concerned with $N = 3$ and $\gamma = 1$. In order to obtain the required regularities of such iterated collision operators, the technical difficulty is to deal with small values of γ . In that case one needs multi-step iteration of Q^+ . In Section 3 we use iteratively the previous multi-step estimates on Q^+ to give a series of positive decompositions $f_t = f_t^n + h_t^n$ for $t \in [t_0, \infty)$ with $t_0 > 0$, for an L^1 mild solution f_t . In this decomposition the f_t^n are bounded (in $L^\infty(\mathbb{R}^N)$) and regular (they belong at least to $H^1(\mathbb{R}^N)$ for instance) when n is large enough; whereas h_t^n decays in L^1 norm exponentially fast as $t \rightarrow +\infty$. By approximation we then extend such positive decompositions to the measure strong solutions F_t . In Section 4 we first use the results of [25] and those obtained in Section 3 to prove Theorem 1.5 for L^1 mild solutions, and then we use approximation by L^1 mild solutions to complete the proof of Theorem 1.5 for measure solutions. The proof of Theorem 1.7 is given in Section 5. In Section 6 we prove Theorem 1.10 which is an application of Theorem 1.2 and Theorem 1.5.

Throughout this paper, unless otherwise stated, we always assume that $N \geq 2$ as already indicated in equation (1.1).

2. L^p -ESTIMATES OF THE ITERATED GAIN TERM

We introduce the weighted Lebesgue spaces $L_s^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty, 0 \leq s < \infty$ as:

$$\begin{cases} f \in L_s^p(\mathbb{R}^N) \iff \|f\|_{L_s^p} = \left(\int_{\mathbb{R}^N} \langle v \rangle^{ps} |f(v)|^p dv \right)^{1/p} < \infty, & 1 \leq p < \infty \\ f \in L_s^\infty(\mathbb{R}^N) \iff \|f\|_{L_s^\infty} = \sup_{v \in \mathbb{R}^N} \langle v \rangle^s |f(v)| < \infty, & p = \infty. \end{cases}$$

In the case $s = 0$, we denote $L_0^p(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ as usual.

We shall use the following formula of change of variables. For any $\mathbf{n} \in \mathbb{S}^{N-1}$ and ψ nonnegative measurable on \mathbb{S}^{N-1} :

$$(2.1) \quad \int_{\mathbb{S}^{N-1}} \psi(\sigma) d\sigma = \int_{-1}^1 (1-t^2)^{(N-3)/2} \left(\int_{\mathbb{S}^{N-2}(\mathbf{n})} \psi(t\mathbf{n} + \sqrt{1-t^2}\omega) d^\perp\omega \right) dt$$

where $\mathbb{S}^{N-2}(\mathbf{n}) = \{\omega \in \mathbb{S}^{N-1} \mid \omega \perp \mathbf{n}\}$ and $d^\perp\omega$ denotes the sphere measure element of $\mathbb{S}^{N-2}(\mathbf{n})$.

For convenience we rewrite (2.1) as follows:

$$(2.2) \quad \int_{\mathbb{S}^{N-1}} \psi(\sigma) d\sigma = \int_{\mathbb{R}} \zeta(t) \left(\int_{\mathbb{S}^{N-2}(\mathbf{n})} \psi(\sigma_{\mathbf{n}}(t, \omega)) d^\perp\omega \right) dt$$

where

$$(2.3) \quad \forall t \in \mathbb{R}, \quad \zeta(t) := (1-t^2)^{\frac{N-3}{2}} \mathbf{1}_{(-1,1)}(t)$$

$$(2.4) \quad \sigma_{\mathbf{n}}(t, \omega) := \begin{cases} -\mathbf{n} & \text{if } t \leq -1 \\ t\mathbf{n} + \sqrt{1-t^2}\omega & \text{if } t \in (-1, 1) \\ \mathbf{n} & \text{if } t \geq 1. \end{cases}$$

Lemma 2.1. *Suppose $N \geq 3$ and let $B(z, \sigma)$ be given by (1.5) with b satisfying (1.45). Let $f \in L_\gamma^1(\mathbb{R}^N)$ and $g, h \in L_{2\gamma}^1(\mathbb{R}^N)$. Then $Q^+(f, Q^+(g, h)) \in L^1(\mathbb{R}^N)$ with the estimate*

$$(2.5) \quad \|Q^+(f, Q^+(g, h))\|_{L^1} \leq A_0^2 \|f\|_{L_\gamma^1} \|g\|_{L_{2\gamma}^1} \|h\|_{L_{2\gamma}^1}.$$

Moreover we have the following representation: for almost every $v \in \mathbb{R}^N$

$$(2.6) \quad Q^+(f, Q^+(g, h))(v) = \iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} K_B(v, v_*, w, w_*) f(v_*) g(w) h(w_*) dv_* dw dw_*$$

where $K_B : \mathbb{R}^{4N} \rightarrow [0, \infty)$ is defined by

$$(2.7) \quad K_B(v, v_*, w, w_*) := \begin{cases} \frac{2^N}{|v-v_*||w-w_*|} \zeta\left(\mathbf{n} \cdot \frac{2v-(w+w_*)}{|w-w_*|}\right) \times \\ \int_{\mathbb{S}^{N-2}(\mathbf{n})} \frac{B(w-w_*, \sigma) B(w'-v_*, \sigma')}{|w'-v_*|^{N-2}} d^\perp\omega \\ \text{if } |v-v_*||w-w_*| \neq 0, \\ 0 \\ \text{if } |w-w_*||v-v_*| = 0, \end{cases}$$

where the function ζ is given by (2.3), and

$$(2.8) \quad \mathbf{n} := \frac{v - v_*}{|v - v_*|}, \quad w' = \frac{w + w_*}{2} + \frac{|w - w_*|}{2}\sigma, \quad \sigma' = \frac{2v - v_* - w'}{|2v - v_* - w'|}$$

with

$$(2.9) \quad \sigma = \sigma(\omega) = \sigma_{\mathbf{n}}(t, \omega) \quad \text{at} \quad t = \mathbf{n} \cdot \left(\frac{2v - (w + w_*)}{|w - w_*|} \right).$$

Remark 2.2. Inserting the formula (1.6) of $B(z, \sigma)$ into (2.7) gives the more detailed expression of K_B :

$$(2.10) \quad K_B(v, v_*, w, w_*) = \frac{2^N}{|w - w_*|^{1-\gamma} |v - v_*|} \zeta \left(\mathbf{n} \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right) \int_{\mathbb{S}^{N-2}(\mathbf{n})} \frac{b \left(\frac{w - w_*}{|w - w_*|} \cdot \sigma \right) b \left(\frac{w' - v_*}{|w' - v_*|} \cdot \sigma' \right)}{|w' - v_*|^{N-2-\gamma}} d^\perp \omega$$

for $|w - w_*||v - v_*| \neq 0$. Also we note that

$$(2.11) \quad K_B(v, v_*, w, w_*) > 0 \implies |v - v_*||w - w_*| \neq 0 \quad \text{and} \quad \left| \mathbf{n} \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right| < 1$$

which implies, by using the formula (2.9) for $\sigma_{\mathbf{n}}(t, \omega)$ in this case and the value of t , that $(v - w') \cdot (v - v_*) = 0$ and therefore by Pythagoras' theorem

$$(2.12) \quad |w' - v_*| = |w' - v + v - v_*| = \sqrt{|v - v_*|^2 + |v - w'|^2},$$

$$(2.13) \quad |w' - v_*| \geq |v - v_*|.$$

Proof of Lemma 2.1. We shall use the following formula of change of variables (see [30, Chapter 1, Sections 4.5-4.6]): For every nonnegative measurable function ψ on \mathbb{R}^{4N} , one has

$$(2.14) \quad \begin{aligned} \iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} B(v - v_*, \sigma) \psi(v', v'_*, v, v_*) d\sigma dv_* dv \\ = \iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} B(v - v_*, \sigma) \psi(v, v_*, v', v'_*) d\sigma dv_* dv. \end{aligned}$$

We can assume that f, g, h are all nonnegative. Applying (1.10), (2.14), and recalling definition of $L_B[\varphi]$ (see (1.17)) we have, for any nonnegative measurable function φ on \mathbb{R}^N ,

$$\begin{cases} \int_{\mathbb{R}^N} Q^+(f, Q^+(g, h))(v) \varphi(v) dv = \int_{\mathbb{R}^N} f(v_*) \left(\int_{\mathbb{R}^N} Q^+(g, h)(w) L_B[\varphi](w, v_*) dw \right) dv_*, \\ \int_{\mathbb{R}^N} Q^+(g, h)(w) L_B[\varphi](w, v_*) dw = \iint_{\mathbb{R}^N \times \mathbb{R}^N} L_B[L_B[\varphi](\cdot, v_*)](w, w_*) g(w) h(w_*) dw dw_*, \end{cases}$$

and so

$$(2.15) \quad \int_{\mathbb{R}^N} Q^+(f, Q^+(g, h))(v) \varphi(v) \, dv \\ = \iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} L_B [L_B[\varphi](\cdot, v_*)] (w, w_*) f(v_*) g(w) h(w_*) \, dv_* \, dw \, dw_*.$$

Taking $\varphi = 1$ and using the inequalities

$$(2.16) \quad \begin{cases} |w - w_*| \leq \langle w \rangle \langle w_* \rangle, \\ |w' - v_*| \leq \frac{|w + w_*|}{2} + \frac{|w - w_*|}{2} + |v_*| \leq \langle w \rangle \langle w_* \rangle \langle v_* \rangle, \end{cases}$$

we obtain

$$L_B [L_B[1](\cdot, v_*)] (w, w_*) \leq A_0^2 \langle v_* \rangle^\gamma \langle w \rangle^{2\gamma} \langle w_* \rangle^{2\gamma}$$

and it follows from (2.15) that

$$0 \leq Q^+(f, Q^+(g, h)) \in L^1(\mathbb{R}^N)$$

and so (2.5) holds true.

Comparing (2.15) with (2.6), it appears that in order to prove the integral representation (2.6) we only need to prove that the following identity

$$(2.17) \quad \forall 0 \leq \varphi \in C_c(\mathbb{R}^N), \quad L_B [L_B[\varphi](\cdot, v_*)] (w, w_*) = \int_{\mathbb{R}^N} K_B(v, v_*, w, w_*) \varphi(v) \, dv$$

holds for all $w, w_*, v_* \in \mathbb{R}^N$ satisfying

$$(2.18) \quad 0 \neq \left| \frac{w + w_*}{2} - v_* \right| \neq \frac{|w - w_*|}{2} \neq 0.$$

Observe that

$$\begin{cases} L_B [L_B[\varphi](\cdot, v_*)] (w, w_*) = L_B [L_B[\varphi(v_* + \cdot)](\cdot, 0)] (w - v_*, w_* - v_*), \\ K_B(v, v_*, w, w_*) = K_B(v - v_*, 0, w - v_*, w_* - v_*). \end{cases}$$

By replacing respectively $\varphi(v_* + \cdot)$, $w - v_*$ and $w_* - v_*$ with $\varphi(\cdot)$, w and w_* , we can assume without loss of generality that $v_* = 0$. That is, in order to prove (2.17), we only need to prove

$$(2.19) \quad \forall 0 \leq \varphi \in C_c(\mathbb{R}^N), \quad L_B [L_B[\varphi](\cdot, 0)] (w, w_*) = \int_{\mathbb{R}^N} K_B(v, 0, w, w_*) \varphi(v) \, dv.$$

To do this we first assume that $b \in C([-1, 1])$ so that the use of the Dirac distribution is fully justified. We compute

$$(2.20) \quad L_B [L_B[\varphi](\cdot, 0)] (w, w_*) = \int_{\mathbb{S}^{N-1}} B(w - w_*, \sigma) \int_{\mathbb{S}^{N-1}} B(w', \omega) \varphi \left(\frac{w'}{2} + \frac{|w'|}{2} \omega \right) \, d\omega \, d\sigma,$$

and, by using (2.8) and (2.18) with $v_* = 0$, we have $w' \neq 0$ for all $\sigma \in \mathbb{S}^{N-1}$. Let $\delta = \delta(x)$ be the one-dimensional Dirac distribution. Applying the integral representation

$$\forall \psi \in C((0, \infty)), \quad \rho > 0, \quad \psi(\rho) = \frac{2}{\rho^{N-2}} \int_0^\infty r^{N-1} \psi(r) \delta(\rho^2 - r^2) dr$$

to the function

$$\psi(\rho) := \varphi\left(\frac{w'}{2} + \rho\omega\right)$$

and then taking $\rho = |w'|/2$ and changing variable $r\omega = z$ we have

$$\begin{aligned} & \int_{\mathbb{S}^{N-1}} B(w', \omega) \varphi\left(\frac{w'}{2} + \frac{|w'|}{2}\omega\right) d\omega \\ &= 2 \left|\frac{w'}{2}\right|^{-(N-2)} \int_{\mathbb{R}^N} B\left(w', \frac{z}{|z|}\right) \varphi\left(\frac{w'}{2} + z\right) \delta\left(\frac{|w'|^2}{4} - |z|^2\right) dz. \end{aligned}$$

We then use the change of variable $z = v - w'/2$ and Fubini theorem:

$$\begin{aligned} & L_B [L_B[\varphi](\cdot, 0)](w, w_*) = \\ & \int_{\mathbb{R}^N} \varphi(v) \left(\int_{\mathbb{S}^{N-1}} 2 \left|\frac{w'}{2}\right|^{-(N-2)} B(w - w_*, \sigma) B\left(w', \frac{v - w'/2}{|v - w'/2|}\right) \delta\left(\left|\frac{w'}{2}\right|^2 - \left|v - \frac{w'}{2}\right|^2\right) d\sigma \right) dv. \end{aligned}$$

We now assume that $v \neq 0$ and $\mathbf{n} = v/|v|$ satisfy

$$\left| \mathbf{n} \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right| \neq 1, \quad \left| v - \frac{w + w_*}{4} \right| \neq \left| \frac{w - w_*}{4} \right|.$$

We deduce that $|v - w'/2| > 0$ for all $\sigma \in \mathbb{S}^{N-1}$ and we compute using (2.2)-(2.3)-(2.4) with $\mathbf{n} = v/|v|$ that

$$\begin{aligned} & \int_{\mathbb{S}^{N-1}} 2 \left|\frac{w'}{2}\right|^{-(N-2)} B(w - w_*, \sigma) B\left(w', \frac{v - w'/2}{|v - w'/2|}\right) \delta\left(\frac{|w'|^2}{4} - \left|v - \frac{w'}{2}\right|^2\right) d\sigma \\ &= \int_{\mathbb{R}} \zeta(t) \left(\int_{\mathbb{S}^{N-2}(\mathbf{n})} 2 \left|\frac{w'}{2}\right|^{-(N-2)} B(w - w_*, \sigma) B\left(w', \frac{v - w'/2}{|v - w'/2|}\right) \Big|_{\sigma=\sigma_{\mathbf{n}}(t, \omega)} d^\perp \omega \right) \\ & \quad \times \delta\left(\left|v\right| \frac{|w - w_*|}{2} t - v \cdot \left(v - \frac{w + w_*}{2}\right)\right) dt \\ &= \frac{2^N}{|v||w - w_*|} \zeta\left(\frac{v \cdot (2v - (w + w_*))}{|v||w - w_*|}\right) \int_{\mathbb{S}^{N-2}(\mathbf{n})} |w'|^{-(N-2)} B(w - w_*, \sigma) B\left(w', \frac{2v - w'}{|2v - w'|}\right) d^\perp \omega \end{aligned}$$

where σ in the last line is given by (2.9). Thus we obtain

$$(2.21) \quad \begin{aligned} L_B [L_B[\varphi](\cdot, 0)](w, w_*) &= \frac{2^N}{|w - w_*|} \int_{\mathbb{R}^N} \frac{\varphi(v)}{|v|} \zeta\left(\mathbf{n} \cdot \frac{2v - (w + w_*)}{|w - w_*|}\right) \\ & \quad \times \int_{\mathbb{S}^{N-2}(\mathbf{n})} |w'|^{-(N-2)} B(w - w_*, \sigma) B\left(w', \frac{2v - w'}{|2v - w'|}\right) d^\perp \omega dv. \end{aligned}$$

This proves (2.19).

Finally, thanks to $N \geq 3$, we use standard approximation arguments in order to prove that (2.19) still holds without the continuity assumption on the function b . We skip these classical calculations. \square

Lemma 2.3. *Suppose $N \geq 3$ and let $B(z, \sigma)$ be defined in (1.5) with b satisfying (1.45). Let $1 \leq p, q \leq \infty$ satisfy $1/p + 1/q = 1$.*

Then, in the case where we have

$$(2.22) \quad 0 < \gamma < N - 2, \quad \frac{N-1}{N-1-\gamma} \leq p < \frac{N}{N-1-\gamma},$$

the following estimate holds

$$(2.23) \quad \left(\int_{\mathbb{R}^N} [K_B(v, v_*, w, w_*)]^p dv \right)^{1/p} \leq C_p \|b\|_{L^\infty}^2 |w - w_*|^{2\gamma - N/q}.$$

Second, in the case where we have

$$(2.24) \quad \gamma \geq N - 2, \quad 1 \leq p < N,$$

then the following estimate holds:

$$(2.25) \quad \left(\int_{\mathbb{R}^N} [K_B(v, v_*, w, w_*)]^p dv \right)^{1/p} \leq C_p \|b\|_{L^\infty}^2 \langle v_* \rangle^{2\gamma - N/q} \langle w \rangle^{2\gamma - N/q} \langle w_* \rangle^{2\gamma - N/q}.$$

The constants C_p only depend on N, γ, p .

Proof. By replacing the function b with $b/\|b\|_{L^\infty}$ we can assume for notation convenience that $\|b\|_{L^\infty} = 1$. Fix $w, w_*, v_* \in \mathbb{R}^N$. To prove the lemma we may assume that $w \neq w_*$. Recall that $N \geq 3$ implies $\zeta(t) \leq \mathbf{1}_{(-1,1)}(t)$. Then from (2.10)-(2.13) we have

$$(2.26) \quad K_B(v, v_*, w, w_*) \leq \frac{2^N |\mathbb{S}^{N-2}|}{|w - w_*|^{1-\gamma}} \cdot \frac{1}{|v - v_*|^{N-1-\gamma}} \mathbf{1}_{(-1,1)} \left(\mathbf{n} \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right) \quad \text{for } 0 < \gamma < N - 2,$$

whereas for $\gamma \geq N - 2$ we have

$$(2.27) \quad K_B(v, v_*, w, w_*) \leq 2^N |\mathbb{S}^{N-2}| \langle v_* \rangle^{\gamma+2-N} \langle w \rangle^{2\gamma+1-N} \langle w_* \rangle^{2\gamma+1-N} \frac{1}{|v - v_*|} \mathbf{1}_{(-1,1)} \left(\mathbf{n} \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right)$$

where we used $N - 2 \geq 1$ and the inequalities in (2.16).

Let us define

$$J_\beta(w, w_*) = \int_{\mathbb{R}^N} \frac{1}{|v - v_*|^\beta} \mathbf{1}_{(-1,1)} \left(\mathbf{n} \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right) dv.$$

We need to prove that

$$(2.28) \quad J_\beta(w, w_*) \leq |\mathbb{S}^{N-1}| (\langle v_* \rangle \langle w \rangle \langle w_* \rangle)^{N-1-\beta} |w - w_*| \quad \text{when } 0 < \beta < N - 1,$$

$$(2.29) \quad J_\beta(w, w_*) \leq \frac{|\mathbb{S}^{N-1}|}{(N - \beta)} |w - w_*|^{N-\beta} \quad \text{when } N - 1 \leq \beta < N.$$

To do this we use the change of variable

$$v = v_* + \frac{|w - w_*|}{2} r \sigma, \quad dv = \left| \frac{w - w_*}{2} \right|^N r^{N-1} dr d\sigma$$

to compute

$$(2.30) \quad J_\beta(w, w_*) = \left| \frac{w - w_*}{2} \right|^{N-\beta} \int_{\mathbb{S}^{N-1}} I(u \cdot \sigma) d\sigma$$

where

$$I(u \cdot \sigma) = \int_0^\infty r^{N-1-\beta} \mathbf{1}_{\{|r+u \cdot \sigma| < 1\}} dr, \quad u = \frac{2v_* - (w + w_*)}{|w - w_*|}.$$

Let us now estimate $I(u \cdot \sigma)$ uniformly in σ . If $u \cdot \sigma \geq 1$, then $I(u \cdot \sigma) = 0$. Suppose $u \cdot \sigma < 1$. If $0 < \beta < N - 1$, then

$$I(u \cdot \sigma) \leq 2(1 + |u|)^{N-1-\beta} \leq 2 \left(\frac{|\frac{w-w_*}{2}| + |\frac{w+w_*}{2}| + |v_*|}{|\frac{w-w_*}{2}|} \right)^{N-1-\beta}$$

which together with (2.30) and the third inequality in (2.16) gives (2.28). If $N-1 \leq \beta < N$, then $0 < N - \beta \leq 1$ so that

$$I(u \cdot \sigma) \leq \frac{2^{N-\beta}}{N-\beta}$$

which implies (2.29).

Now suppose (2.22) is satisfied. Then using (2.26) and (2.28) with

$$N - 1 \leq \beta = (N - 1 - \gamma)p < N$$

we obtain (2.23):

$$\begin{aligned} \left(\int_{\mathbb{R}^N} [K_B(v, v_*, w, w_*)]^p dv \right)^{1/p} &\leq C_p |w - w_*|^{\gamma-1} (J_{(N-1-\gamma)p}(w, w_*))^{1/p} \\ &\leq C_p |w - w_*|^{\gamma-1} |w - w_*|^{\frac{N-p(N-1-\gamma)}{p}} = C_p |w - w_*|^{2\gamma-N/q}. \end{aligned}$$

Next suppose (2.24) is satisfied. If $N - 1 \leq p < N$, then by (2.27)-(2.29) with $\beta = p$ and using $|w - w_*| \leq \langle w \rangle \langle w_* \rangle$ we have

$$\begin{aligned} \left(\int_{\mathbb{R}^N} [K_B(v, v_*, w, w_*)]^p dv \right)^{1/p} &\leq C_p \langle v_* \rangle^{\gamma+2-N} \langle w \rangle^{2\gamma+1-N} \langle w_* \rangle^{2\gamma+1-N} (J_p(w, w_*))^{1/p} \\ &\leq C_p \langle v_* \rangle^{\gamma+2-N} \langle w \rangle^{2\gamma-N/q} \langle w_* \rangle^{2\gamma-N/q}. \end{aligned}$$

Similarly if $1 \leq p < N - 1$, then using (2.27)-(2.28) with $\beta = p$ we have

$$\left(\int_{\mathbb{R}^N} [K_B(v, v_*, w, w_*)]^p dv \right)^{1/p} \leq C_p \langle v_* \rangle^{\gamma-(N-1)/q} \langle w \rangle^{2\gamma-N/q} \langle w_* \rangle^{2\gamma-N/q}.$$

Since $\gamma \geq N - 2 \geq 1$ and $1 \leq p < N$ imply

$$\max \left\{ \gamma + 2 - N, \gamma - \frac{N-1}{q} \right\} \leq 2\gamma - N/q,$$

it follows that

$$\max \left\{ \langle v_* \rangle^{\gamma+2-N}, \langle v_* \rangle^{\gamma-\frac{N-1}{q}} \right\} \leq \langle v_* \rangle^{2\gamma-N/q}.$$

This concludes the proof of (2.25). \square

Lemma 2.4. *Let $K(v, v_*)$ be a measurable function on $\mathbb{R}^N \times \mathbb{R}^N$ and let*

$$\forall v \in \mathbb{R}^N, \quad T(f)(v) := \int_{\mathbb{R}^N} K(v, v_*) f(v_*) dv_*.$$

Assume that $1 \leq r < \infty$, $0 \leq s < \infty$ and that there is $0 < A < \infty$ such that

$$(2.31) \quad \left(\int_{\mathbb{R}^N} |K(v, v_*)|^r dv \right)^{\frac{1}{r}} \leq A \langle v_* \rangle^s \quad \text{a.e. } v_* \in \mathbb{R}^N.$$

Then

$$(2.32) \quad \forall f \in L_s^1(\mathbb{R}^N), \quad \|T(f)\|_{L^r} \leq A \|f\|_{L_s^1}.$$

Furthermore let $1 \leq p, q \leq \infty$ satisfy

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$$

and assume that there is $0 < B < \infty$ such that

$$(2.33) \quad \left(\int_{\mathbb{R}^N} (|K(v, v_*)| \langle v_* \rangle^{-s})^r dv_* \right)^{\frac{1}{r}} \leq B \quad \text{a.e. } v \in \mathbb{R}^N.$$

Then

$$(2.34) \quad \forall f \in L_s^q(\mathbb{R}^N), \quad \|T(f)\|_{L^p} \leq A^{\frac{r}{p}} B^{1-\frac{r}{p}} \|f\|_{L_s^q}.$$

Proof. Let us define

$$\forall v \in \mathbb{R}^N, \quad T_s(f)(v) := \int_{\mathbb{R}^N} K(v, v_*) \langle v_* \rangle^{-s} f(v_*) dv_*.$$

By Minkowski inequality and (2.31) we have

$$(2.35) \quad \forall f \in L^1(\mathbb{R}^N), \quad \|T_s(f)\|_{L^r} \leq A \|f\|_{L^1}.$$

Also by (2.33) we have

$$(2.36) \quad \forall f \in L^{r'}(\mathbb{R}^N), \quad \|T_s(f)\|_{L^\infty} \leq B \|f\|_{L^{r'}}$$

where $1 \leq r' \leq \infty$ satisfies $1/r + 1/r' = 1$.

Let $\theta = 1 - r/p$. By assumption on p, q, r we have $0 \leq \theta \leq 1$ and

$$\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{\infty}, \quad \frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{r'}.$$

So by (2.35), (2.36) and Riesz-Thorin interpolation theorem (see e.g. [29, Chapter 5]) we have

$$(2.37) \quad \forall f \in L^q(\mathbb{R}^N), \quad \|T_s(f)\|_{L^p} \leq A^{1-\theta} B^\theta \|f\|_{L^q}.$$

Now if we set $(f)_s(v_*) = \langle v_* \rangle^s f(v_*)$, then

$$T(f) = T_s((f)_s), \quad \|(f)_s\|_{L^1} = \|f\|_{L^1_s}, \quad \|(f)_s\|_{L^q} = \|f\|_{L^q_s}$$

and thus (2.32)-(2.34) follow from (2.35)-(2.37). \square

In order to highlight structures of inequalities, we adopt the following notional convention: *Functions f, g, h appeared below are arbitrary members in the classes indicated. Whenever the notation (for instance) $\|f\|_{L^p_s}$ appears, it always means that $f \in L^p_s(\mathbb{R}^N)$ with the norm $\|f\|_{L^p_s}$; and if $\|f\|_{L^p_s}, \|f\|_{L^q_k}$ appear simultaneously, it means that $f \in L^p_s(\mathbb{R}^N) \cap L^q_k(\mathbb{R}^N)$.*

Lemma 2.5. *Let $0 < \alpha < N, 1 \leq \alpha q < N, 1 < p \leq \infty$ with $1/p + 1/q = 1$. Then*

$$\sup_{v \in \mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(v_*)|}{|v - v_*|^\alpha} dv_* \leq 2 \left(\frac{|\mathbb{S}^{N-1}|}{N - \alpha q} \right)^{\frac{\alpha}{N}} \|f\|_{L^1}^{1 - \frac{\alpha q}{N}} \|f\|_{L^p}^{\frac{\alpha q}{N}}.$$

Proof. This follows from Hölder inequality and a minimizing argument. \square

Lemma 2.6. *Suppose $N \geq 3$ and let $B(z, \sigma)$ be defined in (1.5) with the condition (1.45). For any $w, w_* \in \mathbb{R}^N$ with $w \neq w_*$, let*

$$(2.38) \quad \forall v \in \mathbb{R}^N, \quad T_{w, w_*}(f)(v) := \int_{\mathbb{R}^N} K_B(v, v_*, w, w_*) f(v_*) dv_*$$

for nonnegative measurable or certain integrable functions f as indicated below.

(i) *Suppose $0 < \gamma < N - 2$. Let $p_1 = (N - 1)/(N - 1 - \gamma)$. Then*

$$(2.39) \quad \|T_{w, w_*}(f)\|_{L^{p_1}} \leq C_{p_1} \|b\|_{L^\infty}^2 \|f\|_{L^1} \langle w \rangle^\gamma \langle w_* \rangle^\gamma.$$

Also if $1 < p, q < \infty$ satisfy

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{p_1} - 1$$

then

$$(2.40) \quad \|T_{w, w_*}(f)\|_{L^p} \leq C_p \|b\|_{L^\infty}^2 \|f\|_{L^1} \frac{\langle w \rangle^{1 - \frac{p_1}{p}} \langle w_* \rangle^{1 - \frac{p_1}{p}}}{|w - w_*|^{1 - \gamma - \frac{1}{p}}}.$$

And if $1 < p \leq \infty, 1 \leq q < N/(N - 1 - \gamma)$ satisfy $1/p + 1/q = 1$, then

$$(2.41) \quad \|T_{w, w_*}(f)\|_{L^\infty} \leq \frac{C_p \|b\|_{L^\infty}^2}{|w - w_*|^{1 - \gamma}} \|f\|_{L^1}^{1 - \frac{N-1-\gamma}{N} q} \|f\|_{L^p}^{\frac{N-1-\gamma}{N} q}.$$

(ii) *Suppose $\gamma \geq N - 2$. Let $1 < p < N, 1/p + 1/q = 1$. Then*

$$(2.42) \quad \|T_{w, w_*}(f)\|_{L^p} \leq C_p \|b\|_{L^\infty}^2 \|f\|_{L^1_{2\gamma-N/q}} \langle w \rangle^{2\gamma-N/q} \langle w_* \rangle^{2\gamma-N/q}.$$

Furthermore if $N/(N - 1) < p < N$, then

$$(2.43) \quad \|T_{w, w_*}(f)\|_{L^\infty} \leq C_p \|b\|_{L^\infty}^2 \|f\|_{L^1_{\gamma+2-N}}^{1 - \frac{q}{N}} \|f\|_{L^p_{\gamma+2-N}}^{\frac{q}{N}} \langle w \rangle^{2\gamma+1-N} \langle w_* \rangle^{2\gamma+1-N}.$$

The constants $C_p < \infty$ only depend on N, γ, p .

Proof. As in the proof of Lemma 2.3 we can assume that $\|b\|_{L^\infty} = 1$.

Case (i). Suppose $0 < \gamma < N - 2$. By Lemma 2.3 we have

$$(2.44) \quad \left(\int_{\mathbb{R}^N} [K_B(v, v_*, w, w_*)]^{p_1} dv \right)^{1/p_1} \leq C_{p_1} |w - w_*|^{2\gamma - N/q_1}$$

where $q_1 = (p_1)/(p_1 - 1) = (N - 1)/\gamma$. Since

$$0 < 2\gamma - N/q_1 < \gamma \quad \text{and} \quad |w - w_*| \leq \langle w \rangle \langle w_* \rangle,$$

(2.39) follows from (2.32) with $r = p_1, s = 0$. Next recalling (2.13) and the second inequality in (2.16) we see that that

$$K_B(v, v_*, w, w_*) > 0 \quad \implies \quad \sqrt{1 + |v - v_*|^2} \leq \sqrt{2} \langle w \rangle \langle w_* \rangle \langle v_* \rangle$$

so that

$$(2.45) \quad K_B(v, v_*, w, w_*) \langle v_* \rangle^{-1} \leq \sqrt{2} \langle w \rangle \langle w_* \rangle \frac{K_B(v, v_*, w, w_*)}{\sqrt{1 + |v - v_*|^2}}.$$

This together with (2.26) and $(N - 1 - \gamma)p_1 = N - 1$ gives

$$(2.46) \quad \left(\int_{\mathbb{R}^N} (K_B(v, v_*, w, w_*) \langle v_* \rangle^{-1})^{p_1} dv_* \right)^{1/p_1} \\ \leq C_{p_1} \frac{\langle w \rangle \langle w_* \rangle}{|w - w_*|^{1-\gamma}} \left(\int_{\mathbb{R}^N} \frac{dv_*}{(1 + |v - v_*|^2)^{p_1/2} |v - v_*|^{N-1}} \right)^{1/p_1} = C_{p_1} \frac{\langle w \rangle \langle w_* \rangle}{|w - w_*|^{1-\gamma}}.$$

Note that the above integral is finite since $p_1 > 1$. If we set

$$A_{w, w_*} := C_{p_1} |w - w_*|^{2\gamma - N/q_1}, \quad B_{w, w_*} := C_{p_1} \frac{\langle w \rangle \langle w_* \rangle}{|w - w_*|^{1-\gamma}}$$

then we see from (2.44) and (2.46) that Lemma 2.4 can be used for $T_{w, w_*}(f)$ with $r = p_1$ and $s = 1$, and thus for all $f \in L_1^q(\mathbb{R}^N)$

$$\|T_{w, w_*}(f)\|_{L^p} \leq (A_{w, w_*})^{\frac{p_1}{p}} (B_{w, w_*})^{1 - \frac{p_1}{p}} \|f\|_{L_1^q} = C_p \frac{\langle w \rangle^{1 - \frac{p_1}{p}} \langle w_* \rangle^{1 - \frac{p_1}{p}}}{|w - w_*|^{1 - \gamma - \frac{1}{p}}} \|f\|_{L_1^q}$$

where we have computed (using the definitions of p_1, q_1)

$$\frac{p_1}{p} \left(2\gamma - \frac{N}{q_1} \right) - \left(1 - \frac{p_1}{p} \right) (1 - \gamma) = \frac{1}{p} + \gamma - 1.$$

This proves (2.40). To prove (2.41) we use (2.26) to get

$$|T_{w, w_*}(f)(v)| \leq \frac{2^N |\mathbb{S}^{N-2}|}{|w - w_*|^{1-\gamma}} \int_{\mathbb{R}^N} \frac{|f(v_*)| dv_*}{|v - v_*|^{N-1-\gamma}}.$$

Since $1 \leq (N - 1 - \gamma)q < N$, it follows from Lemma 2.5 that

$$\|T_{w, w_*}(f)\|_{L^\infty} \leq \frac{C_p}{|w - w_*|^{1-\gamma}} \|f\|_{L^1}^{1 - \frac{N-1-\gamma}{N}q} \|f\|_{L^p}^{\frac{N-1-\gamma}{N}q}.$$

Case (ii). Suppose $\gamma \geq N - 2$. In this case we recall the inequality (2.25). Let $1 < p < N$ and $1/p + 1/q = 1$. Then applying Lemma 2.4 to $T_{w,w_*}(f)$ with $r = p, s = 2\gamma - N/q$ gives (2.42). Finally suppose $N/(N - 1) < p < N$. Recalling (2.38) and using (2.27) we have

$$|T_{w,w_*}(f)(v)| \leq C_{N,\gamma} \langle w \rangle^{2\gamma+1-N} \langle w_* \rangle^{2\gamma+1-N} \int_{\mathbb{R}^N} \frac{\langle v_* \rangle^{\gamma+2-N}}{|v - v_*|} |f(v_*)| dv_*.$$

Since $q = p/(p - 1) < N$, it follows from Lemma 2.5 that

$$\int_{\mathbb{R}^N} \frac{\langle v_* \rangle^{\gamma+2-N}}{|v - v_*|} |f(v_*)| dv_* \leq C_p \|f\|_{L_{\gamma+2-N}^1}^{1-\frac{q}{N}} \|f\|_{L_{\gamma+2-N}^p}^{\frac{q}{N}}.$$

This proves (2.43). \square

Let f, g, h be nonnegative measurable functions on \mathbb{R}^N . Define for any $s \geq 0$

$$(f)_s(v) := \langle v \rangle^s f(v).$$

Then applying the inequality $\langle v \rangle \leq \langle v' \rangle \langle v_*' \rangle$ we have

$$\begin{cases} (Q^+(f, g))_s \leq Q^+((f)_s, (g)_s), \\ (Q^+(f, Q^+(g, h)))_s \leq Q^+((f)_s, Q^+((g)_s, (h)_s)), \end{cases}$$

and so on and so forth. Consequently we have for all $s \geq 0$ and $1 \leq p \leq \infty$:

$$(2.47) \quad \|Q^+(f, Q^+(g, h))\|_{L_s^p} \leq \|Q^+((f)_s, Q^+((g)_s, (h)_s))\|_{L_s^p}$$

provided that the right hand side makes sense.

Now we are going to prove the L^p and L^∞ boundedness of the iterated operator $Q^+(f, Q^+(g, h))$. Let

$$(2.48) \quad N_\gamma = \begin{cases} \left[\frac{N-1}{\gamma} \right] & \text{if } 0 < \gamma < N - 2 \\ 1 & \text{if } \gamma \geq N - 2 \end{cases}$$

where $[x]$ denotes the largest integer not exceeding x .

Theorem 2.7. *Suppose $N \geq 3$ and let $B(z, \sigma)$ be defined by (1.5) with the condition (1.45). Given any $s \geq 0$ we have:*

(i) *Suppose $0 < \gamma < N - 2$. Let N_γ be defined in (2.48) and let*

$$p_n = \frac{N - 1}{N - 1 - n\gamma} \in (1, \infty], \quad n = 1, 2, \dots, N_\gamma.$$

Then

(a) *For all $n = 1, 2, \dots, N_\gamma$, we have*

$$(2.49) \quad \|Q^+(f, Q^+(g, h))\|_{L_s^{p_1}} \leq C_{p_1} \|b\|_{L^\infty}^2 \|f\|_{L_s^1} \|g\|_{L_{s+\gamma}^1} \|h\|_{L_{s+\gamma}^1}.$$

(b) If $1 \leq n \leq N_\gamma - 1$, then

$$(2.50) \quad \left\| Q^+(f, Q^+(g, h)) \right\|_{L_s^{p_{n+1}}} \leq C_{p_{n+1}} \|b\|_{L^\infty}^2 \|f\|_{L_{s+1}^{p_n}} \|g\|_{L_{s+\gamma_1}^1} \|h\|_{L_{s+\gamma_1}^1}^{1-\theta_n} \|h\|_{L_{s+\gamma_1}^{p_n}}^{\theta_n}$$

where

$$(2.51) \quad \gamma_1 = \max\{\gamma, 1\}, \quad \theta_n = \frac{1}{N} \left(1 - \frac{N-2}{n}\right)^+, \quad n \geq 1.$$

(c) Finally if $n = N_\gamma$, then

$$(2.52) \quad \left\| Q^+(f, Q^+(g, h)) \right\|_{L_s^\infty} \leq C_\infty \|b\|_{L^\infty}^2 \|f\|_{L_s^1}^{1-\alpha_1} \|f\|_{L_s^{p_{N_\gamma}}}^{\alpha_1} \|g\|_{L_{s+\gamma_*}^1} \|h\|_{L_{s+\gamma_*}^1}^{1-\alpha_2} \|h\|_{L_s^{p_{N_\gamma}}}^{\alpha_2}$$

where $\gamma_* = (\gamma - 1)^+$ and

$$(2.53) \quad 0 < \alpha_1 := \left(\frac{N-1}{\gamma N_\gamma}\right) \left(\frac{N-1-\gamma}{N}\right) < 1, \quad 0 \leq \alpha_2 := \left(\frac{N-1}{\gamma N_\gamma}\right) \left(\frac{(1-\gamma)^+}{N}\right) < 1.$$

(ii) Suppose $\gamma \geq N - 2$. Let $1 < p < N$, $1/p + 1/q = 1$. Then

$$(2.54) \quad \left\| Q^+(f, Q^+(g, h)) \right\|_{L_s^p} \leq C_p \|b\|_{L^\infty}^2 \|f\|_{L_{s+2\gamma-N/q}^1} \|g\|_{L_{s+2\gamma-N/q}^1} \|h\|_{L_{s+2\gamma-N/q}^1}.$$

Furthermore if $N/(N-1) < p < N$, then

$$(2.55) \quad \left\| Q^+(f, Q^+(g, h)) \right\|_{L_s^\infty} \leq C_p \|b\|_{L^\infty}^2 \|f\|_{L_{s+\gamma+2-N}^1}^{1-\frac{q}{N}} \|f\|_{L_{s+\gamma+2-N}^p}^{\frac{q}{N}} \|g\|_{L_{s+2\gamma+1-N}^1} \|h\|_{L_{s+2\gamma+1-N}^1}.$$

All the constants $C_p < \infty$ only depend on N, γ, p .

Remark 2.8. Observe that in the case $\gamma \geq N - 2$, the iterated operator maps (forgetting about the weights) $L^1 \times L^1 \times L^1$ to L^p for $1 < p < N$. This can be recovered heuristically from Lions' theorem in [19] and [27] that states that Q^+ maps $L^1 \times H^s$ to $H^{s+(N-1)/2}$ for $s \in \mathbb{R}$. Then L^1 is contained in $H^{-N/2-0}$ and applying twice Lions' theorem one gets that the iterated operator maps $L^1 \times L^1 \times L^1$ to $H^{2(N-1)/2-N/2-0} = H^{N/2-1-0}$. And the Sobolev embedding for the space $H^{N/2-1}$ is precisely L^N .

Proof. The proof is a direct application of the inequalities obtained in Lemma 2.5 and Lemma 2.6. We can assume that f, g, h are all nonnegative. And because of (2.47), we need only to prove the theorem for $s = 0$. By the integral representation of $Q^+(f, Q^+(g, h))$ and the definition of T_{w, w_*} we have

$$Q^+(f, Q^+(g, h))(v) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} T_{w, w_*}(f)(v) g(w) h(w_*) dw dw_*$$

and therefore

$$(2.56) \quad \left\| Q^+(f, Q^+(g, h)) \right\|_{L^p} \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \|T_{w, w_*}(f)\|_{L^p} g(w) h(w_*) dw dw_*$$

for all $1 \leq p \leq \infty$.

Case (i). Suppose $0 < \gamma < N - 2$. By (2.39) we have

$$\|T_{w, w_*}(f)\|_{L^{p_1}} \leq C_{p_1} \|b\|_{L^\infty}^2 \|f\|_{L^1} \langle w \rangle^\gamma \langle w_* \rangle^\gamma$$

and so by (2.56)

$$\|Q^+(f, Q^+(g, h))\|_{L^{p_1}} \leq C_{p_1} \|b\|_{L^\infty}^2 \|f\|_{L^1} \|g\|_{L_1^\gamma} \|h\|_{L_1^\gamma}.$$

This proves (2.49) (with $s = 0$).

- Suppose $1 \leq n \leq N_\gamma - 1$. By definition of p_n we have

$$\frac{1}{p_{n+1}} = \frac{1}{p_n} + \frac{1}{p_1} - 1, \quad n = 1, 2, \dots, N_\gamma - 1.$$

By (2.40) and (2.56) we have

$$(2.57) \quad \left\{ \begin{array}{l} \|T_{w, w_*}(f)\|_{L^{p_{n+1}}} \leq C_{p_{n+1}} \|b\|_{L^\infty}^2 \frac{\langle w \rangle^{1-\frac{p_1}{p_{n+1}}} \langle w_* \rangle^{1-\frac{p_1}{p_{n+1}}}}{|w - w_*|^{1-\gamma-\frac{1}{p_{n+1}}}} \|f\|_{L_1^{p_n}}, \\ \|Q^+(f, Q^+(g, h))\|_{L^{p_{n+1}}} \\ \leq C_{p_{n+1}} \|b\|_{L^\infty}^2 \|f\|_{L_1^{p_n}} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\langle w \rangle^{1-\frac{p_1}{p_{n+1}}} \langle w_* \rangle^{1-\frac{p_1}{p_{n+1}}}}{|w - w_*|^{1-\gamma-\frac{1}{p_{n+1}}}} g(w) h(w_*) dw dw_*. \end{array} \right.$$

Let $q_n \geq 1$ be defined by

$$\frac{1}{q_n} + \frac{1}{p_n} = 1, \quad \text{i.e.} \quad q_n = \frac{N-1}{n\gamma}.$$

We have

$$1 - \gamma - \frac{1}{p_{n+1}} = \frac{n+2-N}{N-1} \gamma, \quad q_n \left(1 - \gamma - \frac{1}{p_{n+1}}\right) = 1 - \frac{N-2}{n}.$$

If $n \leq N-2$, then $1 - \gamma - 1/p_{n+1} \leq 0$ and using $|w - w_*| \leq \langle w \rangle \langle w_* \rangle$ we have

$$\frac{\langle w \rangle^{1-\frac{p_1}{p_{n+1}}} \langle w_* \rangle^{1-\frac{p_1}{p_{n+1}}}}{|w - w_*|^{1-\gamma-\frac{1}{p_{n+1}}}} \leq \langle w \rangle^\gamma \langle w_* \rangle^\gamma$$

and so by (2.57) we have

$$\|Q^+(f, Q^+(g, h))\|_{L^{p_{n+1}}} \leq C_{p_{n+1}} \|b\|_{L^\infty}^2 \|f\|_{L_1^{p_n}} \|g\|_{L_1^\gamma} \|h\|_{L_1^\gamma}.$$

If $n > N-2$, then $0 < q_n(1 - \gamma - 1/p_{n+1}) < 1$ so that applying Lemma 2.5 (with $q = q_n$, $\alpha = 1 - \gamma - 1/p_{n+1}$) and recalling the definition of θ_n we have

$$(2.58) \quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\langle w \rangle^{1-\frac{p_1}{p_{n+1}}} \langle w_* \rangle^{1-\frac{p_1}{p_{n+1}}}}{|w - w_*|^{1-\gamma-\frac{1}{p_{n+1}}}} g(w) h(w_*) dw dw_* \leq C_{p_{n+1}} \|g\|_{L_1^1} (\|h\|_{L_1^1})^{1-\theta_n} (\|h\|_{L_1^{p_n}})^{\theta_n}$$

and thus (2.50) (with $s = 0$) follows from (2.57) and (2.58).

- Now let $n = N_\gamma$. Let us recall that

$$q_{N_\gamma} = \frac{N-1}{N_\gamma \gamma}, \quad N_\gamma > \frac{N-1}{\gamma} - 1 > 0,$$

hence

$$(2.59) \quad q_{N_\gamma} (N-1-\gamma) < N-1.$$

Using (2.41) and (2.56) (for the L^∞ norm) we have

$$(2.60) \quad \begin{cases} \|T_{w,w_*}(f)\|_{L^\infty} \leq \frac{C_\infty \|b\|_{L^\infty}^2}{|w-w_*|^{1-\gamma}} \|f\|_{L^1}^{1-\frac{N-1-\gamma}{N}q_{N\gamma}} \|f\|_{L^{p_{N\gamma}}}^{\frac{N-1-\gamma}{N}q_{N\gamma}}, \\ \|Q^+(f, Q^+(g, h))\|_{L^\infty} \\ \leq C_\infty \|b\|_{L^\infty}^2 \|f\|_{L^1}^{1-\frac{N-1-\gamma}{N}q_{N\gamma}} (\|f\|_{L^{p_{N\gamma}}})^{\frac{N-1-\gamma}{N}q_{N\gamma}} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(w)h(w_*)}{|w-w_*|^{1-\gamma}} dw dw_*. \end{cases}$$

If $0 < \gamma < 1$, then from (2.59) and $N \geq 3$ we have $0 < q_{N\gamma}(1-\gamma) < N-1$ so that using Lemma 2.4 gives

$$(2.61) \quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(w)h(w_*)}{|w-w_*|^{1-\gamma}} dw dw_* \leq C_\gamma \|g\|_{L^1} \|h\|_{L^1}^{1-\frac{1-\gamma}{N}q_{N\gamma}} \|h\|_{L^{p_{N\gamma}}}^{\frac{1-\gamma}{N}q_{N\gamma}}.$$

If $1 \leq \gamma < N-2$, then $|w-w_*|^{\gamma-1} \leq \langle w \rangle^{\gamma-1} \langle w_* \rangle^{\gamma-1}$ and so

$$(2.62) \quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(w)h(w_*)}{|w-w_*|^{1-\gamma}} dw dw_* \leq \|g\|_{L^1_{\gamma-1}} \|h\|_{L^1_{\gamma-1}}.$$

Thus (2.52) (with $s=0$) follows from (2.60), (2.61) and (2.62).

Case (ii). Suppose $\gamma \geq N-2$ and let $1 < p < N$, $1/p + 1/q = 1$. Then (2.54) (with $s=0$) follows from (2.56) and (2.42). Furthermore if $N/(N-1) < p < N$, then (2.55) (with $s=0$) follows from (2.56) (for the L^∞ norm) and (2.43). \square

3. ITERATION AND DECOMPOSITION OF SOLUTIONS

We begin by the study of the process of iteration of the collision operator and decomposition of solutions though the following lemma. Roughly speaking the strategy of the decomposition is the following. We use the Duhamel representation formula to decompose the flow associated with the equation into two parts, one of which is more regular than the initial datum, while the amplitude of the other decreases exponentially fast with time, and we repeat this process in order to increase the smoothness, starting each time a new flow having the smoother part of the previous solution as initial datum. Each time we start a new flow, we depart from the true solution, since the initial datum is not the real solution, and we keep track of this error through a Lipschitz stability estimate. Finally the times of the decomposition are chosen in such a way that the time-decay of the non-smooth parts dominates the time-growth in these Lipschitz stability errors.

Lemma 3.1. *Let $B(z, \sigma)$ be defined in (1.5) with $\gamma \in (0, 2]$ and with the condition (1.33). Let $f_t \in L^1_2(\mathbb{R}^N)$ be a mild solution of equation (1.1). Let us define*

$$(3.1) \quad \forall s, t \geq 0, \quad E_s^t(v) := \exp\left(-\int_s^t \int_{\mathbb{R}^N} |v-v_*|^\gamma f_\tau(v_*) dv_* d\tau\right).$$

Given any $t_0 \geq 0$ we also define for all $t \geq t_0$

$$(3.2) \quad f_t^0(v) = f_t(v), \quad h_t^0(v) = 0,$$

$$(3.3) \quad f_t^n(v) = \int_{t_0}^t E_{t_1}^t(v) \int_{t_0}^{t_1} Q^+(f_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1})E_{t_2}^{t_1})(v) dt_2 dt_1,$$

$$(3.4) \quad h_t^1(v) = f_{t_0}(v)E_{t_0}^t(v) + \int_{t_0}^t Q^+(f_{t_1}, f_{t_0}E_{t_0}^{t_1})(v)E_{t_1}^t(v) dt_1,$$

$$(3.5) \quad h_t^n(v) = h_t^1(v) + \int_{t_0}^t E_{t_1}^t(v) \int_{t_0}^{t_1} Q^+(f_{t_1}, Q^+(f_{t_2}, h_{t_2}^{n-1})E_{t_2}^{t_1})(v) dt_2 dt_1 \\ + \int_{t_0}^t E_{t_1}^t(v) \int_{t_0}^{t_1} Q^+(f_{t_1}, Q^+(f_{t_2}^{n-1}, h_{t_2}^{n-1})E_{t_2}^{t_1})(v) dt_2 dt_1 \\ + \int_{t_0}^t E_{t_1}^t(v) \int_{t_0}^{t_1} Q^+(h_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1})E_{t_2}^{t_1})(v) dt_2 dt_1$$

for $n = 1, 2, 3, \dots$

Then $f_t^n \geq 0$, $h_t^n \geq 0$ and there is a null set $Z \subset \mathbb{R}^N$ which is independent of t and n such that for all $v \in \mathbb{R}^N \setminus Z$

$$(3.6) \quad \forall t \in [t_0, \infty), \quad n = 1, 2, 3, \dots, \quad f_t(v) = f_t^n(v) + h_t^n(v).$$

Proof. In the following we denote by $Z_0, Z_1, Z_2, \dots \subset \mathbb{R}^N$ some null sets (i.e. $\text{meas}(Z_n) = 0$) which are independent of the time variable t . The decomposition (3.6) is based on the Duhamel representation formula for the solution f_t : for all $v \in \mathbb{R}^N \setminus Z_0$

$$(3.7) \quad \forall t \geq t_0, \quad f_t(v) = f_{t_0}(v)E_{t_0}^t(v) + \int_{t_0}^t Q^+(f_{t_1}, f_{t_1})(v)E_{t_1}^t(v) dt_1.$$

Here we note that in the definition of $E_s^t(v)$ we have used the assumption (1.33), i.e., $A_0 = 1$. Applying (3.7) to f_t at time $t = t_1$ and inserting it into the second argument of $Q^+(f_{t_1}, f_{t_1})$ we obtain for all $t \geq t_0$ and all $v \in \mathbb{R}^N \setminus Z_1$

$$(3.8) \quad f_t(v) = h_t^1(v) + \int_{t_0}^t E_{t_1}^t(v) \int_{t_0}^{t_1} Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2})E_{t_2}^{t_1})(v) dt_2 dt_1.$$

That is, we have the decomposition

$$\forall t \geq t_0, \quad \forall v \in \mathbb{R}^N \setminus Z_1, \quad f_t(v) = h_t^1(v) + f_t^1(v).$$

Suppose for some $n \geq 1$, the decomposition $f_t(v) = h_t^n(v) + f_t^n(v)$ holds for all $t \geq t_0$ and all $v \in \mathbb{R}^N \setminus Z_n$. Let us insert $f_{t_2} = h_{t_2}^n + f_{t_2}^n$ and $f_{t_1} = h_{t_1}^n + f_{t_1}^n$ into $Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2})E_{t_2}^{t_1})$ in the following way:

$$\begin{cases} Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2})E_{t_2}^{t_1}) = Q^+(f_{t_1}, Q^+(f_{t_2}, h_{t_2}^n)E_{t_2}^{t_1}) + Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2}^n)E_{t_2}^{t_1}), \\ Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2}^n)E_{t_2}^{t_1}) = Q^+(f_{t_1}, Q^+(f_{t_2}, h_{t_2}^n)E_{t_2}^{t_1}) + Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2}^n)E_{t_2}^{t_1}), \\ Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2}^n)E_{t_2}^{t_1}) = Q^+(h_{t_1}^n, Q^+(f_{t_2}, f_{t_2}^n)E_{t_2}^{t_1}) + Q^+(f_{t_1}^n, Q^+(f_{t_2}, f_{t_2}^n)E_{t_2}^{t_1}). \end{cases}$$

Then

$$\begin{aligned} Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2})E_{t_2}^{t_1}) &= Q^+(f_{t_1}, Q^+(f_{t_2}, h_{t_2}^n)E_{t_2}^{t_1}) + Q^+(f_{t_1}, Q^+(f_{t_2}^n, h_{t_2}^n)E_{t_2}^{t_1}) \\ &\quad + Q^+(h_{t_1}^n, Q^+(f_{t_2}^n, f_{t_2}^n)E_{t_2}^{t_1}) + Q^+(f_{t_1}^n, Q^+(f_{t_2}^n, f_{t_2}^n)E_{t_2}^{t_1}). \end{aligned}$$

Inserting this into (3.8) yields

$$\forall t \geq t_0, \quad \forall v \in \mathbb{R}^N \setminus Z_{n+1}, \quad f_t(v) = h_t^{n+1}(v) + f_t^{n+1}(v).$$

This proves the lemma by induction, and the null set Z can be chosen $Z = \bigcup_{n=1}^{\infty} Z_n$. \square

Note that the above iterations make sense since $f_t^0 = f_t \geq 0$ and, by induction, all f_t^n are nonnegative. Note also that if $t_0 > 0$, then, by moment production estimate (Theorem 1.2), we have

$$\forall t \geq t_0, \quad f_t \in \bigcap_{s \geq 0} L_s^1(\mathbb{R}^N).$$

This enables us to use moment estimates for $Q^+(f, Q^+(g, h))$:

$$(3.9) \quad \forall s \geq 0, \quad \|Q^+(f, Q^+(g, h))\|_{L_s^1} \leq \|f\|_{L_{s+\gamma}^1} \|g\|_{L_{s+2\gamma}^1} \|h\|_{L_{s+2\gamma}^1}.$$

Before we can show the regularity property of f_t^n and the exponential decay (in norm) of h_t^n we need further preparation.

Recall that the Sobolev space $H^s(\mathbb{R}^N)$ ($s > 0$) is a subspace of $f \in L^2(\mathbb{R}^N)$ defined by

$$f \in H^s(\mathbb{R}^N) \iff \|f\|_{H^s} = \|\widehat{f}\|_{L_s^2} = \left(\int_{\mathbb{R}^N} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty$$

where $\widehat{f}(\xi)$ is the Fourier transform of f :

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^N} f(v) e^{-i\xi \cdot v} dv.$$

As usual we denote the *homogeneous* seminorm as

$$\|f\|_{\dot{H}^s} = \|\widehat{f}\|_{L_s^2} = \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

The norm and seminorm are related by

$$(3.10) \quad \|f\|_{\dot{H}^s} \leq \|f\|_{H^s} \leq (2\pi)^{N/2} 2^{s/2} \|f\|_{L^2} + 2^{s/2} \|f\|_{\dot{H}^s}.$$

It is easily proved (see [20, pp. 416-417]) that if the angular function b satisfies

$$(3.11) \quad \|b\|_{L^2}^2 := |\mathbb{S}^{N-2}| \int_0^\pi b(\cos \theta)^2 \sin^{N-2} \theta d\theta < \infty$$

then $Q^+ : L_{N+\gamma}^2(\mathbb{R}^N) \times L_{N+\gamma}^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is bounded with

$$(3.12) \quad \|Q^+(f, g)\|_{L^2} \leq C \|b\|_{L^2} \|f\|_{L_{N+\gamma}^2} \|g\|_{L_{N+\gamma}^2}$$

where $C < \infty$ only depends on N and γ . This together with (3.10), (3.12) and the estimate of $\|Q^+(f, g)\|_{\dot{H}^s}$ obtained in [5, 20] for $s = (N-1)/2$ leads to the following lemma.

Lemma 3.2 ([5, 20]). *Let $B(z, \sigma)$ be defined in (1.5) with the condition (3.11). Then $Q^+ : L_{N+\gamma}^2(\mathbb{R}^N) \times L_{N+\gamma}^2(\mathbb{R}^N) \rightarrow H^{\frac{N-1}{2}}(\mathbb{R}^N)$ is bounded with the estimate*

$$\|Q^+(f, g)\|_{H^{\frac{N-1}{2}}} \leq C \|b\|_{L^2} \|f\|_{L_{N+\gamma}^2} \|g\|_{L_{N+\gamma}^2}$$

where $C < \infty$ only depends on N, γ .

The following lemma will be useful to prove the H^1 -regularity of f_t^n in the decomposition $f_t = f_t^n + h_t^n$.

Lemma 3.3. *Let $B(z, \sigma)$ be defined in (1.5) with $\gamma \in (0, 2]$ and with the condition (1.33). Let $F_t \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$ be a conservative measure strong solution of equation (1.1). Then for any $t_0 > 0$ we have*

$$(3.13) \quad \forall t \geq t_0, \forall v \in \mathbb{R}^N, \quad \int_{\mathbb{R}^N} |v - v_*|^\gamma dF_t(v_*) \geq a \langle v \rangle^\gamma \geq a$$

where

$$(3.14) \quad a := \left[\mathcal{K}_4 \left(1 + \max\{1, 1/t_0\} \right)^{2/\gamma} \right]^{-(2-\gamma)/2}$$

and $\mathcal{K}_4 = \mathcal{K}_4(1, 1 + N) (> 1)$ is the constant in (1.23). In particular if $t_0 \geq 1$, then a is independent of t_0 .

Moreover for any $t_0 \leq t_1 \leq t < \infty$, let $E_{t_1}^t(v)$ be defined as in (3.1) for the measure F_τ , i.e.

$$E_{t_1}^t(v) := \exp \left(- \int_{t_1}^t \int_{\mathbb{R}^N} |v - v_*|^\gamma dF_\tau(v_*) d\tau \right).$$

Then for any $f \in L^\infty(\mathbb{R}^N) \cap L_2^1(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ we have $f E_{t_1}^t \in H^1(\mathbb{R}^N)$ and

$$(3.15) \quad \|f E_{t_1}^t\|_{H^1(\mathbb{R}^N)} \leq C \left[\|f\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L_2^1(\mathbb{R}^N)} + \|f\|_{H^1(\mathbb{R}^N)} \right] e^{-a(t-t_1)} (1 + t - t_1)$$

where C only depends on N, γ .

Proof. Let

$$L_s(F_t)(v) := \int_{\mathbb{R}^N} |v - v_*|^s dF_t(v_*).$$

By conservation of mass, momentum and energy, we have $L_2(F_t)(v) = N + |v|^2 > \langle v \rangle^2$ and so the inequality (3.13) is obvious when $\gamma = 2$. Suppose $0 < \gamma < 2$. In this case we use the inequality $|v - v_*| \leq \langle v \rangle \langle v_* \rangle$ and the moment production estimate (1.22) with $s = 4$ to get

$$L_4(F_t)(v) \leq \langle v \rangle^4 \|F_t\|_4 \leq \langle v \rangle^4 \mathcal{K}_4 (1 + 1/t_0)^{\frac{2}{\gamma}}.$$

Then from the decomposition $2 = \gamma \cdot \frac{2}{4-\gamma} + 4 \cdot \frac{2-\gamma}{4-\gamma}$ and using Hölder inequality we have

$$\begin{aligned} \langle v \rangle^2 &< L_2(F_t)(v) \leq [L_\gamma(F_t)(v)]^{\frac{2}{4-\gamma}} [L_4(F_t)(v)]^{\frac{2-\gamma}{4-\gamma}} \\ &\leq [L_\gamma(F_t)(v)]^{\frac{2}{4-\gamma}} \langle v \rangle^{\frac{4(2-\gamma)}{4-\gamma}} \left[\mathcal{K}_4 (1 + 1/t_0)^{\frac{2}{\gamma}} \right]^{\frac{2-\gamma}{4-\gamma}}. \end{aligned}$$

This gives

$$\forall t \geq t_0, \quad \langle v \rangle^\gamma \leq L_\gamma(F_t)(v) \left[\mathcal{K}_4(1 + 1/t_0)^{\frac{2}{\gamma}} \right]^{\frac{2-\gamma}{2}} \leq \frac{1}{a} L_\gamma(F_t)(v)$$

and (3.13) follows.

The proof of (3.15) is based on the following a priori estimates. First of all we have

$$|\partial_{v_j} E_{t_1}^t(v)|^2 \leq \gamma^2 e^{-2a(t-t_1)} (t-t_1) \int_{t_1}^t d\tau \int_{\mathbb{R}^N} |v - v_*|^{2(\gamma-1)} dF_\tau(v_*)$$

where we used Cauchy-Schwartz inequality, $\|F_\tau\|_0 = 1$, and $E_{t_1}^t(v) \leq e^{-a(t-t_1)}$.

Case 1: $0 < \gamma < 1$. In this case we have $-N < 2(\gamma-1) < 0$ so that

$$\int_{\mathbb{R}^N} |f(v)|^2 |v - v_*|^{2(\gamma-1)} dv \leq C \|f\|_{L^\infty}^2 + \|f\|_{L^2}^2$$

hence

$$\sum_{j=1}^N \|f \partial_{v_j} E_{t_1}^t(v)\|_{L^2}^2 \leq C (\|f\|_{L^\infty}^2 + \|f\|_{L^2}^2) e^{-2a(t-t_1)} (t-t_1)^2.$$

Case 2: $\gamma \geq 1$. Since $\gamma \leq 2$, this implies $|v - v_*|^{2(\gamma-1)} \leq \langle v \rangle^2 \langle v_* \rangle^2$. Then recalling $\|F_\tau\|_2 = 1 + N$ and $f \in L^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |f(v)|^2 |v - v_*|^{2(\gamma-1)} dv \right) dF_\tau(v_*) \leq (1+N) \|f\|_{L^\infty} \|f\|_{L^{\frac{1}{2}}}$$

which shows that

$$\sum_{j=1}^N \|f \partial_{v_j} E_{t_1}^t\|_{L^2}^2 \leq C \|f\|_{L^\infty} \|f\|_{L^{\frac{1}{2}}} e^{-2a(t-t_1)} (t-t_1)^2.$$

Combing the two cases and using $\|f\|_{L^\infty} \|f\|_{L^{\frac{1}{2}}} \leq \frac{1}{2} \|f\|_{L^\infty}^2 + \frac{1}{2} \|f\|_{L^{\frac{1}{2}}}^2$ we obtain

$$\|f E_{t_1}^t\|_{H^1(\mathbb{R}^N)}^2 \leq C \left(\|f\|_{L^\infty(\mathbb{R}^N)}^2 + \|f\|_{L^{\frac{1}{2}}(\mathbb{R}^N)}^2 + \|f\|_{H^1(\mathbb{R}^N)}^2 \right) e^{-2a(t-t_1)} (1+t-t_1)^2.$$

A full justification requires standard smooth approximation arguments, for instance one may replace f and $|v - v_*|^\gamma$ with $f * \chi_\varepsilon$ and $(\varepsilon^2 + |v - v_*|^2)^{\gamma/2}$ respectively, and then let $\varepsilon \rightarrow 0^+$, etc., where $\chi_\varepsilon(v) = \varepsilon^{-N} \chi(\varepsilon^{-1}v)$ and $\chi \geq 0$ is a smooth mollifier. We omit the details here. \square

Theorem 3.4. *Suppose $N \geq 3$ and let $B(z, \sigma)$ be defined by (1.5) with $\gamma \in (0, 2]$ and with the conditions (1.33)-(1.45). Let $f_t \in L_{1,0,1}^1(\mathbb{R}^N)$ be a conservative mild solution of equation (1.1).*

Then for any $t_0 > 0$, the positive decomposition $f_t = f_t^n + h_t^n$ given in (3.1)-(3.5) on $[t_0, \infty)$ satisfies the following estimates for all $s \geq 0$:

$$(3.16) \quad \sup_{n \geq N_\gamma + 1, t \geq t_0} \|f_t^n\|_{L_s^\infty} \leq C_{t_0, s}$$

$$(3.17) \quad \sup_{n \geq N_\gamma + 2, t \geq t_0} \|f_t^n\|_{H^1} \leq C_{t_0}$$

$$(3.18) \quad \forall t \geq t_0, \forall n \geq 1, \quad \|h_t^n\|_{L_s^1} \leq C_{t_0, s, n} e^{-\frac{a}{2}(t-t_0)}$$

$$(3.19) \quad \forall t_1, t_2 \geq t_0, \quad \sup_{n \geq 1} \|f_{t_1}^n - f_{t_2}^n\|_{L_s^1}, \quad \sup_{n \geq 1} \|h_{t_1}^n - h_{t_2}^n\|_{L_s^1} \leq C_{t_0, s} |t_1 - t_2|.$$

Here N_γ is defined by (2.48), $a = a_{t_0} > 0$ is given in (3.14), and $C_{t_0}, C_{t_0, s}, C_{t_0, s, n}$ are finite constants depending only on N, γ , the function b , $\max\{1, 1/t_0\}$, s , as well as n in the case of $C_{t_0, s, n}$. In particular if $t_0 \geq 1$, all these constants are independent of t_0 .

Proof. Let

$$\|f\|_s = \sup_{t \geq t_0} \|f_t\|_{L_s^1}, \quad s \geq 0.$$

By using Theorem 1.2, (1.14) and the fact that

$$\|f_0\|_{L_0^1} = 1, \quad \|f_0\|_{L_2^1} = 1 + N$$

we have with $\mathcal{K}_s = \mathcal{K}_s(1, 1 + N)$ that

$$\forall s \geq 0, \quad \|f\|_s \leq \mathcal{K}_s \left(1 + \max\{1, 1/t_0\}\right)^{(s-2)^+/\gamma}.$$

We first prove (3.16) and (3.18). To do this it suffices to prove the following estimates (3.20)-(3.24):

- For all $s \geq 0$ and all $t \geq t_0$

$$(3.20) \quad \forall n \geq 1, \quad \|h_t^n\|_{L_s^1} \leq \|f\|_{s+(2n-1)\gamma}^{2n} (1+t-t_0)^{2n-1} e^{-a(t-t_0)}.$$

- If $0 < \gamma < N - 2$, we then define

$$p_n := \frac{N-1}{N-1-n\gamma}, \quad n = 1, 2, 3, \dots, N_\gamma,$$

and then for all $s \geq 0$

$$(3.21) \quad \max_{1 \leq n \leq N_\gamma} \sup_{t \geq t_0} \|f_t^n\|_{L_s^{p_n}} \leq C_a \|f\|_{s+s_1}^{\beta_*},$$

$$(3.22) \quad \sup_{n \geq N_\gamma + 1} \sup_{t \geq t_0} \|f_t^n\|_{L_s^\infty} \leq C_a \|f\|_{s+s_1}^{\beta^*},$$

where $s_1 = N_\gamma + \gamma$ and $0 < \beta_*, \beta^* < \infty$ depend only on N and γ .

- If $\gamma \geq N - 2$ and $1 < p < N$, then for all $s \geq 0$,

$$(3.23) \quad \sup_{t \geq t_0} \|f_t^1\|_{L_s^p} \leq C_{a,p} \|f\|_{s+2\gamma-N/q}^3,$$

$$(3.24) \quad \sup_{n \geq 2} \sup_{t \geq t_0} \|f_t^n\|_{L_s^\infty} \leq C_a \|f\|_{s+s_1}^{3+4/N},$$

where $s_2 = 3\gamma + 2 - 3N/2$.

In the following we denote by $C, C_*, C_{*,*}$ the finite positive constants (larger than 1) that only depend on $N, \gamma, A_2, \|b\|_{L^\infty}$, and on the arguments “*, *”; they may have different values in different places.

By definition of E_s^t (see (3.1)) and Lemma 3.3 we have

$$(3.25) \quad \forall t_0 \leq t_2 \leq t_1 \leq t, \quad \begin{cases} E_{t_1}^t \leq e^{-a(t-t_1)}, \\ E_{t_2}^{t_1} E_{t_1}^t \leq e^{-a(t-t_2)}. \end{cases}$$

We then deduce from (3.25) and $0 \leq f_t^n \leq f_t$ that

$$(3.26) \quad \begin{aligned} h_t^n(v) &\leq h_t^1(v) + 2 \int_{t_0}^t \int_{t_0}^{t_1} e^{-a(t-t_2)} Q^+(f_{t_1}, Q^+(f_{t_2}, h_{t_2}^{n-1}))(v) dt_2 dt_1 \\ &\quad + \int_{t_0}^t \int_{t_0}^{t_1} e^{-a(t-t_2)} Q^+(h_{t_1}^{n-1}, Q^+(f_{t_2}, f_{t_2}))(v) dt_2 dt_1, \end{aligned}$$

$$(3.27) \quad f_t^n(v) \leq \int_{t_0}^t \int_{t_0}^{t_1} e^{-a(t-t_2)} Q^+(f_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}))(v) dt_2 dt_1.$$

Next by definition of h_t^1 in (3.4) and using (3.25) and $\|f_t\|_{L_s^1} \geq \|f_t\|_{L^1} = 1$ we have

$$\|h_t^1\|_{L_s^1} \leq \|f\|_{s+\gamma}^2 (1+t-t_0) e^{-a(t-t_0)}.$$

Suppose (3.20) holds for some $n \geq 1$. Using (3.26) for h_t^{n+1} we have

$$\begin{aligned} \|h_t^{n+1}\|_{L_s^1} &\leq e^{-a(t-t_0)} \|f\|_{s+\gamma}^2 (1+t-t_0) + 2 \|f\|_{s+2\gamma}^2 \int_{t_0}^t \int_{t_0}^{t_1} \|h_{t_2}^n\|_{L_{s+2\gamma}^1} e^{-a(t-t_2)} dt_2 dt_1 \\ &\quad + \|f\|_{s+2\gamma}^2 \int_{t_0}^t \|h_{t_1}^n\|_{L_{s+\gamma}^1} \int_{t_0}^{t_1} e^{-a(t-t_2)} dt_2 dt_1 \\ &= \|f\|_{s+2\gamma}^2 e^{-a(t-t_0)} \left(1+t-t_0 + \int_{t_0}^t \int_{t_0}^{t_1} e^{a(t_2-t_0)} \left(2 \|h_{t_2}^n\|_{L_{s+2\gamma}^1} + \|h_{t_1}^n\|_{L_{s+\gamma}^1} \right) dt_2 dt_1 \right) \end{aligned}$$

and by inductive hypothesis on h_t^n we have for all $t_0 \leq t_2 \leq t_1$,

$$\begin{aligned} 2 \|h_{t_2}^n\|_{L_{s+2\gamma}^1} + \|h_{t_1}^n\|_{L_{s+\gamma}^1} &\leq 2 (\|f\|_{s+2\gamma+(2n-1)\gamma})^{2n} (1+t_2-t_0)^{2n-1} e^{-a(t_2-t_0)} \\ &\quad + (\|f\|_{s+\gamma+(2n-1)\gamma})^{2n} (1+t_1-t_0)^{2n-1} e^{-a(t_1-t_0)} \\ &\leq 3 \|f\|_{s+(2n+1)\gamma}^{2n} (1+t_1-t_0)^{2n-1} e^{-a(t_2-t_0)}. \end{aligned}$$

So

$$\|h_t^{n+1}\|_{L_s^1} \leq \|f\|_{s+(2n+1)\gamma}^{2(n+1)} e^{-a(t-t_0)} \left(1+t-t_0 + 3 \int_{t_0}^t (1+t_1-t_0)^{2n-1} (t_1-t_0) dt_1 \right).$$

It is easily checked that

$$\forall t \geq t_0, \quad 1+t-t_0 + 3 \int_{t_0}^t (1+t_1-t_0)^{2n-1} (t_1-t_0) dt_1 \leq (1+t-t_0)^{2n+1}.$$

Thus

$$\|h_t^{n+1}\|_{L_s^1} \leq \|f\|_{s+(2n+1)\gamma}^{2(n+1)} (1+t-t_0)^{2n+1} e^{-a(t-t_0)}.$$

This proves (3.20).

Now we are going to prove (3.21)-(3.24). First of all by (3.27) and the inequality

$$\int_{t_0}^t \int_{t_0}^{t_1} e^{-a(t-t_2)} dt_2 dt_1 \leq \frac{1}{a^2}$$

we have

$$(3.28) \quad \sup_{t \geq t_0} \|f_t^n\|_{L_s^p} \leq \frac{1}{a^2} \left(\sup_{t_1 \geq t_2 \geq t_0} \|Q^+(f_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}))\|_{L_s^p} \right)$$

for all $s \geq 0, 1 \leq p \leq \infty$, provided that the right hand side makes sense.

Case 1: $0 < \gamma < N - 2$. We first prove that

$$(3.29) \quad \forall s \geq 0, \quad \sup_{t \geq t_0} \|f_t^n\|_{L_s^{p_n}} \leq C_{a,n} (\|f\|_{s+n-1+\gamma_1})^{\beta_n}, \quad n = 1, 2, \dots, N_\gamma$$

where $\gamma_1 = \max\{\gamma, 1\}$ and

$$\beta_n := 2(N+1) \left(1 + \frac{1}{N}\right)^{n-1} + 1 - 2N.$$

By part (i) of Theorem 2.7 we have

$$\forall t_1 \geq t_2 \geq t_0, \quad \|Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2}))\|_{L_{s_1}^{p_1}} \leq C_1 \|f_{t_1}\|_{L_s^1} \|f_{t_2}\|_{L_{s+\gamma}^1}^2 \leq C_1 \|f\|_{s+\gamma_1}^3.$$

Using (3.28) with $p = p_1$ and $n = 1$ (recalling $f_t^{(0)}(v) = f_t(v)$) gives

$$\forall s \geq 0, \quad \sup_{t \geq t_0} \|f_t^1\|_{L_s^{p_1}} \leq C_{a,1} \|f\|_{s+\gamma_1}^3.$$

Since $\beta_1 = 3$, this proves that the inequality in (3.29) holds for $n = 1$.

Suppose the inequality in (3.29) holds for some $1 \leq n \leq N_\gamma - 1$. Then we compute using $0 \leq f_t^n \leq f_t$ and part (I) of Theorem 2.7 that, for all $s \geq 0$,

$$(3.30) \quad \|Q^+(f_{t_1}^n, Q^+(f_{t_2}^n, f_{t_2}^n))\|_{L_{s_1}^{p_{n+1}}} \leq C_n \|f_{t_1}^n\|_{L_{s+1}^{p_n}} \|f_{t_2}^n\|_{L_{s+\gamma_1}^1} \|f_{t_2}^n\|_{L_{s+\gamma_1}^1}^{1-\theta_n} \|f_{t_2}^n\|_{L_{s+1}^{p_n}}^{\theta_n} \\ \leq C_n \|f\|_{s+\gamma_1}^{2-\theta_n} \left(\sup_{t \geq t_0} \|f_t^n\|_{L_{s+1}^{p_n}} \right)^{1+\theta_n}.$$

By inductive hypothesis on f_t^n we compute

$$(3.31) \quad \|f\|_{s+\gamma_1}^{2-\theta_n} \left(\sup_{t \geq t_0} \|f_t^n\|_{L_{s+1}^{p_n}} \right)^{1+\theta_n} \leq C_{a,n} \|f\|_{s+n+\gamma_1}^{2-\theta_n+\beta_n(1+\theta_n)}.$$

Also by definition of θ_n and β_n it is easily checked that $2 - \theta_n + \beta_n(1 + \theta_n) < \beta_{n+1}$. It then follows from (3.28), (3.30) and (3.31) that

$$\forall s \geq 0, \quad \sup_{t \geq t_0} \|f_t^{n+1}\|_{L_s^{p_{n+1}}} \leq C_{a,n+1} \|f\|_{s+n+\gamma_1}^{\beta_{n+1}}.$$

This proves that the inequality in (3.29) holds for all $n = 1, 2, \dots, N_\gamma$. From (3.29) and $N_\gamma - 1 + \gamma_1 < N_\gamma + \gamma = s_1$, we obtain (3.21) with $\beta_* = \beta_{N_\gamma}$.

Next let us prove (3.22). By Theorem 2.7 (see (2.52),(2.53)) and using (3.21) with $n = N_\gamma$ we have

$$\begin{aligned} \left\| Q^+(f_{t_1}^{N_\gamma}, Q^+(f_{t_2}^{N_\gamma}, f_{t_2}^{N_\gamma})) \right\|_{L_s^\infty} &\leq C \|f_{t_1}^{N_\gamma}\|_{L_s^1}^{1-\alpha_1} \|f_{t_1}^{N_\gamma}\|_{L_s^{p_{N_\gamma}}}^{\alpha_1} \|f_{t_2}^{N_\gamma}\|_{L_{s+\gamma}^1} \|f_{t_2}^{N_\gamma}\|_{L_{s+\gamma}^1}^{1-\alpha_2} \|f_{t_2}^{N_\gamma}\|_{L_s^{p_{N_\gamma}}}^{\alpha_2} \\ &\leq C (\|f\|_{s+s_1})^{3+(\beta_{N_\gamma}-1)(\alpha_1+\alpha_2)}. \end{aligned}$$

This together with (3.28) gives

$$(3.32) \quad \sup_{t \geq t_0} \left\| f_t^{N_\gamma+1} \right\|_{L_s^\infty} \leq C_a \|f\|_{s+s_1}^\eta, \quad \eta := 3 + (\beta_{N_\gamma} - 1)(\alpha_1 + \alpha_2).$$

Using (3.28) with $p = \infty$, Theorem 2.7, and

$$\|f_t^{N_\gamma+k}\|_{L_s^{p_{N_\gamma}}} \leq \|f\|_s^{1/p_{N_\gamma}} \left\| f_t^{N_\gamma+k} \right\|_{L_s^\infty}^{1/q_{N_\gamma}}$$

together with the L_s^∞ -boundedness (3.32) for $k = 1$, we deduce by induction on k that, for all $s \geq 0$,

$$\begin{aligned} (3.33) \quad \sup_{t \geq t_0} \left\| f_t^{N_\gamma+k+1} \right\|_{L_s^\infty} &\leq \frac{1}{a^2} \sup_{t_1 \geq t_2 \geq t_0} \left\| Q^+ \left(f_{t_1}^{N_\gamma+k}, Q^+ \left(f_{t_2}^{N_\gamma+k}, f_{t_2}^{N_\gamma+k} \right) \right) \right\|_{L_s^\infty} \\ &\leq C_a \sup_{t_1 \geq t_2 \geq t_0} \left\{ \left\| f_{t_1}^{N_\gamma+k} \right\|_{L_s^1}^{1-\alpha_1} \left\| f_{t_1}^{N_\gamma+k} \right\|_{L_s^{p_{N_\gamma}}}^{\alpha_1} \right. \\ &\quad \left. \times \left\| f_{t_2}^{N_\gamma+k} \right\|_{L_{s+\gamma}^1} \left\| f_{t_2}^{N_\gamma+k} \right\|_{L_{s+\gamma}^1}^{1-\alpha_2} \left\| f_{t_2}^{N_\gamma+k} \right\|_{L_s^{p_{N_\gamma}}}^{\alpha_2} \right\} \\ &\leq C_a \|f\|_{s+\gamma}^{3-(\alpha_1+\alpha_2)/q_{N_\gamma}} \left(\sup_{t \geq t_0} \|f_t^{N_\gamma+k}\|_{L_s^\infty} \right)^{(\alpha_1+\alpha_2)/q_{N_\gamma}} \\ &= C_a \|f\|_{s+\gamma}^{3-\delta} \left(\sup_{t \geq t_0} \|f_t^{N_\gamma+k}\|_{L_s^\infty} \right)^\delta < \infty, \quad k = 1, 2, 3, \dots \end{aligned}$$

where

$$\delta := \frac{\alpha_1 + \alpha_2}{q_{N_\gamma}} = \frac{N - 1 - \gamma + (1 - \gamma)^+}{N} (< 1).$$

Now fix any $s \geq 0$ and let us define

$$A = C_a \|f\|_{s+\gamma}^{3-\delta} \quad \text{and} \quad Y_k = \sup_{t \geq t_0} \left\| f_t^{N_\gamma+k} \right\|_{L_s^\infty}.$$

Then, from (3.33),

$$Y_{k+1} \leq A Y_k^\delta, \quad k = 1, 2, \dots$$

which gives

$$Y_{k+1} \leq A^{1+\delta+\dots+\delta^{k-1}} Y_1^{\delta^k} = A^{\frac{1-\delta^k}{1-\delta}} Y_1^{\delta^k} \leq A^{\frac{1}{1-\delta}} Y_1, \quad k = 1, 2, \dots$$

It follows from (3.32) and $\gamma < s_1$ that

$$\sup_{t \geq t_0} \left\| f_t^{N_\gamma+k+1} \right\|_{L_s^\infty} = Y_{k+1} \leq \left(C_a \|f\|_{s+\gamma}^{3-\delta} \right)^{\frac{1}{1-\delta}} C_a \|f\|_{s+s_1}^\eta \leq C_a^{1+1/(1-\delta)} \|f\|_{s+s_1}^{\frac{3-\delta}{1-\delta} + \eta}$$

for all $k = 1, 2, 3, \dots$. This gives (3.22) with $\beta^* = (3 - \delta)/(1 - \delta) + \eta$.

Case 2: $\gamma \geq N - 2$. By Theorem 2.7 we have for any $1 < p < N$ and $s \geq 0$

$$\|Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2}))\|_{L_s^p} \leq C_p \|f_{t_1}\|_{L_{s+\gamma-N/q}^1} \|f_{t_2}\|_{L_{s+2\gamma-N/q}^1}^2 \leq C_p \|f\|_{s+2\gamma-N/q}^3.$$

This together with (3.28) with $n = 1$ implies that

$$\forall s \geq 0, \quad \sup_{t \geq t_0} \|f_t^1\|_{L_s^p} \leq C_{a,p} \|f\|_{s+2\gamma-N/q}^3.$$

This proves (3.23). In particular for $p = 2$ we have

$$(3.34) \quad \forall s \geq 0, \quad \sup_{t \geq t_0} \|f_t^1\|_{L_s^2} \leq C_a \|f\|_{s+2\gamma-N/2}^3.$$

Then using (3.28) with $p = \infty$, Theorem 2.7 with

$$p = q = 2 \in (N/(N-1), N),$$

and induction on n starting from $n = 1$ with the L_s^2 -boundedness (3.34) we have, for all $s \geq 0$,

$$(3.35) \quad \begin{aligned} \sup_{t \geq t_0} \|f_t^{n+1}\|_{L_s^\infty} &\leq \frac{1}{a^2} \sup_{t_1 \geq t_2 \geq t_0} \|Q^+(f_{t_1}^n, Q^+(f_{t_2}^n, f_{t_2}^n))\|_{L_s^\infty} \\ &\leq C_a \sup_{t \geq t_1 \geq t_0} \left(\|f_{t_1}^n\|_{L_{s+\gamma+2-N}^1}^{1-2/N} \|f_{t_1}^n\|_{L_{s+\gamma+2-N}^2}^{2/N} \|f_{t_2}^n\|_{L_{s+2\gamma+1-N}^1}^2 \right) \\ &\leq C_a \|f\|_{s+2\gamma+1-N}^{3-2/N} \left(\sup_{t \geq t_0} \|f_t^n\|_{L_{s+\gamma+2-N}^2} \right)^{2/N} < \infty, \quad n = 1, 2, 3, \dots \end{aligned}$$

Taking $n = 1$ and using (3.34) and $2\gamma + 1 - N < 3\gamma + 2 - 3N/2 =: s_2$ we obtain

$$(3.36) \quad \sup_{t \geq t_0} \|f_t^2\|_{L_s^\infty} \leq C_a \|f\|_{s+s_2}^{3+4/N}.$$

Further, using

$$\forall n \geq 2, \quad \|f_t^n\|_{L_{s+\gamma+2-N}^2} \leq \|f\|_{s+2\gamma+4-2N}^{1/2} \|f_t^n\|_{L_s^\infty}^{1/2}$$

and $2\gamma + 4 - 2N \leq 2\gamma + 1 - N \leq s_2$ (because $\gamma \geq N - 2 \geq 1$) we get from (3.35) that

$$\sup_{t \geq t_0} \|f_t^{n+1}\|_{L_s^\infty} \leq C_a \|f\|_{s+s_2}^{3-1/N} \left(\sup_{t \geq t_0} \|f_t^n\|_{L_s^\infty} \right)^{1/N}, \quad n = 2, 3, \dots$$

By iteration we deduce, as shown above with $\delta = 1/N$, and using (3.36) that

$$\begin{aligned} \sup_{t \geq t_0} \|f_t^{n+1}\|_{L_s^\infty} &\leq \left(C_a \|f\|_{s+s_2}^{3-1/N} \right)^{\frac{1-\delta^{n-1}}{1-\delta}} \left(\sup_{t \geq t_0} \|f_t^2\|_{L_s^\infty} \right)^{\delta^{n-1}} \\ &\leq C_a \left(\|f\|_{s+s_2} \right)^{(3-\frac{1}{N})\frac{1-\delta^{n-1}}{1-\delta} + (3+4/N)\delta^{n-1}} \leq C_a \left(\|f\|_{s+s_2} \right)^{3+4/N}, \quad n = 2, 3, \dots \end{aligned}$$

This proves (3.24).

Now let us prove the H^1 -regularity (3.17) of f_t^n for $n \geq N_\gamma + 2$. For notation convenience we denote

$$Q_{t_1, t_2}^{n-1}(v) := Q^+(f_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}) E_{t_2}^{t_1})(v).$$

The iteration formula (3.3) is then written

$$(3.37) \quad \forall t \geq t_0, \quad f_t^n(v) = \int_{t_0}^t E_{t_1}^t(v) \int_{t_0}^{t_1} Q_{t_1, t_2}^{n-1}(v) dt_2 dt_1.$$

Applying Theorem 2.7 and the L_s^∞ estimate in (3.16) we have

$$\|Q_{t_1, t_2}^{n-1}\|_{L^\infty} \leq e^{-a(t_1-t_2)} \|Q^+(f_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}))\|_{L^\infty} \leq C_{t_0} e^{-a(t_1-t_2)}.$$

Also by $f_t^{n-1} \leq f_t$ we have

$$\|Q_{t_1, t_2}^{n-1}\|_{L_2^1} \leq e^{-a(t_1-t_2)} \|Q^+(f_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}))\|_{L_2^1} \leq C_{t_0} e^{-a(t_1-t_2)}.$$

And using Lemma 3.2, (3.12) and the L_s^∞ estimate in (3.16) we have

$$\begin{aligned} \|Q_{t_1, t_2}^{n-1}\|_{H^1} &\leq \|Q_{t_1, t_2}^{n-1}\|_{H^{\frac{N-1}{2}}} \leq C \|f_{t_1}^{n-1}\|_{L_{N+\gamma}^2} \|Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}) E_{t_2}^{t_1}\|_{L_{N+\gamma}^2} \\ &\leq C \|f_{t_1}^{n-1}\|_{L_{N+\gamma}^2} \|f_{t_2}^{n-1}\|_{L_{2(N+\gamma)}^2}^2 e^{-a(t_1-t_2)} \leq C_{t_0} e^{-a(t_1-t_2)}. \end{aligned}$$

Thus we conclude from Lemma 3.3 that $Q_{t_1, t_2}^{n-1} E_{t_1}^t \in H^1(\mathbb{R}^N)$ and

$$(3.38) \quad \|Q_{t_1, t_2}^{n-1} E_{t_1}^t\|_{H^1} \leq C_{t_0} e^{-a(t_1-t_2)} e^{-a(t-t_1)} (1+t-t_1) = C_{t_0} e^{-a(t-t_2)} (1+t-t_1).$$

Using Minkowski inequality to (3.37) we then conclude from (3.38) and the above estimates that $f_t^n \in H^1(\mathbb{R}^N)$ and

$$\|f_t^n\|_{H^1} \leq \int_{t_0}^t \int_{t_0}^{t_1} \|Q_{t_1, t_2}^{n-1} E_{t_1}^t\|_{H^1} dt_2 dt_1 \leq C_{t_0} \int_{t_0}^t \int_{t_0}^{t_1} e^{-a(t-t_2)} (1+t-t_1) dt_2 dt_1 \leq C_{t_0}.$$

This proves (3.17).

Finally let us prove (3.19). To do this we rewrite f_t^n as follows (recall definition of f_t^n in (3.3))

$$f_t^n(v) = E_{t_0}^t(v) \int_{t_0}^t E_{t_1}^{t_0}(v) \int_{t_0}^{t_1} Q^+(f_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}) E_{t_2}^{t_1})(v) dt_2 dt_1$$

and recall that

$$E_s^t(v) = \exp\left(-\int_s^t L_\gamma(f_\tau)(v) d\tau\right), \quad L_\gamma(f_\tau)(v) := \int_{\mathbb{R}^N} |v - v_*|^\gamma f_\tau(v_*) dv_*.$$

Then it is easily seen that the function $t \mapsto f_t^n(v)$ is absolutely continuous on every bounded subinterval of $[t_0, \infty)$ and

$$\frac{\partial}{\partial t} f_t^n(v) = \int_{t_0}^t Q^+(f_t^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}) E_{t_2}^t)(v) dt_2 - L_\gamma(f_t)(v) f_t^n(v), \quad \text{a.e. } t \geq t_0.$$

Since

$$\begin{aligned} \|Q^+(f_t^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}) E_{t_2}^t)\|_{L_s^1} &\leq \|Q^+(f_t, Q^+(f_{t_2}, f_{t_2}))\|_{L_s^1} e^{-a(t-t_2)} \\ &\leq \|f_t\|_{L_{s+\gamma}^1} \|f_{t_2}\|_{L_{s+2\gamma}^1}^2 e^{-a(t-t_2)} \leq C_{t_0, s} e^{-a(t-t_2)} \end{aligned}$$

and

$$\|L_\gamma(f_t) f_t^n\|_{L_s^1} \leq \|f_t\|_\gamma \|f_t\|_{L_{s+\gamma}^1} \leq C_{t_0, s}$$

it follows that

$$\left\| \frac{\partial}{\partial t} f_t^n \right\|_{L_s^1} \leq C_{t_0, s} \quad \text{a.e. } t \geq t_0.$$

Thus, by the absolute continuity of $t \mapsto f_t^n(v)$, we deduce that

$$\forall t_1, t_2 \geq t_0, \quad \|f_{t_1}^n - f_{t_2}^n\|_{L_s^1} \leq C_{t_0, s} |t_1 - t_2|.$$

On the other hand, from

$$\forall t \geq t_0, \quad f_t(v) = f_{t_0}(v) + \int_{t_0}^t [Q^+(f_\tau, f_\tau)(v) - L_\gamma(f_\tau)(v)f_\tau(v)] \, d\tau$$

we also have $\|f_{t_1} - f_{t_2}\|_{L_s^1} \leq C_{t_0, s} |t_1 - t_2|$ for all $t_1, t_2 \geq t_0$. Thus the function $t \mapsto h_t^n = f_t - f_t^n$ satisfies the same estimate. This proves (3.19) and completes the proof of the theorem. \square

Corollary 3.5. *Suppose $N \geq 3$ and let $B(z, \sigma)$ be defined in (1.5) with $\gamma \in (0, 2]$ and with the conditions (1.33), (1.45). Let $F_t \in B_{1,0,1}^+(\mathbb{R}^N)$ be a conservative measure strong solution of equation (1.1). Then for any $t_0 > 0$, F_t can be decomposed as*

$$(3.39) \quad \forall t \geq t_0, \quad dF_t(v) = f_t(v) \, dv + d\mu_t(v),$$

with

$$0 \leq f_t \in \bigcap_{s \geq 0} L_s^\infty \cap H^1(\mathbb{R}^N), \quad \mu_t \in \bigcap_{s \geq 0} \mathcal{B}_s^+(\mathbb{R}^N),$$

satisfying for all $s \geq 0$

$$(3.40) \quad \sup_{t \geq t_0} \|f_t\|_{L_s^\infty} \leq C_{t_0, s}, \quad \sup_{t \geq t_0} \|f_t\|_{H^1} \leq C_{t_0}$$

$$(3.41) \quad \forall t \geq t_0, \quad \|\mu_t\|_s \leq C_{t_0, s} e^{-\frac{a}{2}(t-t_0)},$$

$$(3.42) \quad \forall t_1, t_2 \in [t_0, \infty), \quad \|f_{t_1} - f_{t_2}\|_{L_s^1}, \|\mu_{t_1} - \mu_{t_2}\|_s \leq C_{t_0, s} |t_1 - t_2|,$$

where $a = a_{t_0} > 0$ is given in (3.14) and $C_{t_0}, C_{t_0, s}$ are finite constant depending only on N, γ , the function b , $\max\{1, 1/t_0\}$ and s .

Proof. By Theorem 1.2, there is a sequence $\{f_{k,t}\}_{k=1}^\infty \subset L_{1,0,1}^1(\mathbb{R}^N)$ of conservative mild solutions of equation (1.1) such that

$$(3.43) \quad \forall \varphi \in C_c(\mathbb{R}^N), \quad \forall t \geq 0, \quad \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(v) f_{k,t}(v) \, dv = \int_{\mathbb{R}^N} \varphi(v) \, dF_t(v).$$

Let $n_\gamma = N_\gamma + 2$ with N_γ defined in (2.48) and consider the positive decompositions of $f_{k,t}$

$$\forall k = 1, 2, 3, \dots \quad \forall t \geq t_0, \quad f_{k,t}(v) = f_{k,t}^{n_\gamma}(v) + h_{k,t}^{n_\gamma}(v),$$

given by (3.1)-(3.6). By Theorem 3.4 we have for all $s \geq 0$

$$(3.44) \quad \sup_{k \geq 1, t \geq t_0} \|f_{k,t}^{n_\gamma}\|_{L_s^\infty} \leq C_{t_0,s}, \quad \sup_{k \geq 1, t \geq t_0} \|f_{k,t}^{n_\gamma}\|_{H^1} \leq C_{t_0},$$

$$(3.45) \quad \forall t \geq t_0, \quad \sup_{k \geq 1} \|h_{k,t}^{n_\gamma}\|_{L_s^1} \leq C_{t_0,s} e^{-\frac{\alpha}{2}(t-t_0)},$$

$$(3.46) \quad \forall t_1, t_2 \geq t_0, \quad \sup_{k \geq 1} \|f_{k,t_1}^{n_\gamma} - f_{k,t_2}^{n_\gamma}\|_{L_s^1}, \quad \sup_{k \geq 1} \|h_{k,t_1}^{n_\gamma} - h_{k,t_2}^{n_\gamma}\|_{L_s^1} \leq C_{t_0,s} |t_1 - t_2|.$$

From (3.44), it is easily seen that for every $t \geq t_0$, $\{f_{k,t}^{n_\gamma}\}_{k=1}^\infty$ is relatively compact in $L^1(\mathbb{R}^N)$. Moreover by using the density of rational times, a diagonal process and (3.46), one can prove that there is a common subsequence $\{f_{k_j,t}^{n_\gamma}\}_{j=1}^\infty$ (where $\{k_j\}_{j=1}^\infty$ is independent of t) and a function $0 \leq f_t \in L^1(\mathbb{R}^N)$, such that

$$(3.47) \quad \forall t \geq t_0, \quad \left\| f_{k_j,t}^{n_\gamma} - f_t \right\|_{L^1} \xrightarrow{j \rightarrow \infty} 0.$$

Since $h_{k_j,t}^{n_\gamma} = f_{k_j,t}^{n_\gamma} - f_{k_j,t}^{n_\gamma}$, it follows from (3.47) and the weak convergence (3.43) that for every $t \geq t_0$, $h_{k_j,t}^{n_\gamma}$ converges weakly to some $\mu_t \in \mathcal{B}_0^+(\mathbb{R}^N)$ as $j \rightarrow \infty$, i.e.

$$(3.48) \quad \forall \varphi \in C_c(\mathbb{R}^N), \quad \forall t \geq t_0, \quad \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(v) h_{k_j,t}^{n_\gamma}(v) dv = \int_{\mathbb{R}^N} \varphi(v) d\mu_t(v).$$

This leads to the decomposition (3.39). The inequalities (3.40), (3.41), (3.42) follow easily from (3.44), (3.45), (3.46), (3.47), (3.48) and the equivalent version (1.13) of measure norm $\|\cdot\|_s$. \square

4. RATE OF CONVERGENCE TO EQUILIBRIUM

This section is devoted to the proof of Theorem 1.5. We first recall the results in [25] on the exponential rate of convergence to equilibrium for L^1 mild solutions.

Theorem 4.1 (Cf. Theorem 1.2 of [25]). *Suppose $N \geq 3$ and let $B(z, \sigma)$ be defined in (1.5) with $\gamma \in (0, \min\{2, N-2\}]$ and with the conditions (1.33)-(1.45)-(1.46). Let $\lambda = S_{b,\gamma}(1, 0, 1) > 0$ be the spectral gap of the linear operator L_M in (1.44) associated with $B(z, \sigma)$ and the Maxwellian $M(v) = (2\pi)^{-N/2} e^{-|v|^2/2}$ in $L_{1,0,1}^1(\mathbb{R}^N)$.*

Let $f_0 \in L_{1,0,1}^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and let $f_t \in L_2^1(\mathbb{R}^N)$ be the unique conservative solution of equation (1.1) with the initial datum f_0 . Then there is a constant $0 < C < \infty$, which depends only on N , γ , the function b , and on $\|f_0\|_{L^2}$, such that

$$\forall t \geq 0, \quad \|f_t - M\|_{L^1} \leq C e^{-\lambda t}.$$

In the important case of hard sphere model (i.e. $N = 3$, $\gamma = 1$, and $b = \text{const.}$), the assumption “ $f_0 \in L^1 \cap L^2$ ” can be relaxed to “ $f_0 \in L^1$ ” and the same result holds with the constant C depending only on N , γ , and the function b .

Lemma 4.2 (Cf. Lemma 4.6 of [25]). *Suppose $N \geq 3$ and let $B(z, \sigma)$ be defined in (1.5) with $\gamma \in (0, \min\{2, N - 2\}]$, and with the conditions (1.33)-(1.45)-(1.46). Let $\lambda = S_{b,\gamma}(1, 0, 1) > 0$ be the spectral gap of the linear operator L_M in (1.44) associated with $B(z, \sigma)$ and the Maxwellian $M(v) = (2\pi)^{-N/2} e^{-|v|^2/2}$ in $L^1_{1,0,1}(\mathbb{R}^N)$. Let $\alpha > 0$, $m(v) = e^{-\alpha|v|^\gamma}$.*

Then there are some explicitable finite constants $\varepsilon > 0, C > 0$ depending only on N, γ , the function b , and α , such that if f_t with the initial datum $f_0 \in L^1_{1,0,1}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, m^{-2})$ is a conservative solution to equation (1.1) satisfying

$$\forall t \in [0, \infty), \quad \|f_t - M\|_{L^1(m^{-2})} \leq \varepsilon$$

then

$$\forall t \in [0, \infty), \quad \|f_t - M\|_{L^1(m^{-1})} \leq C \|f_0 - M\|_{L^1(m^{-1})} e^{-\lambda t}.$$

Remarks 4.3. (1) The original version of Theorem 4.1 and Lemma 4.2 in [25] were proved for the class $L^1_{\pi^{N/2}, 0, 1/2}(\mathbb{R}^N)$, i.e. for the Maxwellian $M(v) = M_{\pi^{N/2}, 0, 1/2}(v) = e^{-|v|^2}$. According to Proposition 1.4 (normalization), these are equivalent to the present version. In fact let $g_t \in L^1_{\pi^{N/2}, 0, 1/2}(\mathbb{R}^N)$, $f_t \in L^1_{1,0,1}(\mathbb{R}^N)$ have the relation

$$f_t(v) = (2\pi)^{-N/2} g_{t/c}(v/\sqrt{2}), \quad \text{i.e.} \quad g_t(v) = (2\pi)^{N/2} f_{ct}(\sqrt{2}v), \quad t \geq 0$$

where $c = \pi^{N/2} 2^{-\gamma/2}$. Then f_t is a conservative solution of equation (1.1) if and only if g_t is a conservative solution of equation (1.1). And we have

$$\forall t \geq 0, \quad \|f_t - M_{1,0,1}\|_{L^1} = \pi^{-N/2} \left\| g_{t/c} - M_{\pi^{N/2}, 0, 1/2} \right\|_{L^1},$$

and

$$S_{b,\gamma}(\pi^{N/2}, 0, 1/2) = S_{b,\gamma}(1, 0, 1) \pi^{N/2} 2^{-\gamma/2}.$$

(2) In order to prove the exponential rate of convergence to equilibrium, it was introduced in [25] the modified linearized collision operator

$$\mathcal{L}_m(h) = m^{-1} M L_M (m M^{-1} h), \quad m(v) = e^{-a|v|^s}$$

with $M(v) = e^{-|v|^2}$, $a > 0$ and $0 < s < 2$. It is proved in [25] that \mathcal{L}_m and L_M has the same spectrum, but \mathcal{L}_m is available to connect the exponential moment estimates of solutions. The proof of the original version of Theorem 4.1 in [25] used additional technical assumptions: the angular function b is convex and non-decreasing in $(-1, 1)$, and the constant s in $m(v) = e^{-a|v|^s}$ satisfies $0 < s < \gamma/2$. These assumptions were only used to prove the exponential moment estimate of the form (1.26) (see Lemma 4.7 of [25]). According to Theorem 1.2 in Section 1, these additional assumptions on the function b can be removed and the restriction $0 < s < \gamma/2$ can be relaxed to $0 < s \leq \gamma$. In particular one can choose $s = \gamma$.

(3) In [25] it was actually assumed that $\gamma \in (0, 1]$; however this assumption was only used three times:

- The first and second times are in [25, Proof of Proposition 2.3, pp.643-645]:
 - first in “ $|v - v_*|^\gamma \sin^{N-2} \theta/2 \leq (|v - v_*| \sin \theta/2)^\gamma = |v - v'|^\gamma$ since $N - 2 \geq 1 \geq \gamma$ ” [a key step in obtaining basic properties of the linearized collision operator L_M], but here the condition $0 < \gamma \leq 1$ can be relaxed to $0 < \gamma \leq N - 2$ for any $N \geq 3$;
 - second in “ $\|\mathcal{I}_\delta L^+\|_{L^2(M)} = O(\delta^{2-\gamma}) \rightarrow 0$ as $\delta \rightarrow 0^+$ ”, with $\mathcal{I}_\delta = \tilde{\Theta}_\delta * \mathbf{1}_{\{|\cdot| \leq \delta^{-1}\}}$ as defined in [25, p.639]. In this place, if we assume $0 < \gamma \leq 2$ and let $\mathbf{1}_{\{|\cdot| \leq \delta^{-1}\}}$ be modified as $\mathbf{1}_{\{|\cdot| \leq \delta^{-1/2}\}}$, i.e. we redefine $\mathcal{I}_\delta = \tilde{\Theta}_\delta * \mathbf{1}_{\{|\cdot| \leq \delta^{-1/2}\}}$, then the same proof in [25] also yields $\|\mathcal{I}_\delta L^+\|_{L^2(M)} = O(\delta^{2-\gamma/2}) \rightarrow 0$ as $\delta \rightarrow 0^+$.
- The third time was in applying a regularity result from [27]: in the latter paper the condition $0 < \gamma < 2$ was used to ensure the existence and the uniqueness of the mild solution of equation (1.1) constructed in [23]. However since by Theorem 1.2, the existence and the uniqueness of the mild solution of equation (1.1) have been proven for all $0 < \gamma \leq 2$, the results of [27] mentioned above holds also for $\gamma = 2$.

Therefore the present assumption $0 < \gamma \leq \min\{2, N - 2\}$ satisfies all requirements and so the above Theorem 4.1 and Lemma 4.2 hold true. Of course in the physical case, $N = 3$, there is no improvement on γ .

(4) Let f_t be a conservative mild solution of equation (1.1) on $[\tau, \infty)$ with the initial datum f_τ . Applying Theorem 4.1 and Lemma 4.2 to the solution $t \mapsto f_{t+\tau}$ on $[0, \infty)$, the exponential terms for the decay estimates of $\|f_t - M\|_{L^1}$ on the time interval $[\tau, \infty)$ is given by $e^{-\lambda(t-\tau)}$.

To prove Theorem 1.5, we need further preparation.

Lemma 4.4. *Let $0 \leq f \in L^1_{k+l}(\mathbb{R}^N) \cap L^2_1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ with $k \geq 0$, $l > 0$, $s > 0$. Let*

$$\mathcal{N}(f) = \mathcal{N}_{\rho, u, T}(f) \in L^1_{1, 0, 1}(\mathbb{R}^N)$$

be the normalization of f defined in (1.40), (1.41), and suppose that $|\rho - 1| + |u| + |T - 1| \leq 1/2$. Then

$$(4.1) \quad \|f - \mathcal{N}(f)\|_{L^1_k} \leq C_{N, k, l, s}(f) (|\rho - 1| + |u| + |T - 1|)^{\frac{sl}{(1+s)(k+N+2l)}}$$

where

$$C_{N, k, l, s}(f) := C_{N, k, l} \max \left\{ \|f\|_{L^1_{k+l}}, \|f\|_{L^2_1}, \|f\|_{H^s} \right\}$$

and $C_{N, k, l} < \infty$ only depends on N, k, l .

Proof. Recall that $\mathcal{N}(f) = \rho^{-1}T^{N/2}f(\sqrt{T}v + u)$. Let $\mathcal{N}_1(f) = T^{N/2}f(\sqrt{T}v + u)$. Then

$$\|f - \mathcal{N}(f)\|_{L_k^1} \leq \|f - \mathcal{N}_1(f)\|_{L_k^1} + 2|\rho - 1| \|\mathcal{N}_1(f)\|_{L_k^1}$$

where we used $|1 - \rho^{-1}| \leq 2|\rho - 1|$ because $1/2 \leq \rho \leq 3/2$. We need to prove that

$$(4.2) \quad \|f - \mathcal{N}_1(f)\|_{L_k^1} \leq C_N \|f - \mathcal{N}_1(f)\|_{L_{k+l}^1}^{\frac{k+N}{k+N+2l}} \left\| \widehat{f} - \widehat{\mathcal{N}_1(f)} \right\|_{L^2}^{\frac{2l}{k+N+2l}}.$$

Let $h = f - \mathcal{N}_1(f)$, $R \in (0, \infty)$. We have

$$\|h\|_{L_k^1} = \int_{\langle v \rangle < R} \langle v \rangle^k |h(v)| dv + \int_{\langle v \rangle \geq R} \langle v \rangle^k |h(v)| dv \leq C_N \|h\|_{L^2} R^{\frac{k+N}{2}} + \|h\|_{L_{k+l}^1} \frac{1}{R^l}.$$

Minimizing the right hand side with respect to $R \in (0, \infty)$ leads to

$$\|h\|_{L_k^1} \leq 2C_N^{\frac{2l}{k+N+2l}} \|h\|_{L_{k+l}^1}^{\frac{k+N}{k+N+2l}} \|h\|_{L^2}^{\frac{2l}{k+N+2l}}$$

which gives (4.2) by Plancherel theorem $\|h\|_{L^2} = (2\pi)^{-N/2} \|\widehat{h}\|_{L^2}$.

Since $1/2 \leq T \leq 3/2$ and $|u| \leq 1/2$ imply

$$1 + \left| \frac{v-u}{\sqrt{T}} \right|^2 \leq 4(1 + |v|^2),$$

it follows that

$$\|\mathcal{N}_1(f)\|_{L_{k+l}^1} = \int_{\mathbb{R}^N} \left(1 + \left| \frac{v-u}{\sqrt{T}} \right|^2 \right)^{(k+l)/2} f(v) dv \leq 2^{k+l} \|f\|_{L_{k+l}^1}$$

and thus

$$(4.3) \quad \|f - \mathcal{N}(f)\|_{L_k^1} \leq C_{N,k,l} \|f\|_{L_{k+l}^1}^{\frac{k+N}{k+N+2l}} \left\| \widehat{f} - \widehat{\mathcal{N}_1(f)} \right\|_{L^2}^{\frac{2l}{k+N+2l}} + 2^{k+1} |\rho - 1| \|f\|_{L_k^1}.$$

Next we compute

$$\widehat{\mathcal{N}_1(f)}(\xi) = e^{iT^{-1/2}\xi \cdot u} \widehat{f}(T^{-1/2}\xi),$$

$$\left| 1 - e^{iT^{-1/2}\xi \cdot u} \right| \leq 2 \left(1 + |T^{-1/2}\xi|^2 \right)^{s/2} \max\{|u|, |u|^s\},$$

$$\left| \widehat{f}(\xi) - \widehat{\mathcal{N}_1(f)}(\xi) \right| \leq \left| \widehat{f}(\xi) - \widehat{f}(T^{-1/2}\xi) \right| + 2 \left| \widehat{f}(T^{-1/2}\xi) \right| \left(1 + |T^{-1/2}\xi|^2 \right)^{s/2} \max\{|u|, |u|^s\},$$

hence

$$(4.4) \quad \left\| \widehat{f} - \widehat{\mathcal{N}_1(f)} \right\|_{L^2} \leq \left\| \widehat{f} - \widehat{f}(T^{-1/2}\cdot) \right\|_{L^2} + 2^{1+N/4} \|f\|_{H^s} \max\{|u|, |u|^s\}.$$

Write $\xi = (\xi_1, \xi_2, \dots, \xi_N)$, $v = (v_1, v_2, \dots, v_N)$, and $f_j(v) = v_j f(v)$, $j = 1, 2, \dots, N$. Then

$$\widehat{f}(\xi) - \widehat{f}(T^{-1/2}\xi) = -i \int_{\frac{1}{\sqrt{T}}}^1 \sum_{j=1}^N \widehat{f}_j(t\xi) \xi_j dt.$$

By Cauchy-Schwarz inequality and

$$1/2 \leq T \leq 3/2 \quad \implies \quad \left| \frac{1}{\sqrt{T}} - 1 \right| \leq |T - 1|$$

we have

$$\left| \widehat{f}(\xi) - \widehat{f}(T^{-1/2}\xi) \right| \leq |T-1|^{1/2} \left(\int_{1 \wedge \frac{1}{\sqrt{T}}}^{1 \vee \frac{1}{\sqrt{T}}} \sum_{j=1}^N |\widehat{f}_j(t\xi)|^2 dt \right)^{1/2} |\xi|$$

where $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $1/p + 1/q = 1$ with $p = (1+s)/s$ and $q = 1+s$. Then

$$\begin{aligned} \left| \widehat{f}(\xi) - \widehat{f}(T^{-1/2}\xi) \right|^2 &= \left| \widehat{f}(\xi) - \widehat{f}(T^{-1/2}\xi) \right|^{2/p} \left| \widehat{f}(\xi) - \widehat{f}(T^{-1/2}\xi) \right|^{2/q} \\ &\leq |T-1|^{1/p} \left(\int_{1 \wedge \frac{1}{\sqrt{T}}}^{1 \vee \frac{1}{\sqrt{T}}} \sum_{j=1}^N |\widehat{f}_j(t\xi)|^2 dt \right)^{1/p} |\xi|^{2/p} \left| \widehat{f}(\xi) - \widehat{f}(T^{-1/2}\xi) \right|^{2/q}. \end{aligned}$$

It follows from Hölder inequality and $q/p = s$ that

$$\begin{aligned} \left\| \widehat{f} - \widehat{f}(T^{-1/2}\cdot) \right\|_{L^2}^2 &\leq |T-1|^{1/p} \left(\int_{1 \wedge \frac{1}{\sqrt{T}}}^{1 \vee \frac{1}{\sqrt{T}}} \sum_{j=1}^N \left(\int_{\mathbb{R}^N} |\widehat{f}_j(t\xi)|^2 d\xi \right) dt \right)^{1/p} \left\| \widehat{f} - \widehat{f}(T^{-1/2}\cdot) \right\|_{\dot{L}_s^2}^{2/q} \end{aligned}$$

where \dot{L}_s^2 denotes the weighted L^2 space with the *homogeneous* weight $|\xi|^{2s}$. Since, by Plancherel theorem,

$$\begin{aligned} \int_{1 \wedge \frac{1}{\sqrt{T}}}^{1 \vee \frac{1}{\sqrt{T}}} \left(\sum_{j=1}^N \int_{\mathbb{R}^N} |\widehat{f}_j(t\xi)|^2 d\xi \right) dt &= (2\pi)^N \left(\int_{1 \wedge \frac{1}{\sqrt{T}}}^{1 \vee \frac{1}{\sqrt{T}}} t^{-N} dt \right) \left(\int_{\mathbb{R}^N} |v|^2 f(v)^2 dv \right) \leq C_N \|f\|_{L_1^2}^2 \end{aligned}$$

and

$$\left\| \widehat{f} - \widehat{f}(T^{-1/2}\cdot) \right\|_{\dot{L}_s^2} \leq \left\| \widehat{f} \right\|_{\dot{L}_s^2} + \left\| \widehat{f}(T^{-1/2}\cdot) \right\|_{\dot{L}_s^2} \leq \left(1 + 2^{(N+2s)/4} \right) \|f\|_{H^s},$$

it follows that

$$(4.5) \quad \left\| \widehat{f} - \widehat{f}(T^{-1/2}\cdot) \right\|_{L^2} \leq C_N \|f\|_{L_1^2}^{1/p} \|f\|_{H^s}^{1/q} |T-1|^{1/(2p)}.$$

Thus we get from (4.4), (4.5), and $1/p = s/(1+s)$ that

$$(4.6) \quad \left\| \widehat{f} - \widehat{\mathcal{N}_1(f)} \right\|_{L^2} \leq C_N \max \left\{ \|f\|_{L_1^2}, \|f\|_{H^s} \right\} (|u| + |T-1|)^{\frac{s}{2(1+s)}}$$

and so (4.1) follows from (4.3) and (4.6). \square

In order to apply existing results on L^1 solutions to our measure solutions, we shall use the *Mehler transform*, which is defined as follows:

Definition 4.5. Let $\rho > 0$, $u \in \mathbb{R}^N$ and $T > 0$. The Mehler transform $I_n[F]$ of $F \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ is given by

$$I_n[F](v) = e^{Nn} \int_{\mathbb{R}^N} M_{1,0,T} \left(e^n \left(v - u - \sqrt{1 - e^{-2n}} (v_* - u) \right) \right) dF(v_*), \quad n > 0,$$

where $M_{1,0,T}(v) = (2\pi T)^{-N/2} \exp(-|v|^2/(2T))$.

The following lemma gives some basic properties of the Mehler transform that we shall use in the proof of Theorem 1.5.

Lemma 4.6. *Given any $\rho > 0$, $u \in \mathbb{R}^N$ and $T > 0$. Let $F \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ and let $M \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ be the Maxwellian distribution. Then $I_n[F] \in L_{\rho,u,T}^1(\mathbb{R}^N)$ and for any $0 \leq s \leq 2$*

$$(4.7) \quad \lim_{n \rightarrow \infty} \|I_n[F] - M\|_{L_s^1} = \|F - M\|_s.$$

Proof. Recall the basic formula of $I_n[F]$:

$$(4.8) \quad \int_{\mathbb{R}^N} \psi(v) I_n[F](v) dv = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \psi \left(e^{-n} z + u + \sqrt{1 - e^{-2n}} (v_* - u) \right) M_{1,0,T}(z) dz \right) dF(v_*)$$

where ψ is any Borel function on \mathbb{R}^N satisfying $\sup_{v \in \mathbb{R}^N} |\psi(v)| \langle v \rangle^{-2} < \infty$. This formula (4.8) is easily proved by using Fubini theorem and change of variables. From (4.8) it is easily deduced that $I_n[F] \in L_{\rho,u,T}^1(\mathbb{R}^N)$ for all $n > 0$ and

$$(4.9) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(v) I_n[F](v) dv = \int_{\mathbb{R}^N} \varphi(v) dF(v)$$

for all $\varphi \in C(\mathbb{R}^N)$ satisfying $\sup_{v \in \mathbb{R}^N} |\varphi(v)| \langle v \rangle^{-2} < \infty$.

Let $0 \leq s \leq 2$. Applying the dual version (1.14) of the norm $\|\cdot\|_s$ and the convergence (4.9) we have

$$(4.10) \quad \|F - M\|_s \leq \liminf_{n \rightarrow \infty} \|I_n[F] - M\|_{L_s^1}.$$

On the other hand we shall prove that

$$(4.11) \quad \limsup_{n \rightarrow \infty} \|I_n[F] - M\|_{L_s^1} \leq \|F - M\|_s$$

also holds true, and this together with (4.10) then proves (4.7).

To prove (4.11), we take

$$\psi_n(v) = \langle v \rangle^s \text{sign}(I_n[F](v) - M(v)).$$

Then

$$\begin{aligned}
\|I_n[F] - M\|_s &= \int_{\mathbb{R}^N} \psi_n(v) I_n[F](v) \, dv - \int_{\mathbb{R}^N} \psi_n(v) M(v) \, dv \\
&= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \psi_n \left(e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right) M_{1,0,T}(z) \, dz \right) d(F - M)(v_*) \\
&+ \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \psi_n \left(e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right) M_{1,0,T}(z) \, dz \right) M(v_*) \, dv_* - \int_{\mathbb{R}^N} \psi_n(v) M(v) \, dv \\
&:= I_{n,1} + I_{n,2}.
\end{aligned}$$

Let h be the sign function of $F - M$, i.e., $h : \mathbb{R}^N \rightarrow \mathbb{R}$ is a real Borel function satisfying

$$d(F - M)(v_*) = h(v_*) d|F - M|(v_*) \quad \text{and} \quad h(v_*)^2 \equiv 1$$

(see e.g. [28, Chapter 6]). Then

$$I_{n,1} \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left\langle e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right\rangle^s M_{1,0,T}(z) \, dz \, d|F - M|(v_*).$$

Since

$$\forall (z, v_*) \in \mathbb{R}^N \times \mathbb{R}^N, \quad \lim_{n \rightarrow \infty} \left\langle e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right\rangle^s = \langle v_* \rangle^s$$

and

$$\left\langle e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right\rangle^s \leq 3^s \langle u \rangle^s \langle z \rangle^s \langle v_* \rangle^s$$

so that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 3^s \langle u \rangle^s \langle z \rangle^s \langle v_* \rangle^s M_{1,0,T}(z) \, dz \, d|F - M|(v_*) = 3^s \langle u \rangle^s \|M_{1,0,T}\|_s \|F - M\|_s < \infty,$$

it follows from dominated convergence that

$$(4.12) \quad \limsup_{n \rightarrow \infty} I_{n,1} \leq \|M_{1,0,T}\|_{L^1} \|F - M\|_s = \|F - M\|_s.$$

Next we prove that $\limsup_{n \rightarrow \infty} I_{n,2} \leq 0$. We compute by changing variable that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \psi_n \left(e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right) M_{1,0,T}(z) \, dz \right) M(v_*) \, dv_* \\
&= \frac{1}{(1 - e^{-2n})^{N/2}} \int_{\mathbb{R}^N} M_{1,0,T}(z) \, dz \int_{\mathbb{R}^N} \psi_n(v) M \left(\frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) \, dv.
\end{aligned}$$

So we get

$$\begin{aligned}
& I_{n,2} \\
&= \frac{1}{(1-e^{-2n})^{N/2}} \int_{\mathbb{R}^N} M_{1,0,T}(z) dz \int_{\mathbb{R}^N} \psi_n(v) \left[M \left(\frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) - M(v) \right] dv \\
&\quad + \left(\frac{1}{(1 - e^{-2n})^{N/2}} - 1 \right) \int_{\mathbb{R}^N} \psi_n(v) M(v) dv \\
&\leq \frac{1}{(1 - e^{-2n})^{N/2}} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \langle v \rangle^s \left| M \left(\frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) - M(v) \right| M_{1,0,T}(z) dv dz \\
&\quad + \left(\frac{1}{(1 - e^{-2n})^{N/2}} - 1 \right) \int_{\mathbb{R}^N} \langle v \rangle^s M(v) dv
\end{aligned}$$

and finally (since the last last term above clearly converges to zero)

$$\begin{aligned}
(4.13) \quad & \limsup_{n \rightarrow \infty} I_{n,2} \\
& \leq \limsup_{n \rightarrow \infty} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \langle v \rangle^s \left| M \left(\frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) - M(v) \right| M_{1,0,T}(z) dv dz.
\end{aligned}$$

It is obvious that the integrand in the right hand side of (4.13) tends to zero as $n \rightarrow \infty$.

To find a dominated function for the integrand, we recall that

$$\begin{aligned}
& M \left(\frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) \\
&= \frac{\rho}{(2\pi T)^{N/2}} \exp \left(-\frac{1}{2T} \left| \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} - u \right|^2 \right).
\end{aligned}$$

Elementary calculation shows that

$$\frac{1}{2T} \left| \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} - u \right|^2 \geq \frac{|v - u|^2}{4T} - \frac{|z|^2}{4T}.$$

This gives

$$M \left(\frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) M_{1,0,T}(z) \leq \frac{\sqrt{\rho}}{(2\pi T)^{N/2}} \sqrt{M(v)} \sqrt{M_{1,0,T}(z)}$$

and thus

$$\begin{aligned}
& \langle v \rangle^s \left| M \left(\frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) - M(v) \right| M_{1,0,T}(z) \\
& \leq \frac{\sqrt{\rho}}{(2\pi T)^{N/2}} \langle v \rangle^s \sqrt{M(v)} \sqrt{M_{1,0,T}(z)} + \langle v \rangle^s M(v) M_{1,0,T}(z).
\end{aligned}$$

By dominated convergence we then conclude that the limit in the right hand side of (4.13) is zero. Therefore $\limsup_{n \rightarrow \infty} I_{n,2} \leq 0$ and

$$\limsup_{n \rightarrow \infty} \|I_n[F] - M\|_s \leq \limsup_{n \rightarrow \infty} I_{n,1} + \limsup_{n \rightarrow \infty} I_{n,2} \leq \|F - M\|_s.$$

This proves (4.11) and completes the proof of the lemma. \square

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. We first prove that the theorem holds true for all L^1 mild solutions in $L^1_{1,0,1}(\mathbb{R}^N)$. We shall then use approximation and normalization to extend it to general measure solutions.

Step 1. Let $\lambda = S_{b,\gamma}(1,0,1) > 0$ be the spectral gap of the linearized operator L_M associated with the kernel $B(z, \sigma)$ and the Maxwellian $M(v) = (2\pi)^{-N/2} e^{-|v|^2/2}$ in $L^1_{1,0,1}(\mathbb{R}^N)$. Let $f_0 \in L^1_{1,0,1}(\mathbb{R}^N)$ and let f_t be the unique conservative mild L^1 solution of equation (1.1) with the initial datum f_0 . We shall prove that

$$(4.14) \quad \forall t \geq 0, \quad \|f_t - M\|_{L^1_2} \leq C_0 \|f_0 - M\|_{L^1_2}^{1/2} e^{-\lambda t}$$

where the constant $0 < C_0 < \infty$ depends only on N, γ , and the function b .

To do this we use Theorem 3.4 to consider the positive decomposition:

$$\forall t \geq 1, \quad f_t = g_t + h_t,$$

where $g_t = f_t^n \geq 0$ and $h_t = h_t^n \geq 0$ are given in (3.1)-(3.5) with $n = N_\gamma + 2$ and $t_0 = 1$. In the following we denote by $c_i > 0, C_i > 0$ ($i = 1, 2, \dots$) some finite constants that depend only on N, γ , and the function b . By Theorem 3.4 with $t_0 = 1$ we have

$$(4.15) \quad \forall t \geq 1, \quad \|f_t - g_t\|_{L^1_2} = \|h_t\|_{L^1_2} \leq C_1 e^{-\frac{1}{2}at},$$

$$(4.16) \quad \sup_{t \geq 1} \left\{ \|g_t\|_{L^1_{N+4}}, \|g_t\|_{L^2_1}, \|g_t\|_{H^1} \right\} \leq C_2.$$

We can assume that $C_1 \geq 1$. Let $\tau_0 = (2/a) \log(8C_1) (> 1)$ and let

$$\begin{cases} \rho_t = \int_{\mathbb{R}^N} g_t(v) dv, & u_t = \frac{1}{\rho_t} \int_{\mathbb{R}^N} v g_t(v) dv, & T_t = \frac{1}{N\rho_t} \int_{\mathbb{R}^N} |v - u_t|^2 g_t(v) dv, \\ \mathcal{N}(g_t)(v) = \frac{T_t^{N/2}}{\rho_t} g_t(\sqrt{T_t} v + u_t), & t \geq \tau_0. \end{cases}$$

Using the relation

$$T_t = \frac{1}{N\rho_t} \int_{\mathbb{R}^N} |v|^2 g_t(v) dv - \frac{|u_t|^2}{N}$$

we compute

$$(4.17) \quad \forall t \geq \tau_0, \quad |\rho_t - 1| + |u_t| + |T_t - 1| \leq 4 \|f_t - g_t\|_{L^1_2} \leq 4C_1 e^{-\frac{1}{2}at} \leq \frac{1}{2}.$$

So by (4.16) and applying Lemma 4.4 (with $k = 2, l = N + 2, s = 1$) we have

$$\forall t \geq \tau_0, \quad \|g_t - \mathcal{N}(g_t)\|_{L^1_2} \leq C_3 \left(4C_1 e^{-\frac{1}{2}at} \right)^{1/6} = C_4 e^{-c_1 t}.$$

This together with (4.15) gives

$$(4.18) \quad \forall t \geq \tau_0, \quad \|f_t - \mathcal{N}(g_t)\|_{L^1_2} \leq \|f_t - g_t\|_{L^1_2} + \|g_t - \mathcal{N}(g_t)\|_{L^1_2} \leq C_5 e^{-c_1 t}.$$

Also by (4.17), $\sup_{t \geq 1} \|g_t\|_{L^2} \leq C_2$, $\tau_0 > 1$ and $\mathcal{N}(g_t) \in L^1_{1,0,1}(\mathbb{R}^N)$ we have

$$(4.19) \quad C_6 \leq \inf_{t \geq \tau_0} \|\mathcal{N}(g_t)\|_{L^2}, \quad \sup_{t \geq \tau_0} \|\mathcal{N}(g_t)\|_{L^2} \leq C_7.$$

The second inequality follows from elementary calculations and the bounds $1/2 \leq \rho_t$ and $T_t \leq 3/2$. To prove the first one, we consider some $R > 0$ and write

$$1 = \int_{|v| < R} \mathcal{N}(g_t)(v) dv + \int_{|v| \geq R} \mathcal{N}(g_t)(v) dv \leq |\mathbb{B}^N|^{1/2} R^{N/2} \|\mathcal{N}(g_t)\|_{L^2} + R^{-2} N,$$

where $|\mathbb{B}^N|$ is the volume of the unite ball \mathbb{B}^N . If we now fix $R = \sqrt{2N}$, then

$$\frac{1}{2} |\mathbb{B}^N|^{-1/2} (2N)^{-N/4} \leq \|\mathcal{N}(g_t)\|_{L^2}$$

for all $t \geq \tau_0$ so that the first inequality in (4.19) holds for $C_6 = (1/2)|\mathbb{B}^N|^{-1/2}(2N)^{-N/4}$.

To prove (4.14) we use the following technique of ‘‘moving solutions’’ as used in [1] and [25]. For any $\tau \geq \tau_0$, let $(t, v) \mapsto f_t^{(\tau)}(v)$ be the unique conservative solution on $[\tau, \infty) \times \mathbb{R}^N$ with the initial datum at time $t = \tau$:

$$f_t^{(\tau)}|_{t=\tau} = f_\tau^{(\tau)} = \mathcal{N}(g_\tau).$$

On the one hand, by Theorem 4.1, we have

$$\forall t \geq \tau, \quad \left\| f_t^{(\tau)} - M \right\|_{L^1} \leq C_{f_\tau^{(\tau)}} e^{-\lambda(t-\tau)}$$

where the coefficient $0 < C_{f_\tau^{(\tau)}} < \infty$ depends only on N , γ , the function b , and $\|f_\tau^{(\tau)}\|_{L^2}$. Since (4.19) implies $C_6 \leq \|f_\tau^{(\tau)}\|_{L^2} \leq C_7$ for all $\tau \geq \tau_0$, it follows from Remark 1.6-(3) that $\sup_{\tau \geq \tau_0} C_{f_\tau^{(\tau)}} \leq C_8$, and thus for every $\tau \geq \tau_0$ we have

$$(4.20) \quad \forall t \geq \tau, \quad \left\| f_t^{(\tau)} - M \right\|_{L^1} \leq C_8 e^{-\lambda(t-\tau)}.$$

On the other hand using the stability estimate (1.30) we have

$$(4.21) \quad \forall t \geq \tau, \quad \left\| f_t - f_t^{(\tau)} \right\|_{L^1_2} \leq \left\| f_\tau - f_\tau^{(\tau)} \right\|_{L^1_2} e^{c_2(t-\tau)}.$$

Since (4.18) and $\tau \geq \tau_0$ imply

$$\left\| f_\tau - f_\tau^{(\tau)} \right\|_{L^1_2} = \|f_\tau - \mathcal{N}(g_\tau)\|_{L^1_2} \leq C_5 e^{-c_1 \tau},$$

it follows from (4.20) and (4.21) that

$$(4.22) \quad \forall t \geq \tau, \quad \|f_t - M\|_{L^1} \leq \left\| f_t - f_t^{(\tau)} \right\|_{L^1} + \left\| f_t^{(\tau)} - M \right\|_{L^1} \leq C_5 e^{-c_1 \tau + c_2(t-\tau)} + C_8 e^{-\lambda(t-\tau)}.$$

Now for any

$$t \geq t_1 := \frac{c_1 + c_2 + \lambda}{c_2 + \lambda} \tau_0, \quad \text{we choose } \tau = \tau(t) = \frac{c_2 + \lambda}{c_1 + c_2 + \lambda} t.$$

Then $t > \tau(t) \geq \tau(t_1) = \tau_0$ and

$$-c_1\tau(t) + c_2(t - \tau(t)) = -\frac{c_1\lambda}{c_1 + c_2 + \lambda}t := -c_3t.$$

Thus applying (4.22) with $t > \tau = \tau(t)$ (for all $t \geq t_1$) we obtain

$$(4.23) \quad \forall t \geq t_1, \quad \|f_t - M\|_{L^1} \leq (C_5 + C_8)e^{-c_3t}.$$

Now let

$$m(v) := \exp\left(-\frac{\alpha(1)}{4}|v|^\gamma\right)$$

where $\alpha(t) > 0$ is given in Theorem 1.2 for the initial datum $F_0 \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$ defined by $dF_0(v) = f_0(v) dv$. Then

$$(4.24) \quad \sup_{t \geq 1} \|f_t\|_{L^1(m^{-4})} \leq 2, \quad \|M\|_{L^1(m^{-4})} \leq C_9.$$

Therefore, using Cauchy-Schwarz inequality and (4.23), we get

$$(4.25) \quad \forall t \geq t_1, \quad \|f_t - M\|_{L^1(m^{-2})} \leq \|f_t - M\|_{L^1(m^{-4})}^{1/2} \|f_t - M\|_{L^1}^{1/2} \leq C_{10}e^{-c_4t}.$$

Let $\varepsilon > 0$ be the constant in Theorem 4.1 corresponding to $m(v)$, and let us choose

$$t_2 = \max\left\{t_1, \frac{1}{c_4} \log\left(\frac{C_{10}}{\varepsilon}\right)\right\}.$$

Then we deduce from (4.25) that

$$\forall t \geq t_2, \quad \|f_t - M\|_{L^1(m^{-2})} \leq C_{10}e^{-c_4t} \leq \varepsilon.$$

It follows from Lemma 4.2 (see Remark 4.3-(3)) that

$$(4.26) \quad \forall t \geq t_2, \quad \|f_t - M\|_{L^1(m^{-1})} \leq C_{11} \|f_{t_2} - M\|_{L^1(m^{-1})} e^{-\lambda(t-t_2)}.$$

Next, applying the elementary inequality

$$1 + |v|^2 \leq Ce^{\delta|v|^{2\eta}}, \quad \eta := \frac{\gamma}{2}, \quad \delta := \frac{\alpha(1)}{4}$$

for some constant $C = C_{\eta,\delta} > 0$, we have

$$(4.27) \quad \|f_t - M\|_{L^1_2} \leq C_{12} \|f_t - M\|_{L^1(m^{-1})}.$$

On the other hand, using the bound in (4.24), we have

$$(4.28) \quad \|f_{t_2} - M\|_{L^1(m^{-1})} \leq \|f_{t_2} - M\|_{L^1(m^{-2})}^{1/2} \|f_{t_2} - M\|_{L^1}^{1/2} \leq C_{13} \|f_{t_2} - M\|_{L^1_2}^{1/2}.$$

It follows from (4.27), (4.26) and (4.28) that

$$(4.29) \quad \forall t \geq t_2, \quad \|f_t - M\|_{L^1_2} \leq C_{14} \|f_{t_2} - M\|_{L^1_2}^{1/2} e^{-\lambda t}.$$

It remains to estimate $\|f_t - M\|_{L^1_2}$ in terms of $\|f_0 - M\|_{L^1_2}$ for $t \in [0, t_2]$. To do this we use the estimate in [22, p. 3359 line 4] for the measure $H_t := F_t - G_t$ where F_t, G_t are measure solutions of the equation (1.1). Here we define more precisely F_t, G_t to be

$$dF_t(v) = M(v) dv \quad \text{and} \quad dG_t(v) = f_t(v) dv.$$

Then $\|H_t\|_2 = \|M - f_t\|_{L^1_2}, \|f_t\|_{L^1_2} = \|M\|_{L^1_2}$, and thus (recalling $A_0 = 1$)

$$\begin{aligned} \forall t \in [r, \infty), \quad \|M - f_t\|_{L^1_2} &\leq \\ &2 \|(M - f_r)^+\|_{L^1_2} + 4 \left(\|M\|_{L^1_{2+\gamma}} + \|M\|_{L^1_2} \right) \int_r^t \|M - f_s\|_{L^1_\gamma} ds. \end{aligned}$$

Since $t \mapsto f_t(v) \geq 0$ is continuous on $[0, \infty)$ for a.e. $v \in \mathbb{R}^N$, it follows from dominated convergence that

$$2 \|(M - f_r)^+\|_{L^1_2} \xrightarrow{r \rightarrow 0^+} 2 \|(M - f_0)^+\|_{L^1_2} = \|f_0 - M\|_{L^1_2},$$

where the last equality follows from

$$|f_0 - M| = f_0 - M + 2(M - f_0)^+ \quad \text{and} \quad \|f_0\|_{L^1_2} = \|M\|_{L^1_2}.$$

Thus letting $r \rightarrow 0^+$ gives

$$\forall t \in [0, \infty), \quad \|f_t - M\|_{L^1_2} \leq \|f_0 - M\|_{L^1_2} + c_5 \int_0^t \|f_s - M\|_{L^1_\gamma} ds.$$

Since $\|f_s - M\|_{L^1_\gamma} \leq \|f_s - M\|_{L^1_2}$, it follows from Gronwall lemma that

$$(4.30) \quad \forall t \geq 0, \quad \|f_t - M\|_{L^1_2} \leq \|f_0 - M\|_{L^1_2} e^{c_5 t}.$$

Inserting this estimate with $t = t_2$ into the right hand side of (4.29) gives

$$(4.31) \quad \forall t \geq t_2, \quad \|f_t - M\|_{L^1_2} \leq C_{15} \|f_0 - M\|_{L^1_2}^{1/2} e^{-\lambda t}.$$

Also because of (4.30) and $\|f_0 - M\|_{L^1_2} \leq 2(1 + N)$, we have

$$(4.32) \quad \forall t \in [0, t_2], \quad \|f_t - M\|_{L^1_2} \leq C_{16} \|f_0 - M\|_{L^1_2}^{1/2} e^{-\lambda t}.$$

Combining (4.31), (4.32) we then obtain (4.14) with $C_0 = \max\{C_{15}, C_{16}\}$.

Step 2. Let us now prove that (4.14) holds also true for all measure solutions in $\mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$. Given any $F_0 \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$, let F_t be the unique conservative measure strong solution of equation (1.1) with the initial datum F_0 . By part (e) of Theorem 1.2 and Lemma 4.6, there is a sequence $f_{k,t} \in L^1_{1,0,1}(\mathbb{R}^N)$ of solutions with the initial data $f_{k,0} := I_{n_k}[F_0] \in L^1_{1,0,1}(\mathbb{R}^N)$ such that

$$(4.33) \quad \forall \varphi \in C_b(\mathbb{R}^N), \quad \forall t \geq 0, \quad \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(v) f_{k,t}(v) dv = \int_{\mathbb{R}^N} \varphi(v) dF_t(v),$$

$$(4.34) \quad \lim_{k \rightarrow \infty} \|f_{k,0} - M\|_{L^1_2} = \|F_0 - M\|_2.$$

Using the formulation (1.13) of the norm $\|\cdot\|_2$, we conclude from (4.33) that

$$(4.35) \quad \forall t \geq 0, \quad \|F_t - M\|_2 \leq \limsup_{k \rightarrow \infty} \|f_{k,t} - M\|_{L^1_2}.$$

On the other hand, applying (4.14) to $f_{k,t}$, we have

$$(4.36) \quad \forall t \geq 0, \quad k = 1, 2, 3, \dots, \quad \|f_{k,t} - M\|_{L^1_2} \leq C_0 \|f_{k,0} - M\|_{L^1_2}^{1/2} e^{-\lambda t}.$$

Combining (4.35), (4.36) and (4.34) we obtain

$$(4.37) \quad \forall t \geq 0, \quad \|F_t - M\|_2 \leq C_0 \|F_0 - M\|_2^{1/2} e^{-\lambda t}.$$

Step 3. Finally we show that for any $\rho > 0$, $u \in \mathbb{R}^N$ and $T > 0$, the theorem holds true for all measure solutions in $\mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$. Let $F_0 \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ and let F_t be the unique conservative measure strong solution with the initial datum F_0 . Let $M_{\rho,u,T} \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N)$ be the Maxwellian and let $\mathcal{N} = \mathcal{N}_{\rho,u,T}$ be the normalization operator. By Proposition 1.4, the flow $t \mapsto \mathcal{N}(F_{t/c})$ is the unique conservative measure strong solution of equation (1.1) with the initial datum $\mathcal{N}(F_0) \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$. Here $c = \rho T^{\gamma/2}$. Since $\mathcal{N}(M_{\rho,u,T}) \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$ is the standard Maxwellian, it follows from the above result (4.37) that (writing $\mathcal{N}(F_t) = \mathcal{N}(F_{ct/c})$)

$$\forall t \geq 0, \quad \|\mathcal{N}(F_t) - \mathcal{N}(M_{\rho,u,T})\|_2 \leq C_0 \|\mathcal{N}(F_0) - \mathcal{N}(M_{\rho,u,T})\|_2^{1/2} e^{-\lambda ct}.$$

Then, applying Proposition 1.4, we have

$$\begin{aligned} \forall t \geq 0, \quad \|F_t - M_{\rho,u,T}\|_2 &\leq C_{1/\rho,|u|/\sqrt{T},1/T} \|\mathcal{N}(F_t) - \mathcal{N}(M_{\rho,u,T})\|_2 \\ &\leq C_0 C_{1/\rho,|u|/\sqrt{T},1/T} \|\mathcal{N}(F_0) - \mathcal{N}(M_{\rho,u,T})\|_2^{1/2} e^{-\lambda ct} \\ &\leq C_0 C_{1/\rho,|u|/\sqrt{T},1/T} [C_{\rho,|u|,T}]^{1/2} \|F_0 - M_{\rho,u,T}\|_2^{1/2} e^{-\lambda ct}. \end{aligned}$$

Since $\lambda c = S_{b,\gamma}(1,0,1)\rho T^{\gamma/2} = S_{b,\gamma}(\rho,u,T)$ is the spectral gap of the linearized operator $L_{M_{\rho,u,T}}$, this completes the proof of Theorem 1.5. \square

5. LOWER BOUND OF CONVERGENCE RATE

In this section we prove Theorem 1.7. Recall that we assume here that $\gamma \in (0, 2]$ and that the function b satisfies only (1.33).

Proof of Theorem 1.7. We first prove the theorem for the standard case, i.e. assuming $F_0, M \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$. By

$$\forall t_1, t_2 \in [0, \infty), \quad F_{t_2} = F_{t_1} + \int_{t_1}^{t_2} Q(F_\tau, F_\tau) d\tau$$

and $Q(M, M) = 0$, we have

$$(5.1) \quad \forall t_1, t_2 \in [0, \infty), \quad \left| \|F_{t_2} - M\|_0 - \|F_{t_1} - M\|_0 \right| \leq \left| \int_{t_1}^{t_2} \|Q(F_\tau, F_\tau) - Q(M, M)\|_0 d\tau \right|.$$

Using the inequalities in (1.19), $0 < \gamma \leq 2$, and the conservation of mass and energy (which implies $\|F_t\|_\gamma \leq \|F_t\|_2 = 1 + N$, etc.) we have

$$(5.2) \quad \|Q(F_t, F_t) - Q(M, M)\|_0 \leq 2^4(1 + N) \|F_t - M\|_\gamma.$$

Since $t \mapsto \|F_t - M\|_\gamma$ is bounded and, by Hölder inequality,

$$(5.3) \quad \|F_t - M\|_\gamma \leq \|F_t - M\|_0^{1-\gamma/2} \|F_t - M\|_2^{\gamma/2},$$

it follows from (5.1)-(5.3) that $t \mapsto \|F_t - M\|_0$ is Lipschitz continuous and

$$(5.4) \quad \left| \frac{d}{dt} \|F_t - M\|_0 \right| \leq 2^4(N+1) \|F_t - M\|_0^{1-\gamma/2} \|F_t - M\|_2^{\gamma/2} \quad \text{a.e. } t \in (0, \infty).$$

Next, thanks to the exponential decay of the Maxwellian, we show that $\|F_t - M\|_2$ can be controlled by $\|F_t - M\|_0$ (see e.g. (5.9) below). In fact we show that this property holds for all measure $F \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$. To do this, let $(M - F)^+$ be the positive part of $M - F$, i.e., $(M - F)^+ = \frac{1}{2}(|M - F| + M - F)$. Then $|M - F| = F - M + 2(M - F)^+$. Let h be the sign function of $M - F$, i.e., $h(v)^2 \equiv 1$ such that $d(M - F) = hd|M - F|$. Then $d(M - F)^+ = \frac{1}{2}(1 + h)d(M - F)$. From these we have

$$(5.5) \quad d|M - F| = d(F - M) + 2d(M - F)^+ \quad \text{and} \quad d(M - F)^+ \leq dM$$

where the inequality part is due to $F \geq 0$. Now since F, M have the same mass and energy, it follows from (5.5) and $|F - M| = |M - F|$ that

$$(5.6) \quad \|F - M\|_0 = 2\|(M - F)^+\|_0, \quad \|F - M\|_2 = 2\|(M - F)^+\|_2.$$

Let $0 < \delta < 1$. Applying Jensen inequality to the convex function $x \mapsto \exp(\delta x/2)$ and the measure $(M - F)^+$ and assuming $\|(M - F)^+\|_0 > 0$ we have

$$(5.7) \quad \frac{1}{\|(M - F)^+\|_0} \int_{\mathbb{R}^N} \exp(\delta \langle v \rangle^2 / 2) d(M - F)^+ \geq \exp\left(\frac{\delta}{2} \cdot \frac{\|(M - F)^+\|_2}{\|(M - F)^+\|_0}\right).$$

On the other hand we have

$$d(M - F)^+(v) \leq dM(v) = \frac{1}{(2\pi)^{N/2}} \exp(-|v|^2/2) dv$$

and

$$(5.8) \quad \int_{\mathbb{R}^N} \exp(\delta \langle v \rangle^2 / 2) d(M - F)^+(v) \leq \int_{\mathbb{R}^N} \exp(\delta \langle v \rangle^2 / 2) dM(v) = e^{\delta/2} \left(\frac{1}{1-\delta}\right)^{N/2}.$$

Let us now choose $\delta = \frac{1}{N+1}$. Then

$$e^{\delta/2} \left(\frac{1}{1-\delta}\right)^{N/2} = e^{\frac{1}{2(N+1)}} (1 + 1/N)^{N/2} < 2$$

and thus, from (5.6)-(5.8), we deduce

$$\exp\left(\frac{1}{2(N+1)} \cdot \frac{\|F - M\|_2}{\|F - M\|_0}\right) \leq \frac{4}{\|F - M\|_0},$$

i.e.

$$(5.9) \quad \|F - M\|_2 \leq 2(N+1) \|F - M\|_0 \log\left(\frac{4}{\|F - M\|_0}\right).$$

If we adopt the convention $x \log(1/x) = 0$ for $x = 0$, the inequality (5.9) also holds for $\|F - M\|_0 = 0$.

Now let us go back to the solution F_t . To avoid discussing the case $\|F_t - M\|_0 = 0$ for some t , we consider

$$U_\varepsilon(t) := \frac{\|F_t - M\|_0 + \varepsilon}{4}, \quad 0 < \varepsilon < 1.$$

Then

$$\frac{\|F_t - M\|_0}{4} < U_\varepsilon(t) \leq \frac{2 + \varepsilon}{4} < \frac{2}{3}, \quad 0 < \varepsilon < \frac{2}{3}.$$

Using the inequality

$$\forall 0 \leq x \leq y \leq \frac{2}{3}, \quad x \log \frac{1}{x} \leq 2y \log \frac{1}{y}$$

and (5.9) (with $F = F_t$), we then obtain

$$\frac{\|F_t - M\|_2}{4} \leq 4(N + 1)U_\varepsilon(t) \log \left(\frac{1}{U_\varepsilon(t)} \right).$$

Thus by (5.4) we deduce

$$(5.10) \quad \left| \frac{d}{dt} U_\varepsilon(t) \right| \leq AU_\varepsilon(t) \left[\log \left(\frac{1}{U_\varepsilon(t)} \right) \right]^{\gamma/2} \quad \text{a.e. } t \in (0, \infty)$$

where $A = 2^6(N + 1)^2$.

Case 1: $0 < \gamma < 2$. In this case we have, by (5.10),

$$\frac{d}{dt} \left[\log \left(\frac{1}{U_\varepsilon(t)} \right) \right]^{1-\gamma/2} = -(1 - \gamma/2) \left[\log \left(\frac{1}{U_\varepsilon(t)} \right) \right]^{-\gamma/2} \frac{1}{U_\varepsilon(t)} \cdot \frac{d}{dt} U_\varepsilon(t) \leq (1 - \gamma/2)A$$

for almost every $t \in (0, \infty)$. Observe that the function

$$t \mapsto \left[\log \left(\frac{1}{U_\varepsilon(t)} \right) \right]^{1-\gamma/2}$$

is absolutely continuous on every bounded interval of $[0, \infty)$. It follows that

$$(5.11) \quad \forall t \geq 0, \quad \left[\log \left(\frac{1}{U_\varepsilon(t)} \right) \right]^{1-\gamma/2} \leq \left[\log \left(\frac{1}{U_\varepsilon(0)} \right) \right]^{1-\gamma/2} + (1 - \gamma/2)At.$$

Next, using the convexity inequality

$$\forall x, y \geq 0, \quad \left(x + \left(1 - \frac{\gamma}{2}\right)y \right)^{\frac{1}{1-\gamma/2}} \leq \frac{\gamma}{2} \left(\frac{2x}{\gamma} \right)^{\frac{1}{1-\gamma/2}} + \left(1 - \frac{\gamma}{2}\right) y^{\frac{1}{1-\gamma/2}},$$

we have

$$\left\{ \left[\log \left(\frac{1}{U_\varepsilon(0)} \right) \right]^{1-\gamma/2} + \left(1 - \frac{\gamma}{2}\right) At \right\}^{\frac{1}{1-\gamma/2}} \leq \alpha \log \left(\frac{1}{U_\varepsilon(0)} \right) + \beta_1 t^{\frac{2}{2-\gamma}}$$

where

$$\alpha = \left(\frac{2}{\gamma} \right)^{\frac{\gamma}{2-\gamma}}, \quad \beta_1 = \left(1 - \frac{\gamma}{2}\right) A^{\frac{2}{2-\gamma}} = \left(1 - \frac{\gamma}{2}\right) (2^6(N + 1)^2)^{\frac{2}{2-\gamma}}.$$

Thus, from (5.11), we obtain

$$\forall t \geq 0, \quad U_\varepsilon(t) \geq U_\varepsilon(0)^\alpha \exp \left(-\beta_1 t^{\frac{2}{2-\gamma}} \right).$$

Using the definition of $U_\varepsilon(t)$ and letting $\varepsilon \rightarrow 0+$, we get finally

$$\forall t \geq 0, \quad \frac{\|F_t - M\|_0}{4} \geq \left(\frac{\|F_0 - M\|_0}{4} \right)^\alpha \exp\left(-\beta_1 t^{\frac{2}{2-\gamma}}\right).$$

This concludes the proof of the standard case for $0 < \gamma < 2$.

Case 2: $\gamma = 2$. In this case we have by (5.10) with $\gamma = 2$ that

$$\frac{d}{dt} \log \left(\log \left(\frac{1}{U_\varepsilon(t)} \right) \right) = - \left[\log \left(\frac{1}{U_\varepsilon(t)} \right) \right]^{-1} \frac{1}{U_\varepsilon(t)} \cdot \frac{d}{dt} U_\varepsilon(t) \leq A, \quad \text{a.e. } t \in (0, \infty).$$

Since the function

$$t \mapsto \log \left(\log \left(\frac{1}{U_\varepsilon(t)} \right) \right)$$

is absolutely continuous on every bounded interval of $[0, \infty)$, it follows that for all $t > 0$

$$\log \left(\log \left(\frac{1}{U_\varepsilon(t)} \right) \right) \leq \log \left(\log \left(\frac{1}{U_\varepsilon(0)} \right) \right) + At, \quad \text{i.e. } U_\varepsilon(t) \geq (U_\varepsilon(0))^{e^{At}}.$$

Letting $\varepsilon \rightarrow 0+$ leads to

$$\forall t \geq 0, \quad \frac{\|F_t - M\|_0}{4} \geq \left(\frac{\|F_0 - M\|_0}{4} \right)^{e^{At}}.$$

This prove the standard case for $\gamma = 2$.

General non-normalized setting. The general case can be reduced to the standard case by using normalization. Let $F_0 \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$, $c = \rho T^{\gamma/2}$ and let $M \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$ be the Maxwellian. Then, according to Proposition 1.4, the normalization $t \mapsto \mathcal{N}(F_t/c) \in \mathcal{B}_{1,0,1}^+(\mathbb{R}^N)$ is a conservative measure strong solution of equation (1.1) with the initial datum $\mathcal{N}(F_0)$. Applying the above estimates and $\|F_t - M\|_0 = \rho \|\mathcal{N}(F_t) - \mathcal{N}(M)\|_0$ we obtain that if $0 < \gamma < 2$ then

$$\begin{aligned} \forall t \geq 0, \quad \|F_t - M\|_0 &= \rho \|\mathcal{N}(F_{c^{-1}ct}) - \mathcal{N}(M)\|_0 \\ &\geq 4\rho \left(\frac{\|\mathcal{N}(F_0) - \mathcal{N}(M)\|_0}{4} \right)^\alpha \exp\left(-\beta_1 (ct)^{\frac{2}{2-\gamma}}\right) \\ &= 4\rho \left(\frac{\|F_0 - M\|_0}{4\rho} \right)^\alpha \exp\left(-\beta t^{\frac{2}{2-\gamma}}\right) \end{aligned}$$

with

$$\beta = \beta_1 c^{\frac{2}{2-\gamma}} = \left(1 - \frac{\gamma}{2}\right) \left(2^6(1+N)^2 \rho T^{\gamma/2}\right)^{\frac{2}{2-\gamma}}.$$

Similarly if $\gamma = 2$, then

$$\|F_t - M\|_0 \geq 4\rho \left(\frac{\|\mathcal{N}(F_0) - \mathcal{N}(M)\|_0}{4} \right)^{e^{Act}} = 4\rho \left(\frac{\|F_0 - M\|_0}{4\rho} \right)^{e^{\kappa t}}$$

with $\kappa = Ac = 2^6(N+1)^2 \rho T$. This completes the proof. \square

6. GLOBAL IN TIME STABILITY ESTIMATE

In the last section we prove the the global in time strong stability of the measure strong solutions of equation (1.1).

Proof of Theorem 1.10. Let F_t be a conservative measure strong solution of equation (1.1) with the initial datum $F_0 \in \mathcal{B}_{\rho_0, u_0, T_0}^+(\mathbb{R}^N)$, and let G_t be any conservative measure strong solution of equation (1.1) with the initial datum G_0 . Let

$$(6.1) \quad D_0 := \min \left\{ \frac{\rho_0}{2}, \left(\frac{4 \|F_0\|_2}{N \rho_0^2} + \frac{6}{N} \left(\frac{\|F_0\|_2}{\rho_0^2} \right)^2 \right)^{-1} \frac{T_0}{2} \right\}.$$

If $\|F_0 - G_0\|_2 \geq D_0$, then by conservation of mass and energy we have for all $t \geq 0$,

$$(6.2) \quad \|F_t - G_t\|_2 \leq \|F_0\|_2 + \|G_0\|_2 \leq 2 \|F_0\|_2 + \|G_0 - F_0\|_2 \leq \left(\frac{2 \|F_0\|_2}{D_0} + 1 \right) \|G_0 - F_0\|_2.$$

In the following we assume that $\|F_0 - G_0\|_2 < D_0$. By the uniqueness theorem, we can also assume that $\|F_0 - G_0\|_2 > 0$. Due to our choice of D_0 , we see that G_0 is non-zero and is not a Dirac distribution. Therefore let $\rho > 0$, $u \in \mathbb{R}^N$, $T > 0$ be the mass, mean velocity and temperature corresponding to G_0 , i.e., $G_0 \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$. Using the condition $\|F_0 - G_0\|_2 < D_0$ and elementary estimates we have

$$\left\{ \begin{array}{l} |\rho - \rho_0| \leq \|G_0 - F_0\|_2, \quad 0 < \frac{\rho_0}{2} \leq \rho \leq \frac{3\rho_0}{2}, \\ |u - u_0| \leq \frac{2 \|F_0\|_2}{\rho_0^2} \|G_0 - F_0\|_2, \\ |T - T_0| \leq \left(\frac{4 \|F_0\|_2}{N \rho_0^2} + \frac{6}{N} \left(\frac{\|F_0\|_2}{\rho_0^2} \right)^2 \right) \|G_0 - F_0\|_2, \\ 0 < \frac{T_0}{2} \leq T \leq \frac{3T_0}{2}. \end{array} \right.$$

Let M_{F_0}, M_{G_0} be the Maxwellians associated with F_0, G_0 respectively, i.e. $M_{F_0} \in \mathcal{B}_{\rho_0, u_0, T_0}^+(\mathbb{R}^N)$, $M_{G_0} \in \mathcal{B}_{\rho, u, T}^+(\mathbb{R}^N)$. In the following calculations the constants $0 < C_i < \infty$ ($i = 1, 2, \dots, 9$) only depend on N , the function b , γ , ρ_0 , u_0 and T_0 , and we recall that

$$\|F_0\|_2 = \rho_0(1 + NT_0 + |u_0|^2).$$

We need to estimate $\|M_{G_0} - M_{F_0}\|_2$. Let us define

$$\mathcal{M}(\rho, u, T; v) = (2\pi)^{-N/2} \rho T^{-N/2} \exp\left(-\frac{|v - u|^2}{2T}\right)$$

and let us compute

$$\begin{cases} \frac{\partial}{\partial \rho} \mathcal{M}(\rho, u, T; v) = \mathcal{M}(1, u, T; v), \\ \nabla_u \mathcal{M}(\rho, u, T; v) = \mathcal{M}(\rho, u, T; v) \frac{v - u}{T}, \\ \frac{\partial}{\partial T} \mathcal{M}(\rho, u, T; v) = \left(-\frac{(N/2 + 1)}{T} + \frac{|v - u|^2}{2T^2} \right) \mathcal{M}(\rho, u, T; v). \end{cases}$$

If we set

$$\forall \theta \in [0, 1], \quad \rho(\theta) = \theta\rho + (1 - \theta)\rho_0, \quad u(\theta) = \theta u + (1 - \theta)u_0, \quad T(\theta) = \theta T + (1 - \theta)T_0,$$

then

$$\begin{aligned} |\mathcal{M}(\rho, u, T; v) - \mathcal{M}(\rho_0, u_0, T_0; v)| &\leq |\rho - \rho_0| \int_0^1 \mathcal{M}(1, u(\theta), T(\theta); v) \, d\theta \\ &\quad + |u - u_0| \int_0^1 \mathcal{M}(\rho(\theta), u(\theta), T(\theta); v) \frac{|v - u(\theta)|}{T(\theta)} \, d\theta \\ &\quad + |T - T_0| \int_0^1 \mathcal{M}(\rho(\theta), u(\theta), T(\theta); v) \left(\frac{(N/2 + 1)}{T(\theta)} + \frac{|v - u(\theta)|^2}{2T(\theta)^2} \right) \, d\theta. \end{aligned}$$

We then deduce

$$\begin{aligned} \|M_{G_0} - M_{F_0}\|_2 &= \int_{\mathbb{R}^N} \langle v \rangle^2 |\mathcal{M}(\rho, u, T; v) - \mathcal{M}(\rho_0, u_0, T_0; v)| \, dv \\ &\leq C_1 (|\rho - \rho_0| + |u - u_0| + |T - T_0|) \end{aligned}$$

and thus using the above estimates for $\rho - \rho_0$, $u - u_0$ and $T - T_0$, we obtain

$$(6.3) \quad \|M_{G_0} - M_{F_0}\|_2 \leq C_2 \|G_0 - F_0\|_2.$$

Next from the above estimates we have

$$\lambda_0 = S_{b,\gamma}(1, 0, 1)\rho_0 T_0^{\gamma/2}, \quad \lambda = S_{b,\gamma}(1, 0, 1)\rho T^{\gamma/2} \geq 2^{-1-\gamma/2} \lambda_0.$$

Then using the convergence estimate (1.47) and recalling that

$$C_{1/\rho, |u|/\sqrt{T}, 1/T} = \rho \max \left\{ 1 + |u|^2 + \sqrt{T}|u|, T + \sqrt{T}|u| \right\},$$

we have

$$\begin{cases} \|F_t - M_{F_0}\|_2 \leq C_3 e^{-\lambda_0 t}, \\ \|G_t - M_{G_0}\|_2 \leq C_0 C_{1/\rho, |u|/\sqrt{T}, 1/T} e^{-\lambda t} \leq C_4 \exp \left(-2^{-1-\gamma/2} \lambda_0 t \right). \end{cases}$$

Thus

$$\forall t \geq 0, \quad \|F_t - M_{F_0}\|_2 + \|G_t - M_{G_0}\|_2 \leq C_5 e^{-C_6 t},$$

and it follows from (6.3) that

$$(6.4) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq C_5 e^{-C_6 t} + C_2 \|F_0 - G_0\|_2.$$

On the other hand by the stability estimate (1.28) we have

$$(6.5) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq \Psi_{F_0}(\|F_0 - G_0\|_2) e^{C_7(1+t)}.$$

The remaining of the proof is concerning with balancing properly (6.4) and (6.5).

Case 1: $\Psi_{F_0}(\|F_0 - G_0\|_2) < 1$. Note that $\|F_0 - G_0\|_2 > 0$ implies $\Psi_{F_0}(\|F_0 - G_0\|_2) > 0$. Let

$$t_0 = \log \left\{ \left(\frac{1}{\Psi_{F_0}(\|F_0 - G_0\|_2)} \right)^{\frac{1}{C_6 + C_7}} \right\}.$$

For every $t \geq 0$, if $t \leq t_0$, then, using (6.5),

$$\|F_t - G_t\|_2 \leq \Psi_{F_0}(\|F_0 - G_0\|_2) e^{C_7(1+t_0)} = e^{C_7} [\Psi_{F_0}(\|F_0 - G_0\|_2)]^{\frac{C_6}{C_6 + C_7}}.$$

If $t \geq t_0$, then, using (6.4),

$$\|F_t - G_t\|_2 \leq C_5 \left[\Psi_{F_0}(\|F_0 - G_0\|_2) \right]^{\frac{C_6}{C_6 + C_7}} + C_2 \|F_0 - G_0\|_2.$$

Thus

$$(6.6) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq C_8 \left[\Psi_{F_0}(\|F_0 - G_0\|_2) \right]^{\frac{C_6}{C_6 + C_7}} + C_2 \|F_0 - G_0\|_2.$$

Case 2: $\Psi_{F_0}(\|F_0 - G_0\|_2) \geq 1$. In this case we have, by conservation of mass and energy and $\|F_0 - G_0\|_2 \leq \rho_0/2 \leq \|F_0\|_2/2$ that

$$(6.7) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq \frac{5}{2} \|F_0\|_2 \leq \frac{5}{2} \|F_0\|_2 \left[\Psi_{F_0}(\|F_0 - G_0\|_2) \right]^{\frac{C_6}{C_6 + C_7}}.$$

Combining (6.6), (6.7), and (6.2), we obtain

$$\forall t \geq 0, \quad \|F_t - G_t\|_2 \leq C_9 \left(\left[\Psi_{F_0}(\|F_0 - G_0\|_2) \right]^{\frac{C_6}{C_6 + C_7}} + \|F_0 - G_0\|_2 \right).$$

This proves Theorem 1.10 □

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