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Olivier Ley, Vinh Duc Nguyen

LARGE TIME BEHAVIOR FOR SOME NONLINEAR DEGENERATE PARABOLIC EQUATIONS

OLIVIER LEY AND VINH DUC NGUYEN

Abstract. We study the asymptotic behavior of Lipschitz continuous solutions of nonlinear degenerate parabolic equations in the periodic setting. Our results apply to a large class of Hamilton-Jacobi-Bellman equations. Defining Σ as the set where the diffusion vanishes, i.e., where the equation is totally degenerate, we obtain the convergence when the equation is uniformly parabolic outside Σ and, on Σ, the Hamiltonian is either strictly convex or satisfies an assumption similar of the one introduced by Barles-Souganidis (2000) for first-order Hamilton-Jacobi equations. This latter assumption allows to deal with equations with nonconvex Hamiltonians. We can also release the uniform parabolic requirement outside Σ. As a consequence, we prove the convergence of some everywhere degenerate second-order equations.

1. Introduction

The large time behavior of the solution of

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \sup\limits_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2u) + H_\theta(x,Du)\} = 0, & (x,t)\in\mathbb{T}^N\times(0,+\infty),\\ u(x,0) = u_0(x), & x\in\mathbb{T}^N,\end{cases}$$

in the periodic setting ($\mathbb{T}^N$ is the flat torus) was extensively studied (see the references below) in two frameworks: for first-order Hamilton-Jacobi (HJ in short) equations, i.e., when $A_\theta\equiv 0$, and for uniformly parabolic equations. It appears that there is a gap in the type of results and in their proofs which are different.

In this work, we investigate the situation in between. We obtain a new proof for the large time behavior of fully nonlinear degenerate second order equations which includes most of the two previous type of results and allows to deal with some everywhere degenerate second order equations. According to our knowldege, the only result in this direction is the one of Cagnetti et al. [6] where a particular case of degenerate viscous HJ equation is treated with a completely different approach.

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The precise assumptions and the statements of our results are listed in the next section but let us describe the main ideas. We suppose that there exists a, possibly empty, subset

$$\Sigma = \{x \in \mathbb{T}^N : A_\theta(x) = 0 \text{ for all } \theta \in \Theta\}$$

where the Hamiltonian $H_\theta(x,p)$ satisfies some first-order type assumptions and the equation is uniformly parabolic outside Σ, i.e., for all $\delta > 0$, there exists $\nu_\delta > 0$ such that

$$A_\theta(x) = \sigma_\theta(x)\sigma_\theta(x)^T \geq \nu_\delta I \text{ for } x \in \Sigma_\delta^C := \{\mathrm{dist}(\cdot, \Sigma) > \delta\}.$$

Actually, we are able to replace this assumption with a weaker condition of ellipticity like

$$\left\{ \begin{array}{l} \text{for all } \delta > 0, \text{ there exists } \psi_\delta \in C^2(\mathbb{T}^N) \text{ such that} \\ \sup\limits_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2\psi_\delta)\} - C|D\psi_\delta| > 0 \quad \text{in } \Sigma_\delta^C := \{\mathrm{dist}(\cdot, \Sigma) > \delta\} \end{array} \right. \tag{1.2}$$

(see (2.16) for the more general assumption). It can be interpreted as follows. When considering the exit time stochastic control problem associated with the equation in (1.2), it means that the controlled process leaves Σ_δ^C almost surely in finite time.

Assuming that there exists a solution $(c, v) \in \mathbb{R} \times W^{1,\infty}(\mathbb{T}^N)$ of the ergodic problem associated with (1.1), namely

$$\sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2v) + H_\theta(x, Dv)\} = c, \quad x \in \mathbb{T}^N$$

and that $\{u(\cdot,t)+ct, t \geq 0\}$ enjoys suitable compactness properties in $W^{1,\infty}(\mathbb{T}^N)$, we obtain that

$$u(x,t) + ct \to u_\infty(x) \quad \text{in } C(\mathbb{T}^N) \text{ as } t \to +\infty \tag{1.3}$$

in the two following frameworks.

The first case is when the H_θ's are strictly convex in Σ, uniformly with respect to θ (see (2.10) and Theorems 2.1 and 2.5). A typical example, which includes the mechanical Hamiltonian $|p|^2 + \ell(x)$, is

$$H_\theta(x,p) = a_\theta(x)|p|^{1+\alpha_\theta} + \langle b_\theta(x), p\rangle + \ell_\theta(x), \tag{1.4}$$

$$1 < \underline{\alpha} \leq \alpha_\theta \leq \overline{\alpha}, \quad 0 < \underline{a} \leq a_\theta(x) \leq C,$$

and $a_\theta, b_\theta, \ell_\theta$ are bounded Lipschitz continuous uniformly with respect to θ. Another example is the case of uniformly convex Hamiltonians for which $(H_\theta)_{pp}(x,p) \geq 2\underline{a}I$.

The second case is, roughly speaking, when the H_θ's satisfy

$$\inf_{\theta\in\Theta}\{H_\theta(x,\mu p) - \mu H_\theta(x,p)\} \geq (1-\mu)c \quad \text{for } \mu > 1,\ (x,p) \in \mathbb{T}^N \times \mathbb{R}^N, \tag{1.5}$$

with a strict inequality for $x \in \Sigma$ and $p \neq 0$ (see Assumption (2.11) and Theorems 2.2 and 2.5). This is also a convexity-like assumption close to the one introduced in Barles-Souganidis [3] for first-order HJ equations. This assumption may appear to be restrictive in the sense that, in general, we do not know the exact value of the ergodic constant c which appears in (1.5). The main motivations to deal with such a case are, at first, it holds for some nonconvex cases (see Example 3.4) which are a recurrent difficulty in HJ theory. Secondly, it allows to deal with *Namah-Roquejoffre Hamiltonians* [14] (see Section 3.4)

$H(x,p) = F(x,p) - f(x)$, where F is convex (but may be not strictly convex), $F(x,p) \geq F(x,0) = 0$. When the minimum of f is achieved on Σ, we can calculate explicitely the ergodic constant, check that (1.5) holds and obtain the convergence (1.3).

Detailled examples of applications are given in Section 3 but let us give now a typical control-independent example. Consider

$$\frac{\partial u}{\partial t} - a(x)^2 \operatorname{trace}(\bar{\sigma}\bar{\sigma}^T D^2 u) + H(x, Du) = 0,$$

where $a \in W^{1,\infty}(\mathbb{T}^N)$, $\bar{\sigma} \in \mathcal{M}_N$ is a constant matrix and H is convex on $\mathbb{T}^N$ and strictly convex on Σ.

- When $\bar{\sigma}$ is invertible then the convergence (1.3) holds by Theorem 2.1 without further assumptions on a.
- When a vanishes on $\partial[0,1]^N$, then the convergence (1.3) holds by Theorem 2.5 and Proposition 2.7, for any matrix $\bar{\sigma}$ (degenerate or not).

Let us recall the existing results and compare with ours. The asymptotic behavior of (1.1) was extensively studied for totally degenerate equations, i.e., first-order HJ equations for which $\Sigma = \mathbb{T}^N$, see Namah-Roquejoffre [14], Fathi [10], Davini-Siconolfi [9], Barles-Souganidis [3], Barles-Ishii-Mitake [1] (and the references therein for convergence results in bounded sets with various boundary conditions or in $\mathbb{R}^N$). An assumption similar to (1.5) was introduced in [3] to encompass all the previous works on first-order HJ equations and to extend them to some nonconvex Hamiltonians. The arguments of [3] were recently revisited and simplified in Barles-Ishii-Mitake [1]. Due to the above works, it is therefore natural to assume the strict convexity of H_θ or (1.5) on Σ, which is the area where the equation is totally degenerate, and we recover most of the previous results when taking $\Sigma = \mathbb{T}^N$.

As far as second order parabolic equations are concerned, there are less results in the periodic setting. Barles-Souganidis [4] obtained the asymptotic behavior (1.3) in two contexts for

$$\frac{\partial u}{\partial t} - \Delta u + H(x, Du) = 0 \quad (x,t) \in \mathbb{T}^N \times (0, +\infty).$$

The first one is when the Hamiltonian H is *sublinear*, i.e., typically when $|H(x,p)| \leq C(1+|p|)$. The second one is for *superlinear* Hamiltonians, i.e., typically when $H(x,p)$ is given by (1.4) (see (3.1) for a precise assumption). Some extensions are given when $-\Delta u$ is replaced by $-\operatorname{trace}(A(x,Du)D^2u)$ but the convergence result holds for uniformly parabolic equations. The reason is that the proof of convergence is based on the strong maximum principle and, up to our knowledge, it is the case for all results for second order equations except in the recent work of Cagnetti et al. [6]. In this paper, the authors obtained the convergence (1.3) for (1.1) with assumptions very close to ours in the particular case of control-independent uniformly convex Hamiltonians (see Remark 3.9 for details). Their approach is completely different and relies strongly on the linearity with respect to D^2u of the equation. We refer the reader to Tabet Tchamba [16] and Fujita-Ishii-Loreti [11] and the references therein for related results of convergence for uniformly parabolic equations in different settings (bounded sets, in $\mathbb{R}^N$).

The main step in the proof of our results is the following. We prove that that each $\tilde{u}$ in the ω-limit set of $u + ct$ in $C(\mathbb{T}^N \times [0, +\infty))$ is nonincreasing in t thus $\tilde{u}(x,t) \to u_\infty(x)$ as $t \to +\infty$. The convergence (1.3) then follows easily. To prove this main step, it is enough to show that

$$\sup_{x\in\mathbb{T}^N} P_\eta[\tilde{u}](x,t), \quad \text{with } P_\eta[\tilde{u}](x,t) = \sup_{s\geq t}\{\tilde{u}(x,t) - \tilde{u}(x,s) - \eta(s-t)\}, \tag{1.6}$$

is a nonpositive constant m_η for every $\eta > 0$. We argue by contradiction assuming $m_\eta > 0$. Since, by the stability result, $\tilde{u}$ is still solution of (1.1), we obtain that $P_\eta[\tilde{u}]$ is a subsolution of a linearized equation of the form

$$\frac{\partial U}{\partial t} + \inf_{\theta\in\Theta}\big\{ - \mathrm{tr}(\sigma_\theta(x)\sigma_\theta(x)^T D^2 U)\big\} - C|DU| \leq 0 \quad (x,t) \in \mathbb{T}^N \times (0, +\infty).$$

In the set Σ^C, we use the ellipticity-like condition (1.2) and strong maximum principle arguments to show that the maximum in (1.6) is achieved at $x \in \Sigma$. In the set Σ where $A_\theta = 0$, we have formally a first-order equation. We then apply the first-order type assumptions, H_θ strictly convex or satisfying (1.5), to prove that m_η cannot be positive. The main difficulty at this step is to control the second order terms in (1.1) near Σ, see the proof of Lemma 4.6 for details.

The paper is organized as follows. In Section 2, we start by introducing some steady assumptions for (1.1) which are in force in all the paper. We state Theorem 2.1 (strictly convex Hamiltonians) and Theorem 2.2 (nonconvex cases) when (1.1) is uniformly parabolic outside Σ since it is a more simpler and natural case. Then we extend these results to a more degenerate framework, see Theorem 2.5. Some concrete examples are gathered in Section 3. We also introduce *superlinear Hamiltonians* for which all the steady assumptions of Section 2.1 are satisfied. The rest of the paper is devoted to the proofs. The strategy of proof is the same for the three convergence results. It is why the core of the paper is Section 4 where Theorem 2.1 is proved. It relies on several lemmas. Section 5 and the last Section 6 are devoted, respectively, to the proofs of Theorem 2.2 and Theorem 2.5 and their applications.

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2. Statement of the results

2.1. **Setting of the problem and first assumptions.**

We consider

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup\limits_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2u) + H_\theta(x, Du)\} = 0, & (x,t) \in \mathbb{T}^N \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{T}^N, \end{cases} \tag{2.1}$$

and, for $\lambda > 0$, the associate approximate stationary equation

$$\lambda v_\lambda + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2 v_\lambda) + H_\theta(x, Dv_\lambda)\} = 0, \quad x \in \mathbb{T}^N. \tag{2.2}$$

The following assumptions will be in force in all the paper. The set Θ is a metric space. Let $C > 0$ be a fixed constant (independent of θ).

$$\text{For all } \theta \in \Theta,\ A_\theta = \sigma_\theta\sigma_\theta^T,\ \sigma_\theta \in W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_N) \text{ with } |\sigma_\theta|, |D\sigma_\theta| \le C; \tag{2.3}$$

$$\begin{cases} \text{for all } \theta \in \Theta,\ H_\theta \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{T}^N \times \mathbb{R}^N),\ |H_\theta(x, 0)| \le C, \\ \text{for all } R>0, \text{ there exists } C_R > 0 \text{ independent of } \theta \text{ such that} \\ \quad |H_\theta(x, p) - H_\theta(y, q)| \le C_R(|x - y| + |p - q|),\ x, y \in \mathbb{T}^N, |p|, |q| \le R. \end{cases} \tag{2.4}$$

These assumptions are natural when dealing with Hamilton-Jacobi equations. Notice that (2.4) is automatically satisfied when there is no control. Moreover, we assume

$$\begin{cases} \text{There exists viscosity solutions } u \in C(\mathbb{T}^N \times [0, +\infty)) \text{ and } v_\lambda \in C(\mathbb{T}^N) \\ \text{of (2.1) and (2.2) respectively with} \\ |u(x, t) - u(y, t)|, |v_\lambda(x) - v_\lambda(y)| \le C|x - y|,\ x, y \in \mathbb{T}^N, t \ge 0, \lambda > 0. \end{cases} \tag{2.5}$$

Besides the existence of a continuous viscosity solution of the equation, we assume gradient bounds *independent* of t and λ. This is a crucial point and the first step when trying to prove asymptotic results. Let us give some important consequences of (2.5). At first, we have a comparison principle for (2.1) and (2.2). By the comparison principle (for instance for (2.1)), we mean that, if u_1 and u_2 are respectively USC subsolution and LSC supersolution of (2.1) and either u_1 or u_2 satisfies the Lipschitz continuity of (2.5) then $u_1 - u_2 \le \sup_{\mathbb{T}^N}\{(u_1 - u_2)^+(\cdot, 0)\}$. In particular, we have uniqueness of the solutions of (2.1)-(2.2) in the class of functions satisfying the Lipschitz continuity of (2.5). The second consequence is that we can solve the ergodic problem associated with (2.1). More precisely, there exists a unique $c \in \mathbb{R}$ and $v \in W^{1,\infty}(\mathbb{T}^N)$ solutions of

$$\sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2 v) + H_\theta(x, Dv)\} = c, \quad x \in \mathbb{T}^N. \tag{2.6}$$

A byproduct is $|u(x, t)+ct| \le C$. The proofs of these results are classical (see for instance [4, 13]) so we skip them. In Section 3, we introduce *superlinear Hamiltonians* for which the above assumptions are satisfied. Since the above basic assumptions will be used in all our results, for shortness, we introduce a steady assumption collecting them

$$\text{Assumptions (2.3), (2.4), (2.5) hold.} \tag{2.7}$$

We recall that

$$\Sigma = \{x \in \mathbb{T}^N : A_\theta(x) = \sigma_\theta(x)\sigma_\theta^T(x) = 0 \text{ for all } \theta \in \Theta\}, \tag{2.8}$$

and, for the two first convergence results which follow, we assume a nondegeneracy assumption for σ_θ holds outside Σ:

$$\begin{cases} \text{for all } \delta > 0, \text{ there exists } \nu_\delta > 0 \text{ such that for all } \theta \in \Theta \\ A_\theta(x) = \sigma_\theta(x)\sigma_\theta(x)^T \ge \nu_\delta I \text{ for } x \in \Sigma_\delta^C := \{\mathrm{dist}(\cdot, \Sigma) > \delta\}. \end{cases} \tag{2.9}$$

This assumption is replaced by a weaker one in Section 2.4.

2.2. **A convergence result for strictly convex Hamiltonians.** The main assumption in this section is

$$
\begin{cases} \text{For all } x \in \Sigma,\ 0 < \lambda < 1 \text{ and } p, q \in \mathbb{R}^N \text{ such that } p \neq q, \\ \inf_{\theta\in\Theta} \{\lambda H_\theta(x,p) + (1-\lambda)H_\theta(x,q) - H_\theta(x, \lambda p + (1-\lambda)q)\} > 0. \end{cases} \tag{2.10}
$$

This condition is a strict convexity assumption on the H_θ's on Σ uniformly with respect to θ.

Theorem 2.1. *Suppose* (2.7), (2.9), (2.10) *hold and that* $H_\theta(x,\cdot)$ *is convex for every* $x \in \mathbb{T}^N$. *Then* $u(x,t) + ct \to u_\infty(x)$ *in* $C(\mathbb{T}^N)$ *when* $t \to +\infty$, *where* u *is the solution of* (2.1) *and* u_∞ *is a solution of* (2.6).

Section 4 is devoted to the proof.

2.3. **A convergence result for non necessarily convex Hamiltonians.** We will assume the following for the Hamiltonians H_θ's. Recall that c denotes the ergodic constant in (2.6). There exists $\mu_0 > 1$ such that

$$
\begin{cases} \text{(i) } H_\theta(x,\mu p) - \mu H_\theta(x,p) \geq (1-\mu)c \text{ for all } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N,\ 1 < \mu < \mu_0, \\ \text{(ii) There exists a, possibly empty, compact set } K \text{ of } \Sigma \text{ such that} \\ \quad \text{(a) } H_\theta(x,p) \geq c \text{ for all } (x,p) \in K \times \mathbb{R}^N, \\ \quad \text{(b) for all } x \in \Sigma,\ p \in \mathbb{R}^N,\ 1 < \mu \leq \mu_0, \text{ if } d(x,K) \neq 0,\ p \neq 0, \text{ then} \\ \qquad \inf_{\theta\in\Theta} \{H_\theta(x,\mu p) - \mu H_\theta(x,p)\} > (1-\mu)c. \end{cases} \tag{2.11}
$$

Theorem 2.2. *Suppose that* (2.7), (2.9) *and* (2.11) *hold. Then* $u(x,t) + ct \to u_\infty(x)$ *in* $C(\mathbb{T}^N)$ *when* $t \to +\infty$, *where* u *is the solution of* (2.1) *and* u_∞ *is a solution of* (2.6).

The proof of this theorem is done in Section 5.

We make some comments about the assumptions. Conditions (2.11)(i) and (2.11)(ii)(b) are some kind of convexity requirements but it may apply to some nonconvex Hamiltonians (see Section 3). Taking, $p = 0$ in (2.11)(i), we obtain

$$
H_\theta(x,0) \leq c, \qquad x \in \mathbb{T}^N, \theta \in \Theta, \tag{2.12}
$$

which implies that $v \equiv 0$ is a subsolution of (2.6).

Assumption (2.11) may be seen restrictive. Indeed, in general one does not know the exact value of the ergodic constant c so it is difficult to check that (2.11) holds. We have three motivations to state such a result. At first, there are some interesting cases for which we can calculate the exact value of c and (2.11) holds (see Proposition 2.3). It allows to treat some *Namah-Roquejoffre type* Hamiltonians, see Section 3.4. Secondly, this assumption encompasses nonconvex Hamiltonians (see Section 3) and such nonconvex cases are hard to deal with. Finally, it is worth pointing out that, when there exist C^2 subsolutions of (2.6), then Theorem 2.1 appears as an immediate corollary of Theorem 2.2 (see Remark 2.4).

Proposition 2.3. *Assume* (2.7) *and*

$$H_\theta(x,p) \geq H_\theta(x,0) \text{ for } (x,p) \in \Sigma \times \mathbb{R}^N,\ \theta \in \Theta. \tag{2.13}$$

If, in addition, either

$$\sup_{x\in\mathbb{T}^N,\theta\in\Theta} H_\theta(x,0) = \sup_{x\in\Sigma,\theta\in\Theta} H_\theta(x,0) \tag{2.14}$$

holds or (2.9) *and*

$$H_\theta(x,\mu p) - \mu H_\theta(x,p) \geq (1-\mu)H_\theta(x,0) \text{ for } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N,\ \theta \in \Theta,\ \mu > 1, \tag{2.15}$$

hold, then $c = \sup_{x\in\Sigma,\theta\in\Theta} H_\theta(x,0)$.

This proposition, the proof of which is given in Section 5, is used to apply Theorem 2.2 for Hamiltonians of Namah-Roquejoffre type in Section 3. We see that the value of the ergodic constant is affected by the second-order terms in the sense that it is not the same as for (2.1) with $\sigma_\theta \equiv 0$. Assumption (2.14) requires that the supremum of $H_\theta(\cdot,0)$ is actually achieved where the diffusion vanishes. Assumption (2.15) holds automatically when H_θ is convex.

Remark 2.4. We sketch the proof of the fact that, if there exists a C^2 subsolution of (2.6), then Theorem 2.1 is a corollary of Theorem 2.2. Assuming that u is the solution of (2.1) under the assumptions of Theorem 2.1 and v is a C^2 subsolution of (2.6), we set $w = u + ct - v$. Then w is the bounded solution of

$$\frac{\partial w}{\partial t} - \mathrm{trace}(A(x)D^2w) + H(x, Dv + Dw) - H(x, Dv) - g(x) = 0,$$

where $g(x) := \mathrm{trace}(A(x)D^2v) - H(x,Dv) + c \geq 0$ is continuous since v is C^2. Introducing the new Hamiltonian $G(x,p) = H(x, p+Dv) - H(x,Dv) - g(x)$, it is not difficult to check that the strict convexity assumption (2.10) for H implies that G satisfies (2.11) with $c = 0$ and $K = \emptyset$ (for p bounded which is enough since u and v are Lipschitz continuous in x). We then apply Theorem 2.2 to the new equation to obtain the large time behavior of u. Actually, it is possible to generalize such a proof when there exists a $C^{1,1}$ subsolution of (2.6) but the proof is much more involved. We mention this slight extension because it is known (Bernard [5]) that there exists $C^{1,1}$ subsolutions of (2.6) for first order HJ (i.e., when $\Sigma = \mathbb{T}^N$) under general assumptions.

2.4. **A more general result of convergence.** We now generalize the two previous results when (2.9) is replaced by a weaker assumption. The proof of the results of this section are given in Section 6.

Before stating our main assumption, let us introduce some notations. We denote by $\pi : \mathbb{R}^N \to \mathbb{T}^N$ the canonical projection and we add a superscript $\sim$ to the coset representatives of the objects defined on $\mathbb{T}^N$. For instance, $\tilde{\Sigma}$ is a 1-periodic subset of $\mathbb{R}^N$ such that $\pi(\tilde{\Sigma}) = \Sigma$ and $\tilde{\sigma}_\theta(\tilde{x}) = \sigma_\theta(\pi(\tilde{x}))$ for any $\tilde{x} \in \mathbb{R}^N$.

We assume, for some $C > 0$,

$$
(2.16)\begin{cases}
\text{For all } \delta > 0, \text{ there exists } \tilde{\psi}_\delta \in C^2(\mathbb{R}^N) \text{ and an open set } \Omega_\delta \subset\subset \mathbb{R}^N \text{ such that}\\
[0,1]^N \subset \Omega_\delta, \quad \tilde{\psi}_\delta \le 0 \text{ in } \Omega_\delta, \quad \tilde{\psi}_\delta \ge 0 \text{ in } \Omega_\delta^C,\\
\inf_{\theta\in\Theta}\{-\mathrm{trace}(\tilde{A}_\theta(\tilde{x})D^2\tilde{\psi}_\delta(\tilde{x}))\} - C|D\tilde{\psi}_\delta(\tilde{x})| > 0 \;\text{ for } \tilde{x} \in \tilde{\Sigma}_\delta^C \cap \Omega_\delta,\\
\text{ where } \tilde{\Sigma}_\delta = \{\tilde{x} \in \mathbb{R}^N : \mathrm{dist}(\tilde{x}, \tilde{\Sigma}) \le \delta\}.
\end{cases}
$$

Theorem 2.5. *We assume that either the assumptions of Theorem 2.2 or the assumptions of Theorem 2.1 hold, where* (2.9) *is replaced by* (2.16) *in both cases. Then* $u(x,t) + ct$ *converges uniformly to a solution* $v(x)$ *of* (2.6) *when* $t \to +\infty$.

The difference with the previous theorems is that we do not assume the uniform ellipticity assumption (2.9). We consider the weaker assumption (2.16) instead (see Proposition 2.6). This latter assumption allows to deal with some fully nonlinear everywhere degenerate equations. It is written in a tedious way since, in some cases, we need to construct a supersolution which is not 1-periodic (and therefore it is not a function on $\mathbb{T}^N$).

Proposition 2.6. *If* $\Sigma \neq \emptyset$ *and* (2.9) *holds, then* (2.16) *holds.*

It follows that Theorems 2.1 and 2.2 are corollary of Theorem 2.5 when $\Sigma \neq \emptyset$. We can apply the theorem to obtain the convergence for some everywhere degenerate equations. Let us give an application.

Proposition 2.7. *Assume* (2.3), (2.4),

$$
(x,\theta) \in \mathbb{T}^N \times \Theta \mapsto \sigma_\theta(x) \text{ is continuous}, \quad \Theta \text{ is compact}, \tag{2.17}
$$

$$
\bigcup_{1\le i\le N} \{x = (x_1, \cdots, x_N) \in \mathbb{T}^N : x_i = 0\} \subset \Sigma \tag{2.18}
$$

and

$$
\bigcup_{x\in\Sigma^C, \theta\in\Theta} \ker(\sigma_\theta(x)) \cap \mathbb{S}^{N-1} \neq \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}. \tag{2.19}
$$

Then Assumption (2.16) *holds.*

The assumption (2.18) means that the boundary of the cube $[0,1]^N$ is contained in Σ; more generally, we need the connected components of $\tilde{\Sigma}^C$ to be bounded in $\mathbb{R}^N$. In Σ^C, $\ker(\sigma_\theta(x))$ is at most an hyperplane. The assumption (2.19) means that the union of these hyperplanes does not fulfill the whole space. Some concrete examples of applications are given in Section 3.

3. Applications and examples

3.1. **Superlinear Hamiltonians.** We first introduce an assumption on the H_θ's, called *superlinear* in [4], under which the steady assumptions of Section 2.1 hold.

$$
\left\{
\begin{array}{l}
\text{There exists } L_1 \geq 1 \text{ such that if } |p| \geq L_1, \text{ then} \\
\quad L_1 \left[(H_\theta)_p p - H_\theta - |2\sigma_\theta (\sigma_\theta^T)_x||p| - |H_\theta(\cdot, 0)|_\infty \right] \\
\qquad\qquad -|(H_\theta)_x| - N|(\sigma_\theta)_x|_\infty^2 |p| \} \geq 0, \text{ for a.e. } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N, \theta \in \Theta.
\end{array}
\right. \tag{3.1}
$$

Theorem 3.1. *Assume* (2.3), (2.4) *and* (3.1). *Then* (2.5) *holds, we have a comparison principle for* (2.1) *and* (2.2). *In particular, the ergodic problem* (2.6) *has a solution* $(c, v) \in \mathbb{R} \times W^{1,\infty}(\mathbb{T}^N)$.

The main ingredients in the proof of this result are gradient bounds for the solutions of (2.1) and (2.2) uniform in t and λ respectively. We refer the reader to Barles-Souganidis [4] and [13].

We give some examples of Hamiltonians satisfying both the assumption of Theorem 3.1 and (2.10).

Example 3.2. (strictly convex Hamilton-Jacobi-Bellman equations) We suppose that (2.3) holds and

$$
\begin{aligned}
& H_\theta(x,p) = a_\theta(x)|p|^{1+\alpha_\theta} + \langle b_\theta(x), p\rangle + \ell_\theta(x), \\
& 1 < \underline{\alpha} \leq \alpha_\theta \leq \overline{\alpha}, \quad 0 < \underline{a} \leq a_\theta(x) \leq C, \\
& |a_\theta(x)|, |b_\theta(x)|, |\ell_\theta(x)|, |a_\theta(x) - a_\theta(y)|, |b_\theta(x) - b_\theta(y)|, |\ell_\theta(x) - \ell_\theta(y)| \leq C, \ x, y \in \mathbb{T}^N.
\end{aligned}
$$

Example 3.3. (uniformly convex Hamilton-Jacobi-Bellman equations) We suppose that (2.3)-(2.4) hold and

$$
(H_\theta)_{pp}(x,p) \geq \underline{h} > 0, \qquad |(H_\theta)_x| \leq C(1 + |p|^2).
$$

We now give an example such that the assumptions of Theorem 3.1 still holds but the Hamiltonian is not convex anymore and satisfies (2.11).

Example 3.4. (nonconvex equations) We adapt an example from [3]. We consider (2.1) without control with

$$
H(x,p) = \psi(x,p)F(x, \frac{p}{|p|}) - f(x), \tag{3.2}
$$

where $f \in W^{1,\infty}(\mathbb{T}^N)$ is nonnegative, $F \in W^{1,\infty}(\mathbb{T}^N \times \mathbb{R}^N)$ is strictly positive and $\psi(x,p) = |p + h(x)|^2 - |h(x)|^2$, with $h \in W^{1,\infty}(\mathbb{T}^N; \mathbb{R}^N)$. Notice that $H \in W^{1,\infty}(\mathbb{T}^N \times \mathbb{R}^N)$ is not convex in general. We suppose that $A = \sigma\sigma^T$ satisfies (2.3) and

$$
\{x \in \mathbb{T}^N : f(x) = |h(x)| = |\sigma(x)| = 0\} \neq \emptyset. \tag{3.3}
$$

Arguing as in the proof of Proposition 2.3, we can show that $c = 0$ (we cannot applying Proposition 2.3 directly since (2.13) does not hold).

We now prove that H satisfies (2.11) with $K = \emptyset$. For every $\mu > 1$, we have

$$
H(x, \mu p) - \mu H(x,p) = (\mu^2 - \mu)|p|^2 F(x, \frac{p}{|p|}) + (\mu - 1)f(x) \geq 0,
$$

so (2.11)(i) holds. If $p \neq 0$, the above inequality is strict and therefore (2.11)(ii)(b) holds.

3.2. **Second-order equations satisfying** (2.9) **or** (2.16)**.**

Example 3.5. Without control, we choose $\sigma \in W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_n)$ with for each $x \in \mathbb{T}^N$, either $\sigma(x) = 0$ or $\sigma(x)$ is invertible. Then (2.3) and (2.9) hold. A particular case is $\sigma(x) = a(x)\bar{\sigma}$ where $a \in W^{1,\infty}(\mathbb{T}^N)$ and $\bar{\sigma} \in \mathcal{M}_n$ is a constant invertible matrix.

Example 3.6. With control, we can deal with some cases of fully nonlinear equations. For instance, consider $\sigma_\theta(x) = a(x)\bar{\sigma}_\theta$, where $a \in W^{1,\infty}(\mathbb{T}^N)$ as in Example 3.5 and there exists $\nu > 0$ such that, for all $\theta \in \Theta$, $\bar{\sigma}_\theta \in \mathcal{M}_n$ and $\bar{\sigma}_\theta \bar{\sigma}_\theta^T \geq \nu I$. Then (2.3) and (2.9) hold.

It is worth noticing that, in the two examples above, we may have the two following particular cases: $a > 0$ on $\mathbb{T}^N$ and the equation is uniformly parabolic, or $a = 0$ on $\mathbb{T}^N$ and the equation is a first-order HJ equation.

We now give some examples for which (2.18)-(2.19) hold (and so (2.16) holds thanks to Proposition 2.7).

Example 3.7. A first control-independent case is $\sigma(x) = a(x)\bar{\sigma}$ where $a \in W^{1,\infty}(\mathbb{T}^N)$ is such that $\partial[0,1]^N \subset \{a = 0\}$ and $\bar{\sigma} \in \mathcal{M}_n$ is any nonzero constant degenerate matrix. With control, we can take $\sigma_\theta(x) = a(x)\bar{\sigma}_\theta$ where a satisfies the same assumptions in the control-independent case and $(\bar{\sigma}_\theta)_{\theta\in\Theta}$ is a finite set of nonzero constant degenerate (in this case $\Theta = \{1, 2, \cdots, k\}$). In both case, (2.18)-(2.19) holds.

Example 3.8. For simplicity, we consider a control-independent example. Assume that $\sigma = (\sigma_{ij})_{1\leq i,j\leq N} \in W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_N)$ is such that

$$\partial[0,1]^N \subset \{\sigma_{11} = 0\} \subset \Sigma = \{\sigma = 0\}.$$

Then, for all $x \in \Sigma^C$, $\sigma(x)e_1 \neq 0$, where $e_1 = (1, 0, \cdots 0)$. Therefore (2.19) holds.

3.3. **Application to convergence results.** In the following cases, there exist solutions to (2.1) and (2.6) (i.e., (2.5) holds) and we have a convergence result:

- Applying Theorem 2.1 when the H_θ's are given by Examples 3.2 or 3.3 and the diffusion matrices are given by Examples 3.5 or 3.6.
- Applying Theorem 2.2 when the H_θ's given by Example 3.4 and the diffusion matrices are given by Examples 3.5 or 3.6.
- Applying Theorem 2.5 when the H_θ's are given by Examples 3.2, 3.3 or 3.4 and the diffusion matrices are given by Examples 3.7 or 3.7.

Remark 3.9. When $\Sigma = \mathbb{T}^N$ or $\Sigma = \emptyset$, these convergence results were obtained in [10, 14, 3, 9] and [4] respectively. In the particular case of control-independent C^2 uniformly convex Hamiltonian (see Example 3.3) with $\sigma(x) = a(x)I$ with $a \in C^2(\mathbb{T}^N)$ (see Example 3.5), the result is proven in [6] by using a nonlinear adjoint method. Notice that, on the one side, we can deal with fully nonlinear equations and, on the other side, we only require the Hamiltonians to be uniformly convex on Σ.

When the assumption (3.1) does not hold, we need to prove a priori the existence of Lipschitz solutions to (2.1) and (2.6) before applying a convergence result. For instance, if (2.5) holds for (3.4) in Example 3.10 below, then we have the convergence by applying Theorem 2.1. An other important case is given in Section 3.4.

Example 3.10. Consider

$$\frac{\partial u}{\partial t} - a(x)^2\Delta u + (1-a(x))|Du|^2 = f(x), \quad (x,t)\in\mathbb{T}^N\times(0,+\infty), \tag{3.4}$$

where $a, f \in W^{1,\infty}(\mathbb{T})$ and a is defined by

$$a(x) = \begin{cases} 1-|x| & \text{if } x\in[0,\frac14],\\ 0 & \text{if } x\in[\frac14,\frac34],\\ |x|-1 & \text{if } x\in[\frac34,1]. \end{cases}$$

Then $H(x,p) = (1-a(x))|p|^2$ is striclty convex on $\Sigma = [\frac14,\frac34]$ and (2.10) holds.

We end this section by a counter-example.

Example 3.11. Consider

$$\begin{cases} \frac{\partial u}{\partial t} - \mathrm{trace}(A(x)D^2u) + H(Du) = 0, & (x,t)\in\mathbb{T}^2\times(0,+\infty),\\ u(x,0) = \sin(x_1+x_2), & x=(x_1,x_2)\in\mathbb{T}^2, \end{cases} \tag{3.5}$$

where $A = a(x)^2\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix}$ and $H(p) = \frac{1}{\sqrt{2}}|p+(1,1)| - 1$. The solution of (3.5) is $u(x,t) = \sin(x_1+x_2-t)$ and convergence fails as $t\to+\infty$. In this example, $A(x)$ is degenerate and H is convex but does not satisfy neither (2.10) nor (2.11).

3.4. The Namah-Roquejoffre case.

Consider

$$\frac{\partial u}{\partial t} + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2u) + F_\theta(x,Du) - f_\theta(x)\} = 0, \quad (x,t)\in\mathbb{T}^N\times(0,+\infty), \tag{3.6}$$

where $A_\theta = \sigma_\theta\sigma_\theta^T$ satisfies (2.3),

$$\begin{cases} f_\theta\in W^{1,\infty}(\mathbb{T}^N), \quad F_\theta\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{T}^N\times\mathbb{R}^N) \text{ is convex in } p,\\ F_\theta(x,p)\ge F_\theta(x,0)=0,\\ K := \{x\in\Sigma : f_\theta(x) = \min\limits_{y\in\mathbb{T}^N,\theta\in\Theta} f_\theta(y)\}\neq\emptyset \end{cases} \tag{3.7}$$

and

$$\text{for } x\in\Sigma\setminus K, \quad \inf_{\theta\in\Theta} f_\theta(x) > \inf_{y\in\mathbb{T}^N,\theta\in\Theta} f_\theta(y). \tag{3.8}$$

We call such kind of Hamiltonians of Namah-Roquejoffre type, see [14, 3].

When F is strictly convex in p, then the convergence result for (3.6) can be obtained with the use of Theorem 2.1. Here, we want to deal with the typical Hamiltonian which appears in [14], that is, $F_\theta(x,p) = a_\theta(x)|p|$, which is not strictly convex and does not satisfy (3.1). It is why we assume here a priori that (2.5) holds for (3.6).

From Proposition 2.3, we obtain $c = -\min_{x\in\Sigma,\theta\in\Theta} f_\theta(x)$. Therefore, for $x \in K$, we have $H_\theta(x,0) = -f_\theta(x) = c$ and (2.11)(ii)(a) holds. By (3.7), we have, for all $x \in \mathbb{T}^N$, $\mu > 1$,

$$\begin{array}{rcl} H_\theta(x,\mu p) - \mu H_\theta(x,p) & = & F_\theta(x,\mu p) - \mu F_\theta(x,p) - (1-\mu) f_\theta(x) \\ & \geq & -(1-\mu)\min_{\mathbb{T}^N,\Theta} f_\theta = -(1-\mu)\min_{\Sigma,\Theta} f_\theta. \end{array}$$

Therefore (2.11)(i) holds. By (3.8), (2.11)(ii)(b) holds.

Therefore, assuming (2.5) for (3.6) and (3.7), (3.8), then we obtain the convergence from Theorem 2.2 when A_θ satisfies (2.9) and from Theorem 2.5 when A_θ satisfies the conditions of Examples 3.7 or 3.8.

4. Proof of Theorem 2.1

At first, we notice that we can assume without loss of generality that $c = 0$ in (2.6). Indeed, by a change of function $u(x,t) \to u(x,t) + ct$, the new function satisfies (2.1) where H_θ is replaced with $H_\theta - c$ and, if H_θ satisfies the strict convexity assumption (2.10), then $H_\theta - c$ still satisfies (2.10). So, we suppose that $c = 0$ and the solution $u(x,t)$ of (2.1) is bounded. We aim at proving that $u(x,t)$ converges uniformly to some function $u_\infty(x)$, which is a solution of (2.6) with $c = 0$ by the stability result. In the following, v is a Lipschitz continuous solution of (2.6) with $c = 0$.

Following the ideas of [1, 15], for $\eta > 0, \mu > 1$ and $(x,t) \in \mathbb{T}^N \times (0,+\infty)$, we introduce

$$P_{\eta\mu}[u](x,t) = \sup_{s\geq t}\{u(x,t) - v(x) - \mu(u(x,s) - v(x)) - \mu\eta(s-t)\}, \tag{4.1}$$

and

$$M_{\eta\mu}[u](t) = \sup_{x\in\mathbb{T}^N, s\geq t}\{u(x,t) - v(x) - \mu(u(x,s) - v(x)) - \mu\eta(s-t)\} = \sup_{x\in\mathbb{T}^N} P_{\eta\mu}[u](x,t). \tag{4.2}$$

Lemma 4.1. *The function $P_{\eta\mu}[u](x,t)$ is a subsolution of the Hamilton-Jacobi inequality*

$$\min\left\{U(x,t)\,,\, \frac{\partial U}{\partial t} + \inf_{\theta\in\Theta}\{-\operatorname{trace}(A_\theta(x)D^2U)\} - C|DU|\right\} \leq 0 \quad \text{in } \mathbb{T}^N \times (0,+\infty), \tag{4.3}$$

where C is a constant independent of x,t (given in (4.14)).

Proof of Lemma 4.1. For simplicity, we set $U(x,t) := P_{\eta\mu}[u](x,t)$.

Let any $\phi_0 \in C^2(\mathbb{T}^N \times (0,\infty))$ such that (x_0,t_0), $t_0 > 0$, is a strict maximum point of $U - \phi_0$ in $\mathbb{T}^N \times [t_0-\delta, t_0+\delta]$ for some small $\delta > 0$. If $U(x_0,t_0) \leq 0$, then (4.3) is automatically satisfied. We therefore assume that $U(x_0,t_0) > 0$ to continue.

For $x,y,z \in \mathbb{T}^N$ and $0 \leq t \leq s$, we consider

$$\Phi(x,y,z,t,s) = u(x,t) - v(z) - \mu(u(y,s) - v(z)) - \phi(x,y,z,t,s), \tag{4.4}$$

with

$$\phi(x,y,z,t,s) = \mu\eta(s-t) + \alpha^2(|x-y|^2 + |x-z|^2 + |y-z|^2) + |s-s_0|^2 + \phi_0(x,t), \tag{4.5}$$

where s_0 is the point where the maximum is achieved in (4.1). The function Φ achieves its maximum over $(\mathbb{T}^N)^3 \times \{(t,s) : s \geq t,\ t \in [t_0-\delta, t_0+\delta]\}$ at $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s})$ because u, v are bounded continuous. We obtain some classical estimates when $\alpha \to \infty$,

$$(4.6)\qquad \begin{cases} \Phi(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s}) \to U(x_0,t_0) - \phi_0(x_0,t_0), \\ \alpha(\bar{x}-\bar{y}),\ \alpha(\bar{x}-\bar{z})\, \alpha(\bar{y}-\bar{z}) \ \to 0, \\ (\bar{x}, \bar{t}, \bar{s}) \to (x_0, t_0, s_0) \text{ since } (x_0,t_0,s_0) \text{ is a strict maximum point} \\ \hfill \text{of } U(x,t) - \phi_0(x,t) - |s-s_0|^2, \\ \bar{s} > \bar{t} \text{ since } U(x_0,t_0) > 0. \end{cases}$$

In the sequel, all the derivatives of ϕ are calculated at $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s})$ so we skip this dependence for simplicity.

The theory of second order viscosity [7] yields, for every $\alpha > 1$, the existence of symmetric matrices X, Y, Z such that

$$(4.7)\qquad (\phi_t, D_x\phi, X) \in \overline{J}^{2,+} u(\bar{x}, \bar{t}), \quad (\frac{-\phi_s}{\mu}, \frac{-D_y\phi}{\mu}, \frac{-Y}{\mu}) \in \overline{J}^{2,-} u(\bar{y}, \bar{s}),$$
$$(\frac{D_z\phi}{\mu-1}, \frac{Z}{\mu-1}) \in \overline{J}^{2,+} v(\bar{x}),$$

$$(4.8)\qquad -(\alpha^2 + |A|)I \leq \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \leq A + \frac{1}{\alpha^2}A^2, \quad A = D^2\phi(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s}).$$

We have

$$(4.9)\qquad A = 2\alpha^2\mathcal{A} + \mathcal{B}, \quad \mathcal{A} := \begin{pmatrix} 2I & -I & -I \\ -I & 2I & -I \\ -I & -I & 2I \end{pmatrix}, \ \mathcal{B} := \begin{pmatrix} B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $B = D^2\phi_0(\bar{x}, \bar{t})$. It follows from (4.8) and (4.9) that

$$(4.10)\qquad -C(\alpha^2+|B|)I \leq \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \leq C\alpha^2\mathcal{A}+\mathcal{B} + C(\frac{|B|^2}{\alpha^2}I + \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}).$$

Since $\tilde{u}$ is solution of (2.1) and v is solution of (2.6), the following viscosity inequalities hold,

$$(4.11)\qquad \begin{cases} -\mu\eta + \dfrac{\partial\phi_0}{\partial t}(\bar{x}, \bar{t}) \\ \qquad + \sup\limits_{\theta\in\Theta} \big\{ -\mathrm{trace}(A_\theta(\bar{x})X) + H_\theta\big(\bar{x}, p+q+D\phi_0(\bar{x},\bar{t})\big) \big\} \leq c, \\ -\eta + 2(\bar{s}-s_0) + \sup\limits_{\theta\in\Theta} \{ -\mathrm{trace}(A_\theta(\bar{y})\dfrac{-Y}{\mu}) + H_\theta(\bar{y}, \dfrac{p}{\mu}) \} \geq c, \\ \sup\limits_{\theta\in\Theta} \{ -\mathrm{trace}(A_\theta(\bar{z})\dfrac{Z}{\mu-1}) + H_\theta(\bar{z}, \dfrac{-q}{\mu-1}) \} \leq c, \end{cases}$$

where

$$p = 2\alpha^2(\bar{x} - \bar{y}) + 2\alpha^2(\bar{z} - \bar{y}) \quad \text{and} \quad q = 2\alpha^2(\bar{x} - \bar{z}) + 2\alpha^2(\bar{y} - \bar{z}). \tag{4.12}$$

In the sequel, $o(1) \to 0$ as $\alpha \to +\infty$ uniformly with respect to θ.

Using (2.4), (4.6) and the boundedness of $|p|, |q|$ since u, v are Lipschitz continuous with respect to x (see (2.5)), it follows

$$\begin{cases} \dfrac{\partial \phi_0}{\partial t}(x_0, t_0) - \mu\eta \\ \qquad + \sup\limits_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(\bar{x})X) + H_\theta(x_0, p+q)\} - C|D\phi_0(x_0,t_0)| \le c + o(1), \\ -\mu\eta + \sup\limits_{\theta\in\Theta}\{\mathrm{trace}(A_\theta(\bar{y})Y) + \mu H_\theta(x_0, \dfrac{p}{\mu})\} \ge \mu c + o(1), \\ \sup\limits_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(\bar{z})Z) + (\mu-1)H_\theta(x_0, \dfrac{-q}{\mu-1})\} \le (\mu-1)c + o(1). \end{cases} \tag{4.13}$$

Notice that the above constant C may be chosen as

$$C = \sup_{\theta\in\Theta} |(H_\theta)_p|_{L^\infty(\mathbb{T}^N\times R)}, \qquad \text{with } R = \sup_{t\ge 0} |Du(\cdot,t)|_{L^\infty(\mathbb{T}^N)} + |Dv|_{L^\infty(\mathbb{T}^N)}. \tag{4.14}$$

Summing the inequalities leads to

$$\begin{aligned} &\frac{\partial \phi_0}{\partial t}(x_0,t_0) + \inf_{\theta\in\Theta}\big\{ - \mathrm{trace}(A_\theta(\bar{x})X + A_\theta(\bar{y})Y + A_\theta(\bar{z})Z) \\ &\quad + H_\theta(x_0, p+q) + (\mu-1)H_\theta(x_0, \frac{-q}{\mu-1}) - \mu H_\theta(x_0, \frac{p}{\mu})\big\} - C|D\phi_0(x_0,t_0)| \le o(1). \end{aligned} \tag{4.15}$$

From (4.9) and (4.10), using classical computations [12, p.74], we obtain

$$\begin{aligned} & \mathrm{trace}(A_\theta(\bar{x})X + A_\theta(\bar{y})Y + A_\theta(\bar{z})Z) \\ \le\ & C\alpha^2\, \mathrm{trace}\big((\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{y}))^T(\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{y})) + (\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{z}))^T(\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{z})) \\ & \qquad + (\sigma_\theta(\bar{z}) - \sigma_\theta(\bar{y}))^T(\sigma_\theta(\bar{z}) - \sigma_\theta(\bar{y}))\big) \\ & + \mathrm{trace}\,(A_\theta(\bar{x})B) + C\frac{|B|^2}{\alpha^2} + \mathrm{trace}\,((2A_\theta(\bar{x}) - A_\theta(\bar{y}) - A_\theta(\bar{z}))B)\,. \end{aligned} \tag{4.16}$$

Since $B = D^2\phi_0(x_0,t_0)$, $|B| \le C$, σ_θ is Lispchitz continuous by (2.3) and $\bar{x}, \bar{y}, \bar{z} \to x_0$, we obtain

$$-\mathrm{trace}(A_\theta(\bar{x})X + A_\theta(\bar{y})Y + A_\theta(\bar{z})Z) \ge -\mathrm{tr}(A_\theta(x_0)D^2\phi_0(x_0,t_0)) + o(1). \tag{4.17}$$

Since H_θ is convex and $p/\mu = (p+q)/\mu - q/\mu$, we have

$$H_\theta(x_0, p+q) + (\mu-1)H_\theta(x_0, \frac{-q}{\mu-1}) - \mu H_\theta(x_0, \frac{p}{\mu}) \ge 0. \tag{4.18}$$

Using these previous estimates for (4.15) and letting $\alpha \to +\infty$, we obtain

$$\frac{\partial \phi_0}{\partial t}(x_0,t_0) + \inf_{\theta\in\Theta}\big\{ - \mathrm{tr}(A_\theta(x_0)D^2\phi_0(x_0,t_0))\big\} - C|D\phi_0(x_0,t_0)| \le 0,$$

which is exactly what we need. □

We set $M^+_{\eta\mu}[u](t) = \max\{0, M_{\eta\mu}[u](t)\}$.

Lemma 4.2. *The function $M^+_{\eta\mu}[u](t)$ is nonincreasing, so it converges to some constant $m_{\eta\mu} \geq 0$ as $t \to +\infty$.*

Proof of Lemma 4.2. At first, it is easy to check that $M_{\eta\mu}[u]$ is continuous and, from Lemma 4.1, by classical computations, $M_{\eta\mu}[u]$ is a viscosity subsolution of

$$\min\{M_{\eta\mu}[u](t), M_{\eta\mu}[u]'(t)\} \leq 0 \quad \text{on } (0, +\infty). \tag{4.19}$$

Let $J = \{t \in [0, +\infty) : M_{\eta\mu}[u](t) > 0\}$. If $J = \emptyset$, then $M^+_{\eta\mu}[u](t) = 0$ for all t and the conclusion follows. If $J \neq \emptyset$, then, by continuity, there exists $t_0 < t_1$ such that $[t_0, t_1] \subset J$. By (4.19) $M_{\eta\mu}[u]'(t) \leq 0$ in the viscosity sense on (t_0, t_1). Therefore, $t \mapsto M_{\eta\mu}[u](t)$ is nonincreasing on $[t_0, t_1]$. Necessarily, $\inf J = 0$ and $t \mapsto M_{\eta\mu}[u](t)$ is nonincreasing on $[0, \sup J)$. If $\sup J = +\infty$, then $t \mapsto M_{\eta\mu}[u](t) > 0$ is nonincreasing on $[0, +\infty)$ and the conclusion follows. If $\sup J < +\infty$, then $M^+_{\eta\mu}[u](t) = 0$ on $[\sup J, +\infty)$ and therefore the limit is 0. $\square$

The strategy of the proof of Theorem 2.1 is to obtain $m_{\eta 1} = 0$. An immediate consequence is that $t \mapsto u(x,t)$ is nondecreasing for every x. The conclusion follows easily, see the end of this section.

So, from now on, we argue by contradiction assuming that

$$m_{\eta 1} > 0. \tag{4.20}$$

The following result makes the link between $m_{\eta\mu}$ and $m_{\eta 1}$.

Lemma 4.3. *For all $\epsilon > 0$, there exists $\mu_\epsilon > 1$ such that, for $1 \leq \mu \leq \mu_\epsilon$, we have*

$$P_{\eta\mu}[u](t) \geq P_{\eta 1}[u](t) - \epsilon, \qquad t \geq 0. \tag{4.21}$$

In particular, there exists $\mu_\eta > 1$ such that for $1 < \mu < \mu_\eta$, we have

$$M_{\eta\mu}[u](t) \geq \frac{m_{\eta 1}}{2} > 0, \qquad t \geq 0. \tag{4.22}$$

Proof of Lemma 4.3. Let $x \in \mathbb{T}^N$, $t \geq 0$. There exists $s_1 \geq t$ such that

$$P_{\eta 1}[u](x,t) = u(x,t) - \mu u(x,s_1) - \mu\eta(s_1 - t) \geq u(x,t) - \mu u(x,s_1) \geq -C$$

since u is bounded. We deduce $\eta(s_1 - t) \leq C$. Therefore

$$P_{\eta\mu}[u](x,t) - P_{\eta 1}[u](x,t) \geq (\mu - 1)(v(x) - u(x,s_1) - \eta(s_1 - t)) \geq -C(\mu - 1).$$

To prove (4.22), it is enough to notice that, since $m_{\eta 1} > 0$ by (4.20), then $M_{\eta 1}[u](t)$ is positive nonincreasing and bigger to $m_{\eta 1}$. It is then sufficient to choose $\epsilon = m_{\eta 1}/2$. $\square$

From now on, we choose $1 < \mu < \mu_\eta$, where μ_η is given by Lemma 4.3, in order that $M_{\eta\mu}[u](t) > 0$.

Lemma 4.4. *There exists $t_n \to +\infty$ such that $u(\cdot, \cdot + t_n)$ converges in $W^{1,\infty}(\mathbb{T}^N \times [0, +\infty))$ to a solution $\tilde{u}$ of (2.1). The function $P_{\eta\mu}[\tilde{u}](x,t)$ is a still a subsolution of (4.3) and $M_{\eta\mu}[\tilde{u}](t) = m_{\eta\mu} > 0$ is independent of t.*

Proof of Lemma 4.4. By (2.5), $\{u(\cdot,t), t\geq 0\}$ is relatively compact in $W^{1,\infty}(\mathbb{T}^N)$. Let any sequence $t_n\to+\infty$ such that $u(\cdot,t_n)$ converges. By the comparison principle for (2.1), we have, for any $n,p\geq 0$,

$$|u(x,t+t_n)-u(x,t+t_p)|\leq |u(\cdot,t_n)-u(\cdot,t_p)|_\infty, \qquad x\in\mathbb{T}^N, t\geq 0.$$

Therefore $(u(\cdot,\cdot+t_n))_n$ is a Cauchy sequence in $W^{1,\infty}(\mathbb{T}^N\times[0,+\infty))$. So it converges to some function $\tilde u\in W^{1,\infty}(\mathbb{T}^N\times[0,+\infty))$, which is still a solution of (2.1) by classical stability results.

We observe that

$$M_{\eta\mu}[u](t+t_n)=\sup_{x\in\mathbb{T}^N, s\geq t}\{u(x,t+t_n)-v(x)-\mu(u(x,s+t_n)-v(x))-\mu\eta(s-t)\}.$$

Since $M_{\eta\mu}[u](t+t_n)\to m_{\eta\mu}$ as $n\to+\infty$, $M_{\eta\mu}[\tilde u](t)=m_{\eta\mu}$ is independent of t.

Finally, since $P_{\eta\mu}[u](x,t+t_n)$ converges uniformly to $P_{\eta\mu}[\tilde u](x,t)$ as $n\to+\infty$, we obtain that $P_{\eta\mu}[\tilde u]$ is still a subsolution of (4.3) □

Lemma 4.5. *For any $\tau>0$,*

$$\max_{x\in\mathbb{T}^N, t\geq 0} P_{\eta\mu}[\tilde u](x,t)=M_{\eta\mu}[\tilde u](\tau)=m_{\eta\mu}=P_{\eta\mu}[\tilde u](x_\tau,\tau)\quad \textit{for } x_\tau\in\Sigma \textit{ if } \Sigma\neq\emptyset. \tag{4.23}$$

If $\Sigma=\emptyset$, then $m_{\eta 1}=0$.

The point in this result is that the maximum of $P_{\eta\mu}[\tilde u](\tau)$ is achieved at some point $x_\tau\in\Sigma$.

Proof of Lemma 4.5. Let $\tau>0$ and suppose that x_τ defined by (4.23) lies in $\mathbb{T}^N-\Sigma$. We write $U(x,t)=P_{\eta\mu}[\tilde u](x,t)$ for simplicity. Since $M_{\eta\mu}[\tilde u](t)$ is independent of t, we have

$$U(x_\tau,\tau)=\max_{x\in\mathbb{T}^N,\, t\geq 0} U(x,t)=m_{\eta\mu}.$$

We aim at applying the strong maximum principle of Da Lio [8] for viscosity solutions. Let $\delta>0$ and $\Sigma_\delta=\{\mathrm{dist}(\cdot,\Sigma)\geq\delta\}$. We consider the connected component $\mathcal{C}_\delta$ of x_τ in $\Sigma_\delta^C\cap\{U(\cdot,\tau)>0\}$. From Lemmas 4.3 and 4.4, U is a subsolution of

$$\frac{\partial U}{\partial t}+G(x,DU,D^2U)\leq 0 \qquad \text{in } \mathrm{int}(\mathcal{C}_\delta)\times(\tau-\tau/2,\tau+\tau/2),$$

where

$$G(x,p,X)=\inf_{\theta\in\Theta}\big\{-\mathrm{tr}(A_\theta(x)X)\big\}-C|p|.$$

From (2.9),

$$G(x,p,X+Y)-G(x,p,Y)\leq-\nu_\delta\,\mathrm{trace}(Y),\quad X,Y\in\mathcal{S}_N,\, Y\geq 0,\, x\in\mathcal{C}_\delta,\, p\in\mathbb{R}^N$$

and $G(x,\lambda p,\lambda X)=\lambda G(x,p,X)$ for $\lambda>0$. From [8, Th. 2.1], we infer that U is constant and equal to $m_{\eta\mu}$ in $\partial(\mathcal{C}_\delta\times\{t=\tau\})$. Moreover, since $m_{\eta\mu}>0$, necessarily $\partial\mathcal{C}_\delta\subset\Sigma_\delta$. It follows that there exists $x_{\tau\delta}$ such that $U(x_{\tau\delta},\tau)=m_{\eta\mu}$ and $\mathrm{dist}(x_{\tau\delta},\Sigma)=\delta$. Letting $\delta\to 0$ and extracting subsequences if necessary, we find $y_\tau\in\partial\Sigma$ such that $U(y_\tau,\tau)=P_{\eta\mu}[\tilde u](y_\tau,\tau)=m_{\eta\mu}$.

In the case $\Sigma = \emptyset$, we obtain that $P_{\eta\mu}[\tilde{u}](x,t) = m_{\eta\mu}$ in $\mathbb{T}^N \times [0,+\infty)$. Letting $\mu \to 1$, we get $P_{\eta 1}[\tilde{u}](x,t) = m_{\eta 1} > 0$ in $\mathbb{T}^N \times [0,+\infty)$. Let $s(t)$ be the point where the maximum is achieved in $P_{\eta 1}[\tilde{u}](x,t)$. We have

$$P_{\eta 1}[\tilde{u}](x,t) + P_{\eta 1}[\tilde{u}](x,s(t)) = 2m_{\eta 1} = u(x,t) - u(x,s(s(t))) - \eta(s(s(t)) - t) \leq m_{\eta 1}$$

which leads to a contradiction with (4.20) and implies $m_{\eta 1} = 0$. $\square$

We now obtain the desired contradiction with (4.20). The following result is is one the key step in the proof of Theorem 2.1.

Lemma 4.6. *If, for some $\tau > 0$,*

$$\max_{x \in \mathbb{T}^N, t \geq 0} P_{\eta\mu}[\tilde{u}](x,t) = M_{\eta\mu}[\tilde{u}](\tau) = m_{\eta\mu} = P_{\eta\mu}[\tilde{u}](x_\tau, \tau) \quad \textit{for } x_\tau \in \Sigma,$$

then $m_{\eta\mu} = 0$.

Proof of Lemma 4.6. We fix $\tau > 0$ and we assume that

$$m_{\eta\mu} = M_{\eta\mu}[\tilde{u}](\tau) = \tilde{u}(x_\tau, \tau) - v(x_\tau) - \mu(\tilde{u}(x_\tau, s_\tau) - v(x_\tau)) - \mu\eta(s_\tau - \tau), \quad \text{with } x_\tau \in \Sigma,$$

and we recall that, by contradiction, we assume $m_{\eta\mu} > 0$. Notice that τ is a strict maximum point of $t \mapsto M_{\eta\mu}[\tilde{u}](t) - |t-\tau|^2$ in $(0,+\infty)$ since $M_{\eta\mu}[\tilde{u}](t)$ is constant.

We define Φ, ϕ as in (4.4)-(4.5) by replacing s_0 with s_τ in ϕ and choosing $\phi_0(x,t) = \langle x - x_\tau \rangle + |t-\tau|^2$, where $\langle x \rangle = \sqrt{\epsilon^2 + |x|^2}$ for some fixed $\epsilon > 0$.

Exactly as in the proof of Lemma 4.3, the function Φ achieves its maximum over $(\mathbb{T}^N)^3 \times \{(t,s) : s \geq t,\ t \in [t_0 - \delta, t_0 + \delta]\}$ at $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s})$ and (4.6) are replaced with

$$\begin{cases} \Phi(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s}) \to m_{\eta\mu} - \epsilon, \\ \alpha(\bar{x} - \bar{y}),\ \alpha(\bar{x} - \bar{z})\,,\ \alpha(\bar{y} - \bar{z}) \to 0, \\ (\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s}) \to (x_\tau, x_\tau, x_\tau, \tau, s_\tau) \\ \bar{s} > \bar{t} \text{ since } M_{\eta\mu}[\tilde{u}](\tau) = m_{\eta\mu} > 0. \end{cases} \tag{4.24}$$

Formulas (4.7)–(4.10) still hold with $B = D^2_{xx}\langle \cdot - x_\tau \rangle(\bar{x}) = \langle \bar{x} - x_\tau \rangle^{-1}(I - \frac{\bar{x}-x_\tau}{\langle \bar{x}-x_\tau \rangle} \otimes \frac{\bar{x}-x_\tau}{\langle \bar{x}-x_\tau \rangle})$. Noticing that $|B| \leq \epsilon^{-1}$, we may refine (4.10)

$$-C(\alpha^2 + \frac{1}{\epsilon})I \leq \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \leq C\alpha^2 \begin{pmatrix} 2I & -I & -I \\ -I & 2I & -I \\ -I & -I & 2I \end{pmatrix} + C(\frac{1}{\alpha^2\epsilon^2} + \frac{1}{\epsilon})I. \tag{4.25}$$

In the sequel, $o(1)$ denotes a function which tends to 0 as $\alpha \to +\infty$ for fixed $\epsilon > 0$, uniformly with respect to θ.

The viscosity inequalities (4.11) and (4.13) hold with $\frac{\partial \phi_0}{\partial t}(\bar{x},\bar{t}) = 2(\bar{t}-\tau)$, $D\phi_0(\bar{x},\bar{t}) = \frac{\bar{x}-x_\tau}{\langle \bar{x}-x_\tau \rangle} = o(1)$,

$$(4.26)\qquad \begin{cases} -\mu\eta + \sup\limits_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(\bar{x})X) + H_\theta(x_\tau, p+q)\} \leq o(1), \\ -\mu\eta + \sup\limits_{\theta\in\Theta}\{\mathrm{trace}(A_\theta(\bar{y})Y) + \mu H_\theta(x_\tau, \dfrac{p}{\mu})\} \geq o(1), \\ \sup\limits_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(\bar{z})Z) + (\mu-1) H_\theta(x_\tau, \dfrac{-q}{\mu-1})\} \leq o(1), \end{cases}$$

with p, q defined in (4.12), and (4.15) reads now

$$\inf_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(\bar{x})X + A_\theta(\bar{y})Y + A_\theta(\bar{z})Z) + \mathcal{H}_\theta\} \leq o(1).$$

where we set

$$\mathcal{H}_\theta := H_\theta(x_\tau, p+q) + (\mu-1)H_\theta(x_\tau, \frac{-q}{\mu-1}) - \mu H_\theta(x_\tau, \frac{p}{\mu}).$$

From (4.16), we get

$$(4.27)\qquad \inf_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(\bar{x})X + A_\theta(\bar{y})Y + A_\theta(\bar{z})Z)\} \geq o(1).$$

It then follows

$$(4.28)\qquad \inf_{\theta\in\Theta} \mathcal{H}_\theta \leq o(1).$$

From the convexity of H_θ, we know that $\mathcal{H}_\theta \geq 0$ (see (4.18)) but we need a strict inequality to reach a contradiction.

Up to extract subsequences, we may assume that

$$\lim_{\alpha\to\infty} p = \bar{p} \quad \text{and} \quad \lim_{\alpha\to\infty} q = \bar{q},$$

(recall that p and q are given by (4.12) and are bounded since $\tilde{u}, v$ are Lipschitz continuous). We distinguish two cases depending on the above limit.

First case. We suppose that

$$\frac{\bar{p}}{\mu} + \frac{\bar{q}}{\mu-1} \neq 0.$$

Letting $\alpha \to +\infty$ in (4.28) and recalling that $x_\tau \in \Sigma$, we obtain a contradiction thanks to the strict convexity of H_θ. More precisely, we apply (2.10) with $\lambda := 1/\mu$ and $P \neq Q$ given by $P := \bar{p}+\bar{q}$, $Q := -\bar{q}/(\mu-1)$.

Second case. One necessarily has

$$(4.29)\qquad \frac{\bar{p}}{\mu} = \frac{-\bar{q}}{\mu-1} =: p_\epsilon = p_\epsilon(\eta,\mu,\epsilon).$$

Notice that, in this case, $\lim_{\alpha\to\infty}\mathcal{H}_\theta = 0$ and therefore the strict convexity of the H does not play any role.

From (4.25), we have $|X|, |Y|, |Z| \le C(\alpha^2 + (\alpha\epsilon)^{-2} + \epsilon^{-1})$. Hence

$$\begin{aligned} |\mathrm{trace}(\sigma_\theta(\bar{x})\sigma_\theta(\bar{x})^T X)| &\le |\sigma_\theta(\bar{x})|^2|X| = |\sigma_\theta(\bar{x}) - \sigma_\theta(x_\tau)|^2|X| \\ &\le C(\alpha^2 + \frac{1}{\alpha^2\epsilon^2} + \frac{1}{\epsilon})|\bar{x} - x_\tau|^2, \end{aligned} \tag{4.30}$$

where we used the fact that $\sigma(x_\tau) = 0$ since $x_\tau \in \Sigma$.

We estimate the rate of convergence of the term $|\bar{x} - x_\tau|$. Since Φ achieves its maximum at $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s})$, we have

$$\tilde{u}(\bar{x}, \bar{t}) - v(\bar{z}) - \mu(\tilde{u}(\bar{y}, \bar{s}) - v(\bar{z})) - \mu\eta(\bar{s} - \bar{t}) - \langle \bar{x} - x_\tau \rangle \ge \Phi(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s}) \ge M_{\eta,\mu}[\tilde{u}](\tau) - \epsilon.$$

This implies

$$\begin{aligned} \sqrt{\epsilon^2 + |\bar{x} - x_\tau|^2} &= \langle \bar{x} - x_\tau \rangle \\ &\le \tilde{u}(\bar{x}, \bar{t}) - v(\bar{z}) - \mu(\tilde{u}(\bar{y}, \bar{s}) - v(\bar{z})) - \mu\eta(\bar{s} - \bar{t}) - M_{\eta,\mu}[\tilde{u}](\tau) + \epsilon \\ &= [\tilde{u}(\bar{x}, \bar{t}) - \tilde{u}(\bar{z}, \bar{t})] + \mu[\tilde{u}(\bar{z}, \bar{s}) - \tilde{u}(\bar{y}, \bar{s})] \\ &\quad + [\tilde{u}(\bar{z}, \bar{t}) - v(\bar{z}) - \mu(\tilde{u}(\bar{z}, \bar{s}) - v(\bar{z})) - \mu\eta(\bar{s} - \bar{t})] - M_{\eta,\mu}[\tilde{u}](\bar{t}) + \epsilon \\ &\le [\tilde{u}(\bar{x}, \bar{t}) - \tilde{u}(\bar{z}, \bar{t})] + \mu[\tilde{u}(\bar{z}, \bar{s}) - \tilde{u}(\bar{y}, \bar{s})] + \epsilon \\ &\le C(|\bar{x} - \bar{y}| + |\bar{x} - \bar{z}|) + \epsilon, \end{aligned}$$

where we used the fact that $M_{\eta,\mu}[\tilde{u}](t) = m_{\eta\mu}$ for all $t > 0$ and $\tilde{u}$ is Lipschitz continuous. So,

$$|\bar{x} - x_\tau|^2 \le C(|\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{z}|^2) + C\epsilon(|\bar{x} - \bar{y}| + |\bar{x} - \bar{z}|).$$

It is worth noticing that C depends only on $\tilde{u}$. Recalling that $\alpha^2|\bar{x} - \bar{y}|$, $\alpha^2|\bar{x} - \bar{z}|$ are bounded and plugging the above estimates in (4.30), we get

$$\mathrm{trace}(A_\theta(\bar{x})X) = o(1) + O(\epsilon),$$

where, for fixed $\epsilon > 0$, $o(1) \to 0$ as $\alpha \to +\infty$ and $O(\epsilon) \to 0$ as $\epsilon \to 0$. Both error terms are uniform in θ. In the same way, we obtain

$$\mathrm{trace}(A_\theta(\bar{y})Y)\,,\ \mathrm{trace}(A_\theta(\bar{z})Z) = o(1) + O(\epsilon).$$

Sending α to $+\infty$ in (4.26), we have

$$\begin{cases} -\mu\eta + \sup_{\theta\in\Theta}\{H_\theta(x_\tau, p_\epsilon)\} + O(\epsilon) \le 0, \\ -\mu\eta + \sup_{\theta\in\Theta}\{\mu H_\theta(x_\tau, p_\epsilon)\} + O(\epsilon) \ge 0, \\ \sup_{\theta\in\Theta}\{(\mu - 1)H_\theta(x_\tau, p_\epsilon)\} + O(\epsilon) \le 0 \end{cases}$$

(we recall that p_ϵ is defined in (4.29)). Up to a subsequence if necessary, we can assume that $p_\epsilon \to p_0$ when $\epsilon \to 0$. So, we get

$$\begin{cases} -\mu\eta + \sup_{\theta\in\Theta}\{H_\theta(x_\tau, p_0)\} \le 0, \\ -\mu\eta + \sup_{\theta\in\Theta}\{\mu H_\theta(x_\tau, p_0)\} \ge 0, \\ \sup_{\theta\in\Theta}\{(\mu - 1)H_\theta(x_\tau, p_0)\} \le 0. \end{cases}$$

This implies $\mu\eta = 0$, which is a contradiction. It ends the proof. $\square$

End of the proof of Theorem 2.1. We obtained that $m_{\eta 1} = 0$. From $m_{\eta 1} = 0$, we infer

$$\tilde{u}(x,t) - \tilde{u}(x,s) - \eta(s-t) \leq 0, \text{ for all } x \in \mathbb{T}^N \text{ and } s \geq t \geq 0.$$

Letting η tend to 0, we obtain

$$\tilde{u}(x,t) - \tilde{u}(x,s) \leq 0.$$

The uniform convergence of $(u(\cdot, t_n + \cdot))_n$ to $\tilde{u} \in W^{1,\infty}(\mathbb{T}^N \times [0, +\infty))$ (see Lemma 4.4) yields

$$-o_n(1) + \tilde{u}(x,t) \leq u(x, t+t_n) \leq o_n(1) + \tilde{u}(x,t) \quad \text{in } \mathbb{T}^N \times (0, \infty).$$

Since $\tilde{u}$ is nondecreasing in t, there exists $u_\infty \in W^{1,\infty}(\mathbb{T}^N)$ such that $\tilde{u}(\cdot, t) \to u_\infty(\cdot)$ uniformly as t tends to infinity. Taking Barles-Perthame half relaxed limits, we obtain

$$-o_n(1) + u_\infty(x) \leq \liminf_{t\to+\infty}{}_* u(x,t) \leq \limsup_{t\to+\infty}{}^* u(x,t) \leq o_n(1) + u_\infty(x) \quad x \in \mathbb{T}^N.$$

Letting n tend to infinity, we derive

$$\liminf_{t\to+\infty}{}_* u(x,t) = \limsup_{t\to+\infty}{}^* u(x,t) = u_\infty(x), \quad x \in \mathbb{T}^N,$$

which yields the uniform convergence of $u(\cdot, t)$ to u_∞ in $\mathbb{T}^N$ as t tends to infinity.

By the stability result, u_∞ is a solution of (2.6) with $c = 0$. It ends the proof of Theorem 2.1. $\square$

5. Proof of Theorem 2.2 and Proposition 2.3

The proof of Theorem 2.2 follows the same ideas as the one of Theorem 2.1 with minor adaptations. It is actually easier, since, from (2.12), we choose $v = 0$ in (4.1)-(4.2) which allows to simplify several arguments. We only provide the proof of the main changes which consist, on the one side, in taking into account the set K which appears in (2.11) and, on the other side, in the proof of Lemma 4.6.

As in the proof of Theorem 2.1, we start with a change of function $u \to u + ct$ which allows to deal with bounded functions $u, \tilde{u}$ and $c = 0$.

Lemma 5.1. *For every $x_0 \in K$, The function $t \mapsto u(x_0, t)$ is nonincreasing.*

Proof of Lemma 5.1. Let $x_0 \in K$, $t_0 \geq 0$ and we assume by contradiction that there exists $s_0 > t_0$ such that $u(x_0, s_0) > u(x_0, t_0)$. Consider, for $\epsilon, \alpha > 0$,

$$\sup_{x\in\mathbb{T}^N, t\geq t_0} \{u(x,t) - u(x_0, t_0) - \frac{|x - x_0|^2}{\epsilon^2} - \alpha(t - t_0)\}. \tag{5.1}$$

Since u is bounded, this supremum is positive and is achieved at some $(\bar{x}, \bar{t})$ with $\bar{t} > t_0$ for $\epsilon, \alpha > 0$ small enough. By classical estimates, $\frac{|\bar{x}-x_0|^2}{\epsilon^2} \to 0$ as $\epsilon \to 0$. Since u is a viscosity subsolution of (2.1), we obtain

$$\alpha + \sup_{\theta\in\Theta}\{-\text{trace}(A_\theta(\bar{x})\frac{2I}{\epsilon^2}) + H_\theta(\bar{x}, p)\} \leq 0, \tag{5.2}$$

with $p = 2\frac{\bar{x}-x_0}{\epsilon^2}$. On the one side, since $u(\cdot, t)$ is Lipschitz continuous, p is bounded and, up to extract a subsequence as $\epsilon \to 0$, we may assume that $p \to \bar{p}$. On the other side, since $\bar{x} \to x_0 \in K \subset \Sigma$ and σ_θ satisfies (2.3),

$$|\mathrm{trace}(A_\theta(\bar{x})\frac{2I}{\epsilon^2}| \leq \frac{|\sigma_\theta(\bar{x})|^2}{\epsilon^2} \leq C\frac{|\bar{x}-x_0|^2}{\epsilon^2}.$$

From (5.2), sending $\epsilon \to 0$, we obtain

$$\alpha + \sup_{\theta\in\Theta} H_\theta(\bar{x}, \bar{p}) \leq 0,$$

which is a contradiction with (2.11)(ii)(a) (with $c = 0$).

Therefore, for all $s_0 \geq t_0$, we have $u(x_0, s_0) \leq u(x_0, t_0)$. □

A consequence of Lemma 5.1 is that $u(x, t)$ converges on K and therefore

$$\tilde{u}(x, t) \text{ is independent of } t, \text{ for any } x \in K,$$

where $\tilde{u}$ is defined in the statement of Lemma 4.4. Assuming, as in the proof of Theorem 2.1, that $m_{\eta 1} > 0$ (and therefore $m_{\eta\mu} > 0$ for μ close to 1), we obtain from the very definition of $P_{\eta\mu}[\tilde{u}]$ that

$$\mathrm{dist}(x_\tau, K) \neq 0 \text{ for } \mu \text{ close enough to } 1, \tag{5.3}$$

where $x_\tau \in \mathbb{T}^N$ is the point where the maximum is achieved in $M_{\eta\mu}[\tilde{u}](\tau)$.

Proof of Lemma 4.6 under the assumptions of Theorem 2.2. deptra Let us note that Lemma 4.5 is still true under the assumptions of Theorem 2.2, so we can assume that $x_\tau \in \Sigma$.

Since $v = 0$ in (4.1)-(4.2), we may choose $Z = 0$ in (4.25), and $q = 0$ in (4.12). The viscosity inequalities (4.26) reads

$$\begin{cases} -\mu\eta + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(\bar{x})X) + H_\theta(x_\tau, p)\} \leq o(1), \\ -\mu\eta + \sup_{\theta\in\Theta}\{\mathrm{trace}(A_\theta(\bar{y})Y) + \mu H_\theta(x_\tau, \frac{p}{\mu})\} \geq o(1), \\ \sup_{\theta\in\Theta} H_\theta(x_\tau, 0) \leq o(1). \end{cases} \tag{5.4}$$

Notice that the third inequality is nothing than (2.12) (with $c = 0$ after our change of function). Subtracting the two first inequalities from (4.27) yield

$$\inf_{\theta\in\Theta}\{H_\theta(x_\tau, p) - \mu H_\theta(x_\tau, \frac{p}{\mu})\} \leq o(1). \tag{5.5}$$

As in the corresponding proof in Section 4, we distinguish two cases depending on

$$\lim_{\alpha\to+\infty} p = \bar{p}$$

(up to subsequences if necessary).

First Case. If $\bar{p} \neq 0$. Letting $\alpha \to +\infty$ in (5.5) and recalling (5.3), we obtain a contradiction with (2.11)(ii)(b).

Second Case. If $\bar{p}=0$. Proceeding similarly as in the second case of the proof of Lemma (4.6), we obtain

$$|\mathrm{trace}(A_\theta(\bar{y})Y)| = o(1) + O(\epsilon).$$

Taking into account this estimate, by sending $\alpha \to \infty$ and then $\epsilon \to 0$ in the second inequality in (5.4) , we get

$$-\mu\eta + \sup_{\theta\in\Theta}\{\mu H_\theta(x_\tau, 0)\} \geq 0,$$

which is a contradiction with the third inequality in (5.4). □

Proof of Proposition 2.3. Consider the solution v_λ^ϵ of

$$\lambda v_\lambda^\epsilon + \sup_{|e|\leq\epsilon,\,\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x+e)D^2 v_\lambda^\epsilon) + H_\theta(x+e, Dv_\lambda^\epsilon)\} = 0, \quad x \in \mathbb{T}^N.$$

It follows from [2, Lemma 2.7] that $v_{\lambda\epsilon} = \rho_\epsilon * v_\lambda^\epsilon$, where ρ_ϵ is a standard mollifier, is a C^∞ subsolution of (2.2). Moreover, from [2, Theorem A.1], we have $\lambda|v_\lambda - v_{\lambda\epsilon}| \leq C\epsilon$. Therefore, we have in the classical sense at any $x \in \mathbb{T}^N$,

$$\lambda v_{\lambda\epsilon}(x) + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2 v_{\lambda\epsilon}(x)) + H_\theta(x, Dv_{\lambda\epsilon}(x))\} \leq 0.$$

We can write this inequality at any $\hat{x} \in \Sigma$ where $\mathrm{trace}(A_\theta(\hat{x})D^2 v_{\lambda\epsilon}(\hat{x})) = 0$. It follows

$$-\lambda v_\lambda(\hat{x}) + C\epsilon \geq -\lambda v_{\lambda\epsilon}(\hat{x}) \geq \sup_{\theta\in\Theta} H_\theta(\hat{x}, Dv_{\lambda\epsilon}(\hat{x})) \geq \sup_{\theta\in\Theta} H_\theta(\hat{x}, 0),$$

using (2.13). Sending $\lambda \to 0$ and then $\epsilon \to 0$, we obtain

$$-\lambda v_\lambda(\hat{x}) \to c \geq \sup_{\theta\in\Theta} H_\theta(\hat{x}, 0), \quad \text{for any } \hat{x} \in \Sigma.$$

Hence $c \geq \sup_{x\in\Sigma,\theta\in\Theta} H_\theta(x, 0)$.

We prove now the opposite inequality under either (2.15) or (2.14).

Under Assumption (2.14). Let v_λ be the solution of (2.2) and $x_\lambda \in \mathbb{T}^N$ such that $v_\lambda(x_\lambda) = \min_{\mathbb{T}^N} v_\lambda$. We have $\lambda v_\lambda(x_\lambda) + \sup_\theta H_\theta(x_\lambda, 0) \geq 0$. Taking a subsequence $\lambda \to 0$ such that $\lambda v_\lambda \to -c$, we get

$$c \leq \sup_{x\in\mathbb{T}^N,\theta\in\Theta} H_\theta(x, 0) = \sup_{x\in\Sigma,\theta\in\Theta} H_\theta(x, 0)$$

by (2.13).

Under Assumption (2.15). We set $\hat{H}_\theta(x,p) = H_\theta(x,p) - C$ where $C > 0$ is big enough in order that $\hat{H}_\theta(x, 0) \leq 0$. It follows that, if v_λ is a solution of (2.2), then $\hat{v}_\lambda = v_\lambda + C/\lambda$ is a solution of

$$\lambda \hat{v}_\lambda + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2 \hat{v}_\lambda) + \hat{H}_\theta(x, D\hat{v}_\lambda)\} = 0 \tag{5.6}$$

and $\hat{v}_\lambda \geq 0$. For any $\gamma > 1$, we have

$$\frac{\lambda}{\gamma}\hat{v}_{\lambda/\gamma} + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2 \hat{v}_{\lambda/\gamma}) + \hat{H}_\theta(x, D\hat{v}_{\lambda/\gamma})\} = 0,$$

equivalently,

$$\lambda\hat{v}_{\lambda/\gamma} + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2(\gamma\hat{v}_{\lambda/\gamma})) + \gamma\hat{H}_\theta(x, D\hat{v}_{\lambda/\gamma})\} = 0.$$

Noticing that $\hat{H}_\theta$ still satisfies (2.15), we have

$$\begin{aligned}\lambda(1-\gamma)\min_{\mathbb{T}^N}\hat{v}_{\lambda/\gamma} + \lambda(\gamma\hat{v}_{\lambda/\gamma})\\ + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2(\gamma\hat{v}_{\lambda/\gamma})) + \hat{H}_\theta(x, D(\gamma\hat{v}_{\lambda/\gamma})) - (1-\gamma)\hat{H}_\theta(x,0)\} \geq 0.\end{aligned} \tag{5.7}$$

Subtracting (5.6) and (5.7), we get, for $w_{\lambda\gamma} = \hat{v}_\lambda - \gamma\hat{v}_{\lambda/\gamma}$,

$$\begin{aligned} 0 \ \geq\ & \lambda(\gamma-1)\min_{\mathbb{T}^N}\hat{v}_{\lambda/\gamma} + \lambda w_{\lambda\gamma}\\ & + \inf_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta D^2 w_{\lambda\gamma}) + \hat{H}_\theta(x, D\hat{v}_\lambda) - \hat{H}_\theta(x, D(\gamma\hat{v}_{\lambda/\gamma})) + (1-\gamma)\hat{H}_\theta(x,0)\}\\ \geq\ & \lambda(\gamma-1)\min_{\mathbb{T}^N}\hat{v}_{\lambda/\gamma} + \lambda w_{\lambda\gamma} + \inf_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta D^2 w_{\lambda\gamma}) + (1-\gamma)\hat{H}_\theta(x,0)\} - C|Dw_{\lambda\gamma}|.\end{aligned}$$

Therefore

$$\lambda w_{\lambda\gamma} + \inf_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta D^2 w_{\lambda\gamma})\} - C|Dw_{\lambda\gamma}| \leq (\gamma-1)\left(\sup_{\theta\in\Theta}\hat{H}_\theta(x,0) - \lambda\min_{\mathbb{T}^N}\hat{v}_{\lambda/\gamma}\right). \tag{5.8}$$

Recalling that $\hat{H}_\theta(x,0)\leq 0$ and $\hat{v}_{\lambda/\gamma}\geq 0$, the right-hand side of (5.8) is nonnegative. By the strong maximum principle, we obtain

$$\max_{x\in\mathbb{T}^N} w_{\lambda\gamma} = w_{\lambda\gamma}(x_0) \quad \text{with } x_0\in\Sigma. \tag{5.9}$$

Writing (5.8) at x_0, we obtain

$$\lambda w_{\lambda\gamma}(x_0) \leq (\gamma-1)\left(\sup_{x\in\Sigma,\theta\in\Theta}\hat{H}_\theta(x,0) - \lambda\min_{\mathbb{T}^N}\hat{v}_{\lambda/\gamma}\right).$$

It follows

$$\lambda v_\lambda(x_0) - \gamma^2\frac{\lambda}{\gamma}v_{\lambda/\gamma}(x_0) \leq (\gamma-1)\left(\sup_{x\in\Sigma,\theta\in\Theta}H_\theta(x,0) - \gamma\min_{\mathbb{T}^N}\frac{\lambda}{\gamma}v_{\lambda/\gamma}\right).$$

Sending $\lambda\to 0$, up to take subsequences, we obtain

$$c \leq \sup_{x\in\Sigma,\theta\in\Theta}H_\theta(x,0).$$

□

6. Proof of Theorem 2.5 and Propositions 2.6 and 2.7

Proof of Theorem 2.5. The proof follows exactly the same line as those of Theorems 2.1 and 2.2. The only difference is the proof of Lemma 4.5 which is given below. □

Proof of Lemma 4.5 when (2.16) *holds.* We write $U = P_{\eta\mu}[\tilde{u}]$ for simplicity. Since U is bounded, we can consider the half-relaxed limit

$$\overline{U}(x) = \limsup_{y\to x, t\to+\infty} U(y,t).$$

From Lemma 4.1 and by the stability result, $\overline{U}$ is a viscosity subsolution of

$$\min\{\overline{U}\,,\ \inf_{\theta\in\Theta}\big\{-\operatorname{tr}(A_\theta(x)D^2\overline{U})\big\} - C|D\overline{U}|\} \le 0, \quad x\in\mathbb{T}^N. \tag{6.1}$$

Notice that one still has $\max_{\mathbb{T}^N}\overline{U} = m_{\eta\mu} > 0$.

Step 1. $\operatorname{argmax}\overline{U}\cap\Sigma\neq\emptyset$ *thanks to* (2.16). We argue by contradiction assuming that there exists $\delta>0$ such that $\operatorname{argmax}\overline{U}\subset\Sigma_\delta^C$, where $\Sigma_\delta = \{\operatorname{dist}(\cdot,\Sigma)\le\delta\}$. It follows that there exists $\rho_\delta>0$ such that

$$m_{\eta\mu} = \overline{U}(\hat{x}) = \max_{\mathbb{T}^N}\overline{U} = \max_{\Sigma_\delta^C}\overline{U} \ge \max_{\Sigma_\delta}\overline{U} + \rho_\delta, \quad \text{for some } \hat{x}\in\Sigma_\delta^C. \tag{6.2}$$

Let $\tilde{U}$ be a 1-periodic function of $\mathbb{R}^N$ such that $\overline{U}(\pi(\tilde{x})) = \tilde{U}(\tilde{x})$ for all $\tilde{x}\in\mathbb{R}^N$ and $\tilde{\Sigma}_\delta = \{\operatorname{dist}(\cdot,\tilde{\Sigma})\le\delta\}$. From (6.2) and by 1-periodicity, we infer

$$m_{\eta\mu} = \tilde{U}(\tilde{x}) = \max_{\mathbb{R}^N}\tilde{U} = \sup_{\tilde{\Sigma}_\delta^C}\tilde{U} \ge \sup_{\tilde{\Sigma}_\delta}\tilde{U} + \rho_\delta, \quad \text{for some } \tilde{x}\in\tilde{\Sigma}_\delta^C\cap[0,1]^N.$$

For this $\delta>0$, we consider the C^2 supersolution $\tilde{\psi}_\delta$ and Ω_δ given by (2.16). Notice that, up to divide $\tilde{\psi}_\delta$ by a constant, we can assume that $|\tilde{\psi}_\delta|\le 1$ in $\overline{\Omega}_\delta$. We claim that, for $\varepsilon>0$ small enough,

$$\sup_{\mathbb{R}^N}\{\tilde{U}-\varepsilon\tilde{\psi}_\delta\} = \tilde{U}(\tilde{x}_\delta) - \varepsilon\tilde{\psi}_\delta(\tilde{x}_\delta) \quad \text{with } \tilde{x}_\delta\in\tilde{\Sigma}_\delta^C\cap\Omega_\delta \text{ and } \tilde{U}(\tilde{x}_\delta)>0.$$

Indeed, using that $|\tilde{\psi}_\delta|\le 1$ in $\overline{\Omega}_\delta$ and $\tilde{\psi}_\delta\ge 0$ on Ω_δ^C, we have

$$\begin{aligned}
\sup_{\mathbb{R}^N}\{\tilde{U}-\varepsilon\tilde{\psi}_\delta\} &\ge \tilde{U}(\tilde{x}) - \varepsilon\tilde{\psi}_\delta(\tilde{x}) \\
&\ge \sup_{\tilde{\Sigma}_\delta^C}\tilde{U} - \varepsilon \\
&\ge \sup_{\tilde{\Sigma}_\delta}\tilde{U} + \rho_\delta - \varepsilon \\
&\ge \sup_{\tilde{\Sigma}_\delta}\{\tilde{U}-\varepsilon\tilde{\psi}_\delta\} + \rho_\delta - 2\varepsilon > \sup_{\tilde{\Sigma}_\delta}\{\tilde{U}-\varepsilon\tilde{\psi}_\delta\}
\end{aligned}$$

for ε small enough. Since $\tilde{U}$ is 1-periodic, $\tilde{\psi}_\delta\le 0$ on $\Omega_\delta\supset[0,1]^N$ and $\tilde{\psi}_\delta\ge 0$ on Ω_δ^C, it follows

$$\sup_{\mathbb{R}^N}\{\tilde{U}-\varepsilon\tilde{\psi}_\delta\} = \max_{\Omega_\delta}\{\tilde{U}-\varepsilon\tilde{\psi}_\delta\} = \tilde{U}(\tilde{x}_\delta) - \varepsilon\tilde{\psi}_\delta(\tilde{x}_\delta) \quad \text{with } \tilde{x}_\delta\in\Omega_\delta\cap\tilde{\Sigma}_\delta^C.$$

Moreover

$$\tilde{U}(\tilde{x}_\delta) \ge \tilde{U}(\tilde{x}) - \varepsilon\tilde{\psi}_\delta(\tilde{x}) + \varepsilon\tilde{\psi}_\delta(\tilde{x}_\delta) \ge m_{\eta\mu} - 2\varepsilon.$$

The claim is proved for ε small enough.

Since $\tilde{U}(\tilde{x}_\delta) > 0$, the differential inequality holds in (6.1) in the viscosity sense at $\tilde{x}_\delta$. Using $\varepsilon\tilde{\psi}_\delta$ as a test-function for $\tilde{U}$, we obtain

$$\inf_{\theta\in\Theta}\{-\mathrm{trace}(\tilde{A}_\theta(\tilde{x}_\delta)D^2\tilde{\psi}_\delta(\tilde{x}_\delta))\} - C|D\tilde{\psi}_\delta(\tilde{x}_\delta)| \leq 0,$$

which contradicts (2.16).

Therefore, there exists $\hat{x}_\delta \in \Sigma_\delta$ such that $\overline{U}(\hat{x}_\delta) = m_{\eta\mu}$. Letting $\delta \to 0$ and extracting subsequences if necessary, we can find $\hat{x} \in \mathrm{argmax}\,\overline{U} \cap \Sigma$.

Step 2. Up to replace $\tilde{u}$ *by an accumulation point as in Lemma 4.4, we may assume that* $P_{\eta\mu}[\tilde{u}]$ *achieves its maximum at* $(\hat{x}, 1)$, $\hat{x} \in \Sigma$. From the previous step, we have $\overline{U}(\hat{x}) = m_{\eta\mu}$ for some $\hat{x} \in \Sigma$. By definition of the half-relaxed limit, there exists $t_n \to +\infty$ and $x_n \to \hat{x}$ such that $U(x_n, t_n) \to m_{\eta\mu}$. Let $\hat{t}_n = t_n - 1$. Up to extract subsequences as in the proof of Lemma 4.4, we may assume that $\tilde{u}(x, t+\hat{t}_n)$ converges uniformly in $W^{1,\infty}(\mathbb{T}^N \times [0, +\infty))$ to some function $\hat{u}$. Therefore $P_{\eta\mu}[\tilde{u}](x, t + \hat{t}_n)$ converges uniformly to $P_{\eta\mu}[\hat{u}](x, t)$. It follows

$$P_{\eta\mu}[\tilde{u}](x_n, \hat{t}_n + 1) = U(x_n, t_n) \to P_{\eta\mu}[\hat{u}](\hat{x}, 1) = m_{\eta\mu}.$$

The functions $\hat{u}$, $P_{\eta\mu}[\hat{u}]$ inherit the properties of $\tilde{u}$, $P_{\eta\mu}[\tilde{u}]$ respectively and it is sufficient to prove the convergence of $\hat{u}$ to obtain the convergence of $\tilde{u}$ and u. □

Proof of Proposition 2.6. Since $\Sigma \neq \emptyset$, by translation, we can assume without loss of generality that $0 \in \tilde{\Sigma}$, where $\tilde{\Sigma} \subset \mathbb{R}^N$ is a coset representative of $\Sigma \in \mathbb{T}^N$. Let $\delta > 0$ and $\Sigma_\delta = \{\mathrm{dist}(\cdot, \Sigma) \leq \delta\}$. From (2.9), we have

$$\inf_{\tilde{\Sigma}_\delta^C} |\tilde{\sigma}_\theta(x)x|^2 = \inf_{\tilde{\Sigma}_\delta^C} \nu(x)|x|^2 =: \nu_\delta > 0.$$

We then consider the classical smooth test function which is used to prove the strong maximum principle, that is

$$\tilde{\psi}_\delta(x) = e^{-\gamma_\delta r_\delta^2} - e^{-\gamma_\delta |x|^2},$$

where we fix $r_\delta > \sqrt{N}$, $\Omega_\delta := B(0, r_\delta)$ and $\gamma_\delta > 0$ will be chosen later. We have $\tilde{\psi}_\delta < 0$ in $B(0, r_\delta) \supset [0, 1]^N$, $\tilde{\psi}_\delta \geq 0$ in $B(0, r_\delta)^C$ and $-1 < \tilde{\psi}_\delta \leq e^{-\gamma r_\delta^2}$.

For $x \in \tilde{\Sigma}_\delta^C \cap B(0, r_\delta)$, using (2.3), we have

$$\begin{aligned} & -\mathrm{trace}(\tilde{\sigma}_\theta(x)\tilde{\sigma}_\theta(x)^T D^2\tilde{\psi}_\delta(x)) - C|D\tilde{\psi}_\delta(x)| \\ = \ & 2\gamma_\delta e^{-\gamma_\delta|x|^2}\left(2\gamma_\delta|\tilde{\sigma}_\theta(x)x|^2 - \mathrm{trace}(\tilde{\sigma}_\theta(x)\tilde{\sigma}_\theta(x)^T) - C|x|\right) \\ \geq \ & 2\gamma_\delta e^{-\gamma_\delta|x|^2}\left(2\gamma_\delta\nu_\delta - C^2 - Cr_\delta\right) > 0 \end{aligned}$$

if γ_δ big enough. Therefore (2.16) holds. □

Proof of Proposition 2.7. For $\delta > 0$ and $\Sigma_\delta = \{\mathrm{dist}(\cdot, \Sigma) \leq \delta\}$, we define

$$K_\delta := \bigcup_{x\in\overline{\Sigma_\delta^C}, \theta\in\Theta} \ker(\sigma_\theta(x)) \cap \mathbb{S}^{N-1} \subset K_0 := \bigcup_{x\in\Sigma^C, \theta\in\Theta} \ker(\sigma_\theta(x)) \cap \mathbb{S}^{N-1}.$$

Using (2.17), we check easily that K_δ is a compact subset of $\mathbb{S}^{N-1}$. Since $K_0 \neq \mathbb{S}^{N-1}$ by (2.19), there exists $\xi_\delta \in \mathbb{S}^{N-1}$ and $\epsilon_\delta > 0$ such that

$$\mathcal{C}_\delta \cap K_\delta = \emptyset, \quad \text{with } \mathcal{C}_\delta := \{\zeta \in \mathbb{S}^{N-1} : \langle \zeta, \xi_\delta \rangle \geq 1 - \epsilon_\delta\}. \tag{6.3}$$

For $\lambda > 0$, let $y_\delta = \lambda \xi_\delta \in \mathbb{R}^N$. We have, for all $x \in [0,1]^N$,

$$\langle \frac{y_\delta - x}{|y_\delta - x|}, \xi_\delta \rangle = \frac{\lambda}{|\lambda \xi_\delta - x|} - \frac{\langle x, \xi_\delta \rangle}{|\lambda \xi_\delta - x|} \geq \frac{\lambda}{\lambda + \sqrt{N}} - \frac{\sqrt{N}}{\lambda - \sqrt{N}} \geq 1 - \epsilon_\delta$$

for $\lambda = \lambda_\delta$ big enough. Therefore $\{\frac{y_\delta - x}{|y_\delta - x|} : x \in [0,1]^N\} \subset \mathcal{C}_\delta$. Using (6.3), (2.17) and the periodicity of the coset representatives $\tilde{\sigma}_\theta$, $\tilde{\Sigma}$ of σ_θ, Σ, it follows that

$$\nu_\delta := \inf_{x \in \tilde{\Sigma}_\delta^C \cap [0,1]^N, \theta \in \Theta} |\tilde{\sigma}_\theta(x)(y_\delta - x)| > 0.$$

For $x \in \mathbb{R}^N$, we define

$$\phi(x) = \phi_\delta(x) := -e^{\gamma |x - y_\delta|^2 - \gamma R}, \quad R := 2|y_\delta|^2 + 2N + 1, \ \gamma > 0.$$

Notice that ϕ is smooth on $\mathbb{R}^N$ and $-1 < \phi < 0$ for all $\gamma > 0$. We have, for all $x \in \tilde{\Sigma}_\delta^C \cap [0,1]^N$,

$$\begin{aligned} & -\mathrm{trace}(\tilde{\sigma}_\theta(x)\tilde{\sigma}_\theta(x)^T D^2\phi(x)) - C|D\phi(x)| \\ = \ & 2\gamma|\phi(x)| \big(\mathrm{trace}(\tilde{\sigma}_\theta(x)\tilde{\sigma}_\theta(x)^T) + 2\gamma\, \mathrm{trace}(\tilde{\sigma}_\theta(x)\tilde{\sigma}_\theta(x)^T (x - y_\delta) \otimes (x - y_\delta)) - C|x - y_\delta|\big) \\ \geq \ & 2\gamma|\phi(x)|(2\gamma\nu_\delta^2 - C(r + |y_\delta|)) > 0 \end{aligned}$$

for $\gamma = \gamma_{\delta,r}$ big enough. Therefore ϕ is a smooth supersolution of the equation in (2.16) in $\tilde{\Sigma}_\delta^C \cap (0,1)^N$.

We now define $\tilde{\psi}_\delta$, Ω_δ on the following way. We set $\tilde{\psi}_\delta(x) = \phi(x)$ for $x \in \tilde{\Sigma}_{\delta/2}^C \cap [0,1]^N$ Now, from (2.18), we have $\{\mathrm{dist}(\cdot, \partial[0,1]^N) \leq \delta/4\} \cap \tilde{\Sigma}_{\delta/2}^C = \emptyset$ so we can extend $\tilde{\psi}_\delta$ in a smooth way in $[0,1]^N$ such that $\tilde{\psi}_\delta(x) = 0$ for $x \in \{\mathrm{dist}(\cdot, \partial[0,1]^N) \leq \delta/4\} \cap [0,1]^N$ and $|\tilde{\psi}_\delta| \leq 1$ in $[0,1]^N$. We then extend $\tilde{\psi}$ outside $[0,1]^N$ by 0. We set $\Omega_\delta := \{\mathrm{dist}(\cdot, \partial[0,1]^N) < \delta/4\}$. It is straightforward that the function $\tilde{\psi}_\delta$ satisfies (2.16). $\square$

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