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SOME UNIFORME ESTIMATES FOR SCALAR CURVATURE TYPE EQUATIONS.

SAMY SKANDER BAHOURA

ABSTRACT. We consider the prescribed scalar curvature equation on an open set Ω of \mathbb{R}^n , $\Delta u = Vu^{(n+2)/(n-2)} + u^{n/(n-2)}$ with $V \in C^{1,\alpha}$ ($0 < \alpha \leq 1$) and we prove the inequality $\sup_K u \times \inf_{\Omega} u \leq c$ where K is a compact set of Ω .

In dimension 4, we have an idea on the supremum of the solution of the prescribed scalar curvature if we control the infimum. For this case we suppose the scalar curvature $C^{1,\alpha}$, ($0 < \alpha \leq 1$).

1. INTRODUCTION.

In our work, we denote $\Delta = -\nabla^i \nabla_i$ the Laplace-Beltrami operator in dimension $n \geq 2$.

Here, we study some a priori estimates of type $\sup \times \inf$ for prescribed scalar curvature equations in dimensions 4 and 5, also for perturbed scalar curvature equations in all dimension $n \geq 3$.

The $\sup \times \inf$ inequality is characteristic of those equations like the usual harnack inequalities for harmonic functions.

Note that, the prescribed scalar curvature equation was studied lot of. We can find, see for example, [1], [2], [3], [4], [6], [7], [9], [10], [11], [12], [13] and [14], lot of results about uniform estimates in dimensions $n = 2$ and $n \geq 3$.

In dimension 2, the corresponding equation is:

$$\Delta u = Ve^u \quad (E_0)$$

Note that, Shafir, see [14], have obtained an inequality of type $\sup u + C \inf u < c$ with only L^∞ assumption on V .

To obtain exactly the estimate $\sup u + \inf u < c$, Brezis, Li and Shafir gave a lipschitzian condition on V , see [3]. Later, Chen and Lin have proved that if V is uniformly h\"olderian we can obtain a $\sup + \inf$ inequality, see [7].

In dimension $n \geq 3$, the prescribed curvature equation on general manifold M , is:

$$\Delta u + R_g u = Vu^{(n+2)/(n-2)} \quad (E'_0)$$

When $M = \mathbb{S}_n$, Li, has proved a priori estimates for the solutions of the previous equation. He use the notion of simple isolated points and some flatness conditions on V , see [9] and [10].

If we suppose $n = 3, 4$, we can find in [12] and [13] uniforme estimates for the energy and a $\sup \times \inf$ inequality. Note that, in [13], Li and Zhu have proved the compactness of the solutions of the Yamabe Problem by using the positive mass theorem.

In [2], we can see (on a bounded domain of \mathbb{R}^4) that we have an uniform estimate for the solutions of the equation (E'_0) ($n = 4$ and euclidian case) if we control the infimum of those functions, with only Lipschitzian assumption on the prescribed scalar curvature V .

Here we extend some result of [2] to equations with nonlinear terms or with minimal condition on the prescribed scalar curvature.

For the euclidian case, Chen and Lin gave some a priori estimates for general equations:

$$\Delta u = Vu^{(n+2)/(n-2)} + g(u), \quad (E_0'')$$

with some assumption on g and the Li-flatness conditions on V , see [6].

Here, we give some a priori estimates with some minimal conditions on the prescribed curvature. First, for perturbed scalar curvature equation, in all dimensions $n \geq 3$. Second, for prescribed scalar curvature equation in dimensions 4 and 5.

Note that, we have no assumption on energy. In our work, we use the *blow-up* analysis, the *moving-plane* method. The moving-plane method was developed by Gidas-Ni-Nirenberg, see [8].

2. MAIN RESULTS.

We consider the prescribed scalar curvature equation perturbed by a non linear term:

$$\Delta u = Vu^{(n+2)/(n-2)} + u^{n/(n-2)} \text{ on } \Omega \subset \mathbb{R}^n \quad (E_1).$$

Where $V \in C^{1,\alpha}$, $0 < \alpha \leq 1$, $0 < a \leq V(x) \leq b$ and $\|V\|_{C^{1,\alpha}} \leq A$.

We have,

Theorem 1. *For all $a, b, A, \alpha > 0$ ($0 < \alpha \leq 1$) and all compact set K of Ω , there is a positive constant $c = c(a, b, A, \alpha, K, \Omega, n)$ such that:*

$$\sup_K u \times \inf_{\Omega} u \leq c.$$

If we suppose $V \in C^1(\Omega)$ and $V \geq a > 0$, we have,

Theorem 2. *For all $a > 0$, V and all compact K of Ω , there is a positive constant $c = c(a, V, K, \Omega, n)$ such that:*

$$\sup_K u \times \inf_{\Omega} u \leq c,$$

for all solution u of (E_1) relatively to V .

Now, we suppose $n = 4$, and we consider the following equation (prescribed scalar curvature equation):

$$\Delta u = Vu^3 \text{ on } \Omega \subset \mathbb{R}^4 \quad (E_2)$$

with $0 < a \leq V(x) \leq b$ and $\|V\|_{C^{1,\alpha}} \leq A$, $0 < \alpha \leq 1$.

We have:

Theorem 3. *For all $a, b, m, A, \alpha > 0$, ($0 < \alpha \leq 1$) and all compact K of Ω , there is a positive constant $c = c(a, b, m, A, \alpha, K, \Omega)$ such that:*

$$\sup_K u \leq c \text{ if } \min_{\Omega} u \geq m.$$

If we suppose $n = 4$ and $V \in C^1(\Omega)$ and $V \geq a > 0$ on Ω , we have:

Theorem 4. *For all $a, m > 0$, $V \in C^1(\Omega)$ and all compact $K \in \Omega$, there is a positive constant $c = c(a, m, V, K, \Omega)$ such that:*

$$\sup_K u \leq c \text{ if } \min_{\Omega} u \geq m,$$

for all u solution of (E_2) relatively to V .

3. PROOFS OF THE THEOREMS.

Proof of the Theorems 1 and 2.

Proof of the Theorem 1

Without loss of generality, we suppose $\Omega = B_1$ the unit ball of \mathbb{R}^n . We want to prove an a priori estimate around 0.

Let (u_i) and (V_i) be a sequences of functions on Ω such that:

$$\Delta u_i = V_i u_i^{(n+2)/(n-2)} + u_i^{n/(n-2)}, \quad u_i > 0,$$

with $0 < a \leq V_i(x) \leq b$ and $\|V_i\|_{C^{1,\alpha}} \leq A$.

We argue by contradiction and we suppose that the $\sup \times \inf$ is not bounded.

We have:

$\forall c, R > 0 \exists u_{c,R}$ solution of (E_1) such that:

$$R^{n-2} \sup_{B(0,R)} u_{c,R} \times \inf_M u_{c,R} \geq c, \quad (H)$$

Proposition :(blow-up analysis)

There is a sequence of points $(y_i)_i$, $y_i \rightarrow 0$ and two sequences of positive real numbers $(l_i)_i, (L_i)_i$, $l_i \rightarrow 0, L_i \rightarrow +\infty$, such that if we set $v_i(y) = \frac{u_i(y + y_i)}{u_i(y_i)}$, we have:

$$0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2}, \quad \beta_i \rightarrow 1.$$

$$v_i(y) \rightarrow \left(\frac{1}{1 + |y|^2} \right)^{(n-2)/2}, \quad \text{uniformly on all compact set of } \mathbb{R}^n.$$

$$l_i^{(n-2)/2} u_i(y_i) \times \inf_{B_1} u_i \rightarrow +\infty,$$

Proof of the proposition:

We use the hypothesis (H) , we take two sequences $R_i > 0, R_i \rightarrow 0$ and $c_i \rightarrow +\infty$, such that,

$$R_i^{(n-2)} \sup_{B(0,R_i)} u_i \times \inf_{B_1} u_i \geq c_i \rightarrow +\infty,$$

Let $x_i \in B(x_0, R_i)$ be a point such that $\sup_{B(0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = (R_i - |x - x_i|)^{(n-2)/2} u_i(x)$, $x \in B(x_i, R_i)$. Then, $x_i \rightarrow 0$.

We have:

$$\max_{B(x_i, R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i^{(n-2)/2} u_i(x_i) \geq \sqrt{c_i} \rightarrow +\infty.$$

We set:

$$l_i = R_i - |y_i - x_i|, \quad \bar{u}_i(y) = u_i(y_i + y), \quad v_i(z) = \frac{u_i[y_i + (z/[u_i(y_i)]^{2/(n-2)})]}{u_i(y_i)}.$$

Clearly we have, $y_i \rightarrow x_0$. We also obtain:

$$L_i = \frac{l_i}{(c_i)^{1/2(n-2)}} [u_i(y_i)]^{2/(n-2)} = \frac{[s_i(y_i)]^{2/(n-2)}}{c_i^{1/2(n-2)}} \geq \frac{c_i^{1/(n-2)}}{c_i^{1/2(n-2)}} = c_i^{1/2(n-2)} \rightarrow +\infty.$$

If $|z| \leq L_i$, then $y = [y_i + z/[u_i(y_i)]^{2/(n-2)}] \in B(y_i, \delta_i l_i)$ with $\delta_i = \frac{1}{(c_i)^{1/2(n-2)}}$ and $|y - y_i| < R_i - |y_i - x_i|$, thus, $|y - x_i| < R_i$ and, $s_i(y) \leq s_i(y_i)$. We can write:

$$u_i(y)(R_i - |y - y_i|)^{(n-2)/2} \leq u_i(y_i)(l_i)^{(n-2)/2}.$$

But, $|y - y_i| \leq \delta_i l_i$, $R_i > l_i$ and $R_i - |y - y_i| \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$. We obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[\frac{l_i}{l_i(1 - \delta_i)} \right]^{(n-2)/2} \leq 2^{(n-2)/2}.$$

We set, $\beta_i = \left(\frac{1}{1 - \delta_i} \right)^{(n-2)/2}$, clearly, we have, $\beta_i \rightarrow 1$.

The function v_i satisfies:

$$\Delta v_i = \tilde{V}_i v_i^{(n+2)/(n-2)} + \frac{v_i^{n/(n-2)}}{[u_i(y_i)]^{2/(n-2)}}$$

where, $\tilde{V}_i(y) = V_i [y + y/[u_i(y_i)]^{2/(n-2)}]$. Without loss of generality, we can suppose that $\tilde{V}_i \rightarrow V(0) = n(n-2)$.

We use the elliptic estimates, Ascoli and Ladyzenskaya theorems to have the uniform convergence of (v_i) to v on compact set of \mathbb{R}^n . The function v satisfies:

$$\Delta v = n(n-2)v^{N-1}, \quad v(0) = 1, \quad 0 \leq v \leq 1 \leq 2^{(n-2)/2},$$

By the maximum principle, we have $v > 0$ on \mathbb{R}^n . If we use Caffarelli-Gidas-Spruck result, (see [5]), we obtain, $v(y) = \left(\frac{1}{1 + |y|^2} \right)^{(n-2)/2}$. We have the same properties that in [2].

Remark. When we use the convergence on compact sets of the sequence (v_i) , we can take an increasing sequence of compact sets and we see that, we can obtain, a sequence (ϵ_i) such that $\epsilon_i \rightarrow 0$ and after we choose (\tilde{R}_i) such that $\tilde{R}_i \rightarrow +\infty$ and finally:

$$\tilde{R}_i^{n-2} \|v_i - v\|_{B(0, \tilde{R}_i)} \leq \epsilon_i.$$

We can say that we are in the case of the step 1 of the theorem 1.2 of [6].

Fundamental Point: (a consequence of the blow-up)

According to the work of Chen-Lin, see step 2 of the proof of the theorem 1.3 in [6], in the blow-up point, the prescribed scalar curvature V is such that:

$$\lim_{i \rightarrow +\infty} |\nabla V_i(y_i)| = 0 \quad (P_0)$$

Polar Coordinates (Moving-Plane method)

Now, we must use the same method than in the Theorem 1 of [2]. We will use the moving-plane method.

We must prove the lemma 2 of [2].

We set $t \in]-\infty, -\log 2]$ and $\theta \in \mathbb{S}_{n-1}$:

$$w_i(t, \theta) = e^{(n-2)t/2} u_i(y_i + e^t \theta), \quad \text{and} \quad \bar{V}_i(t, \theta) = V_i(y_i + e^t \theta).$$

We consider the following operator $L = \partial_{tt} - \Delta_\sigma - \frac{(n-2)^2}{4}$, with Δ_σ the Laplace-Baltrami operator on \mathbb{S}_{n-1} .

The function w_i satisfies the following equation:

$$-Lw_i = \bar{V}_i w_i^{N-1} + e^t \times w_i^{n/(n-2)}.$$

For $\lambda \leq 0$, we set :

$$t^\lambda = 2\lambda - t \quad w_i^\lambda(t, \theta) = w_i(t^\lambda, \theta) \quad \text{and} \quad \bar{V}_i^\lambda(t, \theta) = \bar{V}_i(t^\lambda, \theta)$$

First, like in [2], we have the following lemma:

Lemma 1:

Let A_λ be the following property:

$$A_\lambda = \{\lambda \leq 0, \exists (t_\lambda, \theta_\lambda) \in]\lambda, t_i] \times \mathbb{S}_{n-1}, \bar{w}_i^\lambda(t_\lambda, \theta_\lambda) - \bar{w}_i(t_\lambda, \theta_\lambda) \geq 0\}.$$

Then, there is $\nu \leq 0$, such that for $\lambda \leq \nu$, A_λ is not true.

Remark: Here we choose $t_i = \log \sqrt{l_i}$, where l_i is chooses as in the proposition.

Like in the proof of the Theorem 1 of [2], we want to prove the following lemma:

Lemma 2:

For $\lambda \leq 0$ we have :

$$w_i^\lambda - w_i \leq 0 \Rightarrow -L(w_i^\lambda - w_i) \leq 0,$$

on $] \lambda, t_i] \times \mathbb{S}_{n-1}$.

Like in [2], we have:

A useful point:

$\xi_i = \sup\{\lambda \leq \bar{\lambda}_i = 2 + \log \eta_i, w_i^\lambda - w_i < 0, \text{ on }] \lambda, t_i] \times \mathbb{S}_{n-1}\}$. The real ξ_i exists.

First:

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \log \eta_i - 2) + (\log \eta_i + 2)],$$

Proof of the Lemma 2:

In fact, for each i we have $\lambda = \xi_i \leq \log \eta_i + 2$, ($\eta_i = [u_i(y_i)]^{(-2)/(n-2)}$).

Note that,

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \log \eta_i - 2) + (\log \eta_i + 2)],$$

if we use the definition of w_i then for $\xi_i \leq t$:

$$w_i(2\xi_i - t, \theta) = e^{[(n-2)(\xi_i - t + \xi_i - \log \eta_i - 2)]/2} e^{n-2} v_i [\theta e^2 e^{(\xi_i - t) + (\xi_i - \log \eta_i - 2)}] \leq 2^{(n-2)/2} e^{n-2} = \bar{c}.$$

We know that,

$$-L(w_i^{\xi_i} - w_i) = [\bar{V}_i^{\xi_i} (w_i^{\xi_i})^{(n+2)/(n-2)} - \bar{V}_i w_i^{(n+2)/(n-2)}] + [e^{t\xi_i} (w_i^{\xi_i})^{n/(n-2)} - e^t w_i^{n/(n-2)}],$$

We denote by Z_1 and Z_2 the following terms:

$$Z_1 = (\bar{V}_i^{\xi_i} - \bar{V}_i)(w_i^{\xi_i})^{(n+2)/(n-2)} + \bar{V}_i[(w_i^{\xi_i})^{(n+2)/(n-2)} - w_i^{(n+2)/(n-2)}],$$

and,

$$Z_2 = e^{t\xi_i} [(w_i^{\xi_i})^{n/(n-2)} - w_i^{n/(n-2)}] + w_i^{n/(n-2)} (e^{t\xi_i} - e^t).$$

Like in the proof of the Theorem 1 of [2], we have:

$$w_i^{\xi_i} \leq w_i \text{ and } w_i^{\xi_i}(t, \theta) \leq \bar{c} \text{ for all } (t, \theta) \in [\xi_i, -\log 2] \times \mathbb{S}_{n-1},$$

where, \bar{c} is a positive constant independant of i and $w_i^{\xi_i}$ for $\xi_i \leq \log \eta_i + 2$.

The (P_0) hypothesis:

Now we use (P_0) . We write:

$$|\nabla V_i(y_i + e^t \theta) - \nabla V_i(y_i)| \leq Ae^{\alpha t},$$

Thus,

$$|V_i(y_i + e^{t\xi_i}\theta) - V_i(y_i + e^t\theta) - \langle \nabla V_i(y_i)|\theta \rangle (e^{t\xi_i} - e^t)| \leq \frac{A}{1+\alpha} [e^{(1+\alpha)t\xi_i} - e^{(1+\alpha)t}],$$

Then,

$$|V_i^{\xi_i} - V_i| \leq |o(1)|(e^t - e^{t\xi_i}),$$

Thus,

$$Z_1 \leq |o(1)|(w_i^{\xi_i})^{(n+2)/(n-2)}(e^t - e^{t\xi_i}) \quad \text{and} \quad Z_2 \leq (w_i^{\xi_i})^{n/(n-2)} \times (e^{t\xi_i} - e^t).$$

Then,

$$-L(w_i^{\xi_i} - w_i) \leq (w_i^{\xi_i})^{n/(n-2)} [(|o(1)|w_i^{\xi_i 2/(n-2)} - 1)(e^t - e^{t\xi_i})] \leq 0.$$

The lemma is proved.

We set:

$$\xi_i = \sup\{\mu \leq \log \eta_i + 2, w_i^\mu(t, \theta) - w_i(t, \theta) \leq 0, \forall (t, \theta) \in [\mu_i, t_i] \times \mathbb{S}_{n-1}\},$$

with t_0 small enough.

Like in the proof of the Theorem 1 of [2], the maximum principle imply:

$$\min_{\theta \in \mathbb{S}_{n-1}} w_i(t_i, \theta) \leq \max_{\theta \in \mathbb{S}_{n-1}} w_i(2\xi_i - t_i).$$

But,

$$w_i(t_i, \theta) = e^{t_i} u_i(y_i + e^{t_i}\theta) \geq e^{t_i} \min u_i \quad \text{and} \quad w_i(2\xi_i - t_i) \leq \frac{c_0}{u_i(y_i)},$$

thus,

$$l_i^{(n-2)/2} u_i(y_i) \times \min u_i \leq c.$$

The proposition is contradicted.

Proof of the Theorem 2.

The proof of the Theorem 2 is similar to the proof of the Theorem 1. Only the "Fundamental point" change. We have:

According to the work of Chen-Lin, see step 2 of the proof of the theorem 1.1 in [6], in the blow-up point, the prescribed scalar curvature V is such that:

$$\nabla V(0) = 0.$$

The function ∇V is continuous on $B_r(0)$ (r small enough), then it is uniformly continuous and we write (because $y_i \rightarrow 0$):

$$|\nabla V(y_i + y) - \nabla V(y_i)| \leq \epsilon, \quad \text{for } |y| \leq \delta \ll r \quad \forall i$$

Thus,

$$|V^{\xi_i} - V| \leq o(1)(e^t - e^{t\xi_i}),$$

We see that we have the same computations than in the section "Polar Coordinates" in the proof of the Theorem 1.

Proof of the Theorems 3 and 4.

Here, only the section "Polar coordinates" change, the proposition of the first theorem stay true. First, we have:

Fundamental Point: (a consequence of the blow-up)

According to the work of Chen-Lin, see step 2 of the proof of the theorem 1.3 in [6], in the blow-up point, the prescribed scalar curvature V is such that:

Case 1: Theorem 3.

$$\lim_{i \rightarrow +\infty} |\nabla V_i(y_i)| = 0.$$

We write:

$$|\nabla V_i(y_i + e^t \theta) - \nabla V_i(y_i)| \leq A e^{\alpha t},$$

Thus,

$$|V_i^{\xi_i} - V_i| \leq |o(1)|(e^t - e^{t^{\xi_i}}).$$

Case 2: Theorem 4.

$$\nabla V(0) = 0.$$

The function ∇V is continuous on $B_r(0)$ (r small enough), then it is uniformly continuous and we write (because $y_i \rightarrow 0$):

$$|\nabla V(y_i + y) - \nabla V(y_i)| \leq \epsilon, \text{ for } |y| \leq \delta \ll r \forall i$$

Thus,

$$|V^{\xi_i} - V| \leq o(1)(e^t - e^{t^{\xi_i}}),$$

Conclusion for Theorems 3 and 4.

Finally, we can note that we are in the case of the Theorem 2 of [2]. We have the same computations if we consider the following function:

$$\bar{w}_i(t, \theta) = w_i(t, \theta) - \frac{m}{2} e^t.$$

We set, $L = \partial_{tt} - \Delta_\sigma + 1$, where Δ_σ is the Laplace-Beltrami operator on \mathbb{S}_3 and $\bar{V}_i(t, \theta) = V_i(y_i + e^t \theta)$.

Like in [2], we want to prove the following lemma:

Lemma.

$$\bar{w}_i^{\xi_i} - \bar{w}_i \leq 0 \Rightarrow -L(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 0.$$

Proof of the Lemma.

$$-L(\bar{w}_i^{\xi_i} - \bar{w}_i) = \bar{V}_i^{\xi_i} (w_i^{\xi_i})^3 - \bar{V}_i w_i^3.$$

Then,

$$-L(\bar{w}_i^{\xi_i} - \bar{w}_i) = (\bar{V}_i^{\xi_i} - \bar{V}_i)(w_i^{\xi_i})^3 + [(w_i^{\xi_i})^3 - w_i^3] \bar{V}_i.$$

For $t \in [\xi_i, t_i]$ and $\theta \in \mathbb{S}_3$:

$$|\bar{V}_i^{\xi_i}(t, \theta) - \bar{V}_i(t, \theta)| = |V_i(y_i + e^{2\xi_i - t} \theta) - V_i(y_i + e^t \theta)| \leq |o(1)|(e^t - e^{2\xi_i - t}).$$

The real $t_i = \log \sqrt{l_i} \rightarrow -\infty$, where l_i is chooses as in the proposition of the theorem 1.

But, if $\bar{w}_i^{\xi_i} - \bar{w}_i \leq 0$, we obtain:

$$w_i^{\xi_i} - w_i \leq \frac{m}{2} (e^{2\xi_i - t} - e^t) < 0.$$

We use the fact that $0 < w_i^{\xi_i} < w_i$, we have:

$$(w_i^{\xi_i})^3 - w_i^3 = (w_i^{\xi_i} - w_i)[(w_i^{\xi_i})^2 + w_i^{\xi_i} w_i + (w_i)^2] \leq 3(w_i^{\xi_i} - w_i) \times (w_i^{\xi_i})^2.$$

Thus, we have for $t \in [\xi_i, t_i]$ and $\theta \in \mathbb{S}_3$:

$$(w_i^{\xi_i})^3 - w_i^3 \leq 3 \frac{m}{2} (w_i^{\xi_i})^2 (e^{2\xi_i - t} - e^t).$$

We can write,

$$-L(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq (w_i^{\xi_i})^2 \left(\frac{3m}{2} \bar{V}_i - |o(1)| w_i^{\xi_i} \right) (e^{2\xi_i - t} - e^t). \quad (**)$$

We know that for $t \leq \log(l_i) - \log 2 + \log \eta_i$, we have,

$$w_i(t, \theta) = e^t \times \frac{u_i \left(y_i + \frac{e^t \theta}{u_i(y_i)} \right)}{u_i(y_i)} \leq 2e^t.$$

We find,

$$w_i^{\xi_i}(t, \theta) \leq 2e^2 \sqrt{\frac{8}{a}},$$

because, $\xi_i - \log \eta_i \leq 2 + \frac{1}{2} \log \frac{8}{V(0)}$ and $\xi_i \leq t \leq t_i$.

Finally, (**) is negative and the Lemma is proved.

Now, if we use the Hopf maximum principle, we obtain,

$$\min_{\theta \in \mathbb{S}^3} \tilde{w}_i(t_i, \theta) \leq \max_{\theta \in \mathbb{S}^3} \tilde{w}_i(2\xi_i - t_i, \theta).$$

Which imply that,

$$l_i u_i(y_i) \leq c.$$

It is a contradiction.

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