



HAL
open science

Bounds for the (m,n) -mixed chromatic number and the oriented chromatic number

William F. Klostermeyer, Gary Macgillivray, André Raspaud, Eric Sopena

► **To cite this version:**

William F. Klostermeyer, Gary Macgillivray, André Raspaud, Eric Sopena. Bounds for the (m,n) -mixed chromatic number and the oriented chromatic number. Graph Theory Notes of New York, 2013, LXV, pp.10-14. <hal-00828996>

HAL Id: hal-00828996

<https://hal.science/hal-00828996v1>

Submitted on 1 Jun 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Bounds for the (m, n) -mixed chromatic number and the oriented chromatic number

William F. Klostermeyer¹, Gary MacGillivray², André Raspaud³, Eric Sopena³

¹ *School of Computing, University of North Florida,
Jacksonville, FL 32224-2669*

² *Dept. of Mathematics and Statistics, University of Victoria,
Victoria BC, Canada V8W 3R4*

³ *LaBRI, UMR CNRS 5800, Université Bordeaux I,
33405 Talence Cedex, France*

1. INTRODUCTION

A bound for the (n, m) -mixed chromatic number in terms of the chromatic number of the square of the underlying undirected graph is given. A similar bound holds when the chromatic number of the square is replaced by the injective chromatic number. When restricted to $n = 1$ and $m = 0$ (i.e., oriented graphs) this gives a new bound for the oriented chromatic number. In this case, a slightly improved bound is obtained if the chromatic number of the square is replaced the 2-dipath chromatic number (defined in Section 4). In all cases, the method of proof generalizes an argument that has been used to obtain Brooks-type theorems for injective oriented colorings [1, 2, 3, 4, 5, 6, 7, 8]. Similar, though not identical, arguments have appeared in the work of Sopena [9], and Nešetřil and Raspaud [10, 11].

Colorings of mixed graphs were first studied by Nešetřil and Raspaud [11]. They gave a bound on the (n, m) -mixed chromatic number in terms of the acyclic chromatic number of the underlying undirected graph. Bounds for the $(0, 2)$ -mixed chromatic number of planar graphs and other graph families have been found [12]. Some of these have been extended to arbitrary n and m [13].

2. DEFINITIONS

For basic definitions in graph theory, see the text by Bondy and Murty [14].

Definition 2.1. *A (n, m) -colored mixed graph is an $(m + n + 1)$ -tuple $G = (V, A_1, A_2, \dots, A_n, E_1, E_2, \dots, E_m)$ where:*

- (1) *V is a set of vertices;*
- (2) *for $i = 1, 2, \dots, n$ the set A_i is a set of ordered pairs of distinct vertices of G called the arcs of color i ;*
- (3) *for $j = 1, 2, \dots, m$ the set E_j is a set of unordered pairs of distinct vertices of G called the edges of color j ; and*
- (4) *the underlying undirected graph $U[G]$, with vertex set $V(U[G]) = V(G)$ and an edge joining x to y for every i for which the ordered pair $xy \in A_i$ and for every j for which the unordered pair $xy \in E_j$, is simple.*

Observe that a $(0, 1)$ -colored mixed graph is a simple graph, and a $(1, 0)$ -colored mixed graph is an oriented graph.

Homomorphisms between (n, m) -colored mixed graphs are defined in the natural way. For (n, m) -colored mixed graphs G and H , a *homomorphism* of G to H is a function $f : V(G) \rightarrow V(H)$ such that for $1 \leq i \leq n$ we have $f(x)f(y) \in A_i(H)$

whenever $xy \in A_i(G)$, and for $1 \leq j \leq n$ we have $f(x)f(y) \in E_j(H)$ whenever $xy \in E_j(G)$. For more on homomorphisms see [15].

Observe that homomorphisms of (n, m) -colored mixed graphs compose. That is, if F, G , and H are (n, m) -colored mixed graphs, f is a homomorphism of F to G and g is a homomorphism of G to H , then $g \circ f$ is a homomorphism of F to H .

By analogy with the corresponding definitions for other coloring concepts for oriented graphs, a k -coloring of a (n, m) -colored mixed graph is defined to be a homomorphism to a (n, m) -colored mixed graph on k vertices. The smallest k for which there is a k -coloring of a (n, m) -colored graph G is called the (n, m) -mixed chromatic number of G and denoted $\chi_{(n,m)}(G)$.

Informally, a k -coloring of an (n, m) -colored mixed graph is a partition of the vertices of G into k independent sets (sets containing no edges or arcs of any color) X_1, X_2, \dots, X_k such that, for any two independent sets X_p and X_q , there is only one type of adjacency – only arcs of the same color and orientation, or only edges of the same color – between vertices in X_p and X_q .

Example 1. Consider a path on four vertices v_1, v_2, v_3, v_4 with arcs v_1v_2 and v_2v_3 of color 1 and an edge v_3v_4 of color 1. Call this $(1, 1)$ -colored mixed graph G . Then $\chi_{(n,m)}(G) = 3$; color v_1 and v_4 with color 1, v_2 with color 2, and v_3 with color 3.

Example 2. Consider a path on five vertices v_1, v_2, v_3, v_4, v_5 with arcs v_1v_2 and v_2v_3 of color 1, an edge v_3v_4 of color 1, and an edge v_4v_5 of color 2. Call this $(1, 2)$ -colored mixed graph G . Then $\chi_{(n,m)}(G) = 4$; color v_1 and v_4 with color 1, v_2 with color 2, v_3 with color 3, and v_5 with color 4.

We next define a (n, m) -mixed graph. Let $t \geq 3$. For $k = 1, 2, \dots, t$, let I_k be the set of all sequences of length t in which the k^{th} element is “.” and every other entry is a “+ $_i$ ” or “- $_i$ ” ($1 \leq i \leq n$), or “ \sim_j ” ($1 \leq j \leq m$).

Definition 2.2. The graph $H_{(n,m)}^t$ is the (n, m) -colored mixed graph with vertex set $V(H_{(n,m)}^t) = I_1 \cup I_2 \cup \dots \cup I_t$, and

- (1) an arc of color i joining sequence $s_k \in I_k$ to sequence $s_\ell \in I_\ell$ if and only if the ℓ -th entry of s_k is “+ $_i$ ” and the k -th entry of s_ℓ is “- $_i$ ”, and
- (2) an edge of color j joining $s_k \in I_k$ to sequence $s_\ell \in I_\ell$ if and only if the ℓ -th entry of s_k and the k -th entry of s_ℓ are both \sim_j .

In the next section, we bound the (n, m) -mixed chromatic number of $H_{(n,m)}^t$.

3. MAIN RESULT

The graph $H_{(1,0)}^t$ is used to prove bounds for injective oriented colorings [1, 2, 3, 4, 5]. Young has proved that an oriented graph can be properly t -colored so that any two vertices joined by a directed path of length two get different colors if and only if it has a homomorphism to $H_{(1,0)}^t$ [8] (Min and Wang [16] call this a 2-*dipath* k -coloring.) Our bound on the (n, m) -mixed chromatic number of a (n, m) -mixed colored graph G will be obtained by coloring the square of the underlying graph of G with t colors, and then refining this coloring to obtain a homomorphism of G to $H_{(n,m)}^t$.

Lemma 3.1. (The Refinement Lemma)

$$\begin{aligned} \chi_{(n,m)}(H_{(n,m)}^t) &\leq 1 + (2n + m) + (2n + m)^2 + \cdots + (2n + m)^{t-1} \\ &= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ \frac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Let $B = 1 + (2n + m) + (2n + m)^2 + \cdots + (2n + m)^{t-1}$. We describe a (n, m) -mixed B -coloring of $H_{(n,m)}$. For $k = 1, 2, \dots, t$, partition the independent set I_k into $(2n + m)^{k-1}$ subsets $I_{k,\ell}$, $1 \leq \ell \leq (2n + m)^{k-1}$, such that all sequences in $I_{k,\ell}$ agree in the first k places. The total number of sets $I_{p,q}$ is B .

By definition of these sets, if $p < r$ then:

- (1) all arcs joining vertices in $I_{p,q}$ and $I_{r,s}$ are of color i and:
 - (a) oriented towards $I_{p,q}$ if the p -th symbol of all sequences in $I_{r,s}$ is “ $+_i$ ” and the r -th symbol in of all sequence in $I_{r,s}$ is “ $-_i$ ”;
 - (b) oriented towards $I_{r,s}$ if the p -th symbol of all sequences in $I_{r,s}$ is “ $-_i$ ” and the r -th symbol in of all sequence in $I_{r,s}$ is “ $+_i$ ”;
- (2) all edges joining vertices in $I_{p,q}$ and $I_{r,s}$ are of color j if p -th symbol of all sequences in $I_{r,s}$ and the r -th symbol of all sequences in $I_{(p,q)}$ are both \sim_j .

Hence, by identifying all vertices in each set $I_{k,\ell}$, $1 \leq k \leq t$, $1 \leq \ell \leq (2n + m)^{k-1}$, a homomorphism of $H_{(n,m)}^t$ onto the (n, m) -colored mixed graph $\mathcal{O}_{(n,m)}$ on B vertices is obtained. Therefore, $\chi_{(n,m)}(H_{(n,m)}^t) \leq B$. \square

Lemma 3.1 is dubbed “The Refinement Lemma” because the independent sets I_k are refined in order to obtain the homomorphism to $\mathcal{O}_{(n,m)}$.

Let G be an (n, m) -colored mixed graph. Define $\mathcal{S}(G)$ to be square of the underlying undirected graph of G . That is, $\mathcal{S}(G)$ has the same vertex set as G and $xy \in E$ if and only if $1 \leq \text{dist}_G(x, y) \leq 2$.

Lemma 3.2. *Let G be an (n, m) -colored mixed graph. If $\mathcal{S}(G)$ is t -colorable, then there is a homomorphism from G to $H_{(n,m)}^t$.*

Proof. Let C_1, C_2, \dots, C_t be a t -coloring of $\mathcal{S}(G)$. Let $x \in C_k$. Define the sets $S_{x,i} = \{p : xy \in A_i(G) \text{ for some } y \in C_p\}$, $R_{x,i} = \{p : yx \in A_i(G) \text{ for some } y \in C_p\}$, and $T_{x,j} = \{p : xy \in E_j(G) \text{ for some } y \in C_p\}$. These are, respectively, the sets of (vertex) colors to which x sends an arc of color i in D , from which x receives an arc of color i in D , or to which x is joined by an edge of color j in D . Since C_k is an independent set $x \notin S_{x,i} \cup R_{x,i'} \cup T_{x,j}$ for any i, i' and j . By construction of $\mathcal{S}(G)$, for all i, i' and j the intersection between any two of $S_{x,i}, R_{x,i'}$ and $T_{x,j}$ is empty.

Thus, each vertex in C_k can be associated with a sequence with k -th entry is “.” and in which the ℓ -th entry is “ $+_i$ ” if $k \in S_{x,i}$, is “ $-_i$ ” if $k \in R_{x,i}$, is \sim_j if $k \in T_{x,j}$, and is \sim_t otherwise. This is a homomorphism of G to $H_{(n,m)}^t$. \square

Corollary 3.3. *Let G be an (n, m) -colored mixed graph. If $\mathcal{S}(G)$ is t -colorable, then*

$$\begin{aligned} \chi_{(n,m)}(G) &\leq 1 + (2n + m) + (2n + m)^2 + \cdots + (2n + m)^{t-1} \\ &= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ \frac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases} \end{aligned}$$

An *injective k -coloring* of a graph G is an assignment of k colors to the vertices of G so that vertices at distance two are assigned different colors. Adjacent vertices may be assigned the same color. The injective chromatic number of G is the least k for which there exists an injective k -coloring of G . It is bounded above by the chromatic number of $\mathcal{S}(G)$ (ex. see [17]).

The previous arguments work essentially as given with a small modification to take into account the fact that vertices of the same color may be adjacent. The graph $H_{(n,m)}^t$ is redefined by replacing each “.” by one of “+ i ”, “- i ”, or “ \sim_j ”. This increases both the number of vertices of $H_{(n,m)}^t$ and the bound on its (n, m) -mixed chromatic number by a factor of $(2n + m)$. Thus, one obtains:

Corollary 3.4. *Let G be an (n, m) -colored mixed graph. If G has an injective t -coloring, then*

$$\begin{aligned} \chi_{(n,m)}(G) &\leq (2n + m) + (2n + m)^2 + \cdots + (2n + m)^t \\ &= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ (2n + m)^{\frac{(2n+m)^t - 1}{2n+m-1}} & \text{otherwise.} \end{cases} \end{aligned}$$

4. ORIENTED COLORINGS

When $n = 1$ and $m = 0$ we have an oriented graph, and the (n, m) -mixed chromatic number equals the oriented chromatic number. Hence the above results imply bounds for the oriented chromatic number. In particular, if $\mathcal{S}(G)$ has a t -coloring, then $\chi_o(G) \leq 2^t - 1$.

An important property of a coloring of $\mathcal{S}(G)$, or an injective t -coloring of G , used in the above arguments is that vertices joined by a directed path of length two must be assigned different colors. The *2-dipath chromatic number* of G is the smallest k for which there is a 2-dipath k -coloring of G . The 2-dipath chromatic number is at most the chromatic number of the square [16].

The argument in Section 3 goes through unchanged for oriented graphs if a k -coloring of $\mathcal{S}(G)$ is replaced by a 2-dipath k -coloring of G . This is the same as replacing $\mathcal{S}(G)$ by the square of G as a directed graph (two vertices are if they are joined by a directed path of length at most two) in the argument. Doing so, one obtains the following bound:

Corollary 4.1. *Let G be an oriented graph. If G has an 2-dipath t -coloring, then $\chi_o(G) = \chi_{(1,0)}(G) \leq 2^t - 1$*

Finally, if the 2-dipath t -coloring need not assign different colors to adjacent vertices, then proceeding as discussed at the end of the previous section one obtains $\chi_o(G) = \chi_{(1,0)}(G) \leq 2^{t+1} - 1$ (also see [7]).

REFERENCES

- [1] R. Campbell, *Oriented colorings that are injective on neighbourhoods*, M.Sc. Thesis (2009), Department of Mathematics and Statistics, University of Victoria, Canada.
- [2] N. Clarke, and G. MacGillivray, *Injective oriented Colourings: In- and Out- Neighbourhoods separately*. Manuscript, 2009.
- [3] MacGillivray G., A. Raspaud and J. Swarts, *Injective Oriented Colourings*, Lecture Notes in Computer Science 5911, Montpellier, France, 2009, Proceedings of WG.
- [4] MacGillivray G., A. Raspaud and J. Swarts, *Obstructions to Injective Oriented Colouring*, Electronic Notes in Discrete Math., **38** (2011), 597 - 605.

- [5] MacGillivray G., A. Raspaud and J. Swarts, Obstructions to locally injective oriented improper colorings, manuscript 2011.
- [6] MacGillivray G., and J. Swarts, The complexity of locally injective homomorphisms, *Discrete Mathematics* **310** (2010), 2685 - 2696.
- [7] J. Swarts, *The complexity of digraph homomorphisms: Local tournaments, Injective Homomorphisms and Polymorphisms*, Ph.D. Thesis (2008), Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada.
- [8] Young K.M., *2-dipath and Proper 2-dipath Colourings*, M.Sc. Thesis, University of Victoria, 2011.
- [9] E. Sopena, The chromatic number of oriented graphs. *Journal of Graph Theory* **25** (1997), 191-205.
- [10] J. Nešetřil and A. Raspaud, Antisymmetric flows and strong colorings of oriented graphs. *Ann. Inst. Fourier, Grenoble* **49**, 1999. 1037-1056.
- [11] J. Nešetřil and A. Raspaud, Coloured homomorphisms of colored mixed graphs. *Journal of Combinatorial Theory, Series B*, **80** (2000), 147-155.
- [12] A. Montejano, P. Ochem, A. Pinlou, A. Raspaud, and E. Sopena, Homomorphisms of 2-edge-colored graphs. To appear in *Discrete Applied Mathematics*.
- [13] A. Montejano, A. Pinlou, A. Raspaud, and E. Sopena, Chromatic number of sparse colored planar graphs. Manuscript, 2009.
- [14] J. A. Bondy and U. S. R. Murty, *Graph Theory*. Springer, 2007.
- [15] P. Hell and J. Nešetřil, *Graphs and Homomorphisms*, Oxford University Press, 2004.
- [16] C. Min and W. Wang, The 2-dipath chromatic number of Halin graphs. *Information Processing Letters* **99** (2006), 47-53.
- [17] G. Hahn, J. Kratochvíl, J. Širáň, and D. Sotteau, On the injective chromatic number of graphs. *Discrete Math.* **256** (2002), 179-192.