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Bounds for the (m, n) -mixed chromatic number and the oriented chromatic number

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1. INTRODUCTION

A bound for the (n, m) -mixed chromatic number in terms of the chromatic number of the square of the underlying undirected graph is given. A similar bound holds when the chromatic number of the square is replaced by the injective chromatic number. When restricted to $n = 1$ and $m = 0$ (i.e., oriented graphs) this gives a new bound for the oriented chromatic number. In this case, a slightly improved bound is obtained if the chromatic number of the square is replaced by the 2-dipath chromatic number (defined in Section 4). In all cases, the method of proof generalizes an argument that has been used to obtain Brooks-type theorems for injective oriented colorings [1, 2, 3, 4, 5, 6, 7, 8]. Similar, though not identical, arguments have appeared in the work of Sopena [9], and Nešetřil and Raspaud [10, 11].

Colorings of mixed graphs were first studied by Nešetřil and Raspaud [11]. They gave a bound on the (n, m) -mixed chromatic number in terms of the acyclic chromatic number of the underlying undirected graph. Bounds for the $(0, 2)$ -mixed chromatic number of planar graphs and other graph families have been found [12]. Some of these have been extended to arbitrary n and m [13].

2. DEFINITIONS

For basic definitions in graph theory, see the text by Bondy and Murty [14].

Definition 2.1. *A (n, m) -colored mixed graph is an $(m + n + 1)$ -tuple $G = (V, A_1, A_2, \dots, A_n, E_1, E_2, \dots, E_m)$ where:*

- (1) *V is a set of vertices;*
- (2) *for $i = 1, 2, \dots, n$ the set A_i is a set of ordered pairs of distinct vertices of G called the arcs of color i ;*
- (3) *for $j = 1, 2, \dots, m$ the set E_j is a set of unordered pairs of distinct vertices of G called the edges of color j ; and*
- (4) *the underlying undirected graph $U[G]$, with vertex set $V(U[G]) = V(G)$ and an edge joining x to y for every i for which the ordered pair $xy \in A_i$ and for every j for which the unordered pair $xy \in E_j$, is simple.*

Observe that a $(0, 1)$ -colored mixed graph is a simple graph, and a $(1, 0)$ -colored mixed graph is an oriented graph.

Homomorphisms between (n, m) -colored mixed graphs are defined in the natural way. For (n, m) -colored mixed graphs G and H , a *homomorphism* of G to H is a function $f : V(G) \rightarrow V(H)$ such that for $1 \leq i \leq n$ we have $f(x)f(y) \in A_i(H)$

whenever $xy \in A_i(G)$, and for $1 \leq j \leq n$ we have $f(x)f(y) \in E_j(H)$ whenever $xy \in E_j(G)$. For more on homomorphisms see [15].

Observe that homomorphisms of (n, m) -colored mixed graphs compose. That is, if F, G , and H are (n, m) -colored mixed graphs, f is a homomorphism of F to G and g is a homomorphism of G to H , then $g \circ f$ is a homomorphism of F to H .

By analogy with the corresponding definitions for other coloring concepts for oriented graphs, a k -coloring of a (n, m) -colored mixed graph is defined to be a homomorphism to a (n, m) -colored mixed graph on k vertices. The smallest k for which there is a k -coloring of a (n, m) -colored graph G is called the (n, m) -mixed chromatic number of G and denoted $\chi_{(n, m)}(G)$.

Informally, a k -coloring of an (n, m) -colored mixed graph is a partition of the vertices of G into k independent sets (sets containing no edges or arcs of any color) X_1, X_2, \dots, X_k such that, for any two independent sets X_p and X_q , there is only one type of adjacency – only arcs of the same color and orientation, or only edges of the same color – between vertices in X_p and X_q .

Example 1. Consider a path on four vertices v_1, v_2, v_3, v_4 with arcs v_1v_2 and v_2v_3 of color 1 and an edge v_3v_4 of color 1. Call this $(1, 1)$ -colored mixed graph G . Then $\chi_{(n, m)}(G) = 3$; color v_1 and v_4 with color 1, v_2 with color 2, and v_3 with color 3.

Example 2. Consider a path on five vertices v_1, v_2, v_3, v_4, v_5 with arcs v_1v_2 and v_2v_3 of color 1, an edge v_3v_4 of color 1, and an edge v_4v_5 of color 2. Call this $(1, 2)$ -colored mixed graph G . Then $\chi_{(n, m)}(G) = 4$; color v_1 and v_4 with color 1, v_2 with color 2, v_3 with color 3, and v_5 with color 4.

We next define a (n, m) -mixed graph. Let $t \geq 3$. For $k = 1, 2, \dots, t$, let I_k be the set of all sequences of length t in which the k^{th} element is “.” and every other entry is a “ $+_i$ ” or “ $-_i$ ” ($1 \leq i \leq n$), or “ \sim_j ” ($1 \leq j \leq m$).

Definition 2.2. The graph $H_{(n, m)}^t$ is the (n, m) -colored mixed graph with vertex set $V(H_{(n, m)}^t) = I_1 \cup I_2 \cup \dots \cup I_t$, and

- (1) an arc of color i joining sequence $s_k \in I_k$ to sequence $s_\ell \in I_\ell$ if and only if the ℓ -th entry of s_k is “ $+_i$ ” and the k -th entry of s_ℓ is “ $-_i$ ”, and
- (2) an edge of color j joining $s_k \in I_k$ to sequence $s_\ell \in I_\ell$ if and only if the ℓ -th entry of s_k and the k -th entry of s_ℓ are both \sim_j .

In the next section, we bound the (n, m) -mixed chromatic number of $H_{(n, m)}^t$.

3. MAIN RESULT

The graph $H_{(1, 0)}^t$ is used to prove bounds for injective oriented colorings [1, 2, 3, 4, 5]. Young has proved that an oriented graph can be properly t -colored so that any two vertices joined by a directed path of length two get different colors if and only if it has a homomorphism to $H_{(1, 0)}^t$ [8] (Min and Wang [16] call this a 2-dipath k -coloring.) Our bound on the (n, m) -mixed chromatic number of a (n, m) -mixed colored graph G will be obtained by coloring the square of the underlying graph of G with t colors, and then refining this coloring to obtain a homomorphism of G to $H_{(n, m)}^t$.

Lemma 3.1. (The Refinement Lemma)

$$\begin{aligned}\chi_{(n,m)}(H_{(n,m)}^t) &\leq 1 + (2n+m) + (2n+m)^2 + \cdots + (2n+m)^{t-1} \\ &= \begin{cases} t & \text{if } n=0 \text{ and } m=1 \\ \frac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases}\end{aligned}$$

Proof. Let $B = 1 + (2n+m) + (2n+m)^2 + \cdots + (2n+m)^{t-1}$. We describe a (n,m) -mixed B -coloring of $H_{(n,m)}$. For $k = 1, 2, \dots, t$, partition the independent set I_k into $(2n+m)^{k-1}$ subsets $I_{k,\ell}$, $1 \leq \ell \leq (2n+m)^{k-1}$, such that all sequences in $I_{k,\ell}$ agree in the first k places. The total number of sets $I_{p,q}$ is B .

By definition of these sets, if $p < r$ then:

- (1) all arcs joining vertices in $I_{p,q}$ and $I_{r,s}$ are of color i and:
 - (a) oriented towards $I_{p,q}$ if the p -th symbol of all sequences in $I_{r,s}$ is “ $+_i$ ” and the r -th symbol in of all sequence in $I_{r,s}$ is “ $-_i$ ”;
 - (b) oriented towards $I_{r,s}$ if the p -th symbol of all sequences in $I_{r,s}$ is “ $-_i$ ” and the r -th symbol in of all sequence in $I_{r,s}$ is “ $+_i$ ”;
- (2) all edges joining vertices in $I_{p,q}$ and $I_{r,s}$ are of color j if p -th symbol of all sequences in $I_{r,s}$ and the r -th symbol of all sequences in $I_{p,q}$ are both \sim_j .

Hence, by identifying all vertices in each set $I_{k,\ell}$, $1 \leq k \leq t$, $1 \leq \ell \leq (2n+m)^{k-1}$, a homomorphism of $H_{(n,m)}^t$ onto the (n,m) -colored mixed graph $\mathcal{O}_{(n,m)}$ on B vertices is obtained. Therefore, $\chi_{(n,m)}(H_{(n,m)}^t) \leq B$. \square

Lemma 3.1 is dubbed “The Refinement Lemma” because the independent sets I_k are refined in order to obtain the homomorphism to $\mathcal{O}_{(n,m)}$.

Let G be an (n,m) -colored mixed graph. Define $\mathcal{S}(G)$ to be square of the underlying undirected graph of G . That is, $\mathcal{S}(G)$ has the same vertex set as G and $xy \in E$ if and only if $1 \leq \text{dist}_G(x, y) \leq 2$.

Lemma 3.2. *Let G be an (n,m) -colored mixed graph. If $\mathcal{S}(G)$ is t -colorable, then there is a homomorphism from G to $H_{(n,m)}^t$.*

Proof. Let C_1, C_2, \dots, C_t be a t -coloring of $\mathcal{S}(G)$. Let $x \in C_k$. Define the sets $S_{x,i} = \{p : xy \in A_i(G) \text{ for some } y \in C_p\}$, $R_{x,i} = \{p : yx \in A_i(G) \text{ for some } y \in C_p\}$, and $T_{x,j} = \{p : xy \in E_j(G) \text{ for some } y \in C_p\}$. These are, respectively, the sets of (vertex) colors to which x sends an arc of color i in D , from which x receives an arc of color i in D , or to which x is joined by an edge of color j in D . Since C_k is an independent set $x \notin S_{x,i} \cup R_{x,i'} \cup T_{x,j}$ for any i, i' and j . By construction of $\mathcal{S}(G)$, for all i, i' and j the intersection between any two of $S_{x,i}, R_{x,i'}$ and $T_{x,j}$ is empty.

Thus, each vertex in C_k can be associated with a sequence with k -th entry is “ \sim ” and in which the ℓ -th entry is “ $+_i$ ” if $k \in S_{x,i}$, is “ $-_i$ ” if $k \in R_{x,i}$, is \sim_j if $k \in T_{x,j}$, and is \sim_t otherwise. This is a homomorphism of G to $H_{(n,m)}^t$. \square

Corollary 3.3. *Let G be an (n,m) -colored mixed graph. If $\mathcal{S}(G)$ is t -colorable, then*

$$\begin{aligned}\chi_{(n,m)}(G) &\leq 1 + (2n+m) + (2n+m)^2 + \cdots + (2n+m)^{t-1} \\ &= \begin{cases} t & \text{if } n=0 \text{ and } m=1 \\ \frac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases}\end{aligned}$$

An *injective k -coloring* of a graph G is an assignment of k colors to the vertices of G so that vertices at distance two are assigned different colors. Adjacent vertices may be assigned the same color. The injective chromatic number of G is the least k for which there exists an injective k -coloring of G . It is bounded above by the chromatic number of $\mathcal{S}(G)$ (ex. see [17]).

The previous arguments work essentially as given with a small modification to take into account the fact that vertices of the same color may be adjacent. The graph $H_{(n,m)}^t$ is redefined by replacing each “.” by one of “+ _{i} ”, “- _{i} ”, or “~ _{j} ”. This increases both the number of vertices of $H_{(n,m)}^t$ and the bound on its (n, m) -mixed chromatic number by a factor of $(2n + m)$. Thus, one obtains:

Corollary 3.4. *Let G be an (n, m) -colored mixed graph. If G has an injective t -coloring, then*

$$\begin{aligned} \chi_{(n,m)}(G) &\leq (2n + m) + (2n + m)^2 + \cdots + (2n + m)^t \\ &= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ (2n + m)^{\frac{(2n+m)^t - 1}{2n+m-1}} & \text{otherwise.} \end{cases} \end{aligned}$$

4. ORIENTED COLORINGS

When $n = 1$ and $m = 0$ we have an oriented graph, and the (n, m) -mixed chromatic number equals the oriented chromatic number. Hence the above results imply bounds for the oriented chromatic number. In particular, if $\mathcal{S}(G)$ has a t -coloring, then $\chi_o(G) \leq 2^t - 1$.

An important property of a coloring of $\mathcal{S}(G)$, or an injective t -coloring of G , used in the above arguments is that vertices joined by a directed path of length two must be assigned different colors. The *2-dipath chromatic number* of G is the smallest k for which there is a 2-dipath k -coloring of G . The 2-dipath chromatic number is at most the chromatic number of the square [16].

The argument in Section 3 goes through unchanged for oriented graphs if a k -coloring of $\mathcal{S}(G)$ is replaced by a 2-dipath k -coloring of G . This is the same as replacing $\mathcal{S}(G)$ by the square of G as a directed graph (two vertices are if they are joined by a directed path of length at most two) in the argument. Doing so, one obtains the following bound:

Corollary 4.1. *Let G be an oriented graph. If G has an 2-dipath t -coloring, then $\chi_o(G) = \chi_{(1,0)}(G) \leq 2^t - 1$*

Finally, if the 2-dipath t -coloring need not assign different colors to adjacent vertices, then proceeding as discussed at the end of the previous section one obtains $\chi_o(G) = \chi_{(1,0)}(G) \leq 2^{t+1} - 1$ (also see [7]).

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