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1. Introduction

A bound for the (n, m)-mixed chromatic number in terms of the chromatic number of the square of the underlying undirected graph is given. A similar bound holds when the chromatic number of the square is replaced by the injective chromatic number. When restricted to $n = 1$ and $m = 0$ (i.e., oriented graphs) this gives a new bound for the oriented chromatic number. In this case, a slightly improved bound is obtained if the chromatic number of the square is replaced the 2-dipath chromatic number (defined in Section 4). In all cases, the method of proof generalizes an argument that has been used to obtain Brooks-type theorems for injective oriented colorings [1, 2, 3, 4, 5, 6, 7, 8]. Similar, though not identical, arguments have appeared in the work of Sopena [9], and Nešetřil and Raspaud [10, 11].

Colorings of mixed graphs were first studied by Nešetřil and Raspaud [11]. They gave a bound on the (n, m)-mixed chromatic number in terms of the acyclic chromatic number of the underlying undirected graph. Bounds for the $(0, 2)$-mixed chromatic number of planar graphs and other graph families have been found [12]. Some of these have been extended to arbitrary n and m [13].

2. Definitions

For basic definitions in graph theory, see the text by Bondy and Murty [14].

Definition 2.1. *A* (n, m)-colored mixed graph *is an* $(m + n + 1)$*-tuple* $G = (V, A_1, A_2, \ldots, A_n, E_1, E_2, \ldots, E_m)$ *where:*

1. V *is a set of* vertices*;*
2. *for* $i = 1, 2, \ldots, n$ *the set* A_i *is a set of ordered pairs of distinct vertices of* G *called* the arcs of color i*;*
3. *for* $j = 1, 2, \ldots, m$ *the set* E_j *is a set of unordered pairs of distinct vertices of* G *called* the edges of color i*; and*
4. *the* underlying undirected graph $U[G]$*, with vertex set* $V(U[G]) = V(G)$ *and an edge joining* x *to* y *for every* i *for which the ordered pair* $xy \in A_i$ *and for every* j *for which the unordered pair* $xy \in E_j$*, is simple.*

Observe that a $(0, 1)$-colored mixed graph is a simple graph, and a $(1, 0)$-colored mixed graph is an oriented graph.

Homomorphisms between (n, m)-colored mixed graphs are defined in the natural way. For (n, m)-colored mixed graphs G and H, a *homomorphism* of G to H is a function $f : V(G) \to V(H)$ such that for $1 \le i \le n$ we have $f(x)f(y) \in A_i(H)$

whenever $xy \in A_i(G)$, and for $1 \leq j \leq n$ we have $f(x)f(y) \in E_j(H)$ whenever $xy \in E_j(G)$. For more on homomorphisms see [15].

Observe that homomorphisms of (n, m)-colored mixed graphs compose. That is, if F, G, and H are (n, m)-colored mixed graphs, f is a homomorphism of F to G and g is a homomorphism of G to H, then $g \circ f$ is a homomorphism of F to H.

By analogy with the corresponding definitions for other coloring concepts for oriented graphs, a *k-coloring* of a (n, m)-colored mixed graph is defined to be a homomorphism to a (n, m)-colored mixed graph on k vertices. The smallest k for which there is a k-coloring of a (n, m)-colored graph G is called the *(n, m)-mixed chromatic number of* G and denoted $\chi_{(n,m)}(G)$.

Informally, a k-coloring of an (n, m)-colored mixed graph is a partition of the vertices of G into k independent sets (sets containing no edges or arcs of any color) $X_1, X_2, \dots, X_k$ such that, for any two independent sets X_p and X_q, there is only one type of adjacency – only arcs of the same color and orientation, or only edges of the same color – between vertices in X_p and X_q.

Example 1. Consider a path on four vertices v_1, v_2, v_3, v_4 with arcs v_1v_2 and v_2v_3 of color 1 and an edge v_3v_4 of color 1. Call this (1, 1)-colored mixed graph G. Then $\chi_{(n,m)}(G) = 3$; color v_1 and v_4 with color 1, v_2 with color 2, and v_3 with color 3.

Example 2. Consider a path on five vertices v_1, v_2, v_3, v_4, v_5 with arcs v_1v_2 and v_2v_3 of color 1, an edge v_3v_4 of color 1, and an edge v_4v_5 of color 2. Call this (1, 2)-colored mixed graph G. Then $\chi_{(n,m)}(G) = 4$; color v_1 and v_4 with color 1, v_2 with color 2, v_3 with color 3, and v_5 with color 4.

We next define a (n, m)-mixed graph, Let $t \geq 3$. For $k = 1, 2, \dots, t$, let I_k be the set of all sequences of length t in which the k^{th} element is "$\cdot$" and every other entry is a "$+_i$" or "$-_i$" $(1 \leq i \leq n)$, or "$\sim_j$" $(1 \leq j \leq m)$.

Definition 2.2. *The graph $H^t_{(n,m)}$ is the (n, m)-colored mixed graph with vertex set $V(H^t_{(n,m)}) = I_1 \cup I_2 \cup \cdots \cup I_t$, and*

1. *an arc of color i joining sequence $s_k \in I_k$ to sequence $s_\ell \in I_\ell$ if and only if the ℓ-th entry of s_k is "$+_i$" and the k-th entry of s_ℓ is "$-_i$', and*
2. *an edge of color j joining $s_k \in I_k$ to sequence $s_\ell \in I_\ell$ if and only if the ℓ-th entry of s_k and the k-th entry of s_ℓ are both $\sim_j$.*

In the next section, we bound the (n, m)-mixed chromatic number of $H^t_{(n,m)}$.

3. Main result

The graph $H^t_{(1,0)}$ is used to prove bounds for injective oriented colorings [1, 2, 3, 4, 5]. Young has proved that an oriented graph can be properly t-colored so that any two vertices joined by a directed path of length two get different colors if and only if it has a homomorphism to $H^t_{(1,0)}$ [8] (Min and Wang [16] call this a 2-*dipath k-coloring.*) Our bound on the (n, m)-mixed chromatic number of a (n, m)-mixed colored graph G will be obtained by coloring the square of the underlying graph of G with t colors, and then refining this coloring to obtain a homomorphism of G to $H^t_{(n,m)}$.

Lemma 3.1. (The Refinement Lemma)

$$\begin{aligned}\chi_{(n,m)}(H^t_{(n,m)}) &\leq 1 + (2n+m) + (2n+m)^2 + \cdots + (2n+m)^{t-1} \\ &= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ \frac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases}\end{aligned}$$

Proof. Let $B = 1 + (2n+m) + (2n+m)^2 + \cdots + (2n+m)^{t-1}$. We describe a (n,m)-mixed B-coloring of $H_{(n,m)}$. For $k = 1, 2, \ldots, t$, partition the independent set I_k into $(2n+m)^{k-1}$ subsets $I_{k,\ell}$, $1 \leq \ell \leq (2n+m)^{k-1}$, such that all sequences in $I_{k,\ell}$ agree in the first k places. The total number of sets $I_{p,q}$ is B.

By definition of these sets, if $p < r$ then:

1. all arcs joining vertices in $I_{p,q}$ and $I_{r,s}$ are of color i and:
   - (a) oriented towards $I_{p,q}$ if the p-th symbol of all sequences in $I_{r,s}$ is "$+_i$" and the r-th symbol in of all sequence in $I_{r,s}$ is "$-_i$";
   - (b) oriented towards $I_{r,s}$ if the p-th symbol of all sequences in $I_{r,s}$ is "$-_i$" and the r-th symbol in of all sequence in $I_{r,s}$ is "$+_i$";
2. all edges joining vertices in $I_{p,q}$ and $I_{r,s}$ are of color j if p-th symbol of all sequences in $I_{r,s}$ and the r-th symbol of all sequences in $I_{(p,q)}$ are both $\sim_j$.

Hence, by identifying all vertices in each set $I_{k,\ell}$, $1 \leq k \leq t$, $1 \leq \ell \leq (2n+m)^{k-1}$, a homomorphism of $H^t_{(n,m)}$ onto the (n,m)-colored mixed graph $\mathcal{O}_{(n,m)}$ on B vertices is obtained. Therefore, $\chi_{(n,m)}(H^t_{(n,m)}) \leq B$. □

Lemma 3.1 is dubbed "The Refinement Lemma" because the independent sets I_k are refined in order to obtain the homomorphism to $\mathcal{O}_{(n,m)}$.

Let G be an (n,m)-colored mixed graph. Define $\mathcal{S}(G)$ to be square of the underlying undirected graph of G. That is, $\mathcal{S}(G)$ has the same vertex set as G and $xy \in E$ if and only if $1 \leq \text{dist}_G(x,y) \leq 2$.

Lemma 3.2. *Let G be an (n,m)-colored mixed graph. If $\mathcal{S}(G)$ is t-colorable, then there is a homomorphism from G to $H^t_{(n,m)}$.*

Proof. Let $C_1, C_2, \ldots, C_t$ be a t-coloring of $\mathcal{S}(G)$. Let $x \in C_k$. Define the sets $S_{x,i} = \{p : xy \in A_i(G) \text{ for some } y \in C_p\}$, $R_{x,i} = \{p : yx \in A_i(G) \text{ for some } y \in C_p\}$, and $T_{x,j} = \{p : xy \in E_j(G) \text{ for some } y \in C_p\}$. These are, respectively, the sets of (vertex) colors to which x sends an arc of color i in D, from which x receives an arc of color i in D, or to which x is joined by an edge of color j in D. Since C_k is an independent set $x \notin S_{x,i} \cup R_{x,i'} \cup T_{x,j}$ for any i, i' and j. By construction of $\mathcal{S}(G)$, for all i, i' and j the intersection between any two of $S_{x,i}, R_{x,i'}$ and $T_{x,j}$ is empty.

Thus, each vertex in C_k can be associated with a sequence with k-th entry is "$\cdot$" and in which the ℓ-th entry is "$+_i$" if $k \in S_{x,i}$, is "$-_i$" if $k \in R_{x,i}$, is $\sim_j$ if $k \in T_{x,j}$, and is $\sim_t$ otherwise. This is a homomorphism of G to $H^t_{(n,m)}$. □

Corollary 3.3. *Let G be an (n,m)-colored mixed graph. If $\mathcal{S}(G)$ is t-colorable, then*

$$\begin{aligned}\chi_{(n,m)}(G) &\leq 1 + (2n+m) + (2n+m)^2 + \cdots + (2n+m)^{t-1} \\ &= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ \frac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases}\end{aligned}$$

An *injective k-coloring* of a graph G is an assignment of k colors to the vertices of G so that vertices at distance two are assigned different colors. Adjacent vertices may be assigned the same color. The injective chromatic number of G is the least k for which there exists an injective k-coloring of G. It is bounded above by the chromatic number of $\mathcal{S}(G)$ (ex. see [17]).

The previous arguments work essentially as given with a small modification to take into account the fact that vertices of the same color may be adjacent. The graph $H^t_{(n,m)}$ is redefined by replacing replace each "$\cdot$" by one of "$+_i$", "$-_i$", or "$\sim_j$". This increases both the number of vertices of $H^t_{(n,m)}$ and the bound on its (n,m)-mixed chromatic number by a factor of $(2n+m)$. Thus, one obtains:

Corollary 3.4. *Let G be an (n,m)-colored mixed graph. If G has an injective t-coloring, then*

$$\begin{aligned}\chi_{(n,m)}(G) &\leq (2n+m) + (2n+m)^2 + \cdots + (2n+m)^t \\ &= \begin{cases} t & \text{if } n=0 \text{ and } m=1 \\ (2n+m)\frac{(2n+m)^t-1}{2n+m-1} & \text{otherwise.}\end{cases}\end{aligned}$$

4. Oriented colorings

When $n = 1$ and $m = 0$ we have an oriented graph, and the (n,m)-mixed chromatic number equals the oriented chromatic number. Hence the above results imply bounds for the oriented chromatic number. In particular, if $\mathcal{S}(G)$ has a t-coloring, then $\chi_o(G) \leq 2^t - 1$.

An important property of a coloring of $\mathcal{S}(G)$, or an injective t-coloring of G, used in the above arguments is that vertices joined by a directed path of length two must be assigned different colors. The *2-dipath chromatic number* of G is the smallest k for which there is a 2-dipath k-coloring of G. The 2-dipath chromatic number is at most the chromatic number of the square [16].

The argument in Section 3 goes through unchanged for oriented graphs if a k-coloring of $\mathcal{S}(G)$ is replaced by a 2-dipath k-coloring of G. This is the same as replacing $\mathcal{S}(G)$ by the square of G as a directed graph (two vertices are if they are joined by a directed path of length at most two) in the argument. Doing so, one obtains the following bound:

Corollary 4.1. *Let G be an oriented graph. If G has an 2-dipath t-coloring, then* $\chi_o(G) = \chi_{(1,0)}(G) \leq 2^t - 1$

Finally, if the 2-dipath t-coloring need not assign different colors to adjacent vertices, then proceeding as discussed at the end of the previous section one obtains $\chi_o(G) = \chi_{(1,0)}(G) \leq 2^{t+1} - 1$ (also see [7]).

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