Existence theorems for generalized Nash equilibrium problems: an analysis of assumptions

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ABSTRACT: The generalized Nash equilibrium, where the feasible sets of the players depend on other players' action, becomes increasingly popular among academics and practitionners. In this paper, we provide a thorough study of theorems guaranteeing existence of generalized Nash equilibria and analyze the assumptions on practical parametric feasible sets.

KEYWORDS: Noncooperative games; Existence theorem; Nash equilibrium. **MSC2000:** 91A10.

1 Introduction

In noncooperative game theory, solution concepts had been searched for years at the beginning of the XX$^{\text{th}}$ century, cf. [33]. Thankfully, the Nobel prize laureate, John F. Nash, proposed a unified solution concept for noncooperative games, latter called Nash equilibrium, with [26, 27]. Despite some critiscism, this solution concept is widely used among academics to model noncooperative behavior. Classical applications of Nash equilibrium include computer science, telecommunication, energy markets, and many others, see [14] for a recent survey. In this note, we focus on noncooperative games with infinite action space and one-period horizon. Let be N the number of players. The strategy set of player i is denoted by $X_i \subset \mathbb{R}^{n_i}$ and the payoff function by $\theta_i : X \to \mathbb{R}$ (to be maximized), where $X = X_1 \times \cdots \times X_N$. Player i's (pure) strategy is denoted by $x_i \in X_i$ while $x_{-i} \in X_{-i}$ denotes the other players' action, i.e. $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and $X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$. A game is thus described by $(N, X_i, \theta_i(.))$.

Definition 1.1. *A Nash equilibrium is a strategy point $x^\star \in X$ such that no player has an incentive to deviate, i.e. for all $i \in \{1, \ldots, N\}$,*

$$\forall x_i \in X_i,\ \theta_i(x_i, x^\star_{-i}) \leq \theta_i(x^\star_i, x^\star_{-i}). \tag{1}$$

Originally, Nash introduced the equilibrium concept to finite games in [26, 27], i.e. X_i is a finite set. Therefore, he used the mixed strategy concept (i.e. a probability distribution over the pure strategies) and proved the existence of such an equilibrium in that context. We report here the existence theorem of [28] for infinite games.

Theorem 1.2 (Nash). *Let N agents be characterized by an action space X_i and an objective function θ_i. If $\forall i \in \{1, \ldots, N\}$, X_i is nonempty, convex and compact; $\theta_i : X \to \mathbb{R}$ is continuous with $X = X_1 \times \cdots \times X_N$ and $\forall x_{-i} \in X_{-i}, x_i \to \theta_i(x_i, x_{-i})$ is concave on X_i, then there exists a Nash equilibrium.*

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The concavity assumption of the objective function θ_i with respect to x_i is sometimes called player-concavity. When dealing with cost functions rather payoff functions, the concavity assumption has to be replaced by a convexity assumption. Most existence theorems of Nash equilibrium rely on a fixed-point argument, and this from the very beginning. Indeed when Nash introduced his equilibrium concept, a fixed-point theorem is used: the Kakutani theorem in [26] and the Brouwer theorem in [27].

Since the introduction of games $(N, X_i, \theta_i(.))$, many extensions have been proposed in the literature: discontinuous payoffs (e.g. [8]), non concave payoffs (e.g. [4]), topological action spaces (e.g. [24, 30]), constrained strategy sets (e.g. [9, 32]). In the following, we consider the latter extension dealing with games where each player has a range of actions which depends on the actions of other players. This new extension leads to the so-called generalized Nash equilibrium.

Let 2^{X_i} be the family of subsets of X_i. Let $C_i : X_{-i} \to 2^{X_i}$ be constraint correspondence of Player i, i.e. a function mapping a point in X_{-i} to a subset of X_i. Thus, $C_i(x_{-i})$ defines the ith player action space given other players' action x_{-i}. Typically, the constraint correspondence C_i is defined by a parametrized action space as $C_i(x_{-i}) = \{x_i \in X_i,\ g_i(x_i, x_{-i}) \geq 0\}$, where $g_i : X \to \mathbb{R}^{m_i}$ is a constraint function. When g_i does not depend on x_{-i}, we get back to standard game. A generalized game is described by $(N, X_i, C_i(.), \theta_i(.))$ and is also called an abstract economy in reference to Debreu's economic work [1, 9].

Definition 1.3. *The generalized Nash equilibrium for a generalized game (N, X_i, C_i, θ_i) is defined as a point $x^\star$ solving for all $i \in \{1, \ldots, N\}$,*

$$x_i^\star \in \underset{x_i \in C_i(x_{-i}^\star)}{\arg\max}\ \theta_i(x_i, x_{-i}^\star). \tag{2}$$

In the present paper, we provide a self-contained survey of existence theorems for generalized Nash equilibrium. We also emphasize the use of fixed-point theorems in the proof of such theorems. A second purpose of this paper is to analyze the assumptions of those theorems on practical applications, and in particular the assumption on the constraint correspondence. Now, we set the outline of this paper. Section 2 gives the minimum required mathematical tools to study generalized Nash equilibria. Then, Section 3 presents the most recent existence theorems. Finally, Section 4 focuses on the analysis of assumptions when dealing with parametrized constrained sets.

2 Preliminaries

The mathematical tools needed are briefly summarized in this section, so that this paper is self-contained. For further details, we refer readers to the following books [2, 7, 12, 20, 29]. In the following, X and Y are two metric spaces.

2.1 Quasiconcavity

Refinements of concavity are first presented with characterizations based directly on the function f. Special characterizations when f is continuously differentiable or twice continuously differentiable exists but are omitted here, see [10] for a comprehensive study.

Definition 2.1. *A function $f : X \to Y$ is concave (resp. convex) iff $\forall x, y \in X, \forall \lambda \in [0, 1]$,*

$$f(\lambda x + (1-\lambda)y) \geq (\textit{resp.}\ \ \leq)\ \lambda f(x) + (1-\lambda)f(y). \tag{3}$$

Strict convexity/concavity is obtained when Inequality (3) is strict.

Concavity of f can be also be defined in terms of the graph of f. Let $\text{hyp}(f)$, $\text{epi}(f)$ be the hypograph and the epigraph of f, defined as $\text{hyp}(f) = \{(x, y), y \geq f(x)\}$ and $\text{epi}(f) = \{(x, y), y \leq f(x)\}$. The concavity (resp. convexity) of a function f is equivalent to the convexity of $\text{hyp}(f)$ (resp. $\text{epi}(f)$). So, it is immediate that $\text{hyp}(\min(f_1, f_2)) = \text{hyp}(f_1) \cap \text{hyp}(f_2)$: an intersection of two convex sets (resp. $\text{epi}(\max(f_1, f_2)) = \text{epi}(f_1) \cap \text{epi}(f_2)$). The quasiconcavity is now introduced by relaxing Inequality (3).

Definition 2.2. *A function $f : X \to Y$ is quasiconcave (resp. quasiconvex) iff $\forall x, y \in X, \forall \lambda \in]0, 1[$,*

$$f(\lambda x + (1 - \lambda)y) \geq \min(f(x), f(y)), \quad \textit{resp.} \quad f(\lambda x + (1 - \lambda)y) \leq \max(f(x), f(y)). \tag{4}$$

Again, strict quasiconvexity/concavity is obtained when Inequality (4) is strict.

A univariate quasiconvex (resp. quasiconcave) function is either monotone or unimodal. Obviously convexity implies quasiconvexity. To better catch the meaning of quasi-concavity in contrast to concavity, we plot on Figure 1 examples of a concave function, a non-concave quasi-concave function and a non-quasiconcave function.

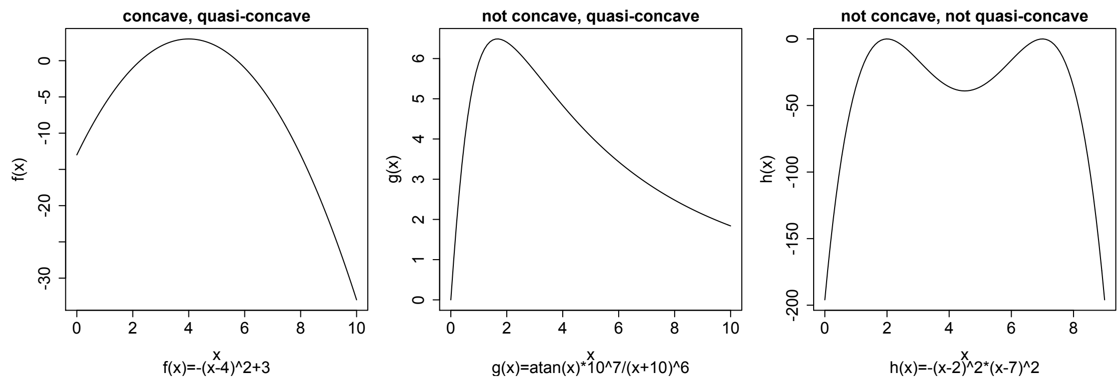

Figure 1: Examples and counter-examples of quasi-concavity

2.2 Correspondences

As unveiled in the introduction, new tools are used to refine the action strategy set from a compact set in Equation (1) to a player-dependent constrained set in Equation (2). Thus, correspondences, also called multi-valued functions, point-to-set maps or set-valued mappings, are introduced.

Definition 2.3. *A correspondence $F : X \to 2^Y$ is an application such that $\forall x \in X$, $F(x)$ is a subset of Y. A correspondence is also denoted by $F : X \to \mathcal{P}(Y)$ or $F : X \rightrightarrows Y$. Given F, the domain is $dom(F) = \{x \in X, F(x) \neq \emptyset\}$, the range is $rg(F) = \bigcup_x F(x)$ and the graph is $Gr(F) = \{(x, y) \in X \times Y, y \in F(x)\}$.*

Example 2.4. *Typical examples of correspondences are the inverse of a function f, $x \to f^{-1}(x)$ (since $f^{-1}(x)$ might be empty, a singleton and a set); a constraint set $x \to \{x, f(x) \leq c\}$; the generalized gradient $x \to \partial f(x)$ or $F : x \to [-|x|, |x|]$.*

Definition 2.5. *For a correspondence $F : X \mapsto 2^Y$, the image of a subset B by F is defined as $F^{-1}(B) = \{x \in X, F(x) = B\}$. The exterior image (also called upper inverse) is $F^+(B) = \{x \in X, F(x) \subset B\}$ whereas the interior image (also called lower inverse) is $F^-(B) = \{x \in X, F(x) \cap B \neq \emptyset\}$.*

A type of continuity for set-valued mappings has been introduced by Bouligand and Kuratowski in 1932: the lower and the upper semicontinuity (abbreviated l.s.c. and u.s.c.). In the literature, there are two concurrent definitions: the semicontinuity in the sense of Berge (e.g. [6, page 109]) and the semicontinuity in the sense of Hausdorff (e.g. [3, page 38-39]). These two definitions depend on the property of the set $F(x)$, yet, they are equivalent if F is compact-valued. In that case, the u.s.c./l.s.c. continuity can be defined on the exterior/interior images of F.

Definition 2.6. *$F : X \to 2^Y$ is upper semicontinuous (u.s.c.) at x, iff $\forall y \in Y$, $F^+(\{y\})$ is an open set in X. $F : X \to 2^Y$ is lower semicontinuous (l.s.c.) at x, iff $\forall y \in Y$, $F^-(\{y\})$ is an open set in X.*

When studying parametrized constrained sets, it is generally more convenient to work with characterizations by sequences. Therefore, the equivalent definition of the semicontinuity in terms of sequences are now given, see e.g. [19].

Definition 2.7. *$F : X \to 2^Y$ is upper semicontinuous (u.s.c.) at x, iff for all sequence $(x_n)_n \in X, x_n \underset{n\to+\infty}{\longrightarrow} x, \forall y_n \in T(x_n)$ and $\forall y \in Y$, $y_n \underset{n\to+\infty}{\longrightarrow} y \Rightarrow y \in T(x)$.*

Definition 2.8. *$F : X \to 2^Y$ is lower semicontinuous (l.s.c.) at x, iff for all sequence $(x_n)_n \in X, x_n \underset{n\to+\infty}{\longrightarrow} x, \forall y \in T(x)$, there exists a sequence $(y_k)_k \in Y$, such that $y_k \underset{n\to+\infty}{\longrightarrow} y$ and $\forall k \in \mathbb{N}, y_k \in T(x_k)$.*

Example 2.9. *Let F defined by $F(x) = [-1, 1]$ if $x \neq 0$ or $\{0\}$ otherwise. F is l.s.c. in 0 but not u.s.c. Let G defined by $G(x) = \{0\}$ if $x \neq 0$ or $[-1, 1]$ otherwise. G is u.s.c. in 0 but not l.s.c. Let H defined by $H(x) = \{0\}$ if $x \neq 0$ or $\{-1, 1\}$ otherwise. H is neither u.s.c. nor l.s.c. in 0.*

2.3 Theorems for correspondences

Thirdly, the necessary fixed-point theorems (for correspondences) are given, namely the Kakutani theorem [21] and the Begle theorem [5]. Let us first recall by the Brouwer theorem that a continuous function f from a finite-dimensional ball into itself admits a fixed-point. The Kakutani theorem is a valuable extention of the Brouwer's theorem to correspondences and is reported from [3]. The original theorem of [21] does not have any ambiguity about upper semicontinuity, since the author works in a finite-dimensional space with compact-valued mapping.

Theorem 2.10 (Kakutani)**.** *Let K be a nonempty compact convex subset of a Banach space (e.g. $\mathbb{R}^n$) and $T : K \to 2^K$ a correspondence. If T is u.s.c. such that $\forall x \in K$, $T(x)$ is nonempty, closed and convex, then T admits a fixed-point theorem.*

The Begle theorem ([5]) is an extension of a fixed-point theorem for locally connected spaces by [11, 23], which in turn extends the Brouwer fixed-point theorem. To introduce this theorem, contractible polyhedrons have first to be defined: contractibility is in a sense related to convexity, see e.g. [22].

Definition 2.11. *A geometric polyhedron is a finite union of convex hulls of finite-point sets.*

Definition 2.12. *A polyhedron is a subset S of $\mathbb{R}^n$ homeomorphic to a geometric polyhedron P, i.e. there exists a bijective function between S and P.*

Definition 2.13. *Contractible sets are nonempty sets deformable into a point by a continuous function (homotopy).*

Example 2.14. *Any star domain of Euclidean spaces is contractible whereas a finite-dimensional sphere is not. Any convex set of Euclidean spaces is contractible.*

The Begle theorem reported here is the version from [9], originally contractible sets are replaced by absolute retracts in [5].

Theorem 2.15 (Begle)**.** *Let Z be a contractible polyhedron and $\phi : Z \to 2^Z$ be upper semicontinuous. If $\forall z \in Z, \phi(z)$ is contractible, then ϕ admits a fixed point.*

Finally, a last theorem needed is the Berge's maximum theorem, see e.g. [29, page 229] and [7, page 64].

Theorem 2.16 (Berge's maximum theorem). *Let X, Y be two metric spaces, $f : X \times Y \to \mathbb{R}$ be an objection function and $F : X \to 2^Y$ a constraint correspondence. Assume that f is continuous, F is both l.s.c. and u.s.c.; and F is nonempty and compact valued. Then we have*

(i) $\phi : x \to \max_{y \in F(x)} f(x, y)$ *is a continuous function from X in $\mathbb{R}$.*

(ii) $\Phi : x \to \arg\max_{y \in F(x)} f(x, y)$ *is u.s.c. correspondence from X in 2^Y and compact-valued.*

As shown in the proof of Theorem 3.1, if in addition f is quasiconcave in y, then Φ is convex valued. Note that $\Phi(x)$ is sometimes written $\{y \in F(x), f(x, y) = \phi(x)\}$. The sequel demonstrates that the maximum theorem and the two fixed-point theorems 2.10 and 2.15 are the base recipes for showing the existence of a generalized Nash equilibrium.

3 State-of-the-art existence theorems

Showing the existence of a generalized Nash equilibrium can be tackled in two different ways: either a direct approach based on fixed-point theorems or a reformulation based on quasi-variational inequalities. Proofs are given to emphasize how the maximum theorem is the link between optimization subproblem (2) and fixed-point theorems.

3.1 The direct approach

Firstly, we investigate the direct approach. Theorem 3.1 was established by [1] in the context of abstract economy, so a simplified version by [20] is reported below. Some equivalent reformulations of Theorem 3.1 using a preference correspondence rather than a payoff function are also available in the following books: Theorem 19.8 in [7] and Theorem 3.7.1 in [12]. Note that [2] propose a different version, called the Arrow-Debreu-Nash theorem, where objective functions are player-concave rather than player-quasiconcave.

Theorem 3.1. *Let N players be characterized by an action space X_i, a constraint correspondence C_i and an objective function $\theta_i : X \to \mathbb{R}$. Assume for all players, we have*

(i) X_i is nonempty, convex and compact subset of a Euclidean space,

(ii) C_i is both u.s.c. and l.s.c. in X_{-i},

(iii) $\forall x_{-i} \in X_{-i}$, $C_i(x_{-i})$ is nonempty, closed, convex,

(iv) θ_i is continuous on the graph $Gr(C_i)$[1],

(v) $\forall x \in X$, $x_i \to \theta_i(x_i, x_{-i})$ is quasiconcave on $C_i(x_{-i})$,

Then there exists a generalized Nash equilibrium.

Proof. Since θ_i is continuous, C_i is both l.s.c. and u.s.c.; and C_i is nonempty and compact valued, the maximum theorem implies that the best response correspondence defined as

$$x_{-i} \overset{P_i}{\longmapsto} \arg\max_{x_i \in C_i(x_{-i})} \theta_i(x_i, x_{-i})$$

is u.s.c. and compact valued. Furthermore, as θ_i is player quasiconcave, P_i is convex valued. Let $z_i, y_i \in P_i(x_{-i})$. By definition of maximal points, $\forall x_i \in C_i(x_{-i})$, we have $\theta_i(y_i, x_{-i}) \geq \theta_i(x_i, x_{-i})$ and $\theta_i(z_i, x_{-i}) \geq \theta_i(x_i, x_{-i})$. Let $\lambda \in]0, 1[$. By the quasiconcaveness assumption, we get

$$\theta_i(\lambda y_i + (1 - \lambda) z_i, x_{-i}) \geq \min\left(\theta_i(y_i, x_{-i}), \theta_i(z_i, x_{-i})\right) \geq \theta_i(x_i, x_{-i}).$$

[1]There is no need for θ_i to be continuous on the whole space X, since only feasible points (i.e. those in $Gr(C_i)$) matters in Equation (2).

Hence, $\lambda y_i + (1-\lambda) z_i \in P_i(x_{-i})$, i.e. $P_i(x_{-i})$ is a convext set. Furthermore, P_i is also nonempty valued since $C_i(x_{-i})$ is nonempty. Now, consider the Cartesian product of $P_i(x_{-i})$ to define Φ as

$$\begin{array}{cccc} \Phi: & X & \to & 2^{X_1} \times \cdots \times 2^{X_N} \\ & x & \to & P_1(x_{-1}) \times \cdots \times P_N(x_{-N}) \end{array}$$

where X is a subset of $\mathbb{R}^n$ with $n = \sum_i n_i$. This multiplayer best response is nonempty, convex and compact valued. In our finite-dimensional setting and with a finite Cartesian product, the upper semicontinuity of each component P_i implies the upper semicontinuity of Φ, see Prop 3.6 of [19]. Finally, the Kakutani theorem gives the existence result. $\square$

The Debreu theorem ([9]) based on contractible sets is now given. Originally, the upper-semicontinuity is replaced by the closedness of the graph $Gr(C_i)$, but this is equivalent since contractible sets are closed and compact sets.

Theorem 3.2. *Let N agents be characterized by an action space X_i and $X = X_1 \times \cdots \times X_N$. Let a payoff function $\theta_i : X \to \mathbb{R}$ and a restricted action space $C_i(x_{-i})$ given other player actions x_{-i}. Each agent i maximizes its payoff on $C_i(x_{-i})$. If for all agents, we have*

(i) X_i is a contractible polyhedron,
(ii) $C_i : X_{-i} \to 2^{X_i}$ is u.s.c.,
(iii) θ_i is continuous from $Gr(C_i)$ to $\mathbb{R}$,
(iv) $\phi_i : x_{-i} \to \max\limits_{x_i \in C_i(x_{-i})} \theta_i(x_i, x_{-i})$ is continuous,
(v) $\forall x_{-i} \in X_{-i}$, the best response set $M_{x_{-i}} = \{x_i \in X_i(x_{-i}),\ \theta_i(x_i, x_{-i}) = \phi_i(x_{-i})\}$ is contractible,

Then there exists a generalized Nash equilibrium.

Proof. Let $G_i = Gr(C_i)$. Again, we work on the best response set, which is defined as

$$M_i = \{(x_i, x_{-i}) \in X_i \times X_{-i},\ x_i \in M_{x_{-i}}\} = \{(x_i, x_{-i}) \in G_i,\ \theta_i(x_i, x_{-i}) = \phi_i(x_{-i})\}.$$

This set is closed since the functions ϕ_i and θ_i are continuous and c_i is u.s.c.. Let Φ be the correspondence defined as $\Phi(x) = M_{x_{-1}} \times \cdots \times M_{x_{-N}}$. Using the Cartesian product, the graph of Φ is given by

$$Gr(\Phi) = \{(x, y) \in X \times X,\ y \in \Phi(x)\} = \bigcap_{i=1}^{N} \{(x, y) \in X \times X,\ (y_i, x_{-i}) \in M_i\},$$

a finite intersection of closed sets. As $Gr(\Phi)$ is closed, Φ is u.s.c.. Moreover for all x, $\Phi(x)$ is contractible as a finite Cartesian product of contractible sets. Applying the Begle fixed-point theorem completes the proof. $\square$

3.2 The QVI reformulation

The generalized Nash equilibrium problem (2) can be reformulated in the quasi-variational inequality (QVI) framework. Variational inequality and QVI are first described and then the existence theorem is given.

Definition 3.3 (Variational Inequality)**.** *Given a function $F : \mathbb{R}^n \to \mathbb{R}^n$ and a set $K \subset \mathbb{R}^n$ a Variational Inequality, denoted by $VI(K, F(.))$, is to find a vector $x^\star$ such that $x^\star \in K$ and*

$$\forall y \in K, (y - x^\star)^T F(x^\star) \geq 0.$$

Definition 3.4 (Quasi-Variational Inequality)**.** *Given a function $F : \mathbb{R}^n \to \mathbb{R}^n$ and a correspondence $K : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, a Quasi-Variational Inequality, denoted by $QVI(K(.), F(.))$, is to find a vector $x^\star$ such that $x^\star \in K(x^\star)$ and*

$$\forall y \in K(x^\star),\ (y - x^\star)^T F(x^\star) \geq 0.$$

For examples of applications of variational inequality problems and links with optimization, see e.g. [15, 16]. Definition 1.3 can be reformulated as the Quasi-Variational Inequality problem $QVI(C(.), \Theta(.))$ with

$$C(x) = C_1(x_{-1}) \times \cdots \times C_N(x_{-N}) \text{ and } \Theta(x) = \begin{pmatrix} \nabla_{x_1}\theta_1(x) \\ \vdots \\ \nabla_{x_N}\theta_N(x) \end{pmatrix}, \tag{5}$$

see e.g. [18] and [14] for a proof. Note that this reformulation assumes the differentiability of objective function θ_i. An existence theorem based on the QVI approach developped in [17] is now given.

Theorem 3.5. *Let N players be characterized by an action space X_i, a constraint correspondence C_i and an objective function $\theta_i : X \to \mathbb{R}$. Assume for all players, θ_i is continuously differentiable on the graph $Gr(C_i)$. Let $C(x) = C_1(x_{-1}) \times \cdots \times C_N(x_{-N})$. Assume there exists a compact convex subset $T \subset \mathbb{R}^n$,*

(i) $\forall x \in T$, $C(x)$ is nonempty, closed, convex subset of T,
(ii) C is both u.s.c. and l.s.c. in T,

Then there exists a generalized Nash equilibrium.

Proof. Using Θ as given in Equation (5), the correspondence $F : T \to 2^T$ is defined as

$$F(x) = \underset{z \in C(x)}{\arg\max} - (z-x)^T\Theta(x).$$

Let $x^\star$ be a fixed-point of F. We have

$$\begin{aligned} x^\star \in F(x^\star) \quad &\Leftrightarrow \quad x^\star \in \underset{z \in C(x^\star)}{\arg\max} - (z-x^\star)^T\Theta(x^\star) \\ &\Leftrightarrow \quad \forall z \in C(x^\star), -(z-x^\star)^T\Theta(x^\star) \leq -(x^\star - x^\star)^T\Theta(x^\star) = 0 \\ &\Leftrightarrow \quad \forall z \in C(x^\star), (z-x^\star)^T\Theta(x^\star) \geq 0. \end{aligned}$$

Thus, the QVI reformulation of (2) turns out to be a fixed-point problem. By assumption, C is nonempty, compact valued and both u.s.c. and l.s.c.. The function $(z, y) \to -(z-y)^T\Theta(y)$ is continuous since θ_i is continuously differentiable. Therefore by the maximum theorem, the correspondence F is u.s.c. and compact valued. Furthermore, the function $z \to -(z-x)^T\Theta(x)$ is a linear function (hence convex). So, F is also convex-valued (by the maximum theorem), hence $F(x)$ is a contractible set for all $x \in T$. Applying the Begle fixed-point theorem completes the proof. □

A significative part of games are such that player strategies are required to satisfy a common coupling constraint (such games are called jointly convex games) see [14] and the references therein. In jointly convex games, the contraint correspondence simplifies to

$$\begin{aligned} C_i : \quad X_{-i} &\to \quad 2^{X_i} \\ x_{-i} &\to \quad \{x_i \in X_i, \ (x_i, x_{-i}) \in K\}, \end{aligned} \tag{6}$$

where $K \subset X_1 \times \cdots \times X_N$ is a nonempty convex set. We are now interested in points solving the (classical) variational inequality problem $VI(K, \Theta(.))$ with K given in Equation (6) and Θ given in Equation (5). As expected, not all solutions of the generalized Nash equilibrium (2) (i.e. solutions of $QVI(C(.), \Theta(.))$) solves this variational inequality problem. Therefore, a special type of generalized Nash equilibrium has been introduced: a variational equilibrium also called a normalized equilibrium. A variational equilibrium has a special interpretation in terms of Lagrange multipliers of the corresponding KKT systems of the GNEP, see e.g. [13, 17].

Definition 3.6 (Variational equilibrium)**.** *A strategy $\bar{x}$ is a variational equilibrium of a generalized game $(N, X_i, C_i(.), \theta_i(.))$ if $\bar{x}$ solves $VI(K, \Theta(.))$ with K given in Equation (6) and Θ given in Equation (5).*

Theorem 3.7. *Let N players be characterized by an action space X_i, a constraint correspondence C_i and an objective function $\theta_i : X \to \mathbb{R}$. Assume for all players, θ_i is continuously differentiable on the graph $Gr(C_i)$ and there exists a nonempty convex compact set $K \subset \mathbb{R}^n$ such that $C_i(x_{-i}) = \{x_i \in X_i, \ (x_i, x_{-i}) \in K\}$, then there exists a variational equilibrium.*

Proof. Same proof as Theorem 3.5 with $C(x)$ replace by K which has the same properties. $\square$

4 Parametrized constrained sets

This final section aims to provide criteria to guarantee the assumptions of previous theorems, as well as, proofs for such criteria. For this purpose, a parametrized constraint set is considered

$$\begin{array}{rrl} C_i: & X_{-i} \to & 2^{X_i} \\ & x_{-i} \to & \{x_i \in X_i,\ g_i(x_i, x_{-i}) \geq 0\}, \end{array} \tag{7}$$

for $x_{-i} \in X_{-i}$, where $g_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$ and $C_i(x_{-i}) = \emptyset$ for $x_{-i} \notin X_{-i}$. The jth component of constraint function g_i is denoted by g_{ij}.

A central assumption of Theorems 3.1, 3.2 and 3.5 is to require C_i to be both lower and upper semicontinuous. Yet, Theorem 3.2 requires C_i to be u.s.c. and ϕ to be continuous, by Berge's maximum theorem, a sufficient condition is that C_i is also l.s.c.. Other assumptions of these theorems are nonemptyness, convexity and closedness of $C_i(x_{-i})$. Theorem 3.2 also requires X_i and M_i to be contractible: a sufficient condition is X_i to be a convex and θ_i to be player-quasiconcave.

4.1 The upper-semicontinuity

[31] devote a full chapter on set-valued analysis. Despite they formulate the outer and inner semicontinuity through superior limit of sets, they work with the Berge's semicontinuity (respectively the lower and upper semicontinuity). Their Theorem 5.7 of [31] gives equivalent reformulations of upper semicontinuity using the graph properties, for which their Example 5.10 is a direct application. A small variant is given here by removing equality constraints.

Proposition 4.1. *Let $C_i : X_{-i} \to 2^{X_i}$ be the feasible set mapping defined in Equation (7). Assume $X_i \subset \mathbb{R}^{n_i}$ is closed and all components g_{ij}'s are continuous on $X_i \times X_{-i} \subset \mathbb{R}^n$, then the correspondence C_i is u.s.c. on X_{-i}.*

Proof. For all $j = \{1, \ldots, m\}$, by the continuity of the jth component g_{ij}, the set of $x_i \in X_i$ such that $g_{ij}(x_i, x_{-i}) \geq 0\}$ is closed. So, $C_i(x_{-i})$ is a finite intersection of closed sets, thus a closed set. Let $(x_{i,n}, x_{-i,n})_n \to x$ and $y_{i,n} \in C_i(x_{-i,n})$, the closedness of $C_i(x_{-i})$ guarantees that $y_{i,n} \to y_i$ implies $y_i \in C_i(x_{-i})$. $\square$

A weaker assumption on the constraint function g_i is given in Theorem 10 of [19]. [19] only assumes that each component g_{ij} is an upper semicontinuous function (i.e. the closedness of the hypograph g_{ij}).

4.2 The lower-semicontinuity

Generally, conditions on g_i in order that the correspondence is l.s.c. are harder to find. Nevertheless, [31, 19] provide conditions for it. An application of Theorem 5.9 of [31] to the constraint correspondence in Equation (7) is presented.

Proposition 4.2. *Let $C_i : X_{-i} \to 2^{X_i}$ be the feasible set mapping defined in Equation (7). Assume g_{ij}'s are continuous and concave in x_i for each x_{-i}. If there exists $\bar{x}$ such that $g_i(\bar{x}_i, \bar{x}_{-i}) > 0$ for all i, then C_i is l.s.c. at $\bar{x}_{-i}$ and in some neighborhood of $\bar{x}_{-i}$ (and also u.s.c.).*

Proof. For ease of notation, the subscript i is removed, x_i and x_{-i} are denoted by x and w, respectively. Let f be the function $f(x,w) = \min(g_1(x,w),\dots,g_m(x,w))$. which is continuous by the continuity of g_j. By the concavity with respect to x, f is also concave with respect to x. The upper level $\text{lev}_{\geq 0} f$ is the graph of C. By the continuity of f, the graph $Gr(f)$ is closed.

$$\text{lev}_{\geq 0} f = \{(x,w), f(x,w) \geq 0\} = \{(x,w), \forall i = 1,\dots,m, g_i(x,w) \geq 0\} = \{(x,w), x \in C(w)\} = Gr(C).$$

So C is u.s.c.. The level set of a convex function is also convex, so is $\text{lev}_{\geq 0} f(.,w)$ with respect to x for all w. As f is continuous and $g(\bar{x},\bar{w}) > 0$, there exists an open set O, such that

$$\forall w \in O, f(\bar{x}, w) > 0.$$

Since the upper level set $\text{lev}_{\geq 0} f(.,w)$ for any w is convex and f is continuous, the interior $\text{int}C(w)$ is nonempty. This guarantees the lower semicontinuity of C at $\bar{w}$. Indeed, for all $\tilde{w} \in O$ and all $\tilde{x} \in \text{int}C(\tilde{w})$, by the continuity of f and the assumption at $(\bar{x},\bar{w})$, there exists a neighborhood $W \subset O \times \text{int}C(\tilde{w})$ such that f is strictly positive on W, and also $W \subset \text{gph}(C)$. As $w \to \tilde{w}$, then certainly $\tilde{x}$ belongs to the inner limit of $C(w)$. This inner limit is a closed set, and so includes $\text{int}C(\tilde{w}) \supset \text{cl}(\text{int}C(\tilde{w}))$. Since $C(\tilde{w})$ is a closed convex set with noempty interior, $\text{cl}(\text{int}C(\tilde{w})) = C(\tilde{w})$. Hence, the inner limit of $C(w)$ contains $C(\tilde{w})$, i.e. C is l.s.c. at $\tilde{w}$ by Theorem 5.9 of [31]. □

The previous property is also given in Theorem 12 of [19] and proved using the sequence characterization of semicontinuity. Theorem 13 of [19] is reported here as it gives weaker conditions for the correspondence to be lower semicontinuous than Theorem 12.

Proposition 4.3. *Let $C_i : X_{-i} \to 2^{X_i}$ be the feasible set mapping as defined above. Let $\widetilde{C}_i$ be the correspondence $\widetilde{C}_i(x_{-i}) = \{x_i \in X_i, g_i(x_i,x_{-i}) > 0\}$. If each component g_{ij} is lower semicontinuous (i.e. closedness of the epigraph) on $\bar{x}_{-i} \times \widetilde{C}_i(\bar{x}_{-i})$ and $C_i(\bar{x}_{-i}) \subset cl(\widetilde{C}_i(\bar{x}_{-i}))$, then C_i is lower semicontinuous at $\bar{x}_{-i}$.*

Proof. For ease of notation, the subscript i is removed, x_i and x_{-i} are denoted by x and w, respectively. If $\widetilde{C}(\bar{w}) = \emptyset$, then by assumption, $C(\bar{w}) = \emptyset$ and the conclusion is trivial. Otherwise, when $\widetilde{C}(\bar{w}) \neq \emptyset$, we choose $\bar{x} \in C(\bar{w})$ and $w_n \to \bar{w}$. Since $C(\bar{w}) \subset \text{cl}(\widetilde{C}(\bar{w}))$, there exists a sequence $(x_m)_m$ of elements in $\widetilde{C}(\bar{w})$ such that $x_m \to \bar{x}$. Construct the sequence n_m such that $n_0 = 0$ and $n_m = \max(n_{m-1}+1, \arg\min_k(\forall l \geq k, g(w_l, x_m) > 0))$. The sequence is well defined by the lower semicontinuity of g. Furthermore, the sequence $(x_{n_m})_{m\geq 0}$ is such that $x_{n_m} \in C(w_{n_m})$ with $x_{n_m} \to \bar{x}$, $w_{n_m} \to \bar{w}$ and $\bar{x} \in C(\bar{w})$, i.e. C is l.s.c. at $\bar{w}$. □

Continuous selections introduced by [25] can be used to further relaxed assumptions on g_i. Originally, [25] works with topological spaces and uses the Berge's semicontinuity. Their Proposition 2.3 is reported below.

Proposition 4.4. *If $\phi : X \to 2^Y$ is l.s.c. and $\phi : X \to 2^Y$ such that for every $x \in X$, $cl(\phi(x)) = cl(\psi(x))$, then ψ is also l.s.c..*

Property 4.4 has strong consequences on the lower semicontinuity of the correspondence C_i and justifies the [19]'s approach to use the correspondence $\widetilde{C}_i$ rather than C_i, since images have the same closure set. Therefore, the lower semicontinuity of each component g_{ij} suffices to get the lower semicontinuity of C_i. With the continuity of g_{ij}, it is even more straightforward to see that for all $x_i \in \widetilde{C}_i(x_{-i})$, there exists a sequence $(x_{-i,n})_n$ and $x_{i,n} \in \widetilde{C}_i(x_{-i,n})$ such that for all $n \geq n_0$, $g_i(x_{i,n}, x_{-i,n}) > 0$. Other types of conditions not based on strict inequalities are given in [19] .

4.3 The nonemptyness, the closedness and the convexity

Finally, we turn our attentions to other assumptions. Theorems 3.1, 3.2 and 3.5 also require C_i to be nonempty, convex and closed valued. The convexity assumption on $C_i(x_{-i})$ is satisfied when g_i is quasi-concave with respect to x_i. This is not immediate with the definition of quasi-concavity given in Section

2. But an equivalent definition for a function f to be quasiconcave is that all upper level sets $U_f(r) = \{x \in X, f(x) \geq r\}$ are convex for all r, see [10]. Thus, if g_i is quasiconcave, then $U_{g_i}(0)$ is convex. The nonemptyness assumption is the most challenging assumption. Except to have a strict inequality condition and the continuity of g_i's, it is hard to find general conditions. Finally, the closedness assumption is satisfied when g_i's are continuous.

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