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# FINITE TIME SINGULARITY IN A FREE BOUNDARY PROBLEM MODELING MEMS

JOACHIM ESCHER, PHILIPPE LAURENÇOT, AND CHRISTOPH WALKER

ABSTRACT. The occurrence of a finite time singularity is shown for a free boundary problem modeling microelectromechanical systems (MEMS) when the applied voltage exceeds some value. The model involves a singular nonlocal reaction term and a nonlinear curvature term accounting for large deformations.

## 1. INTRODUCTION

An idealized electrostatically actuated microelectromechanical system (MEMS) consists of a fixed horizontal ground plate held at zero potential above which an elastic membrane held at potential  $V$  is suspended. A Coulomb force is generated by the potential difference across the device and results in a deformation of the membrane, thereby converting electrostatic energy into mechanical energy, see [1, 4, 7] for a more detailed account and further references. After a suitable scaling and assuming homogeneity in transversal horizontal direction, the ground plate is assumed to be located at  $z = -1$  and the membrane displacement  $u = u(t, x) \in (-1, \infty)$  with  $t > 0$  and  $x \in I := (-1, 1)$  evolves according to

$$\partial_t u - \partial_x \left( \frac{\partial_x u}{\sqrt{1 + \varepsilon^2 (\partial_x u)^2}} \right) = -\lambda (\varepsilon^2 |\partial_x \psi(t, x, u(t, x))|^2 + |\partial_z \psi(t, x, u(t, x))|^2) , \quad (1)$$

for  $t > 0$  and  $x \in I$  with boundary conditions

$$u(t, \pm 1) = 0 , \quad t > 0 , \quad (2)$$

and initial condition

$$u(0, x) = u^0(x) , \quad x \in I . \quad (3)$$

The electrostatic potential  $\psi = \psi(t, x, z)$  satisfies a rescaled Laplace equation in the region

$$\Omega(u(t)) := \{(x, z) \in I \times (-1, \infty) : -1 < z < u(t, x)\}$$

between the plate and the membrane which reads

$$\varepsilon^2 \partial_x^2 \psi + \partial_z^2 \psi = 0 , \quad (x, z) \in \Omega(u(t)) , \quad t > 0 , \quad (4)$$

$$\psi(t, x, z) = \frac{1 + z}{1 + u(t, x)} , \quad (x, z) \in \partial\Omega(u(t)) , \quad t > 0 , \quad (5)$$

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where  $\varepsilon > 0$  denotes the aspect ratio of the device and  $\lambda > 0$  is proportional to the square of the applied voltage. The dynamics of  $(u, \psi)$  is thus given by the coupling of a quasilinear parabolic equation for  $u$  and an elliptic equation in a moving domain for  $\psi$ , the latter being only well-defined as long as the membrane does not touch down on the ground plate, that is,  $u$  does not reach the value  $-1$ . To guarantee optimal operating conditions of the device, this touchdown phenomenon has to be controlled and its occurrence is obviously related to the value of  $\lambda$ .

The main difficulty to be overcome in the analysis of (1)-(5) is the nonlocal and nonlinear implicit dependence on  $u$  of the right-hand side of (1) which is also singular if  $u$  approaches  $-1$ . Except for the singularity, these features disappear when setting  $\varepsilon = 0$  in (1)-(5), a commonly made assumption which reduces (1)-(5) to a singular semilinear reaction-diffusion equation. This so-called small aspect ratio model has received considerable attention in recent years, see [4, 7] and the references therein. In this simplified situation, it has been established that touchdown does not take place if  $\lambda$  is below a certain threshold value  $\lambda_* > 0$ , but occurs if  $\lambda$  exceeds this value [4, 5, 6].

We have recently investigated the well-posedness of (1)-(5) and established the following result [3].

**Theorem 1 (Local Well-Posedness).** *Let  $q \in (2, \infty)$ ,  $\varepsilon > 0$ ,  $\lambda > 0$ , and consider an initial value*

$$u^0 \in W_q^2(I) \quad \text{such that} \quad u^0(\pm 1) = 0 \quad \text{and} \quad 0 \geq u^0(x) > -1 \quad \text{for} \quad x \in I. \quad (6)$$

*Then there is a unique maximal solution  $(u, \psi)$  to (1)-(5) on the maximal interval of existence  $[0, T_m^\varepsilon)$  in the sense that*

$$u \in C^1([0, T_m^\varepsilon), L_q(I)) \cap C([0, T_m^\varepsilon), W_q^2(I))$$

*satisfies (1)-(3) together with*

$$0 \geq u(t, x) > -1, \quad (t, x) \in [0, T_m^\varepsilon) \times I, \quad (7)$$

*and  $\psi(t) \in W_2^2(\Omega(u(t)))$  solves (4)-(5) for each  $t \in [0, T_m^\varepsilon)$ .*

We have also shown in [3] that, if  $\lambda$  and  $u^0$  are sufficiently small, the solution  $(u, \psi)$  to (1)-(5) exists for all times (i.e.  $T_m^\varepsilon = \infty$ ) and touchdown does not take place, not even in infinite time.

**Theorem 2 (Global Existence).** *Let  $q \in (2, \infty)$ ,  $\varepsilon > 0$ , and consider an initial value  $u^0$  satisfying (6). Given  $\kappa \in (0, 1)$ , there are  $\lambda_*(\kappa) > 0$  and  $r(\kappa) > 0$  such that, if  $\lambda \in (0, \lambda_*(\kappa))$  and  $\|u^0\|_{W_q^2(I)} \leq r(\kappa)$ , the maximal solution  $(u, \psi)$  to (1)-(5) exists for all times and  $u(t, x) \geq -1 + \kappa$  for  $(t, x) \in [0, \infty) \times I$ .*

On the other hand, we have been able to prove that no stationary solution to (1)-(5) exists provided  $\lambda$  is sufficiently large. However, whether or not  $T_m^\varepsilon$  is finite in this case has been left as an open question. The purpose of this note is to show that – as expected on physical grounds –  $T_m^\varepsilon$  is indeed finite for  $\lambda$  sufficiently large.

**Theorem 3 (Finite time singularity).** *Let  $q \in (2, \infty)$ ,  $\varepsilon > 0$ , and consider an initial value  $u^0$  satisfying (6). If  $\lambda > 1/\varepsilon$  and  $(u, \psi)$  denotes the maximal solution to (1)-(5) defined on  $[0, T_m^\varepsilon)$ , then  $T_m^\varepsilon < \infty$ .*

The criterion  $\lambda > 1/\varepsilon$  is likely to be far from optimal. As we shall see below, improving it would require to have a better control on  $\partial_x u(\pm 1)$ . The proof of Theorem 3 relies on the derivation of a chain of estimates which allow us to obtain a lower bound on the  $L_1$ -norm of the right-hand side of (1) depending only on  $u$ . The lower bound thus obtained is in fact the mean value of a convex function of  $u$ , and we may then end the proof with the help of Jensen's inequality, an argument which has already been used for the small aspect ratio model, see [5, 6].

We shall point out that, in contrast to the small aspect ratio model, the finiteness of  $T_m^\varepsilon$  does not guarantee that the touchdown phenomenon really takes place as  $t \rightarrow T_m^\varepsilon$ . Indeed, according to [3, Theorem 1.1 (ii)], the finiteness of  $T_m^\varepsilon$  implies that  $\min_{[-1,1]} u(t) \rightarrow -1$  or  $\|u(t)\|_{W_q^2(I)} \rightarrow \infty$  as  $t \rightarrow T_m^\varepsilon$ . While the former corresponds to the touchdown behaviour, the latter is more likely to be interpreted as the membrane being no longer the graph of a function at time  $T_m^\varepsilon$ .

## 2. PROOF OF THEOREM 3

Let  $q \in (2, \infty)$ ,  $\varepsilon > 0$ ,  $\lambda > 0$  and consider an initial value  $u^0$  satisfying (6). We denote the maximal solution to (1)-(5) defined on  $[0, T_m^\varepsilon)$  by  $(u, \psi)$ . Differentiating the boundary conditions (5), we readily obtain

$$\partial_x \psi(t, x, -1) = \partial_x \psi(t, x, u(t, x)) + \partial_x u(t, x) \partial_z \psi(t, x, u(t, x)) = 0, \quad (t, x) \in (0, T_m^\varepsilon) \times I, \quad (8)$$

and

$$\partial_z \psi(t, \pm 1, z) = 1, \quad (t, z) \in (0, T_m^\varepsilon) \times (-1, 0). \quad (9)$$

Additional information on the boundary behaviour of  $\psi$  is provided by the next lemma.

**Lemma 4.** For  $t \in (0, T_m^\varepsilon)$ ,

$$1 + z \leq \psi(t, x, z) \leq 1, \quad (x, z) \in \Omega(u(t)), \quad (10)$$

$$\pm \partial_x \psi(t, \pm 1, z) \leq 0, \quad z \in (-1, 0). \quad (11)$$

**Proof.** Fix  $t \in (0, T_m^\varepsilon)$ . The upper bound in (10) readily follows from the maximum principle. Next, the function  $\sigma$ , defined by  $\sigma(x, z) = 1 + z$ , obviously satisfies  $\varepsilon^2 \partial_x^2 \sigma + \partial_z^2 \sigma = 0$  in  $\Omega(u(t))$  as well as

$$\sigma(\pm 1, z) = 1 + z = \psi(t, \pm 1, z), \quad z \in (-1, 0),$$

$$\sigma(x, -1) = 0 = \psi(t, x, -1), \quad x \in (-1, 1).$$

Owing to the non-positivity (7) of  $u(t)$ , it also satisfies

$$\sigma(x, u(t, x)) = 1 + u(t, x) \leq 1 = \psi(t, x, u(t, x)), \quad x \in (-1, 1),$$

and we infer from the comparison principle that  $\psi(t, x, z) \geq \sigma(x, z)$  for  $(x, z) \in \Omega(u(t))$ . It then follows from (10) that  $\psi(t, x, z) \geq 1 + z = \psi(t, \pm 1, z)$  for  $(x, z) \in \Omega(u(t))$  which readily implies (11).  $\square$

To simplify notations, we set

$$\gamma_m(t, x) := \partial_z \psi(t, x, u(t, x)), \quad \gamma_g(t, x) := \partial_z \psi(t, x, -1), \quad (t, x) \in (0, T_m^\varepsilon) \times (-1, 1), \quad (12)$$

and first derive an upper bound of the  $L_1$ -norm of the right-hand side of (1), observing that, due to (8), it also reads

$$-\lambda \varepsilon^2 |\partial_x \psi(t, x, u(t, x))|^2 + |\partial_z \psi(t, x, u(t, x))|^2 = -\lambda (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 .$$

**Lemma 5.** *For  $t \in (0, T_m^\varepsilon)$ ,*

$$\int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 dx \geq 2 \int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x) dx - 2 . \quad (13)$$

**Proof.** Fix  $t \in (0, T_m^\varepsilon)$ . We multiply (4) by  $\partial_z \psi(t) - 1$  and integrate over  $\Omega(u(t))$ . Using (8), (9), and Green's formula we obtain

$$\begin{aligned} 0 &= -\varepsilon^2 \int_{\Omega(u)} \partial_x \partial_z \psi \partial_x \psi d(x, z) + \varepsilon^2 \int_{-1}^1 (\partial_x u)^2 \gamma_m (\gamma_m - 1) dx \\ &\quad - \frac{1}{2} \int_{-1}^1 (\gamma_g^2 - 2\gamma_g) dx + \frac{1}{2} \int_{-1}^1 (\gamma_m^2 - 2\gamma_m) dx . \end{aligned}$$

Since

$$\int_{\Omega(u)} \partial_x \partial_z \psi \partial_x \psi d(x, z) = \frac{1}{2} \int_{-1}^1 (\partial_x u)^2 \gamma_m^2 dx$$

by (8) and since  $\gamma_g^2 - 2\gamma_g \geq -1$ , we end up with (13).  $\square$

We again use (4) to obtain a lower bound for the boundary integral of the right-hand side of (13) which depends on the Dirichlet energy of  $\psi$ .

**Lemma 6.** *For  $t \in (0, T_m^\varepsilon)$ ,*

$$\int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x) dx \geq \int_{\Omega(u(t))} (\varepsilon^2 |\partial_x \psi(t, x, z)|^2 + |\partial_z \psi(t, x, z)|^2) d(x, z) . \quad (14)$$

**Proof.** Fix  $t \in (0, T_m^\varepsilon)$ . We multiply (4) by  $\psi(t)$  and integrate over  $\Omega(u(t))$ . Using (5), (8), and Green's formula we obtain

$$\begin{aligned} 0 &= - \int_{\Omega(u(t))} (\varepsilon^2 |\partial_x \psi(t, x, z)|^2 + |\partial_z \psi(t, x, z)|^2) d(x, z) + \varepsilon^2 \int_{-1}^0 (1+z) \partial_x \psi(t, 1, z) dz \\ &\quad - \varepsilon^2 \int_{-1}^0 (1+z) \partial_x \psi(t, -1, z) dz + \varepsilon^2 \int_{-1}^1 (\partial_x u(t, x))^2 \gamma_m(t, x) dx + \int_{-1}^1 \gamma_m(t, x) dx . \end{aligned}$$

Owing to (11), the second and third terms of the right-hand side of the above equality are non-positive, whence (14).  $\square$

We finally argue as in [2, Lemma 9] to establish a connection between the Dirichlet energy of  $\psi$  and  $u$ .

**Lemma 7.** For  $t \in (0, T_m^\varepsilon)$ ,

$$\int_{\Omega(u(t))} (\varepsilon^2 |\partial_x \psi(t, x, z)|^2 + |\partial_z \psi(t, x, z)|^2) \, d(x, z) \geq \int_{-1}^1 \frac{dx}{1 + u(t, x)} . \quad (15)$$

**Proof.** Let  $t \in (0, T_m^\varepsilon)$  and  $x \in (-1, 1)$ . We deduce from (5) and the Cauchy-Schwarz inequality that

$$\begin{aligned} \frac{1}{1 + u(t, x)} &= \frac{(\psi(t, x, u(t, x)) - \psi(t, x, -1))^2}{1 + u(t, x)} = \frac{1}{1 + u(t, x)} \left( \int_{-1}^{u(t, x)} \partial_z \psi(t, x, z) \, dz \right)^2 \\ &\leq \int_{-1}^{u(t, x)} (\partial_z \psi(t, x, z))^2 \, dz . \end{aligned} \quad (16)$$

Integrating the above inequality with respect to  $x \in (-1, 1)$  readily gives (15).  $\square$

*Remark 8.* Observe that (16) provides a quantitative estimate on the singularity of  $\partial_z \psi$  generated by  $u$  when touchdown occurs.

Combining the three lemmas above with Jensen's inequality give the following estimate.

**Proposition 9.** For  $t \in (0, T_m^\varepsilon)$ ,

$$\int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 \, dx \geq 4\varphi \left( \frac{1}{2} \int_{-1}^1 u(t, x) \, dx \right) - 2 , \quad (17)$$

where  $\varphi(r) := 1/(1 + r)$ ,  $r \in (-1, \infty)$ .

**Proof.** Fix  $t \in (0, T_m^\varepsilon)$ . We infer from Lemma 5, Lemma 6, and Lemma 7 that

$$\int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 \, dx \geq 2 \int_{-1}^1 \varphi(u(t, x)) \, dx - 2 .$$

To complete the proof, we argue as in [5, 6] and use the convexity of  $\varphi$  and Jensen's inequality to obtain (17).  $\square$

**Proof of Theorem 3.** Introducing

$$E(t) := -\frac{1}{2} \int_{-1}^1 u(t, x) \, dx , \quad t \in [0, T_m^\varepsilon) ,$$

the bounds (7) ensure that

$$0 \leq E(t) < 1 , \quad t \in [0, T_m^\varepsilon) . \quad (18)$$

It follows from (1), (8), and Proposition 9 that

$$\begin{aligned} \frac{dE}{dt}(t) &= -\frac{1}{2} \left[ \frac{\partial_x u(t, x)}{\sqrt{1 + \varepsilon^2 (\partial_x u(t, x))^2}} \right]_{x=-1}^{x=1} + \frac{\lambda}{2} \int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 dx \\ &\geq F_\lambda(E) := 2\lambda\varphi(-E) - \lambda - \frac{1}{\varepsilon}. \end{aligned} \quad (19)$$

If  $\lambda > 1/\varepsilon$ , we note that  $F_\lambda(0) > 0$  and thus  $F_\lambda(r) \geq F_\lambda(0) > 0$  for  $r \in [0, 1)$  due to the monotonicity of  $F_\lambda$ . Since  $E(0) \geq 0$  by (18), it follows from (19) and the properties of  $F_\lambda$  that  $t \mapsto E(t)$  is increasing on  $[0, T_m^\varepsilon)$ . Consequently,

$$\frac{dE}{dt}(t) \geq F_\lambda(E(0)) \geq F_\lambda(0), \quad t \in [0, T_m^\varepsilon).$$

Integrating the previous inequality with respect to time and using (18), we end up with the inequality  $1 \geq E(0) + F_\lambda(0)T_m^\varepsilon$  which provides the claimed finiteness of  $T_m^\varepsilon$ .  $\square$

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