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FINITE TIME SINGULARITY IN A FREE BOUNDARY PROBLEM MODELING MEMS

JOACHIM ESCHER, PHILIPPE LAURENÇOT, AND CHRISTOPH WALKER

ABSTRACT. The occurrence of a finite time singularity is shown for a free boundary problem modeling microelectromechanical systems (MEMS) when the applied voltage exceeds some value. The model involves a singular nonlocal reaction term and a nonlinear curvature term accounting for large deformations.

1. INTRODUCTION

An idealized electrostatically actuated microelectromechanical system (MEMS) consists of a fixed horizontal ground plate held at zero potential above which an elastic membrane held at potential V is suspended. A Coulomb force is generated by the potential difference across the device and results in a deformation of the membrane, thereby converting electrostatic energy into mechanical energy, see [1, 4, 7] for a more detailed account and further references. After a suitable scaling and assuming homogeneity in transversal horizontal direction, the ground plate is assumed to be located at $z = -1$ and the membrane displacement $u = u(t, x) \in (-1, \infty)$ with $t > 0$ and $x \in I := (-1, 1)$ evolves according to

$$\partial_t u - \partial_x \left(\frac{\partial_x u}{\sqrt{1 + \varepsilon^2 (\partial_x u)^2}} \right) = -\lambda \left(\varepsilon^2 |\partial_x \psi(t, x, u(t, x))|^2 + |\partial_z \psi(t, x, u(t, x))|^2 \right) , \quad (1)$$

for $t > 0$ and $x \in I$ with boundary conditions

$$u(t, \pm 1) = 0 , \quad t > 0 , \quad (2)$$

and initial condition

$$u(0, x) = u^0(x) , \quad x \in I . \quad (3)$$

The electrostatic potential $\psi = \psi(t, x, z)$ satisfies a rescaled Laplace equation in the region

$$\Omega(u(t)) := \{(x, z) \in I \times (-1, \infty) : -1 < z < u(t, x)\}$$

between the plate and the membrane which reads

$$\varepsilon^2 \partial_x^2 \psi + \partial_z^2 \psi = 0 , \quad (x, z) \in \Omega(u(t)) , \quad t > 0 , \quad (4)$$

$$\psi(t, x, z) = \frac{1 + z}{1 + u(t, x)} , \quad (x, z) \in \partial\Omega(u(t)) , \quad t > 0 , \quad (5)$$

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where $\varepsilon > 0$ denotes the aspect ratio of the device and $\lambda > 0$ is proportional to the square of the applied voltage. The dynamics of (u, ψ) is thus given by the coupling of a quasilinear parabolic equation for u and an elliptic equation in a moving domain for ψ , the latter being only well-defined as long as the membrane does not touch down on the ground plate, that is, u does not reach the value -1 . To guarantee optimal operating conditions of the device, this touchdown phenomenon has to be controlled and its occurrence is obviously related to the value of λ .

The main difficulty to be overcome in the analysis of (1)-(5) is the nonlocal and nonlinear implicit dependence on u of the right-hand side of (1) which is also singular if u approaches -1 . Except for the singularity, these features disappear when setting $\varepsilon = 0$ in (1)-(5), a commonly made assumption which reduces (1)-(5) to a singular semilinear reaction-diffusion equation. This so-called small aspect ratio model has received considerable attention in recent years, see [4, 7] and the references therein. In this simplified situation, it has been established that touchdown does not take place if λ is below a certain threshold value $\lambda_* > 0$, but occurs if λ exceeds this value [4, 5, 6].

We have recently investigated the well-posedness of (1)-(5) and established the following result [3].

Theorem 1 (Local Well-Posedness). *Let $q \in (2, \infty)$, $\varepsilon > 0$, $\lambda > 0$, and consider an initial value*

$$u^0 \in W_q^2(I) \quad \text{such that} \quad u^0(\pm 1) = 0 \quad \text{and} \quad 0 \geq u^0(x) > -1 \quad \text{for } x \in I. \quad (6)$$

Then there is a unique maximal solution (u, ψ) to (1)-(5) on the maximal interval of existence $[0, T_m^\varepsilon)$ in the sense that

$$u \in C^1([0, T_m^\varepsilon), L_q(I)) \cap C([0, T_m^\varepsilon), W_q^2(I))$$

satisfies (1)-(3) together with

$$0 \geq u(t, x) > -1, \quad (t, x) \in [0, T_m^\varepsilon) \times I, \quad (7)$$

and $\psi(t) \in W_2^2(\Omega(u(t)))$ solves (4)-(5) for each $t \in [0, T_m^\varepsilon)$.

We have also shown in [3] that, if λ and u^0 are sufficiently small, the solution (u, ψ) to (1)-(5) exists for all times (i.e. $T_m^\varepsilon = \infty$) and touchdown does not take place, not even in infinite time.

Theorem 2 (Global Existence). *Let $q \in (2, \infty)$, $\varepsilon > 0$, and consider an initial value u^0 satisfying (6). Given $\kappa \in (0, 1)$, there are $\lambda_*(\kappa) > 0$ and $r(\kappa) > 0$ such that, if $\lambda \in (0, \lambda_*(\kappa))$ and $\|u^0\|_{W_q^2(I)} \leq r(\kappa)$, the maximal solution (u, ψ) to (1)-(5) exists for all times and $u(t, x) \geq -1 + \kappa$ for $(t, x) \in [0, \infty) \times I$.*

On the other hand, we have been able to prove that no stationary solution to (1)-(5) exists provided λ is sufficiently large. However, whether or not T_m^ε is finite in this case has been left as an open question. The purpose of this note is to show that – as expected on physical grounds – T_m^ε is indeed finite for λ sufficiently large.

Theorem 3 (Finite time singularity). *Let $q \in (2, \infty)$, $\varepsilon > 0$, and consider an initial value u^0 satisfying (6). If $\lambda > 1/\varepsilon$ and (u, ψ) denotes the maximal solution to (1)-(5) defined on $[0, T_m^\varepsilon)$, then $T_m^\varepsilon < \infty$.*

The criterion $\lambda > 1/\varepsilon$ is likely to be far from optimal. As we shall see below, improving it would require to have a better control on $\partial_x u(\pm 1)$. The proof of Theorem 3 relies on the derivation of a chain of estimates which allow us to obtain a lower bound on the L_1 -norm of the right-hand side of (1) depending only on u . The lower bound thus obtained is in fact the mean value of a convex function of u , and we may then end the proof with the help of Jensen's inequality, an argument which has already been used for the small aspect ratio model, see [5, 6].

We shall point out that, in contrast to the small aspect ratio model, the finiteness of T_m^ε does not guarantee that the touchdown phenomenon really takes place as $t \rightarrow T_m^\varepsilon$. Indeed, according to [3, Theorem 1.1 (ii)], the finiteness of T_m^ε implies that $\min_{[-1,1]} u(t) \rightarrow -1$ or $\|u(t)\|_{W_q^2(I)} \rightarrow \infty$ as $t \rightarrow T_m^\varepsilon$. While the former corresponds to the touchdown behaviour, the latter is more likely to be interpreted as the membrane being no longer the graph of a function at time T_m^ε .

2. PROOF OF THEOREM 3

Let $q \in (2, \infty)$, $\varepsilon > 0$, $\lambda > 0$ and consider an initial value u^0 satisfying (6). We denote the maximal solution to (1)-(5) defined on $[0, T_m^\varepsilon]$ by (u, ψ) . Differentiating the boundary conditions (5), we readily obtain

$$\partial_x \psi(t, x, -1) = \partial_x \psi(t, x, u(t, x)) + \partial_x u(t, x) \partial_z \psi(t, x, u(t, x)) = 0, \quad (t, x) \in (0, T_m^\varepsilon) \times I, \quad (8)$$

and

$$\partial_z \psi(t, \pm 1, z) = 1, \quad (t, z) \in (0, T_m^\varepsilon) \times (-1, 0). \quad (9)$$

Additional information on the boundary behaviour of ψ is provided by the next lemma.

Lemma 4. For $t \in (0, T_m^\varepsilon)$,

$$1 + z \leq \psi(t, x, z) \leq 1, \quad (x, z) \in \Omega(u(t)), \quad (10)$$

$$\pm \partial_x \psi(t, \pm 1, z) \leq 0, \quad z \in (-1, 0). \quad (11)$$

Proof. Fix $t \in (0, T_m^\varepsilon)$. The upper bound in (10) readily follows from the maximum principle. Next, the function σ , defined by $\sigma(x, z) = 1 + z$, obviously satisfies $\varepsilon^2 \partial_x^2 \sigma + \partial_z^2 \sigma = 0$ in $\Omega(u(t))$ as well as

$$\sigma(\pm 1, z) = 1 + z = \psi(t, \pm 1, z), \quad z \in (-1, 0),$$

$$\sigma(x, -1) = 0 = \psi(t, x, -1), \quad x \in (-1, 1).$$

Owing to the non-positivity (7) of $u(t)$, it also satisfies

$$\sigma(x, u(t, x)) = 1 + u(t, x) \leq 1 = \psi(t, x, u(t, x)), \quad x \in (-1, 1),$$

and we infer from the comparison principle that $\psi(t, x, z) \geq \sigma(x, z)$ for $(x, z) \in \Omega(u(t))$. It then follows from (10) that $\psi(t, x, z) \geq 1 + z = \psi(t, \pm 1, z)$ for $(x, z) \in \Omega(u(t))$ which readily implies (11). $\square$

To simplify notations, we set

$$\gamma_m(t, x) := \partial_z \psi(t, x, u(t, x)), \quad \gamma_g(t, x) := \partial_z \psi(t, x, -1), \quad (t, x) \in (0, T_m^\varepsilon) \times (-1, 1), \quad (12)$$

and first derive an upper bound of the L_1 -norm of the right-hand side of (1), observing that, due to (8), it also reads

$$-\lambda \varepsilon^2 |\partial_x \psi(t, x, u(t, x))|^2 + |\partial_z \psi(t, x, u(t, x))|^2 = -\lambda (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 .$$

Lemma 5. *For $t \in (0, T_m^\varepsilon)$,*

$$\int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 dx \geq 2 \int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x) dx - 2 . \quad (13)$$

Proof. Fix $t \in (0, T_m^\varepsilon)$. We multiply (4) by $\partial_z \psi(t) - 1$ and integrate over $\Omega(u(t))$. Using (8), (9), and Green's formula we obtain

$$\begin{aligned} 0 &= -\varepsilon^2 \int_{\Omega(u)} \partial_x \partial_z \psi \partial_x \psi d(x, z) + \varepsilon^2 \int_{-1}^1 (\partial_x u)^2 \gamma_m (\gamma_m - 1) dx \\ &\quad - \frac{1}{2} \int_{-1}^1 (\gamma_g^2 - 2\gamma_g) dx + \frac{1}{2} \int_{-1}^1 (\gamma_m^2 - 2\gamma_m) dx . \end{aligned}$$

Since

$$\int_{\Omega(u)} \partial_x \partial_z \psi \partial_x \psi d(x, z) = \frac{1}{2} \int_{-1}^1 (\partial_x u)^2 \gamma_m^2 dx$$

by (8) and since $\gamma_g^2 - 2\gamma_g \geq -1$, we end up with (13). $\square$

We again use (4) to obtain a lower bound for the boundary integral of the right-hand side of (13) which depends on the Dirichlet energy of ψ .

Lemma 6. *For $t \in (0, T_m^\varepsilon)$,*

$$\int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x) dx \geq \int_{\Omega(u(t))} (\varepsilon^2 |\partial_x \psi(t, x, z)|^2 + |\partial_z \psi(t, x, z)|^2) d(x, z) . \quad (14)$$

Proof. Fix $t \in (0, T_m^\varepsilon)$. We multiply (4) by $\psi(t)$ and integrate over $\Omega(u(t))$. Using (5), (8), and Green's formula we obtain

$$\begin{aligned} 0 &= - \int_{\Omega(u(t))} (\varepsilon^2 |\partial_x \psi(t, x, z)|^2 + |\partial_z \psi(t, x, z)|^2) d(x, z) + \varepsilon^2 \int_{-1}^0 (1+z) \partial_x \psi(t, 1, z) dz \\ &\quad - \varepsilon^2 \int_{-1}^0 (1+z) \partial_x \psi(t, -1, z) dz + \varepsilon^2 \int_{-1}^1 (\partial_x u(t, x))^2 \gamma_m(t, x) dx + \int_{-1}^1 \gamma_m(t, x) dx . \end{aligned}$$

Owing to (11), the second and third terms of the right-hand side of the above equality are non-positive, whence (14). $\square$

We finally argue as in [2, Lemma 9] to establish a connection between the Dirichlet energy of ψ and u .

Lemma 7. For $t \in (0, T_m^\varepsilon)$,

$$\int_{\Omega(u(t))} (\varepsilon^2 |\partial_x \psi(t, x, z)|^2 + |\partial_z \psi(t, x, z)|^2) \, d(x, z) \geq \int_{-1}^1 \frac{dx}{1 + u(t, x)} . \quad (15)$$

Proof. Let $t \in (0, T_m^\varepsilon)$ and $x \in (-1, 1)$. We deduce from (5) and the Cauchy-Schwarz inequality that

$$\begin{aligned} \frac{1}{1 + u(t, x)} &= \frac{(\psi(t, x, u(t, x)) - \psi(t, x, -1))^2}{1 + u(t, x)} = \frac{1}{1 + u(t, x)} \left(\int_{-1}^{u(t, x)} \partial_z \psi(t, x, z) \, dz \right)^2 \\ &\leq \int_{-1}^{u(t, x)} (\partial_z \psi(t, x, z))^2 \, dz . \end{aligned} \quad (16)$$

Integrating the above inequality with respect to $x \in (-1, 1)$ readily gives (15). $\square$

Remark 8. Observe that (16) provides a quantitative estimate on the singularity of $\partial_z \psi$ generated by u when touchdown occurs.

Combining the three lemmas above with Jensen's inequality give the following estimate.

Proposition 9. For $t \in (0, T_m^\varepsilon)$,

$$\int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 \, dx \geq 4\varphi \left(\frac{1}{2} \int_{-1}^1 u(t, x) \, dx \right) - 2 , \quad (17)$$

where $\varphi(r) := 1/(1 + r)$, $r \in (-1, \infty)$.

Proof. Fix $t \in (0, T_m^\varepsilon)$. We infer from Lemma 5, Lemma 6, and Lemma 7 that

$$\int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 \, dx \geq 2 \int_{-1}^1 \varphi(u(t, x)) \, dx - 2 .$$

To complete the proof, we argue as in [5, 6] and use the convexity of φ and Jensen's inequality to obtain (17). $\square$

Proof of Theorem 3. Introducing

$$E(t) := -\frac{1}{2} \int_{-1}^1 u(t, x) \, dx , \quad t \in [0, T_m^\varepsilon] ,$$

the bounds (7) ensure that

$$0 \leq E(t) < 1 , \quad t \in [0, T_m^\varepsilon] . \quad (18)$$

It follows from (1), (8), and Proposition 9 that

$$\begin{aligned} \frac{dE}{dt}(t) &= -\frac{1}{2} \left[\frac{\partial_x u(t, x)}{\sqrt{1 + \varepsilon^2 (\partial_x u(t, x))^2}} \right]_{x=-1}^{x=1} + \frac{\lambda}{2} \int_{-1}^1 (1 + \varepsilon^2 (\partial_x u(t, x))^2) \gamma_m(t, x)^2 \, dx \\ &\geq F_\lambda(E) := 2\lambda\varphi(-E) - \lambda - \frac{1}{\varepsilon} . \end{aligned} \quad (19)$$

If $\lambda > 1/\varepsilon$, we note that $F_\lambda(0) > 0$ and thus $F_\lambda(r) \geq F_\lambda(0) > 0$ for $r \in [0, 1)$ due to the monotonicity of F_λ . Since $E(0) \geq 0$ by (18), it follows from (19) and the properties of F_λ that $t \mapsto E(t)$ is increasing on $[0, T_m^\varepsilon)$. Consequently,

$$\frac{dE}{dt}(t) \geq F_\lambda(E(0)) \geq F_\lambda(0) , \quad t \in [0, T_m^\varepsilon) .$$

Integrating the previous inequality with respect to time and using (18), we end up with the inequality $1 \geq E(0) + F_\lambda(0)T_m^\varepsilon$ which provides the claimed finiteness of T_m^ε . $\square$

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