

On a conjecture about Dirac's delta representation using q -Gaussians

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A new representation of Dirac's delta-distribution, based on so-called q -exponentials, has been recently conjectured. We prove here this conjecture.

I. INTRODUCTION

Tsallis and Jauregui have recently conjectured a representation of the celebrated Dirac delta distribution, which they call $\delta_q(x)$, based on q -exponential functions. However, they could not prove their conjecture and used numerical experiments that suggest its validity. In this note, we provide a rigorous mathematical approach to this problem and prove their conjecture by recourse to the notion of superstatistics.

II. q -EXPONENTIALS AND SUPERSTATISTICS

Statistical Mechanics' most notorious and renowned probability distribution is that deduced by Gibbs for the canonical ensemble [1, 2], usually referred to as the Boltzmann-Gibbs equilibrium distribution

$$p_G(i) = \frac{\exp(-\beta E_i)}{Z_{BG}}, \quad (1)$$

with E_i the energy of the microstate labeled by i , $\beta = 1/k_B T$ the inverse temperature, k_B Boltzmann's constant, and Z_{BG} the partition function. The exponential term $F_{BG} = \exp(-\beta E)$ is, of course, called the Boltzmann-Gibbs factor. Recently Beck and Cohen [3] have advanced a generalization (called superstatistics) of this BG factor, assuming that the inverse temperature β is a stochastic variable. The generalized statistical factor F_{GS} is thus obtained as the multiplicative convolution

$$F_{GS} = \int_0^\infty \frac{d\beta}{\beta} f(\beta) \exp(-\beta E), \quad (2)$$

where $f(\beta)$ is the density probability of the inverse temperature.

As stated above, β is the inverse temperature, but the integration variable may also be any convenient intensive parameter. Superstatistics, meaning "superposition of statistics", takes into account fluctuations of such intensive parameters.

Beck and Cohen also show that, if $f(\beta)$ is a Gamma distribution, nonextensive thermostatistics is obtained, which is of interest because this thermostatistics is today a very active field, with applications to several scientific disciplines [4–6]. In working in a nonextensive framework one has to deal with power-law distributions, which are certainly ubiquitous in physics (critical phenomena are just a conspicuous example [7]). Indeed, it is well known that power-law distributions arise quite naturally in maximizing Tsallis' information measure (q is a real positive parameter called the "nonextensivity index")

$$H_q(f) = \frac{1}{1-q} \left(1 - \int_{-\infty}^{+\infty} f(x)^q dx \right), \quad (3)$$

subject to appropriate constraints. More precisely, in the case of the canonical distribution, there is only one constraint, the energy E , i.e., $\langle E \rangle = K$ (K a positive constant) and the equilibrium canonical distribution writes

$$f_q(x) = \frac{1}{Z_q} (1 - (1-q)\beta_q E)_+^{\frac{1}{1-q}},$$

with $(x)_+ = \max(0, x)$ and β_q and Z_q stand for the nonextensive counterparts of β and Z_{BG} above. Defining the q -exponential function as

$$e_q(x) = (1 + (1 - q)x)_+^{\frac{1}{1-q}} \quad (4)$$

allows to rewrite the equilibrium distribution in the more natural way

$$f_q(x) = \frac{1}{Z_q} e_q(-\beta E)$$

It is a classical result that as $q \rightarrow 1$, Tsallis entropy reduces to Shannon entropy

$$H_1(f) = - \int_{-\infty}^{+\infty} f(x) \log f(x). \quad (5)$$

Accordingly, the q -exponential function converges to the usual exponential function.

III. PROOF OF JAUREGUI-TSALLIS' CONJECTURE

A. Definitions and Notations

We remind the celebrated formula

$$\delta(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iut} du. \quad (6)$$

We intend to provide a generalization of this formula; namely, we prove the following representation conjectured by Tsallis *et al.* assuming $1 < q < 2$:

$$\delta(t) = \frac{1}{c_q} \int_{\mathbb{R}} e_q(-iut) du. \quad (7)$$

for some constant c_q .

We begin by recalling the mathematical meaning of (6).

Definition 1. A function φ is called rapidly decreasing if φ is C^∞ and if for all integers k, ℓ

$$\lim_{x \rightarrow \pm\infty} x^k \varphi^{(\ell)}(x) = 0.$$

We denote by \mathcal{S} be the set of the rapidly decreasing functions on \mathbb{R} and by \mathcal{S}' the set of the continuous linear functionals over \mathcal{S} . For $\varphi \in \mathcal{S}$, its Fourier transform $\mathcal{F}(\varphi)$ is denoted by $\hat{\varphi}$.

We know from Rudin [8, p. 184 theorem 7.4] the following

Proposition 1. The Fourier transform \mathcal{F} is a continuous linear mapping of \mathcal{S} into \mathcal{S} .

Definition 2. Let f be a bounded measurable function[9]. We let T_f be the linear continuous mapping:

$$\forall \varphi \in \mathcal{S} \quad \langle T_f, \varphi \rangle = \int f(t) \varphi(t) dt$$

B. Proofs

In order to prove the usual representation (6), we simply have to show that for all $\varphi \in \mathcal{S}$:

$$\int du \langle T_{e^{-iut}}, \varphi \rangle = 2\pi \varphi(0).$$

Of course $\langle T_{e^{-iut}}, \varphi \rangle = \hat{\varphi}(u)$. Hence the result.

We can turn to the proof of (7) In this respect, let us pick a $\varphi \in \mathcal{S}$. We have

$$\begin{aligned} \langle T_{e_q(-iut)}, \varphi \rangle &= \int e_q(-iut) \varphi(t) dt \\ &= \int E_W e^{-iut(q-1)W} \varphi(t) dt \end{aligned}$$

where

$$E_W g(W) \triangleq \frac{1}{\Gamma(\frac{1}{q-1})} \int_0^{+\infty} g(w) e^{-w} w^{\frac{1}{q-1}-1} dw$$

is the expectation of $g(W)$, where W is a Gamma distributed random variable with shape parameter $\frac{1}{q-1}$ and g some function such that the above definition makes sense. We remark that the equality

$$e_q(-iut) = E_W e^{-iut(q-1)W}$$

is the expression of the superstatistical theory.

On the other hand, we have

$$\frac{1}{\Gamma(\frac{1}{q-1})} \int \int e^{-w} w^{\frac{1}{q-1}-1} |\varphi(t)| dt dw \leq \int |\varphi(t)| dt.$$

As obviously φ is summable, we can apply the Fubini-Lebesgue theorem and we obtain

$$\begin{aligned} \langle T_{e_q(-iut)}, \varphi \rangle &= E_W \int e^{-iut(q-1)W} \varphi(t) dt \\ &= E_W \hat{\varphi}(u(q-1)W) \\ &= \frac{1}{\Gamma(\frac{1}{q-1})} \int e^{-w} w^{\frac{1}{q-1}-1} \hat{\varphi}(u(q-1)w) dw \end{aligned}$$

Now, consider

$$\int_{\mathbb{R}} \langle T_{e_q(-iut)}, \varphi \rangle du.$$

Since $q < 2$, we have, by the change of variable $u \mapsto v = u(q-1)w$,

$$\int dw \int |e^{-w} w^{\frac{1}{q-1}-1} \hat{\varphi}(u(q-1)w)| du = \frac{\int |\hat{\varphi}(v)| dv}{q-1} \int e^{-w} w^{\frac{1}{q-1}-2} dw < \infty$$

since, by Proposition 1, $\hat{\varphi} \in \mathcal{S}$. Thanks to the Fubini-Lebesgue theorem, we deduce that

$$\begin{aligned} \int_{\mathbb{R}} \langle T_{e_q(-iut)}, \varphi \rangle du &= \frac{1}{\Gamma(\frac{1}{q-1})} \frac{\int \hat{\varphi}(v) dv}{q-1} \int e^{-w} w^{\frac{1}{q-1}-2} dw \\ &= \frac{\Gamma(\frac{1}{q-1} - 1)}{(q-1) \Gamma(\frac{1}{q-1})} 2\pi \varphi(0) \end{aligned}$$

We have proved the result with

$$c_q = \frac{2\pi}{2-q}.$$

IV. CONCLUSION

We have proved the representation of the Dirac delta distribution (7) using q -exponential functions, as conjectured by Tsallis *et al.*

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