HAL
open science

Robust Analysis of Uncertain Dynamic Systems: Combination of the Centre Manifold and Polynomial Chaos theories

Lyes Nechak, Sébastien Berger, Evelyne Aubry

To cite this version:

Lyes Nechak, Sébastien Berger, Evelyne Aubry. Robust Analysis of Uncertain Dynamic Systems: Combination of the Centre Manifold and Polynomial Chaos theories. WSEAS Transactions on Systems, 2010, 9 (4), pp.386-395. hal-00827955

HAL Id: hal-00827955

https://hal.science/hal-00827955v1

Submitted on 30 May 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Robust Analysis of Uncertain Dynamic Systems: Combination of the Centre Manifold and Polynomial Chaos theories

L. NECHAK, S. BERGER, E. AUBRY
MIPS Laboratory
University of Haute Alsace
12 Rue des Frères Lumière, Mulhouse 68093
France
lyes.nechak@uha.fr sebastien.berger@uha.fr evelyne.aubry@uha.fr

Abstract: - A new methodology for a robust analysis of uncertain nonlinear dynamic systems is presented in this paper. The originality of the method proposed lies in the combination of the centre manifold theory with the polynomial chaos approach. The first one is known to be a powerful tool for model reduction of nonlinear dynamic systems in the neighbourhood of the Hopf bifurcation point while the polynomial chaos approach is an efficient tool for uncertainty propagation. Therefore, to couple the two methods can help to overcome computational difficulties due to both the complexity of nonlinear dynamic systems and the cost of the uncertainty propagation with the prohibitive Monte Carlo method. The feasibility and efficiency of the proposed methodology is investigated. So, a two degree of freedom model describing a drum brake system subject to uncertain initial conditions is considered.

Key-Words: - Nonlinear dynamic System, Model Reduction, Centre Manifold, Robust Analysis, Uncertainty Propagation, Polynomial chaos, Non-intrusive methods.

1 Introduction

Uncertainty propagation plays a major role in the robust analysis and design of dynamic systems. It consists of several techniques designed meant to quantify the influence of different kinds of uncertainties (design parameters, initial conditions, inputs) on the process states and outputs or to take uncertainties into account in the design of processes or control systems. There are several theories about this topic in the literature, such as probabilistic approaches [22-24], [4], fuzzy logic [25], neural networks [26] and the interval theory [27-28].

Among the probabilistic approaches, the Monte Carlo method is the most useful. The latter can give the entire probability density function of any system variable, but it is often too expensive since a large number of samples are required for reasonable accuracy. A more efficient probabilistic tool has been presented in the literature. It consists of the polynomial chaos [9]. This theory was pioneered by Ghanem and Spanos who used expansion in Wiener-Hermite polynomials to model stochastic processes with Gaussian random variables [4]. The convergence of such expansion in a mean square sense has been shown [2] and generalized to various continuous and discrete distributions using orthogonal polynomials following the so called Askey-scheme [7]. The capability of polynomial chaos expansions has been tested on numerous applications, such as solving ordinary and partial differential equations [6-7] and [12], observer and controller design problems [10], [15] and [3]. Polynomial chaos gives a mathematical framework to separate the stochastic components of a system response from deterministic ones. The stochastic Galerkin method [4], [1], collocation and regression methods [14] are used to compute the deterministic components called stochastic modes in an intrusive and a non intrusive manner while random components are concentrated in the polynomial basis used. Non intrusive procedures are shown to be more efficient for stochastic dynamic systems since they need only simulations corresponding to particular samples of the random parameters and they need no modifications on the system model contrary to intrusive methods in which Galerkin's techniques are used to generate, from the stochastic system model, a set of deterministic coupled equations which are difficult to implement especially for non-linear systems.

Generally speaking, the analysis of uncertain dynamic systems using both the Monte Carlo and non-intrusive polynomial chaos methods requires deterministic simulations of the system studied. These simulations are based on numerical integration procedures which are too expensive and

difficult tasks with non-linear systems as these have hard non-linearities and numerous degrees of freedom. The model reduction theory proposes a number of solutions consisting of methodologies which approximate complex models with simpler ones while keeping the same dynamic behaviours and the most important properties of the original models. Balanced truncation [16-17], proper orthogonal decomposition (POD) [18] and singular perturbation [19] based methods are well known examples. The centre manifold is another method presented in literature as an efficient tool which helps to simplify a complex dynamic system in a Hopf bifurcation point neighbourhood [20], [5], [11]. This approach is based on the idea that all dynamic characteristics near the equilibrium point are governed by the dynamics on the centre manifold when some eigenvalues have zero real parts and all the other eigenvalues have negative real parts.

The originality of this paper lies in the new methodology proposed to perform a simpler robust analysis of uncertain dynamic systems. The main principle of this methodology is to combine centre manifold theory with the polynomial chaos approach. The first one provides a powerful tool to obtain a reduced model in the Hopf bifurcation neighbourhood, then the main idea is to propagate uncertainty on this reduced model instead of the original one, using a polynomial chaos based approach which is less expensive than the prohibitive Monte Carlo procedure. The objective of this paper is to illustrate the feasibility of the proposed method. So, a two degree of freedom model describing a drum brake system is considered, with uncertainty in the initial conditions. First, the essential principles of both the centre manifold method and the polynomial chaos are summed up in Section 2. The combination of the two methods is tested on a simple self-excited mechanism. All the results are presented in Section 3. Finally, there are some observations and conclusions about perspectives and further research work.

2 Theoretical Methods

Dynamic systems described by the following n-dimensional differential equations are considered in this paper.

$$\dot{x}(t) = f\left(x(t), \mu\right) \tag{1}$$

where $x(t) \in \Re^n$ is defined as the state vector and $\mu \in \Re$ a control parameter.

The vector field f is assumed to be smooth and the origin is the equilibrium point of (1). Additionally, here, only polynomial nonlinearities are considered. This is not restrictive since any smooth nonlinear function can be approximated by a polynomial function using the multi-dimensional Taylor series expansion.

2.1 Centre Manifold

The centre manifold method uses the basic idea that the essential non-linear dynamic system characteristic in the neighbourhood of an equilibrium point is governed by the centre manifold associated with the part of the original system characterized by the eigenvalues with zero real-parts at the Hopf bifurcation point [11].

As the system (1) is polynomial, it can be expressed under the following form:

$$\dot{x} = A(\mu)x + F_1(x,\mu) + F_2(x,\mu) + \cdots + F_k(x,\mu) \tag{2}$$

where $A(\mu)$ is a $n \times n$ matrix and F_k is a vector of degree k polynomial functions in x and μ.

By means of a linear basis transformation $x = Ty$, the system (2) can be put in a canonical form (3) at the Hopf bifurcation point μ_0. The linear basis transformation is given by the $n \times n$ matrix $T = \left[T_1, ..., T_{n_c}, T_{n_c+1}, ..., T_n\right]$ where $T_1, ..., T_{n_c}$ and $T_{n_c+1}, ..., T_n$ are the generalized eigenvectors corresponding respectively to the n_c eigenvalues $\lambda_{i(i=1,...,n_c)}$ of $A(\mu_0)$ with zero real parts and the $(n - n_c)$ eigenvalues $\lambda_{i(i=n_c+1,...,n)}$ of $A(\mu_0)$ with non-zero real parts.

$$\begin{cases} \dot{y}_c = A_c(\mu_0)y_c + F_c(y_c, y_s, \mu_0) \\ \dot{y}_s = A_s(\mu_0)y_s + F_s(y_c, y_s, \mu_0) \end{cases} \tag{3}$$

where $y_c \in \Re^{n_c}$, $y_s \in \Re^{n_s}$ such that $n_c + n_s = n$ and $y = \begin{bmatrix} y_c & y_s \end{bmatrix}^T$, $A_c = \begin{bmatrix} \lambda_1 & 0\cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n_c} \end{bmatrix}$,

$A_c = \begin{bmatrix} \lambda_{n_c+1} & 0\cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$ and

$$\begin{bmatrix} F_c(y_c, y_s, \mu_0) \\ F_s(y_c, y_s, \mu_0) \end{bmatrix} = T^{-1}\left(F_1(Ty, \mu_0) + \cdots F_k(Ty, \mu_0)\right)$$

where, $F_c(0,0,\mu_0)=0$, $F_s(0,0,\mu_0)=0$ and the jacobian matrice, $DF_c(0,0,\mu_0)$ and $DF_s(0,0,\mu_0)$ are matrices with zero entries.

In the neighbourhood of the Hopf bifurcation, the system (3) may be defined by the following augmented dynamics:

$$\begin{cases} \dot{y}_c = A_c(\tilde{\mu})y_c + F_c(y_c, y_s, \tilde{\mu}) \\ \dot{y}_s = A_s(\tilde{\mu})y_s + F_s(y_c, y_s, \tilde{\mu}) \\ \dot{\tilde{\mu}} = 0 \end{cases} \tag{4}$$

where $\tilde{\mu} = (1+\varepsilon)\mu_0$,$\varepsilon << 1$

With the centre manifold theorem [10], [5] and [11], it is demonstrated that for small $\|y_c\|$ and $\|\tilde{\mu}\|$ there is a local centre manifold which helps to express the stable variables y_s as a function of the centre variables $(y_c, \tilde{\mu})$ such that: $y_s = h(y_c, \tilde{\mu})$ where h is a function verifying $h(0,0)=0$ and $Dh(0,0)$ is a matrice with zero entries. Consequently, a reduced order system can be obtained from the system (4) as follows:

$$\begin{cases} \dot{y}_c = A_c(\tilde{\mu})y_c + F_c(y_c, h(y_c, \tilde{\mu}), \tilde{\mu}) \\ \dot{\tilde{\mu}} = 0 \end{cases} \tag{5}$$

The main problem is then to compute the centre manifold h. One simpler manner to perform this computing is to consider h as a polynomial function in the centre variables $(y_c, \tilde{\mu})$ with a fixed order without constant and linear terms (to satisfy the conditions on h given previously) and then to identify its coefficients by resolving the equation (6) equivalent to the equation (7).

$$\dot{y}_s = Dh(y_c, \tilde{\mu}).\begin{bmatrix} y_c \\ \tilde{\mu} \end{bmatrix} \tag{6}$$

$$\begin{aligned} \left(D_{(y_c,\tilde{\mu})}h\right).\left(A_c(\tilde{\mu}).y_c + F_c(y_c, h(y_c,\tilde{\mu}), \tilde{\mu})\right) = \\ A_s(\tilde{\mu})h(y_c,\tilde{\mu}) + F_s(y_c, h(y_c,\tilde{\mu}), \tilde{\mu}) \end{aligned} \tag{7}$$

2.1 polynomial chaos

Polynomial chaos establishes a separation between the stochastic components of a random function and its deterministic components. Here is a brief mathematical framework of this approach. So, from Wiener theory and the Cameron-Martin theorem, any second order random process x can be expanded in a convergent (in the mean square sense) polynomial functions series as:

$$x = \sum_{i=0}^{\infty} \bar{x}_i \phi_i(\xi) \tag{8}$$

ξ is a vector of d independent random variables with known joint density function $W(\xi)$, $\bar{x}_i$ are the stochastic modes of the random process x and ϕ_i are the so-called polynomial chaos which are orthogonal with respect to the joint density function. Thus:

$$\langle \phi_i, \phi_j \rangle = \int \phi_i(\xi)\phi_j(\xi)W(\xi)d\xi = \begin{cases} 0 & i \neq j \\ \langle \phi_i^2 \rangle & i = j \end{cases} \tag{9}$$

$\langle \cdot \rangle$ being the internal product operator.

In practice, polynomial chaos expansion is truncated to a finite number of terms P which is demonstrated to be dependent on the polynomial chaos order r and the stochastic dimension d denoting the number of the uncertain parameters.

$$x \approx \sum_{i=0}^{P} \bar{x}_i \phi_i(\xi) \tag{10}$$

with $$P = \left((d+r)!/d!r!\right) - 1 \tag{11}$$

For random variables ξ with certain distributions, the orthogonal functions ϕ_i can be chosen in such a way that its weight functions has the same form as the probability function $W(\xi)$. In this case, the convergence of the expansion has been shown to be exponential [7].

Random variable ξ	The polynomial family $\phi_i(\xi)$
Gaussian	Hermite
Uniform	Legendre
Gamma	Laguerre
Beta	Jacobi

Table 1. Correspondence between choice of polynomial and given distribution of ξ.

Computing x is then turned into the problem of finding the $\bar{x}_i$ of its truncated expansion (10). Two main approaches are defined to calculate these coefficients called stochastic modes. The non-intrusive approach includes some techniques (non-intrusive spectral projection (NISP), regression) which use simulations corresponding to particular samples of the random parameters. The principal advantage of these techniques is related to the fact that no modification is performed on the system model contrary to the intrusive approach which needs the Galerkin techniques to generate a set of coupled deterministic equations from the uncertain process. Only non-intrusive methods will be considered in this paper and be presented in the next sub-section. So, consider a uniform distribution for the initial condition $x_1(0)$ of the system (1). Consequently, all the state variables also become uncertain. According to the Askey scheme [7], Legendre polynomials are more suitable to deal with uniform uncertainties. So, all the state variables can be expanded in series of Legendre polynomials in the standard stochastic variable ξ uniformly distributed in the orthogonality interval $[-1, 1]$ of the Legendre polynomials.

$$x_i(t,\xi) \approx \sum_{j=0}^{P} \bar{x}_{i,j}(t) L_j(\xi) \quad ,i=1,...,n \tag{12}$$

The main problem now is to compute the stochastic modes $\bar{x}_{i,j}(t)$. Two methods are presented below.

2.1.1 Non-intrusive spectral projection (NISP)

The NISP method uses the inner product of $\bar{x}_i(t,\xi)$ with the polynomial $L_j(\xi)$ and the orthogonality property of the same polynomials to compute the stochastic modes $\bar{x}_{i,j}(t)$ from (12) as follows:

$$\bar{x}_{i,j}(t) = \frac{1}{\langle L_j^2(\xi)\rangle} \int_{-1}^{1} x_i(t,\xi) L_j(\xi) W(\xi) d\xi \tag{13}$$

The denominator of expression (13) is an internal product between the same orthogonal polynomial. It can be calculated analytically and may be tabulated. The problem, however, is related to the calculation of the numerator. Numerical techniques are used for integral computing such as the Monte Carlo or collocation methods [30]. Note that there is sparse grid collocation technique used for multidimensional integral computation in cases of multiple uncertain parameters [31]. In the present paper, the studied system possesses one uncertain initial condition, so the stochastic modes $\bar{x}_{i,j}(t)$ are computed from mono-dimensional integrals. Using the Gauss collocation method, the expression (13) can be approximated by (14).

$$\bar{x}_{i,j}(t) \approx \frac{1}{\langle L_j^2(\xi)\rangle} \sum_{k=1}^{q=P+1} x_i\left(t,\xi^{(k)}\right) L_j\left(\xi^{(k)}\right) W^{(k)} \tag{14}$$

with $i=1,...,n$ and $j=0,...,P$

$\xi^{(k)}$ are the so-called Gauss collocation points. They are given by the roots of the $(P+1)^{th}$ order polynomial while $W^{(k)}$ are the Gausse weights given by:

$$W^{(k)} = \int_{-1}^{1} \prod_{\substack{j=0 \\ k\neq j}}^{P} \frac{\left(\xi - \xi^{(j)}\right)}{\left(\xi^{(k)} - \xi^{(j)}\right)} \tag{15}$$

2.1.2 Regression

The regression method consists in calculating the stochastic modes $\bar{x}_{i,j}(t)$ by minimizing the following least square criterion:

$$\varepsilon = \sum_{k=1}^{\bar{q}} \left[x_i\left(t,\xi^{(k)}\right) - \sum_{j=0}^{P} \bar{x}_{i,j}(t) \phi_j\left(\xi^{(k)}\right) \right]^2 \tag{16}$$

with $\bar{q}$ denoting the number of the deterministic simulations of the system such that $P+1<\bar{q}$.

The regression method doesn't use orthogonality property of the polynomial chaos as the NISP method. It just needs for a sample set of the random parameters $\left\{\xi^{(k)}\right\}$ and the corresponding set of simulation output $\left\{x_i\left(t,\xi^{(k)}\right)\right\}$. The first one can be generated by using Monte Carlo techniques. In this paper, $\xi^{(k)}, k=1,...,\bar{q}$ are chosen as the zeros of a Legendre polynomial with a sufficient order such that $P+1<\bar{q}$. Then the classical solution of the least square problem is given by:

$$\bar{X}_i = \left(Z^T Z\right)^{-1} Z^T X_m^i \tag{17}$$

with $\bar{X}_i = \left[\bar{x}_{i,0}(t) \quad \cdots \quad \bar{x}_{i,P}(t)\right]$ denoting the vector of stochastic modes of the ith state variable,

Z being the matrix of entries defined by: $z_{j,k} = L_j\left(\xi^{(k)}\right)$ $\left(j = 0,...,P\right),\left(k = 1,...,\bar{q}\right)$ and X_m^i the vector the simulation output.

2.1.3 Statistics estimation

Once the stochastic modes are obtained, they are used to determine the statistics of the system variables. The first and second order moments are given respectively by (18).

$$\begin{cases} x_i^{mean}\left(t\right) = \bar{x}_{i,0}\left(t\right) \\ \sigma_i^2\left(t\right) = \sum_{j=0}^{P}\left(\bar{x}_{i,j}\left(t\right)\right)^2\left\langle L_j^2\left(\xi\right)\right\rangle - \left(\bar{x}_{i,0}\left(t\right)\right)^2 \end{cases} \tag{18}$$

3 Application and results

The objective of this application is to show that the combination of the centre manifold and polynomial chaos methods can be an efficient tool to simplify uncertainty propagation problems [29]. Therefore, the estimation of the short term statistics (mean value and standard deviation) of a nonlinear dynamic system subject to uncertain initial conditions has been considered. The estimation of the long term statistics is a more complicated problem which is not dealt with in this paper. A two degree of freedom model describing a drum brake system developing flutter instability (Fig.1) is used.

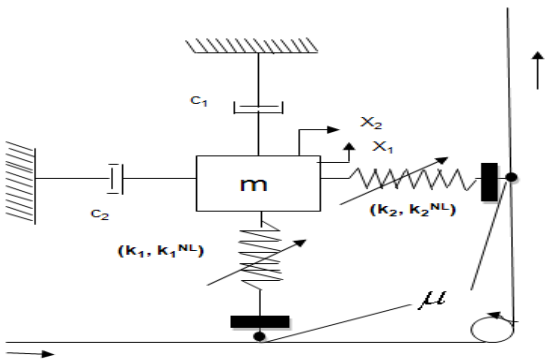

Fig.1. Mechanical system

3.1 System description

Hulten's model is composed of a mass held against a moving band; the contact between the mass and the band is modelled by two plates supported by two different springs. For simplicity's sake, it is assumed that the mass and band surfaces always keep in contact. This assumption is justified by a preload applied to the system. The contact can be expressed by two cubic stiffnesses. Damping is integrated as shown in Fig.1. The friction coefficient at contact is assumed to be constant and the band moves at a constant velocity. Then it is assumed that the direction of the friction force does not change because the relative velocity between the band speed and $\dot{X}_1$ or $\dot{X}_2$ is assumed to be positive.

The tangential force F_T due to friction contact is assumed to be proportional to the normal force F_N as given by Coulomb's law: $F_T = \mu F_N$.

Assuming that the normal force F_N is linearly related to the displacement of the mass normal to the contact surface, the resulting equations of motion can be expressed as: [3], [21].

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{pmatrix} + \begin{bmatrix} \eta_1\omega_1 & 0 \\ 0 & \eta_2\omega_2 \end{bmatrix}\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} + \begin{bmatrix} \omega_1^2 & -\mu\omega_2^2 \\ \mu\omega_1^2 & \omega_2^2 \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -\varphi_1^{NL}X_1^3 + \mu\varphi_2^{NL}X_2^3 \\ -\mu\varphi_1^{NL}X_1^3 - \varphi_2^{NL}X_2^3 \end{pmatrix} \tag{19}$$

where $\eta_i = c_i / \sqrt{mk_i}$ are the relative damping coefficients, $\omega_i = \sqrt{k_i / m}$ are the natural pulsations and $\varphi_i^{NL} = k_i^{NL} / m$ for $i = 1,2$.

For numerical application: $\omega_1 = 2\pi \times 100\, rad/s$, $\omega_2 = 2\pi \times 75\, rad/s$, $\eta_1 = \eta_2 = 0.02$, $\varphi_1^{NL} = \omega_1^2$ and $\varphi_2^{NL} = 0$.

Considering $X_1 = x_1$, $\dot{X}_1 = x_2$, $X_2 = x_3$ and $\dot{X}_2 = x_4$, the system model (19) is expressed with a state space representation as:

$$\dot{x}\left(t\right) = A\left(\mu\right)x\left(t\right) + f_{NL}\left(x\left(t\right),\mu\right) \tag{20}$$

where $x\left(t\right) = \left[x_1\left(t\right) \quad x_2\left(t\right) \quad x_3\left(t\right) \quad x_4\left(t\right)\right]^T$,

$$A\left(\mu\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -\eta_1\omega_1 & \mu\omega_2^2 & 0 \\ 0 & 0 & 0 & 1 \\ -\mu\omega_1^2 & 0 & -\omega_2^2 & -\eta_2\omega_2 \end{bmatrix}$$ and

$$f_{NL}\left(x\left(t\right),\mu\right) = \begin{bmatrix} 0 \\ -\varphi_1^{NL}x_1^3\left(t\right) + \mu\varphi_2^{NL}x_3^3\left(t\right) \\ 0 \\ -\mu\varphi_1^{NL}x_1^3\left(t\right) - \varphi_2^{NL}x_3^3\left(t\right) \end{bmatrix}$$

3.2 Application of the centre manifold

As mentioned in Section 2.1, the centre manifold method gives a powerful formalism which helps to reduce a nonlinear dynamic system near equilibrium in the neighbourhood of a Hopf bifurcation point.

First, it can be noted that the origin is an equilibrium point for the system (20).

The Hopf bifurcation point μ_0 can be obtained by using a classical parametric stability study which consists in the computing of the system eigenvalues near the equilibrium at each value of the control parameter which in our case the friction coefficient. So, the bifurcation point is defined by the following conditions:

$$\begin{cases} \mathrm{Re}al\left(\lambda_{centre}\left(A(\mu)\right)\right)\Big|_{x=0,\,\mu=\mu_0} = 0 \\ \mathrm{Re}al\left(\lambda_{non-centre}\left(A(\mu)\right)\right)\Big|_{x=0,\,\mu=\mu_0} \neq 0 \\ \dfrac{d}{d\mu}\left(\lambda\left(A(\mu)\right)\right)\Big|_{x=0,\,\mu=\mu_0} \neq 0 \end{cases} \tag{21}$$

At the Hopf bifurcation point, the system possesses a pair of purely imaginary eigenvalues λ_{centre}. All the other eigenvalues $\lambda_{non-centre}$ have non-zero real parts. A non-zero speed crossing of the imaginary axis is ensured by the last condition called the transversal condition.

Applying the previous definition, the Hopf bifurcation point is computed for the system (20). It is given by $\mu_0 = 0.289368$. At this point, the linear approximation of (19) possesses a pair of purely imaginary eigenvalues and another pair of stable eigenvalues.

Now, in order to obtain a reduced order model, the centre manifold order is fixed to 3. Equation (7) corresponding to the canonical form of the system (20) is solved. The stable manifold is then expressed as polynomial function h in the centre variables.

After substituting the stables manifolds by the computed centre manifold, a second order model is generated for the neighbourhood defined by $\tilde{\mu} = (1+\varepsilon)\mu_0$ with $\varepsilon = 10^{-4}$.

Matlab's ODE45 solver is used to compute the solutions of the original and reduced models corresponding to the initial condition $x(0) = \begin{bmatrix} 10^{-4} & 0 & 0 & 0 \end{bmatrix}^T$. The speed $\dot{X}_1$ is plotted with respect to the displacement X_1 in Fig.2 while $\dot{X}_2$ is plotted with respect to the displacement X_2 in Fig.3.

As shown in Fig.2 and Fig.3, the reduced model reproduces the same dynamic behaviour as the original model (20). Limit cycle oscillations are distinguished for both the reduced and original models. A small error is observed on the amplitudes of the limit cycles $(X_1, \dot{X}_1)$ and $(X_2, \dot{X}_2)$ of the reduced model comparing to the original ones. This error may be decreased by increasing the centre manifold order.

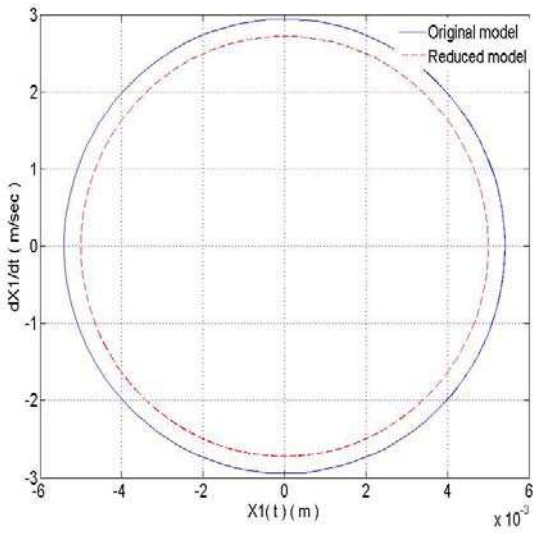

Fig.2. Dynamic behaviours of the original and the reduced models

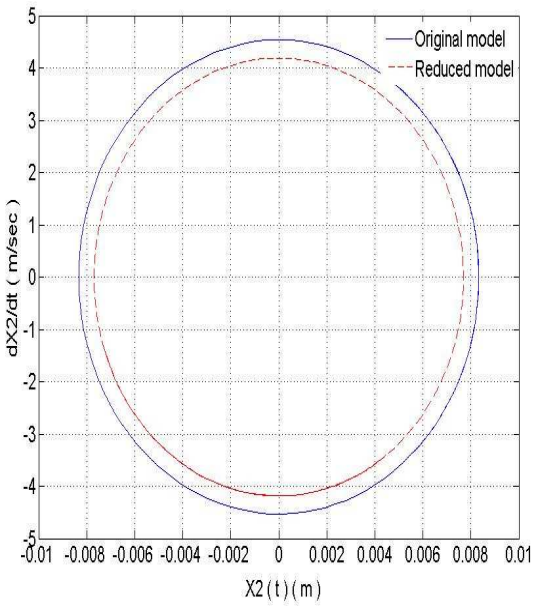

Fig.3. Dynamic behaviours of the original and the reduced models

3.3 Uncertainty propagation

Now, the problem is to analyze short term statistics of the system (20) subject to uncertain initial conditions.

Let consider that uncertainty is in the initial displacement $X_1(0)$ and let assume that this later is governed by a uniform probabilistic law within the interval $[a \;\; b]$. So, $X_1(0)$ can be expressed by a function in random variable ξ uniformly distributed within $[-1,\ 1]$ as:

$$X_1(t=0,\xi) = X_1^{moy}(0) + \alpha\,\xi \tag{22}$$

where $X_1^{moy}(0) = \dfrac{a+b}{2}$ and $\alpha = \dfrac{b-a}{2}$

The next objective is to estimate, for $a = 10^{-4}$ and $b = 3.10^{-4}$, the mean values and the standard deviations of the original system responses X_1 and $\dot{X}_1$ by using the reduced order model obtained in Subsection 3.2. The same task can be performed concerning X_2 and $\dot{X}_2$. Before, note the sensitivity of the systems responses to dispersions of the initial conditions. This is illustrated in Fig.4 where the phase trajectories of the system (20) are plotted for different values of initial conditions with the last 200 points among 50000 points.

We can observe that small variations in the initial condition implies high variations in the amplitude of the displacement X_1 and the corresponding speed $\dot{X}_1$. The same effect has been observed on X_2 and $\dot{X}_2$.

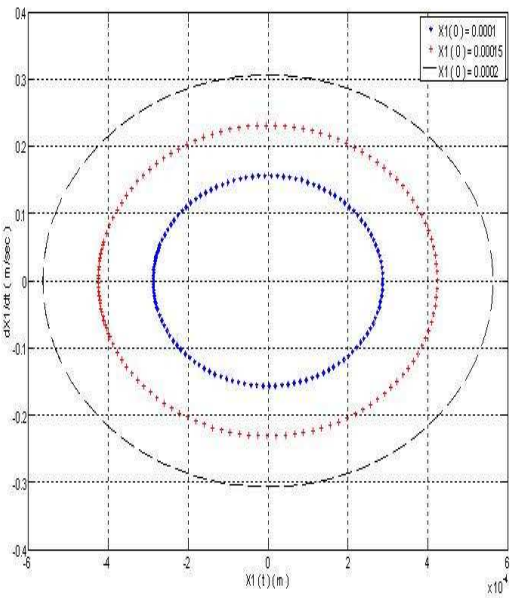

Fig.4. Phase trajectories of the system (20)

The non-intrusive polynomial chaos based methods (NISP and regression) and the Monte Carlo technique are applied to the reduced model.

As the uncertainty is with uniform probabilistic distribution, we must use the Legendre polynomials. So an order equal to 5 is fixed for the Wiener-Legendre expansion which is then used to estimate the mean values and the standard deviations of the original system responses in short term. For the NISP method, only 6 simulations of the reduced order model are needed to compute the stochastic modes. The six samples $\xi^{(k)}$ $(k = 1,...,6)$ correspond to the zeros of the sixth order Legendre polynomial. In the regression method, the number of simulations is fixed arbitrarily to 15 and corresponds to the zeros of the 15th order Legendre polynomial. Both numbers 6 and 15 are much smaller than the one necessary for a Monte Carlo method. The influence of the order of the Wiener-Legendre expansion and the number of the simulations in the regression method is not included here since it is not the aim of the intended application. Recall that the main objective is to show the feasibility of the combination of the centre manifold and the polynomial chaos theories.

The results obtained from applying polynomial chaos based methods are compared to Monte Carlo solutions with 1000 uniformly distributed samples within the fixed interval $\left[10^{-4} \;\; 3.10^{-4}\right]$. The phase trajectory, corresponding to the instantaneous mean values of X_1 and $\dot{X}_1$ is plotted in Fig.5 using the last 200 points while the standard deviations of the previous variables are plotted respectively in Fig.6 and Fig.7.

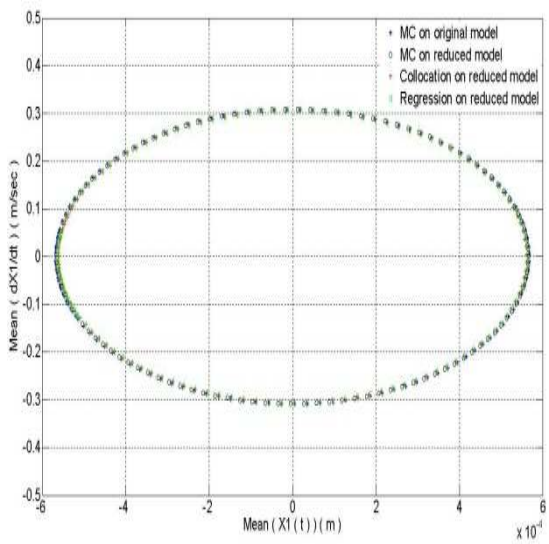

Fig.5. Average trajectory of phase $\left(X_1, \dot{X}_1\right)$

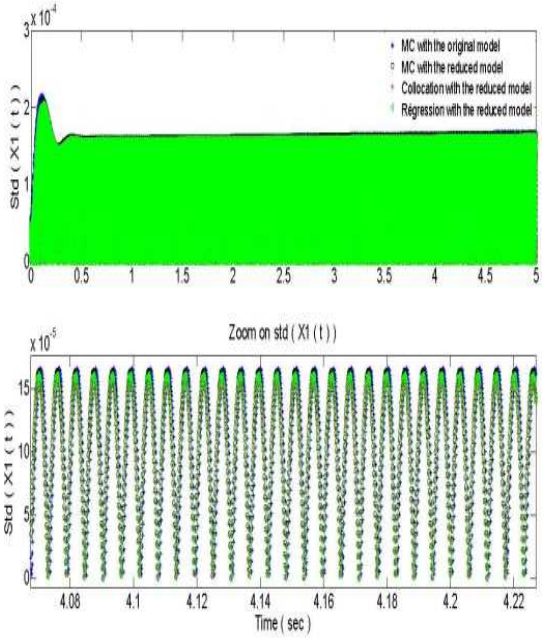

Fig.6. Standard deviation of the displacement X_1

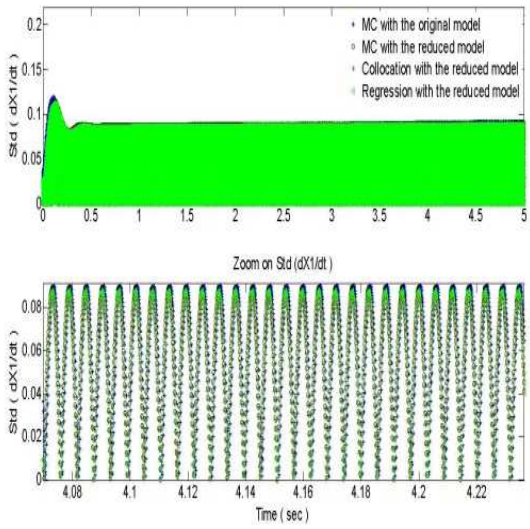

Fig.7. Standard deviation of the speed $\dot{X}_1$

From results shown in Fig.5, Fig.6 and Fig.7, the precision of the reduced model combined with MC and non-intrusive methods can be stated. Indeed, the different combinations well approximate the reference statistics (mean values and standard deviations) obtained with the Monte Carlo applied to the original model. As we are dealing with short term statistics, accuracy of the reduced model is better than the one observed in the subsection 3.2 where both reduced and original models are simulated for longer time (until obtaining limit cycle oscillation).

What makes the polynomial chaos expansion very interesting is also that this expansion can be viewed as a simpler model since theoretically, the system responses can be reconstructed for any value of uncertainty by just evaluating the polynomial chaos at the corresponding value of the random variable ξ. So no simulation of the uncertain system is required. In the context of our application, the original system responses can be built easily for a given value of the initial conditions using the Wiener-Legendre expansion for which the stochastic modes are computed from the reduced order model. Some examples of reconstruction are shown in Fig.8, Fig.9 and Fig.10.

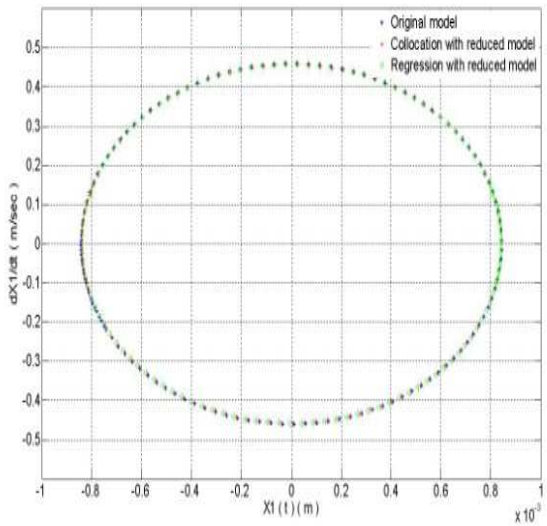

Fig.8. Phase trajectory for $X_1(0) = 3.10^{-4}$

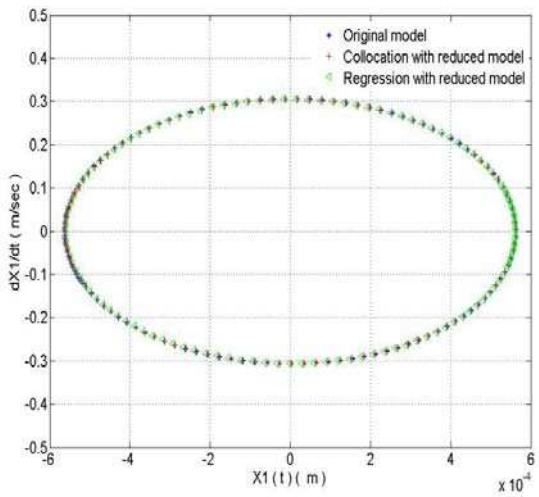

Fig.9. Phase trajectory for $X_1(0) = 2.10^{-4}$

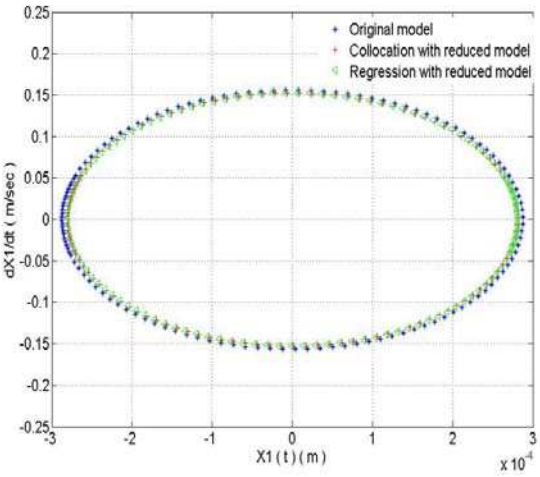

Fig.10. Phase trajectory for $X_1(0) = 10^{-4}$

4 Conclusion

This paper has proposed a new methodology to simplify the uncertainty propagation problem in nonlinear dynamic systems. It consists of central manifold theory combined with the polynomial chaos approach. The first method helps to reduce a parameter dependent system in a Hopf bifurcation neighbourhood while the second allows uncertainties in the analysis of nonlinear systems to be taken into account. Moreover, it helps to avoid the prohibitive MC method. The feasibility and efficiency of the proposed methodology has been verified in a simple self-excited mechanism. A third order centre manifold is computed so a second order model is obtained for the considered system. The reduced model reproduces the same dynamic as the original system model which consists of a limit cycle oscillation. In a second step, the reduced model is used to deal with an uncertainty quantification problem. The aim is to estimate the short term statistics of the considered system with respect to uncertain initial conditions. It has been verified that the proposed methodology is an efficient way for the considered task. The estimation of long term statistics is a more complicated problem due to the polynomial chaos properties which prevent a good estimation of the long term statistics. Other tools can be considered instead of polynomial chaos such as the Haar wavelets expansion. Research work in this context is in progress.

Sébastien Berger gratefully acknowledges the financial support of the French National Research Agency.

References:

[1] R. Babuska, G. Tempone, E. Zouraris, Galerkin finite element approximation of stochastic elliptic partial differential equation, *SIAM Journal of Scientific Computing*, Vol.24,No.2,2004, pp. 619-644.

[2] H. Cameron, W. Martin, Orthogonal developpement of nonlinear functional in series of Fourier-Hermite functional, *Annals of Mathematic*, Vol.48,No.2,1947, pp. 385-392.

[3] F. Hover, S. Michael, S. Triantafillou, Application of polynomial chaos in stability and control, *Automatica*, Vol.42,No.5,2006, pp. 789-795.

[4] R. Ghanem, G. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Springer Verlag, 1991.

[5] A. H. Nayfeh, B. Balachandran, *Applied Nonlinear Dynamics: Analytical*, Computational and Experimental Methods, John Wiley&Sons, 1995.

[6] D. Xiu, G. E. Karniadakis, Modeling Uncertainty in Flow Simulations Via Generalized Polynomial Chaos, *Journal Of Computational Physics*,Vol.187,No.1,2003, pp. 137-167.

[7] D. Xiu, G. E. Karniadakis, Modeling Uncertainty in Steady State Diffusion Problems via Generalized Polynomial Chaos, *Computer Methods in Applied Mechanics and Engineering*,43,2002, pp. 4927-4948.

[8] M. Williams, Polynomial Chaos Functions and stochastic Differential Equations, *Annals of Nuclear Energy*, Vol.33,No.7, 2006, pp. 774-785.

[9] N. Wiener, The Homogeneous Chaos, *American Journal of Mathematics*, Vol.60,No.4, 1938, pp. 897-936.

[10] A. H. Smith, A. Monti, F. Ponci, Indirect Measurement via a Polynomial Chaos Observer, *IEEE Transaction on Instrumentation and Measurment*, Vol.56,No.3, 2007, pp. 743-752.

[11] J. J. Sinou, L. Jezequel, Methods to reduce Non-Linear Mechanical Systems for instability Computation, *Archive of Computational Methods in Enginnering*, Vol.11,No.3, 2004, pp. 257-334.

[12] H. W. Knoblock, Construction of Center Manifolds, *Journal of Applied Mathematics and Mechanics*,Vol.70,No.7,1990, pp. 215-233.

[13] J. Hultèn, Drum Break Squeal- A Self Exciting Mechanism with Constant Friction, *In SAE Truck and Bus Meeting*, Detroit, Mi, USA SAE paper 932965, 1993.

[14] T. Crestaux, O. Le Maitre, J. M. Martinez, Polynomial Chaos Expansion for Sensitivity Analysis, *Relialibility engineering and System Safety*,Vol.94,No.7, 2009, pp. 1161-1172.

[15] Z. K. Nagy, R. D. Braz, Distributional Uncertainty Analysis using Power Series and Polynomial Chaos Expansions, *Journal of Process Control*, Vol.17,No.3, 2006, pp. 229-240.

[16] C. B. Moore, Component Analysis in Linear System: Controllabiity and Observability and Model Reduction, *IEEE Transaction on Automatic Control*, Vol.26,No.1, 1981, pp. 17-32.

[17] J. M. A. Sherpen, Balancing for Non-linear Systems, *Systems and Control Letters*, Vol.21,No.2, 1993, pp. 143-153.

[18] C. W. Rowley, Model reduction for Fluids, Using Balanced Proper Orthogonal Decomposition, *Int. J. On Bifurcation and Chaos*,Vol.15,No.3, 2005, 997-1013.

[19] P. V. Kokotovic, P. Sanutti, Singular Perturbation Method for Reducing the Model Order in Optimal Control Design, *IEEE Trans. On Auto. Control*,Vol.13,No.4, 1968, pp. 377-384.

[20] J. Carr, *Application of Center Manifold*, Springer Verlag, 1981.

[21] J. J. Sinou, L. Jezequel, Mode Coupling Instability in Friction-Induced Vibrations and its Dependency on System Parameters Including Damping, *European Journal of Mechanics A/solid*, Vol.26,No.1, 2007, pp. 106-122.

[22] G. S. Fishman, Monte Carlo, *Concepts, algorithms and Applications*, Springer Verlag, 1996.

[23] M. Kaminski, stochastic Perturbation Approach to Engineering structure Vibration by the Finite Difference Method, *Journal of Sound and Vibration*, Vol.251,No.4, 2002, pp. 651-670.

[24] S. Barmada, Statistical Approach in Complex Circuits by Wavelet Based Thevenin's Theorem, *Proceeding of the 11th WSEAS International Conference on Circuits, Agios Nikolaos, Crete Islad, Greece, July 23-25*, 2007, PP. 185-189.

[25] A. K. Noor, J. H. Starnes, J. M. Peters, Uncertainty Analysis of Stiffened Composite Panels, *Special Symposium Honouring the 70th Birthdays of Professors Charlie Bert and Jack Vinson*, Vol.51,No.2,2002,pp. 139-158.

[26] Z. Zhang, F. Vanderhaegen, P. Millot, Uncertainty analysis in the prediction of human operator violation using neural networks, *WSEAS Transactions on Circuits and systems*,Vol.3,No.2, 2004, pp. 259-265.

[27] R. Moore, W. Lodwick, Interval Analysis and Fuzzy Set Theory, *Fuzzy set and Systems*, Vol.135,No.1, 2003, pp. 5-9.

[28] B. Vasuki, M. Shanmugavalli, Uncertainty Analysis of Instrumentation systems: Classical&Interval Methods, *Proceedings of the 5th WSEAS International Conference on Instrumentation, Measurment, Control, Circuits and Systems, Cancun, Mexico*, May 11-14, 2005, pp.

[29] L. Nechak, S. Berger, E. Aubry, Uncertainty Propagation Using Polynomial Chaos and Centre Manifold theories, *Proceedings of the 9th WSEAS International Conference on Signal Processing, Robotics and Automation, University of Cambridge, UK*, February 20-22, 2010, pp. 46-51.

[30] T. Crestaux, O. Le Maitre, J. M. Martinez, Polynomial Chaos Expansion for Sensitivity Analysis, *Reliability engineering & System Safety*, Vol.94,No.7, 2009, pp. 1161-1172.

[31] B. Ganapathysubramanian, N. Zabaras, Sparse Grid Collocation Schemes for Stochastic Natural Convection Problems, *Journal of Computational Physics*, Vol.25,No.1, 2007, pp. 652-685.