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Ten-parameters deformations of the sixth order Peregrine breather solutions of the NLS equation.

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Abstract

One continues here the study of the solutions of the focusing NLS equation by giving deformations of the solutions to order 6 as a quotient of two determinants. In the case of the order 6, we construct new deformations of the Peregrine breather with 10 real parameters. With this method, we construct new patterns of different types of rogue waves.

1 Introduction

Since fundamental work of Zakharov and Shabat in 1972, and the first expressions of the quasi-rational solutions given by Peregrine in 1983, a considerable number of studies were carried out. Eleonski, Akhmediev and Kulagin obtained the first higher order analogue of the Peregrine breather[3] in 1986. Akhmediev et al. [1, 2] , constructed other families of higher order, using Darboux transformations.
In 2010, rational solutions of the NLS equation have been written as a quotient of two Wronskians in [8]. In 2011, an other representation of the solutions of the NLS equation has been constructed in [10], also in terms of a ratio of two Wronskians determinants of order $2N$.
In 2012, an other representation of the solutions of the focusing NLS equation, as a ratio of two determinants has been given in [13] using generalized

Darboux transform.
Ohta and Yang [17] have given a new approach where solutions of the focusing NLS equation by means of a determinant representation, obtained from Hirota bilinear method.
A the beginning of the year 2012, one obtained a representation in terms of determinants which does not involve limits [12].
These first two formulations given in [10, 12] did depend in fact only on two parameters. Then we found for the order N (for determinants of order 2N), solutions depending on $2N-2$ real parameters.
With this new method, we construct news deformations at order 6 with 10 real parameters.
The analytical expressions of these deformations are completely new and gives by new patterns a best understanding of the NLS equation.

2 Determinant representation of solutions of NLS equation

2.1 Quasi-rational limit solutions of the NLS equation

We recall the results obtained in [10]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

In the following, we consider $2N$ parameters λ_ν, $\nu = 1, \ldots, 2N$ satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N. \tag{2}$$

We define the terms κ_ν, δ_ν, γ_ν by the following equations,

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \tag{3}$$

and

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \ldots N. \tag{4}$$

The terms $x_{r,\nu}$ ($r = 3$, 1) are defined by

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N. \tag{5}$$

The parameters e_ν are defined by

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \le j \le N, \tag{6}$$

where a_j and b_j, for $1 \le j \le N$ are arbitrary real numbers.
The terms ϵ_ν are defined by :

$$\begin{aligned} \epsilon_\nu &= 0, \quad 1 \le \nu \le N \\ \epsilon_\nu &= 1, \quad N+1 \le \nu \le 2N. \end{aligned}$$

We use the following notations :
$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, \quad 1 \le \nu \le 2N.$
We consider the functions

$$\begin{aligned} \phi_{r,\nu}(y) &= \sin\Theta_{r,\nu}, \quad 1 \le \nu \le N, \\ \phi_{r,\nu}(y) &= \cos\Theta_{r,\nu}, \quad N+1 \le \nu \le 2N. \end{aligned} \tag{7}$$

$W_r(y) = W(\phi_{r,1}, \ldots, \phi_{r,2N})$ is the Wronskian

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_{r,\nu})_{\nu,\,\mu\in[1,\ldots,2N]}]. \tag{8}$$

Then we get the following statement [11]

Theorem 2.1 *The function v defined by*

$$v(x,t) = \frac{W_3(0)}{W_1(0)}\exp(2it - i\varphi). \tag{9}$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

To obtain quasi-rational solutions of the NLS equation, we take the limit when the parameters $\lambda_j \to 1$ for $1 \le j \le N$ and $\lambda_j \to -1$ for $N+1 \le j \le 2N$. Fot that, we consider the parameter λ_j written in the form

$$\lambda_j = 1 - 2j^2\epsilon^2, \quad 1 \le j \le N. \tag{10}$$

When ϵ goes to 0, we obtain quasi-rational solutions of the NLS equation given by :

Theorem 2.2 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi)\lim_{\epsilon\to 0}\frac{W_3(0)}{W_1(0)}, \tag{11}$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

2.2 Expression of solutions of NLS equation in terms of a ratio of two determinants

We construct the solutions of the NLS equation expressed as a quotient of two determinants which does not involve a passage to the limit.
We use the following notations :

$$A_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2,$$
$$B_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2,$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (3), (4) and (5).
The parameters e_ν are defined by (6).
Here, the parameters a_j and b_j, for $1 \leq N$ are chosen in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a_k} \epsilon^{2k-1} j^{2k-1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b_k} \epsilon^{2k-1} j^{2k-1}, \quad 1 \leq j \leq N. \tag{12}$$

We consider the following functions :

$$f_{4j+1,k} = \gamma_k^{4j-1} \sin A_k, \quad f_{4j+2,k} = \gamma_k^{4j} \cos A_k,$$
$$f_{4j+3,k} = -\gamma_k^{4j+1} \sin A_k, \quad f_{4j+4,k} = -\gamma_k^{4j+2} \cos A_k, \tag{13}$$

for $1 \leq k \leq N$, and

$$f_{4j+1,k} = \gamma_k^{2N-4j-2} \cos A_k, \quad f_{4j+2,k} = -\gamma_k^{2N-4j-3} \sin A_k,$$
$$f_{4j+3,k} = -\gamma_k^{2N-4j-4} \cos A_k, \quad f_{4j+4,k} = \gamma_k^{2N-4j-5} \sin A_k, \tag{14}$$

for $N+1 \leq k \leq 2N$.
We define the functions $g_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, we replace only the term A_k by B_k.

$$g_{4j+1,k} = \gamma_k^{4j-1} \sin B_k, \quad g_{4j+2,k} = \gamma_k^{4j} \cos B_k,$$
$$g_{4j+3,k} = -\gamma_k^{4j+1} \sin B_k, \quad g_{4j+4,k} = -\gamma_k^{4j+2} \cos B_k, \tag{15}$$

for $1 \leq k \leq N$, and

$$g_{4j+1,k} = \gamma_k^{2N-4j-2} \cos B_k, \quad g_{4j+2,k} = -\gamma_k^{2N-4j-3} \sin B_k,$$
$$g_{4j+3,k} = -\gamma_k^{2N-4j-4} \cos B_k, \quad g_{4j+4,k} = \gamma_k^{2N-4j-5} \sin B_k, \tag{16}$$

for $N+1 \leq k \leq 2N$.
Then we get the following result :

Theorem 2.3 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{17}$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

where

$$n_{j1} = f_{j,1}(x,t,0),\ 1 \le j \le 2N \quad n_{jk} = \frac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0),\ 2 \le k \le N,\ 1 \le j \le 2N$$

$$n_{jN+1} = f_{j,N+1}(x,t,0),\ 1 \le j \le 2N \quad n_{jN+k} = \frac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0),\ 2 \le k \le N,\ 1 \le j \le 2N$$

$$d_{j1} = g_{j,1}(x,t,0),\ 1 \le j \le 2N \quad d_{jk} = \frac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0),\ 2 \le k \le N,\ 1 \le j \le 2N$$

$$d_{jN+1} = g_{j,N+1}(x,t,0),\ 1 \le j \le 2N \quad d_{jN+k} = \frac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0),\ 2 \le k \le N,\ 1 \le j \le 2N$$

The functions f and g are defined in (13),(14), (15), (16).

We don't give here the proof of this result. The ideas are the same as exposed in [12]. We will give the proof in a forthcoming paper.
The solutions of the NLS equation can also be written in the form :

$$v(x,t) = \exp(2it - i\varphi) \times Q(x,t)$$

where $Q(x,t)$ is defined by :

$$Q(x,t) := \frac{\begin{vmatrix} f_{1,1}[0] & \dots & f_{1,1}[N-1] & f_{1,N+1}[0] & \dots & f_{1,N+1}[N-1] \\ f_{2,1}[0] & \dots & f_{2,1}[N-1] & f_{2,N+1}[0] & \dots & f_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{2N,1}[0] & \dots & f_{2N,1}[N-1] & f_{2N,N+1}[0] & \dots & f_{2N,N+1}[N-1] \end{vmatrix}}{\begin{vmatrix} g_{1,1}[0] & \dots & g_{1,1}[N-1] & g_{1,N+1}[0] & \dots & g_{1,N+1}[N-1] \\ g_{2,1}[0] & \dots & g_{2,1}[N-1] & g_{2,N+1}[0] & \dots & g_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{2N,1}[0] & \dots & g_{2N,1}[N-1] & g_{2N,N+1}[0] & \dots & g_{2N,N+1}[N-1] \end{vmatrix}} \tag{18}$$

3 Quasi-rational solutions of order 6 with ten parameters

Wa have already constructed in [10] solutions for the cases from $N = 1$ until $N = 6$, and in [12] with two parameters.
Because of the length of the expression v of the solution of NLS equation with eight parameters, we can't give here. We only construct figures to show deformations of the fifth Peregine breathers.
Conversely to the study with two parameters given in preceding works [10, 12], we get other type of symmetries in the plots in the (x, t) plane. We give some examples of this fact in the following.

3.1 Peregrine breather of order 6

If we choose $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = \tilde{a}_3 = \tilde{b}_3 = \tilde{a}_4 = \tilde{b}_4 = \tilde{a}_5 = \tilde{b}_5 = 0$, we obtain the classical Peregrine breather :
Figure 1

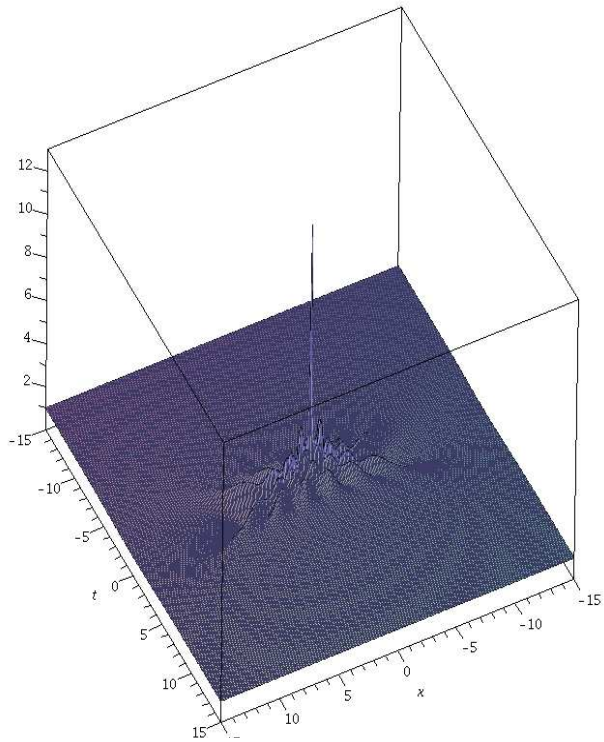

Figure 1: Solution of NLS, N=6, $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = \tilde{a}_3 = \tilde{b}_3 = \tilde{a}_4 = \tilde{b}_4 = 0$.

With other choices of parameters, we obtain all types of configurations : triangles, overlapping triangular-circular and circular configurations with a maximum of 21 peaks.

3.2 Variation of parameters

Figure 2 $a_1 \neq 0$

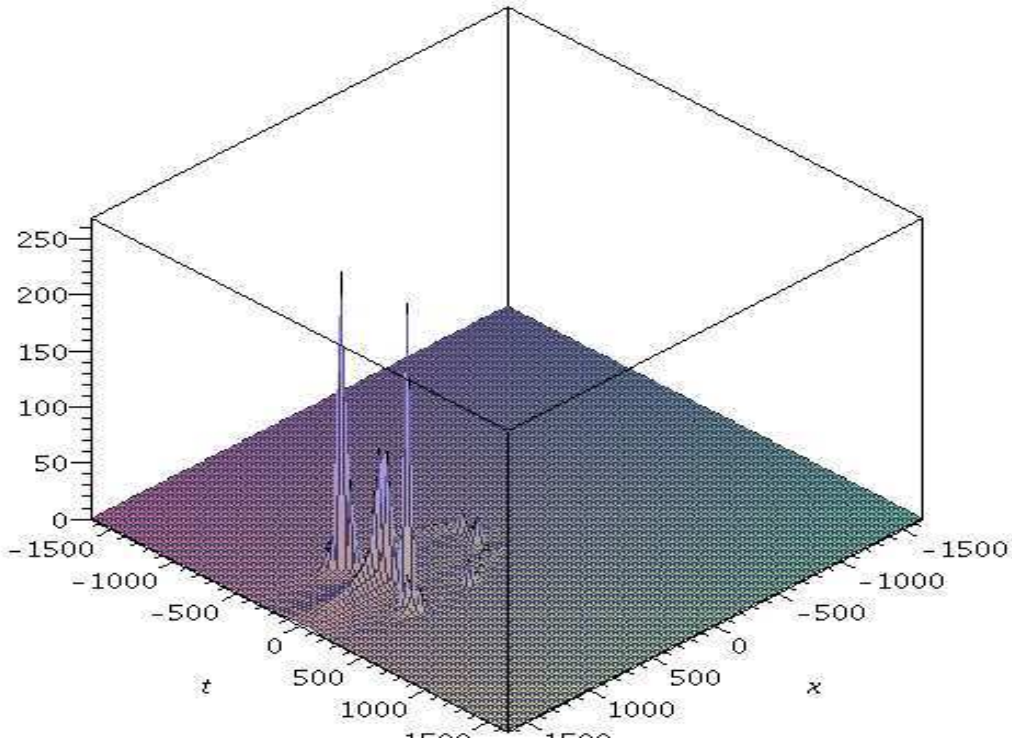

Figure 2: Solution of NLS, N=6, $\tilde{a}_1 = 10^3$.

Figure 3 $b_1 \neq 0$

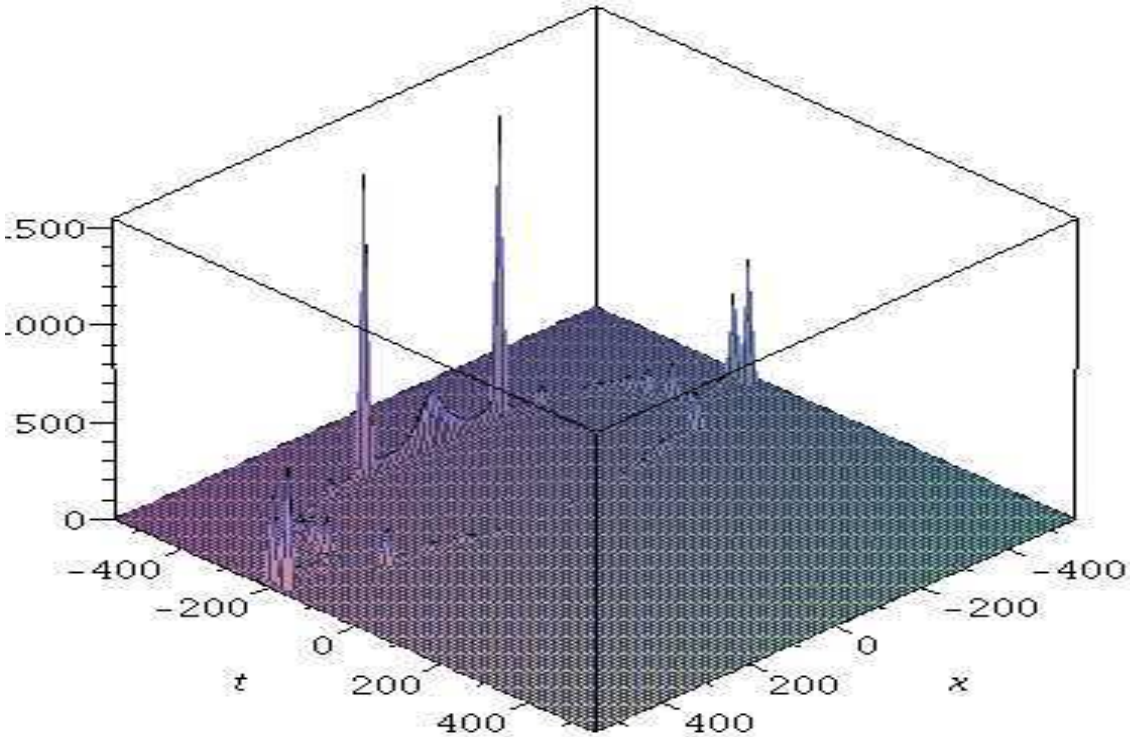

Figure 3: Solution of NLS, N=6, $\tilde{b}_1 = 10^3$.

Figure 4 $a_2 \neq 0$

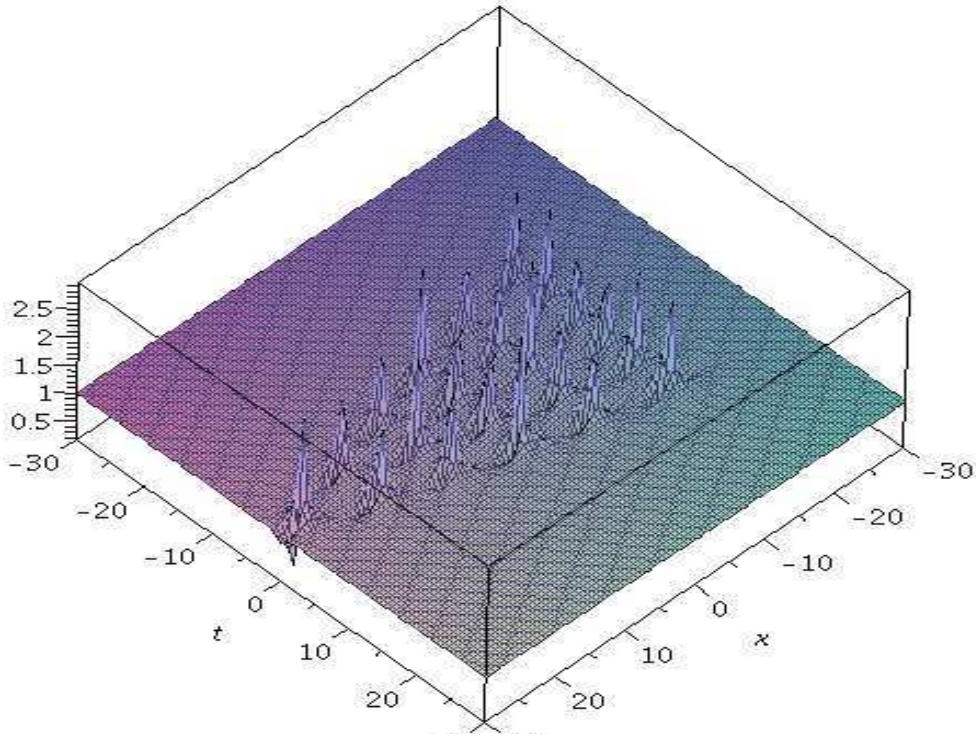

Figure 4: Solution of NLS, N=6, $\tilde{a}_2 = 10^3$.

Figure 5 $b_2 \neq 0$

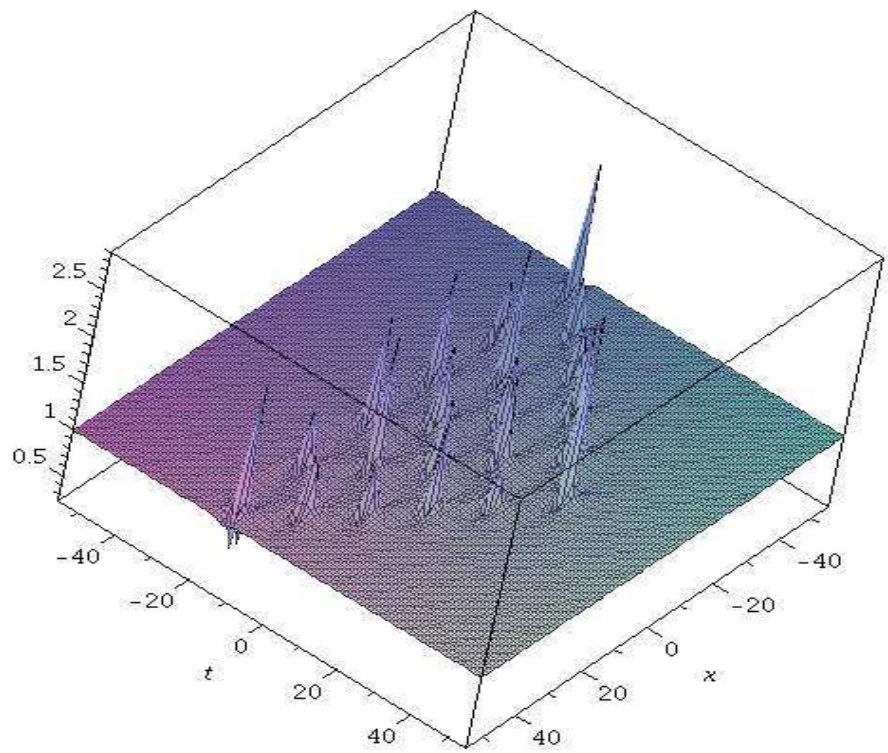

Figure 5: Solution of NLS, N=6, $\tilde{b}_2 = 10^4$.

Figure 6 $a_3 \neq 0$

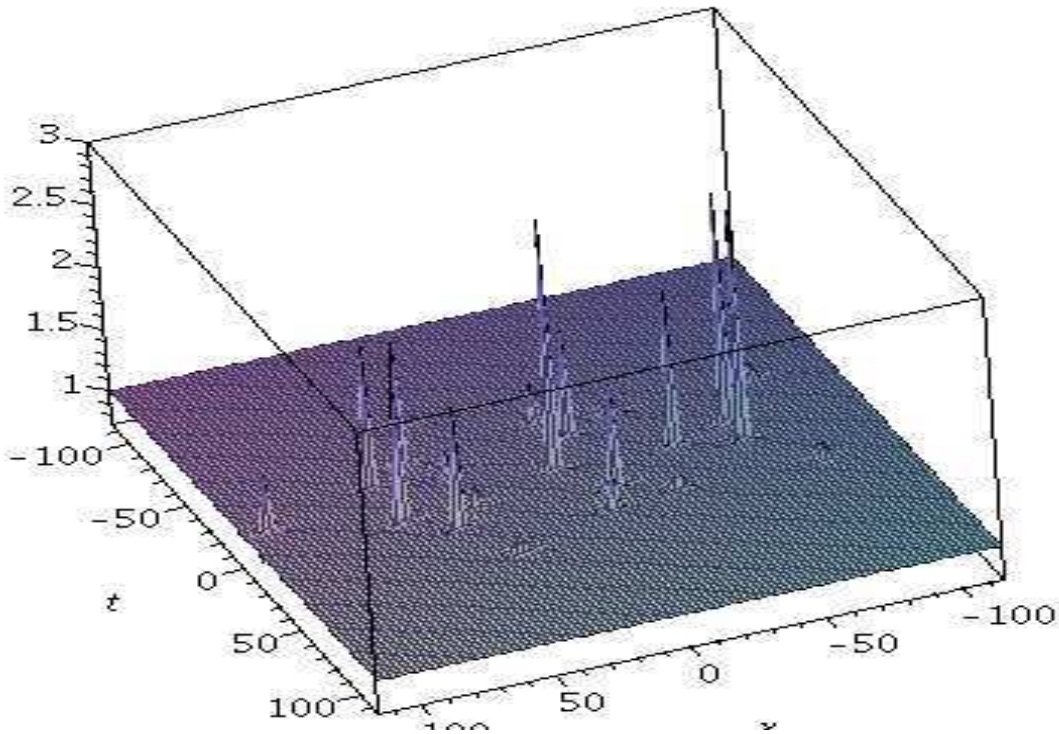

Figure 6: Solution of NLS, N=6, $\tilde{a}_3 = 10^8$.

Figure 7 $b_3 \neq 0$

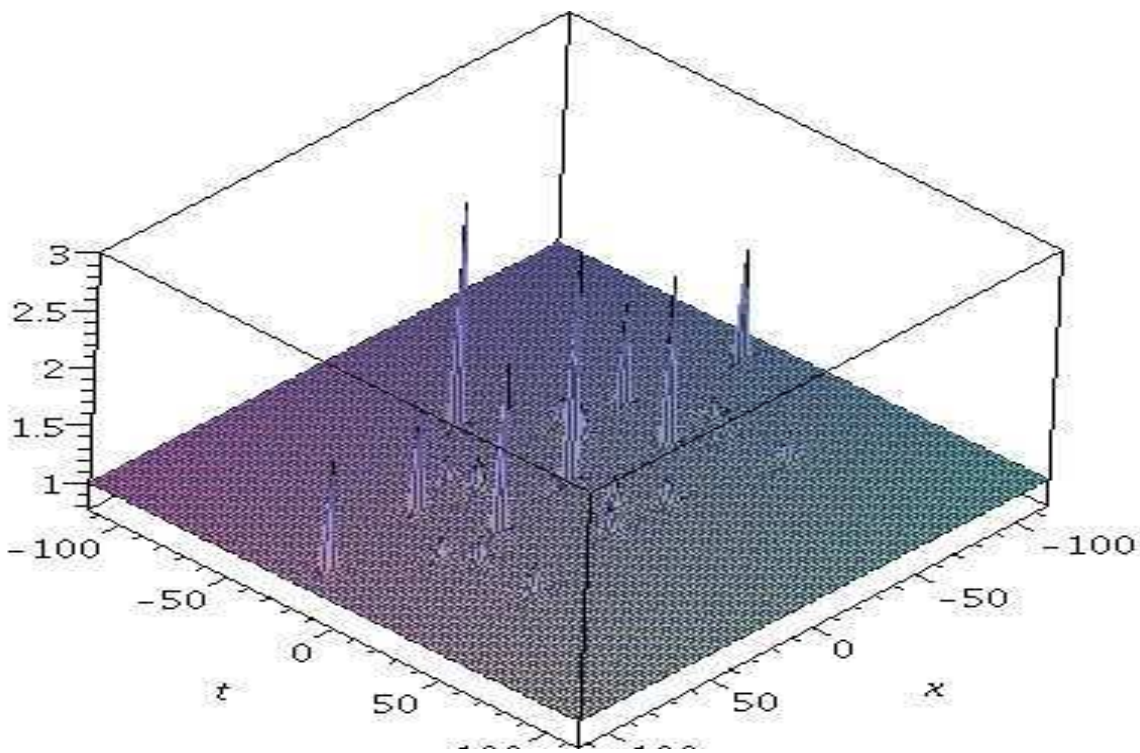

Figure 7: Solution of NLS, N=6, $\tilde{b}_3 = 10^8$.

Figure 8 $a_4 \neq 0$

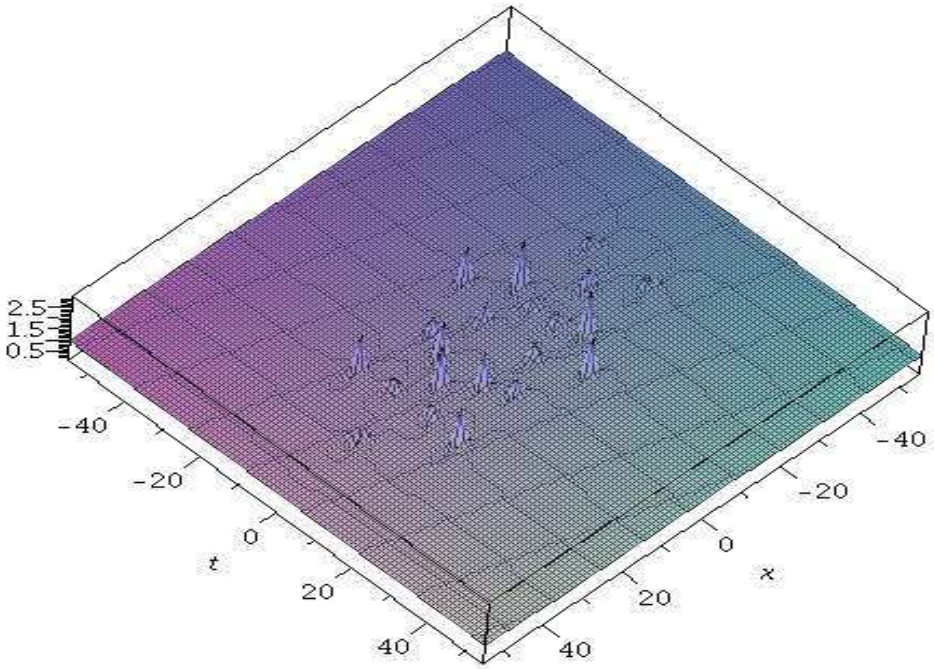

Figure 8: Solution of NLS, N=6, $\tilde{a}_4 = 10^8$.

Figure 9 $b_4 \neq 0$

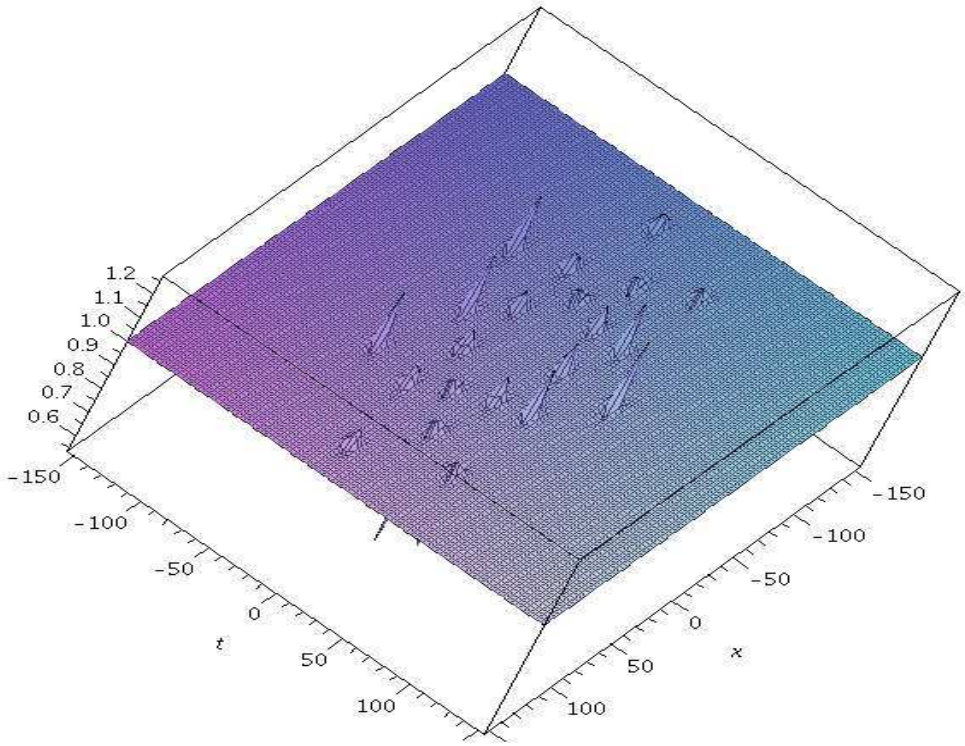

Figure 9: Solution of NLS, N=6, $\tilde{b}_4 = 10^{12}$.

Figure 10 $a_5 \neq 0$

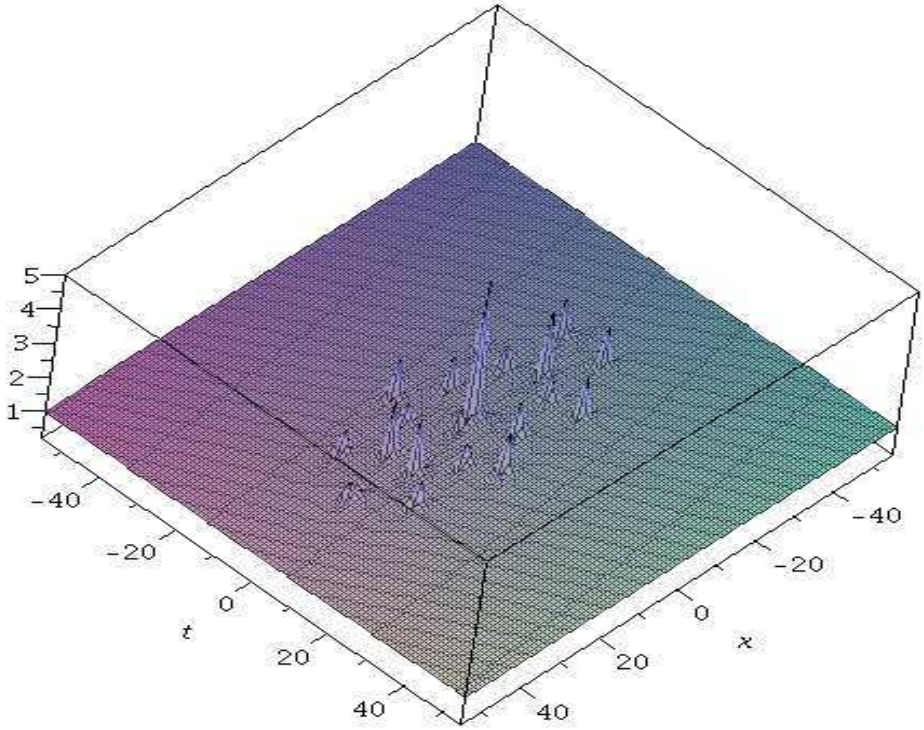

Figure 10: Solution of NLS, N=6, $\tilde{a}_5 = 10^{10}$.

Figure 11 $b_5 \neq 0$

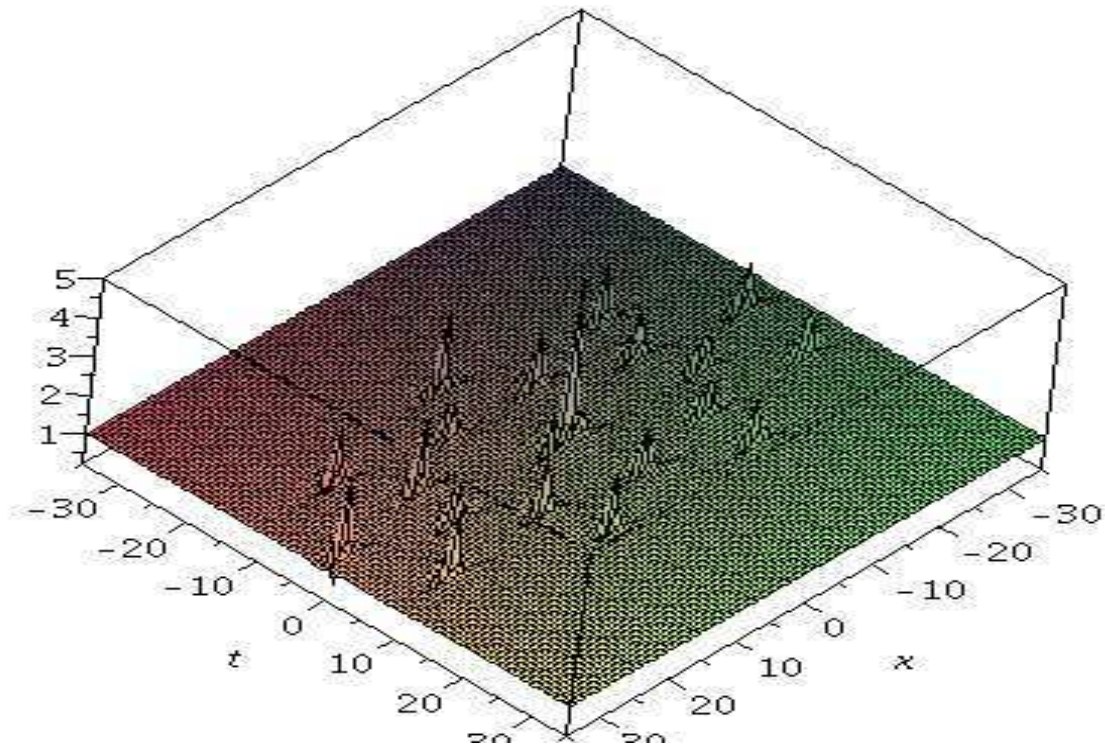

Figure 11: Solution of NLS, N=6, $\tilde{b}_5 = 10^{10}$.

4 Conclusion

In the present paper we construct explicitly solutions of the NLS equation of order N with $2N-2$ real parameters.
By different choices of these parameters, we obtained new patterns in the $(x;t)$ plane; we recognized circular shape as already observed in the case of deformations depending on two parameters [10, 12]. We get news triangular shapes and overlapping circular-triangular shape.
We chose to present here the solutions of the NLS equation in the case $N = 6$ with eight real parameters, only in order not to weigh down the text with . We postpone the presentation of the higher orders in another publication.

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