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# ON THE GROWTH OF HYPERBOLIC GEODESICS IN RANK 1 MANIFOLDS

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ABSTRACT. We give a formula for the topological pressure of the geodesic flow of a compact rank 1 manifold in terms of the growth of the number of closed hyperbolic (rank 1) geodesics. We derive an equidistribution result for these geodesics with respect to equilibrium states. This generalize partially a result of G. Knieper [12] to non constant potentiels.

## 1. INTRODUCTION AND MAIN RESULT

By a result of G Knieper [12], we know that the geodesic flow on a compact rank 1 manifold  $M$  admits a unique measure of maximal entropy, concentrated on the open and dense set of regular vectors in  $T^1M$ . Knieper proved that this measure is approximated by probability measures supported on a finite number of regular closed geodesics. This result generalize previous well known results for negatively curved manifolds [2] [6] [8] [13] [14] [15] [20].

In this paper we consider the case of non constant potentials. We obtain a formula expressing the topological pressure as an exponential growth of the number of weighted regular closed geodesics representing different free homotopy classes. As a consequence we give an equidistribution result for weighted closed regular geodesics to an equilibrium state. These results extend and strengthens previous one by G Knieper ([12] Proposition 6.4) and M Pollicott [19]. The proof uses the Anosov's closing lemma for compact manifolds of nonpositive curvature [5] and also a Riemannian formula for the topological pressure by G P Paterlain [18].

Let  $M = X/\Gamma$  be a compact Riemannian manifold of nonpositive curvature where  $X$  is the universal cover and  $\Gamma$  is the group of deck transformations of  $X$ . The rank of a vector  $v \in T^1M$  is the dimension of the space of all parallel Jacobi fields along the geodesic defined by  $v$ . The rank of the manifold is the minimal rank of all tangeant vectors.

We will assume that  $M$  is a rank 1 manifold (this includes manifolds of negative sectional curvature where all the geodesics are of rank 1). In fact, by a rigidity result of Ballmann [3] and Burns-Spatzier [7] “most of” compact manifolds of nonpositive curvature are rank 1. By a regular vector (resp regular geodesic) we will mean a rank 1 vector (resp a geodesic defined by a rank 1 vector). A geodesic is called hyperbolic if it is regular, extending thus the notion of hyperbolicity to rank 1 manifolds. Let  $\mathcal{R}_{reg}$  be the open subset of  $T^1M$  of regular vectors. It is dense in  $T^1M$  if  $M$  is of finite volume [4]. Let  $\nu$  be the Knieper’s measure of maximal entropy of the geodesic flow of the rank 1 manifold  $M$ . We have  $\nu(\mathcal{R}_{reg}) = 1$  and the complement  $\mathcal{S}_{ing}$  of  $\mathcal{R}_{reg}$  is an invariant closed subset of the unit tangent bundle. The growth of closed geodesics in the “singular part”  $\mathcal{S}_{ing}$  can be exponential [10] as well as subexponential [12]. In this paper we concentrate on the regular set, but it will be interesting to investigate the  $\mathcal{S}_{ing}$ -part.

Two elements  $\alpha, \beta \in \Gamma$  are equivalent if and only if there exists  $n, m \in \mathbb{Z}$  and  $\gamma \in \Gamma$  such that  $\alpha^n = \gamma\beta^m\gamma^{-1}$ . Denote by  $[\Gamma]$  the set of equivalence classes of elements in  $\Gamma$ . Classes in  $[\Gamma]$  are represented by elements in  $\Gamma$  which have a least period (primitive elements):

$$[\alpha] = \{\gamma\alpha_0^m\gamma^{-1} : \alpha_0 \in \Gamma, \alpha_0 \text{ primitive}, \gamma \in \Gamma\}.$$

Let  $x_\alpha$  be the point in  $X$  such that  $d(x_\alpha, \alpha x_\alpha) = \inf_{p \in X} d(p, \alpha p)$  ( $M$  is compact). The axis through  $x_\alpha$  and  $\alpha x_\alpha$  projects onto a closed geodesic in  $M$  with prime period  $d(x_\alpha, \alpha x_\alpha) := l(\alpha_0)$ . We set  $l([\alpha]) := l(\alpha_0)$ , i.e

$$l([\alpha]) = \min\{l(\gamma) : \gamma \in [\alpha]\} = l(\alpha_0).$$

We will denote by  $\Gamma_{hyp} \subset \Gamma$  the subset of those elements with hyperbolic axis. Then  $[\Gamma_{hyp}]$  is the set of conjugacy classes representing geometrically distinct hyperbolic closed geodesics. Finally given a function  $f$  on  $T^1M$  the notation  $\int_{[\alpha]} f$ ,  $[\alpha] \in [\Gamma_{hyp}]$ , stands for the integral of  $f$  along the unique closed geodesic representing the class  $[\alpha]$ . If this geodesic is given by  $\phi_s v_{[\alpha]}$ ,  $0 \leq s \leq l([\alpha])$  for some  $v_{[\alpha]} \in T^1M$ , then,

$$\int_{[\alpha]} f := \int_0^{l([\alpha])} f(\phi_s v_{[\alpha]}) ds := \delta_{[\alpha]}(f).$$

Given a continuous function  $f$  on  $T^1M$ , let  $\mu_t := \mu_t^f$  be the flow invariant probability measures supported on a finite number of hyperbolic closed geodesics defined on continuous functions  $\omega$  by,

$$\mu_t(\omega) := \frac{\sum_{([\alpha] \in \Gamma_{hyp} : l([\alpha]) \leq t)} e^{\int_\alpha f} \delta_{[\alpha]}(\omega)}{\sum_{([\alpha] \in \Gamma_{hyp} : l([\alpha]) \leq t)} e^{\int_\alpha f}}.$$

Here is the main result of the paper.

**Theorem 1.** *Let  $M = X/\Gamma$  be a compact rank 1 manifold equipped with a  $C^\infty$  Riemannian metric and  $f \in \mathcal{C}_\mathbb{R}(T^1M)$ . Then*

(1)

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \sum_{[\alpha] \in [\Gamma_{hyp}]: l([\alpha]) \leq t} e^{\int_\alpha f} = P(f).$$

(2) *The accumulation points of  $\{\mu_t\}$  with respect to the topology of weak convergence of measures, are equilibrium states of the geodesic flow corresponding to the potential  $f$ . Moreover, for any open neighborhood  $V$  in  $\mathcal{P}(T^1M)$  of the subset of equilibrium states  $\mathcal{P}_e(\phi)$  we have,*

$$\lim_{t \rightarrow +\infty} \frac{\sum_{([\alpha] \in \Gamma_{hyp}: l([\alpha]) \leq t, \delta_{[\alpha]} \in V)} e^{\int_\alpha f}}{\sum_{([\alpha] \in \Gamma_{hyp}: l([\alpha]) \leq t)} e^{\int_\alpha f}} = 1,$$

where the convergence is exponential.

In the part (2) of Theorem 1, the condition  $C^\infty$  on the metric is necessary since we need the upper-semicontinuity of the entropy map [17].

Let  $d$  be the distance on  $T^1M$  induced by the Riemannian metric of  $M$ . Consider the metric  $d_t$  on  $T^1M$ , defined for all  $t > 0$  by

$$d_t(u, v) := \sup_{0 \leq s \leq t} d(\phi^s(u), \phi^s(v)).$$

Following [5] we denote by  $P(t, \epsilon)$  the maximal number of regular vectors  $v \in T^1M$  which are  $\epsilon$ -separated in the metric  $d_t$  and for which  $\phi^{t(v)}v = v$  for some  $t(v) \in [t, t + \epsilon]$ . Let  $E(t, \epsilon)$  be the set defined above with  $\#E(t, \epsilon) = P(t, \epsilon)$ .

The following is the Lemma 5.6 from [5] for rank 1 manifolds and continuous potentials (see Lemma 1 (5) below).

**Proposition 1.** *Set  $\int_{c_v} f := \int_0^{l(c_v)} f(\phi^t(v))dt$ , where  $c_v$  is the closed geodesic defined by  $v \in E(t, \epsilon)$  and  $l(c_v)$  is the period of  $v$ . Then,*

$$\lim_{\epsilon \rightarrow 0} \liminf_{t \rightarrow +\infty} \frac{1}{t} \log \sum_{v \in E(t, \epsilon)} e^{\int_{c_v} f} = \lim_{\epsilon \rightarrow 0} \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \sum_{v \in E(t, \epsilon)} e^{\int_{c_v} f} = P(f).$$

## 2. PROOFS

**2.1. Topological pressure.** We recall the notion of topological pressure [21]. Let  $t > 0$  and  $\epsilon > 0$ . A subset  $E \subset T^1M$  is a  $(t, \epsilon)$ -separated

if  $d_t(u, v) > \epsilon$  for  $u \neq v \in E$ . Set

$$r(f; t, \epsilon) := \sup_E \sum_{\theta \in E} e^{\int_0^t f(\phi_t(\theta)) dt}$$

where sup is over all  $(t, \epsilon)$ -separated subsets  $E$ ; and

$$r(f; \epsilon) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log r(f; t, \epsilon).$$

Then the topological pressure of the geodesic flow corresponding to the potential  $f$  is the number,

$$(1) \quad P(f) = \lim_{\epsilon \rightarrow 0} r(f; \epsilon).$$

The topological entropy  $h_{top}$  is  $h_{top} = P(0)$ . We denote by  $\mathcal{P}(T^1M)$  the set of all probability measures on  $T^1M$  with the weak topology of measures, and let  $\mathcal{P}(\phi)$  be the subset of invariant probability measures of the flow. The entropy of a probability measure  $m$  is denoted  $h(m)$  [21]. All these objects satisfy the following variational principle [21]

$$(2) \quad P(f) = \sup_{\mu \in \mathcal{P}(\phi)} \left( h(\mu) + \int_{T^1M} f d\mu \right).$$

An equilibrium state  $\mu_f$  satisfies,

$$(3) \quad h(\mu_f) + \int_{T^1M} f d\mu_f = P(f).$$

When the Riemannian metric of the manifold  $M$  is  $C^\infty$  then by a result of Newhouse [17] the entropy map  $m \rightarrow h(m)$  is upper semicontinuous and then  $h_{top} < \infty$ . Consequently, the set  $\mathcal{P}_e(f)$  of equilibrium states is a non empty closed and convex subset of  $\mathcal{P}(\phi)$ .

**2.2. Proof of Theorem 1 (1).** Let  $\nu$  be the Knieper's measure. Let  $N(t, \epsilon, 1 - \delta, \nu)$  be the minimal number of  $\epsilon$ -balls in the metric  $d_t$  which cover a set of measure at least  $1 - \delta$ . Since  $\nu$  is the unique measure of maximal entropy, we can apply Lemma 5.6 in [5] to this measure.

**Lemma 1 ([5]).** *There exists  $\delta > 0$  such that for all  $\epsilon > 0$ , there exists  $t_1 > 0$  such that*

$$(4) \quad P(t, \epsilon) \geq N(t, \epsilon, 1 - \delta, \nu)$$

for any  $t \geq t_1$ . In particular, we have

$$(5) \quad \lim_{\epsilon \rightarrow 0} \liminf_{t \rightarrow +\infty} \frac{\ln P(t, \epsilon)}{t} = \lim_{\epsilon \rightarrow 0} \limsup_{t \rightarrow +\infty} \frac{\ln P(t, \epsilon)}{t} = h_{top}.$$

We fix  $\epsilon > 0$ . Recall that  $E(t, \epsilon)$  is the maximal set defined above with  $\#E(t, \epsilon) = P(t, \epsilon)$ . From Lemma 1 (4), if  $N(t, \epsilon, 1 - \delta, \nu)$  is the minimal number of  $\epsilon$ -balls  $B_t(v_i, \epsilon)$  in the metric  $d_t$ , which cover the whole space  $T^1M$ , then  $P(t, \epsilon) = N(t, \epsilon, 1 - \delta, \nu)$  for  $t$  sufficiently large (since each  $\epsilon$ -ball  $B_t(v_i, \epsilon)$  contains a unique point from  $E(t, \epsilon)$ ).

Now, it suffices to prove Theorem 1 (1) for Lipschitz functions  $f$ . Suppose then  $f$  Lipschitz and let  $lip(f)$  be it's Lipschitz constant. Consider a  $(2\epsilon, t)$ -separated set  $E_1$  in  $T^1M$ . Thus, two distinct vectors in  $E_1$  lies in two distinct  $\epsilon$ -balls above, so that  $\#E_1 \leq P(t, \epsilon)$ . For each  $\theta \in E_1$ , we associate the unique point  $v_\theta \in E(t, \epsilon)$  such that  $d_t(\theta, v_\theta) \leq \epsilon$ . Let  $\tau_\theta$  the regular closed geodesic corresponding to the regular periodic vector  $v_\theta$  ( $\dot{\tau}_\theta(0) = v_\theta$ ), with period  $l(\tau_\theta) \in [t, t + \epsilon]$ , and  $[\tau_\theta]$  the corresponding free homotopy class. There exists a constant  $C > lip(f)$  such that,

$$\begin{aligned} & \sum_{\theta \in E_1} e^{\int_0^t f(\phi_s(\theta)) ds} \\ & \leq e^{lip(f)\epsilon t} \sum_{[\tau_\theta]: \theta \in E_1, t < l(\tau_\theta) \leq t + \epsilon} e^{\int_0^t f(\tau_\theta(s)) ds} \\ & \leq e^{C\epsilon t} \sum_{[\tau]: t < l(\tau) \leq t + \epsilon} e^{\int_0^{l(\tau)} f(\tau(s)) ds} \\ & \leq e^{C\epsilon t} \sum_{[\tau]: l(\tau) \leq t + 1} e^{\int_0^{l(\tau)} f(\tau(s)) ds} \end{aligned}$$

where the sum is over all the hyperbolic closed geodesics which represent different free homotopy classes and prescribed length. Thus for all  $\epsilon < \epsilon_0$  we obtain that

$$\begin{aligned} P(f; 2\epsilon) &= \limsup_{t \rightarrow \infty} \frac{1}{t} \log \sup_{E_1} \left( \sum_{\theta \in E_1} e^{\int_0^t f(\phi_s(\theta)) ds} \right) \\ &\leq C\epsilon + \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left( \sum_{[\alpha] \in [\Gamma_{hyp}]: l([\alpha]) \leq t} e^{\int_{[\alpha]} f} \right). \end{aligned}$$

We let  $\epsilon \rightarrow 0$  gives,

$$\begin{aligned} P(f) &:= \lim_{\epsilon \rightarrow 0} P(f; 2\epsilon) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left( \sum_{[\alpha] \in [\Gamma_{hyp}]: l([\alpha]) \leq t} e^{\int_{[\alpha]} f} \right). \end{aligned}$$

To show the reverse inequality,  $\limsup \leq P(f)$ , it suffices to observe that the set of hyperbolic closed geodesics with length  $\leq t$  and which represent different free homotopy classes, is  $\epsilon$ -separated for all  $\epsilon < \text{inj}(M)$ .

We will show now that the  $\liminf$  is bounded below by the pressure. For this we use the following result [18].

**Theorem 2** (G P Paternain [18]). *Let  $M$  be a closed connected Riemannian manifold. If the metric of  $M$  is of class  $C^3$  and does not have conjugate points, then for any  $\delta > 0$  we have,*

$$P(f) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \int_{M \times M} \left( \sum_{\gamma_{xy}: t - \delta < l(\gamma_{xy}) \leq t} e^{\int_0^{l(\gamma_{xy})} f(\gamma_{xy}(t), \dot{\gamma}_{xy}(t)) dt} \right) dx dy.$$

For each  $\delta > 0$  and  $(x, y) \in M \times M$  we consider the subset of  $T^1M$  defined by,

$$E_{xy} := \{\dot{\gamma}_{xy}(0) : t - \delta < l(\gamma_{xy}) \leq t\}.$$

It is finite for almost all  $(x, y) \in M \times M$  [?]. As consequence of the nonpositive curvature, the rank 1 manifold  $M$  has no conjugate points. Thus, there exists a constant  $\epsilon_0$ , depending only on  $M$ , such that  $E_{xy}$  is  $(2\epsilon, t)$ -separated for  $\epsilon < \epsilon_0$ . To see this, it suffices to lift every thing to the universal cover of  $M$  and use ([18] p135) and ([11] p375). Thus, the preceding arguments applied to  $E_1 = E_{xy}$  give,

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{t} \log \int_{M \times M} \left( \sum_{\theta \in E_{xy}} e^{\int_0^t f(\phi^s(\theta)) ds} \right) dx dy \\ & \leq \liminf_{t \rightarrow \infty} \frac{1}{t} \log \left( \sum_{[\alpha] \in \Gamma_{hyp}: l([\alpha]) \leq t} e^{\int_{[\alpha]} f} \right). \end{aligned}$$

But by Theorem 2, the left hand side of this inequality is a limit and is equal to  $P(f)$ , which completes the proof.

**Remark (question) 1.** *Find a proof which did not appeal to Paternain's formula in Theorem 2 !*

### 2.3. Proof of Proposition 2.

*Proof.* The proof of Proposition 2 follows from the above arguments.  $\square$

**2.4. Proof of Theorem 1 (2).** Consider the following functional which measures the “distance” of an invariant measure  $m$  to the set of equilibrium states,

$$\rho(m) = P(f) - \left( h(m) + \int f dm \right).$$

Set  $\rho(E) := \inf(\rho(m) : m \in E)$  for  $E \subset \mathcal{P}(\phi)$  and,

$$[\Gamma_{hyp}](t) := \{[\alpha] \in [\Gamma_{hyp}] : l([\alpha]) \leq t\}.$$

**Lemma 2.** *Let  $M$  be a compact smooth rank 1 manifold and  $f$  a continuous potential on  $T^1M$ . Then, for any closed subset  $K$  of  $\mathcal{P}(\phi)$  we have,*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \frac{\sum_{([\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in K)} e^{\int_{[\alpha]} f}}{\sum_{[\alpha] \in [\Gamma_{hyp}](t)} e^{\int_{[\alpha]} f}} \leq -\rho(K).$$

We leave the proof of this lemma for later and finish the proof of Theorem 1. First, let  $V$  be an open neighborhood of  $\mathcal{P}_e(f)$  and set  $K = \mathcal{P}(\phi) \setminus V$ . The set  $K$  is compact and  $\rho(K) > 0$ . For  $t$  sufficiently large we have by Lemma 3,

$$1 \geq \frac{\sum_{([\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in V)} e^{\int_{[\alpha]} f}}{\sum_{[\alpha] \in [\Gamma_{hyp}](t)} e^{\int_{[\alpha]} f}} \geq 1 - e^{-t\rho(K)}.$$

This proves the second assertion in part (2) of the Theorem 1.

**2.4.1. Accumulation measures of  $\mu_t$ .** The proof of the fact that the accumulation measures of  $\mu_t$  are in  $\mathcal{P}_e(\phi)$  follows [1].

We endow  $\mathcal{P}(T^1M)$  with a distance  $d$  compatible with the weak star topology: take a countable base  $\{g_1, g_2, \dots\}$  of the separable space  $C_{\mathbb{R}}(T^1M)$ , where  $\|g_k\| = 1$  for all  $k$ , and set:

$$d(m, m') := \sum_{k=1}^{\infty} 2^{-k} \left| \int g_k dm - \int g_k dm' \right|.$$

Let  $V \subset \mathcal{P}(\phi)$  be a convex open neighborhood of  $\mathcal{P}_e(f)$  and  $\epsilon > 0$ . We consider a finite open cover  $(B_i(\epsilon))_{i \leq N}$  of  $\mathcal{P}_e(f)$  by balls of diameter  $\epsilon$  all contained in  $V$ . Decompose the set  $U := \cup_{i=1}^N B_i(\epsilon)$  as follows,

$$U = \cup_{j=1}^{N'} U_j^\epsilon,$$

where the sets  $U_j^\epsilon$  are disjoint (not necessarily open) and contained in one of the balls  $(B_i(\epsilon))_{i \leq N}$ . We have

$$\mathcal{P}_e(f) \subset U \subset V.$$



We fix in each  $U_j^\epsilon$  an invariant probability measure  $m_j$ ,  $j \leq N'$ , and let  $m_0$  be an invariant probability measure distinct from the above ones; for example take  $m_0 \in V \setminus U$ . Set for convenience,

$$(6) \quad \nu_t(E) := \frac{\sum_{([\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in E)} e^{\int_{[\alpha]} f}}{\sum_{[\alpha] \in [\Gamma_{hyp}](t)} e^{\int_{[\alpha]} f}}, \quad E \subset \mathcal{P}(\phi)$$

and define,

$$(7) \quad \beta_t = \sum_{j=1}^{N'} \nu_t(U_j^\epsilon) m_j + (1 - \nu_t(U)) m_0.$$

We have  $\sum_{j=1}^{N'} \nu_t(U_j^\epsilon) = \nu_t(U)$ . The probability measure  $\beta_t$  lies in  $V$  since it is a convex combination of elements in the convex set  $V$ . Thus

$$d(\mu_t, V) \leq d(\mu_t, \beta_t).$$

We are going to show that

$$d(\mu_t, \beta_t) \leq \epsilon \nu_t(U) + 3\nu_t(U^c),$$

where  $U^c = \mathcal{P}(SM) \setminus U$ .

Consider the measures  $\mu_{t,V}$  on  $SM$  defined by,

$$\mu_{t,V} := \frac{\sum_{([\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in V)} e^{\int_{[\alpha]} f} \delta_{[\alpha]}}{\sum_{[\alpha] \in [\Gamma_{hyp}](t)} e^{\int_{[\alpha]} f}}.$$

By definition of  $\mu_t$  and  $\mu_{t,V}$  and the fact that  $U \subset V$ ,

$$\sum_{k \geq 1} 2^{-k} |\mu_t(g_k) - \mu_{t,V}(g_k)| \leq \nu_t(U^c).$$

It remains to show that

$$\sum_{k \geq 1} 2^{-k} |\mu_{t,V}(g_k) - \beta_t(g_k)| \leq \epsilon \nu_t(U) + \nu_t(U^c).$$

We have for all  $k \geq 1$ ,

$$|\mu_{t,V}(g_k) - \beta_t(g_k)| \leq A + B + C$$

where,

$$A = \frac{\sum_{j=1}^{N'} \sum_{([\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in U_j^\epsilon)} e^{\int_{[\alpha]} f} |\delta_{[\alpha]}(g_k) - m_j(g_k)|}{\sum_{[\alpha] \in [\Gamma_{hyp}](t)} e^{\int_{[\alpha]} f}},$$

$$B = \frac{\sum_{([\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in V \setminus U)} e^{\int_{[\alpha]} f} \delta_{[\alpha]}(g_k)}{\sum_{[\alpha] \in [\Gamma_{hyp}](t)} e^{\int_{[\alpha]} f}},$$

$$C = |(1 - \nu_t(U)) m_0(g_k)|.$$

Thus, since we have for all  $k \geq 1$ ,  $\|g_k\| = 1$ , by definition of  $\nu_t$  we get,

$$\begin{aligned} & \sum_{k \geq 1} 2^{-k} |\mu_{t,V}(g_k) - \beta_t(g_k)| \\ & \leq \epsilon \sum_{j=1}^{N'} \nu_t(U_j^\epsilon) + \nu_t(U^c) + (1 - \nu_t(U)) \\ & = \epsilon \nu_t(U) + \nu_t(U^c). \end{aligned}$$

Finally we have obtained that

$$d(\mu_t, \beta_t) \leq \epsilon \nu_t(U) + \nu_t(U^c).$$

This implies the desired inequality,

$$d(\mu_t, V) \leq \epsilon \nu_t(U) + 3\nu_t(U^c).$$

The set  $U^c$  is closed, so we have  $\lim_{t \rightarrow \infty} \nu_t(U) = 1$ . Since  $\epsilon$  is arbitrary, we conclude that  $\limsup_{t \rightarrow \infty} d(\mu_t, V) = 0$ . The neighborhood  $V$  of  $\mathcal{P}_e(f)$  being arbitrary, this implies that all limit measures of  $\mu_t$  are contained in  $\mathcal{P}_e(f)$ . In particular, if  $\mathcal{P}_e(f)$  is reduced to one measure  $\mu$ , this shows that  $\mu_t$  converges to  $\mu$ .

**2.5. Proof of Lemma 3.** We follow [19]. The functional  $\rho$  is lower semicontinuous (since  $h$  is upper semicontinuous) and  $\rho \geq 0$ . Set for any continuous function  $\omega$  on  $T^1M$ ,

$$(8) \quad Q_f(\omega) := P(f + \omega) - P(f).$$

The fact that  $Q_f$  is a convex and continuous is a consequence of the same properties for  $P$ . Using the variational principle, it is not difficult to see that

$$Q_f(\omega) = \sup_{\mu \in \mathcal{P}(\phi)} \left( \int \omega d\mu - \rho(\mu) \right).$$

By duality we have for any invariant probability measure  $m$ ,

$$\rho(m) = \sup_{\omega \text{ continuous}} \left( \int \omega dm - Q_f(\omega) \right).$$

With the notations introduced above, we have to prove that

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \nu_t(K) \leq -\rho(K).$$

Let  $\epsilon > 0$ . There exists a finite number of continuous functions  $\omega_1, \dots, \omega_l$  such that  $K \subset \cup_{i=1}^l K_i$ , where

$$K_i = \{m \in \mathcal{P}(\phi) : \int \omega_i dm - Q(\omega_i) > \rho(K) - \epsilon\}.$$

We have  $\nu_t(K) \leq \sum_{i=1}^l \nu_t(K_i)$  where

$$\nu_t(K_i) = \frac{\sum_{([\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in K_i)} e^{\int_{[\alpha]} f}}{\sum_{[\alpha] \in [\Gamma_{hyp}](t)} e^{\int_{[\alpha]} f}}.$$

Then,

$$\begin{aligned} & \sum_{[\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in K_i} e^{\int_{[\alpha]} f} \\ & \leq \sum_{[\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in K_i} e^{\int_{[\alpha]} f} e^{l([\alpha])(\int \omega_i d\delta_{[\alpha]} - Q(\omega_i) - (\rho(K) - \epsilon))}. \end{aligned}$$

Set  $C := \sum_{i \leq l} \sup(1, e^{-\delta(-Q(\omega_i) - (\rho(K) - \epsilon))})$ . Thus, by taking into account the sign of  $-Q(\omega_i) - (\rho(K) - \epsilon)$ ,

$$\begin{aligned} & \sum_{\alpha \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in K_i} e^{\int_{[\alpha]} f} \\ & \leq \sum_{[\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in K_i} e^{\int_{[\alpha]} (f + \omega_i)} e^{l([\alpha])(-Q(\omega_i) - (\rho(K) - \epsilon))} \\ & \leq C e^{t(-Q(\omega_i) - (\rho(K) - \epsilon))} \sum_{[\alpha] \in [\Gamma_{hyp}](t) : \delta_{[\alpha]} \in K_i} e^{\int_{[\alpha]} (f + \omega_i)}. \end{aligned}$$

For  $t$  sufficiently large, it follows from Theorem 1 (1),

$$\begin{aligned} \nu_t(K) & \leq C \sum_{i=1}^l e^{t(P(f + \omega_i) + \epsilon)} e^{-t(P(f) - \epsilon)} e^{t(-Q(\omega_i) - (\rho(K) - \epsilon))} \\ & = C l e^{t(-\rho(K) + 3\epsilon)}. \end{aligned}$$

Take the logarithme, divide by  $t$  and take the lim sup,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \nu_t(K) \leq -\rho(K) + 3\epsilon.$$

$\epsilon$  being arbitrary, this proves Lemma 3.

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