



**HAL**  
open science

## Measures maximizing topological pressure

Abdelhamid Amroun

► **To cite this version:**

| Abdelhamid Amroun. Measures maximizing topological pressure. 2013. hal-00827783

**HAL Id: hal-00827783**

**<https://hal.science/hal-00827783>**

Preprint submitted on 29 May 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MEASURES MAXIMIZING TOPOLOGICAL PRESSURE

AMROUN ABDELHAMID

Université Paris-Sud, Département de Mathématiques, CNRS UMR  
8628, 91405 Orsay Cedex France

ABSTRACT. We give a general method on the way of approximating equilibrium states for a dynamical system of a compact metric space.

## 1. INTRODUCTION

We show in this paper how to build equilibrium states for a dynamical system on a compact metric space  $X$  starting from the definition of the topological pressure. For each finite set  $E$  of “sufficiently separated” points, we consider the probability measure  $\mu_E$  with support  $E$ . We then prove that the weak limits of  $\mu_E$ , as  $|E| \rightarrow \infty$ , are equilibrium states. In fact,  $|E|$  increases exponentially fast and  $|E| \approx e^{th_{top}}$ , where  $h_{top}$  is the topological entropy of the dynamical system. The result works for flows as well as for maps of  $X$ . We begin with a general and abstract result which gives a sufficient condition under which the weak limits of a sequence of probability measures are contained in a given closed and convex subset of  $\mathcal{P}(X)$ , the space of probability measures of  $X$ . We apply this result to convex and lower semicontinuous functionals on  $\mathcal{P}(X)$ . It results a way of approximating zeros of such functionals. In the setting of dynamical systems, these functionals are given by the Legendre transform of the topological pressure considered as a functional on continuous potentials on  $X$ . In certain cases, the result follows from a direct application of the ergodic theorem. We give an example of such situation, by considering probability measures supported on finite trajectories of a diffusion process associated with a second order differential operators.

## 2. AN ABSTRACT MODEL AND MAIN RESULT

Let  $X$  be a compact metric space and denote by  $\mathcal{P}(X)$  the space of probability measures on  $X$  equipped with the topology of weak convergence of measures. Let  $\beta_t$ ,  $t \in \mathbb{R}$ , be a probability measure on  $X$  and

consider a random variable  $\delta_t$  on the measure space  $(X, \beta_t)$  with values in  $\mathcal{P}(X)$ . Thus, for each  $x \in X$ ,  $\delta_t(x)$  is a probability measure on  $X$ .

Consider the law  $\nu_t$  of  $\delta_t$  with respect to  $\beta_t$ , that is,  $\nu_t := \beta_t \circ \delta_t^{-1}$ . Also, the expectation of  $\delta_t$  is given by the probability measure

$$(1) \quad \mu_t := E_{\beta_t}(\delta_t) = \int_X \delta_t(x) d\beta_t.$$

We state the main result of this section.

**Theorem 1.** *Let  $(\beta_t)_{t \geq 0}$  be a family of probability measures on  $X$ , and for each  $t \geq 0$  a map  $\delta_t$  on  $(X, \beta_t)$  with values in the space  $\mathcal{P}(X)$ . Suppose there is a nonempty closed and convex subset  $\Gamma$  of  $\mathcal{P}(X)$  such that: for any open neighborhood  $U \subset \mathcal{P}(X)$  of  $\Gamma$  we have,*

$$(2) \quad \lim_{t \rightarrow +\infty} \nu_t(U) = 1.$$

*Then, any weak limit  $\mu$  of  $\mu_t := E_{\beta_t}(\delta_t)$ , as  $t \rightarrow +\infty$  is contained in  $\Gamma$ . In particular, if  $\Gamma = \{m\}$  then  $\mu_t$  converges to  $m$ .*

Note that the condition (2) in the theorem is equivalent to assume that  $\lim_{t \rightarrow +\infty} \nu_t(U^c) = 0$  since  $\nu_t(U) + \nu_t(U^c) = 1$ . This condition means that the proportion of probability measures of the type  $\delta_t$  which are close to  $\Gamma$  is asymptotically 1. On the other hand, in the case where  $\beta_t$  do not depend on  $t$ , there are non trivial examples where the convergence of  $E_{\beta}(\delta_t)$  to some measure is a consequence of the ergodic theorem (see section 4).

The applications given below are based on the following special case of the main Theorem 1. Let  $J : \mathcal{P}(X) \rightarrow [0, \infty]$  be a lower semicontinuous functional and set  $\Gamma := \{J = 0\}$ . This is a compact subset of  $\mathcal{P}(X)$  and furthermore, the convexity of  $\Gamma$  follows from the convexity of the functionl  $J$ .

In practice, to obtain (2) one can prove a stronger result, namely a large deviation upper bound for the process  $\nu_t$ : for any closed subset  $K$  of  $\mathcal{P}(X)$ ,

$$(3) \quad \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \nu_t(K) \leq - \inf_{m \in K} J(m).$$

A sufficient condition which imply this upper bound is given in [11] (condition (1.3) p506). In the notations of Theorem 1, this condition is equivalent to say that the limit

$$(4) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \log E_{\beta_t}(e^{t \int_X f d\delta_t})$$

exists for any continuous function  $f$  on  $X$ . When this limit exists, it coincides with the topological pressure  $P(f)$  introduced in the sequel.

Furthermore, the functional  $J$  in this case will be the convex conjugate of  $P$  in a certain sens (see section 3).

Let then  $U$  be an open neighborhood of  $\Gamma := \{J = 0\}$ . Since  $J$  is lower semicontinuous and the set  $U^c$  is compact, there will exist  $m_0 \in U^c$  such that  $\inf_{m \in U^c} J(m) = J(m_0) > 0$ . Thus, for  $t$  sufficiently large we will have  $\nu_t(U) \geq 1 - e^{-tJ(m_0)}$ , which implies (2). We summarize this discussion in the following result which is a corollary of Theorem 1.

**Theorem 2.** *Let  $(\beta_t)_{t \geq 0}$  be a family of probability measures on  $X$ , and for each  $t \geq 0$  a map  $\delta_t$  on  $(X, \beta_t)$  with values in the space  $\mathcal{P}(X)$ . Suppose there exists a lower semicontinuous convex functional  $J : \mathcal{P}(X) \rightarrow [0, \infty]$  such that, for any open neighborhood  $U$  of  $\{J = 0\}$  we have,*

$$(5) \quad \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \nu_t(U^c) \leq - \inf_{m \in U^c} J(m).$$

*Then,  $J(\mu) = 0$  for any weak limit  $\mu$  of  $\mu_t := E_{\beta_t}(\delta_t)$ , as  $t \rightarrow +\infty$ .*

### 2.1. Proof of the main result.

*Proof.* We endow  $\mathcal{P}(X)$  with a distance  $d$  compatible with this topology: take a countable base  $\{g_1, g_2, \dots\}$  of the separable space  $C_{\mathbb{R}}(X)$ , where  $\|g_k\| = 1$  for all  $k$ , and set:

$$d(m, m') := \sum_{k=1}^{\infty} 2^{-k} \left| \int g_k dm - \int g_k dm' \right|.$$

Let  $V \subset \mathcal{P}(X)$  be a convex open neighborhood of  $\Gamma$  and  $\gamma > 0$ . Consider a finite open cover  $(B_i(\gamma))_{i \leq N}$  of  $\Gamma$  by balls of diameter  $\gamma$  all contained in  $V$  (if  $B_i(\gamma)$  is not entirely contained in  $V$  we just take its restriction to  $V$ ).

Decompose the set  $U := \cup_{i=1}^N B_i(\gamma)$  as follows,

$$U = \cup_{j=1}^{N'} U_j^\gamma, \text{ for some } N' \geq N,$$

where the sets  $U_j^\gamma$  (which are not necessarily open) are disjoint and contained in one of the balls  $(B_i(\gamma))_{i \leq N}$ . We have

$$\Gamma \subset U \subset V,$$

and  $\sum_{j=1}^{N'} \nu_n(U_j^\gamma) = \nu_n(U)$ . We fix in each  $U_j^\gamma$  a probability measure  $p_j$ ,  $j \leq N'$ , and let  $p_0$  be a probability measure distinct from the above ones (for example take  $p_0 \in V \setminus U$ ). Set,

$$\omega_t := \sum_{j=1}^{N'} p_j 1_{\delta_t^{-1}(U_j^\gamma)} + (1 - \nu_t(U)) p_0.$$

We have,

$$(6) \quad E_{\beta_t}(\omega_t) = \sum_{j=1}^{N'} \nu_t(U_j^\gamma) p_j + (1 - \nu_t(U)) p_0.$$

The probability measure  $E_{\beta_t}(\omega_t)$  are contained in  $V$  since it is a convex combination of elements of the convex set  $V$ . We have then  $d(\mu_t, V) \leq d(\mu_t, E_{\beta_t}(\omega_t))$ . We will show that

$$d(\mu_t, E_{\beta_t}(\omega_t)) \leq \gamma \nu_t(U) + 3\nu_t(U^c),$$

where  $U^c = \mathcal{P}(X) \setminus U$ .

Consider the finite positive measures,

$$\mu_{t,V} := E_{\beta_t}((1_V \circ \delta_t)\delta_t).$$

Evaluated on a function  $g$  this gives,

$$\mu_{t,V}(g) := E_{\beta_t}((1_V \circ \delta_t)\delta_t(g)) = \int_X 1_V(\delta_t(x)) \left( \int_X g(y) d\delta_t(x)(y) \right) d\beta_t(x).$$

Note that for any continuous function  $g$  on  $X$  with  $\|g\| = 1$  and any subset  $E$  of  $\mathcal{P}(X)$  we have

$$(7) \quad |\mu_{t,E}(g)| \leq E_{\beta_t}(1_E \circ \delta_t) = \nu_t(E).$$

By the definition of  $d$  we have to estimate:

$$d(\mu_t, E_{\beta_t}(\omega_t)) = \sum_{k=1}^{\infty} 2^{-k} |\mu_t(g_k) - E_{\beta_t}(\omega_t)(g_k)|.$$

First write,

$$\begin{aligned} & d(\mu_t, E_{\beta_t}(\omega_t)) \\ & \leq \sum_{k \geq 1} 2^{-k} |\mu_t(g_k) - \mu_{t,V}(g_k)| + \sum_{k \geq 1} 2^{-k} |\mu_{t,V}(g_k) - E_{\beta_t}(\omega_t)(g_k)|. \end{aligned}$$

From (6) – (7), the definition of  $\mu_t$  and  $\mu_{t,V}$  and the fact that  $U \subset V$ , we get

$$(8) \quad \sum_{k \geq 1} 2^{-k} |\mu_t(g_k) - \mu_{t,V}(g_k)| = \sum_{k \geq 1} 2^{-k} |\mu_{t,V^c}(g_k)| \leq \nu_t(U^c).$$

It remains to show that

$$\sum_{k \geq 1} 2^{-k} |\mu_{t,V}(g_k) - E_{\beta_t}(\omega_t)(g_k)| \leq \gamma \nu_t(U) + \nu_t(U^c).$$

We have

$$\sum_{k \geq 1} 2^{-k} |\mu_{t,V}(g_k) - E_{\beta_t}(\omega_t)(g_k)| = \sum_{k \geq 1} 2^{-k} |A - B|$$

where,

$$A := \sum_{j=1}^{N'} \left( E_{\beta_t} \left( (1_{U_j^\gamma} \circ \delta_t) \delta_t \right) (g_k) - \nu_t(U_j^\gamma) p_j(g_k) \right), \text{ and}$$

$$B := E_{\beta_t} \left( (1_{V \setminus U} \circ \delta_t) \delta_t \right) (g_k) + (1 - \nu_t(U)) p_0(g_k).$$

We have for all  $k \geq 1$ ,

$$(9) \quad \sum_{k \geq 1} 2^{-k} A \leq \gamma \sum_{j=1}^{N'} \nu_t(U_j^\gamma) = \gamma \nu_t(U).$$

and

$$(10) \quad \sum_{k \geq 1} 2^{-k} B \leq \nu_t(V \setminus U) + (1 - \nu_t(U)) \leq 2\nu_t(U^c).$$

Putting together (8) – (10) we get,

$$d(\mu_t, E_{\beta_t}(\omega_t)) \leq \gamma \nu_t(U) + 3\nu_t(U^c).$$

This implies the desired inequality,

$$d(\mu_t, V) \leq \gamma \nu_t(U) + 3\nu_t(U^c).$$

Now, since  $U$  is open, we have by assumption,  $\lim_{t \rightarrow \infty} \nu_t(U) = 1$  and  $\lim_{t \rightarrow \infty} \nu_t(U^c) = 0$ . Thus,  $\limsup_{t \rightarrow \infty} d(\mu_t, V) \leq \gamma$ , for all  $\gamma > 0$ . We conclude that  $\limsup_{t \rightarrow \infty} d(\mu_t, V) = 0$ . The neighborhood  $V$  of  $\Gamma$  being arbitrary, this implies that the all limit measures of  $\mu_t$  are contained in  $\Gamma$ .  $\square$

**Remark 1.** *It is not necessary for  $V$  to be convex. Indeed, let  $V$  be an open neighborhood of  $\Gamma$ , and consider finite open cover  $(B_i(\gamma))_{i \leq N}$  of  $\Gamma$  by balls of diameter  $\gamma$  all contained in  $V$  where each ball  $B_i(\gamma)$  is centred at a measure  $m_i \in \Gamma$ . Fix also a measures  $m_0 \in \Gamma$  distinct from the above ones (since  $\Gamma$  is convex, if it contains two elements it contains the segment formed by them). Now define  $E_{\beta_t}(\delta_t)$  as in (6). The difference here resides in the fact that each  $m_i$  can appear more than one time in (6). However,  $E_{\beta_t}(\delta_t)$  is still a convex combination of elements in  $\Gamma$ , so that  $E_{\beta_t}(\delta_t) \in \Gamma$ .*

### 3. APPLICATION TO DYNAMICAL SYSTEMS

Let us explain how to apply the above results in the context of dynamical systems. Leaving the details and the precise statements for later, we begin by a general description of the method. Let  $X$  be a compact metric space and  $\phi : \mathbb{R} \times X \rightarrow X$  a continuous (smooth) flow

acting on it. The same can be done for maps of  $X$ . Set  $\phi^t := \phi(\cdot, t) : X \rightarrow X$ . The functional  $J$  has then the following form,

$$(11) \quad J(\mu) = \sup_{\omega \in C_{\mathbb{R}}(X)} \left( \int \omega d\mu - Q(\omega) \right),$$

where  $Q : C_{\mathbb{R}}(X) \rightarrow \mathbb{R}$  is continuous and convex,  $\phi$ -invariant i.e,  $Q(\omega \circ \phi^t) = Q(\omega)$  for  $t \in \mathbb{R}$ , and  $Q(0) = 0$ . Thus  $J \geq 0$  and  $J$  is lower semicontinuous. Since for any  $\omega$ , the map  $\mu \rightarrow (\int \omega d\mu - Q(\omega))$  is convex, we deduce by duality that,

$$(12) \quad Q(\omega) = \sup_{\mu \in \mathcal{P}(X)} \left( \int \omega d\mu - J(\mu) \right).$$

We are interested in the subset  $\mathcal{P}_0(X)$  of  $\mathcal{P}(X)$  of probability measures  $\mu$  such that  $J(\mu) = 0$ ,

$$(13) \quad \mathcal{P}_0(X) := \{\mu \in \mathcal{P}(X) : J(\mu) = 0\}.$$

Elements of  $\mathcal{P}_0(X)$  which are invariant by  $A$ , are called equilibrium states of the dynamical system. Theorem 2 in this case says that, if we have the upper bound (5), then  $\mu \in \mathcal{P}_0(X)$  for any weak limit  $\mu$  of  $\mu_t = E_{\beta_t}(\delta_t)$ .

The objective in what follows is to specify all the ingredients of Theorem 2 and to establish (5).

A probability measure  $\mu$  on  $X$  is said to be invariant by  $\phi$ , if for all continuous function  $\omega$  and all  $t \in \mathbb{R}$  we have  $\int_X \omega(\phi^t(x)) d\mu = \int_X \omega(x) d\mu$ . Let  $\mathcal{P}_{inv}(X)$  be the closed and convex subspace of  $\mathcal{P}(X)$  which are invariant by the flow  $\phi$ .

**3.1. Topological pressure and the functional  $J$ .** The topological pressure  $P(f)$  of a continuous function  $f : X \rightarrow X$  is defined using separated sets of  $X$  (see [17]) as follows:

$$(14) \quad P(f) = \lim_{\epsilon \rightarrow 0} \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \sup_E \sum_{x \in E} e^{\int_0^t f(\phi^s x) ds},$$

where the sup is taken over the  $(t, \epsilon)$ -separated sets  $E \subset X$ . Since  $X$  is a compact, in formula (14) we can take the sup over maximal separated sets [17]. There is also an equivalent definition based on spanning sets [17].

It is well known that  $P(f)$  satisfies the following variational principle [17],

$$(15) \quad P(f) = \sup_{\mu \in \mathcal{P}_{inv}(X)} \left( h(\mu) + \int f d\mu \right).$$

For  $f = 0$  formula (12) gives the usual variational principle for the topological entropy  $h_{top}$  of the flow  $\phi$ ,

$$(16) \quad h_{top} = \sup_{\mu \in \mathcal{P}_{inv}(X)} h(\mu).$$

Let  $\mathcal{P}_e(f)$  be the subset of measures  $\mu \in \mathcal{P}_{inv}(X)$  realizing the sup in (15). These maximizing measures are called equilibrium states corresponding to the potential  $f$ .

We assume in the sequel that the entropy map  $m \rightarrow h(m)$  is upper semicontinuous (u.s.c). Then  $h_{top} < +\infty$  and  $\mathcal{P}_e(f)$  is a nonempty closed and convex subset of  $\mathcal{P}_{inv}(X)$  (see [17]). For example, by an important result of Newhouse [15], if  $X$  is a compact manifold equipped with a  $C^\infty$  Riemannian metric then the entropy map is u.s.c. In this context, the main example of flow we have in mind is the geodesic flow [1].

Define the functional  $Q_f$  by,

$$(17) \quad Q_f(\omega) := P(f + \omega) - P(f), \text{ for } \omega \in C_X(\mathbb{R}).$$

The functional  $J_f$  is then defined on the space of probability measures on  $X$  by,

$$(18) \quad J_f(\mu) := \sup_{\omega} \left( \int \omega d\mu - Q_f(\omega) \right),$$

where the sup is over the continuous functions  $\omega$  on  $X$ . Since  $Q_f(0) = 0$ , we have  $J_f \geq 0$ . In [1] (Lemma 1,2) we proved,

**Lemma 1.**

- (1)  $Q_f$  is  $\phi$ -invariant, that is  $Q_f(\omega \circ \phi_t) = Q_f(\omega)$  for all continuous function  $\omega$  and  $t \in \mathbb{R}$ . Moreover,  $Q_f$  is convex and continuous on continuous functions.  $J_f$  is convex and lower semicontinuous functional and by duality,

$$Q_f(\omega) = \sup_{\mu \in \mathcal{P}(X)} \left( \int \omega d\mu - J_f(\mu) \right).$$

- (2) Set for any invariant probability measure  $\mu$ ,

$$I(\mu) := P(f) - \left( h(\mu) + \int_X f d\mu \right).$$

Then  $Q_f(\omega) = \sup_{\mu \in \mathcal{P}_{inv}(X)} \left( \int \omega d\mu - I(\mu) \right)$ . In other words, the functionals  $I$  and  $J_f$  agree on invariant measures.

- (3) We have,

$$\mathcal{P}_e(f) = \{J_f = 0\} \cap \mathcal{P}_{inv}(X).$$



- (4) *The uniqueness of the solution of the equation  $J_f = 0$  in  $\mathcal{P}(X)$  is equivalent to the uniqueness of the equilibrium state corresponding to  $f$ .*

Note that a probability measure  $m$  for which we have  $J_f(m) = 0$  is not necessarily invariant, however the set  $\{J_f = 0\}$  is invariant by the flow. Given  $x \in X$  and  $t > 0$ , we define a probability measure  $\delta_t(x)$  on  $X$  by:

$$(19) \quad \int \omega d\delta_t(x) := \frac{1}{t} \int_0^t \omega(\phi^s x) ds.$$

The main result of this section is,

**Theorem 3.** *For any real continuous function  $f$  on  $X$  we have:*

- (1) *Let  $\gamma > 0$ . There exists  $\epsilon_0 > 0$  such that for all  $\epsilon < \epsilon_0$ , there exists a sequence  $(E_n, t_n)_{n \geq 0}$  where  $E_n$  is a  $(t_n, \epsilon)$ -separated set in  $X$  and  $t_n \geq n$  such that,*

$$\sum_{x \in E_n} e^{\int_0^{t_n} f(\phi^s x) ds} \geq e^{t_n(P(f) - \gamma)}, \text{ for all } n \geq 0.$$

- (2) *Let  $\mu_n$  be the probability measures defined by,*

$$\mu_n := \mu_{t_n} = \frac{\sum_{x \in E_n} e^{\int_0^{t_n} f(\phi^s x) ds} \delta_{t_n}(x)}{\sum_{x \in E_n} e^{\int_0^{t_n} f(\phi^s x) ds}},$$

*where  $(t_n, E_n)$  is the sequence determined in (1). Then any weak limit  $\mu$  of  $\mu_n$ , as  $n \rightarrow \infty$  is invariant by the flow and satisfies  $J_f(\mu) = 0$ . The measure  $\mu$  is then an equilibrium state.*

As the proof will show, the conclusion in part (2) of Theorem 3 is true for any sequence  $E_n$  of  $(t_n, \epsilon)$ -separated sets in  $X$  for which Theorem 3 (1) holds. We describe now examples of such situations.

Consider a smooth compact Riemannian manifold  $M$  with negative curvature. The set of primitive closed geodesic (which represent different free homotopy class) of period  $\leq t$  is  $(\epsilon, t)$ -separated whenever  $\epsilon < \text{inj}(M)$ , where  $\text{inj}(M)$  is the injectivity radius of  $M$ . Moreover, condition (1) of Theorem 3 holds here since the exponential growth rate of these geodesics is given by the topological pressure [4] [3] [5] [14],

$$P(f) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \sum_{c: l(c) \leq t} e^{\int_c f},$$

where the sum is over closed geodesics (primitive) of length at most  $t$ . The precise statement is as follows.

**Corollary 1.** *Let  $M$  be a compact manifold equipped with a  $C^\infty$  Riemannian metric of negative curvature. Let  $\phi^t : X \rightarrow X$  be the geodesic flow acting on the unit tangent bundle  $X = T^1M$  of  $M$ . Then, for any Hölder continuous function  $f : X \rightarrow \mathbb{R}$ , the probability measures defined by*

$$\mu_t(\omega) := \frac{\sum_{c: l(c) \leq t} e^{\int_c f} \left( \frac{1}{l(c)} \int_c \omega \right)}{\sum_{c: l(c) \leq t} e^{\int_c f}},$$

converge to the unique equilibrium state of the geodesic flow corresponding to  $f$ .

### 3.2. Proof of Theorem 3.

#### 3.2.1. Proof of Part (1).

*Proof.* The proof is a consequence of the definition (1). Indeed, let  $\eta > 0$  be fixed. There exists  $\epsilon_0 > 0$  such that for all  $0 < \epsilon < \epsilon_0$ :

$$\inf_{n \geq 0} \sup_{t \geq n} \frac{1}{t} \log \sup_{E: (t, \epsilon)\text{-separated}} \sum_{x \in E} e^{\int_0^t f(\phi^s x) ds} \geq P(f) - \eta.$$

From this we deduce that for each  $n \geq 0$  there exists  $t_n \geq n$  and an  $(t_n, \epsilon)$ -separated set  $E_n$  such that,

$$\sum_{x \in E_n} e^{\int_0^{t_n} f(\phi^s x) ds} \geq e^{t_n(P(f) - \eta)}.$$

□

#### 3.2.2. Proof of Part (2).

*Proof.* Define the atomic probability measures with support in  $E_n$ :

$$(20) \quad \beta_n(B) = \beta_{t_n}(B) := \frac{\sum_{x \in B \cap E_n} e^{\int_0^{t_n} f(\phi^s x) ds}}{\sum_{x \in E_n} e^{\int_0^{t_n} f(\phi^s x) ds}}.$$

Then, the measure  $\delta_{t_n}$  defined in (19), can be seen as random variable on the measure space  $(X, \beta_n)$  and the measures  $\mu_n$  in Theorem 3 as the expectation  $\mu_n = E_{\beta_n}(\delta_{t_n})$ . Consider the image  $\nu_n$  of  $\beta_n$  under the map  $\delta_{t_n}$ :

$$(21) \quad \nu_n := \beta_n \circ \delta_{t_n}^{-1}.$$

To prove part (2) we have to check the upper bound in Theorem 2 for  $\{\nu_n\}$  namely, for any closed subset  $K$  of  $\mathcal{P}(X)$ ,

$$\limsup_{n \rightarrow +\infty} \frac{1}{t_n} \log \nu_n(K) \leq -J_f(K) := - \inf_{m \in K} J_f(m).$$

We follow [16], [11] and [1]. Let  $\eta > 0$ . Observe that the compact set  $K$  is contained the union of open sets,

$$K \subset \cup_{\omega} \{ \mu \in \mathcal{P}(X) : \int \omega dm - Q_f(\omega) > J_f(K) - \eta \}.$$

There exists then a finite number of continuous functions  $\omega_1, \dots, \omega_l$  such that  $K \subset \cup_{i=1}^l K_i$ , where

$$K_i = \{ m \in \mathcal{P}(X) : \int \omega_i dm - Q_f(\omega_i) > J_f(K) - \eta \}.$$

We have  $\nu_n(K) \leq \sum_{i=0}^l \nu_n(K_i)$  where,

$$(22) \quad \nu_n(K_i) = \frac{\sum_{x \in E_n : \delta_{t_n}(x) \in K_i} e^{\int_0^{t_n} f(\phi^s x) ds}}{\sum_{x \in E_n} e^{\int_0^{t_n} f(\phi^s x) ds}}.$$

For  $n$  sufficiently large we have,

$$\begin{aligned} & \sum_{x \in E_n : \delta_{t_n}(x) \in K_i} e^{\int_0^{t_n} f(\phi^s x) ds} \\ \leq & \sum_{x \in E_n : \delta_{t_n}(x) \in K_i} e^{\int_0^{t_n} f(\phi^s x) ds} e^{t_n(\int \omega_i d\delta_{t_n}(x) - Q_f(\omega_i) - (J_f(K) - \eta))} \\ \leq & e^{t_n(-Q_f(\omega_i) - (J_f(K) - \eta))} \sum_{x \in E_n} e^{\int_0^{t_n} (f + \omega_i)(\phi^s x) ds}. \end{aligned}$$

By the definition of the pressure and the fact that  $t_n \geq n$  we also have,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{t_n} \log \sum_{x \in E_n} e^{\int_0^{t_n} (f + \omega_i)(\phi^s x) ds} \\ \leq & \limsup_{n \rightarrow \infty} \frac{1}{t_n} \log \sup_{E:(t_n, \epsilon)\text{-separated}} \sum_{x \in E} e^{\int_0^{t_n} (f + \omega_i)(\phi^s x) ds} \\ \leq & \limsup_{t \rightarrow \infty} \frac{1}{t} \log \sup_{E:(t, \epsilon)\text{-separated}} \sum_{x \in E} e^{\int_0^t (f + \omega_i)(\phi^s x) ds} \\ = & P(f + \omega_i). \end{aligned}$$

Thus, for  $n$  sufficiently large we obtain,

$$\sum_{x \in E_n} e^{\int_0^{t_n} (f + \omega_i)(\phi^s x) ds} \leq e^{t_n(P(f + \omega_i) + \eta)}.$$

Consequently we obtain for  $\nu_n(K)$ ,

$$\begin{aligned} \nu_n(K) &\leq \sum_{i=1}^l \nu_n(K_i) \\ &\leq \sum_{i=1}^l \left( e^{t_n(-P(f)+\eta-Q_f(\omega_i)-(J_f(K)-\epsilon))} \sum_{x \in E_n} e^{\int_0^{t_n} (f+\omega_i)(\phi^s x) ds} \right) \\ &\leq \sum_{i=1}^l \left( e^{t_n(-P(f)+\eta-Q_f(\omega_i)-(J_f(K)-\eta))} e^{t_n(P(f+\omega_i)+\eta)} \right). \end{aligned}$$

But by (17) we have  $P(f + \omega_i) - P(f) = Q_f(\omega_i)$ . We deduce finally that,

$$\nu_n(K) \leq e^{t_n(-J_f(K)+2\eta)}$$

Take the logarithme, divide by  $t_n$  and the lim sup,

$$\limsup_{n \rightarrow \infty} \frac{1}{t_n} \log \nu_n(K) \leq -J_f(K) + 2\eta.$$

$\eta$  being arbitrary, this proves part (2).  $\square$

#### 4. SECOND ORDER DIFFERENTIAL OPERATORS

The main futers of this section is to give an example of what happend if the measures  $\beta_t$  do not depend on  $t$  (Theorem 1). As we will see, the convergence of  $\mu_t$  towards the unique equilibrium state is a consequence of the ergodic theorem. The materials of this section are taken from [11] and [12].

Let  $M$  be a locally compact finite dimensional manifold  $M$  equipped with a  $C^2$  Riemannian metric and  $G$  a connected open subset of  $M$  with smooth boundary  $\partial G$  such that  $\overline{G} = G \cup \partial G$ . Let  $L$  be a second order elliptic differential operator with  $C^2$  coefficients on  $M$  if  $G = M$  (i.e  $M$  compact) or in some neighborhood of  $\overline{G}$  if  $G \neq M$  and  $L1 = 0$ . This last condition means that  $L$  has locally the form

$$L = \sum_{ij} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b^i(x) \frac{\partial}{\partial x_i}.$$

We denote by  $\mathcal{P}(\overline{G})$  the space of probability measures on  $\overline{G}$ . For any continuous function  $V$  on  $M$  define the operator

$$L_V := L + V.$$

Let  $\Sigma(V)$  be the spectrum of  $L_V$  corresponding to the Dirichlet boundary conditions on  $\partial G$  (with no boundary conditions if  $G = M$  is compact). Consider,

$$\lambda_V := \sup\{Re(\lambda) : \lambda \in \Sigma(V)\}.$$

Then  $\lambda_V$  is a spectral value for  $L_V$  i.e  $\lambda_V \in \Sigma(V)$  and by [6, 7] one has the representation

$$(23) \quad \lambda_V = \sup_{\mu \in \mathcal{P}(\overline{G})} \left( \int V d\mu - I(\mu) \right)$$

where  $I(\mu)$  is the entropy of the measure  $\mu$  given by

$$(24) \quad I(\mu) = - \sup_{u \in D_+(L)} \int \frac{Lu}{u} d\mu$$

and  $D_+(L)$  is the set of functions from the domain of  $L$  having positive lower bounds.

For Hölder continuous function  $V$  it is well known [13] that  $\lambda_V$  is an eigenvalue of  $L_V$ , in fact it is the leading eigenvalue. The corresponding eigenspace is one dimensional and the corresponding eigenfunction  $r_V$  is positive. Moreover [12], there exists a unique probability measure  $\mu_V$  on  $\overline{G}$  solving the variational principle (23),

$$(25) \quad \lambda_V = \int V d\mu_V - I(\mu_V).$$

Let  $X_V(t)$  be the diffusion process with values in  $M$  and generator  $L_V$  [9], defined on a probability space  $(\Omega, \mathcal{F}, P)$ . Consider the semigroup of operators (Perron-Frobenius operators) acting on  $C(\overline{G})$

$$(26) \quad T_V(t)g(x) := E_x \left( 1_{\tau_G > t} g(X_V(t)) e^{\int_0^t V(X_V(s)) ds} \right)$$

where  $E_x$  denotes the expectation for  $X_V(0) = x$  and

$$\tau_G := \inf\{t \geq 0 : X_V(t) \notin G\}.$$

Set  $P_x(A) = E_x(1_A)$ .

If  $G = M$  is compact just set  $\tau_G = \infty$ . In case of Hölder continuous function  $V$  then  $e^{\lambda_V}$  is an eigenvalue of  $T_V(1)$  and in fact its leading eigenvalue and we have

$$(27) \quad \lambda_V = \lim_{t \rightarrow +\infty} \frac{1}{t} \log (T_V(1)), \text{ where } 1(x) = 1.$$

Furthermore, the measure  $\mu_V$  maximizes (20) if and only if it is an invariant measure for the Markov process with transition operators

$$\mathcal{T}_V(t)g(x) := e^{-\lambda_V} r_V(x)^{-1} T_V(t)(g r_V)(x).$$

Then for any continuous function  $u$  on  $\overline{G}$

$$\int_{\overline{G}} \mathcal{T}_V(t)u(x)d\mu_V(x) = \int_{\overline{G}} u(x)d\mu_V(x).$$

Thus for Hölder continuous potentials  $V$ , the maximizing measure  $\mu_V$  is the the only invariant measure for the Markov process above.  $\mu_V$  is then ergodic.

Consider now the random probability measures supported on the finite trajectories of the diffusion  $X_V(t)$

$$(28) \quad \delta_t := \frac{1}{t} \int_0^t \delta_{X_V(s)} ds$$

where  $\delta_{X_V(s)}$  is the Dirac measure at  $X_V(s)$ . By the ergodic theorem, for  $\mu_V$  a.e  $x \in M$ , and  $P_x$  a.e  $\omega \in \Omega$ , the probability measures  $\delta_t$  converge weakly towards  $\mu_V$ . Thus, it is easy to see that the measures  $E_x(\delta_t) = \int \delta_t dP_x$  converge towards  $\mu_V$ .

Now it follows from [11] that the measures  $\nu_t := P_x \circ \delta_t^{-1}$  satisfy a large deviation principle in particular the following upper bound : for any  $x \in G$  and any closed subset  $K \in \mathcal{P}(\overline{G})$  we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x(\delta_t \in K) \leq - \inf_{m \in K} J(m)$$

where  $J(m) = J_V(m) := I(m) - \int V dm + \lambda_V$ . Note that for  $V \equiv 0$  this is just the conclusion of Theorem 4.1 from [11]. The functional  $J$  is non negative and  $J(m) = 0$  if and only if  $m = \mu_V$  (recall that  $V$  is Hölder continuous). Since  $\mu_V$  is unique, Theorem 2 tell us that  $E_x(\delta_t) = \int \delta_t dP_x$  converge towards  $\mu_V$ , which is the statement proved above.

## REFERENCES

- [1] Amroun A. Equidistribution results for geodesic flows. To appear in Ergod. Th. Dynam. Sys (2013).
- [2] Bowen R. Maximizing entropy for hyperbolic flow. Math Syst Theory 7 (1973), 300-303.
- [3] Bowen R. The equidistribution of closed geodesics. Amer J Math 94, (1972) 413-423.
- [4] Bowen R. Periodic orbits for hyperbolic flows. Amer J Math 94, (1972) 1-30.
- [5] Bowen R, D Ruelle. The ergodic theory for Axiom A flows. Invent Math 29, (1975), 181-202.
- [6] Donsker M D and Varadhan S R S. On a variational formula for the principal eigenvalue for operators with maximum principle. Proc. Natl. Acad. Sci. USA 72 (1975), 780-783.
- [7] Donsker M D and Varadhan S R S. On the principal eigenvalue of second-order elliptic differential operators. Commun. Pure Appl. Math 29, (1976), 595-621.

- [8] Franco E. Flows with unique equilibrium state. *Am J Math.* 99, 486-514 (1977).
- [9] Friedman A. *Stochastic differential equations and applications.* New York, Academic Press 1975.
- [10] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems.* Modern theory of dynamical systems, *Encyclopedia of mathematics and its applications*, Cambridge, Cambridge university press 1995.
- [11] Kifer Y. Large deviations in dynamical systems stochastic processes. *Trans. Amer. Math. Soc.* 321, 505-525, 1988.
- [12] Kifer Y. Principle eigenvalues, topological pressure, and stochastic stability of equilibrium states. *Israel. J. Math* (1990), vol 70, N1, 1-47.
- [13] Krasnoseleskii M A. *Positive solutions of operators equations.* Noordhoff, Groningen, 1964.
- [14] Ledrappier F. Structure au bord des variétés à courbure négative. *Séminaire de théorie spectrale et géométrie.* Grenoble, 1994-1995, 92-122.
- [15] Newhouse S. Continuity properties of entropy. *Ann. Math.* 129, 215-235, 1989.
- [16] Pollicott M. Closed geodesic distribution for manifolds of non-positive curvature. *Disc And Cont Dyn Sys.* Vol 2, Number 2 (1996) 153-151.
- [17] Walters P. *An introduction to ergodic theory* G.T.M. 79 Springer Berlin 1982.