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Acceleration Waves in Micropolar Elastic Media

V. A. Eremeyev

For nonlinearly elastic micropolar media, a condition of the existence of weak discontinuous solutions of equations of motion has been obtained. For these solutions, which constitute acceleration waves, the continuity of the second derivatives of displacement and microrotation fields is broken on certain singular surfaces. In the framework of the micropolar-medium model (Cosserat continuum), each particle has the degrees of freedom of an absolutely rigid body, their rotation interaction may be taken into account, and couple stresses exist in addition to standard stresses. The Cosserat model is used to describe granulated, powder-like, and loose media, as well as polycrystalline bodies, composites, and nanostructures. It is also applied to develop non-classical models of thin-walled constructions: bars, plates, and shells.

An analogue of the Fresnel–Hadamard–Duhem theorem on the existence of acoustic axes and real acoustic numbers has been proved. It has been shown that the condition of the existence of an acceleration wave is equivalent to the requirement of a strong ellipticity of equilibrium equations, as in the case of simple materials that are free of couple stresses.

BASIC RELATIONS

The equations of motion of the micropolar medium (Cosserat continuum), which present the balance of the momentum and angular momentum for an arbitrary body part, have the form [1–11]

$$\begin{aligned} \operatorname{div} \mathbf{T} + \rho \mathbf{f} &= \rho \frac{d\mathbf{v}}{dt}, \\ \operatorname{div} \mathbf{M} + (\mathbf{F}^T \cdot \mathbf{T})_{\times} + \rho \mathbf{m} &= \rho \gamma \frac{d\boldsymbol{\omega}}{dt}. \end{aligned} \quad (1)$$

Here, \mathbf{T} and \mathbf{M} are the Piola tensors of stresses and couple stresses, respectively; $\mathbf{F} = \operatorname{grad} \mathbf{R}$ is the strain gradient; div and grad are, respectively, the divergence and gradient operators in the Lagrangian coordinates; ρ is

the medium density in the reference configuration; \mathbf{f} and \mathbf{m} are the mass force and mass moment, respectively; $\rho \gamma$ is the scalar measure of the rotation inertia of continuum particles; $\mathbf{v} = \frac{d\mathbf{R}}{dt}$ is the particle velocity; the radius-vector $\mathbf{R}(t)$ specifies the position of particles of the micropolar medium at time t ; the orthogonal microrotation tensor $\mathbf{H}(t)$ determines its orientation; $\boldsymbol{\omega} = \left(\frac{d\mathbf{H}}{dt} \cdot \mathbf{H}^T \right)_{\times}$ is the angular velocity; and \mathbf{T}_{\times} represents the vector invariant of the second-rank tensor \mathbf{T} [12].

The constitutive equations of the micropolar elastic medium can be written in terms of the specific (per unit volume in the reference configuration) potential energy of deformation $W = W(\mathbf{E}, \mathbf{K})$ as [10]

$$\begin{aligned} \mathbf{T} &= W_{,\mathbf{E}} \cdot \mathbf{H}, \quad \mathbf{M} = W_{,\mathbf{K}} \cdot \mathbf{H}, \\ \mathbf{E} &= \mathbf{F} \cdot \mathbf{H}^T, \quad \mathbf{K} \times \mathbf{I} = -(\operatorname{grad} \mathbf{H}) \cdot \mathbf{H}^T, \end{aligned} \quad (2)$$

where \mathbf{E} and \mathbf{K} are the measures of the metric and bending deformations, respectively; \mathbf{I} is the unit tensor; W is assumed to be twice continuously differentiable; and

$$W_{,\mathbf{E}} = \frac{\partial W}{\partial \mathbf{E}}, \quad W_{,\mathbf{K}} = \frac{\partial W}{\partial \mathbf{K}}.$$

PROPAGATION OF WEAK DISCONTINUITIES (ACCELERATION WAVES)

We consider a medium motion that may be accompanied by a break in the continuity of kinematic and dynamic quantities on a certain smooth surface $S(t)$, which is called *singular*. We assume that the limiting values of these quantities exist on S and that they are generally different on the opposite sides of S . The jump of Ψ on S is denoted as

$$[\Psi] = \Psi^+ - \Psi^-.$$

The acceleration wave (weak-discontinuity wave or second-order singular surface) is a moving singular surface on which the second derivatives (with respect to the spatial coordinates and time) of the radius-vector \mathbf{R} and microrotation tensor \mathbf{H} are discontinuous, while the

quantities themselves and their first derivatives are continuous; i.e., the equalities

$$\llbracket \mathbf{F} \rrbracket = \mathbf{0}, \quad \llbracket \text{grad} \mathbf{H} \rrbracket = \mathbf{0}, \quad \llbracket \mathbf{v} \rrbracket = \mathbf{0}, \quad \llbracket \boldsymbol{\omega} \rrbracket = \mathbf{0} \quad (3)$$

are valid on S . According to Eqs. (3), deformation measures \mathbf{E} and \mathbf{K} are continuous near S , and, in view of constitutive equations (2), jumps of the tensors \mathbf{T} and \mathbf{M} are absent.

The application of the Maxwell theorem [13] to continuous fields of velocities \mathbf{v} and $\boldsymbol{\omega}$, stresses, and couple stresses \mathbf{T} and \mathbf{M} yields a system of equations that relate the jumps of their derivatives with respect to the spatial coordinates and time:

$$\begin{aligned} \llbracket \dot{\mathbf{v}} \rrbracket &= -V\mathbf{a}, \quad \llbracket \text{grad} \dot{\mathbf{v}} \rrbracket = \mathbf{n} \otimes \mathbf{a}, \quad \llbracket \dot{\boldsymbol{\omega}} \rrbracket = -V\mathbf{b}, \\ \llbracket \text{grad} \dot{\boldsymbol{\omega}} \rrbracket &= \mathbf{n} \otimes \mathbf{b}, \end{aligned} \quad (4)$$

$$V\llbracket \text{div} \mathbf{T} \rrbracket = -\mathbf{n} \cdot \llbracket \dot{\mathbf{T}} \rrbracket, \quad V\llbracket \text{div} \mathbf{M} \rrbracket = -\mathbf{n} \cdot \llbracket \dot{\mathbf{M}} \rrbracket.$$

Here, \mathbf{a} and \mathbf{b} are the vector amplitudes for the jumps of the linear and angular accelerations, \mathbf{n} is the unit normal vector to S , and V is the velocity of the surface S in the direction \mathbf{n} [13]. If mass forces and moments are continuous, the relations

$$\llbracket \text{div} \mathbf{T} \rrbracket = \rho \llbracket \dot{\mathbf{v}} \rrbracket, \quad \llbracket \text{div} \mathbf{M} \rrbracket = \rho \gamma \llbracket \dot{\boldsymbol{\omega}} \rrbracket$$

follow from the equations of motion (1). Differentiating constitutive equations (2) and using Eqs. (4), we express these relations only in terms of the vector amplitudes \mathbf{a} and \mathbf{b} :

$$\begin{aligned} \mathbf{n} \cdot W_{,EE} \cdot (\mathbf{H} \cdot \mathbf{a} \otimes \mathbf{n}) + \mathbf{n} \cdot W_{,EK} \cdot (\mathbf{H} \cdot \mathbf{b} \otimes \mathbf{n}) \\ = \rho V^2 \mathbf{a} \cdot \mathbf{H}^T, \\ \mathbf{n} \cdot W_{,KE} \cdot (\mathbf{H} \cdot \mathbf{a} \otimes \mathbf{n}) + \mathbf{n} \cdot W_{,KK} \cdot (\mathbf{H} \cdot \mathbf{b} \otimes \mathbf{n}) \\ = \rho \gamma V^2 \mathbf{b} \cdot \mathbf{H}^T. \end{aligned}$$

These relations can be written in a more compact form using the matrix notation

$$\mathbb{A}(\mathbf{n}) \cdot \boldsymbol{\xi} = \rho V^2 \mathbb{B} \cdot \boldsymbol{\xi}, \quad (5)$$

where $\boldsymbol{\xi} = (\mathbf{a}', \mathbf{b}') \in \mathbb{R}^6$, $\mathbf{a}' = \mathbf{a} \cdot \mathbf{H}^T$, $\mathbf{b}' = \mathbf{b} \cdot \mathbf{H}^T$, \mathbb{A} and \mathbb{B} are the matrices with tensor elements

$$\mathbb{A}(\mathbf{n}) \equiv \begin{bmatrix} W_{,EE}\{\mathbf{n}\} & W_{,EK}\{\mathbf{n}\} \\ W_{,KE}\{\mathbf{n}\} & W_{,KK}\{\mathbf{n}\} \end{bmatrix}, \quad \mathbb{B} \equiv \rho V^2 \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \gamma \mathbf{I} \end{bmatrix}$$

and the notation $\mathbf{G}\{\mathbf{n}\} \equiv G_{klmn} n_k n_m \mathbf{i}_l \otimes \mathbf{i}_n$ is introduced for any fourth-rank tensor \mathbf{G} and any vector \mathbf{n} that are represented in terms of their decompositions in the arbitrary Cartesian basis \mathbf{i}_k , $k = 1, 2, 3$.

Thus, the problem of acceleration wave propagation in the micropolar medium has been reduced to the spectral problem given by Eq. (5). Owing to the existence of the potential-energy function W , $\mathbb{A}(\mathbf{n})$ is symmetric. This property enables one to formulate an analogue of

the Fresnel–Hadamard–Duhem theorem for micropolar media.

Theorem 1. *The squares of the velocities of second-order singular surfaces (acceleration waves) in the micropolar elastic medium are real for arbitrary propagation directions specified by the vector \mathbf{n} .*

The positive definiteness of $\mathbb{A}(\mathbf{n})$, which is both necessary and sufficient for the wave velocity V to be real, i.e., the inequality

$$\begin{aligned} \boldsymbol{\xi} \cdot \mathbb{A}(\mathbf{n}) \cdot \boldsymbol{\xi} > 0, \\ \forall \boldsymbol{\xi} \in \mathbb{R}^6, \quad \mathbf{n} \neq \mathbf{0}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^6, \quad \boldsymbol{\xi} \neq \mathbf{0}, \end{aligned} \quad (6)$$

is an additional constraint imposed on constitutive relationships (2).

Following [7], one can prove Theorem 2.

Theorem 2. *Condition (6) for the existence of the acceleration wave in the micropolar elastic medium is equivalent the condition of strong ellipticity of the equilibrium equation.*

Weak inequality (6) is an analogue of the Hadamard inequality from the spatial theory of elasticity [12, 13]. As in the case of simple materials, a break in inequality (6) means the possibility of existing nonsmooth solutions of the equilibrium equations.

As an example, we consider the equation of state of a physically linear micropolar medium [7]

$$W = W_1(\mathbf{E}) + W_2(\mathbf{K}). \quad (7)$$

Here,

$$\begin{aligned} W_1(\mathbf{E}) &= \alpha_1 \text{tr}((\mathbf{E} - \mathbf{I}) \cdot (\mathbf{E} - \mathbf{I})^T) \\ &+ \alpha_2 \text{tr}((\mathbf{E} - \mathbf{I})^2) + \alpha_3 \text{tr}^2(\mathbf{E} - \mathbf{I}), \end{aligned}$$

$$W_2(\mathbf{K}) = \beta_1 \text{tr}(\mathbf{K} \cdot \mathbf{K}^T) + \beta_2 \text{tr}(\mathbf{K}^2) + \beta_3 \text{tr}^2(\mathbf{K}),$$

where α_k and β_k , $k = 1, 2, 3$, are constants. In this case, inequality (6) is equivalent to the inequalities [7]

$$\alpha_1 > 0, \quad \alpha_1 + \alpha_2 + \alpha_3 > 0, \quad \beta_1 > 0, \quad \beta_1 + \beta_2 + \beta_3 > 0.$$

Under these conditions, the solutions of Eq. (5) are given by the expressions

$$V_{1,2} = \sqrt{\frac{\alpha_1}{\rho}}, \quad \boldsymbol{\xi}_{1,2} = (\mathbf{e}_{1,2}, \mathbf{0}), \quad (8)$$

$$V_3 = \sqrt{\frac{\alpha_1 + \alpha_2 + \alpha_3}{\rho}}, \quad \boldsymbol{\xi}_3 = (\mathbf{0}, \mathbf{n}),$$

$$V_{4,5} = \sqrt{\frac{\beta_1}{\gamma \rho}}, \quad \boldsymbol{\xi}_{4,5} = (\mathbf{e}_{4,5}, \mathbf{0}), \quad (9)$$

$$V_6 = \sqrt{\frac{\beta_1 + \beta_2 + \beta_3}{\gamma \rho}}, \quad \boldsymbol{\xi}_6 = (\mathbf{0}, \mathbf{n}),$$

where \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_4 , and \mathbf{e}_5 are arbitrary unit vectors lying in the plane tangent to S ($\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{n} = 0$,

$\mathbf{e}_4 \cdot \mathbf{e}_5 = \mathbf{e}_4 \cdot \mathbf{n} = \mathbf{e}_5 \cdot \mathbf{n} = 0$). Solutions (8) describe the transverse and longitudinal acceleration waves, and solutions (9), the transverse and longitudinal microrotation acceleration waves. The velocities obtained for the acceleration waves given by Eqs. (8) and (9) coincide with the limiting phase velocities of plane harmonic waves propagating in the linearly elastic micropolar medium [2, 9] when the wave frequency tends to infinity.

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