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Julien Roth

► **To cite this version:**

| Julien Roth. A note on biharmonic submanifolds of product spaces. 2013. hal-00827336v1

**HAL Id: hal-00827336**

**<https://hal.science/hal-00827336v1>**

Preprint submitted on 29 May 2013 (v1), last revised 12 Jun 2013 (v2)

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# A NOTE ON BIHARMONIC SUBMANIFOLDS OF PRODUCT SPACES

JULIEN ROTH

ABSTRACT. We investigate biharmonic submanifolds of the product of two space forms. We prove a necessary and sufficient condition for biharmonic submanifolds in these product spaces. Then, we obtain mean curvature estimates for proper-biharmonic submanifold of a product of two unit spheres. We also prove a non-existence result in the case of the product of a sphere and a hyperbolic space.

## 1. INTRODUCTION

A harmonic map  $\psi$  between two Riemannian manifolds  $(M, g)$  and  $(N, h)$  is defined as a critical point of the energy functional

$$E(\psi) = \frac{1}{2} \int_M |d\psi|^2 dv_g.$$

In [3], Eells and Sampson proposed a natural generalization of harmonic maps. The map  $\psi$  is said to be *biharmonic* if it is a critical point of the bienergy functional

$$E_2(\psi) = \frac{1}{2} \int_M |\tau(\psi)|^2 dv_g,$$

where  $\tau(\psi) = \text{tr}(\nabla d\psi)$  is the tension field which vanishes for harmonic map. The Euler-Lagrange equation for the bienergy functional is given by  $\tau_2(\psi) = 0$  (see [10]), where  $\tau_2(\psi)$  is the bitension field and is defined by

$$\tau_2(\psi) = \Delta \tau(\psi) - \text{tr}(R^N(d\psi, \tau(\psi))d\psi).$$

Here,  $\Delta$  stands for the rough Laplacian acting on sections of  $\psi^{-1}(TN)$  and  $R^N$  is the curvature tensor of  $N$ .

Since any harmonic map is biharmonic, we will focus on biharmonic maps that are not harmonic, which are called *proper-harmonic*.

A *biharmonic submanifold* of a Riemannian manifold is a submanifold for which the inclusion map is biharmonic.

Biharmonic submanifolds of different ambient spaces have been intensively studied in the last decade, in particular for real space forms [2], complex space forms [4], 3-dimensional homogeneous manifolds [9], Sasakian space forms [5] or the product of a sphere with the real line [6]. In the present paper, we investigate the biharmonic submanifolds of product manifolds. Precisely, we will consider submanifolds into the

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*Date:* May 29, 2013.

*2000 Mathematics Subject Classification.* 53C42, 53C43.

*Key words and phrases.* biharmonic submanifolds, product spaces.

products of two real space forms, with a focus on the case of the product of two spheres, or the product of a sphere with a hyperbolic space.

## 2. PRELIMINARIES

Let  $(M^m, g)$  be a Riemannian manifold isometrically immersed into the product space  $(\mathbb{M}^{n_1}(c_1) \times \mathbb{M}^{n_2}(c_2), \tilde{g})$ , where  $\mathbb{M}^{n_j}(c_j)$  is the  $n_j$ -dimensional real space form of curvature  $c_j$ , for  $j = 1, 2$ . We denote by  $F$  the product structure of  $\mathbb{M}^{n_1}(c_1) \times \mathbb{M}^{n_2}(c_2)$ . Namely, the map  $F : T\mathbb{M}^{n_1}(c_1) \times T\mathbb{M}^{n_2}(c_2) \longrightarrow T\mathbb{M}^{n_1}(c_1) \times T\mathbb{M}^{n_2}(c_2)$  is defined by  $F(X_1 + X_2) = X_1 - X_2$ , where  $X_j \in T\mathbb{M}^{n_j}(c_j)$ . Obviously,  $F$  satisfies

$$(1) \quad F^2 = Id \quad (\text{and } F \neq Id),$$

$$(2) \quad \tilde{g}(FX, Y) = \tilde{g}(X, FY),$$

$$(3) \quad \tilde{\nabla}F = 0.$$

Moreover, we recall (see [7, 8] for instance) that the curvature tensor of  $(\mathbb{M}^{n_1}(c_1) \times \mathbb{M}^{n_2}(c_2), \tilde{g})$  is given by

$$(4) \quad \begin{aligned} \tilde{R}(X, Y)Z &= a \left[ \langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle FY, Z \rangle FX - \langle FX, Z \rangle FY \right] \\ &+ b \left[ \langle Y, Z \rangle FX - \langle X, Z \rangle FY + \langle Y, FZ \rangle X - \langle X, FZ \rangle Y \right], \end{aligned}$$

with  $a = \frac{c_1+c_2}{2}$  and  $\frac{c_1-c_2}{2}$ . The product structure  $F$  induces the existence of the following four operators

$$f : TM \longrightarrow TM, \quad h : TM \longrightarrow NM, \quad s : NM \longrightarrow TM \quad \text{and} \quad t : NM \longrightarrow NM$$

defined for any  $X \in TM$  and  $\xi \in NM$  by

$$(5) \quad FX = fX + hX \quad \text{and} \quad F\xi = s\xi + t\xi.$$

From Equations (1) and (2), we deduce that  $f$  and  $t$  are symmetric and we have the following relations between these four operators

$$(6) \quad f^2X = X - shX,$$

$$(7) \quad t^2\xi = \xi - hs\xi,$$

$$(8) \quad fs\xi + st\xi = 0,$$

$$(9) \quad hfX + thX = 0,$$

$$(10) \quad \tilde{g}(hX, \xi) = \tilde{g}(X, s\xi),$$

for any  $X \in \Gamma(TM)$  and  $\xi \in \Gamma(NM)$ .

We have now all the ingredients to state our results.

### 3. MAIN RESULTS

First, we give a necessary and sufficient condition characterizing biharmonicity in product spaces.

**Theorem 3.1.** *Let  $M$  a submanifold of  $\mathbb{M}^{n_1}(c_1) \times \mathbb{M}^{n_2}(c_2)$  of dimension  $m$ . Then  $M$  is biharmonic if and only if the two following equations hold*

$$(11) -\Delta^\perp H + \text{tr}B(\cdot, A_H \cdot) = a[mH - hsH + \text{tr}(f)tH] + b[mtH + \text{tr}(f)H]$$

$$(12) \frac{m}{2} \text{grad}|H|^2 + 2\text{tr}A_{\nabla^\perp H}(\cdot) = a[-fsH + \text{tr}(f)sH] + b(m-1)sH.$$

*Proof:* It is a classical fact (see [1, 6] for instance) that, after decomposition into tangential and normal parts,  $\tau_2(\psi) = 0$  is equivalent to

$$\begin{cases} -\Delta^\perp H + \text{tr}B(\cdot, A_H \cdot) + \text{tr}(\tilde{R}(\cdot, H) \cdot)^\perp = 0, \\ \frac{m}{2} \text{grad}|H|^2 + 2\text{tr}A_{\nabla^\perp H}(\cdot) + 2\text{tr}(\tilde{R}(\cdot, H) \cdot)^\top = 0. \end{cases}$$

From the expression of the curvature tensor  $\tilde{R}$ , we get

$$\begin{aligned} \text{tr}(\tilde{R}(\cdot, H) \cdot) &= a[-mH + F((FH)^\top) - \text{tr}(f)FH] \\ &\quad + b[-mFH + (FH)^\top - \text{tr}(f)H]. \end{aligned}$$

Now, we use the decomposition of  $F$  into the 4 operators  $f, h, s$  and  $t$  to obtain

$$\begin{aligned} \text{tr}(\tilde{R}(\cdot, H) \cdot) &= a[-mH + fsH + hsH - \text{tr}(f)sH - \text{tr}(f)tH] \\ &\quad + b[-msH - mtH + sH - \text{tr}(f)H]. \end{aligned}$$

By identification of the tangential and normal parts, we get the two equations of the theorem.  $\square$

In particular, we have the following particular cases.

**Corollary 3.2.** *Let  $M$  a submanifold of  $\mathbb{M}^{n_1}(c_1) \times \mathbb{M}^{n_2}(c_2)$  of dimension  $m$ .*

(1) *If  $FH$  is tangent to  $M$ , then  $M$  is biharmonic if and only if*

$$\begin{cases} -\Delta^\perp H + \text{tr}B(\cdot, A_H \cdot) - [a(m-1) + b\text{tr}(f)]H = 0, \\ \frac{m}{2} \text{grad}|H|^2 + 2\text{tr}A_{\nabla^\perp H}(\cdot) + [a\text{tr}(f) - b(m-1)]FH = 0. \end{cases}$$

(2) *If  $FH$  is normal to  $M$ , then  $M$  is biharmonic if and only if*

$$\begin{cases} -\Delta^\perp H + \text{tr}B(\cdot, A_H \cdot) - [am + b\text{tr}(f)]H - [bm + a\text{tr}(f)]FH = 0, \\ \frac{m}{2} \text{grad}|H|^2 + 2\text{tr}A_{\nabla^\perp H}(\cdot) = 0. \end{cases}$$

*Proof:* If  $FH$  is tangent, then by the definition of the operators  $s$  and  $t$ , we have  $sH = H$  and  $tH = 0$ . Hence, by (7), we have  $hsH = H$  and we get the wanted equations, from Theorem 3.1.

Now, if  $FH$  is normal, then  $sH = 0$  and  $tH = H$  and we conclude immediately from Theorem 3.1.  $\square$

From now on, we will focus on the case where the ambient space is a product of two spheres of same radius. Note that in this case, we have  $a = \frac{c_1+c_2}{2} = 1$  and  $b = \frac{c_1-c_2}{2} = 0$ . For a sake of simplicity, we will assume that the common radius is 1. First, we have an estimate for the length of the mean curvature vector when  $FH$  is tangent to the submanifold.

**Corollary 3.3.** *Let  $M$  be a submanifold of  $\mathbb{S}^p \times \mathbb{S}^{n-p}$ , of dimension  $m \geq 2$  with non-zero constant mean curvature and such that  $FH$  is tangent. Then we have*

- (1) *If  $M$  is proper-biharmonic, then  $0 < |H|^2 \leq \frac{m-1}{m}$ .*
- (2) *If  $|H|^2 = \frac{(m-1)}{m}$ , then  $M$  is proper-biharmonic if and only if it is pseudo-umbilical,  $\nabla^\perp H = 0$  and  $\text{tr}(f) = 0$ .*

*Proof:* Let  $M$  be a biharmonic submanifold of  $\mathbb{S}^p \times \mathbb{S}^{n-p}$  with non-zero constant mean curvature. Since we assume that  $FH$  is tangent, by the first assertion of Corollary 3.2, we have

$$-\Delta^\perp H + \text{tr}B(\cdot, A_H \cdot) - (m-1)H = 0.$$

Hence, by taking the scalar product with  $H$ , we have

$$\langle \Delta^\perp H, H \rangle = (m-1)|H|^2 - |A_H|^2.$$

Using the Bochner formula and the fact that  $|H|$  is constant, we get

$$(m-1)|H|^2 = |A_H|^2 + |\nabla^\perp H|^2.$$

Since  $A_H$  is the shape operator associated with the mean curvature, by Cauchy-Schwarz inequality, we get  $|A_H|^2 \geq m|H|^4$ . Therefore, we have

$$(m-1)|H|^2 \geq m|H|^4 + |\nabla^\perp H|^2 \geq m|H|^4.$$

Since  $|H|$  is a non-zero constant, we have  $0 < |H|^2 \leq \frac{m-1}{m}$ .

Now, if  $|H|^2 = \frac{(m-1)}{m}$ , if  $M$  is proper-biharmonic, then the inequalities above become equalities. First, we have  $\nabla^\perp H = 0$ . Moreover, equality occurs in the Cauchy-Schwarz inequality  $|A_H|^2 \geq m|H|^4$ , that is,  $M$  is pseudo-umbilical. Finally since  $\nabla^\perp H = 0$ , the second equation in Corollary 3.2 becomes  $\text{tr}(f)FH = 0$ . Since  $H$  has constant positive norm and since  $F$  is an isometry, then  $FH$  vanishes nowhere. Hence, we have  $\text{tr}(f) = 0$ .

Conversly, if  $M$  is pseudo-umbilical, with  $\nabla^\perp H = 0$  and  $\text{tr}(f) = 0$ , then, we have obviously

$$\begin{cases} -\Delta^\perp H + \text{tr}B(\cdot, A_H \cdot) - (m-1)H = 0, \\ \frac{m}{2} \text{grad}|H|^2 + 2\text{tr}A_{\nabla^\perp H}(\cdot) + \text{tr}(f)FH = 0, \end{cases}$$

where all terms are zero in the second equality. This concludes the proof.  $\square$

When  $M$  is a hypersurface, the characterization of biharmonicity can be expressed in a simpler way. Namely, for biharmonic hypersurfaces with  $FH$  tangent to  $M$ , we have the following result.

**Proposition 3.4.** *Let  $M$  a hypersurface of  $\mathbb{S}^p \times \mathbb{S}^{n-p}$  with non-zero constant mean curvature and such that  $FH$  is tangent to  $M$ . Then  $M$  is biharmonic if and only if*

$$\operatorname{tr}(f) = 0 \quad \text{and} \quad |B|^2 = (n-2).$$

*Proof:* This is a direct consequence of Corollary 3.2. When  $M$  is a hypersurface of non-zero constant mean curvature with  $FH$  tangent to  $M$ , the first equation becomes  $\operatorname{tr}(B(\cdot, A_H \cdot)) - (n-2)H = 0$ , that is,  $|B|^2 = (n-2)$  since  $\operatorname{tr}(B(\cdot, A_H \cdot)) = |B|^2|H|$ . The second equation of assertion (1) in Corollary 3.2 is reduced to  $\operatorname{tr}(f) = 0$ .  $\square$

Using this proposition, we can compute the scalar curvature of biharmonic hypersurfaces with  $FH$  tangent to  $M$ .

**Proposition 3.5.** *Let  $M$  be a proper-biharmonic hypersurface of  $\mathbb{S}^p \times \mathbb{S}^{n-p}$  with non-zero constant mean curvature and such that  $FH$  is tangent to  $M$ . Then the scalar curvature of  $M$  is constant and given by*

$$\operatorname{Scal}_M = (n-1)(n-2) + (n-1)^2|H|^2.$$

*Proof:* First, by the Gauss equation, we have

$$\operatorname{Scal}_M = \sum_{i,j=1}^{n-1} \left\langle \tilde{R}(X_i, X_j)X_j, X_i \right\rangle - |B|^2 + (n-1)H^2.$$

We recall that the curvature  $\tilde{R}$  of  $\mathbb{S}^p \times \mathbb{S}^{n-p}$  is given by (4) with  $a = 1$  and  $b = 0$ . Hence, we have

$$\left\langle \tilde{R}(X_i, X_j)X_j, X_i \right\rangle = 1 - \langle X_i, X_j \rangle + \langle fX_i, X_i \rangle \langle fX_j, X_j \rangle + \langle fX_i, X_j \rangle^2.$$

By summation, we get

$$\sum_{i,j=1}^{n-1} \left\langle \tilde{R}(X_i, X_j)X_j, X_i \right\rangle = (n-1)(n-2) + \operatorname{tr}(f)^2 + \sum_{i,j=1}^{n-1} \langle fX_i, X_j \rangle^2.$$

Since  $f$  is symmetric and using (6), we have

$$\begin{aligned} \sum_{i,j=1}^{n-1} \langle fX_i, X_j \rangle^2 &= \sum_{j=1}^{n-1} \langle fX_j, fX_j \rangle \\ &= \sum_{j=1}^{n-1} \langle FX_j, FX_j \rangle - \langle X_j, F\nu \rangle^2 \\ &= (n-1) - |s\nu|^2. \end{aligned}$$

Now, we will use the hypothesis that  $FH$  is tangent to  $M$ . Since  $M$  is a hypersurface, this is equivalent to the fact that  $F\nu$  is tangent to  $M$ , where  $\nu$  is the unit normal to the hypersurface. Then this implies that  $s\nu = F\nu$  and therefore  $|s\nu| = 1$ . Hence, we have

$$\sum_{i,j=1}^{n-1} \langle fX_i, X_j \rangle^2 = n - 2.$$

Finally, since by the previous Proposition, we have  $\text{tr}(f) = 0$  and  $|B|^2 = (n - 2)$ , we get

$$\begin{aligned} \text{Scal}_M &= (n - 1)(n - 2) + \text{tr}(f)^2 + (n - 2) - |B|^2 - (n - 1)^2|H|^2 \\ &= (n - 1)(n - 2) + (n - 1)^2|H|^2. \end{aligned}$$

□

**Remark 3.6.** *One could obtain results comparable to Corollary 3.3, Proposition 3.4 and 3.5 for submanifolds with  $FH$  normal. However, all the estimates would depend on  $\text{tr}(f)$ . We will not write these results here.*

We finish this note by a non-existence results in the case of the product of a sphere with a hyperbolic space. We recall that in this case, we have  $a = 0$  and  $b = 1$ .

**Proposition 3.7.** *There exists no proper-biharmonic submanifolds of  $\mathbb{S}^p \times \mathbb{H}^{n-p}$  of dimension  $m \geq 2$  with parallel mean curvature and such that  $FH$  is not normal.*

*Proof:* First, we remark that since  $H$  is parallel and not identically zero ( $M$  is proper-biharmonic), then  $H$  has no zeros. Moreover,  $F$  is an isometry, then  $FH$  also has no zeros. If  $M$  is biharmonic, by Theorem 3.1, we have

$$\frac{m}{2} \text{grad}|H|^2 + 2\text{tr}A_{\nabla^\perp H}(\cdot) - (m - 1)sH = 0.$$

If  $M$  has parallel mean curvature, then this equation becomes  $sH = 0$ . If, in addition,  $FH$  is not normal, then its tangential part  $sH$  does not vanish identically, which is impossible. □

**Remark 3.8.** *In particular, there exists no parallel mean curvature proper-biharmonic submanifolds of  $\mathbb{S}^p \times \mathbb{H}^{n-p}$  such that  $FH$  is tangent.*

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(J. Roth) LAMA, UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE, CITÉ DESCARTES, CHAMPS SUR MARNE, 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE

*E-mail address:* julien.roth@univ-mlv.fr