

Sufficient conditions for topological invariance of 2D images under rigid transformations

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Hugues TALBOT



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— PARIS-EST

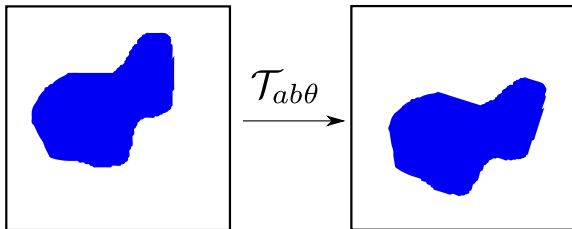
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Rigid transformation

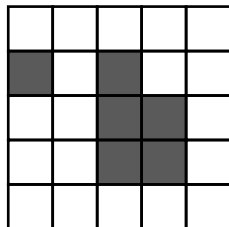
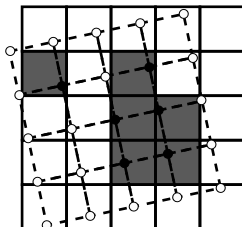
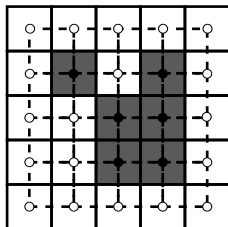
Rigid transformation is a function $\mathcal{T}_{ab\theta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p \cos \theta - q \sin \theta + a \\ p \sin \theta + q \cos \theta + b \end{pmatrix}$$

where $a, b \in \mathbb{R}$, $\theta \in [0, 2\pi[$ and $(p, q), (p', q') \in \mathbb{R}^2$.



Topology issues



Problem statement

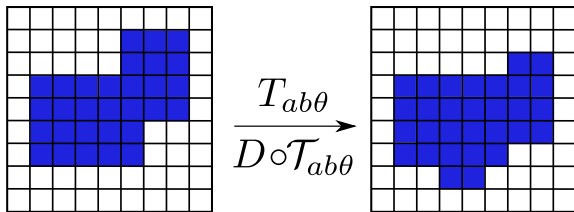
Topology is not preserved due to digitization process from \mathbb{R}^2 to \mathbb{Z}^2 .

Digital rigid transformation

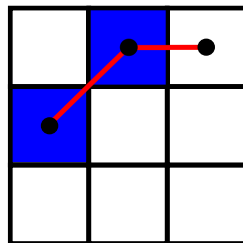
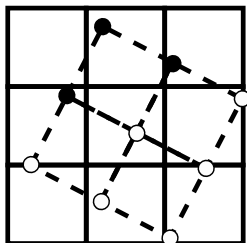
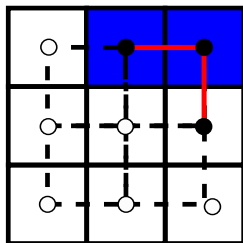
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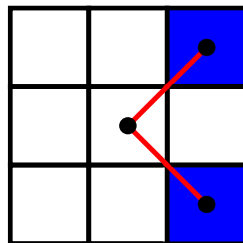
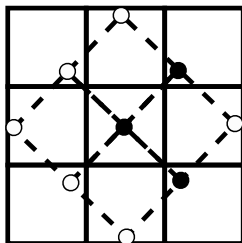
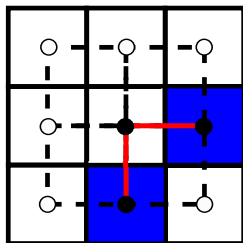
Geometrical alteration: Distance and angle



Distance	
Before	After
1	$\sqrt{2}$

Angle	
Before	After
90°	135°

Geometrical alteration: Distance and angle

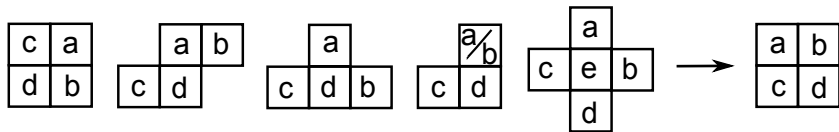


Distance	
Before	After
1	$\sqrt{2}$
$\sqrt{2}$	2

Angle	
Before	After
90°	135°
90°	90°

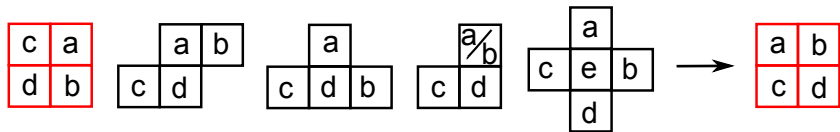
Topological alteration

The following configurations (up to symmetries) can generate a 2×2 sample under digital rigid transformations.



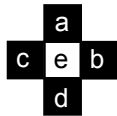
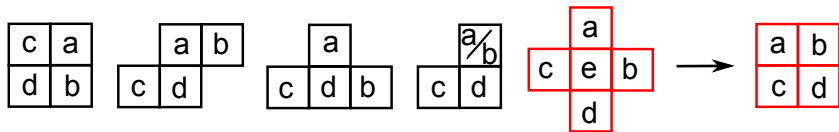
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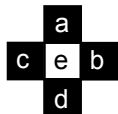
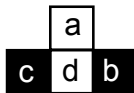
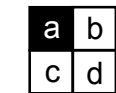
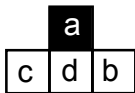
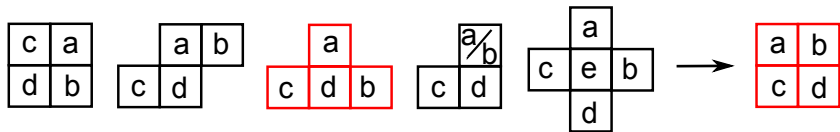
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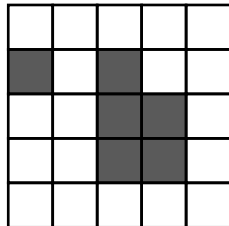
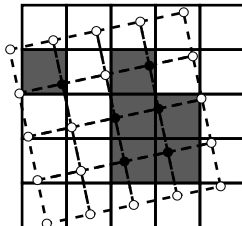
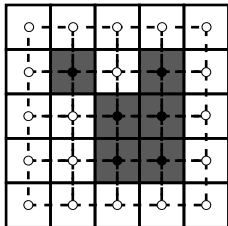


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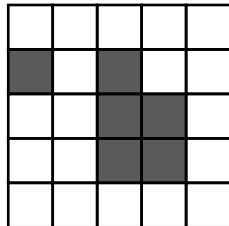
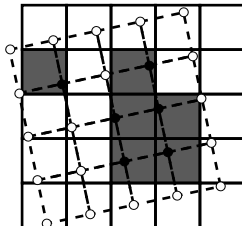
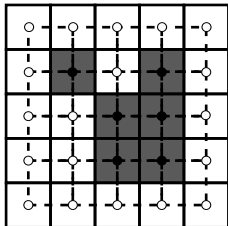
Motivations



Our questions

- Do binary images exist that preserve their topology under any rigid transformations?

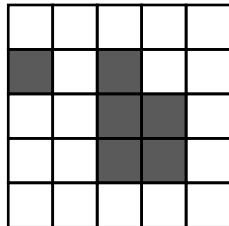
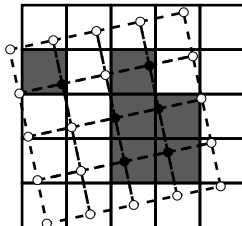
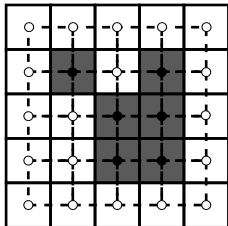
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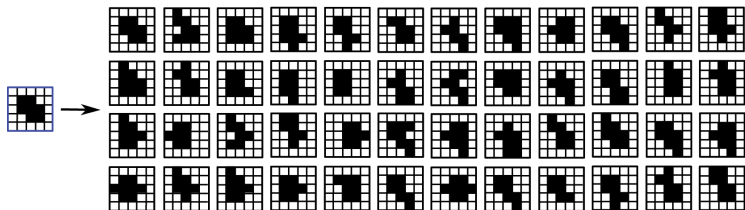
Our questions

- Do binary images exist that preserve their topology under any rigid transformations?
- Can we verify topological preservation for a given image?
- What the conditions for images to preserve their topology?

Topological invariance

Definition

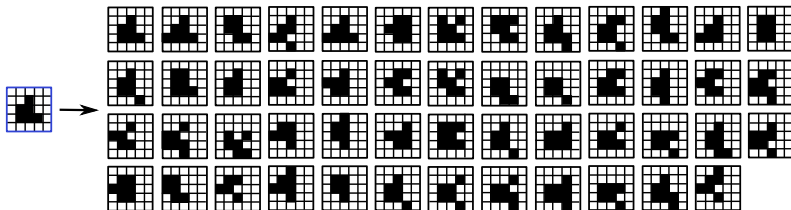
An image I is *topologically invariant* if its transformed images under any rigid transformations have the same homotopy-type as I .



Topological invariance

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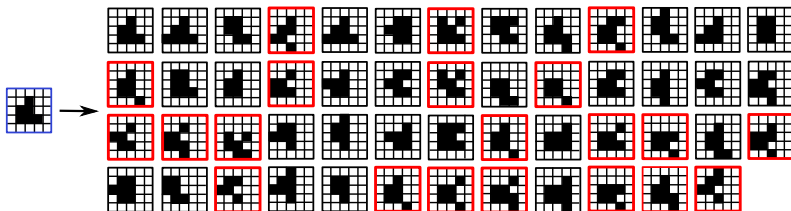
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Tools for topological invariance verification

Methodological tools

- DRT graph, and

Tools for topological invariance verification

Methodological tools

- DRT graph, and
- simple points.

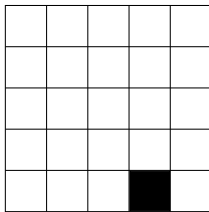
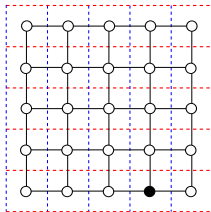
Critical transformation

Digital rigid transformation $T_{ab\theta} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} [p \cos \theta - q \sin \theta + a] \\ [p \sin \theta + q \cos \theta + b] \end{pmatrix}$$

Definition

A **critical rigid transformation** moves at least one integer point into either a **vertical** or a **horizontal** half-grid point.



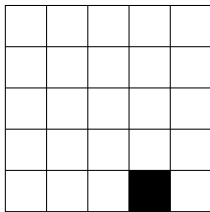
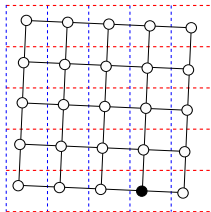
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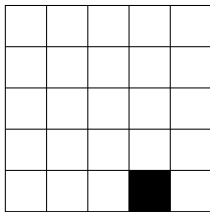
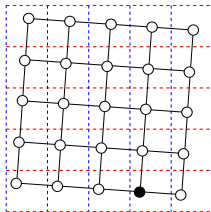
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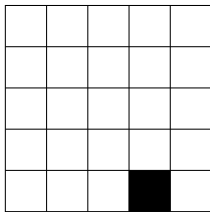
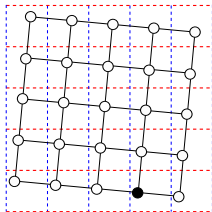
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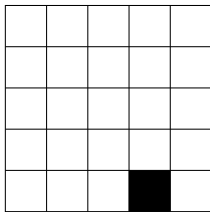
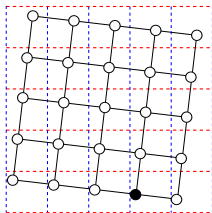
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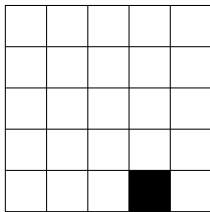
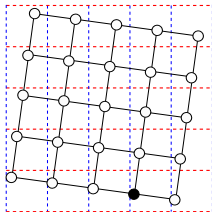
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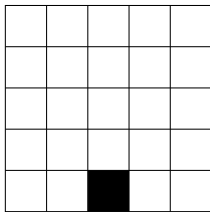
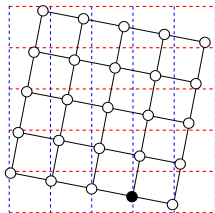
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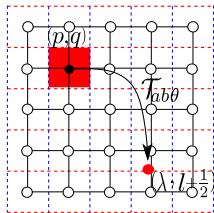
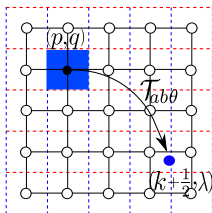
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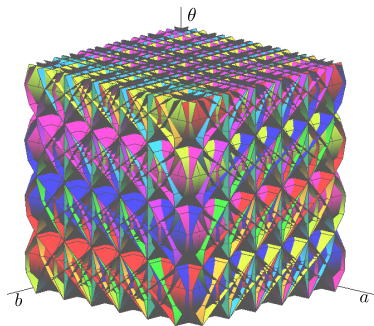
Tipping surfaces are the surfaces associated to the critical transformations in the parameter space (a, b, θ) .

$$\left| \begin{array}{lll} \Phi_{pqk} : & \mathbb{R}^2 & \longrightarrow \mathbb{R} \\ & (b, \theta) & \longmapsto a = k + \frac{1}{2} + q \sin \theta - p \cos \theta \quad (\text{vertical}) \end{array} \right.$$

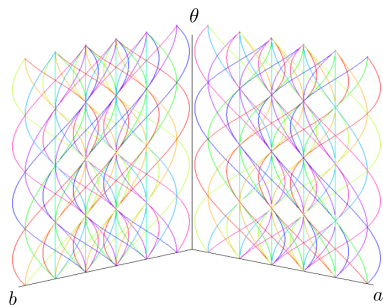
$$\left| \begin{array}{lll} \Psi_{pql} : & \mathbb{R}^2 & \longrightarrow \mathbb{R} \\ & (a, \theta) & \longmapsto b = l + \frac{1}{2} - p \sin \theta - q \cos \theta \quad (\text{horizontal}) \end{array} \right.$$

for $p, q, k, l \in \mathbb{Z}$.

Example of tipping surfaces



(a) Tipping surfaces



(b) Orthogonal sections

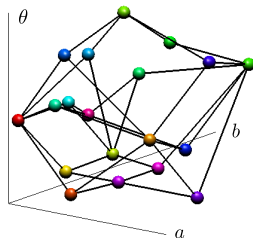
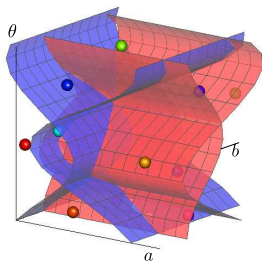
All tipping surfaces Φ_{pqk} and Ψ_{pql} with $p, q \in [0, 4]$ and $k, l \in [0, 5]$.

DRT graph

Definition

A **discrete rigid transformation graph** (DRT graph) is a graph $G = (V, E)$ such that

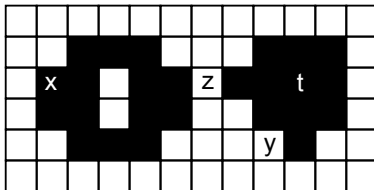
- each vertex in V corresponds to a DRT,
- each edge in E connects two vertices sharing a tipping surface.



Simple point

Definition

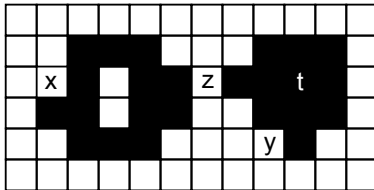
A *simple point* is a pixel whose value can be modified without changing the digital topology of the associated image.



Simple point

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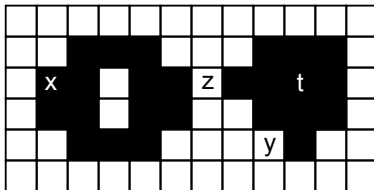
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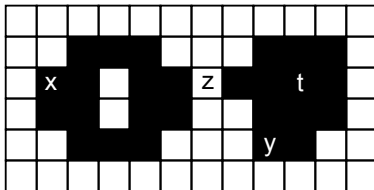
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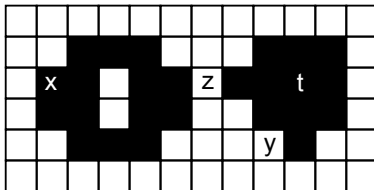
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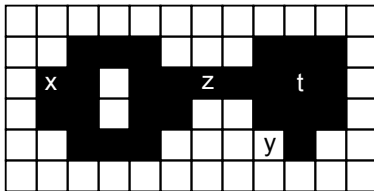
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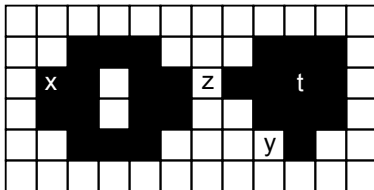
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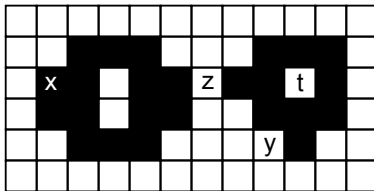
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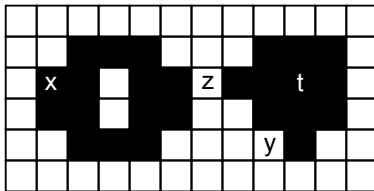
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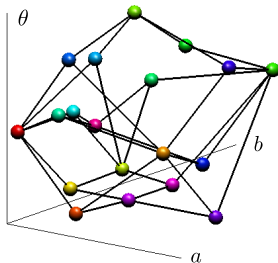
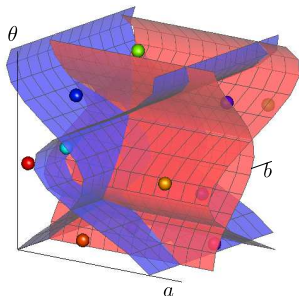


The simplicity can be tested locally in constant time using 3×3 neighbourhood.

Principles of the naive method

We use

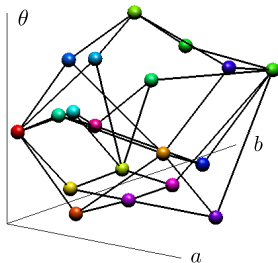
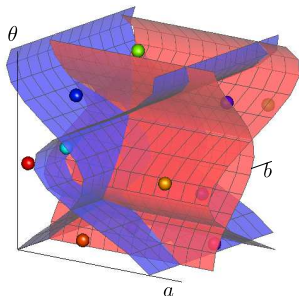
- **DRT graph** to generate incrementally all transformed images,



Principles of the naive method

We use

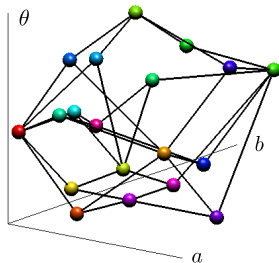
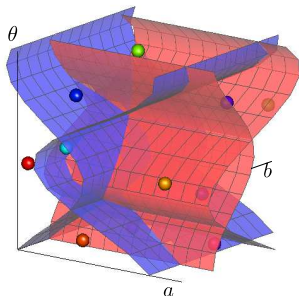
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- **simplicity test** to verify the topological invariance.



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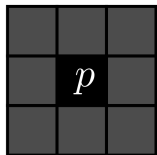
The DRT graph has a high complexity of $O(N^9)$, for $N \times N$ is the image size.

Local property

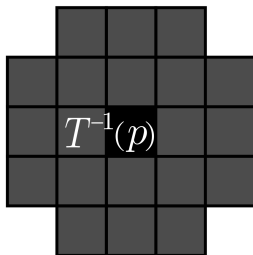
Property

Let $\mathbf{p} \in \mathbb{Z}^2$ be a pixel and $\mathbf{q} \in N_8(\mathbf{p})$, we have

$$T^{-1}(\mathbf{q}) \in N_{20}(T^{-1}(\mathbf{p})),$$



$N_8(\mathbf{p})$



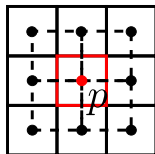
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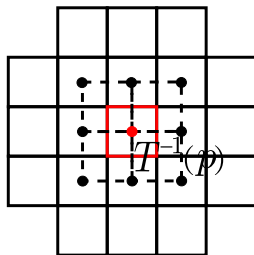
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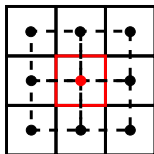
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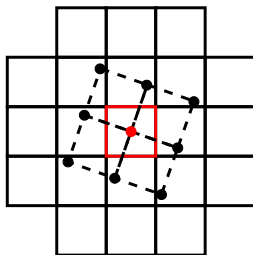
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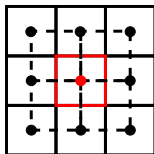
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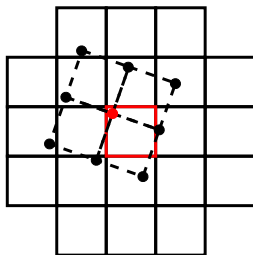
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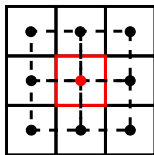
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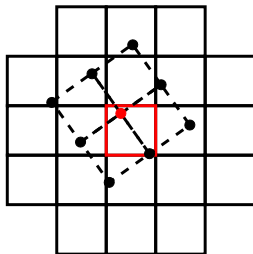
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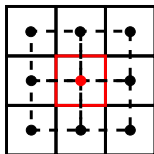
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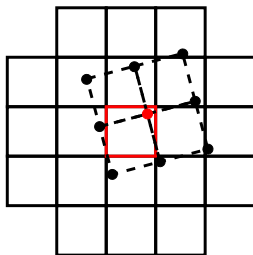
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Method using local configurations

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- generate all 20-neighborhood samples;

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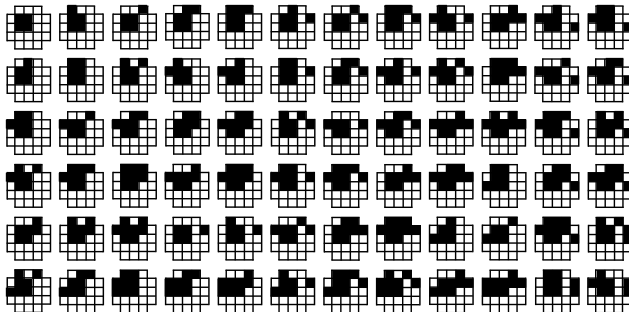
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The verification is in linear time with respect to the image size.

Samples in the Look-Up-Tables

Among 124 260 binary configurations up to symmetries, there are

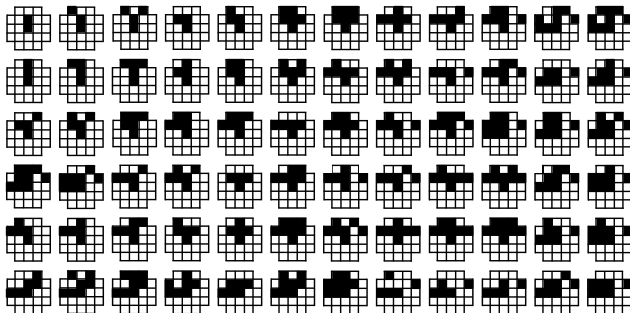
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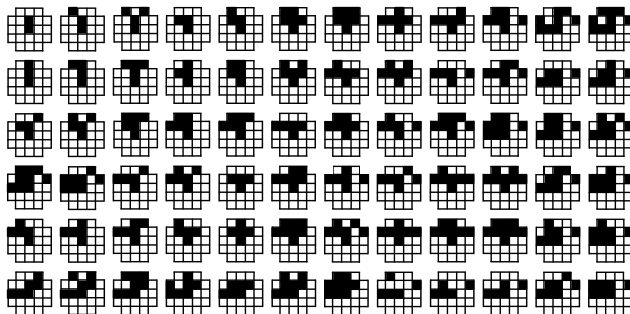
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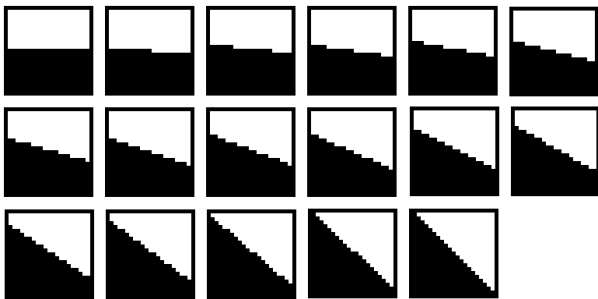
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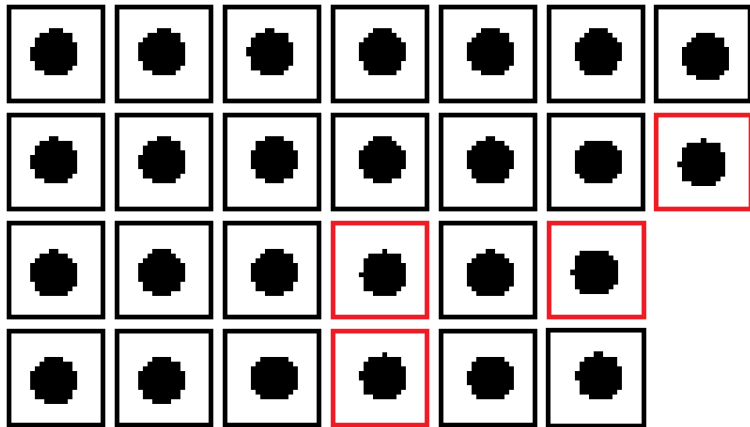
Sufficient conditions of topologically invariant images.

Experiment for half-planes



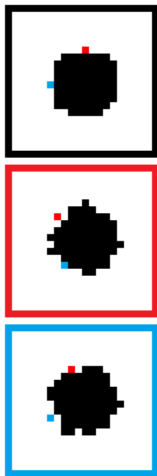
All centered half-planes generated in images of size 21×21 .

Experiment for disks

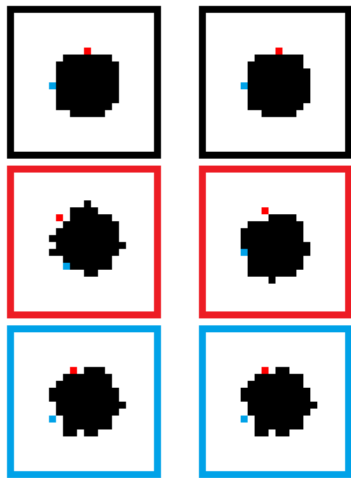


Digitized disks of radius 11 whose center is anywhere inside the center pixel.

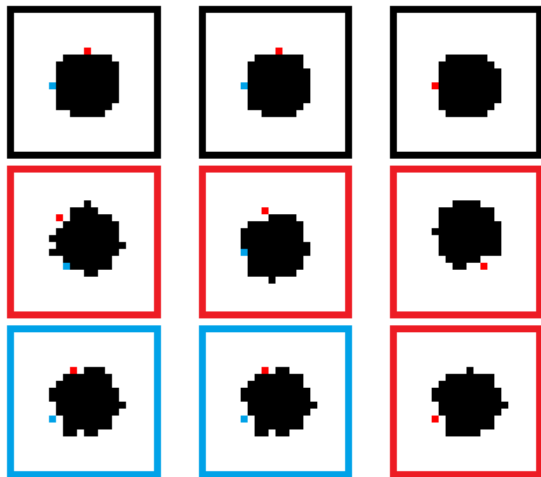
Experiment for disks



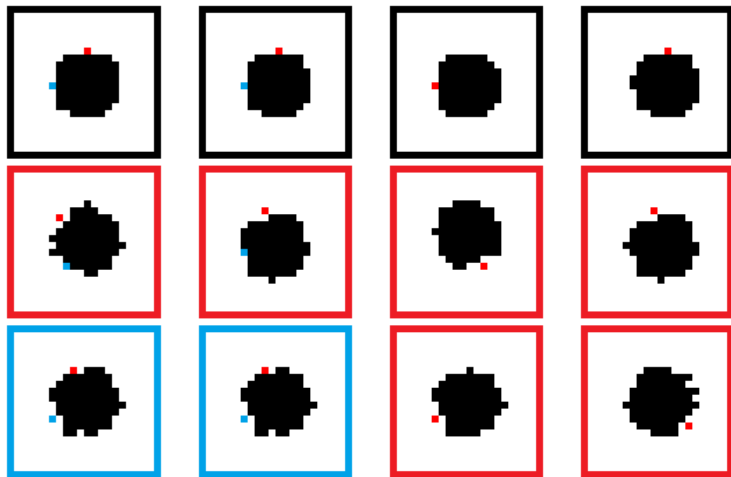
Experiment for disks



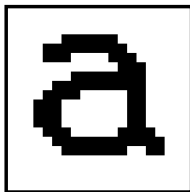
Experiment for disks



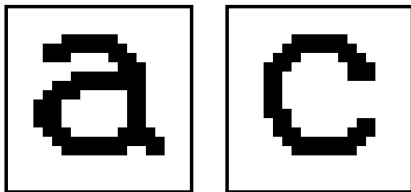
Experiment for disks



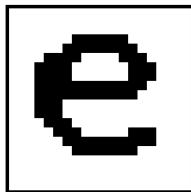
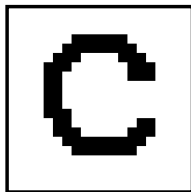
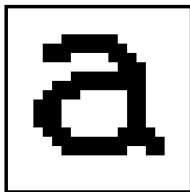
Experiment for letters: Topological invariance



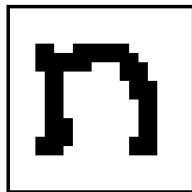
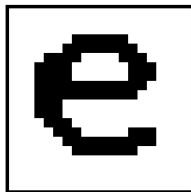
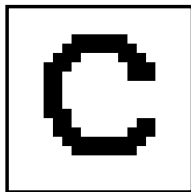
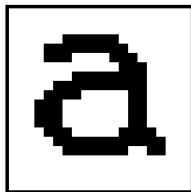
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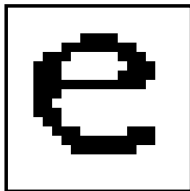
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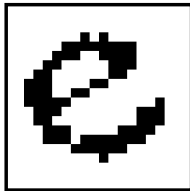
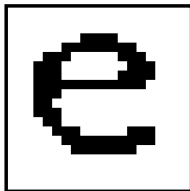
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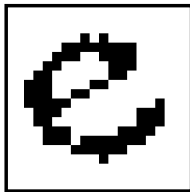
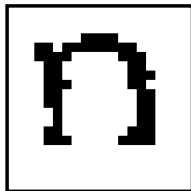
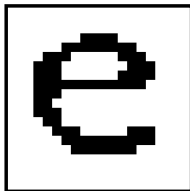
Experiment for letters: Topological variance



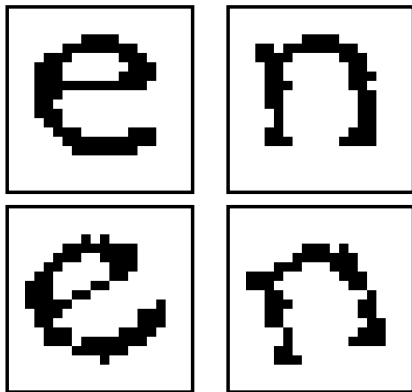
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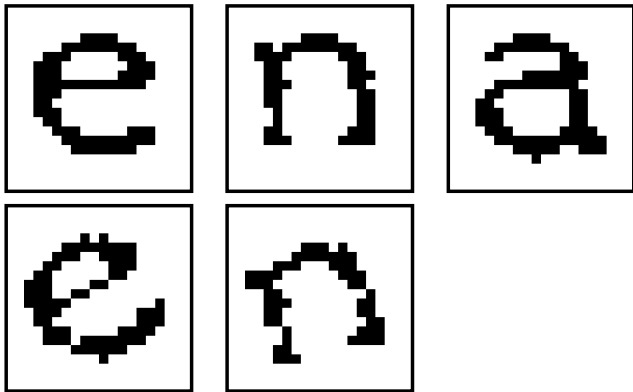
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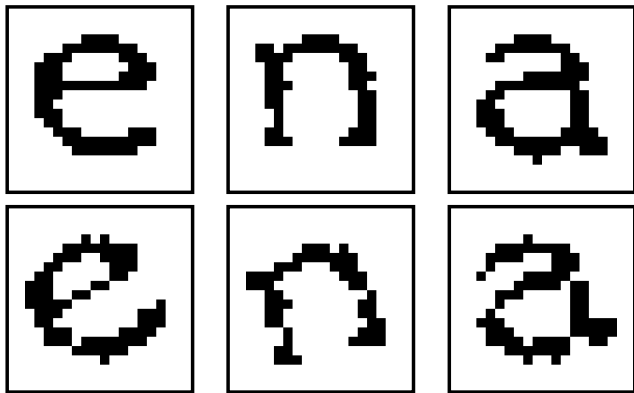
Experiment for letters: Topological variance



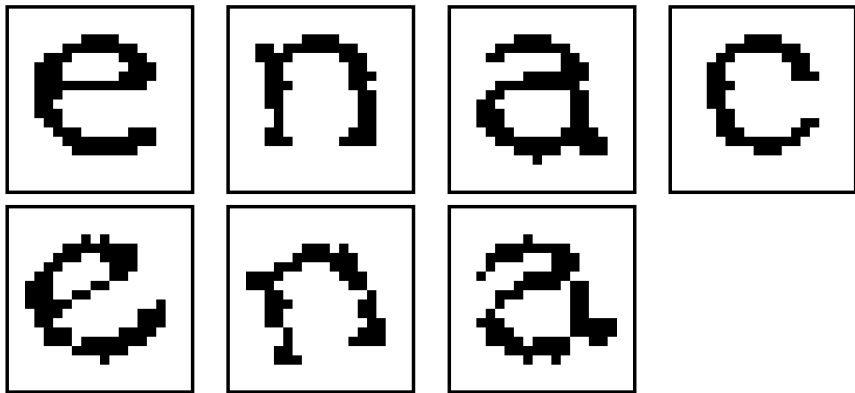
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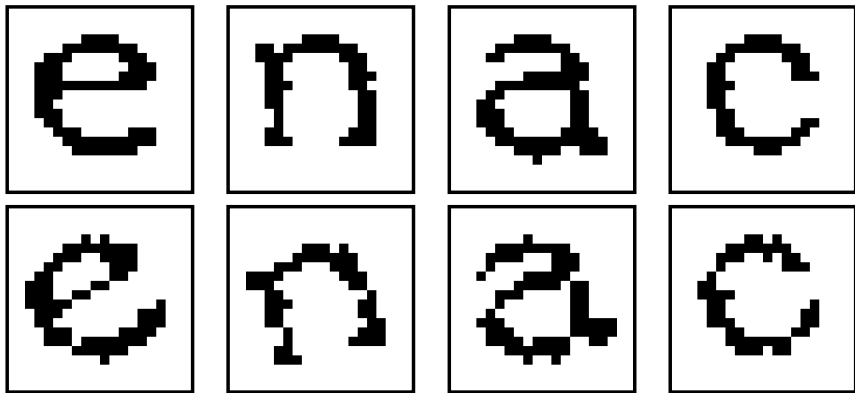
Experiment for letters: Topological variance



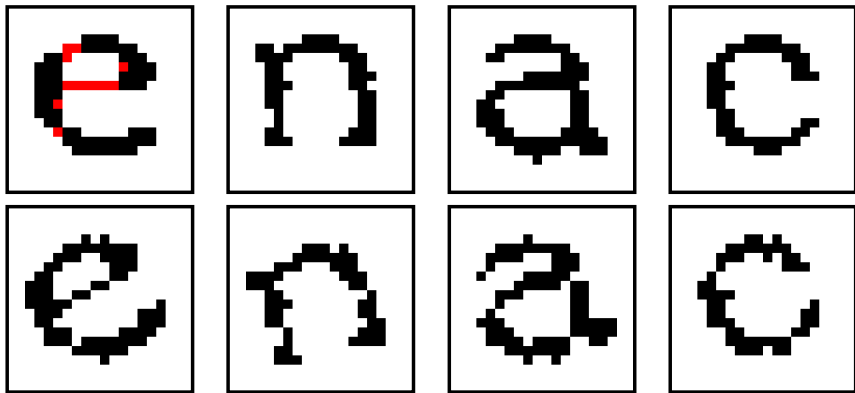
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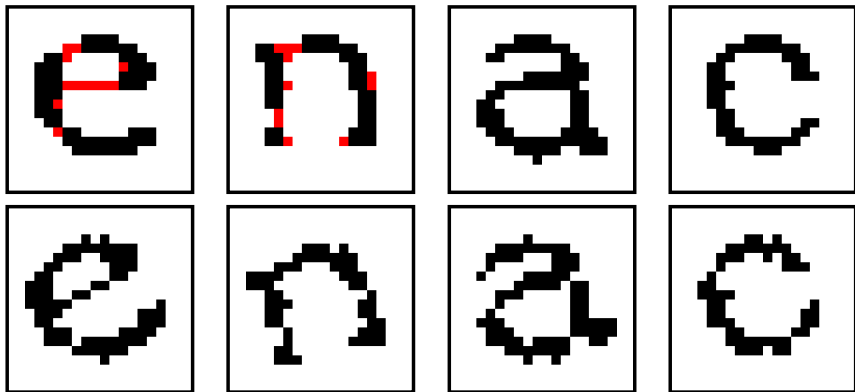
Experiment for letters: Topological variance



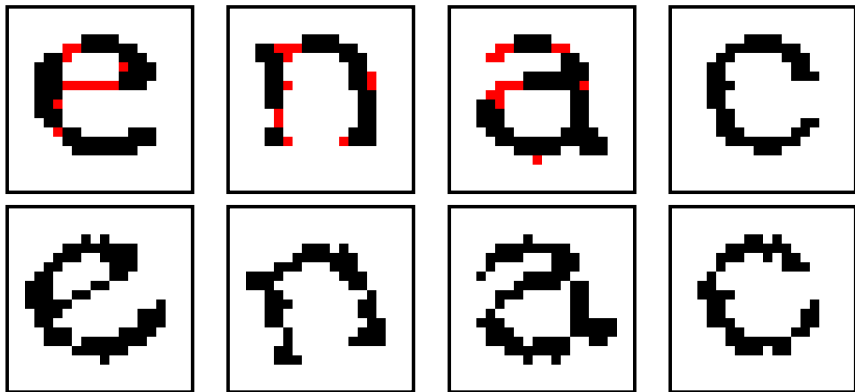
Experiment for letters: Topological variance



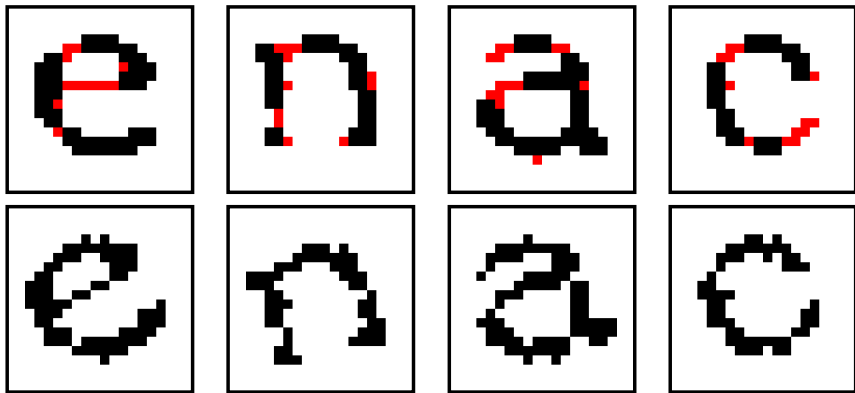
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Conclusion

Results:

- study on the relationships between geometry and topology in the framework of computer imagery,
- proposition of sufficient conditions and a linear algorithm to evaluate the topological invariance of digital images,
- classification of points in binary images.

Conclusion

Perspectives:

- repairing images to be topologically invariant,
- proving the necessary conditions,
- finding topologically invariant conditions, similarly to regularity for digitization,
- extending to well-composed sets, labelled images, *etc.*

Thank you for your attention

We welcome your questions, suggestions and comments!

I'm looking for a post-doc position 😊

Annexes

