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Abstract

We establish the global asymptotic equivalence between a pure jumps Lévy process with unknown Lévy measure ν and a sequence of independent Poisson random variables with parameters depending on ν. Combining this result with the one in Brown and Low (1996), we deduce an asymptotic equivalence between an additive process with unknown drift and Lévy measure and a discrete model composed by a non-parametric regression plus some independent Poisson random variables.

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Non-parametric experiments, deficiency distance, asymptotic equivalence, Lévy processes, additive processes, non-parametric regression.

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1. Introduction

In recent years, the Le Cam theory on the asymptotic equivalence between statistical models has aroused great interest and a large number of works has been published on this subject. Roughly speaking, asymptotic equivalence means that any statistical inference procedure can be transferred from one experiment to the other in such a way that the asymptotic risk remains the same, at least for bounded loss functions. One can use this property in order to obtain asymptotic results working in a simpler but equivalent setting. For the basic concepts and a detailed description of the

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notion of asymptotic equivalence, we refer to [18, 19]. A short review on this topic will be given in Subsection 2.2.

The main result of this paper is that the observation in the time interval $[0,T]$ of a pure jumps Lévy process with unknown Lévy measure ν is asymptotically equivalent to observe independent Poisson random variables whose parameters are of the form $T\nu\big(\big]k+\frac{j-1}{m},k+\frac{j}{m}\big]\big)$. More precisely, in Theorem 2.12 below, we prove that, under certain conditions on the class of admissible Lévy measures ν, observing the $2m^2+2$ independent Poisson random variables

$$\begin{cases} R_{-\infty} \sim \mathscr{P}\Big(T\nu\big(\big]-\infty,-m\big]\big)\Big), \\ R_{j,k} \sim \mathscr{P}\Big(T\nu\Big(\big]k+\frac{j-1}{m},k+\frac{j}{m}\big]\Big)\Big), \\ R_{\infty} \sim \mathscr{P}\Big(T\nu\big(\big]m,\infty\big[\big)\Big), \end{cases}$$

where $j=1,\dots,m$, $k=-m,\dots,m-1$, is asymptotically equivalent, as m goes to infinity, to observe a trajectory $\{x_t\}$ of a pure jumps Lévy process with Lévy measure ν, that is to say a Lévy process whose characteristic function is given by

$$\mathbb{E}\big[e^{iux_t}\big]=\exp\Big(t\int_{\mathbb{R}}\big(e^{iuy}-1\big)\nu(dy)\Big)\quad\forall u\in\mathbb{R},\quad\forall t\in[0,T].$$

Generalizing this result, we consider an additive process $\{x_t\}$ whose characteristic function is given by

$$\mathbb{E}\big[e^{iux_t}\big]=\exp\Big(iu\int_0^t f(r)dr-\frac{u^2}{2}\int_0^t\frac{\sigma^2(r)}{n}dr-t\int_{\mathbb{R}}\big(1-e^{iuy}\big)\nu(dy)\Big). \quad (1)$$

Here $f(\cdot)$ and ν (the latter is supposed to be homogeneous in time) are unknown whereas $\sigma^2(\cdot)$ is supposed to be known. An introduction to additive processes can be found in [7, 25]. We recall the notions which we will be interested in in Subsection 2.1.

Brown and Low [2] have already treated the case $\nu\equiv 0$, which corresponds to the study of the experiments:

$$dy_t=f(t)dt+\frac{\sigma(t)}{\sqrt{n}}dB_t,\quad t\in[0,T], \quad (2)$$

where B_t is a standard Brownian motion. They found out that the continuous model (2) is asymptotically equivalent to its discrete counterpart, i.e.,

the non-parametric regression

$$Y_i = f(t_i) + \sigma(t_i)\xi_i, \quad i = 1, \dots, n, \tag{3}$$

with a uniform grid $t_i = \frac{T(i-1)}{n}$ and standard normal variables ξ_i. They required f to vary in a non-parametric subset of $L_2([0,T])$, defined by a moderate smoothness condition and the discretization $\bar{f}_n$ of f to tend to f quickly enough in the L_2 norm.

Combining our results with the one of Brown and Low, we prove the global asymptotic equivalence between the statistical models associated to (1) and its discrete counterpart, that turns out to be:

$$\begin{cases} Y_i \sim \mathcal{N}\Big(f\Big(\frac{T(i-1)}{n}\Big), \sigma^2\Big(\frac{T(i-1)}{n}\Big)\Big) \\ R_{-\infty} \sim \mathscr{P}\Big(T\nu\Big(\big]-\infty, -m\big]\Big)\Big) \\ R_{j,k} \sim \mathscr{P}\Big(T\nu\Big(\big]k + \frac{j-1}{m}, k + \frac{j}{m}\big]\Big)\Big) \\ R_{\infty} \sim \mathscr{P}\Big(T\nu\Big(\big]m, \infty\big[\Big)\Big) \\ (Y_1, \dots, Y_n) \amalg (R_{-\infty}, R_{1,-m}, \dots, R_{m,-m}, \dots, R_{1,m-1}, \dots, R_{m,m-1}, R_{\infty}), \end{cases} \tag{4}$$

where $i = 1, \dots, n$, $j = 1, \dots, m$, $k = -m, \dots, m-1$, $\mathcal{N}(\cdot)$ stands for the Gaussian distribution, $\mathscr{P}(\cdot)$ for the Poisson one and $\amalg$ means independent. Heuristically, one wants to treat the continuous and discontinuous parts of $\{x_t\}$ as in (1) separately, because they are independent. Then the former is settled by the result of Brown and Low and the latter by Theorem 2.12 below. However, some care is needed to combine those two results. Intrinsic properties of the Le Cam distance, proved in Subsection 2.2, allow us to do so, thus proving the equivalence between models (1) and (4) (Proposition 2.15 below).

As a "toy example", one can think about the case where $\nu = \lambda\delta_1$; then, Proposition 2.15 below states that a process obtained adding a white noise with drift problem to a homogeneous Poisson process with intensity λ is asymptotically equivalent to $(Y_1, \dots, Y_n, R)$, where R has a Poisson distribution of parameter λT and is independent of $(Y_1, \dots, Y_n)$.

In parametric statistics, Le Cam's theory has successfully been applied to a huge variety of experiments, because in this case it usually reduces to the property of local asymptotic normality (LAN) and its modification (see [19]). The asymptotic equivalence for non-parametric experiments is conceptually more demanding, and has been the object of several recent papers.

In particular, asymptotic equivalence theory has been developed for nonparametric regression in [2, 4, 5, 13, 21, 24], nonparametric density estimation models in [1, 23], generalized linear models in [12], nonparametric autoregression in [14, 22], diffusion models in [8, 9, 10], GARCH model in [3], functional linear regression in [20] and spectral density estimation in [11].

The estimation of the characteristic triplet of a Lévy process has been the subject of numerous papers. More generally, recent years have witnessed a great revival of interest in Lévy processes, which form the fundamental building block for stochastic continuous-time models with jumps. There is an important trend using Lévy models in finance, see e.g. [7] for a detailed treatment and many references, but also many recent models in queueing, telecommunications, extreme value theory, quantum theory or biology rely on Lévy processes. To our knowledge, this is the first work studying the case of a local characteristics $(f(\cdot), \sigma^2(\cdot), \nu)$ where f and σ are inhomogeneous in the time and ν possibly infinite. However, we must require that σ is known *a priori*.

We refer e.g. to [6, 15, 16] and references therein for a non-parametric approach to inference for Lévy processes.

The paper is organized as follows. Subsections 2.1 to 2.4 fix assumptions and notation while the main results, as well as examples, are given in Subsection 2.5. The proofs are postponed to Section 3 and, in part, to the Appendix.

2. Assumptions and main results

2.1. Additive processes

Definition 2.1. A stochastic process $\{X_t\} = \{X_t : t \in [0,T]\}$ on $\mathbb{R}$ defined on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ is an *additive process* if the following conditions are satisfied.

1. $X_0 = 0$ $\mathbb{P}$-a.s.
2. For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \ldots < t_n$, random variables X_{t_0}, $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
3. There is $\Omega_0 \in \mathscr{A}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.
4. It is stochastically continuous.

Thanks to the *Lévy-Khintchine formula* (see [7], Theorem 14.1), the characteristic function of any additive process $\{X_t\}$ can be expressed, for all u in $\mathbb{R}$, as:

$$\mathbb{E}\big[e^{iuX_t}\big] = \exp\Big(iu\int_0^t f(r)dr - \frac{u^2}{2}\int_0^t \sigma^2(r)dr - t\int_{\mathbb{R}}(1-e^{iuy}+iuy\mathbb{I}_{|y|\leq 1})\nu(dy)\Big), \tag{5}$$

where f is a real function on $[0,T]$ with finite variation, σ^2 belongs to $L_1[0,T]$ and ν is a positive measure on $\mathbb{R}$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}}(y^2 \wedge 1)\nu(dy) < \infty.$$

In the sequel we shall refer to $(f(\cdot), \sigma^2(\cdot), \nu)$ as the local characteristics of the process $\{X_t\}$ and ν as above will be called a *Lévy measure*. This data characterizes uniquely the law of the process $\{X_t\}$. When $f(\cdot)$ and $\sigma(\cdot)$ are constant functions the increments of the process $\{X_t\}$ satisfying (5) are stationary; in this case the process $\{X_t\}$ is called a *Lévy process* of characteristic triplet (f, σ^2, ν).

Let $D = D([0,T], \mathbb{R})$ be the space of mappings ω from $[0,T]$ into $\mathbb{R}$ that are right-continuous with left limits. Define the *canonical process* $x : D \to D$ by

$$\forall \omega \in D, \quad x_t(\omega) = \omega_t, \ \ \forall t \in [0,T].$$

Let $\mathscr{D}$ be the smallest σ-algebra of parts of D that makes x_s, s in $[0,T]$, measurable. Further, for any $t \in [0,T]$, let $\mathscr{D}_t$ be the smallest σ-algebra that makes x_s, s in $[0,t]$, measurable.

Let $\{X_t\}$ be an additive process defined on $(\Omega, \mathscr{A}, \mathbb{P})$ having local characteristics $(f(\cdot), \sigma^2(\cdot), \nu)$. It is well known that it induces a probability measure $P^{(f,\sigma^2,\nu)}$ on (D,$\mathscr{D}$) such that $\{x_t\}$ defined on $\big(D, \mathscr{D}, P^{(f,\sigma^2,\nu)}\big)$ is an additive process identical in law with $(\{X_t\}, \mathbb{P})$ (that is the local characteristics of $\{x_t\}$ under $P^{(f,\sigma^2,\nu)}$ is $(f(\cdot), \sigma^2(\cdot), \nu)$).

In the sequel we will denote by $\big(\{x_t\}, P^{(f,\sigma^2,\nu)}\big)$ such an additive process, stressing the probability measure.

Further, for every function x in D, we will denote by Δx_r its jump at the time r and by x^c, x^d its continuous and discontinuous part, respectively:

$$\Delta x_r = x_r - \lim_{s\uparrow r} x_s, \ \ x_t^d = \sum_{r\leq t} \Delta x_r, \ \ x_t^c = x_t - x_t^d.$$

In the case where $\int_{|y|\leq 1} |y|\nu(dy) < \infty$, we let

$$\eta_\nu := \int_{|y|\leq 1} y\nu(dy) \tag{6}$$

and we adopt the notation from [25], pages 38-39

$$(f(\cdot), \sigma^2(\cdot), \nu)_0 = (f(\cdot) + \eta_\nu, \sigma^2(\cdot), \nu).$$

Note that, if ν is a finite Lévy measure, then the process $(\{x_t\}, P^{(\eta_\nu, 0, \nu)})$ of characteristic triplet $(0, 0, \nu)_0$ is a compound Poisson process.

We now recall the *Lévy-Itô decomposition*, i.e. the decomposition in continuous and discontinuous parts of an additive process.

Theorem 2.2 (See [25], Theorem 19.3)**.** *Suppose that* $\int_{|y|\leq 1} |y|\nu(dy) < \infty$ *and consider* $\left(\{x_t\}, P^{(f+\eta_\nu, \sigma^2, \nu)}\right)$*. Then the following holds.*

(i) The process $\{x_t^d\}$ *is a Lévy process on* $\mathbb{R}$ *with characteristic triplet* $(\eta_\nu, 0, \nu)$*.*

(ii) The process $\{x_t^c\}$ *is an additive process on* $\mathbb{R}$ *with local characteristics* $(f(\cdot), \sigma^2(\cdot), 0)$*.*

(iii) The two processes $\{x_t^d\}$ *and* $\{x_t^c\}$ *are independent.*

For the proofs of our results we also need the following theorem on the equivalence of measures for Lévy processes. We will use the notation $P|_{\mathscr{D}_t}$ for the restriction of the probability P to $\mathscr{D}_t$ and we will write $\nu \approx \tilde{\nu}$ to indicate that the measures ν and $\tilde{\nu}$ are equivalent.

Theorem 2.3 (See [25], Theorems 33.1–33.2)**.** *Let* $\left(\{x_t\}, P^{(0,0,\tilde{\nu})}\right)$ *and* $\left(\{x_t\}, P^{(\gamma^\nu, 0, \nu)}\right)$ *be two Lévy processes on* $\mathbb{R}$*, where*

$$\gamma^\nu := \int_{|y|\leq 1} y(\nu - \tilde{\nu})(dy) \tag{7}$$

is supposed to be finite. Then $P^{(\gamma^\nu, 0, \nu)}$ *is locally equivalent to* $P^{(0,0,\tilde{\nu})}$ *if and only if* $\nu \approx \tilde{\nu}$ *and the density* $\frac{d\nu}{d\tilde{\nu}} = \rho$ *satisfies*

$$\int (\sqrt{\rho(y)} - 1)^2 \tilde{\nu}(dy) < \infty. \tag{8}$$

When $P^{(\gamma^\nu, 0, \nu)}$ *is locally equivalent to* $P^{(0,0,\tilde{\nu})}$*, the density is*

$$\frac{dP^{(\gamma^\nu, 0, \nu)}}{dP^{(0,0,\tilde{\nu})}}\Big|_{\mathscr{D}_t}(x) = \exp(U_t^\rho(x)),$$

with

$$U_t^\rho(x) = \lim_{\varepsilon \to 0} \Big(\sum_{r \leq t} \ln \rho(\Delta x_r) \mathbb{I}_{|\Delta x_r| > \varepsilon} - \int_{|y|>\varepsilon} t(\rho(y) - 1)\tilde{\nu}(dy) \Big), P^{(0,0,\tilde{\nu})}\text{-a.s.} \tag{9}$$

The convergence in (9) *is uniform in* t *on any bounded interval,* $P^{(0,0,\tilde{\nu})}$*-a.s. Besides,* $U^{\rho}(x)$ *defined by* (9) *is a Lévy process satisfying* $\mathbb{E}_{P^{(0,0,\tilde{\nu})}}[e^{U_t^{\rho}(x)}] = 1$, $\forall t \in [0,T]$.

Remark that the finiteness of the integral appearing in (7) follows from (8).

2.2. Some properties of the Le Cam Δ*-distance*

The concept of asymptotic equivalence that we shall adopt in this paper is tightly related to the Le Cam Δ-distance between statistical experiments [17]. A *statistical model* or *statistical experiment* is a triplet $\mathscr{P}_j = (\mathscr{X}_j, \mathscr{A}_j, \{P_{j,\theta}; \theta \in \Theta\})$ where $\{P_{j,\theta}; \theta \in \Theta\}$ is a family of probability distributions all defined on the same σ-field $\mathscr{A}_j$ over the *sample space* $\mathscr{X}_j$ and Θ is the *parameter space*. For two statistical models $\mathscr{P}_1$ and $\mathscr{P}_2$ indexed by the same parameter space, Le Cam (1964) introduced a quantity, the *deficiency* $\delta(\mathscr{P}_1, \mathscr{P}_2)$ of $\mathscr{P}_1$ with respect to $\mathscr{P}_2$, which quantifies "how much information we lose" by using $\mathscr{P}_1$ instead of $\mathscr{P}_2$. Closely associated with the notion of deficiency or, equivalently, for the (undefined) amounts of information carried by the experiments, is the so called Δ-distance, i.e. the pseudo metric defined by:

$$\Delta(\mathscr{P}_1, \mathscr{P}_2) := \max(\delta(\mathscr{P}_1, \mathscr{P}_2), \delta(\mathscr{P}_2, \mathscr{P}_1)).$$

The original definition of the Le Cam deficiency involves "transitions". More specifically, the deficiency $\delta(\mathscr{P}_1, \mathscr{P}_2)$ of $\mathscr{P}_1$ with respect to $\mathscr{P}_2$ is defined as $\delta(\mathscr{P}_1, \mathscr{P}_2) = \inf_K \sup_{\theta\in\Theta} ||KP_{1,\theta} - P_{2,\theta}||_{TV}$, where TV stands for "total variation" and the infimum is taken over all "transitions" K (see [18], page 18). In our setting, however, the general notion of "transitions" can be replaced with the notion of Markov kernels. Indeed, when the model $\mathscr{P}_1$ is dominated and the sample space $(\mathscr{X}_2, \mathscr{A}_2)$ of the experiment $\mathscr{P}_2$ is a Polish space, the infimum appearing on the definition of the deficiency δ can be taken over all Markov kernels K on $\mathscr{X}_1 \times \mathscr{A}_2$ (see [23], Proposition 10.2), i.e.

$$\delta(\mathscr{P}_1, \mathscr{P}_2) = \inf_K \sup_{\theta\in\Theta} \sup_{A\in\mathscr{A}_2} \left| \int_{\mathscr{X}_1} K(x, A) P_{1,\theta}(dx) - P_{2,\theta}(A) \right|.$$

Two sequences of statistical models $(\mathscr{P}_1^n)_{n\in\mathbb{N}}$ and $(\mathscr{P}_2^n)_{n\in\mathbb{N}}$ are called *asymptotically equivalent* if $\Delta(\mathscr{P}_1^n, \mathscr{P}_2^n)$ tends to zero as n goes to infinity. As a corollary of such an equivalence, every procedure which is asymptotically minimax in $(\mathscr{P}_1^n)_{n\in\mathbb{N}}$ yields a corresponding minimax procedure in $(\mathscr{P}_2^n)_{n\in\mathbb{N}}$. There are various techniques to bound the Δ-distance. In our context we will only use the following two well-known properties (see [18]):

Property 2.4. *Let, for $i = 1, 2$, $\mathscr{P}_i = (\mathscr{X}, \mathscr{A}, \{P_{i,\theta}, \theta \in \Theta\})$ be two dominated statistical models with the same sample space $\mathscr{X}$ and parameter space Θ. Let ξ be a common dominating measure and $g_{i,\theta} = \frac{dP_{i,\theta}}{d\xi}$. Define*

$$L_1(\mathscr{P}_1, \mathscr{P}_2) = \sup_{\theta \in \Theta} \int_{\mathscr{X}} |g_{1,\theta}(x) - g_{2,\theta}(x)| \xi(dx).$$

Then,

$$\Delta(\mathscr{P}_1, \mathscr{P}_2) \leq L_1(\mathscr{P}_1, \mathscr{P}_2).$$

Property 2.5. *Let, for $i = 1, 2$, $\mathscr{P}_i = (\mathscr{X}_i, \mathscr{A}_i, \{P_{i,\theta}, \theta \in \Theta\})$ be two statistical models and let $(\mathscr{X}_1, \mathscr{A}_1)$ be a Polish space. Let $S : \mathscr{X}_1 \to \mathscr{X}_2$ be a sufficient statistics such that the distribution of S under $P_{1,\theta}$ is equal to $P_{2,\theta}$. Then $\Delta(\mathscr{P}_1, \mathscr{P}_2) = 0$.*

In order to prove our statements we shall also need the following two properties. Their proofs can be found at Appendix A .

Property 2.6. *Let $\mathscr{F}$ be the space of functions on $[0, T]$ with finite variation and $\mathscr{M}^{(4)}$ be the space of the Lévy measures satisfying Assumptions (M1)–(M4) below. Consider the statistical models*

$$\mathscr{P}^* = \Big(D, \mathscr{D}, \big\{P^{(f+\eta_\nu, \sigma^2, \nu)}; (f, \nu) \in \mathscr{F} \times \mathscr{M}^{(4)}\big\}\Big),$$
$$\mathscr{P}^{\otimes} = \Big(D^2, \mathscr{D}^{\otimes 2}, \big\{P^{(f,\sigma^2,0)} \otimes P^{(\eta_\nu, 0, \nu)}; (f, \nu) \in \mathscr{F} \times \mathscr{M}^{(4)}\big\}\Big).$$

Then $\Delta(\mathscr{P}^*, \mathscr{P}^{\otimes})$=0.

Property 2.7. *Let us consider, for $i = 1, 2$, the following statistical models,*

$$\mathscr{P}_i^n = \Big(\mathscr{X}_i^n, \mathscr{A}_i^n, \big\{P_\theta^{i,n}; \theta \in \Theta\big\}\Big)$$
$$\mathscr{P}_3 = \Big(\mathscr{X}_3, \mathscr{A}_3, \big\{P_\theta^3; \theta \in \tilde{\Theta}\big\}\Big)$$
$$\mathscr{P}_{i,3}^n = \Big(\mathscr{X}_{i,3}^n, \mathscr{A}_{i,3}^n, \big\{P_\theta^{i,3,n} = P_\theta^{i,n} \otimes P_{\tilde{\theta}}^3; (\theta, \tilde{\theta}) \in \Theta \times \tilde{\Theta}\big\}\Big),$$

where $\mathscr{X}_{i,3}^n = \mathscr{X}_i^n \times \mathscr{X}_3$, $\mathscr{A}_{i,3}^n = \mathscr{A}_i^n \otimes \mathscr{A}_3$ and $(\mathscr{X}_i^n, \mathscr{A}_i^n)$ are dominated Polish spaces. Then, $\lim_{n\to\infty} \Delta(\mathscr{P}_1^n, \mathscr{P}_2^n) = 0$ entails $\lim_{n\to\infty} \Delta(\mathscr{P}_{1,3}^n, \mathscr{P}_{2,3}^n) = 0$.

2.3. The parameter spaces

In order to handle the added aspect of an unknown Lévy measure, the parameter space will be expanded to include both the smooth function f and the Lévy measure ν, namely $\Theta = \mathscr{F} \times \mathscr{M}$. To formulate our results we need to impose the following regularity assumptions on the elements of $\mathscr{F}$ and those of $\mathscr{M}$. We will require that $\mathscr{F}$ is a class of real functions satisfying the following conditions:

(F1)
$$\sup_{f\in\mathscr{F}} \big\{|f(t)| : t \in [0,T]\big\} = B < \infty.$$

(F2) Denote, for $j = 1, \ldots, n$, $I_j = \big[\frac{(j-1)T}{n}, \frac{jT}{n}\big[$. Define the step function $\bar{f}_n(\cdot)$ as
$$\bar{f}_n(t) = \sum_{j=1}^{n} f\Big(\frac{(j-1)T}{n}\Big)\mathbb{I}_{I_j}(t).$$
We ask that:
$$\lim_{n\to\infty} \sup_{f\in\mathscr{F}} n \int_0^T (f(t) - \bar{f}_n(t))^2 dt = 0.$$

The hypotheses (F_1) and (F_2) were already present in the work of Brown and Low [2], where one can also find examples of some functional classes satisfying them. Concerning conditions on the class of measures $\mathscr{M}$, we will require that:

(M1) There exists a Lévy measure $\tilde{\nu}$ such that $\nu \approx \tilde{\nu}$ for all ν in $\mathscr{M}$ (we will write ρ^ν for the density $\frac{d\nu}{d\tilde{\nu}}$).

(M2) $\int_{\mathbb{R}} (\sqrt{\rho^\nu(y)} - 1)^2 \tilde{\nu}(dy) < \infty$, for all ν in $\mathscr{M}$.

Moreover, following the same principle as in [2], we introduce a discretization of the measure ν. To that aim define

$$\bar{\rho}_m^\nu(y) := \begin{cases} \frac{\nu(J_{-\infty})}{\tilde{\nu}(J_{-\infty})} & \text{if } y \in J_{-\infty} \\ \quad\vdots & \\ \frac{\nu(J_{j,k})}{\tilde{\nu}(J_{j,k})} & \text{if } y \in J_{j,k} \text{ and } J_{j,k} \not\subseteq]-\frac{1}{m}, \frac{1}{m}], \\ \quad\vdots & \\ 1 & \text{if } y \in]-\frac{1}{m}, \frac{1}{m}], \\ \quad\vdots & \\ \frac{\nu(J_{\infty})}{\tilde{\nu}(J_{\infty})} & \text{if } y \in J_{\infty}, \end{cases} \tag{10}$$

where, for $j = 1, \dots, m$ and $k = -m, \dots, m-1$,

$$J_{-\infty} = \Big]-\infty, -m\Big], \quad J_{j,k} = \Big]k + \frac{j-1}{m}, k + \frac{j}{m}\Big], \quad J_{\infty} = \Big]m, \infty\Big[.$$

Define two more conditions as:

(M3) $\lim_{m\to\infty} \sup_{\nu\in\mathscr{M}} \int_{\mathbb{R}} |\rho^{\nu}(y) - \bar{\rho}^{\nu}_m(y)|\tilde{\nu}(dy) = 0,$

(M4) $\sup_{\nu\in\mathscr{M}} \int_{|y|\le 1} |y|\nu(dy) < \infty$ and $\int_{|y|\le 1} |y|\tilde{\nu}(dy) < \infty,$

where we have denoted by $\bar{\nu}_m$ the measure having $\bar{\rho}^{\nu}_m$ in (10) as a density with respect to $\tilde{\nu}$. For brevity's sake, in the sequel we will omit the symbol ν simply writing $\rho = \rho^{\nu}$ or $\bar{\rho}_n = \bar{\rho}^{\nu}_n$, when this causes no confusion.

Notation 2.8. In the following, we will denote by $\mathscr{M}^{(3)}$ a class of Lévy measures satisfying (M1)–(M3) and by $\mathscr{M}^{(4)}$ a class of Lévy measures satisfying (M1)–(M4). Further, $\mathscr{F}$ will always denote a class of functions on $[0, T]$ satisfying (F1) and (F2).

Here are some examples of possible choices of $\mathscr{M}^{(4)}$. Proofs can be found at Appendix B.

Example 2.9. Let $\tilde{\nu}$ be a known finite Lévy measure and L, K be fixed finite positive real numbers. Consider

$$\mathscr{M}^{(4)}_{\tilde{\nu},L,K} = \Big\{\nu \text{ Lévy measure } : \rho = \frac{d\nu}{d\tilde{\nu}} > 0 \text{ exists, differentiable } \tilde{\nu}\text{-a.e.,} \\ L\text{-Liptschtiz and } |\rho(0)| \le K\Big\}.$$

Remark that the stated conditions on $\tilde{\nu}$ and ρ imply in particular that ν must be finite, i.e. $\nu(\mathbb{R}) < \infty$.

Example 2.10. Let ε, C and M be fixed positive real numbers, with ε and C known. Consider the class of all Gamma Lévy measures, defined by:

$$\mathscr{M}^{\Gamma,(4)}_{\varepsilon,C,M} = \Big\{\nu \text{ Lévy measure } : \text{its density with respect to Lebesgue is} \\ g(y) = Ce^{-\lambda y}y^{-1}\mathbb{I}_{y>0} \text{ where } \varepsilon \le \lambda \le M\Big\}.$$

Example 2.11. Let $\alpha, \varepsilon, C_1, C_2$ be known positive real numbers such that $\alpha < 1$ and let M be a positive number. Consider the class of all tempered stable Lévy measures, that is

$$\mathscr{M}^{T,(4)}_{\alpha,C_1,C_2,M,\varepsilon} = \Big\{\nu \text{ Lévy measure } : \text{its density with respect to Lebesgue is } g(y) = \frac{C_1}{|y|^{1+\alpha}} e^{-\lambda_1 |y|} \mathbb{I}_{y<0} + \frac{C_2}{y^{1+\alpha}} e^{-\lambda_2 y} \mathbb{I}_{y\geq 0} \text{ where } \varepsilon \leq \lambda_j \leq M,\ j = 1, 2\Big\}.$$

2.4. Definition of the experiments

Let us now introduce the classes of experiments that we shall consider. From now on we assume that $\sigma^2(\cdot)$ is a function in $L_1[0,T]$ supposed to be known. Further we consider $2m^2 + 2$ observations of the form

$$\begin{cases} R_{-\infty} \sim \mathscr{P}\Big(T\nu\Big(\big]-\infty, -m\big]\Big)\Big), \\ \quad \vdots \\ R_{j,k} \sim \mathscr{P}\Big(T\nu\Big(\big]k + \frac{j-1}{m}, k + \frac{j}{m}\big]\Big)\Big), \\ \quad \vdots \\ R_{\infty} \sim \mathscr{P}\Big(T\nu\Big(\big]m, \infty\big[\Big)\Big). \end{cases}$$

where $\mathscr{P}(\cdot)$ stands for the Poisson distribution and the components of the vector $R = (R_{-\infty}, \ldots, R_{j,k}, \ldots, R_{\infty})$, $j = 1, \ldots, m$, $k = -m, \ldots, m-1$, are independent. Let $Q_\nu^{m,R}$ be the law of R. Then, the first pair of statistical experiments is described by

$$\mathscr{P}^{(\gamma^\nu, 0, \nu)} = \Big(D, \mathscr{D}, \big\{P^{(\gamma^\nu,0,\nu)}; \nu \in \mathscr{M}^{(3)}\big\}\Big) \tag{11}$$

$$\mathscr{Q}^R_m = \Big(\mathbb{N}^{2m^2+2}, \mathcal{P}(\mathbb{N}^{2m^2+2}), \big\{Q^{m,R}_\nu; \nu \in \mathscr{M}^{(3)}\big\}\Big). \tag{12}$$

The second pair consists of the experiments

$$\mathscr{P}'^{(\eta_\nu, 0, \nu)} = \Big(D, \mathscr{D}, \big\{P^{(\eta_\nu,0,\nu)}; \nu \in \mathscr{M}^{(4)}\big\}\Big) \tag{13}$$

$$\mathscr{Q}'^R_m = \Big(\mathbb{N}^{2m^2+2}, \mathcal{P}(\mathbb{N}^{2m^2+2}), \big\{Q^{m,R}_\nu; \nu \in \mathscr{M}^{(4)}\big\}\Big). \tag{14}$$

The asymptotic equivalence between the statistical models $\mathscr{P}^{(\gamma^\nu,0,\nu)}$, $\mathscr{Q}_m^R$ and between $\mathscr{P}'^{(\eta_\nu,0,\nu)}$, $\mathscr{Q}_m'^R$ will be the subject of Theorem 2.12 below. Furthermore, we will consider the non-parametric regression:

$$Y_i = f(t_i) + \sigma(t_i)\xi_i, \quad i = 1, \dots, n, \tag{15}$$

for a uniform grid $t_i = \frac{T(i-1)}{n}$ and standard i.i.d. normal variables ξ_i. Formally,

$$\mathscr{Q}_n^Y = \Big(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \big\{Q_f^{n,Y}; f \in \mathscr{F}\big\}\Big), \tag{16}$$

where $Q_f^{n,Y}$ is the law of $(Y_1, \dots, Y_n)$. Asymptotic equivalence between $\mathscr{Q}_n^Y$ and

$$\mathscr{P}^{(f,\sigma^2/n,0)} = \Big(D, \mathscr{D}, \big\{P^{(f,\sigma^2/n,0)}; f \in \mathscr{F}\big\}\Big)$$

has already been proved by Brown and Low [2].

Finally, we will be interested in the product statistical model between $\mathscr{Q}_m'^R$ and $\mathscr{Q}_n^Y$, i.e.

$$\mathscr{Q}_n^Y \otimes \mathscr{Q}_m'^R = \Big(\mathbb{R}^n \times \mathbb{N}^{2m^2+2}, \mathscr{B}(\mathbb{R}^n) \otimes \mathcal{P}(\mathbb{N}^{2m^2+2}), \big\{Q_f^{n,Y} \otimes Q_\nu^{m,R}; (f,\nu) \in \mathscr{F} \times \mathscr{M}^{(4)}\big\}\Big). \tag{17}$$

Asymptotic equivalence of statistical models $\mathscr{Q}_n^Y \otimes \mathscr{Q}_m'^R$ and

$$\mathscr{P}'^{(f+\eta_\nu,\sigma^2/n,\nu)} = \Big(D, \mathscr{D}, \big\{P^{(f+\eta_\nu,\sigma^2/n,\nu)}; (f,\nu) \in \mathscr{F} \times \mathscr{M}^{(4)}\big\}\Big)$$

will be treated in Proposition 2.15.

2.5. Main results and Examples

Notations will be kept as in Notation 2.8 and Subsection 2.4.

Theorem 2.12. *The experiments $\mathscr{P}^{(\gamma^\nu,0,\nu)}$ and $\mathscr{Q}_m^R$ are asymptotically equivalent, that is*

$$\lim_{m\to\infty} \Delta\big(\mathscr{P}^{(\gamma^\nu,0,\nu)}, \mathscr{Q}_m^R\big) = 0. \tag{18}$$

We also have

$$\lim_{m\to\infty} \Delta\big(\mathscr{P}'^{(\eta_\nu,0,\nu)}, \mathscr{Q}_m'^R\big) = 0. \tag{19}$$

Remark 2.13. The asymptotic equivalence in (19) involves the parameter space $\mathscr{M}^{(4)}$ which is smaller than that in (18), but this allows us to treat the model $\mathscr{P}'^{(\eta_\nu,0,\nu)}$, that is, the case of pure jumps Lévy processes. This will be helpful for Proposition 2.15 stated below.

Loosely speaking, the main interest in Le Cam's asymptotic decision theory lies in the approximation of general statistical experiments by simpler ones. Adopting this point of view, we can reformulate Theorem 2.12 as follows.

Corollary 2.14. *As a by-product of Theorem 2.12, we obtain that, so far the study of ν is concerned, observing a Lévy process $\{X_t\}$ of characteristic triplet $(\gamma^\nu, 0, \nu)$ (or $(\eta_\nu, 0, \nu)$) asymptotically gives the same amount of information as the* a priori *coarser process $\big\{\sum_{t\le T} \mathbb{I}_A(\Delta X_t)\big\}_{A\in\mathscr{A}_m}$, where $\mathscr{A}^m$ is the set defined by $\mathscr{A}^m = \Big\{J_{\pm\infty}, J_{j,k}; k = -m, \dots, m-1, j = 1, \dots, m\Big\}$.*

Combining the result of Brown and Low [2] with Theorem 2.12 we obtain the following statement.

Proposition 2.15. *Let $\sigma^2(\cdot) > 0$ be a known absolutely continuous function on $[0, T]$ such that*

$$\left|\frac{d}{dt}\ln\sigma(t)\right| \le C_1, \qquad t \in [0, T], \tag{20}$$

for some $C_1 < \infty$. Then the experiments $\mathscr{P}'^{(f+\eta_\nu, \sigma^2/n, \nu)}$ and $\mathscr{Q}_n^Y \otimes \mathscr{Q}_m'^R$ are asymptotically equivalent, that is

$$\lim_{n,m\to\infty} \Delta\big(\mathscr{P}'^{(f+\eta_\nu, \sigma^2/n, \nu)}, \mathscr{Q}_n^Y \otimes \mathscr{Q}_m'^R\big) = 0. \tag{21}$$

Remark 2.16. An important consequence of the asymptotic equivalence between $\mathscr{P}^{(\gamma^\nu, 0, \nu)}$ and $\mathscr{Q}_m^R$ is that every inference procedure (such as estimators or tests) available in $\mathscr{P}^{(\gamma^\nu, 0, \nu)}$ have a corresponding procedure in $\mathscr{Q}_m^R$ that performs nearly as well and vice versa.

Theorem 2.12 allows us to rewrite the asymptotic equivalence (18) between the statistical models $\mathscr{P}^{(\gamma^\nu, 0, \nu)}$ and $\mathscr{Q}_m^R$ in terms of decision theory as follows.

If there exists a procedure τ_1 in $\mathscr{P}^{(\gamma^\nu, 0, \nu)}$ with risk $P^{(\gamma^\nu, 0, \nu)} L_m(\tau_1)$ then, for bounded loss functions L_m, there exists a procedure τ_2^m relative to $\mathscr{Q}_m^R$ such that

$$\lim_{m\to\infty} \sup_{(f,\nu)\in\mathscr{F}\times\mathscr{M}^{(3)}} \left|P^{(\gamma^\nu, 0, \nu)} L_m(\tau_1) - Q_\nu^{m,R} L_m(\tau_2^m)\right| = 0.$$

Clearly, a similar interpretation also exists for the asymptotic equivalences (19) and (21).

Remark 2.17. Hypotheses (M1)–(M2) ensure good convergence properties for an appropriate transformation of $\big(\{x_t^d\}, P^{(f,\sigma^2,\nu)}\big)$ which plays an important role in proving Theorem 2.12. Hypotheses (M3) and (M4) are uniform smoothness conditions on ν. The four hypotheses (M1)–(M4) are satisfied in a wide variety of examples, in particular in Examples 2.9, 2.10 and 2.11. For a discussion about hypotheses (F1) and (F2) we refer to Brown and Low [2] who proved (21) in the case $\nu \equiv 0$, under hypothesis (20) on $\sigma(\cdot)$.

We now propose two different examples fitting in the hypotheses of Theorem 2.12. The first one treats the class of the compound Poisson processes, hence it is an example where the class of Lévy measures $\mathscr{M}^{(4)}$ consists of finite measures. The second one enlightens how the Theorem 2.12 can be applied in situations where the Lévy measure is not finite by treating the class of the tempered stable processes.

Compound Poisson Process

Consider the process

$$X_t = \sum_{i=1}^{N_t} Y_i, \quad \mathbb{P}\text{-a.s.}$$

where jumps size Y_i are i.i.d. random variables with unknown distribution μ ($\mu(\cdot) = \mathbb{P}(Y_1 \in \cdot)$) and $(\{N_t\}, \mathbb{P})$ is a Poisson process with unknown intensity λ, independent from $(Y_i)_{i\geq 1}$. Thus, $\{X_t\}$ is a Lévy process with characteristic triplet $\big(\lambda \int_{-1}^{1} y\mu(dy), 0, \lambda\mu(dy)\big)$. Moreover, let us assume that:

- the law μ of Y_1 varies in a family dominated by a given finite measure $\tilde{\mu}$ such that $\frac{d\mu}{d\tilde{\mu}} > 0$,
- the density $r := \frac{d\mu}{d\tilde{\mu}}$ is L-Lipschitz, $|r(0)| \leq K$ and $\sup\limits_{\lambda,\mu} \lambda \int_{|y|\leq 1} |y|\mu(dy) < \infty$, for some $L, K < \infty$.

Those assumptions entail that the Lévy measure $\nu(dy) = \lambda\mu(dy)$ of $\{X_t\}$ belongs to the class $\mathscr{M}^{(4)}_{\tilde{\mu},L,K}$ of Example 2.9. Hence we can deduce the asymptotic equivalence between the statistical models associated to the process $\{X_t\}$ and to the variables:

$$\begin{cases} R_{\pm\infty} \sim \mathscr{P}\big(T\lambda\mu(J_{\pm\infty})\big) \\ \vdots \\ R_{j,k} \sim \mathscr{P}\big(T\lambda\mu(J_{j,k})\big) \end{cases}$$

where $j = 1, \dots, m$ and $k = -m, \dots, m-1$.

We can also reinterpret this result in light of Corollary 2.14 and Remark 2.16 that allow us to construct an asymptotically minimax estimator for the parameter (λ, μ) in the statistical model associated to the process $\{X_t\}$ from a minimax estimator of the same parameter in the submodel associated to the process $\{\sum_{i=1}^{N_T} \mathbb{I}_A(Y_i)\}_{A \in \mathscr{A}_m}$. Note that the latter is simply a sum of independent Bernoulli random variables counting the number of jumps whose amplitude falls in a given set A in $\mathscr{A}_m$.

Tempered Stable Processes

Let $\{T_t\}$ be a tempered stable process, that is, a Lévy process on $\mathbb{R}$ with no Gaussian component and such that its Lévy measure has a density of the form

$$\rho(y) = \frac{C_-}{|y|^{1+\alpha}} e^{-\lambda_- |y|} \mathbb{I}_{y<0} + \frac{C_+}{y^{1+\alpha}} e^{-\lambda_+ y} \mathbb{I}_{y>0},$$

where the parameters satisfy $C_\pm > 0$, $\lambda_\pm > 0$ and $\alpha < 2$, with $\lambda_\pm$ unknown.

Further, consider the statistical model associated to:

$$\begin{cases} R_{\pm\infty} \sim \mathscr{P}\left(T \int_{J_{\pm\infty}} \rho(y) dy\right) \\ R_{j,k} \sim \mathscr{P}\left(T \int_{J_{j,k}} \rho(y) dy\right) \end{cases} \tag{22}$$

where $j = 1, \dots, m$ and $k = -m, \dots, m-1$.

If we suppose that:

- $0 < \alpha < 1$ and $C_\pm < \infty$ are kept fixed and known;
- there exist fixed $\varepsilon > 0$ and $M < \infty$ such that $\varepsilon \leq \lambda_\pm \leq M$, with ε known;
- the quantity $C_+ \lambda_+^{\alpha-1} - C_- \lambda_-^{\alpha-1}$ is supposed to be known (that is, η_ν is known);

then, thanks to Example 2.11, the statistical models associated to $\{T_t\}$ and to (22) are asymptotically equivalent.

Here is an example of a model fitting in the framework of Proposition 2.15.

Merton model with inhomogeneous drift

Consider:

$$X_t = \int_0^t f(r)dr + \int_0^t \frac{\sigma(r)}{\sqrt{n}} dB_r + \sum_{i=1}^{N_t} Y_i \quad \mathbb{P}\text{-a.s.} \tag{23}$$

where $(\{B_t\}, \mathbb{P})$ is a standard Brownian motion, $(\{N_t\}, \mathbb{P})$ is a Poisson process of unknown intensity λ and $(Y_i)_i$ is a sequence, independent of the process $\{N_t\}$, of i.i.d. random variables distributed as $\mathcal{N}(\mu, \xi^2)$, μ and ξ^2 unknown.

Further, note by $\phi_{\mu,\xi^2}(A) := \mathbb{P}(X \in A)$, $\forall A \in \mathscr{B}(\mathbb{R})$ where $X \sim \mathcal{N}(\mu, \xi^2)$. Consider the statistical model associated to the variables:

$$\begin{cases} Y_i \sim \mathcal{N}\Big(f\Big(\frac{T(j-1)}{n}\Big), \sigma^2\Big(\frac{T(j-1)}{n}\Big)\Big) \\ R_{\pm\infty} \sim \mathscr{P}\big(T\lambda\phi_{\mu,\xi^2}\big(J_{\pm\infty}\big)\big) \\ R_{j,k} \sim \mathscr{P}\big(T\lambda\mathbb{P}\phi_{\mu,\xi^2}\big(J_{j,k}\big)\big) \\ (Y_1, \ldots, Y_n) \amalg (R_{-\infty}, R_{1,-m}, \ldots, R_{m,-m}, \ldots, R_{1,m-1}, \ldots, R_{m,m-1}, R_\infty), \end{cases} \tag{24}$$

where $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = -m, \ldots, m-1$.

Then, if we suppose that

- $f(\cdot)$ satisfies (F1) and (F2) and $\sigma(\cdot)$ verifies (20),
- there exist real numbers $\varepsilon > 0$ and $L, M < \infty$ such that $\xi^2 \geq 2\varepsilon^2$, $\mu \leq M$ and $\lambda \leq L$ with ε and L known,

the statistical models associated to (23) and to (24) are asymptotically equivalent. To prove this fact, simply remark that the Lévy measure of $\{X_t\}$ belongs to the class $\mathscr{M}^{(4)}_{\tilde{\nu},C,K}$ of Example 2.9 taking $\tilde{\nu}(A) = L\phi_{0,\varepsilon^2}(A)$, $C = \frac{1}{2\varepsilon}\exp\big(\frac{M^2}{\varepsilon^2} - \frac{1}{2}\big)$ and $K = \frac{1}{\sqrt{2}}$. Indeed, with such a choice of $\tilde{\nu}$, we have $\rho(y) := \frac{d\nu}{d\tilde{\nu}}(y) = \frac{\lambda\varepsilon}{L|\xi|}\exp\Big(\frac{(y-\mu)^2}{2\xi^2} - \frac{y^2}{2L^2}\Big)$, hence $\Big|\Big|\frac{d\rho(y)}{dy}\Big|\Big|_\infty = \frac{\lambda\varepsilon}{L|\xi|}\frac{\sqrt{\xi^2-\varepsilon^2}}{\xi^2}\exp\big(\frac{\varepsilon^2+\mu^2-\xi^2}{2(\xi^2-\varepsilon^2)}\big) \leq \frac{1}{2\sqrt{2}\varepsilon}\exp\big(\frac{M^2}{\varepsilon^2} - \frac{1}{2}\big)$.

3. Proofs

This section is devoted to the proofs of our main results.

3.1. Proof of Theorem 2.12

We will proceed in three steps.

STEP 1. The task is to prove that: $\lim_{m\to\infty} \Delta\big(\mathscr{P}^{(\gamma^\nu,0,\nu)}, \mathscr{P}^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}\big) = 0$. Recall that $\bar\nu_m$ is the Lévy measure defined in Subsection 2.3 and recall also the notation:

$$\gamma^\nu = \int_{|y|\le 1} y(\nu(dy) - \tilde\nu(dy)),$$

(this quantity is finite thanks to Assumption (M2), see [25], Remark 33.3). Moreover, $\gamma^{\bar\nu_m}$ is finite thanks to Hypothesis (8) and the definition of $\bar\rho_m$. We know, by Theorem 2.3, that the L_1 distance between $P^{(\gamma^\nu,0,\nu)}$ and $P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}$ is given by

$$\begin{aligned} L_1\Big(P^{(\gamma^\nu,0,\nu)}, P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}\Big) &= \mathbb{E}_{P^{(0,0,\tilde\nu)}}\left[\Big|\frac{dP^{(\gamma^\nu,0,\nu)}}{dP^{(0,0,\tilde\nu)}}(x) - \frac{dP^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}}{dP^{(0,0,\tilde\nu)}}(x)\Big|\right] \\ &= \mathbb{E}_{P^{(\gamma^\nu,0,\nu)}}\left[\Big|1 - \frac{dP^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}}{dP^{(\gamma^\nu,0,\nu)}}(x)\Big|\right] \\ &= \mathbb{E}_{P^{(\gamma^\nu,0,\nu)}}\Big[\big|1 - \exp\big(U_T^{\bar\rho_m}(x) - U_T^{\rho}(x)\big)\big|\Big], \end{aligned}$$

with $U_T^\rho(x)$ defined as in (9). Introduce the quantity $R_T^m(x) := \exp\big(U_T^{\bar\rho_m}(x) - U_T^\rho(x)\big)$ and observe that, by definition, $P^{(\gamma^\nu,0,\nu)}$ a.s., we have:

$$R_T^m(x) = \exp\left(\lim_{\varepsilon\to 0}\Big(\sum_{r\le T} \ln\frac{d\bar\nu_m}{d\nu}(\Delta x_r)\mathbb{I}_{|\Delta x_r|>\varepsilon} - T\int_{|y|>\varepsilon}\Big(\frac{d\bar\nu_m}{d\nu}(y) - 1\Big)\nu(dy)\Big)\right),$$

so, by Lemma Appendix A.2, we get:

$$L_1\Big(P^{(\gamma^\nu,0,\nu)}, P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}\Big) \le 2\sinh\Big(T\int_{\mathbb{R}} |\rho(y) - \bar\rho_m(y)|\tilde\nu(dy)\Big). \tag{25}$$

Thus, thanks to Assumption (M3), we have

$$\lim_{m\to\infty}\sup_{\nu\in\mathscr{M}^{(3)}} L_1\Big(P^{(\gamma^\nu,0,\nu)}, P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}\Big) = 0.$$

By Property 2.4, we conclude that the models $\mathscr{P}^{(\gamma^\nu,0,\nu)}$ and $\mathscr{P}^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}$ are asymptotically equivalent as m goes to infinity.

STEP 2. The goal is to prove that: $\Delta\big(\mathscr{P}^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}, \mathscr{Q}_m^R\big) = 0$ for all m. Consider the statistics $S : (D, \mathscr{D}) \to \big(\bar{\mathbb{N}}^{2m^2+2}, \mathcal{P}(\bar{\mathbb{N}}^{2m^2+2})\big)$ defined by

$$S(x) = \Big(N_T^{x;-\infty}, N_T^{x;\,1,-m}, \dots, N_T^{x;\,m,m-1}, N_T^{x;\infty}\Big),$$

where

$$N_T^{x;\pm\infty} = \sum_{r\le T} \mathbb{I}_{J_{\pm\infty}}(\Delta x_r), \quad N_T^{x;\,j,k} = \sum_{r\le T} \mathbb{I}_{J_{j,k}}(\Delta x_r).$$

We shall prove, using the Fisher's factorization theorem, that S is a sufficient statistics for the family of probability measures $\{P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}; \nu \in \mathscr{M}^{(3)}\}$. To that aim, we need to insure the existence of a density function for $P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}$ with respect to a dominating measure, that can be written as a function of ν and $S(x)$. The existence of such a density is guaranteed by Theorem 2.3 applied to $\big(\{x_t\}, P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}\big)$ and $(\{x_t\}, P^{(0,0,\tilde\nu)})$. The fact that the hypotheses of the mentioned theorem are satisfied is guaranteed by the definition of $\bar\rho_m$. Indeed, the finiteness of $\int(\sqrt{\bar\rho_m}-1)^2 d\tilde\nu < \infty$ follows from the fact that $\bar\rho_m \equiv 1$ close to zero and that $\bar\nu_m$ and $\tilde\nu$ are Lévy measures. Hence, with the usual notations, we have

$$\begin{aligned}\frac{dP^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}}{dP^{(0,0,\tilde\nu)}}(x) = e^{U_T^{\bar\rho_m}(x)} = \exp\Bigg(&\ln\frac{\nu(J_{-\infty})}{\tilde\nu(J_{-\infty})} N_T^{x;-\infty} + \ln\frac{\nu(J_\infty)}{\tilde\nu(J_\infty)} N_T^{x;\infty} \\ &+ \sum_{\substack{j=1,\dots,m\\ k=-m,\dots,m-1}} \ln\frac{\nu(J_{j,k})}{\tilde\nu(J_{j,k})} N_T^{x;\,j,k} + T\int_{\mathbb{R}} (\bar\rho_m(y)-1)\tilde\nu(dy)\Bigg).\end{aligned}$$

Now note that, by definition, $\gamma^{\bar\nu_m}$ depends only on the parameters $\tilde\nu$ and ν, hence we conclude that S is a sufficient statistics for $\{P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}; \nu \in \mathscr{M}^{(3)}\}$. Furthermore, under $P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}$, the random variables $N_T^{x;j,k}$ (resp. $N_T^{x;-\infty}$ and $N_T^{x;\infty}$) have Poisson distributions with parameters $T\bar\nu_m\big(J_{j,k}\big)$ (resp. $T\bar\nu_m\big(J_{-\infty}\big)$ and $T\bar\nu_m\big(J_\infty\big)$), which, by the definition of $\bar\nu_m$, is equal to $T\nu\big(J_{j,k}\big)$ (resp. $T\nu\big(J_{-\infty}\big)$, $T\nu\big(J_\infty\big)$). Moreover, since the J's intervals are disjoint, the random variables $N_T^{x,\cdot}$ are independent. It follows that the law of S under $P^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}$ is $Q_\nu^{m,R}$. Then, by means of Property 2.5, we get

$$\Delta(\mathscr{P}^{(\gamma^{\bar\nu_m},0,\bar\nu_m)}, \mathscr{Q}_m^R) = 0, \text{ for all } m.$$

STEP 3. The purpose is to prove that: if ν belongs to $\mathscr{M}^{(4)}$ then,

$$\Delta(\mathscr{P}^{(\gamma^\nu,0,\nu)}, \mathscr{P}'^{(\eta_\nu,0,\nu)}) = 0.$$

To that aim, consider the Markov kernels π_1, π_2 defined as follows

$$\begin{aligned}&\pi_1(x,A) = \mathbb{I}_A(x^d),\\ &\pi_2(x,A) = \mathbb{I}_A(x^d - \eta_{\tilde\nu}),\end{aligned} \qquad x\in D, A\in\mathscr{D}.$$

On the one hand we have:

$$\begin{aligned}\pi_1 P^{(\gamma^\nu,0,\nu)}(A) &= \int_D \pi_1(x,A) P^{(\gamma^\nu,0,\nu)}(dx) = \int_D \mathbb{I}_A(x^d) P^{(\gamma^\nu,0,\nu)}(dx) \\ &= P^{(\eta_\nu,0,\nu)}(A),\end{aligned}$$

where in the last equality we have used the fact that, under $P^{(\gamma^\nu,0,\nu)}$, $\{x_t^d\}$ is a Lévy process with characteristic triplet $(\eta_\nu, 0, \nu)$ (see point (i) of Theorem 2.2). On the other hand:

$$\begin{aligned}\pi_2 P^{(\eta_\nu,0,\nu)}(A) &= \int_D \pi_2(x,A) P^{(\eta_\nu,0,\nu)}(dx) = \int_D \mathbb{I}_A(x^d - \eta_{\tilde{\nu}}) P^{(\eta_\nu,0,\nu)}(dx) \\ &= P^{(\gamma^\nu,0,\nu)}(A),\end{aligned}$$

since, by definition, γ^ν is equal to $\eta_\nu - \eta_{\tilde{\nu}}$. The conclusion follows by the definition of the Δ-distance. □

3.2. Proof of Corollary 2.14

It is enough to note that, for all A in $\mathscr{A}_m$, the random variable $\sum_{t\leq T} \mathbb{I}_A(\Delta x_t)$ has a Poisson distribution of parameter $T\nu(A)$ under both $P^{(\gamma^\nu,0,\nu)}$ and $P^{(\eta_\nu,0,\nu)}$. □

3.3. Proof of Proposition 2.15

Recall that, using our notations, Brown and Low have proved in [2] the asymptotic equivalence between $\mathscr{P}^{(f,\sigma^2/n,0)}$ and $\mathscr{Q}_n^Y$. Making use of the previous results, we conclude:

$$\Delta\Big(\mathscr{P}'^{(f+\eta_\nu,\sigma^2/n,\nu)}, \mathscr{P}^{(f,\sigma^2/n,0)} \otimes \mathscr{P}'^{(\eta_\nu,0,\nu)}\Big) = 0 \text{ (by Property 2.6)},$$

$$\lim_{n\to\infty} \Delta\Big(\mathscr{P}^{(f,\sigma^2/n,0)} \otimes \mathscr{P}'^{(\eta_\nu,0,\nu)}, \mathscr{Q}_n^Y \otimes \mathscr{P}'^{(\eta_\nu,0,\nu)}\Big) = 0 \text{ (by [2] and Property 2.7)},$$

$$\lim_{m\to\infty} \Delta\Big(\mathscr{Q}_n^Y \otimes \mathscr{P}'^{(\eta_\nu,0,\nu)}, \mathscr{Q}_n^Y \otimes \mathscr{Q}_m'^R\Big) = 0 \text{ (by Theorem 2.12 and Property 2.7)}.$$

Remark that, in the last expression, the convergence is uniform in n, therefore Proposition 2.15 is proved. □

Appendix A. Some technical proofs

In the following, we will limit ourselves to defining Markov kernels on the rectangular sets since they generate the product σ-algebra.

Proof of Property 2.6. Let us consider the Markov kernels π^*, $\pi^{\otimes}$ defined as follows, for $x_j \in D$, $A_j \in \mathscr{D}$:

$$\pi^{\otimes}((x_1, x_2), A_1) = \mathbb{I}_{A_1}(x_1 + x_2),$$
$$\pi^*(x_1, A_1 \times A_2) = \mathbb{I}_{A_1}(x_1^c)\mathbb{I}_{A_2}(x_1^d).$$

Then, on the one hand we have:

$$\begin{aligned}
\pi^{\otimes} P^{(f,\sigma^2,0)} \otimes P^{(\eta_\nu,0,\nu)}(A) &= \iint_{D^2} \pi^{\otimes}((x_1, x_2), A) P^{(f,\sigma^2,0)}(dx_1) P^{(\eta_\nu,0,\nu)}(dx_2) \\
&= \iint_{D^2} \mathbb{I}_{A_1}(x_1 + x_2) P^{(f,\sigma^2,0)}(dx_1) P^{(\eta_\nu,0,\nu)}(dx_2) \\
&= P^{(f,\sigma^2,0)} * P^{(\eta_\nu,0,\nu)}(A) \\
&= P^{(f+\eta_\nu,\sigma^2,\nu)}(A) \qquad \forall A \in \mathscr{D}.
\end{aligned}$$

On the other hand we have:

$$\begin{aligned}
\pi^* P^{(f+\eta_\nu,\sigma^2,\nu)}(A_1 \times A_2) &= \int_D \pi^*(x, A_1 \times A_2) P^{(f,\sigma^2,0)} * P^{(\eta_\nu,0,\nu)}(dx) \\
&= \iint_{D^2} \pi^*(x + y, A_1 \times A_2) P^{(f,\sigma^2,0)}(dx) P^{(\eta_\nu,0,\nu)}(dy) \\
&= \iint_{D^2} \mathbb{I}_{A_1}(x^c)\mathbb{I}_{A_2}(x^d) P^{(f,\sigma^2,0)}(dx) P^{(\eta_\nu,0,\nu)}(dy) \\
&= P^{(f,\sigma^2,0)}(A_1) P^{(\eta_\nu,0,\nu)}(A_2) \qquad \forall A_1, A_2 \in \mathscr{D},
\end{aligned}$$

that concludes the proof by the definition of the Δ-distance. □

Proof of Property 2.7. By hypothesis, there exist $\pi_{i,j}^n : \mathscr{X}_i^n \to \mathscr{A}_j^n$, $(i,j) = (1,2), (2,1)$, Markov kernels which do not depend on the parameter space Θ such that

$$\lim_{n\to\infty} \sup_{A \in \mathscr{A}_j^n} |\pi_i^n P_\theta^{i,n}(A) - P_\theta^{j,n}(A)| = 0.$$

We conclude by considering the Markov kernels $\pi_{i,j,3}^n : \mathscr{X}_{i,3}^n \to \mathscr{A}_{j,3}^n$ defined, for $(i,j) = (1,2), (2,1)$, by:

$$\pi_{i,j,3}^n\big((x_i, x_3); A_j \times A_3\big) := \pi_i^n(x_i; A_j)\mathbb{I}_{A_3}(x_3).$$

□

We now prove two technical lemmas needed for the proof of Theorem 2.12.

Lemma Appendix A.1. *For all x, y in $\mathbb{R}$ we have:*

$$|1 - e^{x+y}| \leq \frac{1+e^x}{2}|1 - e^y| + \frac{1+e^y}{2}|1 - e^x|. \tag{A.1}$$

Proof. Consider the cases:

- $x, y \geq 0$: In this case we have that $|1 - e^{x+y}|$ is exactly equal to $\frac{1+e^x}{2}|1 - e^y| + \frac{1+e^y}{2}|1 - e^x|$.
- $x \geq 0, y \leq 0, x + y \geq 0$: Then the member on the right hand side of (A.1) is equal to $e^x - e^y \geq e^x - 1 \geq e^{x+y} - 1$.
- $x \geq 0, y \leq 0, x + y \leq 0$: In this case the member on the right of (A.1) is equal to $e^x - e^y \geq 1 - e^y \geq 1 - e^{x+y}$.
- $x, y \leq 0$: Also in this case we have that $|1 - e^{x+y}|$ is exactly equal to $\frac{1+e^x}{2}|1 - e^y| + \frac{1+e^y}{2}|1 - e^x|$.

We handle the remaining cases by symmetrical arguments. □

Lemma Appendix A.2. *The following limit*

$$R_T^m(x) := \lim_{\varepsilon \to 0} \left(\exp\Big(\sum_{r \leq T} \ln \frac{\bar{\rho}_m}{\rho}(\Delta x_r) \mathbb{I}_{|\Delta x_r| > \varepsilon} - T \int_{|y| > \varepsilon} (\bar{\nu}_m - \nu)(dy) \Big) \right), \tag{A.2}$$

with ρ, $\bar{\rho}_m$, γ^ν and $\bar{\nu}_m$ as in Section 2, exists uniformly in t in any bounded interval, $P^{(\gamma^\nu, 0, \nu)}$-a.s. and

$$\mathbb{E}_{P^{(\gamma^\nu,0,\nu)}} \Big[\big| 1 - R_T^m(x) \big| \Big] \leq 2 \sinh \left(T \int_{\mathbb{R}} |\rho(y) - \bar{\rho}_m(y)| \tilde{\nu}(dy) \right). \tag{A.3}$$

Proof. To prove the existence of the limit in (A.2) we want to apply Theorem 2.3. To that aim just note that Assumption (M2) implies the finiteness of the integral $\int_{\mathbb{R}} \left(\sqrt{\frac{d\bar{\nu}_m}{d\nu}(y)} - 1 \right)^2 \nu(dy)$. Indeed, integrating the last quantity

separately over the intervals $\big[-\frac{1}{m},\frac{1}{m}\big]$, $\big]\frac{1}{m},\infty\big[$ and $\big]-\infty,-\frac{1}{m}\big[$, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\left(\sqrt{\frac{d\bar{\nu}_m}{d\nu}(y)}-1\right)^2\nu(dy) &= \int_{\mathbb{R}}\big(\sqrt{\bar{\rho}_m(y)}-\sqrt{\rho(y)}\big)^2\tilde{\nu}(dy)\\ &= \int_{-\frac{1}{m}}^{\frac{1}{m}}\big(1-\sqrt{\rho(y)}\big)^2\tilde{\nu}(dy)+\int_{\frac{1}{m}}^{\infty}\big(\sqrt{\bar{\rho}_m(y)}-\sqrt{\rho(y)}\big)^2\tilde{\nu}(dy)\\ &\quad+\int_{-\infty}^{-\frac{1}{m}}\big(\sqrt{\bar{\rho}_m(y)}-\sqrt{\rho(y)}\big)^2\tilde{\nu}(dy)\\ &\le \int_{-\frac{1}{m}}^{\frac{1}{m}}\big(1-\sqrt{\rho(y)}\big)^2\tilde{\nu}(dy)+4\nu\left(\Big]\frac{1}{m},\infty\Big[\cup\Big]-\infty,-\frac{1}{m}\Big[\right),\end{aligned}$$

that is finite thanks to Assumption (M2) and the fact that ν is a Lévy measure (in the last inequality we have used the elementary inequality: for $a,b\ge 0$, $(\sqrt{a}-\sqrt{b})^2\le 2a+2b$).
In order to simplify the notations let us write

$$A^{\pm}(x):=\lim_{\varepsilon\to 0}\left(\sum_{r\le T}\ln f^{\pm}(\Delta x_r)\mathbb{I}_{|\Delta(x_r)|>\varepsilon}-T\int_{|y|>\varepsilon}(f^{\mp}(y)-1)\nu(dy)\right)$$

with $f^+=\left(\frac{\bar{\rho}_m}{\rho}\right)^{\mathbb{I}_{\bar{\rho}_m\ge\rho}}$ and $f^-=\left(\frac{\bar{\rho}_m}{\rho}\right)^{\mathbb{I}_{\rho>\bar{\rho}_m}}$, so that

$$R_T^m(x)=\exp(A_+(x)+A_-(x)).$$

Then, using Lemma Appendix A.1 and the fact that $A_+(x)\ge 0$ and $A_-(x)\le 0$ we get:

$$\begin{aligned}\mathbb{E}_{P(\gamma^\nu,0,\nu)}\big[|1-R_T^m(x)|\big] &= \mathbb{E}_{P(\gamma^\nu,0,\nu)}\big|1-\exp(A^+(x)+A^-(x))\big|\\ &\le \mathbb{E}_{P(\gamma^\nu,0,\nu)}\left[\frac{1+e^{A^+(x)}}{2}\Big|1-e^{A^-(x)}\Big|+\frac{1+e^{A^-(x)}}{2}\Big|1-e^{A^+(x)}\Big|\right]\\ &= \mathbb{E}_{P(\gamma^\nu,0,\nu)}\Big[e^{A^+(x)}-e^{A^-(x)}\Big].\end{aligned}$$

In order to compute the last quantity we apply Theorem 2.3 and the fact

that both $A^+(x)$ and $A^-(x)$ have the same law under $P^{(\gamma^\nu,0,\nu)}$ and $P^{(0,0,\nu)}$:

$$\begin{aligned}\mathbb{E}_{P^{(\gamma^\nu,0,\nu)}}\left[e^{A^+(x)}-e^{A^-(x)}\right] &= \exp\left(T\int_{\mathbb{R}}(f^+(y)-f^-(y))\nu(dy)\right.\\ &\quad\left.-\exp\left(T\int_{\mathbb{R}}(f^-(y)-f^+(y))\nu(dy)\right)\right.\\ &= 2\sinh\left(T\int_{\mathbb{R}}(f^+(y)-f^-(y)\nu(dy)\right)\\ &= 2\sinh\left(T\int_{\mathbb{R}}|\rho(y)-\bar\rho_m(y)|\tilde\nu(dy)\right).\end{aligned}$$

□

Appendix B. Proofs of the examples

Proof of example 2.9. Assumption (M1) is obvious by construction, Assumptions (M2) and (M4) follow from the finiteness of the measures ν and $\tilde\nu$ plus the inequality $(\sqrt{\rho}-1)^2 \le \rho+1$. Finally, Assumption (M3) is a straightforward consequence of the inequality:

$$|\rho(y)-\bar\rho_m(y)| \le \frac{1}{m}\left\|\frac{d\rho(y)}{dy}\right\|_\infty \quad \forall y \in \left]-m,-\frac{1}{m}\right] \cup \left]\frac{1}{m},m\right]. \tag{B.1}$$

Indeed, since $\left\|\frac{d\rho(y)}{dy}\right\|_\infty \le L$, we have

$$\begin{aligned}&\int_{\mathbb{R}}|\rho(y)-\bar\rho_m(y)|\tilde\nu(dy) = \int\limits_{\frac{1}{m}\le|y|<m}|\rho(y)-\bar\rho_m(y)|\tilde\nu(dy)\\ &+\int\limits_{|y|\ge m}|\rho(y)-\bar\rho_m(y)|\tilde\nu(dy)+\int\limits_{|y|<\frac{1}{m}}|\rho(y)-1|\tilde\nu(dy)\\ &\le \frac{L}{m}\tilde\nu\Big(\frac{1}{m}\le|y|<m\Big)+2\nu\big(|y|\ge m\big)+\int_{|y|<\frac{1}{m}}(K+L|y|+1)\tilde\nu(dy)\\ &\le \frac{L}{m}\tilde\nu\Big(\frac{1}{m}\le|y|<m\Big)+2\int_{|y|\ge m}(K+L|y|)\tilde\nu(dy)+\int_{|y|<\frac{1}{m}}(K+L|y|+1)\tilde\nu(dy),\end{aligned}$$

that tends to zero, uniformly on ν, as m goes to infinity, since $\tilde\nu$ is a finite Lévy measure.
In order to prove (B.1), fix an interval $J_{j,k}$, $j=1,\dots,m$ and $k=-m,\dots,m-1$ and note that, by construction of $\bar\rho_m$, there always exist y_1, y_2 verifying

$$\bar\rho_m(y) \le \rho(y_1) \text{ and } \bar\rho_m(y) \ge \rho(y_2) \quad \forall y \in J_{j,k}.$$

Thus, by the continuity of ρ, we conclude that, for all y in $J_{j,k}$, there exists $\hat{y}$ such that $\bar{\rho}_m(y) = \rho(\hat{y})$. Then we apply the mean value theorem to bound $|\rho(y) - \bar{\rho}_m(y)|$. □

Proof of example 2.10. Consider ν in $\mathscr{M}^{\Gamma,(4)}_{\varepsilon,C,M}$ and define a Lévy measure $\tilde{\nu}$, absolutely continuous with respect to Lebesgue, whose density is $Ce^{-\varepsilon y}y^{-1}\mathbb{I}_{y>0}$. Consequently, $\rho(y) = \frac{d\nu}{d\tilde{\nu}}(y) = e^{y(\varepsilon-\lambda)}\mathbb{I}_{y>0}$. Condition (M2) writes

$$\begin{aligned}\int_{\mathbb{R}}(\sqrt{\rho(y)}-1)^2\tilde{\nu}(dy) &= \int_0^\infty \left(e^{\frac{y}{2}(\varepsilon-\lambda)}-1\right)^2 e^{-\varepsilon y}y^{-1}dy \\ &\leq \int_0^1 \Big(\frac{y^2}{4}(\varepsilon-\lambda)^2 + O(y^3)\Big)y^{-1}dy + \frac{1}{\varepsilon} < \infty.\end{aligned}$$

To verify condition (M3), treat separately the integral $\int_0^\infty |\rho(y)-\bar{\rho}_m(y)|\tilde{\nu}(dy)$ over the intervals $\left[0,\frac{1}{m}\right]$, $\left]\frac{1}{m},m\right]$, $]m,\infty[$:

$$\begin{aligned}&C\int_0^{\frac{1}{m}} |\rho(y)-1|e^{-\varepsilon y}y^{-1}dy \leq C\frac{\lambda-\varepsilon}{m} + O\Big(\frac{1}{m^2}\Big) \leq C\frac{M-\varepsilon}{m} + O\Big(\frac{1}{m^2}\Big),\\ &C\int_{\frac{1}{m}}^m |\rho(y)-\bar{\rho}_m(y)|e^{-\varepsilon y}y^{-1}dy \leq 2C\frac{M-\varepsilon}{m}\log(m), \quad \text{using (B.1)}\\ &\int_m^\infty |\rho(y)-\bar{\rho}_m(y)|\tilde{\nu}(dy) \leq \int_m^\infty (\rho(y)+\bar{\rho}_m(y))\tilde{\nu}(dy) = 2\tilde{\nu}([m,\infty)),\end{aligned}$$

we conclude since the quantities above tend to zero, uniformly in ν, as m goes to infinity. Finally, condition (M4) follows from

$$\begin{aligned}\int_{|y|\leq 1}|y|\nu(dy) &= \int_0^1 Ce^{-\lambda y}dy = \frac{C}{\lambda}(1-e^{-\lambda}) \leq \frac{C}{M}(1-e^{-\varepsilon}),\\ \int_{|y|\leq 1}|y|\tilde{\nu}(dy) &= \int_0^1 Ce^{-\varepsilon y}dy = \frac{C}{\varepsilon}(1-e^{-\varepsilon}).\end{aligned}$$

□

Proof of Example 2.11. Let ν belong to $\mathscr{M}^{T,(4)}_{\alpha,C_1,C_2,M,\varepsilon}$ and consider the measure $\tilde{\nu}$ having density (with respect to Lebesgue) given by $C_1\frac{e^{-\varepsilon|y|}}{|y|^{1+\alpha}}\mathbb{I}_{y<0} + C_2\frac{e^{-\varepsilon y}}{y^{1+\alpha}}\mathbb{I}_{y>0}$. Hence, $\rho(y) = \frac{d\nu}{d\tilde{\nu}}(y) = e^{-|y|(\lambda_1-\varepsilon)}\mathbb{I}_{y<0} + e^{-y(\lambda_2-\varepsilon)}\mathbb{I}_{y>0}$. Condition (M2) writes:

$$C_1\int_{-\infty}^0 \left(e^{\frac{y}{2}(\lambda_1-\varepsilon)}-1\right)^2\frac{e^{\varepsilon y}}{(-y)^{1+\alpha}}dy + C_2\int_0^\infty \left(e^{\frac{y}{2}(\varepsilon-\lambda_2)}-1\right)^2\frac{e^{-\varepsilon y}}{y^{1+\alpha}}dy < \infty,$$

which is true since, near zero, the integrands are equivalent to $\frac{1}{|y|^{\alpha-1}}$ and are integrable for $\alpha < 1$. To verify condition (M3) treat again separately the integral $\int_0^\infty |\rho(y) - \bar{\rho}_m(y)|\tilde{\nu}(dy)$ over the intervals $\pm\Big[0, \frac{1}{m}\Big]$, $\pm\Big]\frac{1}{m}, m\Big]$, $\pm]m, \infty[$. The computations are essentially the same as in the previous proof, thanks to the integrability of $y^{-\alpha}$ between $-\frac{1}{m}$ and $\frac{1}{m}$. The integrals between $\frac{1}{m}$ and m and between $-m$ and $-\frac{1}{m}$ are bounded by $\max\{C_1, C_2\}(M - \varepsilon)\big(\frac{m^{\alpha-1}}{\alpha} - \frac{1}{\alpha m^{\alpha+1}}\big)$, hence tend to zero. Condition (M4) follows as above. □

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