



HAL
open science

Scaling limit of the path leading to the leftmost particle in a branching random walk

Xinxin Chen

► **To cite this version:**

Xinxin Chen. Scaling limit of the path leading to the leftmost particle in a branching random walk. SIAM Theory of Probability and its Applications, 2013, 59 (4), pp.567-589. hal-00827040

HAL Id: hal-00827040

<https://hal.science/hal-00827040>

Submitted on 28 May 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Scaling limit of the path leading to the leftmost particle in a branching random walk

Xinxin CHEN

Université Paris VI

Summary. We consider a discrete-time branching random walk defined on the real line, which is assumed to be supercritical and in the boundary case. It is known that its leftmost position of the n -th generation behaves asymptotically like $\frac{3}{2} \ln n$, provided the non-extinction of the system. The main goal of this paper, is to prove that the path from the root to the leftmost particle, after a suitable normalization, converges weakly to a Brownian excursion in $D([0, 1], \mathbb{R})$.

Keywords. Branching random walk; spinal decomposition.

1 Introduction

We consider a branching random walk, which is constructed according to a point process \mathcal{L} on the line. Precisely speaking, the system is started with one initial particle at the origin. This particle is called the root, denoted by \emptyset . At time 1, the root dies and gives birth to some new particles, which form the first generation. Their positions constitute a point process distributed as \mathcal{L} . At time 2, each of these particles dies and gives birth to new particles whose positions – relative to that of their parent – constitute a new independent copy of \mathcal{L} . The system grows according to the same mechanism.

We denote by \mathbb{T} the genealogical tree of the system, which is clearly a Galton-Watson tree rooted at \emptyset . If a vertex $u \in \mathbb{T}$ is in the n -th generation, we write $|u| = n$ and denote its position by $V(u)$. Then $\{V(u), |u| = 1\}$ follows the same law as \mathcal{L} . The family of positions $(V(u); u \in \mathbb{T})$ is viewed as our branching random walk.

Throughout the paper, the branching random walk is assumed to be in the boundary case (Biggins and Kyprianou [5]):

$$(1.1) \quad \mathbf{E} \left[\sum_{|u|=1} 1 \right] > 1, \quad \mathbf{E} \left[\sum_{|x|=1} e^{-V(x)} \right] = 1, \quad \mathbf{E} \left[\sum_{|x|=1} V(x) e^{-V(x)} \right] = 0.$$

For any $y \in \mathbb{R}$, let $y_+ := \max\{y, 0\}$ and $\log_+ y := \log(\max\{y, 1\})$. We also assume the following integrability conditions:

$$(1.2) \quad \mathbf{E} \left[\sum_{|u|=1} V(u)^2 e^{-V(u)} \right] < \infty,$$

$$(1.3) \quad \mathbf{E}[X(\log_+ X)^2] < \infty, \quad \mathbf{E}[\tilde{X} \log_+ \tilde{X}] < \infty,$$

where

$$X := \sum_{|u|=1} e^{-V(u)}, \quad \tilde{X} := \sum_{|u|=1} V(u)_+ e^{-V(u)}.$$

We define I_n to be the leftmost position in the n -th generation, i.e.

$$(1.4) \quad I_n := \inf\{V(u), |u| = n\},$$

with $\inf \emptyset := \infty$. If $I_n < \infty$, we choose a vertex uniformly in the set $\{u : |u| = n, V(u) = I_n\}$ of leftmost particles at time n and denote it by $m^{(n)}$. We let $[\emptyset, m^{(n)}] = \{\emptyset =: m_0^{(n)}, m_1^{(n)}, \dots, m_n^{(n)} := m^{(n)}\}$ be the shortest path in \mathbb{T} relating the root \emptyset to $m^{(n)}$, and introduce the path from the root to $m^{(n)}$ as follows

$$(I_n(k); 0 \leq k \leq n) := (V(m_k^{(n)}); 0 \leq k \leq n).$$

In particular, $I_n(0) = 0$ and $I_n(n) = I_n$. Let σ be the positive real number such that $\sigma^2 = \mathbf{E} \left[\sum_{|u|=1} V(u)^2 e^{-V(u)} \right]$. Our main result is as follows.

Theorem 1.1 *The rescaled path $(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1)$ converges in law in $D([0, 1], \mathbb{R})$, to a normalized Brownian excursion $(e_s; 0 \leq s \leq 1)$.*

Remark 1.2 *It has been proved in [1], [11] and [2] that I_n is around $\frac{3}{2} \ln n$. In [3], the authors proved that, for the model of branching Brownian motion, the time reversed path followed by the leftmost particle converges in law to a certain stochastic process.*

Let us say a few words about the proof of Theorem 1.1. We first consider the path leading to $m^{(n)}$, by conditioning that its ending point I_n is located atypically below $\frac{3}{2} \ln n - z$

with large z . Then we apply the well-known spinal decomposition to show that this path, conditioned to $\{I_n \leq \frac{3}{2} \ln n - z\}$, behaves like a simple random walk staying positive but tied down at the end. Such a random walk, being rescaled, converges in law to the Brownian excursion (see [9]). We then prove our main result by removing the condition of I_n . The main strategy is borrowed from [2], but with appropriate refinements.

The rest of the paper is organized as follows. In Section 2, we recall the spinal decomposition by a change of measures, which implies the useful many-to-one lemma. We prove a conditioned version of Theorem 1.1 in Section 3. In Section 4, we remove the conditioning and prove the theorem.

Throughout the paper, we use $a_n \sim b_n$ ($n \rightarrow \infty$) to denote $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$; and let $(c_i)_{i \geq 0}$ denote finite and positive constants. We write $\mathbf{E}[f; A]$ for $\mathbf{E}[f \mathbf{1}_A]$. Moreover, $\sum_{\emptyset} := 0$ and $\prod_{\emptyset} := 1$.

2 Lyons' change of measures and spinal decomposition

For any $a \in \mathbb{R}$, let \mathbf{P}_a be the probability measure such that $\mathbf{P}_a((V(u), u \in \mathbb{T}) \in \cdot) = \mathbf{P}((V(u) + a, u \in \mathbb{T}) \in \cdot)$. The corresponding expectation is denoted by \mathbf{E}_a . Let $(\mathcal{F}_n, n \geq 0)$ be the natural filtration generated by the branching random walk and let $\mathcal{F}_{\infty} := \bigvee_{n \geq 0} \mathcal{F}_n$. We introduce the following random variables:

$$(2.1) \quad W_n := \sum_{|u|=n} e^{-V(u)}, \quad n \geq 0.$$

It follows immediately from (1.1) that $(W_n, n \geq 0)$ is a non-negative martingale with respect to (\mathcal{F}_n) . It is usually referred as the additive martingale. We define a probability measure \mathbf{Q}_a on \mathcal{F}_{∞} such that for any $n \geq 0$,

$$(2.2) \quad \left. \frac{d\mathbf{Q}_a}{d\mathbf{P}_a} \right|_{\mathcal{F}_n} := e^a W_n.$$

For convenience, we write \mathbf{Q} for \mathbf{Q}_0 .

Let us give the description of the branching random walk under \mathbf{Q}_a in an intuitive way, which is known as the spinal decomposition. We introduce another point process $\widehat{\mathcal{L}}$ with Radon-Nykodim derivative $\sum_{x \in \mathcal{L}} e^{-x}$ with respect to the law of \mathcal{L} . Under \mathbf{Q}_a , the branching random walk evolves as follows. Initially, there is one particle w_0 located at $V(w_0) = a$. At each step n , particles at generation n die and give birth to new particles independently according to the law of \mathcal{L} , except for the particle w_n which generates its children according

to the law of $\widehat{\mathcal{L}}$. The particle w_{n+1} is chosen proportionally to $e^{-V(u)}$ among the children u of w_n . We still call \mathbb{T} the genealogical tree of the process, so that $(w_n)_{n \geq 0}$ is a ray in \mathbb{T} , which is called the spine. This change of probabilities was presented in various forms; see, for example [15], [11] and [8].

It is convenient to use the following notation. For any $u \in \mathbb{T} \setminus \{\emptyset\}$, let \overleftarrow{u} be the parent of u , and

$$\Delta V(u) := V(u) - V(\overleftarrow{u}).$$

Let $\Omega(u)$ be the set of brothers of u , i.e. $\Omega(u) := \{v \in \mathbb{T} : \overleftarrow{v} = \overleftarrow{u}, v \neq u\}$. Let δ denote the Dirac measure. Then under \mathbf{Q}_a , $\sum_{|u|=1} \delta_{\Delta V(u)}$ follows the law of $\widehat{\mathcal{L}}$. Further, We recall the following proposition, from [11] and [15].

Proposition 2.1 (1) For any $|u| = n$, we have

$$(2.3) \quad \mathbf{Q}_a[w_n = u | \mathcal{F}_n] = \frac{e^{-V(u)}}{W_n}.$$

(2) Under \mathbf{Q}_a , the random variables $\left(\sum_{v \in \Omega(w_n)} \delta_{\Delta V(v)}, \Delta V(w_n) \right)$, $n \geq 1$ are i.i.d..

As a consequence of this proposition, we get the many-to-one lemma as follows:

Lemma 2.2 There exists a centered random walk $(S_n; n \geq 0)$ with $\mathbf{P}_a(S_0 = a) = 1$ such that for any $n \geq 1$ and any measurable function $g : \mathbb{R}^n \rightarrow [0, \infty)$, we have

$$(2.4) \quad \mathbf{E}_a \left[\sum_{|u|=n} g(V(u_1), \dots, V(u_n)) \right] = \mathbf{E}_a [e^{S_n - a} g(S_1, \dots, S_n)],$$

where we denote by $[[\emptyset, u]] = \{\emptyset =: u_0, u_1, \dots, u_{|u|} := u\}$ the ancestral line of u in \mathbb{T} .

Note that by (1.3), S_1 has the finite variance $\sigma^2 = \mathbf{E}[S_1^2] = \mathbf{E}[\sum_{|u|=1} V(u)^2 e^{-V(u)}]$.

2.1 Convergence in law for the one-dimensional random walk

Let us introduce some results about the centered random walk (S_n) with finite variance, which will be used later. For any $0 \leq m \leq n$, we define $\underline{S}_{[m,n]} := \min_{m \leq j \leq n} S_j$, and $\underline{S}_n = \underline{S}_{[0,n]}$. We denote by $R(x)$ the renewal function of (S_n) , which is defined as follows:

$$(2.5) \quad R(x) = \mathbf{1}_{\{x=0\}} + \mathbf{1}_{\{x>0\}} \sum_{k \geq 0} \mathbf{P}(-x \leq S_k < \underline{S}_{n-1}).$$

For the random walk $(-S_n)$, we define $\underline{S}_{[m,n]}^-$, \underline{S}_n^- and $R_-(x)$ similarly. It is known (see [10] p. 360) that there exists $c_0 > 0$ such that

$$(2.6) \quad \lim_{x \rightarrow \infty} \frac{R(x)}{x} = c_0.$$

Moreover, it is shown in [13] that there exist $C_+, C_- > 0$ such that for any $a \geq 0$,

$$(2.7) \quad \mathbf{P}_a(\underline{S}_n \geq 0) \sim \frac{C_+}{\sqrt{n}} R(a);$$

$$(2.8) \quad \mathbf{P}_a(\underline{S}_n^- \geq 0) \sim \frac{C_-}{\sqrt{n}} R_-(a).$$

We also state the following inequalities (see Lemmas 2.2 and 2.4 in [4], respectively).

Fact 2.3 (i) *There exists a constant $c_1 > 0$ such that for any $b \geq a \geq 0$, $x \geq 0$ and $n \geq 1$,*

$$(2.9) \quad \mathbf{P}(\underline{S}_n \geq -x; S_n \in [a-x, b-x]) \leq c_1(1+x)(1+b-a)(1+b)n^{-3/2}.$$

(ii) *Let $0 < \lambda < 1$. There exists a constant $c_2 > 0$ such that for any $b \geq a \geq 0$, $x, y \geq 0$ and $n \geq 1$,*

$$(2.10) \quad \mathbf{P}_x(S_n \in [y+a, y+b], \underline{S}_n \geq 0, \underline{S}_{[\lambda n, n]} \geq y) \leq c_2(1+x)(1+b-a)(1+b)n^{-3/2}.$$

Before we give the next lemma, we recall the definition of lattice distribution (see [10], p. 138). The distribution of a random variable X_1 is lattice, if it is concentrated on a set of points $\alpha + \beta\mathbb{Z}$, with α arbitrary. The largest β satisfying this property is called the span of X_1 . Otherwise, the distribution of X_1 is called non-lattice.

Lemma 2.4 *Let $(r_n)_{n \geq 0}$ be a sequence of real numbers such that $\lim_{n \rightarrow \infty} \frac{r_n}{\sqrt{n}} = 0$. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a Riemann integrable function. We suppose that there exists a non-increasing function $\bar{f} : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $|f(x)| \leq \bar{f}(x)$ for any $x \geq 0$ and $\int_{x \geq 0} x \bar{f}(x) dx < \infty$. For $0 < \Delta < 1$, let $F : D([0, \Delta], \mathbb{R}) \rightarrow [0, 1]$ be continuous. Let $a \geq 0$.*

(I) Non-lattice case. *If the distribution of $(S_1 - S_0)$ is non-lattice, then there exists a constant $C_1 > 0$ such that*

$$(2.11) \quad \lim_{n \rightarrow \infty} n^{3/2} \mathbf{E} \left[F \left(\frac{S_{[sn]}}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) f(S_n - y); \underline{S}_n \geq -a, \underline{S}_{[\Delta n, n]} \geq y \right] \\ = C_1 R(a) \int_{x \geq 0} f(x) R_-(x) dx \mathbf{E}[F(e_s; 0 \leq s \leq \Delta)],$$

uniformly in $y \in [0, r_n]$.

(II) Lattice case. *If the distribution of $(S_1 - S_0)$ is supported in $(\alpha + \beta\mathbb{Z})$ with span β , then for any $d \in \mathbb{R}$,*

$$(2.12) \quad \lim_{n \rightarrow \infty} n^{3/2} \mathbf{E} \left[F \left(\frac{S_{\lfloor sn \rfloor}}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) f(S_n - y + d); \underline{S}_n \geq -a, \underline{S}_{[\Delta n, n]} \geq y - d \right] \\ = C_1 R(a) \beta \sum_{j \geq \lceil -\frac{d}{\beta} \rceil} f(\beta j + d) R_-(\beta j + d) \mathbf{E}[F(e_s; 0 \leq s \leq \Delta)].$$

uniformly in $y \in [0, r_n] \cap \{\alpha n + \beta\mathbb{Z}\}$.

Proof of Lemma 2.4. The lemma is a refinement of Lemma 2.3 in [2], which proved the convergence in the non-lattice case when $a = 0$ and $F \equiv 1$. We consider the non-lattice case first. We denote the expectation on the left-hand side of (2.11) by $\chi(F, f)$. Observe that for any $K \in \mathbb{N}_+$,

$$\chi(F, f) = \chi(F, f(x)1_{(0 \leq x \leq K)}) + \chi(F, f(x)1_{(x > K)}).$$

Since $0 \leq F \leq 1$, we have $\chi(F, f(x)1_{(x > K)}) \leq \chi(1, f(x)1_{(x > K)})$, which is bounded by

$$\sum_{j \geq K} \mathbf{E}_a \left[f(S_n - y - a); \underline{S}_n \geq 0, \underline{S}_{[\Delta n, n]} \geq y + a, S_n \in [y + a + j, y + a + j + 1] \right].$$

Recall that $|f(x)| \leq \bar{f}(x)$ with \bar{f} non-increasing. We get that

$$\chi(1, f(x)1_{(x > K)}) \leq \sum_{j \geq K} \bar{f}(j) \mathbf{P}_a \left[\underline{S}_n \geq 0, \underline{S}_{[\Delta n, n]} \geq y + a, S_n \in [y + a + j, y + a + j + 1] \right].$$

It then follows from (2.10) that

$$(2.13) \quad \chi(1, f(x)1_{(x > K)}) \leq 2c_2(1 + a) \left(\sum_{j \geq K} \bar{f}(j)(2 + j) \right) n^{-3/2}.$$

Since $\int_0^\infty x \bar{f}(x) dx < \infty$, the sum $\sum_{j \geq K} \bar{f}(j)(2 + j)$ decreases to zero as $K \uparrow \infty$. We thus only need to estimate $\chi(F, f(x)1_{(0 \leq x \leq K)})$. Note that f is Riemann integrable. It suffices to consider $\chi(F, 1_{(0 \leq x \leq K)})$ with K a positive constant.

Applying the Markov property at time $[\Delta n]$ shows that

$$(2.14) \quad \chi(F, 1_{(0 \leq x \leq K)}) = \mathbf{E}_a \left[F \left(\frac{S_{\lfloor sn \rfloor} - a}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right); S_n \leq y + a + K, \underline{S}_n \geq 0, \underline{S}_{[\Delta n, n]} \geq y + a \right] \\ = \mathbf{E}_a \left[F \left(\frac{S_{\lfloor sn \rfloor} - a}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \Psi_K(S_{[\Delta n]}); \underline{S}_{[\Delta n]} \geq 0 \right],$$

where $\Psi_K(x) := \mathbf{P}_x \left[S_{n-\lfloor \Delta n \rfloor} \leq y + a + K, \underline{S}_{n-\lfloor \Delta n \rfloor} \geq y + a \right]$. By reversing time, we obtain that $\Psi_K(x) = \mathbf{P} \left[\underline{S}_m^- \geq (-S_m) + (y + a - x) \geq -K \right]$ with $m := n - \lfloor \Delta n \rfloor$.

We define τ_n as the first time when the random walk $(-S)$ hits the minimal level during $[0, n]$, namely, $\tau_n := \inf \{ k \in [0, n] : -S_k = \underline{S}_n^- \}$. Define also $\varkappa(z, \zeta; n) := \mathbf{P}(-S_n \in [z, z + \zeta], \underline{S}_n^- \geq 0)$ for any $z, \zeta \geq 0$. Then,

$$(2.15) \quad \begin{aligned} \Psi_K(x) &= \sum_{k=0}^m \mathbf{P} \left[\tau_m = k; \underline{S}_m^- \geq (-S_m) + (y + a - x) \geq -K \right] \\ &= \sum_{k=0}^m \mathbf{P} \left[-S_k = \underline{S}_k^- \geq -K; \varkappa(x - y - a, \underline{S}_k^- + K; m - k) \right], \end{aligned}$$

where the last equality follows from the Markov property.

Let $\psi(x) := xe^{-x^2/2} \mathbf{1}_{(x \geq 0)}$. Combining Theorem 1 of [6] with (2.7) yields that

$$(2.16) \quad \varkappa(z, \zeta; n) = \mathbf{P}_0 \left[-S_n \in [z, z + \zeta]; \underline{S}_n^- \geq 0 \right] = \frac{C_- \zeta}{\sigma n} \psi \left(\frac{z}{\sigma \sqrt{n}} \right) + o(n^{-1}),$$

uniformly in $z \in \mathbb{R}_+$ and ζ in compact sets of \mathbb{R}_+ . Note that ψ is bounded on \mathbb{R}_+ . Therefore, there exists a constant $c_3 > 0$ such that for any $\zeta \in [0, K]$, $z \geq 0$ and $n \geq 0$,

$$(2.17) \quad \varkappa(z, \zeta; n) \leq c_3 \frac{(1 + K)}{n + 1}.$$

Let $k_n := \lfloor \sqrt{n} \rfloor$. We divide the sum on the right-hand side of (2.15) into two parts:

$$(2.18) \quad \Psi_K(x) = \sum_{k=0}^{k_n} + \sum_{k=k_n+1}^m \mathbf{P} \left[-S_k = \underline{S}_k^- \geq -K; \varkappa(x - y - a, \underline{S}_k^- + K; m - k) \right].$$

By (2.16), under the assumption that $y = o(\sqrt{n})$, the first part becomes that

$$(2.19) \quad \begin{aligned} &\frac{C_-}{\sigma m} \psi \left(\frac{x - a}{\sigma \sqrt{m}} \right) \sum_{k=0}^{k_n} \mathbf{E} \left[\underline{S}_k^- + K; -S_k = \underline{S}_k^- \geq -K \right] + o(n^{-1}) \sum_{k=0}^{k_n} \mathbf{P} \left[-S_k = \underline{S}_k^- \geq -K \right] \\ &= \frac{C_-}{\sigma m} \psi \left(\frac{x - a}{\sigma \sqrt{m}} \right) \int_0^K R_-(u) du + o(n^{-1}), \end{aligned}$$

where the last equation comes from the fact that $\sum_{k \geq 0} \mathbf{E} \left[\underline{S}_k^- + K; -S_k = \underline{S}_k^- \geq -K \right] = \int_0^K R_-(u) du$. On the other hand, using (2.17) for $\varkappa(x - y - a, \underline{S}_k^- + K; m - k)$ and then applying (i) of Fact 2.3 imply that for n large enough, the second part of (2.18) is bounded

by

$$(2.20) \quad \begin{aligned} & \sum_{k=k_n+1}^m c_3 \frac{1+K}{m+1-k} \mathbf{P}(S_k^- \geq -K, -S_k \in [-K, 0]) \\ & \leq c_4 \sum_{k=k_n+1}^m \frac{(1+K)^3}{(m+1-k)k^{3/2}} = o(n^{-1}). \end{aligned}$$

By (2.19) and (2.20), we obtain that as n goes to infinity,

$$(2.21) \quad \Psi_K(x) = o(n^{-1}) + \frac{C_-}{\sigma(n - \lfloor \Delta n \rfloor)} \psi\left(\frac{x-a}{\sigma\sqrt{n - \lfloor \Delta n \rfloor}}\right) \int_0^K R_-(u) du,$$

uniformly in $x \geq 0$ and $y \in [0, r_n]$. Plugging it into (2.14) and then combining with (2.7) yield that

$$\begin{aligned} \chi(F, 1_{(0 \leq x \leq K)}) &= o(n^{-3/2}) + \frac{C_-}{\sigma(1-\Delta)n} \int_0^K R_-(u) du \\ &\quad \times \frac{C_+ R(a)}{\sqrt{\Delta n}} \mathbf{E}_a \left[F\left(\frac{S_{\lfloor sn \rfloor} - a}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\right) \psi\left(\frac{S_{\Delta n} - a}{\sigma\sqrt{(1-\Delta)n}}\right) \Big| S_{\Delta n} \geq 0 \right]. \end{aligned}$$

Theorem 1.1 of [7] says that under the conditioned probability $\mathbf{P}_a\left(\cdot \Big| S_{\Delta n} \geq 0\right), \left(\frac{S_{\lfloor r\Delta n \rfloor}}{\sigma\sqrt{\Delta n}}; 0 \leq r \leq 1\right)$ converges in law to a Brownian meander, denoted by $(\mathcal{M}_r; 0 \leq r \leq 1)$. Therefore,

$$\chi(F, 1_{(0 \leq x \leq K)}) \sim \frac{C_- C_+ R(a)}{\sigma n^{3/2} (1-\Delta) \sqrt{\Delta}} \int_0^K R_-(u) du \mathbf{E} \left[F\left(\sqrt{\Delta} \mathcal{M}_{s/\Delta}; 0 \leq s \leq \Delta\right) \psi\left(\frac{\sqrt{\Delta} \mathcal{M}_1}{\sqrt{1-\Delta}}\right) \right].$$

It remains to check that

$$(2.22) \quad \frac{1}{(1-\Delta)\sqrt{\Delta}} \mathbf{E} \left[F\left(\sqrt{\Delta} \mathcal{M}_{s/\Delta}; 0 \leq s \leq \Delta\right) \psi\left(\frac{\sqrt{\Delta} \mathcal{M}_1}{\sqrt{1-\Delta}}\right) \right] = \sqrt{\frac{\pi}{2}} \mathbf{E} \left[F(e_s; 0 \leq s \leq \Delta) \right].$$

Let $(R_s; 0 \leq s \leq 1)$ be a standard three-dimensional Bessel process. Then, as is shown in [12],

$$\begin{aligned} & \frac{1}{(1-\Delta)\sqrt{\Delta}} \mathbf{E} \left[F\left(\sqrt{\Delta} \mathcal{M}_{s/\Delta}; 0 \leq s \leq \Delta\right) \psi\left(\frac{\sqrt{\Delta} \mathcal{M}_1}{\sqrt{1-\Delta}}\right) \right] \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{(1-\Delta)\sqrt{\Delta}} \mathbf{E} \left[\frac{1}{R_1} F\left(\sqrt{\Delta} R_{s/\Delta}; 0 \leq s \leq \Delta\right) \psi\left(\frac{\sqrt{\Delta} R_1}{\sqrt{1-\Delta}}\right) \right], \\ &= \sqrt{\frac{\pi}{2}} \mathbf{E} \left[\frac{1}{(1-\Delta)^{3/2}} e^{-\frac{R_1^2}{2(1-\Delta)}} F\left(R_s; 0 \leq s \leq \Delta\right) \right], \end{aligned}$$

where the last equation follows from the scaling property of Bessel process. Let $(r_s; 0 \leq s \leq 1)$ be a standard three-dimensional Bessel bridge. Note that for any $\Delta < 1$, $(r_s; 0 \leq s \leq \Delta)$ is equivalent to $(R_s; 0 \leq s \leq \Delta)$, with density $(1 - \Delta)^{-3/2} \exp(-\frac{R_\Delta^2}{2(1-\Delta)})$ (see p. 468 (3.11) of [16]). Thus,

$$\frac{1}{(1-\Delta)\sqrt{\Delta}} \mathbf{E} \left[F \left(\sqrt{\Delta} \mathcal{M}_{s/\Delta}; 0 \leq s \leq \Delta \right) \psi \left(\frac{\sqrt{\Delta} \mathcal{M}_1}{\sqrt{1-\Delta}} \right) \right] = \sqrt{\frac{\pi}{2}} \mathbf{E} \left[F(r_s; 0 \leq s \leq \Delta) \right].$$

Since a normalized Brownian excursion is exactly a standard three-dimensional Bessel bridge, this yields (2.22). Therefore, (2.11) is proved by taking $C_1 = \sqrt{\frac{\pi}{2}} \frac{C_- C_+}{\sigma}$.

The proof of the lemma in the lattice case is along the same lines, except that we use Theorem 2 (instead of Theorem 1) of [6]. \square

3 Conditioning on the event $\{I_n \leq \frac{3}{2} \ln n - z\}$

On the event $\{I_n \leq \frac{3}{2} \ln n - z\}$, we analyze the sample path leading to a particle located at the leftmost position at the n th generation. For $z \geq 0$ and $n \geq 1$, let $a_n(z) := \frac{3}{2} \ln n - z$ if the distribution of \mathcal{L} is non-lattice and let $a_n(z) := \alpha n + \beta \lfloor \frac{\frac{3}{2} \ln n - \alpha n}{\beta} \rfloor - z$ if the distribution of \mathcal{L} is supported by $\alpha + \beta\mathbb{Z}$. This section is devoted to the proof of the following proposition.

Proposition 3.1 *For any $\Delta \in (0, 1]$ and any continuous functional $F : D([0, \Delta], \mathbb{R}) \rightarrow [0, 1]$,*

$$(3.1) \quad \lim_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E} \left[F(e_s; 0 \leq s \leq \Delta) \right] \right| = 0.$$

We begin with some preliminary results.

For any $0 < \Delta < 1$ and $L, K \geq 0$, we denote by $J_{z,K,L}^\Delta(n)$ the following collection of particles:

$$(3.2) \quad \left\{ u \in \mathbb{T} : |u| = n, V(u) \leq a_n(z), \min_{0 \leq k \leq n} V(u_k) \geq -z + K, \min_{\Delta n \leq k \leq n} V(u_k) \geq a_n(z + L) \right\}.$$

Lemma 3.2 *For any $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that for any $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,*

$$(3.3) \quad \mathbf{P} \left(m^{(n)} \notin J_{z,K,L}^\Delta(n), I_n \leq a_n(z) \right) \leq \left(e^K + \varepsilon(1 + z - K) \right) e^{-z}.$$

Proof. It suffices to show that for any $\varepsilon \in (0, 1)$, there exists $L_\varepsilon \geq 1$ such that for any $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,

$$(3.4) \quad \mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), u \notin J_{z,K,L}^\Delta(n)\right) \leq \left(e^K + \varepsilon(1 + z - K)\right)e^{-z}.$$

We observe that

$$(3.5) \quad \mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), u \notin J_{z,K,L}^\Delta(n)\right) \leq \mathbf{P}\left(\exists u \in \mathbb{T} : V(u) \leq -z + K\right) \\ + \mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), \min_{0 \leq k \leq n} V(u_k) \geq -z + K, \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(z + L)\right).$$

On the one hand, by (2.4),

$$(3.6) \quad \mathbf{P}\left(\exists u \in \mathbb{T} : V(u) \leq -z + k\right) \leq \sum_{n \geq 0} \mathbf{E}\left[\sum_{|u|=n} \mathbf{1}_{\{V(u) \leq -z + K < \min_{k < n} V(u_k)\}}\right] \\ = \sum_{n \geq 0} \mathbf{E}[e^{S_n}; S_n \leq -z + K < \underline{S}_{n-1}] \leq e^{-z+K}.$$

On the other hand, denoting $A_n(z) := [a_n(z) - 1, a_n(z)]$ for any $z \geq 0$,

$$\mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), \min_{0 \leq k \leq n} V(u_k) \geq -z + K, \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(z + L)\right) \\ = \mathbf{P}_{z-K}\left(\exists |u| = n : V(u) \leq a_n(K), \min_{0 \leq k \leq n} V(u_k) \geq 0, \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(K + L)\right) \\ \leq \sum_{\ell \geq L+K} \sum_{j=K}^{j=K+\ell} \mathbf{P}_{z-K}\left(\exists |u| = n : V(u) \in A_n(j), \min_{0 \leq k \leq n} V(u_k) \geq 0, \min_{\Delta n \leq k \leq n} V(u_k) \in A_n(\ell)\right).$$

According to Lemma 3.3 in [2], there exist constants $1 > c_5 > 0$ and $c_6 > 0$ such that for any $n \geq 1$, $L \geq 0$ and $x, z \geq 0$,

$$(3.7) \quad \mathbf{P}_x\left(\exists u \in \mathbb{T} : |u| = n, V(u) \in A_n(z), \min_{0 \leq k \leq n} V(u_k) \geq 0, \min_{\Delta n \leq k \leq n} V(u_k) \in A_n(z + L)\right) \\ \leq c_6(1 + x)e^{-c_5 L}e^{-x-z}.$$

Hence, combining (3.6) with (3.5) yields that

$$\mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), u \notin J_{z,K,L}^\Delta(n)\right) \\ \leq e^{-z+K} + \sum_{\ell \geq L} \sum_{0 \leq j \leq \ell} c_6(1 + z - K)e^{-c_5(\ell-j)}e^{-z-j} \\ \leq \left(e^K + c_7 \sum_{\ell \geq L} e^{-c_5 \ell}(1 + z - K)\right)e^{-z},$$

where the last inequality comes from the fact that $\sum_{j \geq 0} e^{-(1-c_5)j} < \infty$. We take $L_\varepsilon = -c_8 \ln \varepsilon$ so that $c_7 \sum_{\ell \geq L} e^{-c_5 \ell} \leq \varepsilon$ for all $L \geq L_\varepsilon$. Therefore, for any $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,

$$(3.8) \quad \mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), u \notin J_{z,K,L}^\Delta(n)\right) \leq \left(e^K + \varepsilon(1+z-K)\right)e^{-z},$$

which completes the proof. \square

For $b \in \mathbb{Z}_+$, we define

$$(3.9) \quad \mathcal{E}_n = \mathcal{E}_n(z, b) := \{\forall k \leq n - b, \min_{u \geq w_k, |u|=n} V(u) > a_n(z)\}.$$

We note that on the event $\mathcal{E}_n \cap \{I_n \leq a_n(z)\}$, any particle located at the leftmost position must be separated from the spine after time $n - b$.

Lemma 3.3 *For any $\eta > 0$ and $L > 0$, there exist $K(\eta) > 0$, $B(L, \eta) \geq 1$ and $N(\eta) \geq 1$ such that for any $b \geq B(L, \eta)$, $n \geq N(\eta)$ and $z \geq K \geq K(\eta)$,*

$$(3.10) \quad \mathbf{Q}\left(\mathcal{E}_n^c, w_n \in J_{z,K,L}^\Delta(n)\right) \leq \eta(1+L)^2(1+z-K)n^{-3/2}.$$

We feel free to omit the proof of Lemma 3.3 since it is just a slightly stronger version of Lemma 3.8 in [2]. It follows from the same arguments.

Let us turn to the proof of Proposition 3.1. We break it up into 3 steps.

Step (I) (The conditioned convergence of $(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta)$ for $\Delta < 1$ in the non-lattice case)

Assume that the distribution of \mathcal{L} is non-lattice in this step. Recall that $a_n(z) = \frac{3}{2} \ln n - z$. The tail distribution of I_n has been given in Propositions 1.3 and 4.1 of [2], recalled as follows.

Fact 3.4 ([2]) *There exists a constant $C > 0$ such that*

$$(3.11) \quad \lim_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) - C \right| = 0.$$

Furthermore, for any $\varepsilon > 0$, there exist $N_\varepsilon \geq 1$ and $\Lambda_\varepsilon > 0$ such that for any $n \geq N_\varepsilon$ and $\Lambda_\varepsilon \leq z \leq \frac{3}{2} \ln n - \Lambda_\varepsilon$,

$$(3.12) \quad \left| \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) - C \right| \leq \varepsilon. \quad \square$$

For any continuous functional $F : D([0, \Delta], \mathbb{R}) \rightarrow [0, 1]$, it is convenient to write that

$$(3.13) \quad \Sigma_n(F, z) := \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \mathbf{1}_{\{I_n \leq a_n(z)\}} \right].$$

In particular, if $F \equiv 1$, $\Sigma_n(1, z) = \mathbf{P}(I_n \leq a_n(z))$. Thus,

$$(3.14) \quad \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} = \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \middle| I_n \leq a_n(z) \right].$$

Let us prove the following convergence for $0 < \Delta < 1$,

$$(3.15) \quad \lim_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| = 0.$$

Proof of (3.15). For any $n \geq 1$, $L \geq 0$ and $z \geq K \geq 0$, let

$$(3.16) \quad \Pi_n(F) = \Pi_n(F, z, K, L) := \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \mathbf{1}_{\{m^{(n)} \in J_{z, K, L}^\Delta(n)\}} \right].$$

By Lemma 3.2, we obtain that for $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,

$$(3.17) \quad \left| \Sigma_n(F, z) - \Pi_n(F) \right| \leq \left(e^K + \varepsilon(1 + z - K) \right) e^{-z}.$$

Note that $m^{(n)}$ is chosen uniformly among the particles located at the leftmost position. Thus,

$$\begin{aligned} \Pi_n(F) &= \mathbf{E} \left[\sum_{|u|=n} \mathbf{1}_{(u=m^{(n)}, u \in J_{z, K, L}^\Delta(n))} F \left(\frac{V(u_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \right] \\ &= \mathbf{E} \left[\frac{1}{\sum_{|u|=n} \mathbf{1}_{(V(u)=I_n)}} \sum_{|u|=n} \mathbf{1}_{(V(u)=I_n, u \in J_{z, K, L}^\Delta(n))} F \left(\frac{V(u_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \right]. \end{aligned}$$

Applying the change of measures given in (2.2), it follows from Proposition 2.1 that

$$(3.18) \quad \Pi_n(F) = \mathbf{E}_{\mathbf{Q}} \left[\frac{e^{V(w_n)}}{\sum_{|u|=n} \mathbf{1}_{(V(u)=I_n)}} \mathbf{1}_{(V(w_n)=I_n, w_n \in J_{z, K, L}^\Delta(n))} F \left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \right].$$

In order to estimate Π_n , we restrict ourselves to the event \mathcal{E}_n . Define

$$\Lambda_n(F) := \mathbf{E}_{\mathbf{Q}} \left[\frac{e^{V(w_n)}}{\sum_{|u|=n} \mathbf{1}_{(V(u)=I_n)}} \mathbf{1}_{(V(w_n)=I_n, w_n \in J_{z, K, L}^\Delta(n))} F \left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right); \mathcal{E}_n \right].$$

In view of Lemma 3.3, for any $b \geq B(L, \eta)$, $n \geq N(\eta)$ and $z \geq K \geq K(\eta)$,

$$(3.19) \quad \begin{aligned} \left| \Pi_n(F) - \Lambda_n(F) \right| &\leq \mathbf{E}_{\mathbf{Q}} \left[e^{V(w_n)}; w_n \in J_{z, K, L}^\Delta(n), \mathcal{E}_n^c \right] \\ &\leq e^{-z} n^{-3/2} \mathbf{Q} \left(\mathcal{E}_n^c, w_n \in J_{z, K, L}^\Delta(n) \right) \\ &\leq \eta(1 + L)^2 (1 + z - K) e^{-z}. \end{aligned}$$

On the event $\mathcal{E}_n \cap \{I_n \leq a_n(z)\}$, $\Lambda_n(F)$ equals

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{e^{V(w_n)}}{\sum_{u > w_{n-b}, |u|=n} \mathbf{1}_{\{V(u)=I_n\}}} \mathbf{1}_{\{V(w_n)=I_n, w_n \in J_{z,K,L}^\Delta(n)\}} F\left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\right); \mathcal{E}_n \right].$$

Let, for $x \geq 0$, $L > 0$, and $b \geq 1$,

$$\begin{aligned} f_{L,b}(x) &:= \mathbf{E}_{\mathbf{Q}_x} \left[\frac{e^{V(w_b)-L} \mathbf{1}_{\{V(w_b)=I_b\}}}{\sum_{|u|=b} \mathbf{1}_{\{V(u)=I_b\}}}, \min_{0 \leq k \leq b} V(w_k) \geq 0, V(w_b) \leq L \right] \\ (3.20) \quad &\leq \mathbf{Q}_x \left(\min_{0 \leq k \leq b} V(w_k) \geq 0, V(w_b) \leq L \right). \end{aligned}$$

We choose n large enough so that $\Delta n \leq n - b$. Thus, applying the Markov property at time $n - b$ yields that

$$\begin{aligned} (3.21) \quad \Lambda_n(F) &= n^{3/2} e^{-z} \mathbf{E}_{\mathbf{Q}} \left[F\left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\right) f_{L,b}(V(w_{n-b}) - a_n(z+L)); \right. \\ &\quad \left. \min_{0 \leq k \leq n-b} V(w_k) \geq -z+K, \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L), \mathcal{E}_n \right]. \end{aligned}$$

Let us introduce the following quantity by removing the restriction to \mathcal{E}_n :

$$\begin{aligned} (3.22) \quad \Lambda_n^I(F) &:= n^{3/2} e^{-z} \mathbf{E}_{\mathbf{Q}} \left[F\left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\right) f_{L,b}(V(w_{n-b}) - a_n(z+L)); \right. \\ &\quad \left. \min_{0 \leq k \leq n-b} V(w_k) \geq -z+K, \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L) \right]. \end{aligned}$$

We immediately observe that

$$\begin{aligned} (3.23) \quad \left| \Lambda_n(F) - \Lambda_n^I(F) \right| &\leq n^{3/2} e^{-z} \mathbf{Q} \left(f_{L,b}(V(w_{n-b}) - a_n(z+L)), \right. \\ &\quad \left. \min_{0 \leq k \leq n-b} V(w_k) \geq -z+K, \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L); (\mathcal{E}_n)^c \right). \end{aligned}$$

By (3.20), we check that $\left| \Lambda_n(F) - \Lambda_n^I(F) \right| \leq n^{3/2} e^{-z} \mathbf{Q}(w_n \in J_{z,K,L}^\Delta(n), (\mathcal{E}_n)^c)$. Applying Lemma 3.3 again implies that

$$(3.24) \quad \left| \Lambda_n(F) - \Lambda_n^I(F) \right| \leq \eta(1+L)^2(1+z-K)e^{-z}.$$

Combining with (3.19), we obtain that for any $b \geq B(L, \eta)$, $z \geq K \geq K(\eta)$ and n large enough,

$$(3.25) \quad \left| \Pi_n(F) - \Lambda_n^I(F) \right| \leq 2\eta(1+L)^2(1+z-K)e^{-z}.$$

Note that $(V(w_k); k \geq 1)$ is a centered random walk under \mathbf{Q} and that it is proved in [2] that $f_{L,b}$ satisfies the conditions of Lemma 2.4. By (I) of Lemma 2.4, we get that

$$(3.26) \quad \lim_{n \rightarrow \infty} \Lambda_n^I(F) = \alpha_{L,b}^I R(z-K) e^{-z} \mathbf{E}[F(e_s, 0 \leq s \leq \delta)],$$

where $\alpha_{L,b}^I := C_1 \int_{x \geq 0} f_{L,b}(x) R_-(x) dx \in [0, \infty)$. Thus, by (3.25), one sees that for any $b \geq B(L, \eta)$ and $z \geq K \geq K(\eta)$,

$$(3.27) \quad \limsup_{n \rightarrow \infty} \left| \Pi_n(F) - \alpha_{L,b}^I R(z-K) e^{-z} \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq 2\eta(1+L)^2(1+z-K)e^{-z}.$$

Going back to (3.17), we deduce that for any $L \geq L_\varepsilon$, $b \geq B(L, \eta)$ and $z \geq K \geq K(\eta)$,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \Sigma_n(F, z) - \alpha_{L,b}^I R(z-K) e^{-z} \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \\ & \leq 2\eta(1+L)^2(1+z-K)e^{-z} + \left(e^K + \varepsilon(1+z-K) \right) e^{-z}. \end{aligned}$$

Recall that $\lim_{z \rightarrow \infty} \frac{R(z)}{z} = c_0$. We multiply each term by $\frac{e^z}{z}$, and then let z go to infinity to conclude that

$$(3.28) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{e^z}{z} \Sigma_n(F, z) - \alpha_{L,b}^I c_0 \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq 2\eta(1+L)^2 + \varepsilon.$$

In particular, taking $F \equiv 1$ gives that

$$(3.29) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) - \alpha_{L,b}^I c_0 \right| \leq 2\eta(1+L)^2 + \varepsilon.$$

It follows from Fact 3.4 that $|C - \alpha_{L,b}^I c_0| \leq 2\eta(1+L)^2 + \varepsilon$. We thus choose $0 < \varepsilon < C/10$ and $0 < \eta \leq \frac{\varepsilon}{2(1+L_\varepsilon)^2}$ so that $2C > \alpha_{L_\varepsilon, b}^I c_0 > C/2 > 0$.

Therefore, for any $\varepsilon \in (0, C/10)$, $0 < \eta \leq \frac{\varepsilon}{2(1+L_\varepsilon)^2}$, $L = L_\varepsilon$ and $b \geq B(L_\varepsilon, \eta)$,

$$(3.30) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq \frac{4\varepsilon}{C/2 - 2\varepsilon},$$

which completes the proof of (3.15) in the non-lattice case.

Step (II) (The conditioned convergence of $(\frac{I_n(s)}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta)$ for $\Delta < 1$ in the lattice case) Assume that the law of \mathcal{L} is supported by $\alpha + \beta\mathbb{Z}$ with span β . Recall that $a_n(0) = \alpha n + \beta \lfloor \frac{3}{2} \frac{\ln n - \alpha n}{\beta} \rfloor$ and that $a_n(z) = a_n(0) - z$. We use the same notation of Step (I). Let us prove

$$(3.31) \quad \lim_{\beta\mathbb{Z} \ni z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| = 0.$$

Suppose that $z \in \beta\mathbb{Z}$. Whereas the arguments of Step (I), we obtain that for any $L \geq L_\varepsilon$, $b \geq B(L, \eta)$, $z \geq K \geq K(\eta)$ and n sufficiently large,

$$(3.32) \quad \left| \Sigma_n(F, z) - \Lambda_n^{II}(F) \right| \leq 2\eta(1+L)^2(1+z-K)e^{-z} + \left(e^K + \varepsilon(1+z-K) \right) e^{-z},$$

where

$$\Lambda_n^{II}(F) = \Lambda^{II}(F, z, K, L, b) := e^{a_n(0)} e^{-z} \mathbf{E}_{\mathbf{Q}} \left[F \left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \times \right. \\ \left. f_{L,b}(V(w_{n-b} - a_n(z+L))) \right]; \quad \min_{0 \leq k \leq n-b} V(w_k) \geq -z + K, \quad \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L).$$

Under \mathbf{Q} , the distribution of $V(w_1) - V(w_0)$ is also supported by $\alpha + \beta\mathbb{Z}$. Let $d = d(L, b) := \beta \lceil \frac{\alpha b - L}{\beta} \rceil - \alpha b + L$ and $\lambda_n := n^{3/2} e^{-a_n(0)}$. Recall that $f_{L,b}$ is well defined in (3.20), it follows from (II) of Lemma 2.4 that

$$(3.33) \quad \lim_{n \rightarrow \infty} \lambda_n \Lambda_n^{II}(F) = \alpha_{L,b}^{II} R(z-K) e^{-z} \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)].$$

where $\alpha_{L,b}^{II} := C_1 \beta \sum_{j \geq 0} f_{L,b}(\beta j + d) R_-(\beta j + d) \in [0, \infty)$. Observe that $1 \leq \lambda_n \leq e^\beta$. Combining with (3.32), we conclude that

$$(3.34) \quad \limsup_{\beta\mathbb{Z} \ni z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{e^z}{z} \lambda_n \Sigma_n(F, z) - \alpha_{L,b}^{II} c_0 \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq e^\beta (2\eta(1+L)^2 + \varepsilon).$$

We admit for the moment that there exist $0 < c_9 < c_{10} < \infty$ such that $\alpha_{L,b}^{II} \in [c_9, c_{10}]$ for all L, b large enough. Then take $\varepsilon < \frac{c_9 c_0}{4e^\beta}$, $L = L_\varepsilon$, $\eta = \frac{\varepsilon}{2(1+L_\varepsilon)^2}$ and $b \geq B(L_\varepsilon, \eta)$ so that $e^\beta (2\eta(1+L)^2 + \varepsilon) < c_9 c_0 / 2 \leq \alpha_{L_\varepsilon, b}^{II} c_0 / 2 \leq 2c_{10} c_0$. Note that $\frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} = \frac{e^{-z} \lambda_n \Sigma_n(F, z)}{e^{-z} \lambda_n \Sigma_n(1, z)}$. We thus deduce from (3.34) that

$$(3.35) \quad \limsup_{\beta\mathbb{Z} \ni z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq \frac{4\varepsilon}{c_9 c_0 / e^\beta - 2\varepsilon},$$

which tends to zero as $\varepsilon \downarrow 0$.

It remains to prove that $\alpha_{L,b}^{II} \in [c_9, c_{10}]$ for all L, b large enough. Instead of investigating the entire system, we consider the branching random walk killed at 0. Define

$$(3.36) \quad I_n^{kill} := \inf \{ V(u) : |u| = n, V(u_k) \geq 0, \forall 0 \leq k \leq n \},$$

and we get the following fact from Corollary 3.4 and Lemma 3.6 of [2].

Fact 3.5 ([2]) *There exists a constant $c_{11} > 0$ such that for any $n \geq 1$ and $x, z \geq 0$,*

$$(3.37) \quad \mathbf{P}_x(I_n^{kill} \leq a_n(z)) \leq c_{11}(1+x)e^{-x-z}.$$

Moreover, there exists $c_{12} > 0$ such that for any $n \geq 1$ and $z \in [0, a_n(1)]$,

$$(3.38) \quad \mathbf{P}(I_n^{kill} \leq a_n(z)) \geq c_{12}e^{-z}.$$

Even though Fact 3.5 is proved in [2] under the assumption that the distribution of \mathcal{L} is non-lattice, the lattice case is actually recovered from that proof.

Analogously, let $m^{kill,(n)}$ be the particle chosen uniformly in the set $\{u : |u| = n, V(u) = I_n^{kill}, \min_{0 \leq k \leq n} V(u_k) \geq 0\}$. Moreover, let $\Sigma_n^{kill}(1, z) := \mathbf{P}[I_n^{kill} \leq a_n(z)]$ and $\Pi_n^{kill}(1, z, z, L) := \mathbf{P}[I_n^{kill} \leq a_n(z), m^{kill,(n)} \in J_{z,z,L}^\Delta(n)]$. By (3.7) again, we check that for all $L \geq L_\varepsilon$,

$$(3.39) \quad \begin{aligned} & \left| \Sigma_n^{kill}(1, z) - \Pi_n^{kill}(1, z, z, L) \right| \\ & \leq \mathbf{P}\left[\exists |u| = n : V(u) \leq a_n(z); \min_{0 \leq k \leq n} V(u_k) \geq 0; \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(z+L)\right] \\ & \leq \varepsilon e^{-z}. \end{aligned}$$

Recounting the arguments of Step (1), one sees that for any $L \geq L_\varepsilon, b \geq B(L, \eta), z \geq K(\eta)$ and n sufficiently large,

$$(3.40) \quad \left| \Pi_n^{kill}(1, z, z, L) - \Lambda_n^{kill} \right| \leq 2\eta(1+L)^2 e^{-z},$$

where

$$(3.41) \quad \Lambda_n^{kill} := \mathbf{E}_{\mathbf{Q}} \left[f^{kill}(V(w_{n-b})); \min_{0 \leq k \leq n-b} V(w_k) \geq 0, \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L) \right],$$

with $f^{kill}(x) := \mathbf{E}_{\mathbf{Q}_x} \left[\frac{e^{V(w_b)} \mathbf{1}_{\{V(w_b) = I_b^{kill}\}}}{\sum_{|u|=b} \mathbf{1}_{\{V(u) = I_b^{kill}, \min_{0 \leq j \leq b} V(u_j) \geq 0\}}}; \min_{0 \leq k \leq b} V(w_k) \geq a_n(z+L), V(w_b) \leq a_n(z) \right]$. For $\varepsilon > 0$ and n sufficiently large, it has been proved in [2] that

$$(3.42) \quad \left| e^z \Lambda_n^{II}(1, z, z, L, b) - \Lambda_n^{kill} \right| \leq \varepsilon.$$

Recalling the convergence (3.33) with $K = z$ and $F \equiv 1$, we deduce from (3.39), (3.40) and (3.42) that for any $L \geq L_\varepsilon, b \geq B(L, \eta)$ and $z \geq K(\eta)$,

$$(3.43) \quad \limsup_{n \rightarrow \infty} \left| \lambda_n \Sigma_n^{kill}(1, z) - \alpha_{L,b}^{II} e^{-z} \right| \leq e^\beta \left(2\eta(1+L)^2 + 2\varepsilon \right) e^{-z},$$

since $R(0) = 1$ and $1 \leq \lambda_n \leq e^\beta$. Fact 3.5 implies that $c_{12} \leq e^z \lambda_n \mathbf{P}(I_n^{kill} \leq a_n(z)) \leq c_{11} e^\beta$. Hence, we obtain that

$$(3.44) \quad c_{12} - e^\beta \left(2\eta(1+L)^2 + 2\varepsilon \right) \leq \alpha_{L,b}^{II} \leq e^\beta c_{11} + e^\beta \left(2\eta(1+L)^2 + 2\varepsilon \right).$$

Let $c_{10} := c_{11} e^\beta + c_{12}$ and $c_9 := 3c_{12}/4 > 0$. For any $\varepsilon < e^{-\beta} c_{12}/12$, we take $L = L_\varepsilon$ and $\eta \leq \varepsilon/2(1+L_\varepsilon)^2$. Then $c_{10} > \alpha_{L,b}^{II} \geq c_9 > 0$ for $b \geq B(L_\varepsilon, \eta)$. This completes the second step.

Step (III)(The tightness) Actually, it suffices to prove the following proposition.

Proposition 3.6 *For any $\eta > 0$,*

$$(3.45) \quad \lim_{\delta \rightarrow 0} \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta \sigma \sqrt{n} \mid I_n \leq a_n(z) \right) = 0.$$

The first two steps allow us to obtain the following fact whether the distribution is lattice or non-lattice.

Fact 3.7 *There exist constants $c_{13}, c_{14} \in (0, \infty)$ such that*

$$(3.46) \quad c_{13} \leq \liminf_{z \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) \leq \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) \leq c_{14}.$$

Proof of Proposition 3.6. First, we observe that for any $M \geq 1$ and $\delta \in (0, 1/2)$,

$$\begin{aligned} & \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \delta \sigma \sqrt{n}, I_n \leq a_n(z) \right) \\ & \leq \mathbf{P} \left(m_n^{(n)} \notin J_{z,0,L}^{1/2}(n), I_n \leq a_n(z) \right) + \mathbf{P} \left(I_n(n - \lfloor \delta n \rfloor) \geq M \sigma \sqrt{\delta n}, I_n \leq a_n(z) \right) + \chi(\delta, z, n). \end{aligned}$$

where $\chi(\delta, z, n) := \mathbf{P} \left(m_n^{(n)} \in J_{z,0,L}^{1/2}(n), I_n(n - \lfloor \delta n \rfloor) \leq M \sigma \sqrt{\delta n}, \sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta \sigma \sqrt{n} \right)$.

It follows from Lemma 3.2 that for any $\varepsilon > 0$, if $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq 0$,

$$(3.47) \quad \mathbf{P} \left(m_n^{(n)} \notin J_{z,0,L}^{1/2}(n), I_n \leq a_n(z) \right) \leq (1 + \varepsilon(1+z))e^{-z}.$$

Then dividing each term of (3.47) by $\mathbf{P}(I_n \leq a_n(z))$ yields that

$$(3.48) \quad \begin{aligned} & \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta \sigma \sqrt{n} \mid I_n \leq a_n(z) \right) \\ & \leq \frac{(1 + \varepsilon(1+z))e^{-z}}{\mathbf{P}(I_n \leq a_n(z))} + \mathbf{P} \left(I_n(n - \lfloor \delta n \rfloor) \geq M \sigma \sqrt{\delta n} \mid I_n \leq a_n(z) \right) + \frac{\chi(\delta, z, n)}{\mathbf{P}(I_n \leq a_n(z))}. \end{aligned}$$

On the one hand, by Fact 3.7,

$$(3.49) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{(1 + \varepsilon(1 + z))e^{-z}}{\mathbf{P}(I_n \leq a_n(z))} \leq \frac{\varepsilon}{c_{13}}.$$

On the other hand, Steps (I) and (II) tell us that for any $1 > \delta > 0$ and $M \geq 1$,

$$(3.50) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left[I_n(n - \lfloor \delta n \rfloor) \geq M\sigma\sqrt{\delta n} \mid I_n \leq a_n(z) \right] = \mathbf{P}[e_{1-\delta} \geq M\sqrt{\delta}],$$

which, by Chebyshev's inequality, is bounded by $\frac{\mathbf{E}[e_{1-\delta}]}{M\sqrt{\delta}} = \frac{4\sqrt{1-\delta}}{M\sqrt{2\pi}}$. Consequently,

$$(3.51) \quad \begin{aligned} & \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n - k) - I_n| \geq \eta\sigma\sqrt{n} \mid I_n \leq a_n(z) \right) \\ & \leq \frac{\varepsilon}{c_{13}} + \frac{2}{M} + \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\chi(\delta, z, n)}{\mathbf{P}(I_n \leq a_n(z))}. \end{aligned}$$

Let us estimate $\chi(\delta, z, n)$. One sees that

$$\chi(\delta, z, n) \leq \mathbf{E} \left[\sum_{|u|=n} \mathbf{1}_{\{u \in J_{z,L}^{1/2}(n); \sup_{0 \leq k \leq \delta n} |V(u_{n-k}) - V(u)| \geq \eta\sigma\sqrt{n}; V(u_{n-\lfloor \delta n \rfloor}) \leq M\sigma\sqrt{\delta n}\} \right].$$

By Lemma 2.4, it becomes that

$$\begin{aligned} \chi(\delta, z, n) & \leq \mathbf{E} \left[e^{S_n}; S_n \leq a_n(z), \underline{S}_n \geq -z, \underline{S}_{[n/2, n]} \geq a_n(z + L), \right. \\ & \quad \left. S_{n-\lfloor \delta n \rfloor} \leq M\sigma\sqrt{\delta n}, \sup_{0 \leq k \leq \delta n} |S_{n-k} - S_n| \geq \eta\sigma\sqrt{n} \right] \\ & \leq n^{3/2} e^{-z} \Upsilon(\delta, z, n), \end{aligned}$$

where $\Upsilon(\delta, z, n) := \mathbf{P} \left(S_n \leq a_n(z), \underline{S}_n \geq -z, \underline{S}_{[n/2, n]} \geq a_n(z + L), S_{n-\lfloor \delta n \rfloor} \leq M\sigma\sqrt{\delta n}, \sup_{0 \leq k \leq \delta n} |S_{n-k} - S_n| \geq \eta\sigma\sqrt{n}, S_{n-\lfloor \delta n \rfloor} \leq M\sigma\sqrt{\delta n} \right)$.

Reversing time yields that

$$(3.52) \quad \begin{aligned} \Upsilon(\delta, z, n) & \leq \mathbf{P} \left(\underline{S}_n^- \geq -a_n(0), \underline{S}_{n/2}^- \geq -L, -S_n \in [-a_n(z), -a_n(z + L)], \right. \\ & \quad \left. \sup_{0 \leq k \leq \delta n} | -S_k | \geq \eta\sigma\sqrt{n}, -S_{\lfloor \delta n \rfloor} \leq M\sigma\sqrt{\delta n} - a_n(z + L) \right). \end{aligned}$$

Applying the Markov property at time $\lfloor \delta n \rfloor$, we obtain that

$$(3.53) \quad \Upsilon(\delta, z, n) = \mathbf{E} \left[\Theta(-S_{\lfloor \delta n \rfloor}); \underline{S}_{\delta n}^- \geq -L, \sup_{0 \leq k \leq \delta n} | -S_k | \geq \eta\sigma\sqrt{n} \right],$$

where $\Theta(x) := \mathbf{1}_{\{x \leq M\sigma\sqrt{\delta n} - a_n(z+L)\}} \mathbf{P}_x \left(\underline{S}_{(1/2-\delta)n}^- \geq -L, \underline{S}_{(1-\delta)n}^- \geq -a_n(0), -S_{n-[\delta n]} \in [-a_n(z), -a_n(z+L)] \right)$. Reversing time again implies that

$$\Theta(x) \leq \mathbf{1}_{\{x \leq M\sigma\sqrt{\delta n}\}} \mathbf{P} \left(\underline{S}_{(1-\delta)n} \geq -z - L, \underline{S}_{[n/2, (1-\delta)n]} \geq a_n(z+2L), S_{n-[\delta n]} \in [x + a_n(z+L), x + a_n(z)] \right).$$

By (2.10), $\Theta(x) \leq c_{15}(1+z+L)(1+L)(1+M\sigma\sqrt{\delta n}+2L)n^{-3/2}$. Plugging it into (3.53) and taking n large enough so that $1+2L < \eta\sigma\sqrt{\delta n}$, we get that

$$\Upsilon(\delta, z, n) \leq c_{15}(1+z)(1+L)^2 n^{-3/2} (M+\eta)\sigma\sqrt{\delta n} \mathbf{E} \left[\underline{S}_{\delta n}^- \geq -L, \sup_{0 \leq k \leq \delta n} |-S_k| \geq \eta\sigma\sqrt{n} \right].$$

Recall that $\chi(\delta, z, n) \leq e^{-z} n^{3/2} \Upsilon(\delta, z, n)$. We check that

$$(3.54) \quad \chi(\delta, z, n) \leq c_{15} e^{-z} (1+z)(1+L)^2 (M+\eta)\sigma \times \mathbf{E}_L \left[\sup_{0 \leq k \leq \delta n} (-S_k) \geq \eta\sigma\sqrt{n} \mid \underline{S}_{\delta n}^- \geq 0 \right] \left(\sqrt{\delta n} \mathbf{P}_L \left[\underline{S}_{\delta n}^- \geq 0 \right] \right).$$

On the one hand, by Theorem 1.1 of [7], $\mathbf{E}_L \left[\sup_{0 \leq k \leq \delta n} (-S_k) \geq \eta\sigma\sqrt{n} \mid \underline{S}_{\delta n}^- \geq 0 \right]$ converges to $\mathbf{P}(\sup_{0 \leq s \leq 1} \mathcal{M}_s \geq \eta/\sqrt{\delta})$ as $n \rightarrow \infty$. On the other hand, (2.7) shows that $\sqrt{\delta n} \mathbf{P}_L \left[\underline{S}_{\delta n}^- \geq 0 \right]$ converges to $C_- R_-(L)$ as $n \rightarrow \infty$. Therefore,

$$\limsup_{n \rightarrow \infty} \chi(\delta, z, n) \leq c_{15} e^{-z} (1+z)(1+L)^2 (M+\eta)\sigma C_- R_-(L) \times \mathbf{P} \left(\sup_{0 \leq s \leq 1} \mathcal{M}_s \geq \eta/\sqrt{\delta} \right).$$

Going back to (3.51) and letting $z \rightarrow \infty$, we deduce from Fact 3.7 that

$$(3.55) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta\sigma\sqrt{n} \mid I_n \leq a_n(z) \right) \leq \frac{\varepsilon}{c_{13}} + \frac{2}{M} + \frac{c_{15}(1+L)^2 (M+\eta)\sigma C_- R_-(L) \times \mathbf{P}(\sup_{0 \leq s \leq 1} \mathcal{M}_s \geq \eta/\sqrt{\delta})}{c_{13}}.$$

Notice that $\mathbf{P}(\sup_{0 \leq s \leq 1} \mathcal{M}_s \geq \eta/\sqrt{\delta})$ decreases to 0 as $\delta \downarrow 0$. Take $M \geq 2/\varepsilon$. We conclude that for any $0 < \varepsilon < c_{13}$,

$$(3.56) \quad \limsup_{\delta \rightarrow 0} \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta\sigma\sqrt{n} \mid I_n \leq a_n(z) \right) \leq \frac{\varepsilon}{c_{13}} + \varepsilon,$$

which completes the proof of Proposition 3.6. And Proposition 3.1 is thus proved. \square

4 Proof of Theorem 1.1

Let us prove the main theorem now. It suffices to prove that for any continuous functional $F : D([0, 1], \mathbb{R}) \rightarrow [0, 1]$, we have

$$(4.1) \quad \lim_{n \rightarrow \infty} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \right] - \mathbf{E} \left[F(e_s, 0 \leq s \leq 1) \right] \right| = 0.$$

Proof of (4.1). Define for $A \geq 0$,

$$(4.2) \quad \mathcal{Z}[A] := \{u \in \mathbb{T} : V(u) \geq A > \max_{k < |u|} V(u_k)\}.$$

For any particle $u \in \mathcal{Z}[A]$, there is a subtree rooted at u . If $|u| \leq n$, let

$$I_n(u) := \min_{v \geq u, |v|=n} V(v).$$

Moreover, assume m_n^u is the particle uniformly chosen in the set $\{|v| = n : v \geq u, V(v) = I_n(u)\}$. Similarly, we write $[\emptyset, m_n^u] := \{\emptyset =: m_0^u, m_1^u, \dots, m_n^u\}$. The trajectory leading to m_n^u is denoted by $\{V(m_k^u); 0 \leq k \leq n\}$. Let ω_A be the particle uniformly chosen in $\{u \in \mathcal{Z}[A] : |u| \leq n, I_n(u) = I_n\}$.

Let $\mathcal{Y}_A := \{\max_{u \in \mathcal{Z}[A]} |u| \leq M, \max_{u \in \mathcal{Z}[A]} V(u) \leq M\}$. Then for any $\varepsilon > 0$, there exist $M := M(A, \varepsilon)$ large enough such that $\mathbf{P}(\mathcal{Y}_A^c) \leq \varepsilon$. It follows that

$$(4.3) \quad \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \right] - \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \right| \\ \leq \varepsilon + \mathbf{P}[|I_n - a_n(0)| \geq A/2].$$

We then check that for $n \geq M$,

$$(4.4) \quad \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \\ = \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} F \left(\frac{V(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right].$$

Define another trajectory $\{\tilde{V}(m_k^u); 0 \leq k \leq n\}$ as follows.

$$(4.5) \quad \tilde{V}(m_k^u) := \begin{cases} V(u) & \text{if } k < |u|; \\ V(m_k^u) & \text{if } |u| \leq k \leq n. \end{cases}$$

It follows that

$$\begin{aligned}
(4.6) \quad & \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \\
&= \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} F \left(\frac{\tilde{V}(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] + o_n(1),
\end{aligned}$$

where $o_n(1) \rightarrow 0$ as n goes to infinity.

Define the sigma-field $\mathcal{G}_A := \sigma\{(u, V(u), I_n(u)); u \in \mathcal{Z}[A]\}$. Note that on \mathcal{Y}_A , $I_n = \min_{u \in \mathcal{Z}[A]} I_n(u)$ as long as $n \geq M$. One sees that $\mathcal{Y}_A \cap \{|I_n - a_n(0)| \leq A/2\}$ is \mathcal{G}_A -measurable for all n large enough. Thus,

$$\begin{aligned}
(4.7) \quad & \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} F \left(\frac{\tilde{V}(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \\
&= \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} \mathbf{E} \left[F \left(\frac{\tilde{V}(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| \mathcal{G}_A, u = \omega_A \right]; \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right].
\end{aligned}$$

Further, we notice by the branching property that conditioned on $\{(u, V(u)); u \in \mathcal{Z}[A]\}$, the subtrees generated by $u \in \mathcal{Z}[A]$ are independent copies of the original one, started from $V(u)$, respectively. Therefore, given $\mathcal{Y}_A \cap \{|I_n - a_n(0)| \leq A/2\}$,

$$\begin{aligned}
& \mathbf{1}_{(u=\omega_A)} \mathbf{E} \left[F \left(\frac{\tilde{V}(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| \mathcal{G}_A, u = \omega_A \right] \\
&= \mathbf{1}_{(u=\omega_A)} \mathbf{E} \left[F \left(\frac{I(\lfloor s(n - |u|) \rfloor)}{\sigma \sqrt{n - |u|}}; 0 \leq s \leq 1 \right) \middle| I_{n-|u|} \leq a_n(-r_u) \right] + o_n(1),
\end{aligned}$$

where $r_u := \min\{\min_{v \in \mathcal{Z}[A] \setminus \{u\}} I_n(v) - a_n(0), A/2\} - V(u)$ is independent of $I_{n-|u|}$. Thus, (4.6) becomes that

$$\begin{aligned}
(4.8) \quad & \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \\
&= \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} \mathbf{E} \left[F \left(\frac{I(\lfloor s(n - |u|) \rfloor)}{\sigma \sqrt{n - |u|}}; 0 \leq s \leq 1 \right) \middle| I_{n-|u|} \leq a_n(-r_u) \right]; \right. \\
& \quad \left. \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] + o_n(1).
\end{aligned}$$

The event $\mathcal{Y}_A \cap \{|I_n - a_n(0)| \leq A/2\}$ ensures that $A/2 + M \geq -r_u \geq A/2$. The conditioned convergence has been given in Proposition 3.1. We need a slightly stronger version here.

According to Proposition 3.1, for any $\varepsilon > 0$, there exists $z_\varepsilon > 0$ such that for all $z \geq z_\varepsilon$,

$$(4.9) \quad \limsup_{n \rightarrow \infty} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| < \varepsilon.$$

Thus, for any $z \geq z_\varepsilon$, there exists $N_z \geq 1$ such that for any $n \geq N_z$,

$$(4.10) \quad \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| < 2\varepsilon.$$

Take $A = 2z_\varepsilon$ and $K = M$. We say that for n sufficiently large,

$$(4.11) \quad \sup_{z \in [z_\varepsilon, z_\varepsilon + K]} \left| \mathbf{E} \left[F \left(\frac{I(\lfloor s(n) \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \leq 3\varepsilon.$$

In the lattice case, (4.11) follows immediately. We only need to prove it in the non-lattice case.

Recall that $\Sigma_n(F, z) = \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); I_n \leq a_n(z) \right]$ with $0 \leq F \leq 1$. Then, for any $\ell > 0$ and $z \geq 0$,

$$(4.12) \quad \begin{aligned} & \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \right| \\ & \leq \left| \frac{\Sigma_n(F, z) - \Sigma_n(F, z + \ell)}{\Sigma_n(1, z)} \right| + \left| \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \right| \\ & = \frac{1}{\Sigma_n(1, z)} \left(\left| \Sigma_n(F, z) - \Sigma_n(F, z + \ell) \right| + \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \left| \Sigma_n(1, z + \ell) - \Sigma_n(1, z) \right| \right). \end{aligned}$$

Since $0 \leq F \leq 1$, the two following inequalities

$$\begin{aligned} \left| \Sigma_n(F, z) - \Sigma_n(F, z + \ell) \right| &= \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); a_n(z + \ell) < I_n \leq a_n(z) \right] \\ &\leq \mathbf{P}(a_n(z + \ell) < I_n \leq a_n(z)), \end{aligned}$$

and $\frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \leq 1$ hold. Note also that $|\Sigma_n(1, z + \ell) - \Sigma_n(1, z)| = \mathbf{P}(a_n(z + \ell) < I_n \leq a_n(z))$.

It follows that

$$(4.13) \quad \begin{aligned} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \right| &\leq 2 \frac{\mathbf{P}(a_n(z + \ell) < I_n \leq a_n(z))}{\mathbf{P}(I_n \leq a_n(z))} \\ &= 2 - 2 \frac{\mathbf{P}(I_n \leq a_n(z + \ell))}{\mathbf{P}(I_n \leq a_n(z))}. \end{aligned}$$

In view of Fact 3.4, we take $\frac{3}{2} \ln n - \Lambda_{\varepsilon'} \geq \ell + z > z \geq \Lambda_{\varepsilon'}$ so that for any $n \geq N_{\varepsilon'}$,

$$(4.14) \quad \frac{\mathbf{P}(I_n \leq a_n(z + \ell))}{\mathbf{P}(I_n \leq a_n(z))} \geq \frac{(C - \varepsilon')(z + \ell)e^{-z - \ell}}{(C + \varepsilon')ze^{-z}} \geq \frac{C - \varepsilon'}{C + \varepsilon'} e^{-\ell}.$$

For $\varepsilon' = C\varepsilon/8 > 0$, we choose $\zeta = \frac{\varepsilon}{4}$ so that $\frac{C-\varepsilon'}{C+\varepsilon'}e^{-\zeta} \geq 1 - \frac{\varepsilon}{2}$. As a consequence, for any $\Lambda_{\varepsilon'} \leq z \leq \frac{3}{2} \ln n - \Lambda_{\varepsilon'} - \zeta$, $0 \leq \ell \leq \zeta$ and $n \geq N_{\varepsilon'}$,

$$(4.15) \quad \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \right| \leq 2 \left(1 - \frac{C - \varepsilon'}{C + \varepsilon'} e^{-\ell} \right) \leq \varepsilon.$$

For $\varepsilon > 0$, z_ε can be chosen so that $[z_\varepsilon, z_\varepsilon + K] \subset [\Lambda_{\varepsilon'}, \frac{3}{2} \ln n - \Lambda_{\varepsilon'}]$ for $n \geq e^K N_{\varepsilon'}$. For any integer $0 \leq j \leq \lceil K/\zeta \rceil$, let $z_j := z_\varepsilon + j\zeta$. Then $[z_\varepsilon, z_\varepsilon + K] \subset \cup_{0 \leq j \leq \lceil K/\zeta \rceil} [z_j, z_{j+1}]$. Take $N'_\varepsilon = \max_{0 \leq j \leq \lceil K/\zeta \rceil} \{N_{z_j}, e^K N_{\varepsilon'}\}$. By (4.10) and (4.15), we conclude that for any $n \geq N'_\varepsilon$,

$$\begin{aligned} & \sup_{z \in [z_\varepsilon, z_\varepsilon + K]} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \\ & \leq \sup_{0 \leq j \leq \lceil K/\zeta \rceil} \left| \frac{\Sigma_n(F, z_j)}{\Sigma_n(1, z_j)} - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| + \sup_{0 \leq j < \lceil K/\zeta \rceil} \sup_{z_j \leq z \leq z_{j+1}} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z_j)}{\Sigma_n(1, z_j)} \right| \\ & \leq 3\varepsilon. \end{aligned}$$

We continue to prove the main theorem. Since $\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} = 1$, we deduce from (4.8) and (4.11) that for n sufficiently large,

$$\begin{aligned} & \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \\ & \leq 3\varepsilon \mathbf{P}(\mathcal{Y}_A; |I_n - a_n(0)| \leq A/2) + o_n(1) + \mathbf{P}(\mathcal{Y}_A^c) + \mathbf{P}(|I_n - a_n(0)| \geq A/2) \\ & \leq 4\varepsilon + o_n(1) + \mathbf{P}(|I_n - a_n(0)| \geq A/2). \end{aligned}$$

Going back to (4.3), we conclude that for n large enough,

$$\left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \leq 5\varepsilon + 2\mathbf{P}(|I_n - a_n(0)| \geq A/2) + o_n(1).$$

Let n go to infinity and then make $\varepsilon \downarrow 0$. Therefore,

$$(4.16) \quad \begin{aligned} & \limsup_{n \rightarrow \infty} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \\ & \leq \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} 2\mathbf{P}(|I_n - a_n(0)| \geq z). \end{aligned}$$

It remains to show that $\limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}(|I_n - a_n(0)| \geq z) = 0$. Because of Fact (3.7), it suffices to prove that

$$(4.17) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}(I_n \geq a_n(0) + z) = 0.$$

In the non-lattice case, Theorem 1.1 of [2] implies it directly. In the lattice case, we see that for n large enough,

$$(4.18) \quad \mathbf{P}(I_n \geq a_n(0) + z) \leq \mathbf{E} \left[\prod_{u \in \mathcal{Z}[A]} (1 - \Phi_u(z, n)); \mathcal{Y}_A \right] + \varepsilon,$$

with $\Phi_u(z, n) := \mathbf{P}(I_{n-|u|} \leq a_n(V(u) - z))$. Take $A = 2z$ here. Then it follows from Fact 3.7 that for n large enough and for any particle $u \in \mathcal{Z}[A]$,

$$(4.19) \quad \Phi_u(z, n) \geq c_{13}/2(V(u) - z)e^{z-V(u)} \geq \frac{c_{13}}{4}V(u)e^{z-V(u)}.$$

(4.18) hence becomes that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{P}(I_n \geq a_n(0) + z) &\leq \mathbf{E} \left[\prod_{u \in \mathcal{Z}[A]} (1 - \frac{c_{13}}{4}V(u)e^{z-V(u)}); \mathcal{Y}_A \right] + \varepsilon \\ &\leq \mathbf{E} \left[\exp \left(- \frac{c_{13}}{4}e^z \sum_{u \in \mathcal{Z}[A]} V(u)e^{-V(u)} \right) \right] + \varepsilon. \end{aligned}$$

It has been proved that as A goes to infinity, $\sum_{u \in \mathcal{Z}[A]} V(u)e^{-V(u)}$ converges almost surely to some limit D_∞ , which is strictly positive on the set of non-extinction of \mathbb{T} , (see (5.2) in [2]). We end up with

$$(4.20) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}(I_n \geq a_n(0) + z) \leq \varepsilon,$$

which completes the proof of Theorem 1.1. \square

Acknowledgments

I am grateful to my Ph.D. advisor Zhan Shi for his advice and encouragement.

References

- [1] Addario-Berry, L. and Reed, B. (2009). Minima in branching random walks. *Ann. Probab.* **37**, 1044–1079.
- [2] Aidékon, E. (2011+). Weak convergence of the minimum of a branching random walk. [ArXiv:1101.1810](#)
- [3] Aidékon, E., Berestycki, J., Brunet, É. and Shi, Z. (2011) The branching Brownian motion seen from its tip. [Arxiv:1104.3738v2](#)
- [4] Aidékon, E. and Shi, Z. (2010+). The Seneta-Heyde scaling for the branching random walk. (preprint)

- [5] Biggins, J.D. and Kyprianou, A.E. (2005). Fixed points of the smoothing transform: the boundary case. *Electron. J. Probab.* **10**, Paper no. 17, 609–631.
- [6] Caravenna, F. (2005). A local limit theorem for random walks conditioned to stay positive. *Probab. Theory Related Fields* **133**, 508–530.
- [7] Caravenna, F. and Chaumont, L. (2008) Invariance principles for random walks conditioned to stay positive. *Annales de l’Institut Henri Poincaré - Probabilités et Statistiques* **44**, No. 1, 170–190.
- [8] Chauvin, B. and Rouault, A. (1988). KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Related Fields* **80**, 299–314.
- [9] Durrett, R. Iglehart, D. and Miller, D. (1977). Weak convergence to Brownian meander and Brownian excursion. *Ann. Probab.* **5**, No. 1, 117–129.
- [10] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications II*, 2nd ed. Wiley, New York.
- [11] Hu, Y. and Shi, Z. (2009). Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Probab.* **37**, 742–789.
- [12] Imhof, J.-P. (1984) Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.* **21**, 500-510.
- [13] Kozlov, M.V. (1976). The asymptotic behavior of the probability of non-extinction of critical branching processes in a random environment. *Theory Probab. Appl.* **21**, 791–804.
- [14] Kyprianou, A. (2004). Travelling wave solutions to the K-P-P equation: alternatives to Simon Harris’ probabilistic analysis. *Ann. Inst. H. Poincaré Probab. Statist.* **40**, 53–72.
- [15] Lyons, R. (1997). A simple path to Biggins’ martingale convergence for branching random walk. In: *Classical and Modern Branching Processes* (Eds.: K.B. Athreya and P. Jagers). *IMA Volumes in Mathematics and its Applications* **84**, 217–221. Springer, New York.
- [16] Revuz, D. and Yor, M. (2005) *Continuous Martingales and Brownian Motion*. 3rd ed, Springer-Verlag.