Improving Hölder's inequality

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IMPROVING HÖLDER'S INEQUALITY

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Abstract. We show that the remainder in Hölder's inequality may be computed exactly. It satisfies functional equations, and possesses monotonicity and scaling properties. Improved inequalities follow.

1. Introduction

For any two measurable functions $f(x)$, $g(x)$ on a measure space $(\Omega, \mathcal{A}, \mu)$, and any $p \in (1, +\infty)$, Hölder's inequality [8, 5, 7, 4] states that

$$\int_\Omega fg\, d\mu \leq \|f\|_p \|g\|_q, \tag{1.1}$$

where $q = p/(p-1)$, and $\|f\|_p = \left(\int_\Omega |f(x)|^p\, d\mu\right)^{1/p}$ is the L^p norm. It plays in Nonlinear Analysis the rôle of the Cauchy-Schwarz inequality. It is a consequence of the inequality

$$a^\alpha b^\beta \leq \alpha a + \beta b, \tag{1.2}$$

for a, b, α, β nonnegative, with $0 < \alpha < 1$ and $\beta = 1 - \alpha$. Indeed, it follows from (1.2) with $\alpha = 1/p$, $a = F$ and $b = G$, where

$$F = |f(x)|^p / \|f\|_p^p, \quad G = |g(x)|^q / \|g\|_q^q.$$

Inequality (1.2) may be deduced from the convexity inequality $j(\alpha t + \beta s) \leq \alpha j(t) + \beta j(s)$ by taking j to be the exponential, $t = \ln a$ and $s = \ln b$. However, this does not give a very precise expression for the difference between the two

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sides of the inequality. On the other hand, for $\alpha = \beta = 1/2$, (1.2) follows from the identity

$$\sqrt{ab} = \frac{1}{2}(a+b) - \frac{1}{2}(\sqrt{a} - \sqrt{b})^2.$$

This suggests the search for identities that imply (1.2). Define the remainder function R by

$$R(\alpha\,;\,a,b) = \alpha a + \beta b - a^\alpha b^\beta, \tag{1.3}$$

where a and b are positive, and $\beta = 1-\alpha$. Thus, for $t > 0$,

$$t^\alpha = 1 + \alpha(t-1) - R(\alpha\,;\,t,1).$$

In particular, $R(0\,;\,a,b) = R(1\,;\,a,b) = 0$ and

$$R(\frac{1}{2}\,;\,a,b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2. \tag{1.4}$$

The function R extends by continuity to $a = 0$ if $\alpha > 0$, and to $b = 0$ if $\beta > 0$. The convexity of the exponential and the concavity of t^α yield, for $0 < \alpha < 1$,

$$1 + \alpha \ln t \le t^\alpha \le 1 + \alpha(t-1),$$

hence the inequality

$$0 \le R(\alpha\,;\,t,1) \le \alpha(t - 1 - \ln t). \tag{1.5}$$

We are interested in expressing R as a manifestly nonnegative expression that vanishes for $a = b$.

In terms of the remainder function, Hölder's inequality follows from the identity

$$\frac{\int_\Omega |fg|\,d\mu}{\|f\|_p \|g\|_q} = 1 - \int_\Omega R(\frac{1}{p}\,;\,F,G)\,d\mu, \tag{1.6}$$

and the positivity of R. Any lower bound on R yields a sharpening of Hölder's inequality. Such lower bounds will be obtained as a consequence of the following properties of the remainder, proved in Section 2.

- It satisfies scaling, monotonicity and symmetry properties (Theorem 2.1). These properties are valid also for $\alpha > 1$, or negative. In particular, it follows from the symmetry properties that R is nonpositive for $\alpha \notin [0,1]$.
- It may be computed for exponents $\alpha\alpha'$ and $\alpha + \alpha'$ if it is known for exponents α and α' (Theorem 2.2).
- It may be computed in closed form for rational α (Theorem 2.3). In addition, it admits an integral representation (Theorem 2.4).

It is convenient to relax, in some of the general results, the restriction $0 < \alpha < 1$; due to the symmetry relation (2.3) below, inequalities for $0 < \alpha < 1$ imply counterparts for $\alpha > 1$, corresponding to $0 < p < 1$. Section 3 develops the consequences of the properties of R:

- A two-sided estimate of R derived from the addition theorem (§ 3.1).
- A sequence of more and more precise estimates of R derived from the multiplication theorem and the integral representation. The first of these estimates recovers a result of [1] (§ 3.2).
- Another estimate derived from the multiplication formula, improving an inequality discussed in [6] (§ 3.3).
- An improvement of Clarkson's inequality suggested by the exact formulae for R (§ 3.4).

2. Properties of R

In this section, a and b are positive. It is not assumed that α lies between 0 and 1.

2.1. **Scaling, symmetry and monotonicity.**

Theorem 2.1. *The remainder function possesses the scaling property:*

$$R(\alpha\,;\,\lambda a, \lambda b) = \lambda R(\alpha\,;\,a, b). \tag{2.1}$$

if $\lambda > 0$. It possesses two symmetry properties:

$$R(\alpha\,;\,a, b) = R(\beta\,;\,b, a), \tag{2.2}$$

and, for $\alpha \neq 0$,

$$R(\frac{1}{\alpha}\,;\,a^\alpha, b^\alpha) = -\frac{1}{\alpha} b^{-\beta} R(\alpha\,;\,a, b). \tag{2.3}$$

Finally, it possesses the monotonicity property:

$$\alpha_0^{-1} R(\alpha_0\,;\,a, b) \le \alpha^{-1} R(\alpha\,;\,a, b) \tag{2.4}$$

for $\alpha \le \alpha_0$, both nonzero.

Remark. Since $R(\alpha\,;\,a, b) \ge 0$ for $0 < \alpha < 1$, (2.3) implies that $R(\alpha\,;\,a, b) \le 0$ for $\alpha > 1$; this in turn implies, by (2.2), that the same holds for $\alpha < 0$.

Proof. Equations (2.1) and (2.2) are readily checked using the definition of R.

To prove (2.3), let $t = a/b$ and write

$$\begin{aligned} R(\frac{1}{\alpha}\,;\,\frac{a^\alpha}{b^\alpha},1) = R(\frac{1}{\alpha}\,;\,t^\alpha,1) &= \frac{t^\alpha}{\alpha} + (1-\frac{1}{\alpha}) - t \\ &= \frac{1}{\alpha}\left[t^\alpha - (1-\alpha) - \alpha t\right] \\ &= -\frac{1}{\alpha}R(\alpha\,;\,t,1) = -\frac{1}{\alpha b}R(\alpha\,;\,a,b) \end{aligned}$$

by the scaling property. Multiplying through by b^α, and using scaling once more, the result follows.

To prove (2.4), consider $\varphi(\alpha,t) := R(\alpha\,;\,t,1)/\alpha = (t-1) - \alpha^{-1}(t^\alpha - 1)$ for $\alpha \neq 0$, extended by continuity to $\alpha = 0$, so that $\varphi(0,t) = t - 1 - \ln t$. We have, for $\alpha \neq 0$,

$$\frac{\partial\varphi}{\partial\alpha} = \frac{t^\alpha - 1}{\alpha^2} - \frac{t^\alpha}{\alpha}\ln t = -\frac{t^\alpha}{\alpha^2}\left[t^{-\alpha} - 1 + \alpha\ln t\right].$$

The inequality $e^u \geq 1 + u$, with $u = -\alpha\ln t$, implies that $\partial\varphi/\partial\alpha \leq 0$, as desired. $\square$

2.2. **Functional equations.**

Theorem 2.2. *The remainder satisfies the functional equation*

$$R(\alpha\alpha'\,;\,a,b) = \alpha' R(\alpha\,;\,a,b) + b^{1-\alpha}R(\alpha'\,;\,a^\alpha,b^\alpha). \tag{2.5}$$

It also satisfies an addition theorem: writing R_α for $R(\alpha\,;\,a,b)$, we have

$$R_{\alpha+\alpha'} = \left[1 + \alpha'\frac{a-b}{b}\right]R_\alpha + (a/b)^\alpha R_{\alpha'} - \frac{\alpha\alpha'}{b}(a-b)^2. \tag{2.6}$$

Remark. Since the left-hand side of (2.5) is symmetric with respect to α and α', we have the identity

$$\alpha' R(\alpha\,;\,a,b) - \alpha R(\alpha'\,;\,a,b) = b^{1-\alpha'}R(\alpha\,;\,a^{\alpha'},b^{\alpha'}) - b^{1-\alpha}R(\alpha'\,;\,a^\alpha,b^\alpha).$$

The addition theorem may be given a form symmetric in α and α', by replacing $(a/b)^\alpha = (a^\alpha b^\beta)/b$ by $1 + \alpha(a-b)/b - R_\alpha/b$; the result is:

$$R(\alpha+\alpha'\,;\,a,b) = R_\alpha + R_{\alpha'} - b^{-1}R_\alpha R_{\alpha'} + \frac{a-b}{b}\left[\alpha R_{\alpha'} + \alpha' R_\alpha\right] - \frac{\alpha\alpha'}{b}(a-b)^2.$$

Proof. Writing $t^{\alpha\alpha'} = (t^\alpha)^{\alpha'}$, we obtain

$$\begin{aligned} t^{\alpha\alpha'} &= 1 + \alpha'(t^\alpha - 1) - R(\alpha'\,;\,t^\alpha,1) \\ &= 1 + \alpha'\left[\alpha(t-1) - R(\alpha\,;\,t,1)\right] - R(\alpha'\,;\,t^\alpha,1) \\ &= 1 + \alpha\alpha'(t-1) - \alpha' R(\alpha\,;\,t,1) - R(\alpha'\,;\,t^\alpha,1). \end{aligned}$$

Therefore,

$$R(\alpha\alpha'\,;\,t,1) = \alpha' R(\alpha\,;\,t,1) + R(\alpha'\,;\,t^{\alpha},1). \tag{2.7}$$

Replacing t by a/b and using the scaling property, equation (2.5) follows.

For the addition theorem, one writes, letting $R_{\alpha} = R(\alpha\,;\,t,1)$ and $R_{\alpha'} = R(\alpha'\,;\,t,1)$,

$$\begin{aligned}
1 - t^{\alpha+\alpha'} &= 1 + t^{\alpha}(-t^{\alpha'}) \\
&= 1 + t^{\alpha}(R_{\alpha'} - 1 - \alpha'(t-1)) \\
&= 1 + t^{\alpha}R_{\alpha'} - t^{\alpha}(1 + \alpha'(t-1)) \\
&= 1 + t^{\alpha}R_{\alpha'} - (1 + \alpha(t-1) - R_{\alpha})(1 + \alpha'(t-1)) \\
&= \left[R_{\alpha}(1 + \alpha'(t-1)) + t^{\alpha}R_{\alpha'} - \alpha\alpha'(t-1)^2\right] - (\alpha+\alpha')(t-1).
\end{aligned}$$

Since $R_{\alpha+\alpha'} = 1+(\alpha+\alpha')(t-1)-t^{\alpha+\alpha'}$, the result follows, using again scaling. □

2.3. **Closed-form representations of R.** For rational α, we now show that the remainder may be expressed using polynomials with positive integral coefficients.

Theorem 2.3. *Let k and n be positive integers, with $1 \le k \le n$, and $\alpha = \dfrac{k}{n+1}$. We then have*

$$R(\alpha\,;\,a^{n+1},b^{n+1}) = \frac{(a-b)^2}{n+1}(\alpha I_{n,k} + \beta J_{n,k}), \tag{2.8}$$

where

$$I_{n,k} = \sum_{j=k-1}^{n-1}(n-j)a^j b^{n-1-j},$$

$J_{n,1} = 0$ and, for $2 \le k \le n$,

$$J_{n,k} = \sum_{j=0}^{k-2}(j+1)a^j b^{n-1-j}.$$

Thus, the remainder is the product of $(a-b)^2/(n+1)$ by a polynomial with positive integral coefficients.

Proof. By scaling, it suffices to determine $(n+1)R(\alpha\,;\,t^{n+1},1)$, with $\alpha = \frac{k}{n+1}$ (so that $\beta = (n+1-k)/(n+1)$). It may be written

$$kt^{n+1} + (n+1-k) - (n+1)t^k = k(t^{n+1}-1) - (n+1)(t^k-1).$$

Consider first the case $k = 1$. We have

$$t^{n+1} - (n+1)t + n = (t-1)(t^n + t^{n-1} + \cdots + t - n).$$

Furthermore,

$$t^n + t^{n-1} + \cdots + t - n = (t-1)(t^{n-1} + 2t^{n-2} + \cdots + (n-1)t + n).$$

Since

$$I_{n,1} = \sum_{j=0}^{n-1} (n-j)t^j = t^{n-1} + 2t^{n-1} + \cdots + (n-1)t + n,$$

we have

$$t^{n+1} + n - (n+1)t = (t-1)^2 I_{n,1}. \tag{2.9}$$

Since $J_{n,1} = 0$, this proves the result for $k = 1$.

Assume now $2 \le k \le n$. Equation (2.9), multiplied by k, may be written

$$k(t^{n+1} - 1) - k(n+1)(t-1) = (t-1)^2 \sum_{j=0}^{n-1} k(n-j)t^j.$$

Exchanging the rôles of k and $n+1$,

$$(n+1)(t^k - 1) - (n+1)k(t-1) = (t-1)^2 \sum_{j=0}^{k-2} (n+1)(k-1-j)t^j.$$

Subtracting, we obtain, since $k \le n$,

$$\begin{aligned} k(t^{n+1} - 1) - (n+1)(t^k - 1) &= (t-1)^2 \\ &\times \left\{ \sum_{j=0}^{k-2} \left[k(n-j) - (n+1)(k-1-j) \right] t^j + \sum_{j=k-1}^{n-1} k(n-j)t^j \right\}. \end{aligned}$$

Since

$$k(n-j) - (n+1)(k-1-j) = (n+1-k)(j+1) = (n+1)\beta(j+1) \ge 0,$$

we have proved (2.8). $\square$

For general exponents, we have:

Theorem 2.4. *For $0 < \alpha < 1$, the remainder admits the general expression*

$$R(\alpha\,;\, a, b) = \left(\ln \frac{a}{b} \right)^2 \int_0^1 \min[\alpha(1-\tau), \beta\tau] a^\tau b^{1-\tau} d\tau. \tag{2.10}$$

As a consequence,

$$R(\alpha\,;\, a, b) \ge \frac{1}{2} \alpha\beta \min(a, b) \left(\ln \frac{a}{b} \right)^2. \tag{2.11}$$

Proof. Let $j(t)$ be twice continuously differentiable for $t > 0$, and x, y positive. Consider

$$u(t) := tj(x) + (1-t)j(y) - j(tx + (1-t)y).$$

We have $u(0) = u(1) = 0$, and

$$-\frac{d^2u}{dt^2} = h(t) := (x-y)^2 j''(tx + (1-t)y).$$

This implies

$$u(t) = \int_0^1 g(t,\tau)h(\tau)d\tau,$$

where $g(t,\tau) = \min[(1-t)\tau, (1-\tau)t]$ is the Green's function for this problem; indeed, one checks directly that $-\partial^2 g(t,\tau)/\partial t^2 = \delta(t-\tau)$, and that g vanishes for $t = 0$ or 1. It follows that

$$\alpha j(x) + \beta j(y) - j(\alpha x + \beta y) = (x-y)^2 \int_0^1 g(\alpha,\tau) j''(\tau x + (1-\tau)y)d\tau.$$

We now let $x = \ln a$, $y = \ln b$, and $j(t) = \exp(t)$ to obtain (2.10). For $0 < \tau < 1$, $a^\tau b^{1-\tau} \geq \min(a,b)$. Since g is nonnegative, and $\int_0^1 g(\alpha,\tau)d\tau = \frac{1}{2}\alpha\beta$, (2.11) follows. □

3. Applications

3.1. **A consequence of the monotonicity property.** Monotonicity, together with the exact formulae of Theorem 2.3, yields precise estimates of the remainder. Indeed, if $t > 0$, and k and n are integers such that $1 \leq k \leq n$ and $k/(n+1) < \alpha < (k+1)/(n+1)$, (2.4) yields

$$\frac{(n+1)\alpha}{k+1} R(\frac{k+1}{n+1}\,;\, t, 1) \leq R(\alpha\,;\, t, 1) \leq \frac{(n+1)\alpha}{k} R(\frac{k}{n+1}\,;\, t, 1).$$

One may also obtain an exact formula: write $\alpha = k/(n+1) + \varepsilon$, $R_\alpha = R(\alpha\,;\, t, 1)$, and $R_{k,n} = R(\frac{k}{n+1}\,;\, t, 1)$; the addition theorem (2.6) then yields

$$R_\alpha \quad = \quad [1 + \varepsilon(t-1)]\, R_{k,n} + t^{k/(n+1)} R_\varepsilon - \frac{k\varepsilon}{n+1}(t-1)^2.$$

Combining with estimate (1.5), we obtain

$$0 \leq R_\alpha - [1 + \varepsilon(t-1)]\, R_{k,n} + \frac{k\varepsilon}{n+1}(t-1)^2 \leq \varepsilon t^{k/(n+1)}(t - 1 - \ln t).$$

3.2. **First consequence of the multiplication theorem.** The functional equation (2.5) may be used in two different ways, depending on whether R_α or $R_{\alpha'}$ is known. This section discusses the first.

Since Theorem 2.3 gives the remainder for $\alpha = 1/m$, with integral m, equation (2.5), where the pair (α, α') is replaced by $(\frac{1}{m}, m\alpha)$, yields

$$R(\alpha\,;\,t,1) = m\alpha R(\frac{1}{m}\,;\,t,1) + R(m\alpha\,;\,t^{1/m},1). \tag{3.1}$$

For example, for $m = 2$, using (1.4), $R(\alpha\,;\,t,1) = \alpha(\sqrt{t}-1)^2 + R(2\alpha\,;\,\sqrt{t},1)$, and (2.11) gives

$$R(\alpha\,;\,t,1) \geq \alpha(\sqrt{t}-1)^2 + \frac{1}{4}\alpha(1-2\alpha)\min(\sqrt{t},1)\,(\ln t)^2 \tag{3.2}$$

for $0 < \alpha \leq 1/2$. This improves the inequality [1]:

$$R(\alpha\,;\,t,1) \geq \alpha(\sqrt{t}-1)^2 \text{ for } 0 < \alpha \leq \frac{1}{2}.$$

If $1/2 \leq \alpha < 1$, one may apply this result to $R(\beta\,;\,t,1) = R(\alpha\,;\,1,t)$. More and more precise expressions may be obtained by iterating the process: thus, applying the same argument to $R(2\alpha\,;\,\sqrt{t},1)$, we obtain

$$R(\alpha\,;\,t,1) = \alpha(\sqrt{t}-1)^2 + 2\alpha(t^{1/4}-1)^2 + R(4\alpha\,;\,t^{1/4},1).$$

Therefore, for $0 < \alpha \leq \frac{1}{4}$,

$$R(\alpha\,;\,t,1) \geq \alpha(\sqrt{t}-1)^2 + 2\alpha(t^{1/4}-1)^2 + \frac{1}{8}\alpha(1-4\alpha)\min(t^{1/4},1)(\ln t)^2.$$

3.3. **Second consequence of the multiplication theorem.** The second way to take advantage of (2.5) runs as follows. Since $R(\frac{1}{2}\,;\,a,b)$ is known (by (1.4)), we apply (2.5), where the pair (α, α') is replaced by $(m\alpha, \frac{1}{m})$; this yields, instead of (3.1),

$$R(\alpha\,;\,t,1) = \frac{1}{m}R(m\alpha\,;\,t,1) + R(\frac{1}{m}\,;\,t^{m\alpha},1). \tag{3.3}$$

Consider now two conjugate exponents p and q, and $\alpha = 1/q$. Taking $m = 2$ and $\alpha = 1/q$, we obtain in particular

$$R(\frac{1}{q}\,;\,t,1) = \frac{1}{2}R(\frac{2}{q}\,;\,t,1) + R(\frac{1}{2}\,;\,t^{2/q},1),$$

hence

$$R(\frac{1}{q}\,;\,a,b) = \frac{1}{2}R(\frac{2}{q}\,;\,a,b) + b^{1-2/q}R(\frac{1}{2}\,;\,a^{2/q},b^{2/q}).$$

Therefore, for f and g positive, using (2.2) and the relation $p/q = p-1$,

$$\begin{aligned} R(\frac{1}{p}\,;\,f^p,g^q) &= R(\frac{1}{q}\,;\,g^q,f^p) \\ &= \frac{1}{2}R(\frac{2}{q}\,;\,g^q,f^p) + f^{p-(2p/q)}R(\frac{1}{2}\,;\,g^2,f^{2p/q}) \\ &= \frac{1}{2}R(\frac{2}{q}\,;\,g^q,f^p) + \frac{1}{2}f^{2-p}\left(g-f^{p-1}\right)^2. \end{aligned}$$

For $1 < q \le 2$, the first term is nonpositive (see the remark following Th. 2.1), and we recover the inequality [6]

$$\frac{f^p}{p} + \frac{g^q}{q} - fg \le \frac{1}{2}f^{2-p}(g-f^{p-1})^2.$$

If $q > 2$, the inequality is reversed, since R changes sign. More precisely, applying the symmetry property (2.3) with $\alpha = 2/q$, we have

$$R(\frac{q}{2}\,;\,g^2,f^{2p/q}) = -\frac{q}{2}f^{(2p/q)-p}R(\frac{2}{q}\,;\,g^q,f^p),$$

hence, after substitution and simplification using the scaling property,

$$R(\frac{1}{p}\,;\,f^p,g^q) = \frac{1}{2}f^{2-p}\left(g-f^{p-1}\right)^2 - \frac{1}{q}R(\frac{q}{2}\,;\,f^{2-p}g^2,f^p).$$

As before, (2.11) leads to sharpened inequalities. For example, for $q \ge 2$,

$$\begin{aligned} \frac{f^p}{p} + \frac{g^q}{q} - fg &= \frac{1}{2}R(\frac{2}{q}\,;\,g^q,f^p) + \frac{1}{2}f^{2-p}\left(g-f^{p-1}\right)^2 \\ &\ge \frac{1}{2}f^{2-p}\left(g-f^{p-1}\right)^2 + \frac{1}{q}(1-\frac{2}{q})\min(f^p,g^q)(\ln\frac{f^p}{g^q})^2. \end{aligned}$$

3.4. **Consequences of the exact expression for R.** We now turn to examples illustrating the identities in Theorem 2.3. They suggest improvements of other classical inequalities. We begin with the simplest examples. The formulae below have been obtained by multiplying (2.8) by $(n+1)$, and letting $a = f^{n+1}$, $b = g^{n+1}$.

For $k = 1 = n$, we recover $2fg = f^2 + g^2 - (f-g)^2$.

For $k = 1$ et $n = 2$, we obtain

$$3fg^2 = f^3 + 2g^3 - (f-g)^2(f+2g). \tag{3.4}$$

For $k = 1$ or 2, and $n = 4$,

$$\begin{aligned} 5fg^4 &= f^5 + 4g^5 - (f-g)^2(f^3 + 2f^2g + 3fg^2 + 4g^3), \\ 5f^2g^3 &= 2f^5 + 3g^5 - (f-g)^2(2f^3 + 4f^2g + 6fg^2 + 3g^3). \end{aligned}$$

Therefore, $R(\frac{1}{3}\,;\,t,1) = \frac{1}{3}(t^{1/3}-1)^2(t^{1/3}+2)$. Equation (3.4) implies the identity

$$(f+g)^3 + 3(f-g)^3 = 4(f^3+g^3) - 6g(f-g)^2;$$

it may be compared with Clarkson's inequality [3, 2]

$$|f+g|^3 + |f-g|^3 \le 4(|f|^3+|g|^3).$$

More generally, we have the following.

Theorem 3.1. *If $p>2$ and $f>g>0$,*

$$(f+g)^p + (2^{p-1}-1)(f-g)^p \le 2^{p-1}(f^p+g^p).$$

Proof. Let $t = f/g$. Consider $\varphi(t) = \left[2^{p-1}(t^p+1)-(t+1)^p\right](t-1)^{-p}$, for $t>1$. We obtain

$$\begin{aligned}
\frac{1}{p}(t-1)^{p+1}\frac{d\varphi}{dt} &= (t-1)\left\{(2t)^{p-1}-(t+1)^{p-1}\right\} - \left\{2^{p-1}(t^p+1)-(t+1)^p\right\} \\
&= (2t)^{p-1}[t-1-t] - 2^{p-1} + (t+1)^{p-1}[-t+1+t+1] \\
&= 2\left[(t+1)^{p-1} - 2^{p-2}(t^{p-1}+1)\right].
\end{aligned}$$

Since $p-1>1$, $(t+1)^{p-1} \le 2^{p-2}(t^{p-1}+1)$, by Hölder's inequality. Therefore, $\varphi(t)$ is decreasing over $(1,+\infty)$, and its infimum is its limit as $t\to\infty$, namely $2^{p-1}-1$. The theorem follows. $\square$

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