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Michael Friedman

► **To cite this version:**

Michael Friedman. Conjugation-free groups, lower central series and line arrangements. 2013. hal-00826941

HAL Id: hal-00826941

<https://hal.science/hal-00826941>

Preprint submitted on 28 May 2013

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CONJUGATION-FREE GROUPS, LOWER CENTRAL SERIES AND LINE ARRANGEMENTS

MICHAEL FRIEDMAN

ABSTRACT. The quotients G_k/G_{k+1} of the lower central series of a finitely presented group G are an important invariant of this group. In this work we investigate the ranks of these quotients in the case of a certain class of conjugation-free groups, which are groups generated by x_1, \dots, x_n and having only cyclic relations:

$$x_{i_t}x_{i_{t-1}} \cdots x_{i_1} = x_{i_{t-1}} \cdots x_{i_1}x_{i_t} = \cdots = x_{i_1}x_{i_t} \cdots x_{i_2}.$$

Using tools from group theory and from the theory of line arrangements we explicitly find these ranks, which depend only at the number and length of these cyclic relations. It follows that for these groups the associated graded Lie algebra $gr(G)$ decomposes, in any degree, as a direct product of local components.

1. INTRODUCTION

Let G be a group and consider its lower central series defined by $G_1 = G, G_k = [G_{k-1}, G], k \geq 2$. The graded sum

$$gr(G) \doteq \bigoplus_{k \geq 1} G_k/G_{k+1}$$

has a graded Lie algebra structure induced by the commutator bracket on G . In general, $gr(G)$ reflects many properties of the group G . For example, if G is finitely generated, then the abelian groups G_k/G_{k+1} are also finitely generated. It is, however, difficult to find out the structure of G_k/G_{k+1} even in the case when G is finitely generated. Thus, determining this structure, or even the ranks of the quotients $\phi_k(G) \doteq \text{rank}(G_k/G_{k+1})$, for a certain class of finitely generated groups is a significant problem.

In the case of $G = \mathbb{F}_n$, the free group of rank n , Magnus (see e.g. [16]) showed that $gr(\mathbb{F}_n)$ is the free Lie algebra on n generators, whose ranks were computed by Witt [24]. Hall [12] introduced the basic commutators of \mathbb{F}_n , showing that the coset classes of weight k form a basis of G_k/G_{k+1} . Therefore, for the free group, the structure of $gr(\mathbb{F}_n)$ is completely known. Another well-known example is the work of Chen [1], computing the structure of $gr(\mathbb{F}_n/(\mathbb{F}_n)'')$.

In this paper we concentrate on a particular class of finitely presented groups: groups having a presentation where all the relations are cyclic relations; that is, of the form:

$$x_{i_t}^{s_t} x_{i_{t-1}}^{s_{t-1}} \cdots x_{i_1}^{s_1} = x_{i_{t-1}}^{s_{t-1}} \cdots x_{i_1}^{s_1} x_{i_t}^{s_t} = \cdots = x_{i_1}^{s_1} x_{i_t}^{s_t} \cdots x_{i_2}^{s_2},$$

where $\{x_1, \dots, x_n\}$ are the generators of the group, $2 \leq t$, $\{i_1, i_2, \dots, i_t\} \subseteq \{1, \dots, n\}$ is an increasing subsequence of indices and $s_j \in \langle x_1, \dots, x_n \rangle$. We call such groups *cyclic-related* groups (see Definition 2.1) and the above relation a *cyclic relation* of length t . When all the conjugating elements are equal to the identity, i.e. $s_j = e$, then we call such a group *conjugation-free* (see Definition 2.6). The simplest example of such a group is the group $\mathbb{F}_n/\langle R_n \rangle$ generated by n generators with one cyclic relation of length n :

$$R_n : x_n x_{n-1} \cdots x_1 = x_{n-1} x_{n-2} \cdots x_1 x_n = \cdots = x_1 x_n \cdots x_2.$$

Obviously, $\mathbb{F}_n/\langle R_n \rangle \simeq \mathbb{F}_{n-1} \oplus \mathbb{Z}$, and thus the structure of $gr(\mathbb{F}_n/\langle R_n \rangle)$ is known.

A less-known example of a cyclic-related group is the pure braid group PB_n (see [17, Theorem 2.3] for a cyclic-related presentation of this group). Note that given an arrangement of hyperplanes

$\mathcal{A} = \{H_1, \dots, H_n\} \subset \mathbb{C}^m$, the fundamental group $\pi_1(\mathbb{C}^m - \mathcal{A})$ is a cyclic-related group (see Remark 4.9); thus, from this perspective, it is clear why the pure braid group is a cyclic-related group, as it is the fundamental group of the complement of a hyperplane arrangement known as the braid arrangement. In general, if W is a real reflection group, $\mathcal{A}_W \subset \mathbb{C}^m$ the associated hyperplane arrangement and B_W the Artin group associated to W , then $\pi_1(\mathbb{C}^m - \mathcal{A}_W) \simeq \ker(B_W \rightarrow W)$ is a cyclic-related group. Note that the study of the lower central series quotients of the pure braid group was initiated by Kohno [14].

As an arrangement of lines in \mathbb{C}^2 is an example of a hyperplane arrangement, for any line arrangement \mathcal{L} the associated fundamental group $G = \pi_1(\mathbb{C}^2 - \mathcal{L})$ is a cyclic-related group. Kohno [13] proved, using the mixed Hodge structure on $H^1(\mathbb{C}^2 - \mathcal{L}, \mathbb{Q})$, that $gr(G) \otimes \mathbb{Q}$ is isomorphic to the nilpotent completion of the holonomy Lie algebra of $\mathbb{C}^2 - \mathcal{L}$. Falk proved [6], using Sullivan's 1-minimal models, that the lower central series ranks $\phi_k(G)$ are determined only by the combinatorics of \mathcal{L} . However, a precise combinatorial formula for $\phi_k(G)$, and even for $\phi_3(G)$, is not known (for the general formula for ϕ_3 , see e.g. [21, Corollary 3.6] or Remark 3.12 below).

These motivations lead us to investigate these ranks for cyclic-related and conjugation-free groups. While it is fairly easy to find out the rank ϕ_2 for any cyclic-related group (see Section 3.1.1), it is a harder task when considering the rank ϕ_3 . We find that for a certain class of conjugation-free groups, there is an upper bound on ϕ_3 (see Proposition 3.9). Explicitly, for such a group G , using only group-theoretic arguments, we prove that $\phi_3(G) \leq \sum_{i \geq 3} n_i \phi_3(\mathbb{F}_{i-1})$, where n_i is the number of cyclic relations of length i of G .

We then associate to this group G a line arrangement $\mathcal{L}(G)$, such that $\pi_1(\mathbb{C}^2 - \mathcal{L}(G)) \simeq G$. By [7], given any line arrangement \mathcal{L} , $\phi_3(\pi_1(\mathbb{C}^2 - \mathcal{L}))$ is bounded from below by the above upper bound. Thus, for this class of conjugation-free groups, one can calculate ϕ_3 directly, i.e. the third quotient of the lower central series behaves as if the group is a direct product of free groups. From this it follows, by [20], that these conjugation-free groups are *decomposable* (see Definition 4.19), that is, the quotient G_k/G_{k+1} decomposes, for $k \geq 2$, as a direct product of local components and that $\phi_k(G) = \sum_{i \geq 3} n_i \phi_k(\mathbb{F}_{i-1})$ for every $k \geq 2$ (see Theorem 4.18). Hence, for this class of groups, we have found the complete structure of $gr(G)$.

The paper is organized as follows. In Section 2 we define the main object of our research: cyclic-related and conjugation-free groups. We prove in Section 3 that for a certain class of conjugation-free groups, there is an upper bound on the third rank ϕ_3 . Associating to such a group a line arrangement, we show in Section 4 that this upper bound is obtained. This leads us to find an explicit formula on the ranks ϕ_k for $k \geq 3$ and to find a new series of examples of decomposable groups.

Acknowledgements: The author would like to thank Michael Falk, Uzi Vishne, Roland Bacher and Mikhail Zaidenberg for stimulating talks, and the Fourier Institut in Grenoble for the warm hospitality and support.

2. CYCLIC-RELATED AND CONJUGATION-FREE GROUPS

In this section, we define the main object of our research: cyclic-related groups and conjugation-free groups. Let G be a group, and denote $[a, b] \doteq a^{-1}b^{-1}ab$, $a^b = b^{-1}ab$ for $a, b \in G$.

Definition 2.1. Let G be a finitely presented group. We say that G is *cyclic-related* if G has a presentation such that it is generated by x_1, \dots, x_n with relations R_1, \dots, R_w and the following requirements hold:

- (1) All the relations are of the following form:

$$R_p = R_{p, (i_t, \dots, i_1)} : x_{i_t}^{s_{p,t}} x_{i_{t-1}}^{s_{p,t-1}} \cdot \dots \cdot x_{i_1}^{s_{p,1}} = x_{i_{t-1}}^{s_{p,t-1}} \cdot \dots \cdot x_{i_1}^{s_{p,1}} x_{i_t}^{s_{p,t}} = \dots = x_{i_1}^{s_{p,1}} x_{i_t}^{s_{p,t}} \cdot \dots \cdot x_{i_2}^{s_{p,2}},$$

where $1 \leq p \leq w$, $2 \leq t$, $\{i_1, i_2, \dots, i_t\} \subseteq \{1, \dots, n\}$ is an increasing subsequence of indices, $s_{p,j} \in \langle x_1, \dots, x_n \rangle$ for $1 \leq j \leq t$. These relations are called *cyclic relations of length t* . Note that when $t = 2$ we get the commutator $[x_{i_1}^s, x_{i_2}^s] = e$.

- (2) For every pair of indices $j_1, j_2 \in \{1, \dots, n\}$, $j_1 \neq j_2$, there is a unique relation $R_{p,(i_t, \dots, i_1)}$ such that $\{j_1, j_2\} \subseteq \{i_1, i_2, \dots, i_t\}$.
- (3) For every two relations $R_{p,(i_t, \dots, i_1)}$, $R_{p',(j_s, \dots, j_1)}$, $p \neq p'$ we have that:

$$|\{i_1, i_2, \dots, i_t\} \cap \{j_1, j_2, \dots, j_s\}| \leq 1.$$

Example 2.2. (1) The pure braid group PB_n is an example of a cyclic-related group; see [17, Theorem 2.3] for a presentation of PB_n with generators and relations satisfying the requirements above.

(2) For any hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^k$, $\pi_1(\mathbb{C}^k - \mathcal{A})$ is a cyclic-related group (see Remark 4.9(1)).

Remark 2.3. Note that $R_{p,(i_t, \dots, i_1)}$ a cyclic relation of length t can be written as a list of $t - 1$ commutators:

$$[x_{i_k}^{s_{p,k}}, x_{i_{k-1}}^{s_{p,k-1}} \dots \cdot x_{i_1}^{s_{p,1}} \cdot x_{i_t}^{s_{p,t}} \dots \cdot x_{i_{k+1}}^{s_{p,k+1}}] = e,$$

where $1 \leq k \leq t$.

Definition 2.4. We say that a cyclic relation of length t is *multiple* if $t \geq 3$.

Definition 2.5. Given a cyclic-related group G , we define its *associated graph* $Gr(G)$ in the following way:

- Vertices: for every multiple cyclic relation $R_p = R_{p,(i_t, \dots, i_1)}$ (where $t \geq 3$) associate a vertex $v_p = v_{p,(i_t, \dots, i_1)}$. When it is clear what are the corresponding generators to $v_{p,(i_t, \dots, i_1)}$ we write only v_p .
- Edges: two vertices $v_p = v_{p,(i_t, \dots, i_1)}$, $v_{p'} = v_{p',(j_s, \dots, j_1)}$ are connected by a edge e_{i_v} if

$$\{i_1, i_2, \dots, i_t\} \cap \{j_1, j_2, \dots, j_s\} = \{i_v\} = \{j_u\},$$

where $1 \leq v \leq t, 1 \leq u \leq s$.

Definition 2.6. Let G be a cyclic-related group. If every cyclic relation is of the form

$$R_{p,(i_t, \dots, i_1)} : x_{i_t} x_{i_{t-1}} \dots \cdot x_{i_1} = x_{i_{t-1}} \dots \cdot x_{i_1} x_{i_t} = \dots = x_{i_1} x_{i_t} \dots \cdot x_{i_2},$$

(that is, there are no conjugations on the generators appearing in the relation) then we say that G is *conjugation-free* and that its relations are *conjugation-free*.

Example 2.7. (1) If the graph $Gr(G)$ of a conjugation-free group G is empty, then all the relations of G are commutators between all the generators and hence G is a free abelian group, i.e. if G is generated by n generators then $G \simeq \mathbb{Z}^n$. This is since, by requirement (2) of Definition 2.1, every generator commutes with all other generators. Note that this is not true any more if G is cyclic-related, but not conjugation-free. For example, for the cyclic-related group

$$G_1 \doteq \langle x_1, x_2, x_3 : [x_1, x_2^{x_3}] = [x_1, x_3^{x_2}] = [x_2, x_3^{x_1}] = e \rangle,$$

the graph $Gr(G_1)$ is empty (as there are no multiple relations) but G_1 is not abelian (one can check that there exists an epimorphism to Sym_3).

- (2) Let G be a conjugation-free group with n generators and one cyclic relation of length n , i.e.

$$G \simeq \langle x_1, \dots, x_n : x_n x_{n-1} \dots \cdot x_1 = \dots = x_1 x_n \dots \cdot x_2 \rangle.$$

Then $G \simeq \mathbb{F}_{n-1} \oplus \mathbb{Z}$ and the graph consists of only one vertex.

(3) Let G be a conjugation-free group such that $Gr(G)$ is a disjoint union of two graphs. Then G is a direct sum of two conjugation-free groups, whose graphs correspond to the two graphs which are the components of $Gr(G)$.

(4) If the graph of a conjugation-free group consists of only vertices (i.e. there are no edges), then, by the previous examples, this group is isomorphic to a direct sum of a free abelian group and free groups. See remark 4.17 for a generalization of this when $Gr(G)$ does not have cycles.

(5) Let G_2 be a cyclic-related group generated by x_1, \dots, x_6 with the following relations:

- cyclic relations of length 3:

$$R_1 : x_3^{x_4} x_2 x_1 = x_2 x_1 x_3^{x_4} = x_1 x_3^{x_4} x_2, \quad R_2 : x_6 x_5 x_1 = x_5 x_1 x_6 = x_1 x_6 x_5, \quad R_3 : x_5 x_4 x_3 = x_4 x_3 x_5 = x_3 x_5 x_4.$$

- cyclic relations of length 2:

$$R_4 : x_4 x_2 = x_2 x_4, \quad R_5 : x_5 x_2^{x_3} = x_2^{x_3} x_5, \quad R_6 : x_6 x_2 = x_2 x_6, \quad R_7 : x_6 x_4 = x_4 x_6$$

Then, the vertices of $Gr(G_2)$ are v_1, v_2, v_3 (associated to the relations R_1, R_2, R_3) and the edges are e_1, e_3 and e_5 (see Figure 1). Note that with this presentation G_2 is not conjugation-free.

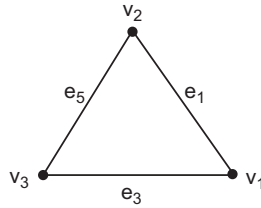


FIGURE 1. The graph $Gr(G_2)$ associated to the cyclic-related group G_2 .

Remark 2.8. Let G be a cyclic-related group. By requirement (3) in Definition 2.1, every two vertices of $Gr(G)$ are connected with at most one edge.

3. LOWER CENTRAL SERIES AND CONJUGATION-FREE GROUPS

In this section we prove the main result: that for a certain class of conjugation-free groups, the rank of the third quotient of the lower central series is bounded from above. We will see in Section 4.3 that this inequality is in fact an equality, using tools arising from line arrangements.

First, we give in Section 3.1 a short survey of some of the known results concerning the quotients of the lower central series, afterwards proving the main theorem in Section 3.2.

3.1. The lower central series. We begin with some notations. Let G be a group generated by x_1, \dots, x_n and consider its lower central series G_k , where $G_1 = G$ and $G_k = [G_{k-1}, G]$, $k \geq 2$. Denote $\phi_k(G) = \text{rank}(G_k/G_{k+1})$. If $G = \mathbb{F}_n$ the free group with n generators, then we denote $\omega_k(n) \doteq \phi_k(\mathbb{F}_n)$.

Notation 3.1. (1) $x_{i,j} \doteq [x_i, x_j] \in G_2$ for $i, j \in \{1, \dots, n\}$.

(2) $x_{i,j,k} \doteq [x_i, x_j x_k], x_{i,j,k} \doteq [x_i x_j, x_k] \in G_2$ for $i, j, k \in \{1, \dots, n\}$.

(3) $x_{i,j,k} \doteq [x_{i,j}, x_k] = [[x_i, x_j], x_k] \in G_3$ for $i, j, k \in \{1, \dots, n\}$.

(4) $x_{i,j,kl} \doteq [x_{i,j}, x_k x_l] = [[x_i, x_j], x_k x_l] \in G_3$ for $i, j, k, l \in \{1, \dots, n\}$.

(5) More generally, if $\alpha \in G$, then we denote $x_{i,\alpha} = [x_i, \alpha]$ and $x_{i,\alpha,k} = [x_{i,\alpha}, x_k] \in G_3$ and in the same way for expressions of the form $x_{\alpha,j}, x_{\alpha,j,k}$ or $x_{i,j,\alpha}$.

Remark 3.2. If G is a group generated by x_1, \dots, x_n , then it is well-known that G_2/G_3 is generated by $x_{j,i}$ where $j > i$ and G_3/G_4 is generated by $x_{j,i,k}$ where $j > i$ and $k \geq i$ (see [12]).

Concentrating on the groups G_2/G_3 and G_3/G_4 , we recall a few of their well-known properties.

Proposition 3.3. (I) For every $i, j, k \in \{1, \dots, n\}$, the following equivalences hold in G_2/G_3 :

$$[x_i, x_j^{x_k}] \equiv [x_i, x_j], \quad x_{i,jk} \equiv x_{i,k}x_{i,j}.$$

(II) The following equivalences hold in G_3/G_4 :

$$(1) \quad x_{i,j,kl} \equiv x_{i,j,l}x_{i,j,k}, \quad x_{i,jk,l} \equiv x_{i,k,l}x_{i,j,l}, \quad x_{ij,k,l} \equiv x_{j,k,l}x_{i,k,l}.$$

$$(2) \quad x_{i,j,k} \equiv x_{j,i,k}^{-1},$$

where i, j, k, l are either words in G or indices in $\{1, \dots, n\}$.

Proof. (I) Obvious.

(II)(1) We give the proof only for the expression $x_{i,jk,l}$. The proof for the other expressions is similar.

$$x_{i,jk,l} = [[x_i, x_j x_k], x_l] = [[x_i, x_k][x_i, x_j]^{x_k}, x_l] = [[x_i, x_k][x_k, [x_i, x_j]^{-1}][x_i, x_j], x_l].$$

When expanding the commutator brackets of the right hand side, we see that both the expression $[x_k, [x_i, x_j]^{-1}]$ and its inverse appear. As $[x_l, [x_i, x_j]^{-1}] \in G_3$, it commutes in the group G_3/G_4 with any other element. Therefore,

$$x_{i,jk,l} \equiv [[x_i, x_k][x_i, x_j], x_l] \equiv [[x_i, x_k], x_l] \cdot [[x_i, x_j], x_l] \pmod{G_4},$$

where in the last equivalence we used that $[ab, c] = [a, c]^b[b, c] = [b, [a, c]^{-1}][a, c][b, c]$, and that $[b, [a, c]^{-1}] \in G_4$ if $a \in G_2$.

(2) Let $a = x_{i,j}$; thus $a^{-1} = x_{j,i}$. Therefore $x_{i,j,k} = [a, x_k] = a^{-1}x_k^{-1}ax_k$ and $x_{j,i,k} = [a^{-1}, x_k] = ax_k^{-1}a^{-1}x_k$. Hence, $a^{-1}x_{j,i,k}a = x_k^{-1}a^{-1}x_ka = x_{i,j,k}^{-1}$. Thus, in G_3/G_4 , we get that $x_{i,j,k} \equiv x_{j,i,k}^{-1}$. \square

3.1.1. The rank of G_2/G_3 . From now on, let G be a cyclic-related group with relations R_1, \dots, R_w . In this subsection we give a combinatorial description of $\phi_2(G)$ (see also Remark 4.21(1) for a description of $\phi_2(G)$ via topological invariants). Recall again that G_2/G_3 is generated by $x_{j,i}$ when $j > i$.

Note that by Remark 2.3 and Proposition 3.3(I), every cyclic relation of length t in G is equivalent, in G_2/G_3 , to a list of $t - 1$ commutators, where the generators appear without conjugations. That is, the relation R_p of G :

$$R_p = R_{p,(i_t, \dots, i_1)} : x_{i_t}^{s_{p,t}} x_{i_{t-1}}^{s_{p,t-1}} \cdot \dots \cdot x_{i_1}^{s_{p,1}} = x_{i_{t-1}}^{s_{p,t-1}} \cdot \dots \cdot x_{i_1}^{s_{p,1}} x_{i_t}^{s_{p,t}} = \dots = x_{i_1}^{s_{p,1}} x_{i_t}^{s_{p,t}} \cdot \dots \cdot x_{i_2}^{s_{p,2}},$$

is equivalent in G_2/G_3 to the following list of commutators:

$$R_{p,k} : [x_{i_k}, x_{i_{k-1}} \cdot \dots \cdot x_{i_1} \cdot x_{i_t} \cdot \dots \cdot x_{i_{k+1}}] = e,$$

where $1 \leq k \leq t$. Moreover, by Proposition 3.3(I) we see that every relation $R_{p,k}$, where $1 \leq k \leq t$, is equivalent to

$$(3) \quad x_{i_k, i_{k+1}} \cdot \dots \cdot x_{i_k, i_t} \cdot x_{i_k, i_1} \cdot \dots \cdot x_{i_k, i_{k-1}} = e.$$

By requirement (2) of Definition 2.1, for every pair of indices i, j , the generator $x_{j,i}$ of G_2/G_3 appears as a term in the relations of G_2/G_3 only once. Therefore, for every two different relations $R_p, R_{p'}$ of G , when considering these relations in G_2/G_3 (as in Equation (3)), the generators that appear in these relations are different.

For example, every cyclic relation of length 3: $x_k^\alpha x_j^\beta x_i^\gamma = x_j^\beta x_i^\gamma x_k^\alpha = x_i^\gamma x_k^\alpha x_j^\beta$ (where $\alpha, \beta, \gamma \in G$ and $k > j > i$) is equivalent to equalities of the form $x_{j,i} \equiv x_{k,j} \equiv x_{k,i}^{-1}$ in G_2/G_3 . Hence, while in every such cyclic relation appear three generators of G_2/G_3 , two of them can be expressed as the third (or as its inverse).

In the same way, while in a cyclic relation R_p (in G_2/G_3) of length m appear $\binom{m}{2}$ generators of G_2/G_3 , $m - 1$ of them can be expressed as a product of the others; thus a cyclic relation

R_p of length m contributes $v(p) \doteq \binom{m}{2} - m + 1$ independent generators to G_2/G_3 . Note that $v(p) = \binom{m-1}{2} = \omega_2(m-1)$.

Hence $\sum_{p=1}^w v(p) = \sum_{i \geq 3} n_i \omega_2(i-1)$, where n_i is the number of cyclic relations of length i . We therefore see that

$$\text{rank}(G_2/G_3) = \phi_2(G) = \sum_{i \geq 3} n_i \omega_2(i-1).$$

3.2. The group G_3/G_4 . Here we would like to generalize the above equality, proved for the rank of G_2/G_3 , for the group G_3/G_4 , for a special class of conjugation-free groups. We first introduce some notations.

Notation 3.4. Let G be a cyclic-related group.

- (1) Let n_m be the number of cyclic relations of length m .
- (2) Let $\beta(\text{Gr}(G))$ be the first Betti number of the graph $\text{Gr}(G)$.

Thus, we want to prove the following theorem:

Theorem 3.5. *Let G be a conjugation-free group such that $\beta(\text{Gr}(G)) \leq 1$. Then:*

$$\phi_3(G) = \sum_{i \geq 3} n_i \omega_3(i-1).$$

The proof of this theorem is divided into two parts. The first part, given in Proposition 3.9, proves that $\phi_3(G) \leq \sum_{i \geq 3} n_i \omega_3(i-1)$, using only arguments from group-theory. The second part, given in section 4.3, proves that $\phi_3(G) \geq \sum_{i \geq 3} n_i \omega_3(i-1)$ and uses arguments from the theory of line arrangements. In fact, as will be seen in Remark 3.13, the condition that $\beta(\text{Gr}(G)) \leq 1$ can be relaxed. Before proving these propositions, we need to examine closely the generators of G_3/G_4 .

Example 3.6. Let $G \simeq \langle x_1, \dots, x_n : x_n x_{n-1} \cdots x_1 = \cdots = x_1 x_n \cdots x_2 \rangle$. As $G \cong \mathbb{F}_{n-1} \oplus \mathbb{Z}$, we get that $\phi_k(G) = \omega_k(n-1)$.

Definition 3.7. Let G be a conjugation-free group generated by x_1, \dots, x_n , with the relations R_1, \dots, R_w .

- (1) A generator of G_3/G_4 of the form $x_{j,i,k} = [[x_j, x_i], x_k]$ is called *local* if there is a multiple relation $R_p = R_{p,(u_1, \dots, u_t)}$ ($2 < t, 1 \leq p \leq w$) such that $\{i, j, k\} \subseteq \{u_r\}_{r=1}^t$.
- (2) We say that a generator x_k *participates* in a cyclic relation $R_p = R_{p,(u_1, \dots, u_t)}$ if $k \in \{u_r\}_{r=1}^t$.

Remark 3.8. Every conjugation-free cyclic relation of length t induces $t^3 - t$ local generators of G_3/G_4 , which are not induced from other relations (since we subtract the number of generators of the form $x_{i,i,i}$. Note also that $x_{i,i,i} = e$). However, by Example 3.6, from such relation there are only $\omega_3(t-1)$ non-trivial independent local generators, that is, the other local generators can be expressed as a multiplication of powers of the other basic local generators.

We are now ready to prove the first part of Theorem 3.5:

Proposition 3.9. *Let G be a conjugation-free group such that $\beta(\text{Gr}(G)) \leq 1$. Then:*

$$\phi_3(G) \leq \sum_{i \geq 3} n_i \omega_3(i-1).$$

Proof. Assume that G is generated by n generators x_1, \dots, x_n . In order to prove that $\phi_3(G) \leq \sum_{i \geq 3} n_i \omega_3(i-1)$, we prove that every *non-local* generator of the form $x_{j,i,k} = [[x_j, x_i], x_k]$ (where $j > i$ and $k \geq i$) is equivalent (modulo G_4) to the identity element e . From now on, we denote by \equiv equalities that take place in G_3 , modulo the group G_4 .

We prove this claim by splitting our proof into cases. If the generators x_i, x_j commute, then $x_{j,i} = e$ and thus $x_{j,i,k} = e$. Assume thus that the generators x_j, x_i participate in the same multiple relation R_1 , and let v_1 be the corresponding vertex to this relation in $Gr(G)$. Explicitly, as $j > i$,

$$R_1 = R_{1,(y''_{k'}, \dots, y''_j, y'_{k'}, \dots, y'_0, i, y_k, \dots, y_0)}.$$

Remark 3.10. Note that some (but not all) of the sets $\{y_0, \dots, y_k\}, \{y'_0, \dots, y'_{k'}\}, \{y''_0, \dots, y''_{k''}\}$ may be empty.

If $x_{i,k} = x_{j,k} = e$, that is, the generator x_k commutes with the generators x_i and x_j , then $x_{j,i,k} = e$. We therefore examine the cases when the generator x_k participates with at least one of the generators, x_i or x_j , in the same multiple relation. We consider three cases: when $x_{i,k} = e$ but $x_{j,k} \neq e$; when $x_{j,k} = e$ but $x_{i,k} \neq e$; and when $x_{i,k} \neq e$ and $x_{j,k} \neq e$.

First case: In this case x_k, x_i commute, i.e. $x_{k,i} = e$, and x_k, x_j participate in the same multiple relation R_2 ; let v_2 be the corresponding vertex to this relation in $Gr(G)$ (see Figure 2).

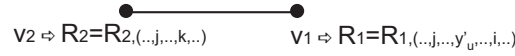


FIGURE 2. Part of the graph $Gr(G)$: The generators x_i, x_j participate in a multiple relation R_1 (to whom the vertex v_1 is associated) and the generators x_k, x_j participate in a multiple relation R_2 (to whom the vertex v_2 is associated when we assume that $k > j$). The generator $x_{y'_u}$ is another generator participating in R_1 .

Let us prove that $x_{j,i,k} \equiv e$.

$$(4) \quad x_{j,i,k}^{-1} \stackrel{Eqn.(2)}{\equiv} x_{j,i,k} = x_j^{-1} x_i^{-1} x_j x_i x_k^{-1} x_i^{-1} x_j^{-1} x_i x_j x_k \stackrel{x_{k,i}=e}{\equiv} x_j^{-1} x_i^{-1} x_j x_k^{-1} x_j^{-1} x_i x_j x_k.$$

Denote $Y = x_{y_k} \cdot \dots \cdot x_{y_0}, Y' = x_{y'_{k'}} \cdot \dots \cdot x_{y'_0}, Y'' = x_{y''_{k''}} \cdot \dots \cdot x_{y''_0}$. Therefore, since G is conjugation-free, by Definition 2.6 part of the relations of R_1 are the following:

$$(5) \quad Y'' x_j Y' x_i Y = x_j Y' x_i Y Y'' = Y' x_i Y Y'' x_j = x_i Y Y'' x_j Y' = Y Y'' x_j Y' x_i.$$

Remark 3.11. Let us note that if $a \in G_3$ then for every $b \in G$, $a \equiv b^{-1} a b \pmod{(G_4)}$, that is, we can conjugate an element in G_3 by any element in G and remain in the same conjugacy class.

We now consider two cases: either that $[x_k, Y] = [x_k, Y'] = [x_k, Y''] = e$ or that at least one of these equalities does not hold.

Case (1): Assume that $[x_k, Y] = [x_k, Y'] = [x_k, Y''] = e$. This happens when all the generators participating in R_1 , except the generator x_j , commute with x_k (for example, this situation occurs when the vertex v_1 is not a part of the cycle of the graph $Gr(G)$).

By Remark 3.11, we can conjugate $x_{j,i,k}^{-1}$ by any element. Thus, following Equation (4):

$$(6) \quad x_{j,i,k}^{-1} \stackrel{Conj. by Y Y'' Y'}{\equiv} (Y Y'' Y')^{-1} \cdot x_j^{-1} x_i^{-1} x_j x_k^{-1} x_j^{-1} x_i x_j x_k \cdot Y Y'' Y'$$

$$\stackrel{[x_k, Y Y'' Y'] = e}{=} (Y Y'' Y')^{-1} \cdot x_j^{-1} x_i^{-1} x_j x_k^{-1} x_j^{-1} x_i x_j \cdot Y Y'' Y' \cdot x_k.$$

We now examine the following expression: $x_i x_j \cdot Y Y'' Y'$.

$$x_i x_j \cdot Y Y'' Y' = x_i Y Y'' x_j [x_j, Y Y''] Y' = x_i Y Y'' x_j Y' [x_j, Y Y''] [[x_j, Y Y''], Y']$$

$$\stackrel{Rel.(5)}{=} x_j Y' x_i Y Y'' [x_j, Y Y''] [[x_j, Y Y''], Y'] = x_j Y' x_i Y Y'' [x_j, Y Y''] m,$$

where we denote $m = [[x_j, YY''], Y']$. Note that $m \in G_3$ and therefore commutes in G_3/G_4 with any other element that belongs to G . Thus, Equation (6) turns to:

$$(7) \quad x_{j,i,k}^{-1} \equiv m^{-1}[x_j, YY'']^{-1}Y''^{-1}Y^{-1}x_i^{-1}Y'^{-1}x_j^{-1} \cdot x_jx_k^{-1}x_j^{-1} \cdot x_jY'x_iYY''[x_j, YY'']mx_k \\ \equiv [x_j, YY'']^{-1}Y''^{-1}Y^{-1}x_i^{-1}Y'^{-1}x_k^{-1}Y'x_iYY''[x_j, YY'']x_k.$$

Since $[x_k, x_i] = [x_k, Y] = [x_k, Y'] = [x_k, Y''] = e$, then Equation (7) becomes:

$$x_{j,i,k}^{-1} \equiv [x_j, YY'']^{-1}x_k^{-1}[x_j, YY'']x_k.$$

Denoting $\eta = YY''$, we get:

$$x_{j,i,k}^{-1} \equiv x_{j,\eta,k} \stackrel{Eqn.(2)}{\equiv} x_{\eta,j,k}^{-1}.$$

Thus:

$$(8) \quad x_{j,i,k} \equiv x_{\eta,j,k} = x_j^{-1}\eta^{-1}x_j\eta x_k^{-1}\eta^{-1}x_j^{-1}\eta x_j x_k \stackrel{x_k,\eta=e, Conj. by Y'x_i}{=} \\ (Y'x_i)^{-1} \cdot x_j^{-1}\eta^{-1}x_jx_k^{-1}x_j^{-1}\eta x_j x_k \cdot Y'x_i \stackrel{x_k, Y'=x_k, i=e}{=} (Y'x_i)^{-1}x_j^{-1}\eta^{-1}x_jx_k^{-1}x_j^{-1}\eta x_j Y'x_i x_k.$$

Now, $\eta x_j Y' x_i \stackrel{Rel.(5)}{=} x_j Y' x_i \eta$. Therefore, Equation (8) becomes:

$$x_{j,i,k} \equiv (x_j Y' x_i \eta)^{-1} \cdot x_j x_k^{-1} x_j^{-1} \cdot x_j Y' x_i \eta x_k = \eta^{-1} x_i^{-1} Y'^{-1} x_k^{-1} Y' x_i \eta x_k = e,$$

when in the last equality we use the fact that x_k commutes with all the other terms in the product.

Case (2): Let us now check when $[x_k, y_0] \neq e$, when $y_0 \in \{Y, Y', Y''\}$. This happens when the generator x_k participates in a multiple relation R_3 , where $R_3 \neq R_2$, and one of the generators in the set $\{x_v\}_{v \in \mathcal{Y}}$, where $\mathcal{Y} = \{y_0, \dots, y_k\} \cup \{y'_0, \dots, y'_k\} \cup \{y''_0, \dots, y''_k\}$, also participates in R_3 (see Figure 3(a)). Note that x_k participates in (at least) two multiple relations: R_2 and R_3 .

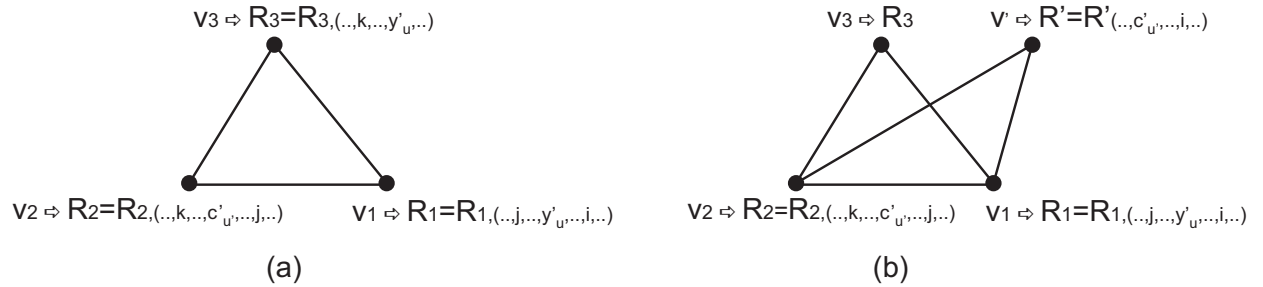


FIGURE 3. Part (a): a part of the graph $Gr(G)$: The generators x_i, x_j participate in a multiple relation R_1 (to whom the vertex v_1 is associated) and the generator x_k participates in two multiple relations R_2, R_3 (to whom the vertices v_2, v_3 are associated, when we assume that $k > j$). The generator $x_{y'_u}$ is another generator participating in R_1 and R_3 , and $x_{c'_u}$ is another generator participating in R_2 . Part (b): a forbidden case when $x_{c'_u}$ and x_i participate in the same multiple relation R' . In this case, $\beta(Gr(G)) > 1$.

Assume that $k > j$ (the proof for $k < j$ is similar and we leave it to the reader; note that if $k = j$ then $x_{j,i,j}$ is a local generator). Examining closely R_2 , we see that:

$$R_2 = R_{2,(c''_{k'}, \dots, c''_0, k, c'_{k'}, \dots, c'_0, j, c_k, \dots, c_0)}.$$

(when we take into consideration Remark 3.10 with respect to the sets $\{c_i\}, \{c'_i\}, \{c''_i\}$). Denote $C = x_{c_k} \cdot \dots \cdot x_{c_0}$, $C' = x_{c'_{k'}} \cdot \dots \cdot x_{c'_0}$, $C'' = x_{c''_{k'}} \cdot \dots \cdot x_{c''_0}$. By Definition 2.6, part of the relations of R_2 are the following:

$$(9) \quad C''x_kC'x_jC = x_kC'x_jCC'' = C'x_jCC''x_k = x_jCC''x_kC' = CC''x_kC'x_j.$$

By the Hall-Witt identity in G_3/G_4 , we get that $[[x_j, x_i], x_k] \cdot [[x_i^{-1}, x_k^{-1}], x_j] \cdot [[x_k, x_j^{-1}], x_i^{-1}] \equiv e$. As $[x_i, x_k] = e$, we get that $[[x_j, x_i], x_k] \cdot [[x_k, x_j^{-1}], x_i^{-1}] \equiv e$. It is easy to see that in G_3/G_4 , $[[x_k, x_j^{-1}], x_i^{-1}] \equiv [[x_j, x_k], x_i]^{-1}$, and thus $x_{j,i,k} = [[x_j, x_i], x_k] \equiv [[x_j, x_k], x_i] = x_{j,k,i}$. Since $x_{j,k,i} \stackrel{Eqn.(2)}{\equiv} x_{k,j,i}^{-1}$, we prove that $x_{k,j,i} \equiv e$.

$$(10) \quad x_{k,j,i} = x_j^{-1}x_k^{-1}x_jx_kx_i^{-1}x_k^{-1}x_j^{-1}x_kx_jx_i \stackrel{x_{k,i}=e, Conj. by C'CC''}{\equiv} (C'CC'')^{-1} \cdot x_j^{-1}x_k^{-1}x_jx_i^{-1}x_j^{-1}x_kx_jx_i \cdot C'CC''.$$

Let $\mathcal{C} = \{c_0, \dots, c_k\} \cup \{c'_0, \dots, c'_{k'}\} \cup \{c''_0, \dots, c''_{k''}\}$. Note that x_i commutes with every generator in the set $\{x_v\}_{v \in \mathcal{C}}$, since otherwise $\beta(Gr(G))$ would be greater than 1. Indeed, if both x_i and x_c participated in the same multiple relation R' for an index $c \in \mathcal{C}$, then $Gr(G)$ would have (at least) two cycles (see Figure 3(b)). Therefore $[x_i, C] = [x_i, C'] = [x_i, C''] = e$ and

$$(11) \quad x_{k,j,i} \equiv (C'CC'')^{-1} \cdot x_j^{-1}x_k^{-1}x_jx_i^{-1}x_j^{-1}x_kx_j \cdot C'CC'' \cdot x_i.$$

Let us examine the expression $x_kx_jC'CC''$:

$$\begin{aligned} x_kx_jC'CC'' &= x_kC'x_j[x_j, C']CC'' \stackrel{t \doteq [[x_j, C'], CC'']}{=} \\ &= x_kC'x_jCC''[x_j, C']t \stackrel{Rel.(9)}{=} x_jCC''x_kC'[x_j, C']t. \end{aligned}$$

Note that $t \in G_3$ and thus, in G_3/G_4 , it commutes with any other element. Therefore, Equation (11) becomes:

$$(12) \quad \begin{aligned} x_{k,j,i} &\equiv t^{-1}[x_j, C']^{-1}C'^{-1}x_k^{-1}C''^{-1}C^{-1}x_j^{-1} \cdot x_jx_i^{-1}x_j^{-1} \cdot x_jCC''x_kC'[x_j, C']tx_i \\ &\equiv [x_j, C']^{-1}C'^{-1}x_k^{-1}C''^{-1}C^{-1}x_i^{-1}CC''x_kC'[x_j, C']x_i = [x_j, C']^{-1}x_i^{-1}[x_j, C']x_i, \end{aligned}$$

when in the last equality we used that $[x_i, x_k] = [x_i, C] = [x_i, C'] = [x_i, C''] = e$. Thus:

$$\begin{aligned} x_{k,j,i} &\stackrel{Eqn.(12)}{\equiv} x_{j,C',i} \stackrel{Eqn.(2)}{\equiv} x_{C',j,i}^{-1} \Rightarrow \\ x_{C',j,i} &= x_j^{-1}C'^{-1}x_jC'x_i^{-1}C'^{-1}x_j^{-1}C'x_jx_i \stackrel{[x_i, C']=e, Conj. by CC''x_k}{\equiv} \\ &= (CC''x_k)^{-1} \cdot x_j^{-1}C'^{-1}x_jx_i^{-1}x_j^{-1}C'x_jx_i \cdot CC''x_k = \\ &\stackrel{[x_i, CC''] \equiv [x_i, x_k] = e}{=} (x_k^{-1}C''^{-1}C^{-1}x_j^{-1}C'^{-1})x_jx_i^{-1}x_j^{-1}(C'x_jCC''x_k)x_i = e, \end{aligned}$$

where in the last equality we used the relation:

$$C'x_jCC''x_ix_k \stackrel{Rel.(9)}{=} x_jCC''kC'$$

and then the fact that x_i commutes with C, C', C'' and x_k .

Second case: In this case x_k, x_j commute, i.e. $x_{j,k} = e$, and x_k, x_i participate in the same multiple relation R_4 , whose associated vertex in $Gr(G)$ is denoted by v_4 . The proof that $x_{j,i,k} \equiv e$ is similar to the proof presented in the First case. Thus, we only outline the main steps and leave the detailed checks to the reader. We use the same notations, regarding the relation R_1 and generators participating in it, that appeared in the First case. Again, we consider two cases: either that $[x_k, Y] = [x_k, Y'] = [x_k, Y''] = e$ or that at least one of these equalities is not true.

Case (1): Assuming that $[x_k, Y] = [x_k, Y'] = [x_k, Y''] = e$, we conjugate $x_{j,i,k}$ by $Y'Y''$ and get, using Relation (5), that $x_{j,i,k} \equiv x_{i,Y',k}$. Thus $x_{j,i,k} \equiv x_{Y',i,k}^{-1}$. Conjugating $x_{Y',i,k}$ by $YY''x_j$ and again using Relation (5), we see that $x_{Y',i,k} \equiv e$.

Case (2): This case happens when the generator x_k participates in another multiple relation R_3 , where $R_4 \neq R_3$, and in R_3 participate also one of the generators in the set $\{x_v\}_{v \in \mathcal{Y}}$, where $\mathcal{Y} = \{y_0, \dots, y_k\} \cup \{y'_0, \dots, y'_{k'}\} \cup \{y''_0, \dots, y''_{k''}\}$ (see Figure 4).

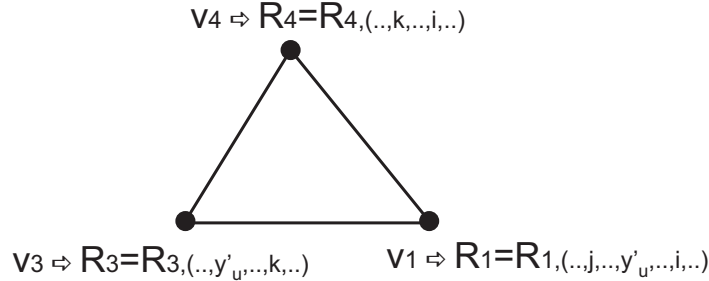


FIGURE 4. a portion of the graph $Gr(G)$: The generators x_i, x_j participate in a multiple relation R_1 (to whom the vertex v_1 is associated) and the generator x_k participates in two multiple relations R_3, R_4 (to whom the vertices v_3, v_4 are associated). The generator $x_{y'_u}$ is another generator participating in R_1 and R_3 .

Note that it is known that $i < k$. Examining closely R_3 , we see that:

$$R_3 = R_{3,(z''_{k''}, \dots, z''_0, k, z'_{k'}, \dots, z'_0, i, z_k, \dots, z_0)}.$$

(when we take into consideration Remark 3.10 with respect to the sets $\{z_i\}, \{z'_i\}, \{z''_i\}$). Denote $Z = x_{z_k} \cdot \dots \cdot x_{z_0}, Z' = x_{z'_{k'}} \cdot \dots \cdot x_{z'_0}, Z'' = x_{z''_{k''}} \cdot \dots \cdot x_{z''_0}$. Therefore, by Remark 4.9, part of the relations of R_3 are the following:

$$(13) \quad Z''x_kZ'x_iZ = x_kZ'x_iZ'' = Z'x_iZ''x_k = x_iZZ''x_kZ' = ZZ''x_kZ'x_i.$$

Again, by the Hall-Witt identity in G_3/G_4 , we get that $x_{j,i,k} \equiv x_{i,k,j}^{-1}$ and therefore $x_{j,i,k} \equiv x_{k,i,j}$. Noting that $[x_j, Z] = [x_j, Z'] = [x_j, Z''] = e$ (otherwise $\beta(Gr(G)) > 1$), we conjugate $x_{k,i,j}$ by $Z'Z''$, and using Relation (13), we get that $x_{k,i,j} \equiv x_{i,Z',j}$. Thus $x_{k,i,j} \equiv x_{Z',i,j}^{-1}$. Conjugating $x_{Z',i,j}$ by $ZZ''x_k$ and again using Relation (13), we get that $x_{Z',i,j} \equiv e$.

Third case: In this case $x_{i,k} \neq e$ and $x_{j,k} \neq e$. There are two cases when this may occur: the first is when all of the three generators x_i, x_j, x_k participate in the same multiple relation. In this case, the generator $x_{j,i,k} \in G_3/G_4$ is a local generator, so we are done. The second case, of a cycle of length 3 in the graph $Gr(G)$, occurs when the vertices of the cycle correspond to the different multiple relations R_1, R_2 and R_4 , where x_i, x_j participate in R_1 , x_k, x_j participate in R_2 and x_i, x_k participate in R_4 (see Figure 5(a)).

We use the notations of the First case, i.e. that

$$R_1 = R_{1,(y''_{k''}, \dots, y''_0, j, y'_{k'}, \dots, y'_0, i, y_k, \dots, y_0)}.$$

We also use the notations Y, Y', Y'' . Note that $[x_k, Y] = [x_k, Y'] = [x_k, Y''] = e$, as otherwise x_k would participate in (at least) three multiple relations and $\beta(Gr(G)) > 1$ (see Figure 5(b)). Thus:

$$(14) \quad x_{j,i,k} = x_i^{-1}x_j^{-1}x_ix_jx_k^{-1}x_j^{-1}x_i^{-1}x_jx_ix_k \stackrel{\text{Conj. by } Y'Y''}{\equiv} [x_k, Y'Y''] = e$$

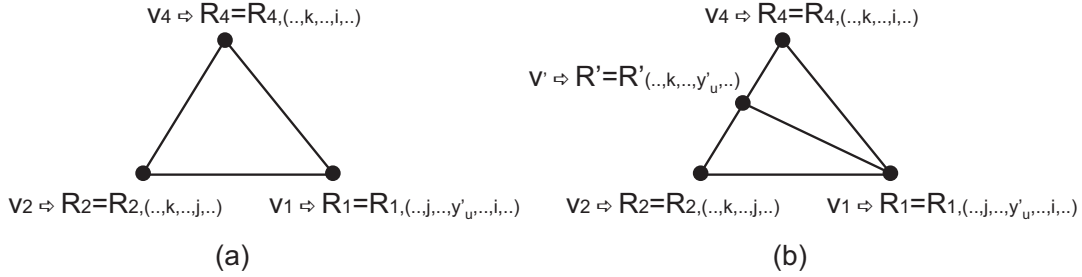


FIGURE 5. Part (a): A cycle of length three: all pairs of generators participate in different multiple relations. The generator $x_{y'_u}$ participates in R_1 . Part (b): a forbidden case when $x_{y'_u}$ and x_k participate in the same multiple relation R' . In this case, $\beta(Gr(G)) > 1$.

$$\equiv (Y'YY'')^{-1}x_i^{-1}x_j^{-1}x_i x_j x_k^{-1}x_j^{-1}x_i^{-1}x_j x_i Y'YY''x_k.$$

Let us examine the expression $x_j x_i Y'YY''$.

$$x_j x_i Y'YY'' = x_j Y' x_i [x_i, Y'] YY'' = x_j Y' x_i YY'' [x_i, Y'] [[x_i, Y'], YY''] \stackrel{Rel.(5)}{=} x_i YY'' x_j Y' [x_i, Y'] r,$$

where we set $r = [[x_i, Y'], YY''] \in G_3$, which commutes in G_3/G_4 with every other element. Thus Equation (14) becomes:

$$(15) \quad x_{j,i,k} \equiv r^{-1} [x_i, Y']^{-1} Y'^{-1} x_j^{-1} Y''^{-1} Y^{-1} x_i^{-1} \cdot x_i x_j x_k^{-1} x_j^{-1} x_i^{-1} \cdot x_i YY'' x_j Y' [x_i, Y'] r x_k \equiv [x_i, Y']^{-1} Y'^{-1} x_j^{-1} Y''^{-1} Y^{-1} x_j x_k^{-1} x_j^{-1} YY'' x_j Y' [x_i, Y'] x_k.$$

Noting that $[x_i, Y'] x_k = x_k [x_i, Y'] [[x_i, Y'], x_k] \stackrel{x_i, Y', k \in G_3}{\equiv} x_k [[x_i, Y'], x_k] [x_i, Y']$, we denote

$$f = Y'^{-1} x_j^{-1} Y''^{-1} Y^{-1} x_j x_k^{-1} x_j^{-1} YY'' x_j Y' x_k,$$

and get from Equation (15) that:

$$(16) \quad x_{j,i,k} \equiv (f \cdot [[x_i, Y'], x_k])^{[x_i, Y']} \stackrel{x_j, i, k \in G_3, Rem. 3.11}{\equiv} f \cdot [[x_i, Y'], x_k].$$

Since $x_{j,i,k}, [x_i, Y'], x_k \in G_3/G_4$, then also $f \in G_3/G_4$. Denoting $\eta = YY''$, we examine the expression f :

$$f \stackrel{Conj. by x_i}{\equiv} x_i^{-1} Y'^{-1} x_j^{-1} \eta^{-1} x_j x_k^{-1} x_j^{-1} \eta x_j Y' x_k x_i = x_i^{-1} Y'^{-1} x_j^{-1} \eta^{-1} x_j x_k^{-1} x_j^{-1} \cdot \eta x_j Y' x_i \cdot x_k [x_k, x_i].$$

Now,

$$\eta x_j Y' x_i \stackrel{Rel.(5)}{=} x_j Y' x_i \eta.$$

Thus,

$$f \equiv \eta^{-1} x_i^{-1} Y'^{-1} x_j^{-1} \cdot x_j x_k^{-1} x_j^{-1} \cdot x_j Y' x_i \eta x_k [x_k, x_i] = \eta^{-1} x_i^{-1} Y'^{-1} x_k^{-1} Y' x_i \eta x_k [x_k, x_i] \stackrel{[x_k, Y'] = [x_k, \eta] = e}{=} \eta^{-1} x_i^{-1} x_k^{-1} x_i x_k \eta [x_k, x_i] = \eta^{-1} [x_k, x_i]^{-1} \eta [x_k, x_i] = x_{k,i,\eta}^{-1}.$$

Moreover, using the Hall-Witt identity in G_3/G_4 (as in the First case), we see that $[[x_i, Y'], x_k] \equiv [[x_i, x_k], Y']$ and thus $[[x_i, Y'], x_k] \equiv x_{k,i,Y'}^{-1}$. Therefore, Equation (16) becomes

$$x_{j,i,k} \equiv x_{k,i,\eta}^{-1} \cdot x_{k,i,Y'}^{-1}.$$

By Proposition 3.3(II) (see Equation (1)), the expression on the right hand side is a product of inverses of terms of the form $x_{k,i,y}$, where $y \in \mathcal{Y} \doteq \{y_i\} \cup \{y'_i\} \cup \{y''_i\}$. However, as x_k commutes with x_y (for every $y \in \mathcal{Y}$), this case is already handled in the Second case. Therefore $x_{j,i,k} \equiv e$. \square

Remark 3.12. Obviously there are cyclic-related groups, which may not be conjugation-free, whose ϕ_3 is not bounded by the above upper bound. For example, for the pure braid group BP_4 it is known that $\phi_3 = 10$ (the ranks of the lower central series of the pure braid group were first studied by Kohno [14]). Moreover, it has a cyclic-related presentation, under which it is generated by 6 generators with 4 cyclic relations of length 3 and 3 cyclic relations of length 2 (see [17]). However, were BP_4 a conjugation-free group with $\beta(\text{Gr}(PB_4)) \leq 1$, then $\phi_3 \leq 8$.

Remark 3.13. The restriction on the conjugation-free group G in Theorem 3.5, i.e. that $\beta(\text{Gr}(G)) \leq 1$, can be relaxed. Assume that $\beta(\text{Gr}(G)) = 1$ and that the length of the cycle is t . Let v_1, \dots, v_t be the vertices of the graph on the cycle, corresponding to the multiple relations R_1, \dots, R_t . Let $e_{i_1}, e_{i_2}, \dots, e_{i_t}$ be the edges of the cycle, when e_{i_j} connects the vertices v_j and $v_{j+1(\text{mod } t)}$. Every edge e_{i_j} , $1 \leq j \leq t$ corresponds to a generator x_{i_j} which participates both in R_j and in $R_{j+1(\text{mod } t)}$. Let $X_0 = \{x_{i_1}, \dots, x_{i_t}\}$. For each relation R_i , $2 \leq i \leq t$, let X_{p_i} be the set of generators of G participating in R_i , and let $X_2 = \cup_{i=2}^t X_{p_i}$. Denote $X_1 = X_2 - X_0$.

By examining the proof of Theorem 3.5, note that if a generator x_r participates, for example, in R_1 , then in order to prove that a non-local generator $x_{j,i,k}$ is equivalent to e , when one of the indices j, i or k is r , we only have to demand that x_r commutes with all the generators in X_1 .

This means that in order that Theorem 3.5 could be applied, we can omit the requirement that $\beta(\text{Gr}(G)) \leq 1$ and require that if there are a few cycles (where v_1 is a vertex, corresponding to the relation R_1 , in a cycle composed of the vertices v_1, \dots, v_t and x_r participates in R_1), then x_r can participate in other multiple relations R such that $R \neq R_i$, $1 \leq i \leq t$, as long as in this multiple relation R , there are no other generators, belonging to X_1 , which participate in it.

Definition 3.14. A planar, simple connected graph H is called *cycle-separated* if every two cycles of H do not have a vertex in common and all the edges of H are straight segments.

Obviously, a graph H with $\beta(H) \leq 1$ is cycle-separated. An example to a cycle-separated graph with two cycles is presented in Figure 6(a).

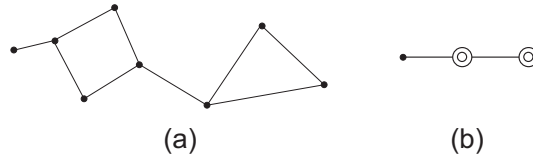


FIGURE 6. Part (a): An example to a cycle-separated graph H . Part (b): The graph H after the identification of its cycles into cycled-vertices.

Remark 3.15. Let us find the general form of a cycle-separated graph. Given a cycle in a cycle-separated graph, we identify all its vertices and edges connecting them and call the resulting object a *cycled-vertex*, depicted by \odot (in contrast to ordinary vertices, which are depicted by \bullet). For example, after this identification, the graph H in Figure 6(a) is transformed to the graph presented in Figure 6(b).

We claim that given a cycle-separated graph, after this identification, the new graph will be a tree (whose vertices are either \odot or \bullet). Indeed, if there is a cycle in the new graph, then the original graph would have two cycles with common vertices.

Thus we have the following:

Proposition 3.16. *If G is a conjugation-free group whose graph is a disjoint union of cycle-separated graphs, then*

$$\phi_3(G) \leq \sum_{i \geq 3} n_i \omega_3(i-1).$$

Proof. The group G , whose graph is cycle-separated, satisfies the (relaxed) conditions posed in Remark 3.13, and since G is conjugation-free, Proposition 3.9 can be applied for the group G . By Example 2.7(3), this claim is true when $Gr(G)$ is a disjoint union of cycle-separated graphs. \square

4. LINE ARRANGEMENTS AND CONJUGATION-FREE GRAPHS

In this section we prove that given a conjugation-free group G such that $Gr(G)$ is a cycle-separated graph, $\phi_3(G) = \sum_{i \geq 3} n_i \omega_3(i-1)$, and in fact $\phi_k(G) = \sum_{i \geq 3} n_i \omega_k(i-1)$ for every $k \geq 3$. In order to do so, we have to use tools arising from the theory of line arrangement. For the reader who is not familiar with line arrangements, let us outline the main constructions of this section.

In section 4.1 we associate for every conjugation-free group G (with an associated cycle-separated graph) a line arrangement $\mathcal{L}(G)$ in \mathbb{C}^2 (see Definition 4.1), later showing in Section 4.2 that the fundamental group $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$ is isomorphic to G . In Section 4.3 we see that given any line arrangement \mathcal{L} , it is known by Falk [7] that $\phi_3(\pi_1(\mathbb{C}^2 - \mathcal{L})) \geq \sum_{i \geq 3} n_i \omega_3(i-1)$, where n_i is the number of the singular points of \mathcal{L} with multiplicity i . Combining this with Proposition 3.16, we get that $\phi_3(G) = \sum_{i \geq 3} n_i \omega_3(i-1)$. Finally, Papadima-Suciu [20, Theorem 2.4] shows that if ϕ_3 attains this equality, then ϕ_k attains it for every $k \geq 3$.

4.1. The associated line arrangement. An *affine line arrangement* in \mathbb{C}^2 is a union of copies of \mathbb{C}^1 in \mathbb{C}^2 . Such an arrangement is called *real* if the defining equations of all its lines are written with real coefficients, and *complex* otherwise. A *multiple point* $p \in \mathcal{L}$ is an intersection point where more than two lines intersect (i.e. an intersection point with intersection multiplicity greater than 2).

We construct an affine line arrangement, associated to a conjugation-free group.

Definition 4.1. Let G be a conjugation-free group, generated by x_1, \dots, x_n , with a planar graph $Gr(G)$ such that all its edges are straight segments. We assume that $Gr(G)$ is connected (see Remark 4.3 for the case when $Gr(G)$ has several components). Let v_1, \dots, v_m be the set of vertices of $Gr(G)$, corresponding to the set of multiple relations R_1, \dots, R_m of G . We construct a real line arrangement $\mathcal{L}(G)$ from $Gr(G)$.

If $Gr(G)$ is empty, then we define $\mathcal{L}(G)$ as a real generic line arrangement (that is, all the singular points are nodes), consisting of n lines.

Hence assume that $Gr(G)$ is not an empty graph. If $Gr(G)$ consists of only one vertex v_1 and no edges, then v_1 corresponds in G to a multiple relation of length t . In that case, fix a point $p_1 \in \mathbb{R}^2$ and pass through it t different real lines.

Assume thus that there is at least one edge in $Gr(G)$. First, we extend the edges of $Gr(G)$ to be infinite straight lines, and call the union of all these lines \mathcal{L} . Let I be the set of intersection points of \mathcal{L} , i.e., at this step, $I = \{p_1, \dots, p_m\}$, where the p_i 's are the points corresponding to the vertices of $Gr(G)$ (for an example, see Figures 7(a) and 7(b.I)). We now build the rest of the arrangement inductively.

Looking at the vertex v_1 , it corresponds in G to a multiple relation $R_1 = R_{1,(i_1, \dots, i_1)}$, $t \geq 3$, and in \mathcal{L} to the singular point p_1 . $Gr(G)$ is connected, so there is an edge e connecting v_1 with another vertex. Renumerate the generators if needed, this edge corresponds to a generator x_{i_1} , which participates in the relation R_1 and in another multiple relation; this generator corresponds to the already drawn line $\ell_{i_1} \in \mathcal{L}$ (this is the extended edge $e = e_{i_1}$). In addition, there might be other generators x_{i_2}, \dots, x_{i_u} , for which the corresponding lines were already drawn (see Figure 7(b.I)). Thus, assume that for the generators $x_{i_{u+1}}, \dots, x_{i_t}$, the corresponding lines were not drawn. We draw a line $\ell_{i_{u+1}}$ which corresponds to the generator $x_{i_{u+1}}$, in the following way: the line $\ell_{i_{u+1}}$ passes through the singular point p_1 and intersects all the other lines in \mathcal{L} at nodes, i.e. it does not pass through the singular points in $I - \{p_1\}$. By abuse of notation, let \mathcal{L} be the new arrangement $\mathcal{L} \cup \ell_{i_{u+1}}$ and let I be the set of all intersection points of the new arrangement \mathcal{L} . We now repeat the same process for all the other generators $x_{i_{u+2}}, \dots, x_{i_t}$ (that is, the lines $\ell_{i_{u+2}}, \dots, \ell_{i_t}$ pass through

p_1 and intersect all the lines of \mathcal{L} at nodes). In this way, we have attached to every generator that participates in R_1 a line in \mathcal{L} (see Figure 7(b.II)).

We now look at the other vertices which are neighbors of v_1 . If v_2 is such a vertex (corresponding to the singular point p_2 in \mathcal{L}), corresponding to the relation $R_2 = R_{2,(j_s,\dots,j_1)}$ in G , we know that already for some of the generators in $\{x_{j_k}\}_{k=1}^s$, the corresponding lines were drawn. Thus, for the rest of the generators in this set, we draw the corresponding lines, in the same way as above: these lines pass only through the singular point p_2 and intersect all the other lines in \mathcal{L} at nodes (see Figure 7(III)).

In the same way, we go over all the vertices of $Gr(G)$ (see Figure 7(b.IV)). Eventually we get that \mathcal{L} is a real line arrangement in \mathbb{R}^2 with n' lines. In $n' < n$ then there are $n - n'$ generators of G : $x_{k_1}, \dots, x_{k_{n-n'}}$, which do not participate in any multiple relation, that is, $[x_i, x_{k_u}] = e$ for every $1 \leq i \leq n$ and $1 \leq u \leq n - n'$.

We draw a line ℓ_{k_1} which corresponds to the generator x_{k_1} , in the following way: the line ℓ_{k_1} does not pass through any singular point of \mathcal{L} , hence it intersects all the lines of \mathcal{L} at nodes (see Figure 7(b.V)). By abuse of notation, let \mathcal{L} be the new arrangement $\mathcal{L} \cup \ell_{k_1}$. We now repeat the same process for all the other generators $x_{k_2}, \dots, x_{k_{n-n'}}$. Eventually we get that \mathcal{L} is a real line arrangement with n lines.

Complexifying the equations of the lines, we get an affine real line arrangement $\mathcal{L}(G)$ in \mathbb{C}^2 with n lines. Note that the number of multiple points of $\mathcal{L}(G)$ is the number of multiple relations.

Example 4.2. Let H be a conjugation-free group, generated by 7 generators x_1, \dots, x_7 with the following relations:

$R_1 : x_3x_2x_1 = x_2x_1x_3 = x_1x_3x_2$, $R_2 : x_5x_4x_3 = x_4x_3x_5 = x_3x_5x_4$, $R_3 : x_6x_5x_1 = x_5x_1x_6 = x_1x_6x_5$,
and

$R_4 : x_4x_2 = x_2x_4$, $R_5 : x_5x_2 = x_2x_5$, $R_6 : x_6x_2 = x_2x_6$, $R_7 : x_6x_4 = x_4x_6$, $R_{7+j} : [x_7, x_i] = e$, $1 \leq j \leq 6$.

The graph $Gr(H)$ is presented in Figure 7(a), where next to each vertex we write the associated multiple relation. The edges of $Gr(H)$ are denoted by e_1, e_3 and e_5 and the vertices by v_1, v_2 and v_3 (recall that the index of e_i corresponds to the common generator x_i that participate in the two relations, that correspond to the vertices connected by e_i).

In order to construct the corresponding line arrangement $\mathcal{L}(H)$ we first extend the edges e_1, e_3 and e_5 to infinite straight lines ℓ_1, ℓ_3 and ℓ_5 . These lines intersect each other at the points p_1, p_2 and p_3 , which correspond to the vertices v_1, v_2 and v_3 (see Figure 7(b.I)). Next, look at the vertex v_1 , which is associated to the relation $R_{1,(3,2,1)}$. Since the lines ℓ_1 and ℓ_3 were already drawn, we only have to draw the line ℓ_2 , which correspond to the generator x_2 . We draw it such that it would pass through p_1 and intersect the other lines at nodes (see Figure 7(b.II)). Then we do the same for the vertices v_2 and v_3 : we first draw the line ℓ_4 (passing through p_2) and finally, the line ℓ_6 that pass through p_3 (see Figure 7(b.III) and 7(b.IV) respectively). We get a line arrangement with 6 lines, corresponding to the generators x_1, \dots, x_6 . As for the generator x_7 , we draw a generic line ℓ_7 which intersects the lines ℓ_1, \dots, ℓ_6 at nodes (see Figure 7(b.V)).

Remark 4.3. Let G be a conjugation-free group such that $Gr(G)$ has several components, that is $Gr(G) = \cup_{i=1}^v Gr(G)_i$. In order to build $\mathcal{L}(G)$ we just apply Definition 4.1 to every connected component $Gr(G)_i$ separately, when the addition of the lines corresponding to generators which do not participate in any multiple relation is done at the end, after we have drawn all the lines that are associated to all the components of $Gr(G)$.

4.2. Fundamental groups of complements of line arrangements. For a real or complex line arrangement \mathcal{L} , Fan [10] defined a graph $Gr_L(\mathcal{L})$ which is associated to the multiple points of \mathcal{L} . We give here its version for real arrangements (the general version is more delicate to explain and will be omitted): Given a real line arrangement $\mathcal{L} \subset \mathbb{C}^2$, the graph $Gr_L(\mathcal{L})$ lies on the real part of \mathcal{L} .

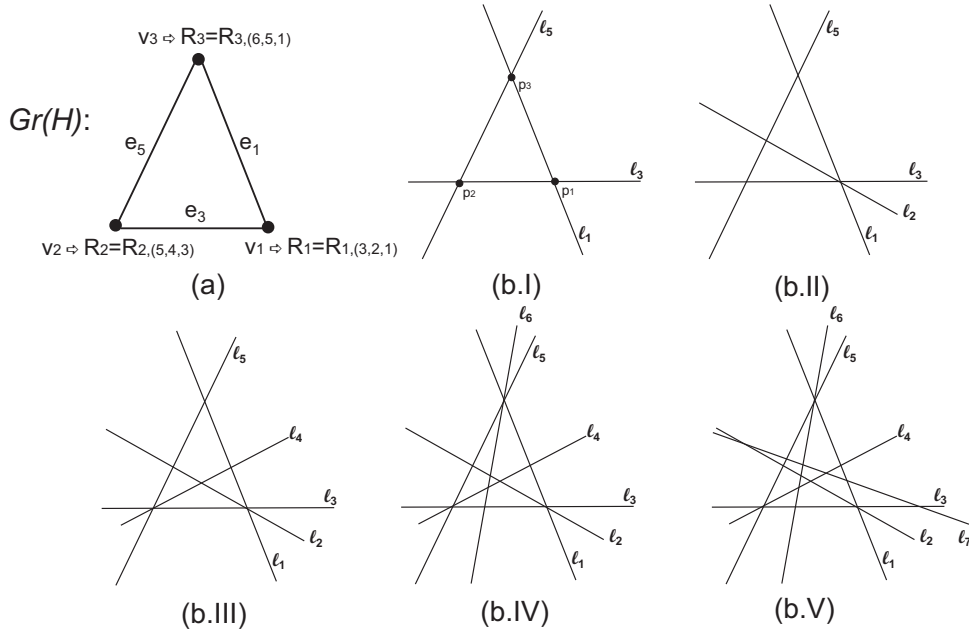


FIGURE 7. An example to a construction of a line arrangement from a conjugation-free group H and its cycle-separated graph $Gr(H)$.

Its vertices are the multiple points of \mathcal{L} and its edges are the segments between the multiple points on lines which have at least two multiple points. Note that if the arrangement consists of three multiple points on the same line, then $Gr_L(\mathcal{L})$ has three vertices on the same edge. If two such lines happen to intersect in a *simple* point (i.e. a point where exactly two lines are intersected), it is ignored (i.e. there is no corresponding vertex in the graph). See an example in Figure 8.

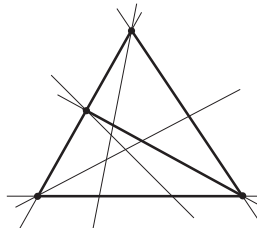


FIGURE 8. An example for the graph $Gr_L(\mathcal{L})$ associated to a line arrangement \mathcal{L} .

We have now the following obvious proposition:

Proposition 4.4. *Let G be a conjugation-free group with a planar graph $Gr(G)$. Let $\mathcal{L}(G)$ be the associated line arrangement. Then $Gr(G) = Gr_L(\mathcal{L}(G))$.*

Proof. Indeed, through every vertex of $Gr(G)$ we draw more than two lines, and thus every vertex corresponds to a multiple point on $\mathcal{L}(G)$, which in turn corresponds to a vertex (found in the same position) of $Gr_L(\mathcal{L}(G))$. The edges connecting these vertices in $Gr(G)$ are only extended to infinite straight lines in $\mathcal{L}(G)$, now connecting between multiple points, and hence considered again as edges in $Gr_L(G)$. \square

Remark 4.5. Given a complex line arrangement \mathcal{L} , it is known that $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is a free abelian group if and only if \mathcal{L} has only nodes as intersection points (see for example [2, Example 1.6(a)]).

Moreover, Fan [9, 10] proved that if the graph $Gr_L(\mathcal{L})$ has no cycles, i.e. $\beta(Gr_L(\mathcal{L})) = 0$, then $\pi_1(\mathbb{C}\mathbb{P}^2 - \mathcal{L})$, and therefore also $\pi_1(\mathbb{C}^2 - \mathcal{L})$, are isomorphic to a direct sum of a free abelian group and free groups. Eliyahu et al. [5] proved the inverse direction to Fan's result, i.e. if the fundamental group of the arrangement is a direct sum of a free abelian group and free groups, then the graph $Gr_L(\mathcal{L})$ has no cycles.

Recall that for computing the fundamental group of a complement of a curve C in \mathbb{C}^2 , one can use the Zariski-van Kampen theorem [23]. This theorem uses a generic projection $\pi : \mathbb{C}^2 \rightarrow \ell = \mathbb{C}^1$ (or a projection $\pi : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^1$ with a center O) to a generic line ℓ in order to induce in the fiber $\mathbb{C}_{p_0}^1 = \pi^{-1}(p_0)$ generators of the group $\pi_1(\mathbb{C}_{p_0}^1 - (\mathbb{C}_{p_0}^1 \cap C), p_0)$, where p_0 is a generic point on ℓ . These generators also generate $\pi_1(\mathbb{C}^2 - C, p_0)$.

Using these notations, we recall the notion of a *conjugation-free geometric presentation* for the fundamental group of the complement of line arrangements (see [3, 11]):

Definition 4.6. Let $G = \pi_1(\mathbb{C}^2 - \mathcal{L})$ be the fundamental group of the affine complement of a real line arrangement with n lines $\{\ell_1, \dots, \ell_n\}$. We say that G has a conjugation-free geometric presentation if G has a presentation with the following properties:

- (1) The generators $\{x_1, \dots, x_n\}$ are the meridians of the lines at $\mathbb{C}_{p_0}^1$.
- (2) The induced relations are of the following type:

$$x_{i_t} x_{i_{t-1}} \cdots x_{i_1} = x_{i_{t-1}} \cdots x_{i_1} x_{i_t} = \cdots = x_{i_1} x_{i_t} \cdots x_{i_2}$$

where $\{i_1, i_2, \dots, i_t\} \subseteq \{1, \dots, n\}$ is an increasing subsequence of indices. This relation is the only relation induced by an intersection point of multiplicity t , i.e. the intersection of the lines $\{\ell_{i_1}, \ell_{i_2}, \dots, \ell_{i_t}\}$. Note that if $t = 2$ we get the usual commutator and also that there are no other relations in G .

Note that we claim that with respect to particular choices of the line ℓ (the line to which we project the arrangement), the point p_0 (being the basepoint for the meridians in the fiber $\mathbb{C}_{p_0}^1$ and also for $\pi_1(\mathbb{C}^2 - \mathcal{L})$) and the projection point O , the conjugation-free property holds.

Definition 4.7. A line arrangement \mathcal{L} is called *conjugation-free* if $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

Proposition 4.8. If a line arrangement \mathcal{L} is conjugation-free then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is a conjugation-free group.

Proof. In order to check that $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is a conjugation-free group, we check the three requirements listed in Definition 2.1. Requirement (1) (that every relation is a conjugation-free relation, i.e. a cyclic-relation without conjugations) is fulfilled due to Property (2) of Definition 4.6. If ℓ_{j_1}, ℓ_{j_2} are two different lines in \mathcal{L} , then they meet only at one point p , whose multiplicity is $m(p)$. Therefore, the associated generators x_{j_1}, x_{j_2} to these lines participate in the cyclic relation R of length $m(p)$, induced from the point p , which is the only relation in which both of these generators participate. Thus requirement (2) of Definition 2.1 is fulfilled. As for requirement (3), assume that there are two relations $R_{p, (i_t, \dots, i_1)}, R_{p', (j_s, \dots, j_1)}$ of $\pi_1(\mathbb{C}^2 - \mathcal{L})$, induced from the singular points p, p' , such that:

$$|\{i_1, i_2, \dots, i_t\} \cap \{j_1, j_2, \dots, j_s\}| \geq 2.$$

Let $\{i', i''\} \subseteq \{i_1, i_2, \dots, i_t\}$ be two of the indices contained in this intersection. As every relation is induced from an intersection point, this means that the lines $\ell_{i'}$ and $\ell_{i''}$ intersect at two different points, which is impossible. \square

Remark 4.9. (1) Given any line arrangement \mathcal{L} , The Zariski-van Kampen theorem shows that an intersection point p of multiplicity t induces the following relation in $\pi_1(\mathbb{C}^2 - \mathcal{L})$:

$$x_{i_t}^{s_{p,t}} x_{i_{t-1}}^{s_{p,t-1}} \cdots x_{i_1}^{s_{p,1}} = x_{i_{t-1}}^{s_{p,t-1}} \cdots x_{i_1}^{s_{p,1}} x_{i_t}^{s_{p,t}} = \cdots = x_{i_1}^{s_{p,1}} x_{i_t}^{s_{p,t}} \cdots x_{i_2}^{s_{p,2}},$$

where the x_{i_j} , $1 \leq j \leq t$ are the meridians of the lines l_{i_j} and $s_{p,i} \in \langle x_1, \dots, x_n \rangle$. Therefore, for every line arrangement \mathcal{L} , the group $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is a cyclic-related group (the proof of that is along the same lines as the proof of Proposition 4.8).

(2) Note that if for every singular point p in the line arrangement \mathcal{L} , $s_{p,i} = e$ (in the induced relation) for every i , then \mathcal{L} is conjugation-free (and thus $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is a conjugation-free group).

(3) The Zariski-Lefschetz hyperplane section theorem (see [18]) states that $\pi_1(\mathbb{P}^N - S) \simeq \pi_1(H - (H \cap S))$, where S is a hypersurface and H is a generic 2-plane. When S is a hyperplane arrangement, $H \cap S$ is a line arrangement in \mathbb{P}^2 . Thus, one can investigate the topology of hyperplane arrangements via the fundamental groups $\pi_1(\mathbb{P}^2 - \mathcal{L})$ and $\pi_1(\mathbb{C}^2 - \mathcal{L})$, where \mathcal{L} is an arrangement of lines.

We can now state the opposite statement to Proposition 4.4:

Proposition 4.10. *Let \mathcal{L} be a conjugation-free real line arrangement, $G = \pi_1(\mathbb{C}^2 - \mathcal{L})$. Then $Gr_L(\mathcal{L}) \simeq Gr(G)$.*

Proof. Indeed, the vertices of $Gr_L(\mathcal{L})$ correspond to multiple points, which induce multiple conjugation-free relations in G , and thus correspond to vertices in $Gr(G)$. The edges of $Gr_L(\mathcal{L})$ correspond to lines connecting two multiple points, which correspond in G to generators participating in the relations (which correspond to these points), and hence these generators correspond to edges in $Gr(G)$ connecting the corresponding vertices there. \square

Note that the importance of the family of line arrangements whose fundamental group has a conjugation-free geometric presentation is that such a presentation of the fundamental group can be read directly from the arrangement (in particular, from its intersection lattice) without any additional computations, and thus depends on the combinatorics of the arrangement.

Denote by $\beta(\mathcal{L})$ the first Betti number of the graph $Gr_L(\mathcal{L})$. We recall the following proposition from [4] (whose complete proof is given at [11]):

Proposition 4.11. *Let \mathcal{L} be a real line arrangement with $\beta(\mathcal{L}) \leq 1$. Then \mathcal{L} is conjugation-free.*

The class of conjugation-free real line arrangement is in fact bigger than the class of line arrangements with $\beta(\mathcal{L}) \leq 1$. We cite the following definition from [11, Definition 6.4]:

Definition 4.12. Let G be a planar connected graph and denote by $\deg(v)$ the number of edges exiting from a vertex $v \in G$. The graph G is called a *conjugation-free graph* if:

- (1) $\beta(G) \leq 1$, or
- (2) Let $\{v_i\}_{i=1}^m$ be the set of vertices in G satisfying $\deg(v_i) \leq 2$. For each v_i , $1 \leq i \leq m$, denote by V_i the subset of G , composed of the vertex v_i and the edge(s) exiting from it. Let $X = X(G) \doteq \bigcup_{i=1}^m V_i$. Then G is a conjugation-free graph if $G - X$ is.

For example, the graph in Figure 6(a) is a conjugation-free graph with two cycles. In [11] the following result was proved:

Theorem 4.13. [11, Theorem 6.5] *Let \mathcal{L} be a real line arrangement. If $Gr_L(\mathcal{L})$ is a disjoint union of conjugation-free graphs, then \mathcal{L} is a conjugation-free line arrangement.*

Remark 4.14. By looking at the general form of a cycle-separated graph (see Remark 3.15), it is easy to see that a cycle-separated graph is a conjugation-free graph: since if G is a cycle-separated graph, $X = X(G)$ as in Definition 4.12, then $\beta(G - X) \leq 1$.

We therefore have the following connection between a conjugation-free group G and its associated line arrangement.

Proposition 4.15. *Let G be a conjugation-free group with a planar conjugation-free connected graph $Gr(G)$. Let $\mathcal{L}(G)$ be the associated line arrangement. Then $\pi_1(\mathbb{C}^2 - \mathcal{L}(G)) \simeq G$.*

Proof. Assume that all the edges of $Gr(G)$ are straight segments. We know that $Gr(G) \simeq Gr_L(\mathcal{L}(G))$ and thus $Gr_L(\mathcal{L}(G))$ is a conjugation-free graph. Hence, $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$ is a conjugation-free group (by Theorem 4.13) such that $Gr(G) \simeq Gr(\pi_1(\mathbb{C}^2 - \mathcal{L}(G)))$ (by Proposition 4.10), where the multiplicities of the isomorphic vertices (i.e. the length of the cyclic relations) are the same.

Let x_1, \dots, x_n be the generators of $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$, y_1, \dots, y_n the generators of G . We claim that the isomorphism $f_L : \pi_1(\mathbb{C}^2 - \mathcal{L}(G)) \rightarrow G$ is determined by the given isomorphism on the graphs $f_G : Gr_L(\mathcal{L}(G)) \rightarrow Gr(G)$. Explicitly, let $Gr_L(\mathcal{L}(G)) = (V_{\mathcal{L}}, E_{\mathcal{L}})$ and $Gr(G) = (V_G, E_G)$, where to each vertex and edge in both graphs, the data regarding the corresponding relations and generators is also included.

If $Gr_L(\mathcal{L}(G)) \simeq Gr(G)$ is empty, then G is a free abelian group (see Example 2.7(1)), isomorphic to \mathbb{Z}^n , and by Definition 4.1, $\mathcal{L}(G)$ is a generic line arrangement; by Remark 4.5, $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$ is also a free abelian group of the same rank, i.e. $G \simeq \pi_1(\mathbb{C}^2 - \mathcal{L}(G))$.

If $Gr_L(\mathcal{L}(G))$ consists of one vertex $v_{1,\mathcal{L}} = v_{1,\mathcal{L},(i_1, \dots, i_1)}$, let $f_G(v_{1,\mathcal{L}}) = v_{1,G} = v_{1,G,(j_1, \dots, j_1)}$ be the isomorphic vertex. In this case we define $f_L(x_{i_k}) = y_{j_k}$ for $1 \leq k \leq t$.

Assume thus that $Gr(G)$ has at least one edge. For every edge $e_{i,\mathcal{L}} \in E_{\mathcal{L}}$, let $f_G(e_{i,\mathcal{L}}) = e_{j,G} \in E_G$ be the isomorphic edge. Thus we define $f_L(x_i) \doteq y_j$. Consider now a vertex $v_{g,\mathcal{L}} = v_{g,\mathcal{L},(i_1, \dots, i_1)} \in V_{\mathcal{L}}$, and let $f_G(v_{g,\mathcal{L}}) = v_{h,G} = v_{h,G,(j_1, \dots, j_1)} \in V_G$ be the isomorphic vertex.

Every vertex corresponds to a multiple relation, and is connected by several edges to other vertices, such that these edges correspond to generators participating in this multiple relation. Let $\{t_m, \dots, t_1\} \subseteq \{1, \dots, t\}$ (resp. $\{s_m, \dots, s_1\} \subseteq \{1, \dots, t\}$) be a decreasing sequence of indices such that $x_{i_{t_k}}$ corresponds to an edge $e_{i_{t_k},\mathcal{L}}$ for every $1 \leq k \leq m$ (resp. $y_{j_{s_k}}$ corresponds to an edge $e_{j_{s_k},G}$). Note that by the above construction, $f_L(x_{i_{t_k}}) = y_{j_{s_k}}$ for $1 \leq k \leq m$. Let $t' = t_m$, $t'' = s_m$; therefore $f_L(x_{i_{t'}}) = y_{j_{t''}}$.

Hence, we define

$$(17) \quad f_L(x_{i_{t'-k}}) \doteq y_{j_{(t''-k) \bmod t}} \quad \text{for } 0 \leq k < t' \quad \text{and} \quad f_L(x_{i_{t'+k}}) \doteq y_{j_{(t''+k) \bmod t}} \quad \text{for } 1 \leq k \leq t - t'.$$

The above defined function is compatible with the already defined function on the generators $x_{i_{t_k}}$ as otherwise the graphs $Gr_L(\mathcal{L}(G)), Gr(G)$ would not be isomorphic, when considering the data attached to every vertex.

The defined function f_L so far was defined on the generators (both in G and in $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$) that participate in a multiple relation. If there are generators that do not participate in any multiple relation, then they commute with all the other generators (this is since every two generators participate in a mutual conjugation-free cyclic relation. If this relation is not multiple, then it is of length 2, hence a commutator). Assume that these generators in $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$ are $\{x_{w_1}, \dots, x_{w_u}\}$, and that these generators in G are $\{y_{w'_1}, \dots, y_{w'_u}\}$. Hence define $f_L(x_{w_i}) = y_{w'_i}$ for every $1 \leq i \leq u$.

We have defined f_L on every generator of $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$. We now have to check that f_L is indeed an isomorphism, that is, we have to check that the relations are preserved. Let us note that if n_i is the number of cyclic relation of length i in G , then it is also the number of cyclic relation of length i in $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$.

Obviously, all the commutator relations are transferred to commutator relations. Let $R_{g,\mathcal{L},(i_1, \dots, i_1)}$ be a multiple relation in $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$, $v_{g,\mathcal{L}} \in V_{\mathcal{L}}$ be the corresponding vertex, $f_G(v_{g,\mathcal{L}}) = v_{h,G} \in V_G$ the isomorphic vertex and $R_{h,G,(j_1, \dots, j_1)}$ the corresponding relation in G .

Remark 4.16. Assume that H is a group with cyclic conjugation-free relation R_t of length t . Let $\sigma = (1 \ 2 \ \dots \ t) \in \text{Sym}_t$, $z_1, \dots, z_t \in H$. Then it is easy to check that the cyclic relation of length t :

$$R_t : z_t z_{t-1} \cdots z_1 = \cdots = z_1 z_t \cdots z_2$$

is invariant under the action of σ on the indices, i.e., for every $0 \leq i < t$, if

$$R_t^i : z_{\sigma^i(t)} z_{\sigma^i(t-1)} \cdots z_{\sigma^i(1)} = \cdots = z_{\sigma^i(1)} z_{\sigma^i(t)} \cdots z_{\sigma^i(2)},$$

then $R_t = R_t^i$.

We have to show that $f_L(R_{g,\mathcal{L},(i_t,\dots,i_1)}) = R_{h,G,(j_t,\dots,j_1)}$. Examining the isomorphism (defined at Equation (17)) on the generators $\{x_{i_m}\}_{m=1}^t$, we see that there exists $0 \leq r < t$ such that $f_L(x_{i_m}) = y_{j_{\sigma^r(m)}}$, hence

$$f_L(R_{g,\mathcal{L},(i_t,\dots,i_1)}) = R_{h,G,(j_{\sigma^r(t)},\dots,j_{\sigma^r(1)})} \stackrel{\text{Rem. 4.16}}{=} R_{h,G,(j_t,\dots,j_1)}.$$

This proves that multiple relations are transferred to the corresponding multiple relations, and since the number of relations of any length is equal in G and in $\pi_1(\mathbb{C}^2 - \mathcal{L}(G))$, we are done. \square

Remark 4.17. Let G be a conjugation-free group such that $\beta(Gr(G)) = 0$. Then $\beta(Gr_L(\mathcal{L}(G))) = 0$ and $G \simeq \pi_1(\mathbb{C}^2 - \mathcal{L}(G))$. Hence, by Remark 4.5, G is isomorphic to a direct sum of a free abelian group and free groups.

4.3. The lower central series and line arrangements. Let G be a conjugation-free group with a conjugation-free graph and let $\mathcal{L}(G)$ be the associated line arrangement. We know that $G \simeq \pi_1(\mathbb{C}^2 - \mathcal{L}(G))$ by Proposition 4.15. Moreover, note that if n_i is the number of cyclic relation of length i in G , then it is the number of singular points of multiplicity i of $\mathcal{L}(G)$.

Given any line arrangement \mathcal{L} , denote $\phi_k(\mathcal{L}) \doteq \phi_k(\pi_1(\mathbb{C}^2 - \mathcal{L}))$. Falk [7, Proposition 3.8] proved that

$$\phi_3(\mathcal{L}) \geq \sum_{i \geq 3} n_i \omega_3(i-1),$$

where n_i is the number of singular points of multiplicity i (Moreover, it is proven in [7] that $\phi_k(\mathcal{L})$ are determined by the combinatorics of \mathcal{L}). Therefore,

$$\phi_3(G) = \phi_3(\mathcal{L}(G)) \geq \sum_{i \geq 3} n_i \omega_3(i-1).$$

By Proposition 3.9, we know that if $Gr(G)$ is a cycle-separated graph, then

$$\phi_3(\mathcal{L}(G)) = \phi_3(G) \leq \sum_{i \geq 3} n_i \omega_3(i-1).$$

Therefore, we can prove the following theorem:

Theorem 4.18. *Let G be a conjugation-free group whose graph is a cycle-separated graph. Then:*

$$\phi_k(G) = \sum_{i \geq 3} n_i \omega_k(i-1), \text{ for every } k \geq 2.$$

Proof. For $k = 2$ the equality was proved in Section 3.1.1 and it is clear from the discussion above that the equality holds for $k = 3$. Moreover, Papadima-Suciu [20, Theorem 2.4] proves that given a line arrangement \mathcal{L} such that $\phi_3(\mathcal{L}) = \sum_{i \geq 3} n_i \omega_3(i-1)$ then $\phi_k(\mathcal{L}) = \sum_{i \geq 3} n_i \omega_k(i-1)$ for every $k \geq 3$. \square

Definition 4.19. Given a line arrangement \mathcal{L} such that $\phi_k(\mathcal{L}) = \sum_{i \geq 3} n_i \omega_k(i-1)$ for every $k \geq 3$, Papadima-Suciu [20] calls these arrangements *decomposable*, since the associated graded Lie algebra

$$\bigoplus_{i \geq 2} \pi_1(\mathbb{C}^2 - \mathcal{L})_i / \pi_1(\mathbb{C}^2 - \mathcal{L})_{i+1}$$

behaves as if $\pi_1(\mathbb{C}^2 - \mathcal{L})$ were a direct product of free groups. Following this definition, we call a group G *decomposable* if $\bigoplus_{i \geq 2} G_i / G_{i+1}$ behaves as if G were a direct product of free groups, i.e. if $\phi_k(G) = \sum_{i \geq 3} n_i \omega_k(i-1)$, for every $k \geq 2$.

Remark 4.20. A group G can be decomposable even though it might be that G cannot be isomorphic to any finite product of free groups of finite rank. For example, let us look at the arrangement at Figure 7(b.IV), which is also called X_3 .

The graph of this arrangement is a cycle of length 3, and thus $G_{X_3} = \pi_1(\mathbb{C}^2 - X_3)$ is a conjugation-free group with a cycle-separated graph, i.e. it is decomposable. However, as it is shown, for example, in [20, Section 1], G_{X_3} cannot be isomorphic to any such product.

Remark 4.21. Given any line arrangement \mathcal{L} with n lines, the invariants $\phi_2(\mathcal{L}), \phi_3(\mathcal{L})$ can be computed via topological invariants.

(1) It is known that $\phi_2 = a_2$, where a_i is the number of minimal generators of degree i in the Orlik-Solomon ideal I (see [19, Definition 3.5]), or equivalently that $a_2 = \binom{n}{2} - b_2$, where b_2 is the second Betti number of $\mathbb{C}^2 - \mathcal{A}$ (see e.g. [6] or Equation (18)). Moreover, if G is a conjugation-free group with a cycle-separated graph, let $\mathcal{L}(G)$ be the associated line arrangement. Then by the discussion above $\phi_2(G) = \phi_2(\mathcal{L}(G))$.

(2) By [21, Corollary 3.6], $\phi_3 = b_3 + b_1 \binom{b_1}{2} - \binom{b_1}{3} + a_3$, where the b_i 's are the Betti numbers of the central 3-arrangement, which is the cone of our line arrangement in \mathbb{C}^2 . While the b_i 's can be read directly from the intersection lattice of \mathcal{L} , a_3 cannot.

Example 4.22. In [20, Section 7] a series of examples of hyperplane arrangements, consisting of graphic arrangements, was given, as an example of decomposable arrangements. Let us give a different example of decomposable arrangements which are not graphic arrangements.

Let A_k be the braid arrangement in \mathbb{C}^k ; i.e. $A_k = \{\prod_{1 \leq i < j \leq k} (x_i - x_j) = 0\}$. Let $A_k^0 \subset \mathbb{C}^2$ be the intersection of A_k with a generic plane of dimension 2. In order to find a decomposable line arrangement which is not a graphic arrangement, it is enough to find a conjugation-free line arrangement which is not a sub-arrangement of A_k^0 (as every graphic arrangement is a sub-arrangement of A_k). The graph $Gr_L(A_k^0)$ has only vertices associated to multiple points of multiplicity 3. This is since the intersection lattice of A_k is the partition lattice (see e.g. [19, Proposition 2.9]), and hence the codimension-two intersections have an intersection lattice equal to the rank-three truncation of the partition lattice. As can be easily checked, this truncation shows that the lines of A_k^0 intersect only at nodes or singular points with multiplicity 3.

Therefore, a real line arrangement \mathcal{L} , whose graph $Gr_L(\mathcal{L})$ is cycle-separated with some vertices associated to singular points whose multiplicity is greater or equal to 4, cannot be a graphic arrangement, though it is decomposable (as $Gr_L(\mathcal{L})$ is cycle-separated).

Note also that the arrangement X_2 (see Figure 9(a) and e.g. [22, Example 10.4]) is decomposable but does not have a conjugation-free graph (see Figure 9(b)).

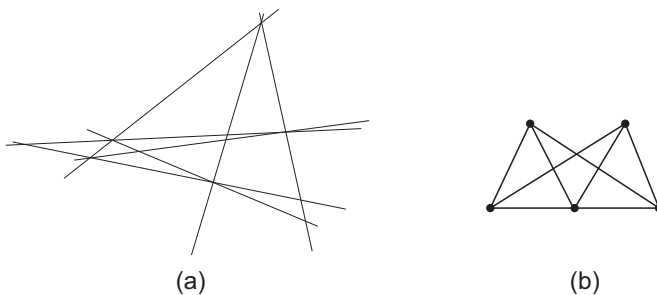


FIGURE 9. Part (a): The arrangement X_2 . Part (b): The graph $Gr_L(X_2)$ of the arrangement.

Remark 4.23. The LCS formula of Falk and Randell [8] expresses the ranks $\phi_k(\mathcal{L})$ of a fiber-type arrangement \mathcal{L} in terms of the Poincaré polynomial $P(M, t) = \sum b_i t^i$ of the complement $M = \mathbb{C}^2 - \mathcal{L}$:

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(\mathcal{L})} = P(M, -t).$$

Note that there are arrangements for which the LCS formula holds but the arrangements are not decomposable (e.g. the deleted B_3 -arrangement, see [22, Example 10.6]), and there are arrangements for which the LCS formula does not hold but the arrangements are decomposable (e.g. the X_3 arrangement). Other LCS formulas were also obtained, for other types of arrangements; for example, for graphic arrangements [15].

Moreover, for decomposable line arrangements \mathcal{L} with n lines there is another LCS formula, found by Papadima-Suciu [20, Corollary 2.6]:

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(\mathcal{L})} = (1 - t)^{n - b_2(\mathcal{L})} \cdot \prod_{p \in \text{Sing}(\mathcal{L})} (1 - (m(p) - 1)t),$$

where $m(p)$ is the multiplicity of the singular point p and

$$(18) \quad b_2(\mathcal{L}) = \sum_{p \in \text{Sing}(\mathcal{L})} (m(p) - 1).$$

Note that this equation can also be stated for a conjugation-free group G with a cycle-separated graph, generated by n generators:

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(G)} = (1 - t)^{n - b_2(G)} \cdot \prod_{R \in \text{Rel}(G)} (1 - (\text{len}(R) - 1)t),$$

where $\text{Rel}(G)$ is the set of relations of G , $\text{len}(R)$ is the length of the cyclic relation R and $b_2(G) = \sum_{R \in \text{Rel}(G)} (\text{len}(R) - 1)$.

Based on computations made for other arrangements, whose graph is not cycle-separated, we conclude this paper with the following conjecture:

Conjecture 4.24. *Let \mathcal{L} (resp. G) be a conjugation-free line arrangement (resp. group). Then \mathcal{L} (resp. G) is decomposable.*

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MICHAEL FRIEDMAN, INSTITUT FOURIER, 100 RUE DES MATHS, BP 74, 38402 ST MARTIN D'HÉRES CEDEX, FRANCE

E-mail address: Michael.Friedman@ujf-grenoble.fr