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# A splitting algorithm for coupled system of primal–dual monotone inclusions

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## Abstract

We propose a splitting algorithm for solving a coupled system of primal-dual monotone inclusions in real Hilbert spaces. The weak convergence of the algorithm proposed is proved. Applications to minimization problems is demonstrated.

**Keywords:** coupled system, monotone inclusion, monotone operator, operator splitting, cocoercivity, forward-backward algorithm, composite operator, duality, primal-dual algorithm

**Mathematics Subject Classifications (2010)** 47H05, 49M29, 49M27, 90C25

## 1 Introduction

Various problems in applied mathematics such as evolution inclusions [2], partial differential equations [1, 30, 32], mechanics [31], variational inequalities [10, 29], Nash equilibria [4], and optimization problems [6, 16, 22, 27, 37, 43], reduce to solving monotone inclusions. The simplest monotone inclusion is to find a zero point of a maximally monotone operator  $A$  acting on a real Hilbert space  $\mathcal{H}$ . This problem can be solved efficiently by the proximal point algorithm when the resolvent of  $A$  is easy to implement numerically [41] (see [11, 13, 14, 25, 34, 35, 38] in the context of variable metric). This problem was then extended to the problem of finding a zero of the sum of a maximally monotone operator  $A$  and a cocoercive operator  $B$ . In this case, we can use the forward-backward splitting algorithm [2, 18, 32, 43] (see [26] in the context of variable metric).

When  $A$  has a structure, for examples, mixtures of composite, Lipschitzian or cocoercive, and parallel-sum type monotone operators as in [23, 26, 44, 45], existing purely primal splitting methods do not offer satisfactory options to solve the problem due to the appearance of the composite components and hence alternative primal-dual strategies must be explored. Very recently, these

frameworks are unified into a system of monotone inclusions with mixtures of composite, Lipschitzian, and parallel-sum type monotone operators in [19]. In this paper, we address to the numerical solutions of coupled system of primal-dual inclusions in real Hilbert spaces.

**Problem 1.1** Let  $m, s$  be strictly positive integers. For every  $i \in \{1, \dots, m\}$ , let  $(\mathcal{H}_i, \langle \cdot | \cdot \rangle)$  be a real Hilbert space, let  $z_i \in \mathcal{H}_i$ , let  $A_i: \mathcal{H}_i \rightarrow 2^{\mathcal{H}_i}$  be maximally monotone, let  $C_i: \mathcal{H}_1 \times \dots \times \mathcal{H}_m \rightarrow \mathcal{H}_i$  be such that

$$\begin{aligned} & \left( \exists \nu_0 \in ]0, +\infty[ \right) \left( \forall (x_i)_{1 \leq i \leq m} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m \right) \left( \forall (y_i)_{1 \leq i \leq m} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m \right) \\ & \sum_{i=1}^m \langle x_i - y_i | C_i(x_1, \dots, x_m) - C_i(y_1, \dots, y_m) \rangle \geq \nu_0 \sum_{i=1}^m \|C_i(x_1, \dots, x_m) - C_i(y_1, \dots, y_m)\|^2. \end{aligned} \quad (1.1)$$

For every  $k \in \{1, \dots, s\}$ , let  $(\mathcal{G}_k, \langle \cdot | \cdot \rangle)$  be a real Hilbert space, let  $r_k \in \mathcal{G}_k$ , let  $B_k: \mathcal{G}_k \rightarrow 2^{\mathcal{G}_k}$  be maximally monotone, let  $S_k: \mathcal{G}_1 \times \dots \times \mathcal{G}_s \rightarrow \mathcal{G}_k$  be such that

$$\begin{aligned} & \left( \exists \mu_0 \in ]0, +\infty[ \right) \left( \forall (v_k)_{1 \leq k \leq s} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s \right) \left( \forall (w_k)_{1 \leq k \leq s} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s \right) \\ & \sum_{k=1}^s \langle v_k - w_k | S_k(v_1, \dots, v_s) - S_k(w_1, \dots, w_s) \rangle \geq \mu_0 \sum_{k=1}^s \|S_k(v_1, \dots, v_s) - S_k(w_1, \dots, w_s)\|^2. \end{aligned} \quad (1.2)$$

For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , let  $L_{k,i}: \mathcal{H}_i \rightarrow \mathcal{G}_k$  be a bounded linear operator. The problem is to solve the following system of primal-dual inclusions:

find  $\bar{x}_1 \in \mathcal{H}_1, \dots, \bar{x}_m \in \mathcal{H}_m$  and  $\bar{v}_1 \in \mathcal{G}_1, \dots, \bar{v}_s \in \mathcal{G}_s$  such that

$$\begin{cases} z_1 - \sum_{k=1}^s L_{k,1}^* \bar{v}_k \in A_1 \bar{x}_1 + C_1(\bar{x}_1, \dots, \bar{x}_m) \\ \vdots \\ z_m - \sum_{k=1}^s L_{k,m}^* \bar{v}_k \in A_m \bar{x}_m + C_m(\bar{x}_1, \dots, \bar{x}_m), \end{cases} \quad \text{and} \quad \begin{cases} \sum_{i=1}^m L_{1,i} \bar{x}_i - r_1 \in B_1 \bar{v}_1 + S_1(\bar{v}_1, \dots, \bar{v}_s) \\ \vdots \\ \sum_{i=1}^m L_{s,i} \bar{x}_i - r_s \in B_s \bar{v}_s + S_s(\bar{v}_1, \dots, \bar{v}_s). \end{cases} \quad (1.3)$$

We denote by  $\Omega$  the set of solutions to (1.3).

In the case when every linear operators  $((L_{k,i})_{1 \leq k \leq s})_{1 \leq i \leq m}$  are zeros, we can use the algorithm in [2] to solve the inclusions in the left hand side and in the right hand side of (1.3) separately. Let us note that the non-linear coupling terms  $(C_i)_{1 \leq i \leq m}$  and  $(S_k)_{1 \leq k \leq s}$  are introduced in [2] and they are cocoercive operators which often play a central role; see for instance [2, 10, 17, 18, 29, 30, 31, 32, 42, 43, 46]. Let us add that the general algorithm in [19] can solve Problem 1.1 for the case when  $C_i$  and  $S_k$  are univariate, monotone and Lipschitzian. Furthermore, the primal-dual algorithm in [26, Section 6] can solve Problem 1.1 for the case when  $m = 1$  and each  $S_k$  are univariate, monotone and Lipschitzian. To sum up, the recent general frameworks can solve special cases of the above problem and no existing algorithm can solve it in the general case.

In the present paper, we propose a primal-dual splitting algorithm for solving Problem 1.1 in Section 3. We recall some notations and background on the monotone operator theory in Section 2. In Section 4, we provide application to coupled system of monotone inclusions in duality. Section 5 is devoted to applications to minimization problems. In the last section, an application to multi-dictionary signal representation is presented.

## 2 Notation and background, and technical results

### 2.1 Notation and background

Throughout,  $\mathcal{H}$ ,  $\mathcal{G}$ , and  $(\mathcal{G}_i)_{1 \leq i \leq m}$  are real Hilbert spaces. Their scalar products and associated norms are respectively denoted by  $\langle \cdot | \cdot \rangle$  and  $\|\cdot\|$ . We denote by  $\mathcal{B}(\mathcal{H}, \mathcal{G})$  the space of bounded linear operators from  $\mathcal{H}$  to  $\mathcal{G}$ . The adjoint of  $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$  is denoted by  $L^*$ . We set  $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ . The symbols  $\rightharpoonup$  and  $\rightarrow$  denote respectively weak and strong convergence, and  $\text{Id}$  denotes the identity operator, we denote by  $\ell_+^1(\mathbb{N})$  the set of summable sequences in  $[0, +\infty[$  and by  $\ell^2(\mathbb{K})$  ( $\emptyset \neq \mathbb{K} \subset \mathbb{N}$ ) the set of square summable sequences, indexed by  $\mathbb{K}$ , in  $\mathbb{R}$ .

Let  $M_1$  and  $M_2$  be self-adjoint operators in  $\mathcal{B}(\mathcal{H})$ , we write  $M_1 \succcurlyeq M_2$  if and only if  $(\forall x \in \mathcal{H}) \langle M_1 x | x \rangle \geq \langle M_2 x | x \rangle$ . Let  $\alpha \in ]0, +\infty[$ . We set

$$\mathcal{P}_\alpha(\mathcal{H}) = \{M \in \mathcal{B}(\mathcal{H}) \mid M^* = M \text{ and } M \succcurlyeq \alpha \text{Id}\}. \quad (2.1)$$

The square root of  $M$  in  $\mathcal{P}_\alpha(\mathcal{H})$  is denoted by  $\sqrt{M}$ . Moreover, for every  $M \in \mathcal{P}_\alpha(\mathcal{H})$ , we define respectively a scalar product and a norm by

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x | y \rangle_M = \langle Mx | y \rangle \quad \text{and} \quad \|x\|_M = \sqrt{\langle Mx | x \rangle}, \quad (2.2)$$

and, for any  $L \in \mathcal{B}(\mathcal{H})$ , we define

$$\|L\|_M = \sup_{\|x\|_M \leq 1} \|Lx\|_M. \quad (2.3)$$

Let  $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be a set-valued operator. The domain of  $A$  is  $\text{dom } A = \{x \in \mathcal{H} \mid Ax \neq \emptyset\}$ , and the graph of  $A$  is  $\text{gra } A = \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}$ . The set of zeros of  $A$  is  $\text{zer } A = \{x \in \mathcal{H} \mid 0 \in Ax\}$ , and the range of  $A$  is  $\text{ran } A = \{u \in \mathcal{H} \mid (\exists x \in \mathcal{H}) u \in Ax\}$ . The inverse of  $A$  is  $A^{-1}: \mathcal{H} \mapsto 2^{\mathcal{H}}: u \mapsto \{x \in \mathcal{H} \mid u \in Ax\}$ , and the resolvent of  $A$  is

$$J_A = (\text{Id} + A)^{-1}. \quad (2.4)$$

Moreover,  $A$  is monotone if

$$(\forall (x, y) \in \mathcal{H} \times \mathcal{H})(\forall (u, v) \in Ax \times Ay) \quad \langle x - y | u - v \rangle \geq 0, \quad (2.5)$$

and maximally monotone if it is monotone and there exists no monotone operator  $B: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  such that  $\text{gra } A \subset \text{gra } B$  and  $A \neq B$ . A single-valued operator  $B: \mathcal{H} \rightarrow \mathcal{H}$  is  $\beta$ -cocoercive, for some  $\beta \in ]0, +\infty[$ , if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y | Bx - By \rangle \geq \beta \|Bx - By\|^2. \quad (2.6)$$

The parallel sum of  $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  and  $B: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  is

$$A \square B = (A^{-1} + B^{-1})^{-1}. \quad (2.7)$$

Let  $\Gamma_0(\mathcal{H})$  be the class of proper lower semicontinuous convex functions from  $\mathcal{H}$  to  $]-\infty, +\infty]$ . For any  $U \in \mathcal{P}_\alpha(\mathcal{H})$  and  $f \in \Gamma_0(\mathcal{H})$ , we define

$$J_{U^{-1}\partial f} = \text{prox}_f^U: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto \underset{y \in \mathcal{H}}{\text{argmin}} \left( f(y) + \frac{1}{2} \|x - y\|_U^2 \right), \quad (2.8)$$

and

$$J_{\partial f} = \text{prox}_f: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto \underset{y \in \mathcal{H}}{\text{argmin}} \left( f(y) + \frac{1}{2} \|x - y\|^2 \right), \quad (2.9)$$

and the conjugate function of  $f$  is

$$f^*: a \mapsto \sup_{x \in \mathcal{H}} (\langle a | x \rangle - f(x)). \quad (2.10)$$

Note that,

$$(\forall f \in \Gamma_0(\mathcal{H}))(x \in \mathcal{H})(y \in \mathcal{H}) \quad y \in \partial f(x) \Leftrightarrow x \in \partial f^*(y), \quad (2.11)$$

or equivalently,

$$(\forall f \in \Gamma_0(\mathcal{H})) \quad (\partial f)^{-1} = \partial f^*. \quad (2.12)$$

The infimal convolution of the two functions  $f$  and  $g$  from  $\mathcal{H}$  to  $] -\infty, +\infty ]$  is

$$f \square g: x \mapsto \inf_{y \in \mathcal{H}} (f(y) + g(x - y)). \quad (2.13)$$

The indicator function of a nonempty closed convex set  $C$  is denoted by  $\iota_C$ , its dual function is the support function  $\sigma_C$ , the distance function of  $C$  is denoted by  $d_C$ . Finally, the strong relative interior of a subset  $C$  of  $\mathcal{H}$  is the set of points  $x \in C$  such that the cone generated by  $-x + C$  is a closed vector subspace of  $\mathcal{H}$ .

## 2.2 Technical results

We recall some results on monotone operators.

**Definition 2.1** [2, Definition 2.3] *An operator  $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  is demiregular at  $x \in \text{dom } A$  if, for every sequence  $((x_n, u_n))_{n \in \mathbb{N}}$  in  $\text{gra } A$  and every  $u \in Ax$  such that  $x_n \rightarrow x$  and  $u_n \rightarrow u$ , we have  $x_n \rightarrow x$ .*

**Lemma 2.2** [2, Proposition 2.4] *Let  $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be monotone and suppose that  $x \in \text{dom } A$ . Then  $A$  is demiregular at  $x$  in each of the following cases.*

- (i)  *$A$  is uniformly monotone at  $x$ , i.e., there exists an increasing function  $\phi: [0, +\infty[ \rightarrow [0, +\infty]$  that vanishes only at 0 such that  $(\forall u \in Ax)(\forall (y, v) \in \text{gra } A) \langle x - y | u - v \rangle \geq \phi(\|x - y\|)$ .*
- (ii)  *$A$  is strongly monotone, i.e., there exists  $\alpha \in ]0, +\infty[$  such that  $A - \alpha \text{Id}$  is monotone.*
- (iii)  *$J_A$  is compact, i.e., for every bounded set  $C \subset \mathcal{H}$ , the closure of  $J_A(C)$  is compact. In particular,  $\text{dom } A$  is boundedly relatively compact, i.e., the intersection of its closure with every closed ball is compact.*
- (iv)  *$A: \mathcal{H} \rightarrow \mathcal{H}$  is single-valued with a single-valued continuous inverse.*
- (v)  *$A$  is single-valued on  $\text{dom } A$  and  $\text{Id} - A$  is demicompact, i.e., for every bounded sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\text{dom } A$  such that  $(Ax_n)_{n \in \mathbb{N}}$  converges strongly,  $(x_n)_{n \in \mathbb{N}}$  admits a strong cluster point.*

(vi)  $A = \partial f$ , where  $f \in \Gamma_0(\mathcal{H})$  is uniformly convex at  $x$ , i.e., there exists an increasing function  $\phi: [0, +\infty[ \rightarrow [0, +\infty]$  that vanishes only at 0 such that  $(\forall \alpha \in ]0, 1[)(\forall y \in \text{dom } f)$   $f(\alpha x + (1 - \alpha)y) + \alpha(1 - \alpha)\phi(\|x - y\|) \leq \alpha f(x) + (1 - \alpha)f(y)$ .

(vii)  $A = \partial f$ , where  $f \in \Gamma_0(\mathcal{H})$  and, for every  $\xi \in \mathbb{R}$ ,  $\{x \in \mathcal{H} \mid f(x) \leq \xi\}$  is boundedly compact.

**Lemma 2.3** [26, Lemma 3.7] *Let  $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be maximally monotone, let  $\alpha \in ]0, +\infty[$ , let  $U \in \mathcal{P}_\alpha(\mathcal{H})$ , and let  $\mathcal{G}$  be the real Hilbert space obtained by endowing  $\mathcal{H}$  with the scalar product  $(x, y) \mapsto \langle x \mid y \rangle_{U^{-1}} = \langle x \mid U^{-1}y \rangle$ . Then the following hold.*

(i)  $UA: \mathcal{G} \rightarrow 2^{\mathcal{G}}$  is maximally monotone.

(ii)  $J_{UA}: \mathcal{G} \rightarrow \mathcal{G}$  is 1-cocoercive, i.e., firmly nonexpansive, hence nonexpansive.

(iii)  $J_{UA} = (U^{-1} + A)^{-1} \circ U^{-1}$ .

**Lemma 2.4** *Let  $\alpha$  and  $\beta$  be strictly positive reals, let  $B: \mathcal{H} \rightarrow \mathcal{H}$  be  $\beta$ -cocoercive, let  $U \in \mathcal{P}_\alpha(\mathcal{H})$  such that  $\|U^{-1}\| < 2\beta$  and set  $P = \text{Id} - U^{-1}B$ . Then,*

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \|Px - Py\|_U^2 \leq \|x - y\|_U^2 - (2\beta - \|U^{-1}\|)\|Bx - By\|^2. \quad (2.14)$$

Hence,  $P$  is nonexpansive with respect to the norm  $\|\cdot\|_U$ .

*Proof.* Let  $x \in \mathcal{H}$  and  $y \in \mathcal{H}$ . Then using the cocoercivity of  $B$ , we have

$$\begin{aligned} \|Px - Py\|_U^2 &= \|x - y\|_U^2 - 2\langle x - y \mid Bx - By \rangle + \|U^{-1}(Bx - By)\|_U^2 \\ &\leq \|x - y\|_U^2 - 2\beta\|Bx - By\|^2 + \langle Bx - By \mid U^{-1}(Bx - By) \rangle \\ &\leq \|x - y\|_U^2 - (2\beta - \|U^{-1}\|)\|Bx - By\|^2, \end{aligned} \quad (2.15)$$

which proves (2.14).  $\square$

**Theorem 2.5** [26, Theorem 4.1] *Let  $\mathcal{K}$  be a real Hilbert space with scalar product  $\langle \cdot \mid \cdot \rangle$  and the associated norm  $\|\cdot\|$ . Let  $A: \mathcal{K} \rightarrow 2^{\mathcal{K}}$  be maximally monotone, let  $\alpha \in ]0, +\infty[$ , let  $\beta \in ]0, +\infty[$ , let  $B: \mathcal{K} \rightarrow \mathcal{K}$  be  $\beta$ -cocoercive, let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , and let  $(U_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{K})$  such that*

$$\mu = \sup_{n \in \mathbb{N}} \|U_n\| < +\infty \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad (1 + \eta_n)U_{n+1} \succcurlyeq U_n. \quad (2.16)$$

Let  $\varepsilon \in ]0, \min\{1, 2\beta/(\mu + 1)\}]$ , let  $(\lambda_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, 1]$ , let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, (2\beta - \varepsilon)/\mu]$ , let  $x_0 \in \mathcal{K}$ , and let  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  be absolutely summable sequences in  $\mathcal{K}$ . Suppose that

$$Z = \text{zer}(A + B) \neq \emptyset, \quad (2.17)$$

and set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n = x_n - \gamma_n U_n(Bx_n + b_n) \\ x_{n+1} = x_n + \lambda_n (J_{\gamma_n U_n A}(y_n) + a_n - x_n). \end{cases} \quad (2.18)$$

Then the following hold for some  $\bar{x} \in Z$ .

(i)  $x_n \rightarrow \bar{x}$ .

(ii)  $\sum_{n \in \mathbb{N}} \|Bx_n - B\bar{x}\|^2 < +\infty$ .

(iii) Suppose that at every point in  $Z$ ,  $A$  or  $B$  is demiregular, then  $x_n \rightarrow \bar{x}$ .

### 3 Algorithm and convergence

We propose the following algorithm for solving Problem 1.1.

**Algorithm 3.1** Let  $\alpha \in ]0, +\infty[$  and, for every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , let  $(U_{i,n})_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{H}_i)$  and let  $(V_{k,n})_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{G}_k)$ . Set  $\beta = \min\{\nu_0, \mu_0\}$ , and let  $\varepsilon \in ]0, \min\{1, \beta\}[$ , let  $(\lambda_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, 1]$ . Let  $(x_{i,0})_{1 \leq i \leq m} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m$  and  $(v_{k,0})_{1 \leq k \leq s} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s$ . Set

$$\begin{array}{l}
 \text{For } n = 0, 1, \dots \\
 \left[ \begin{array}{l}
 \text{For } i = 1, \dots, m \\
 \left[ \begin{array}{l}
 p_{i,n} = J_{U_{i,n}A_i} \left( x_{i,n} - U_{i,n} \left( \sum_{k=1}^s L_{k,i}^* v_{k,n} + C_{i,n}(x_{1,n}, \dots, x_{m,n}) + c_{i,n} - z_i \right) \right) + a_{i,n} \\
 y_{i,n} = 2p_{i,n} - x_{i,n} \\
 x_{i,n+1} = x_{i,n} + \lambda_{i,n}(p_{i,n} - x_{i,n})
 \end{array} \right. \\
 \text{For } k = 1, \dots, s \\
 \left[ \begin{array}{l}
 q_{k,n} = J_{V_{k,n}B_k} \left( v_{k,n} + V_{k,n} \left( \sum_{i=1}^m L_{k,i} y_{i,n} - S_{k,n}(v_{1,n}, \dots, v_{s,n}) - d_{k,n} - r_k \right) \right) + b_{k,n} \\
 v_{k,n+1} = v_{k,n} + \lambda_{m+k,n}(q_{k,n} - v_{k,n}),
 \end{array} \right.
 \end{array} \right.
 \end{array} \quad (3.1)$$

where, for every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , the following conditions hold

- (i)  $(\forall n \in \mathbb{N}) U_{i,n+1} \succeq U_{i,n}$  and  $V_{k,n+1} \succeq V_{k,n}$ , and

$$\mu = \sup_{n \in \mathbb{N}} \{ \|U_{1,n}\|, \dots, \|U_{m,n}\|, \|V_{1,n}\|, \dots, \|V_{s,n}\| \} < +\infty. \quad (3.2)$$

- (ii)  $(C_{i,n})_{n \in \mathbb{N}}$  are operators from  $\mathcal{H}_1 \times \dots \times \mathcal{H}_m$  to  $\mathcal{H}_i$  such that

- (a)  $(C_{i,n} - C_i)_{n \in \mathbb{N}}$  are Lipschitz continuous with respective constants  $(\kappa_{i,n})_{n \in \mathbb{N}} \in ]0, +\infty[$  satisfying  $\sum_{n \in \mathbb{N}} \kappa_{i,n} < +\infty$ .

- (b) There exists  $\bar{s} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m$  not depending  $i$  such that  $(\forall n \in \mathbb{N}) C_{i,n} \bar{s} = C_i \bar{s}$ .

- (iii)  $(S_{k,n})_{n \in \mathbb{N}}$  are operators from  $\mathcal{G}_1 \times \dots \times \mathcal{G}_s$  to  $\mathcal{G}_k$  such that

- (a)  $(S_{k,n} - S_k)_{n \in \mathbb{N}}$  are Lipschitz continuous with respective constants  $(\eta_{k,n})_{n \in \mathbb{N}} \in ]0, +\infty[$  satisfying  $\sum_{n \in \mathbb{N}} \eta_{k,n} < +\infty$ .

- (b) There exists  $\bar{w} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s$  not depending  $k$  such that  $(\forall n \in \mathbb{N}) S_{k,n} \bar{w} = S_k \bar{w}$ .

- (iv)  $(a_{i,n})_{n \in \mathbb{N}}$  and  $(c_{i,n})_{n \in \mathbb{N}}$  are absolutely summable sequences in  $\mathcal{H}_i$ .

- (v)  $(b_{k,n})_{n \in \mathbb{N}}$  and  $(d_{k,n})_{n \in \mathbb{N}}$  are absolutely summable sequences in  $\mathcal{G}_k$ .

- (vi)  $(\lambda_{i,n})_{n \in \mathbb{N}}$  and  $(\lambda_{m+k,n})_{n \in \mathbb{N}}$  are in  $]0, 1]$  such that

$$\sum_{n \in \mathbb{N}} \left( |\lambda_{i,n} - \lambda_n| + |\lambda_{m+k,n} - \lambda_n| \right) < +\infty. \quad (3.3)$$

**Remark 3.2** Here are some remarks

- (i) Our algorithm has basically a structure of the variable metric forward-backward splitting since the multi-valued operators are used individually in the backward steps via their resolvents, the single-valued operators are used individually in the forward steps via their values.
- (ii) The algorithm allows the metric to vary over the course of the iterations. Even when restricted to the constant metric case (which is the case where  $(U_{i,n})_{1 \leq i \leq m}$  and  $(V_{k,n})_{1 \leq k \leq s}$  are identity operators), the algorithm is new.
- (iii) Condition (i) is used in [26, 45] while conditions (ii), (iii) and (vi) are used in [2], and conditions (iv) and (v) which quantify the tolerance allowed in the inexact implementation of the resolvents and the approximations of single-valued are widely used in the literature.
- (iv) Algorithm 3.1 is an extension of [26, Corollary 6.2] where  $m = 1$  and every  $n \in \mathbb{N}$ :  $C_{1,n} = C$  and for every  $k \in \{1, \dots, s\}$ ,  $S_{k,n} = D_k^{-1}$  are restricted to univariate and cocoercive, and  $B_k$  is replaced by  $B_k^{-1}$ , and for every  $j \in \{1, \dots, m + s\}$ ,  $\lambda_{j,n} = \lambda_n$ .

The main result of the paper can be now stated.

**Theorem 3.3** *Suppose in Problem 1.1 that  $\Omega \neq \emptyset$  and there exists  $L_{k_0, i_0} \neq 0$  for some  $i_0 \in \{1, \dots, m\}$  and  $k_0 \in \{1, \dots, s\}$ . For every  $n \in \mathbb{N}$ , set*

$$\delta_n = \left( \sqrt{\sum_{i=1}^m \sum_{k=1}^s \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\|^2} \right)^{-1} - 1, \quad (3.4)$$

and suppose that

$$\zeta_n = \frac{\delta_n}{(1 + \delta_n) \max_{1 \leq i \leq m, 1 \leq k \leq s} \{\|U_{i,n}\|, \|V_{k,n}\|\}} \geq \frac{1}{2\beta - \varepsilon}. \quad (3.5)$$

For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , let  $(x_{i,n})_{n \in \mathbb{N}}$  and  $(v_{k,n})_{n \in \mathbb{N}}$  be sequences generated by Algorithm 3.1. Then the following hold for some  $(\bar{x}_1, \dots, \bar{x}_m, \bar{v}_1, \dots, \bar{v}_s) \in \Omega$ .

- (i)  $(\forall i \in \{1, \dots, m\}) \ x_{i,n} \rightarrow \bar{x}_i$  and  $(\forall k \in \{1, \dots, s\}) \ v_{k,n} \rightarrow \bar{v}_k$ .
- (ii) Suppose that the operator  $(x_i)_{1 \leq i \leq m} \mapsto (C_j(x_i)_{1 \leq i \leq m})_{1 \leq j \leq m}$  is demiregular (see Lemma 2.2 for special cases) at  $(\bar{x}_1, \dots, \bar{x}_m)$ , then  $(\forall i \in \{1, \dots, m\}) \ x_{i,n} \rightarrow \bar{x}_i$ .
- (iii) Suppose that the operator  $(v_k)_{1 \leq k \leq s} \mapsto (S_j(v_k)_{1 \leq k \leq s})_{1 \leq j \leq s}$  is demiregular (see Lemma 2.2 for special cases) at  $(\bar{v}_1, \dots, \bar{v}_s)$ , then  $(\forall k \in \{1, \dots, s\}) \ v_{k,n} \rightarrow \bar{v}_k$ .
- (iv) Suppose that there exists  $j \in \{1, \dots, m\}$  and an operator  $C: \mathcal{H}_j \rightarrow \mathcal{H}_j$  such that  $(\forall (x_i)_{1 \leq i \leq m} \in (\mathcal{H}_i)_{1 \leq i \leq m}) \ C_j(x_1, \dots, x_m) = Cx_j$  and  $C$  is demiregular (see Lemma 2.2 for special cases) at  $\bar{x}_j$ , then  $x_{j,n} \rightarrow \bar{x}_j$ .
- (v) Suppose that there exists  $j \in \{1, \dots, s\}$  and an operator  $D: \mathcal{G}_j \rightarrow \mathcal{G}_j$  such that  $(\forall (v_k)_{1 \leq k \leq s} \in (\mathcal{G}_k)_{1 \leq k \leq s}) \ S_j(v_1, \dots, v_s) = Dv_j$  and  $D$  is demiregular (see Lemma 2.2 for special cases) at  $\bar{v}_j$ , then  $v_{j,n} \rightarrow \bar{v}_j$ .



*Proof.* Let us introduce the Hilbert direct sums

$$\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_m, \quad \mathcal{G} = \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_s, \quad \text{and} \quad \mathcal{K} = \mathcal{H} \oplus \mathcal{G}. \quad (3.6)$$

We denote by  $\mathbf{x} = (x_i)_{1 \leq i \leq m}$ ,  $\mathbf{y} = (y_i)_{1 \leq i \leq m}$  and  $\mathbf{v} = (v_k)_{1 \leq k \leq s}$ ,  $\mathbf{w} = (w_k)_{1 \leq k \leq s}$  the generic elements in  $\mathcal{H}$  and  $\mathcal{G}$ , respectively. The generic elements in  $\mathcal{K}$  will be in the form  $\mathbf{p} = (\mathbf{x}, \mathbf{v})$ . The scalar product and the norm of  $\mathcal{H}$  are respectively defined by

$$\langle \cdot | \cdot \rangle : (\mathbf{x}, \mathbf{y}) \mapsto \sum_{i=1}^m \langle x_i | y_i \rangle, \quad (3.7)$$

and

$$\| \cdot \| : \mathbf{x} \mapsto \sqrt{\langle \mathbf{x} | \mathbf{x} \rangle}. \quad (3.8)$$

The scalar product and the norm of  $\mathcal{G}$  are defined by the same fashion as those of  $\mathcal{H}$ ,

$$\langle \cdot | \cdot \rangle : (\mathbf{v}, \mathbf{w}) \mapsto \sum_{k=1}^s \langle v_k | w_k \rangle, \quad (3.9)$$

and

$$\| \cdot \| : \mathbf{v} \mapsto \sqrt{\langle \mathbf{v} | \mathbf{v} \rangle}. \quad (3.10)$$

We next define the scalar product and the norm of  $\mathcal{K}$  are respectively defined by

$$\langle \langle \cdot | \cdot \rangle \rangle : ((\mathbf{x}, \mathbf{v}), (\mathbf{y}, \mathbf{w})) \mapsto \sum_{i=1}^m \langle x_i | y_i \rangle + \sum_{k=1}^s \langle v_k | w_k \rangle \quad (3.11)$$

and

$$\| \cdot \| : (\mathbf{x}, \mathbf{v}) \mapsto \sqrt{\langle \langle \mathbf{x}, \mathbf{v} | \mathbf{x}, \mathbf{v} \rangle \rangle}. \quad (3.12)$$

Set

$$\left\{ \begin{array}{l} \mathbf{A}: \mathcal{H} \rightarrow 2^{\mathcal{H}}: \mathbf{x} \mapsto \times_{i=1}^m A_i x_i \\ \mathbf{C}: \mathcal{H} \rightarrow \mathcal{H}: \mathbf{x} \mapsto (C_i \mathbf{x})_{1 \leq i \leq m} \\ \mathbf{L}: \mathcal{H} \rightarrow \mathcal{G}: \mathbf{x} \mapsto (\sum_{i=1}^m L_{k,i} x_i)_{1 \leq k \leq s} \\ \mathbf{z} = (z_1, \dots, z_m), \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \mathbf{B}: \mathcal{G} \rightarrow 2^{\mathcal{G}}: \mathbf{v} \mapsto \times_{k=1}^s B_k v_k \\ \mathbf{D}: \mathcal{G} \rightarrow \mathcal{G}: \mathbf{v} \mapsto (S_k \mathbf{v})_{1 \leq k \leq s} \\ \mathbf{r} = (r_1, \dots, r_s), \end{array} \right. \quad (3.13)$$

and

$$(\forall n \in \mathbb{N}) \quad \mathbf{C}_n: \mathcal{H} \rightarrow \mathcal{H}: \mathbf{x} \mapsto (C_{i,n} \mathbf{x})_{1 \leq i \leq m} \quad \text{and} \quad \mathbf{D}_n: \mathcal{G} \rightarrow \mathcal{G}: \mathbf{v} \mapsto (S_{k,n} \mathbf{v})_{1 \leq k \leq s}. \quad (3.14)$$

Then, it follows from (1.1) that

$$(\forall \mathbf{x} \in \mathcal{H})(\forall \mathbf{y} \in \mathcal{H}) \quad \langle \mathbf{x} - \mathbf{y} | \mathbf{C}\mathbf{x} - \mathbf{C}\mathbf{y} \rangle \geq \nu_0 \|\mathbf{C}\mathbf{x} - \mathbf{C}\mathbf{y}\|^2, \quad (3.15)$$

from (1.2) that

$$(\forall \mathbf{v} \in \mathcal{G})(\forall \mathbf{w} \in \mathcal{G}) \quad \langle \mathbf{v} - \mathbf{w} | \mathbf{D}\mathbf{v} - \mathbf{D}\mathbf{w} \rangle \geq \mu_0 \|\mathbf{D}\mathbf{v} - \mathbf{D}\mathbf{w}\|^2, \quad (3.16)$$

which shows that  $\mathbf{C}$  and  $\mathbf{D}$  are respectively  $\nu_0$ -cocoercive and  $\mu_0$ -cocoercive and hence they are maximally monotone [10, Example 20.28]. Moreover, it follows from [10, Proposition 20.23] that  $\mathbf{A}$  and  $\mathbf{B}$  are maximally monotone. Furthermore,

$$\mathbf{L}^*: \mathcal{G} \rightarrow \mathcal{H}: \mathbf{v} \mapsto \left( \sum_{k=1}^s L_{k,i}^* v_k \right)_{1 \leq i \leq m}. \quad (3.17)$$

Then, using (3.13), we can rewrite the system of monotone inclusions (1.3) as a monotone inclusion in  $\mathcal{K}$ ,

$$\text{find } (\bar{\mathbf{x}}, \bar{\mathbf{v}}) \in \mathcal{K} \text{ such that } \mathbf{z} - \mathbf{L}^* \bar{\mathbf{v}} \in (\mathbf{A} + \mathbf{C}) \bar{\mathbf{x}} \quad \text{and} \quad \mathbf{L} \bar{\mathbf{x}} - \mathbf{r} \in (\mathbf{B} + \mathbf{D}) \bar{\mathbf{v}}. \quad (3.18)$$

Set

$$\begin{cases} \mathbf{M}: \mathcal{K} \rightarrow 2^{\mathcal{K}}: (\mathbf{x}, \mathbf{v}) \mapsto (-\mathbf{z} + \mathbf{A}\mathbf{x}, \mathbf{r} + \mathbf{B}\mathbf{v}) \\ \mathbf{S}: \mathcal{K} \rightarrow \mathcal{K}: (\mathbf{x}, \mathbf{v}) \mapsto (\mathbf{L}^* \mathbf{v}, -\mathbf{L}\mathbf{x}) \\ \mathbf{Q}: \mathcal{K} \rightarrow \mathcal{K}: (\mathbf{x}, \mathbf{v}) \mapsto (\mathbf{C}\mathbf{x}, \mathbf{D}\mathbf{v}), \end{cases} \quad (3.19)$$

and

$$(\forall n \in \mathbb{N}) \begin{cases} \mathbf{Q}_n: \mathcal{K} \rightarrow \mathcal{K}: (\mathbf{x}, \mathbf{v}) \mapsto (\mathbf{C}_n \mathbf{x}, \mathbf{D}_n \mathbf{v}) \\ \mathbf{\Lambda}_n: \mathcal{K} \rightarrow \mathcal{K}: (\mathbf{x}, \mathbf{v}) \mapsto ((\lambda_{i,n} x_i)_{1 \leq i \leq m}, (\lambda_{m+k,n} v_k)_{1 \leq k \leq s}) \\ \mathbf{U}_n: \mathcal{K} \rightarrow \mathcal{K}: (\mathbf{x}, \mathbf{v}) \mapsto ((U_{i,n} x_i)_{1 \leq i \leq m}, (V_{k,n} v_k)_{1 \leq k \leq s}) \\ \mathbf{V}_n: \mathcal{K} \rightarrow \mathcal{K}: (\mathbf{x}, \mathbf{v}) \mapsto \mathbf{U}_n^{-1}(\mathbf{x}, \mathbf{v}) - (\mathbf{L}^* \mathbf{v}, \mathbf{L}\mathbf{x}). \end{cases} \quad (3.20)$$

Then  $\mathbf{M}, \mathbf{S}$  are maximally monotone operators and (3.15), (3.16) implies that  $\mathbf{Q}$  is  $\beta$ -cocoercive and hence it is maximally monotone [10, Example 20.28]. Therefore,  $\mathbf{M} + \mathbf{S} + \mathbf{Q}$  is maximally monotone [10, Corollary 24.4]. Furthermore, the problem (3.18) is reduced to find a zero point of  $\mathbf{M} + \mathbf{S} + \mathbf{Q}$ . Note that  $\Omega \neq \emptyset$  implies that

$$\text{zer}(\mathbf{M} + \mathbf{S} + \mathbf{Q}) \neq \emptyset \quad (3.21)$$

Moreover, we also have

$$(\forall n \in \mathbb{N}) \quad \|\mathbf{\Lambda}_n\|_{\mathbf{V}_n} = \max_{1 \leq j \leq m+s} \lambda_{j,n} \leq 1 \quad \text{and} \quad \|\text{Id} - \mathbf{\Lambda}_n\|_{\mathbf{V}_n} = 1 - \min_{1 \leq j \leq m+s} \lambda_{j,n} \leq 1. \quad (3.22)$$

Hence,

$$\|\mathbf{\Lambda}_n\|_{\mathbf{V}_n} + \|\text{Id} - \mathbf{\Lambda}_n\|_{\mathbf{V}_n} = 1 + \max_{1 \leq j \leq m+s} (\lambda_{j,n} - \lambda_n) - \min_{1 \leq j \leq m+s} (\lambda_{j,n} - \lambda_n) \leq 1 + \tau_n, \quad (3.23)$$

where

$$(\forall n \in \mathbb{N}) \quad \tau_n = 2 \max_{1 \leq j \leq m+s} |\lambda_{j,n} - \lambda_n|. \quad (3.24)$$

We derive from the condition (vi) in Algorithm 3.1 that

$$\sum_{n \in \mathbb{N}} \tau_n \leq 2 \sum_{j=1}^{m+s} \sum_{n \in \mathbb{N}} |\lambda_{j,n} - \lambda_n| < +\infty. \quad (3.25)$$

We next derive from the condition (i) in Algorithm 3.1 that

$$\mu = \sup_{n \in \mathbb{N}} \|\mathbf{U}_n\| < +\infty, \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \mathbf{U}_{n+1} \succeq \mathbf{U}_n \in \mathcal{P}_\alpha(\mathcal{K}), \quad (3.26)$$

and it follows from (3.12) and [25, Lemma 2.1(ii)] that

$$\begin{aligned}
(\forall n \in \mathbb{N})(\forall \mathbf{p} = (\mathbf{x}, \mathbf{v}) \in \mathcal{K}) \quad \|\mathbf{p}\|_{\mathcal{U}_n^{-1}}^2 &= \sum_{i=1}^m \|x_i\|_{U_{i,n}^{-1}}^2 + \sum_{k=1}^s \|v_k\|_{V_{k,n}^{-1}}^2 \\
&\geq \sum_{i=1}^m \|x_i\|^2 \|U_{i,n}^{-1}\| + \sum_{k=1}^s \|v_k\|^2 \|V_{k,n}^{-1}\| \\
&\geq \|\mathbf{p}\|^2 \min_{1 \leq i \leq m, 1 \leq k \leq s} \{\|U_{i,n}\|^{-1}, \|V_{k,n}\|^{-1}\}. \tag{3.27}
\end{aligned}$$

Note that  $(\mathbf{V}_n)_{n \in \mathbb{N}}$  are self-adjoint, let us check that  $(\mathbf{V}_n)_{n \in \mathbb{N}}$  are strongly monotone. To this end, let us introduce

$$(\forall n \in \mathbb{N}) \begin{cases} \mathbf{T}_n: \mathcal{H} \rightarrow \mathcal{G}: \mathbf{x} \mapsto \left( \sum_{i=1}^m \sqrt{V_{k,n}} L_{k,i} x_i \right)_{1 \leq k \leq s} \\ \mathbf{R}_n: \mathcal{G} \rightarrow \mathcal{G}: \mathbf{v} \mapsto \left( \sqrt{V_{1,n}}^{-1} v_1, \dots, \sqrt{V_{s,n}}^{-1} v_s \right). \end{cases} \tag{3.28}$$

Then, by using Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
(\forall n \in \mathbb{N})(\mathbf{x} \in \mathcal{H}) \quad \|\mathbf{T}_n \mathbf{x}\|^2 &= \sum_{k=1}^s \left\| \sum_{i=1}^m \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \sqrt{U_{i,n}}^{-1} x_i \right\|^2 \\
&\leq \sum_{k=1}^s \left( \sum_{i=1}^m \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\| \left\| \sqrt{U_{i,n}}^{-1} x_i \right\| \right)^2 \\
&\leq \sum_{k=1}^s \left( \sum_{i=1}^m \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\|^2 \right) \left( \sum_{i=1}^m \left\| \sqrt{U_{i,n}}^{-1} x_i \right\|^2 \right) \\
&= \left( \sum_{i=1}^m \|x_i\|_{U_{i,n}^{-1}}^2 \right) \sum_{k=1}^s \sum_{i=1}^m \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\|^2 \\
&= \beta_n \sum_{i=1}^m \|x_i\|_{U_{i,n}^{-1}}^2, \tag{3.29}
\end{aligned}$$

where we set

$$(\forall n \in \mathbb{N}) \quad \beta_n = \sum_{k=1}^s \sum_{i=1}^m \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\|^2, \tag{3.30}$$

which together with (3.4) imply that

$$(\forall n \in \mathbb{N}) \quad (1 + \delta_n) \beta_n = \frac{1}{1 + \delta_n}. \tag{3.31}$$

Moreover,

$$\begin{aligned}
(\forall n \in \mathbb{N})(\mathbf{v} \in \mathcal{G}) \quad \|\mathbf{R}_n \mathbf{v}\|^2 &= \sum_{k=1}^s \left\| \sqrt{V_{k,n}}^{-1} v_k \right\|^2 \\
&= \sum_{k=1}^s \|v_k\|_{V_{k,n}^{-1}}^2. \tag{3.32}
\end{aligned}$$

Therefore, for every  $\mathbf{p} = (\mathbf{x}, \mathbf{v}) \in \mathcal{K}$ , and every  $n \in \mathbb{N}$ , it follows from (3.20), (3.28), (3.31), (3.32) and (3.27), (3.5) that

$$\begin{aligned}
\langle \langle \mathbf{p} \mid \mathbf{V}_n \mathbf{p} \rangle \rangle &= \|\mathbf{p}\|_{\mathbf{U}_n^{-1}}^2 - 2 \langle \mathbf{L} \mathbf{x} \mid \mathbf{v} \rangle \\
&= \|\mathbf{p}\|_{\mathbf{U}_n^{-1}}^2 - 2 \sum_{k=1}^s \left\langle \sum_{i=1}^m \sqrt{V_{k,n}} L_{k,i} x_i \mid \sqrt{V_{k,n}^{-1}} v_k \right\rangle \\
&= \|\mathbf{p}\|_{\mathbf{U}_n^{-1}}^2 - 2 \left\langle \sqrt{(1+\delta_n)\beta_n}^{-1} \mathbf{T}_n \mathbf{x} \mid \sqrt{(1+\delta_n)\beta_n} \mathbf{R}_n \mathbf{v} \right\rangle \\
&\geq \|\mathbf{p}\|_{\mathbf{U}_n^{-1}}^2 - \left( \frac{\|\mathbf{T}_n \mathbf{x}\|^2}{(1+\delta_n)\beta_n} + (1+\delta_n)\beta_n \|\mathbf{R}_n \mathbf{v}\|^2 \right) \\
&\geq \|\mathbf{p}\|_{\mathbf{U}_n^{-1}}^2 - \left( \frac{\sum_{i=1}^m \|x_i\|_{U_{i,n}^{-1}}^2}{(1+\delta_n)} + (1+\delta_n)\beta_n \sum_{k=1}^s \|v_k\|_{V_{k,n}^{-1}}^2 \right) \\
&= \frac{\delta_n}{1+\delta_n} \left( \sum_{i=1}^m \|x_i\|_{U_{i,n}^{-1}}^2 + \sum_{k=1}^s \|v_k\|_{V_{k,n}^{-1}}^2 \right) \\
&\geq \zeta_n \|\mathbf{p}\|^2.
\end{aligned} \tag{3.33}$$

In turn,  $(\mathbf{V}_n)_{n \in \mathbb{N}}$  are invertible, by [25, Lemma 2.1(iii)] and (3.5),

$$(\forall n \in \mathbb{N}) \quad \|\mathbf{V}_n^{-1}\| \leq \frac{1}{\zeta_n} \leq 2\beta - \varepsilon, \tag{3.34}$$

and by [25, Lemma 2.1(i)], (3.26),  $(\forall n \in \mathbb{N}) (\mathbf{U}_{n+1} \succcurlyeq \mathbf{U}_n \Rightarrow \mathbf{U}_n^{-1} \succcurlyeq \mathbf{U}_{n+1}^{-1} \Rightarrow \mathbf{V}_n \succcurlyeq \mathbf{V}_{n+1} \Rightarrow \mathbf{V}_{n+1}^{-1} \succcurlyeq \mathbf{V}_n^{-1})$ . Furthermore, we derive from [25, Lemma 2.1(ii)] that

$$(\forall \mathbf{p} \in \mathcal{K}) \quad \langle \langle \mathbf{V}_n^{-1} \mathbf{p} \mid \mathbf{p} \rangle \rangle \geq \|\mathbf{V}_n\|^{-1} \|\mathbf{p}\|^2 \geq \frac{1}{\rho} \|\mathbf{p}\|^2, \quad \text{where } \rho = \alpha^{-1} + \|\mathbf{S}\|. \tag{3.35}$$

Altogether,

$$\sup_{n \in \mathbb{N}} \|\mathbf{V}_n^{-1}\| \leq 2\beta - \varepsilon \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \mathbf{V}_{n+1}^{-1} \succcurlyeq \mathbf{V}_n^{-1} \in \mathcal{P}_{1/\rho}(\mathcal{K}). \tag{3.36}$$

Moreover, using [25, Lemma 2.1(i)(ii)] and (3.36), we obtain

$$(\forall (\mathbf{z}_n)_{n \in \mathbb{N}} \in \mathcal{K}^{\mathbb{N}}) \quad \sum_{n \in \mathbb{N}} \|\mathbf{z}_n\| < +\infty \Leftrightarrow \sum_{n \in \mathbb{N}} \|\mathbf{z}_n\|_{\mathbf{V}_n^{-1}} < +\infty \tag{3.37}$$

and

$$(\forall (\mathbf{z}_n)_{n \in \mathbb{N}} \in \mathcal{K}^{\mathbb{N}}) \quad \sum_{n \in \mathbb{N}} \|\mathbf{z}_n\| < +\infty \Leftrightarrow \sum_{n \in \mathbb{N}} \|\mathbf{z}_n\|_{\mathbf{V}_n} < +\infty, \tag{3.38}$$

and

$$(\forall \mathbf{p} \in \mathcal{K}) \quad \sup_{n \in \mathbb{N}} \|\mathbf{p}\|_{\mathbf{V}_n} < +\infty. \tag{3.39}$$

Now we can reformulate the algorithm (3.1) as iterations in the space  $\mathcal{K}$ . We first observe that (3.1) is equivalent to

$$\begin{array}{l} \text{For } n = 0, 1 \dots \\ \left[ \begin{array}{l} \text{For } i = 1, \dots, m \\ \left[ \begin{array}{l} U_{i,n}^{-1}(x_{i,n} - p_{i,n}) - \sum_{k=1}^s L_{k,i}^* v_{k,n} - C_{i,n}(x_{1,n}, \dots, x_{m,n}) \in \\ \quad -z_i + A_i(p_{i,n} - a_{i,n}) + c_{i,n} - U_{i,n}^{-1} a_{i,n} \\ x_{i,n+1} = x_{i,n} + \lambda_{i,n}(p_{i,n} - x_{i,n}) \end{array} \right. \\ \text{For } k = 1, \dots, s \\ \left[ \begin{array}{l} V_{k,n}^{-1}(v_{k,n} - q_{k,n}) - \sum_{i=1}^m L_{k,i}(x_{i,n} - p_{i,n}) - S_{k,n}(v_{1,n}, \dots, v_{s,n}) \in \\ \quad r_k + B_k(q_{k,n} - b_{k,n}) - \sum_{i=1}^m L_{k,i} p_{i,n} + d_{k,n} - V_{k,n}^{-1} b_{k,n} \\ v_{k,n+1} = v_{k,n} + \lambda_{m+k,n}(q_{k,n} - v_{k,n}). \end{array} \right. \end{array} \right. \end{array} \quad (3.40)$$

Set

$$\begin{cases} \mathbf{p}_n = (x_{1,n}, \dots, x_{m,n}, v_{1,n}, \dots, v_{s,n}) \\ \mathbf{y}_n = (p_{1,n}, \dots, p_{m,n}, q_{1,n}, \dots, q_{s,n}) \\ \mathbf{a}_n = (a_{1,n}, \dots, a_{m,n}, b_{1,n}, \dots, b_{s,n}) \end{cases} \quad \begin{cases} \mathbf{c}_n = (c_{1,n}, \dots, c_{m,n}, d_{1,n}, \dots, d_{s,n}) \\ \mathbf{d}_n = (U_{1,n}^{-1} a_{1,n}, \dots, U_{m,n}^{-1} a_{m,n}, V_{1,n}^{-1} b_{1,n}, \dots, V_{s,n}^{-1} b_{s,n}) \\ \mathbf{b}_n = (\mathbf{S} + \mathbf{V}_n) \mathbf{a}_n + \mathbf{c}_n - \mathbf{d}_n. \end{cases}$$

Then, using the same arguments as in [44, Eqs. (3.22)–(3.35)], using (3.19), (3.20), (3.40) yields

$$(\forall n \in \mathbb{N}) \quad \left[ \begin{array}{l} \mathbf{V}_n(\mathbf{p}_n - \mathbf{y}_n) - \mathbf{Q}_n \mathbf{p}_n \in (\mathbf{M} + \mathbf{S})(\mathbf{y}_n - \mathbf{a}_n) + \mathbf{S} \mathbf{a}_n + \mathbf{c}_n - \mathbf{d}_n \\ \mathbf{p}_{n+1} = \mathbf{p}_n + \mathbf{\Lambda}_n(\mathbf{y}_n - \mathbf{p}_n). \end{array} \right. \quad (3.41)$$

We have

$$\begin{aligned} & (\forall n \in \mathbb{N}) \quad \mathbf{V}_n(\mathbf{p}_n - \mathbf{y}_n) - \mathbf{Q}_n \mathbf{p}_n \in (\mathbf{M} + \mathbf{S})(\mathbf{y}_n - \mathbf{a}_n) + \mathbf{S} \mathbf{a}_n + \mathbf{c}_n - \mathbf{d}_n \\ \Leftrightarrow & (\forall n \in \mathbb{N}) \quad (\mathbf{V}_n - \mathbf{Q}_n) \mathbf{p}_n \in (\mathbf{M} + \mathbf{S} + \mathbf{V}_n)(\mathbf{y}_n - \mathbf{a}_n) + (\mathbf{S} + \mathbf{V}_n) \mathbf{a}_n + \mathbf{c}_n - \mathbf{d}_n \\ \Leftrightarrow & (\forall n \in \mathbb{N}) \quad \mathbf{y}_n = (\mathbf{M} + \mathbf{S} + \mathbf{V}_n)^{-1} \left( (\mathbf{V}_n - \mathbf{Q}_n) \mathbf{p}_n - (\mathbf{S} + \mathbf{V}_n) \mathbf{a}_n - \mathbf{c}_n + \mathbf{d}_n \right) + \mathbf{a}_n \\ \Leftrightarrow & (\forall n \in \mathbb{N}) \quad \mathbf{y}_n = \left( \mathbf{Id} + \mathbf{V}_n^{-1}(\mathbf{M} + \mathbf{S}) \right)^{-1} \left( (\mathbf{Id} - \mathbf{V}_n^{-1} \mathbf{Q}_n) \mathbf{p}_n - \mathbf{V}_n^{-1} \mathbf{b}_n \right) + \mathbf{a}_n \\ \Leftrightarrow & (\forall n \in \mathbb{N}) \quad \mathbf{y}_n = J_{\mathbf{V}_n^{-1}(\mathbf{M} + \mathbf{S})} \left( (\mathbf{Id} - \mathbf{V}_n^{-1} \mathbf{Q}_n) \mathbf{p}_n - \mathbf{V}_n^{-1} \mathbf{b}_n \right) + \mathbf{a}_n. \end{aligned} \quad (3.42)$$

Therefore, (3.41) becomes

$$(\forall n \in \mathbb{N}) \quad \mathbf{p}_{n+1} = \mathbf{p}_n + \mathbf{\Lambda}_n \left( J_{\mathbf{V}_n^{-1}(\mathbf{M} + \mathbf{S})} \left( \mathbf{p}_n - \mathbf{V}_n^{-1}(\mathbf{Q}_n \mathbf{p}_n + \mathbf{b}_n) \right) + \mathbf{a}_n - \mathbf{p}_n \right). \quad (3.43)$$

By setting

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \tilde{\mathbf{M}}: \mathcal{K} \rightarrow 2^{\mathcal{K}}: (\mathbf{x}, \mathbf{v}) \mapsto \mathbf{M}(\mathbf{x}, \mathbf{v}) + \mathbf{S}(\mathbf{x}, \mathbf{v}), \\ \tilde{\mathbf{P}}_n = \mathbf{Id} - \mathbf{V}_n^{-1} \mathbf{Q}_n \quad \text{and} \quad \mathbf{P}_n = \mathbf{Id} - \mathbf{V}_n^{-1} \mathbf{Q}, \\ \tilde{\mathbf{E}}_n = \mathbf{Q}_n - \mathbf{Q} \quad \text{and} \quad \tilde{\mathbf{Q}}_n = \mathbf{V}_n^{-1} \mathbf{E}_n, \\ \mathbf{e}_{1,n} = \tilde{\mathbf{Q}}_n \mathbf{p}_n + \mathbf{V}_n^{-1} \mathbf{b}_n, \\ \mathbf{e}_n = \mathbf{a}_n + \frac{1}{\lambda_n} (\lambda_n \mathbf{Id} - \mathbf{\Lambda}_n) \left( \mathbf{p}_n - J_{\mathbf{V}_n^{-1} \tilde{\mathbf{M}}}(\mathbf{P}_n \mathbf{p}_n - \mathbf{e}_{1,n}) - \mathbf{a}_n \right), \end{cases} \quad (3.44)$$

we have

$$\begin{aligned}
(3.43) \Leftrightarrow (\forall n \in \mathbb{N}) \quad \mathbf{p}_{n+1} &= \mathbf{p}_n + \mathbf{\Lambda}_n \left( J_{\mathbf{V}_n^{-1} \widetilde{\mathbf{M}}}(\widetilde{\mathbf{P}}_n \mathbf{p}_n - \mathbf{V}_n^{-1} \mathbf{b}_n) + \mathbf{a}_n - \mathbf{p}_n \right) \\
&= (\text{Id} - \mathbf{\Lambda}_n) \mathbf{p}_n + \mathbf{\Lambda}_n J_{\mathbf{V}_n^{-1} \widetilde{\mathbf{M}}}(\widetilde{\mathbf{P}}_n \mathbf{p}_n - \mathbf{V}_n^{-1} \mathbf{b}_n) + \mathbf{\Lambda}_n \mathbf{a}_n \\
&= (\text{Id} - \mathbf{\Lambda}_n) \mathbf{p}_n + \mathbf{\Lambda}_n J_{\mathbf{V}_n^{-1} \widetilde{\mathbf{M}}}(\mathbf{P}_n \mathbf{p}_n - \mathbf{e}_{1,n}) + \mathbf{\Lambda}_n \mathbf{a}_n \tag{3.45} \\
&= (1 - \lambda_n) \mathbf{p}_n + \lambda_n \left( J_{\mathbf{V}_n^{-1} \widetilde{\mathbf{M}}}(\mathbf{P}_n \mathbf{p}_n - \mathbf{e}_{1,n}) + \mathbf{e}_n \right). \tag{3.46}
\end{aligned}$$

Algorithm (3.46) is a special instance of the variable metric forward-backward splitting (2.18) with

$$(\forall n \in \mathbb{N}) \quad \gamma_n = 1 \in \left[ \varepsilon, (2\beta - \varepsilon) / (\sup_{n \in \mathbb{N}} \|\mathbf{V}_n^{-1}\|) \right] \quad (\text{see (3.36)}). \tag{3.47}$$

Moreover, since  $\widetilde{\mathbf{M}}$  is maximally monotone,  $\mathbf{Q}$  is  $\beta$ -cocoercive, and  $(\forall n \in \mathbb{N}) \lambda_n \in [\varepsilon, 1]$ , since (3.36) and (3.21) respectively show that (2.16) and (2.17) are satisfied. In view of Theorem 2.5, it is sufficient to prove that  $(\mathbf{e}_{1,n})_{n \in \mathbb{N}}$  and  $(\mathbf{e}_n)_{n \in \mathbb{N}}$  are absolutely summable in  $\mathcal{K}$ , i.e, we prove that

$$\sum_{n \in \mathbb{N}} \|\|\|\mathbf{e}_{1,n}\|\|\| < +\infty, \tag{3.48}$$

and

$$\sum_{n \in \mathbb{N}} \|\|\|\mathbf{e}_n\|\|\| < +\infty. \tag{3.49}$$

For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , since  $(a_{i,n})_{n \in \mathbb{N}}$ ,  $(c_{i,n})_{n \in \mathbb{N}}$  and  $(b_{k,n})_{n \in \mathbb{N}}$  and  $(d_{k,n})_{n \in \mathbb{N}}$  are absolutely summable, we have

$$\sum_{n \in \mathbb{N}} \|\|\|\mathbf{a}_n\|\|\| \leq \sum_{n \in \mathbb{N}} \left( \sum_{i=1}^m \|a_{i,n}\| + \sum_{k=1}^s \|b_{k,n}\| \right) < +\infty, \tag{3.50}$$

and

$$\sum_{n \in \mathbb{N}} \|\|\|\mathbf{c}_n\|\|\| \leq \sum_{n \in \mathbb{N}} \left( \sum_{i=1}^m \|c_{i,n}\| + \sum_{k=1}^s \|d_{k,n}\| \right) < +\infty. \tag{3.51}$$

Moreover, for every  $n \in \mathbb{N}$ ,  $\mathbf{U}_n \in \mathcal{P}_\alpha(\mathcal{K})$ , it follows from [25, Lemma 2.1(iii)] that  $\|\mathbf{U}_n^{-1}\| \leq \alpha^{-1}$ . Hence,

$$\sum_{n \in \mathbb{N}} \|\|\|\mathbf{d}_n\|\|\| \leq \alpha^{-1} \sum_{n \in \mathbb{N}} \|\|\|\mathbf{a}_n\|\|\| < +\infty. \tag{3.52}$$

and

$$\sum_{n \in \mathbb{N}} \|\|\|\mathbf{b}_n\|\|\| \leq \rho \sum_{n \in \mathbb{N}} \|\|\|\mathbf{a}_n\|\|\| + \sum_{n \in \mathbb{N}} (\|\|\|\mathbf{c}_n\|\|\| + \|\|\|\mathbf{d}_n\|\|\|) < +\infty. \tag{3.53}$$

Therefore,  $(\mathbf{a}_n)_{n \in \mathbb{N}}$ ,  $(\mathbf{b}_n)_{n \in \mathbb{N}}$ ,  $(\mathbf{c}_n)_{n \in \mathbb{N}}$  and  $(\mathbf{d}_n)_{n \in \mathbb{N}}$  are absolutely summable in  $\mathcal{K}$ . Next it follows from the conditions (ii), (iii) in Algorithm 3.1 and (3.36), (3.33), (3.5) that, for every  $\mathbf{p} = (\mathbf{x}, \mathbf{v}) \in \mathcal{K}$

and  $\mathbf{q} = (\mathbf{y}, \mathbf{w}) \in \mathcal{K}$ ,

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad & \|\tilde{\mathbf{Q}}_n \mathbf{p} - \tilde{\mathbf{Q}}_n \mathbf{q}\|_{\mathbf{V}_n}^2 = \langle \langle \tilde{\mathbf{Q}}_n \mathbf{p} - \tilde{\mathbf{Q}}_n \mathbf{q} \mid \mathbf{V}_n (\tilde{\mathbf{Q}}_n \mathbf{p} - \tilde{\mathbf{Q}}_n \mathbf{q}) \rangle \rangle \\
& = \langle \langle \mathbf{E}_n \mathbf{p} - \mathbf{E}_n \mathbf{q} \mid \mathbf{V}_n^{-1} \mathbf{E}_n \mathbf{p} - \mathbf{V}_n^{-1} \mathbf{E}_n \mathbf{q} \rangle \rangle \\
& \leq \|\mathbf{V}_n^{-1}\| \|\mathbf{E}_n \mathbf{p} - \mathbf{E}_n \mathbf{q}\|^2 \\
& \leq (2\beta - \varepsilon) \left( \|(\mathbf{C}_n - \mathbf{C})\mathbf{x} - (\mathbf{C}_n - \mathbf{C})\mathbf{y}\|^2 \right. \\
& \quad \left. + \|(\mathbf{D}_n - \mathbf{D})\mathbf{v} - (\mathbf{D}_n - \mathbf{D})\mathbf{w}\|^2 \right) \\
& = (2\beta - \varepsilon) \left( \sum_{i=1}^m \|(C_{i,n} - C_i)\mathbf{x} - (C_{i,n} - C_i)\mathbf{y}\|^2 \right. \\
& \quad \left. + \sum_{k=1}^s \|(S_{k,n} - S_k)\mathbf{v} - (S_{k,n} - S_k)\mathbf{w}\|^2 \right) \\
& \leq (2\beta - \varepsilon) \left( \sum_{i=1}^m \kappa_{i,n}^2 \|\mathbf{x} - \mathbf{y}\|^2 + \sum_{k=1}^s \eta_{k,n}^2 \|\mathbf{v} - \mathbf{w}\|^2 \right) \\
& \leq (2\beta - \varepsilon) \left( \sum_{i=1}^m \kappa_{i,n}^2 + \sum_{k=1}^s \eta_{k,n}^2 \right) \|\mathbf{p} - \mathbf{q}\|^2 \\
& \leq (2\beta - \varepsilon) \zeta_n^{-1} \left( \sum_{i=1}^m \kappa_{i,n}^2 + \sum_{k=1}^s \eta_{k,n}^2 \right) \|\mathbf{p} - \mathbf{q}\|_{\mathbf{V}_n}^2 \\
& \leq (2\beta - \varepsilon)^2 \left( \sum_{i=1}^m \kappa_{i,n}^2 + \sum_{k=1}^s \eta_{k,n}^2 \right) \|\mathbf{p} - \mathbf{q}\|_{\mathbf{V}_n}^2, \tag{3.54}
\end{aligned}$$

which implies that  $\tilde{\mathbf{Q}}_n$  is Lipschitz continuous (in the norm  $\|\cdot\|_{\mathbf{V}_n}$ ) with respectively constant

$$\kappa_n = (2\beta - \varepsilon) \sqrt{\sum_{i=1}^m \kappa_{i,n}^2 + \sum_{k=1}^s \eta_{k,n}^2}, \tag{3.55}$$

that satisfies

$$\sum_{n \in \mathbb{N}} \kappa_n < +\infty. \tag{3.56}$$

Let  $\mathbf{p} = (\mathbf{x}, \mathbf{v}) \in \text{zer}(\mathbf{M} + \mathbf{S} + \mathbf{Q})$  and noting that  $(\forall n \in \mathbb{N}) \tilde{\mathbf{Q}}_n(\bar{\mathbf{s}}, \bar{\mathbf{w}}) = 0$ ,

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad & \|e_{1,n}\|_{\mathbf{V}_n} \leq \|\tilde{\mathbf{Q}}_n \mathbf{p}_n\|_{\mathbf{V}_n} + \|\mathbf{V}_n^{-1} \mathbf{b}_n\|_{\mathbf{V}_n} \\
& \leq \|\tilde{\mathbf{Q}}_n \mathbf{p}_n - \tilde{\mathbf{Q}}_n \mathbf{p}\|_{\mathbf{V}_n} + \|\tilde{\mathbf{Q}}_n \mathbf{p} - \tilde{\mathbf{Q}}_n(\bar{\mathbf{s}}, \bar{\mathbf{w}})\|_{\mathbf{V}_n} + \|\mathbf{V}_n^{-1} \mathbf{b}_n\|_{\mathbf{V}_n} \\
& \leq \kappa_n \|\mathbf{p}_n - \mathbf{p}\|_{\mathbf{V}_n} + \kappa_n \|\mathbf{p} - (\bar{\mathbf{s}}, \bar{\mathbf{w}})\|_{\mathbf{V}_n} + \|\mathbf{V}_n^{-1} \mathbf{b}_n\|_{\mathbf{V}_n} \\
& = \kappa_n \|\mathbf{p}_n - \mathbf{p}\|_{\mathbf{V}_n} + \kappa_n \|\mathbf{p} - (\bar{\mathbf{s}}, \bar{\mathbf{w}})\|_{\mathbf{V}_n} + \|\mathbf{b}_n\|_{\mathbf{V}_n^{-1}}. \tag{3.57}
\end{aligned}$$

Since  $\mathbf{p} \in \text{zer}(\mathbf{M} + \mathbf{S} + \mathbf{Q})$ , we have

$$(\forall n \in \mathbb{N}) \quad \mathbf{p} = J_{\mathbf{V}_n^{-1} \tilde{\mathbf{M}}}(\mathbf{P}_n \mathbf{p}). \tag{3.58}$$

Hence, since  $J_{\mathbf{V}_n^{-1}\widetilde{M}}$  and  $\mathbf{P}_n$  are nonexpansive with respect to the norm  $\|\cdot\|_{\mathbf{V}_n}$  by Lemma 2.3(ii) and Lemma 2.4, on one hand, we have

$$\begin{aligned} \|\|J_{\mathbf{V}_n^{-1}\widetilde{M}}(\mathbf{P}_n\mathbf{p}_n - \mathbf{e}_{1,n}) - \mathbf{p}\|\|_{\mathbf{V}_n} &= \|\|J_{\mathbf{V}_n^{-1}\widetilde{M}}(\mathbf{P}_n\mathbf{p}_n - \mathbf{e}_{1,n}) - J_{\mathbf{V}_n^{-1}\widetilde{M}}(\mathbf{P}_n\mathbf{p})\|\|_{\mathbf{V}_n} \\ &\leq \|\|\mathbf{p}_n - \mathbf{p}\|\|_{\mathbf{V}_n} + \|\|\mathbf{e}_{1,n}\|\|_{\mathbf{V}_n}, \end{aligned} \quad (3.59)$$

which and (3.45), (3.22), (3.23) imply that

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad \|\|\mathbf{p}_{n+1} - \mathbf{p}\|\|_{\mathbf{V}_n} &\leq \|\|(\text{Id} - \mathbf{\Lambda}_n)(\mathbf{p}_n - \mathbf{p})\|\|_{\mathbf{V}_n} \\ &\quad + \|\|\mathbf{\Lambda}_n(J_{\mathbf{V}_n^{-1}\widetilde{M}}(\mathbf{P}_n\mathbf{p}_n - \mathbf{e}_{1,n}) - \mathbf{p})\|\|_{\mathbf{V}_n} + \|\|\mathbf{\Lambda}_n\mathbf{a}_n\|\|_{\mathbf{V}_n} \\ &\leq \left(\|\|(\text{Id} - \mathbf{\Lambda}_n)\|\|_{\mathbf{V}_n} + \|\|\mathbf{\Lambda}_n\|\|_{\mathbf{V}_n}\right) \|\|\mathbf{p}_n - \mathbf{p}\|\|_{\mathbf{V}_n} + \|\|\mathbf{\Lambda}_n\|\|_{\mathbf{V}_n} \|\|\mathbf{e}_{1,n}\|\|_{\mathbf{V}_n} + \|\|\mathbf{\Lambda}_n\mathbf{a}_n\|\|_{\mathbf{V}_n} \\ &\leq (1 + \tau_n + \kappa_n) \|\|\mathbf{p}_n - \mathbf{p}\|\|_{\mathbf{V}_n} + \alpha_n, \end{aligned} \quad (3.60)$$

where

$$(\forall n \in \mathbb{N}) \quad \alpha_n = \kappa_n \|\|\mathbf{p} - (\overline{\mathbf{s}}, \overline{\mathbf{w}})\|\|_{\mathbf{V}_n} + \|\|\mathbf{b}_n\|\|_{\mathbf{V}_n^{-1}} + \|\|\mathbf{a}_n\|\|_{\mathbf{V}_n}. \quad (3.61)$$

Noting that, by (3.25), (3.56), (3.39), (3.38), (3.37) and (3.53), (3.50), we have

$$\sum_{n \in \mathbb{N}} \alpha_n < +\infty \quad \text{and} \quad \sum_{n \in \mathbb{N}} (\tau_n + \kappa_n) < +\infty. \quad (3.62)$$

Therefore, we derive from (3.60) and  $(\forall n \in \mathbb{N}) \mathbf{V}_n \succeq \mathbf{V}_{n+1}$  that

$$(\forall n \in \mathbb{N}) \quad \|\|\mathbf{p}_{n+1} - \mathbf{p}\|\|_{\mathbf{V}_{n+1}} \leq (1 + \tau_n + \kappa_n) \|\|\mathbf{p}_n - \mathbf{p}\|\|_{\mathbf{V}_n} + \alpha_n, \quad (3.63)$$

and hence, by [36, Lemma 2.2.2],

$$\sup_{n \in \mathbb{N}} \|\|\mathbf{p}_n - \mathbf{p}\|\|_{\mathbf{V}_n} < +\infty, \quad (3.64)$$

which and (3.57), (3.56), (3.62), (3.53), (3.37), (3.38), (3.39) imply that

$$\sum_{n \in \mathbb{N}} \|\|\mathbf{e}_{1,n}\|\|_{\mathbf{V}_n} < +\infty. \quad (3.65)$$

On the other hand,

$$\|\|\mathbf{p}_n - J_{\mathbf{V}_n^{-1}\widetilde{M}}(\mathbf{P}_n\mathbf{p}_n - \mathbf{e}_{1,n}) - \mathbf{a}_n\|\|_{\mathbf{V}_n} \leq 2 \|\|\mathbf{p}_n - \mathbf{p}\|\|_{\mathbf{V}_n} + \|\|\mathbf{e}_{1,n}\|\|_{\mathbf{V}_n} + \|\|\mathbf{a}_n\|\|_{\mathbf{V}_n}, \quad (3.66)$$

which and (3.64), (3.65) imply that

$$\nu = \sup_{n \in \mathbb{N}} \|\|\mathbf{p}_n - J_{\mathbf{V}_n^{-1}\widetilde{M}}(\mathbf{P}_n\mathbf{p}_n - \mathbf{e}_{1,n}) - \mathbf{a}_n\|\|_{\mathbf{V}_n} < +\infty. \quad (3.67)$$

Now using the condition (vi), (3.67) and the definition of  $(\mathbf{e}_{1,n})_{n \in \mathbb{N}}$  in (3.44), we obtain

$$\sum_{n \in \mathbb{N}} \|\|\mathbf{e}_n\|\|_{\mathbf{V}_n} \leq \sum_{n \in \mathbb{N}} \|\|\mathbf{a}_n\|\|_{\mathbf{V}_n} + \nu \varepsilon^{-1} \sum_{n \in \mathbb{N}} \left( \sum_{i=1}^m |\lambda_{i,n} - \lambda_n| + \sum_{k=1}^s |\lambda_{m+k,n} - \lambda_n| \right) < +\infty. \quad (3.68)$$



By using (3.38), we derive from (3.68) and (3.65) that

$$\sum_{n \in \mathbb{N}} \|e_n\| < +\infty \quad \text{and} \quad \sum_{n \in \mathbb{N}} \|e_{1,n}\| < +\infty, \quad (3.69)$$

which prove (3.48) and (3.49).

(i): By Theorem 2.5(i),  $p_n \rightarrow \bar{p} \in \text{zer}(M + S + Q)$ .

(ii)(iii): By Theorem 2.5(ii) and (iii),

$$\|Qp_n - Q\bar{p}\| \rightarrow 0, \quad (3.70)$$

which implies that, for every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ ,

$$C_i(x_{1,n}, \dots, x_{m,n}) - C_i(\bar{x}_1, \dots, \bar{x}_m) \rightarrow 0 \quad \text{and} \quad S_k(v_{1,n}, \dots, v_{s,n}) - S_k(\bar{v}_1, \dots, \bar{v}_s) \rightarrow 0. \quad (3.71)$$

Moreover, by (i),  $(\forall i \in \{1, \dots, m\}) x_{i,n} \rightarrow \bar{x}_i$  and  $(\forall k \in \{1, \dots, s\}) v_{k,n} \rightarrow \bar{v}_k$ . Therefore, the conclusions follow from the definition of the demiregular operators.

(iv)(v): The conclusions follow from our assumptions the definition of the demiregular operators.

□

## 4 Application to coupled system of monotone inclusions in duality

We provide an application to coupled system of monotone inclusions.

**Problem 4.1** Let  $m, s$  be strictly positive integers. For every  $i \in \{1, \dots, m\}$ , let  $(\mathcal{H}_i, \langle \cdot | \cdot \rangle)$  be a real Hilbert space, let  $z_i \in \mathcal{H}_i$ , let  $A_i: \mathcal{H}_i \rightarrow 2^{\mathcal{H}_i}$  be maximally monotone, let  $C_i: \mathcal{H}_1 \times \dots \times \mathcal{H}_m \rightarrow \mathcal{H}_i$  be such that

$$\left( \exists \nu_0 \in ]0, +\infty[ \right) \left( \forall (x_i)_{1 \leq i \leq m} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m \right) \left( \forall (y_i)_{1 \leq i \leq m} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m \right) \\ \sum_{i=1}^m \langle x_i - y_i | C_i(x_1, \dots, x_m) - C_i(y_1, \dots, y_m) \rangle \geq \nu_0 \sum_{i=1}^m \|C_i(x_1, \dots, x_m) - C_i(y_1, \dots, y_m)\|^2. \quad (4.1)$$

For every  $k \in \{1, \dots, s\}$ , let  $(\mathcal{G}_k, \langle \cdot | \cdot \rangle)$  be a real Hilbert space, let  $r_k \in \mathcal{G}_k$ , let  $D_k: \mathcal{G}_k \rightarrow 2^{\mathcal{G}_k}$  be maximally monotone and  $\nu_k$ -strongly monotone for some  $\nu_k \in ]0, +\infty[$ , let  $B_k: \mathcal{G}_k \rightarrow 2^{\mathcal{G}_k}$  be maximally monotone. For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , let  $L_{k,i}: \mathcal{H}_i \rightarrow \mathcal{G}_k$  be a bounded linear operator. The primal problem is to solve the primal inclusion:

$$\text{find } \bar{x}_1 \in \mathcal{H}_1, \dots, \bar{x}_m \in \mathcal{H}_m \text{ such that} \\ \begin{cases} z_1 & \in A_1 \bar{x}_1 + \sum_{k=1}^s L_{k,1}^* \left( (D_k \square B_k) \left( \sum_{i=1}^m L_{k,i} \bar{x}_i - r_k \right) \right) + C_1(\bar{x}_1, \dots, \bar{x}_m) \\ & \vdots \\ z_m & \in A_m \bar{x}_m + \sum_{k=1}^s L_{k,m}^* \left( (D_k \square B_k) \left( \sum_{i=1}^m L_{k,i} \bar{x}_i - r_k \right) \right) + C_m(\bar{x}_1, \dots, \bar{x}_m). \end{cases} \quad (4.2)$$

We denote by  $\mathcal{P}$  the set of solutions to (4.2). The dual problem is to solve the dual inclusion:

$$\begin{aligned} & \text{find } \bar{v}_1 \in \mathcal{G}_1, \dots, \bar{v}_s \in \mathcal{G}_s \text{ such that } (\exists (x_i)_{1 \leq i \leq m} \in (\mathcal{H}_i)_{1 \leq i \leq m}) \\ & \begin{cases} z_1 - \sum_{k=1}^s L_{k,1}^* \bar{v}_k \in A_1 x_1 + C_1(x_1, \dots, x_m) \\ \vdots \\ z_m - \sum_{k=1}^s L_{k,m}^* \bar{v}_k \in A_m x_m + C_m(x_1, \dots, x_m), \end{cases} \quad \text{and} \quad \begin{cases} \sum_{i=1}^m L_{1,i} x_i - r_1 \in B_1^{-1} \bar{v}_1 + D_1^{-1} \bar{v}_1 \\ \vdots \\ \sum_{i=1}^m L_{s,i} x_i - r_s \in B_s^{-1} \bar{v}_s + D_s^{-1} \bar{v}_s. \end{cases} \end{aligned} \quad (4.3)$$

The set of solutions to (4.3) is denoted by  $\mathcal{D}$ .

Problem 4.1 covers not only a wide class of monotone inclusions and duality frameworks in the literature [3, 5, 6, 12, 17, 18, 26, 28, 33, 37, 39, 40, 42, 43, 44] and coupled system of monotone inclusions unified in [2] and the references therein, but also a wide class of minimization formulations, in particular, in the multi-component signal decomposition and recovery [2, 5, 7] and the references therein.

**Algorithm 4.2** Let  $\alpha \in ]0, +\infty[$  and, for every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , let  $(U_{i,n})_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{H}_i)$  and let  $(V_{k,n})_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{G}_k)$ . Set  $\beta = \min\{\nu_0, \nu_1, \dots, \nu_s\}$ , and let  $\varepsilon \in ]0, \min\{1, \beta\}[$ , let  $(\lambda_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, 1]$ . Let  $(x_{i,0})_{1 \leq i \leq m} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m$  and  $(v_{k,0})_{1 \leq k \leq s} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s$ . Set

$$\begin{aligned} & \text{For } n = 0, 1, \dots \\ & \left[ \begin{array}{l} \text{For } i = 1, \dots, m \\ \left[ \begin{array}{l} p_{i,n} = J_{U_{i,n} A_i} \left( x_{i,n} - U_{i,n} \left( \sum_{k=1}^s L_{k,i}^* v_{k,n} + C_i(x_{1,n}, \dots, x_{m,n}) + c_{i,n} - z_i \right) \right) + a_{i,n} \\ y_{i,n} = 2p_{i,n} - x_{i,n} \\ x_{i,n+1} = x_{i,n} + \lambda_{i,n} (p_{i,n} - x_{i,n}) \end{array} \right. \\ \text{For } k = 1, \dots, s \\ \left[ \begin{array}{l} q_{k,n} = J_{V_{k,n} B_k^{-1}} \left( v_{k,n} + V_{k,n} \left( \sum_{i=1}^m L_{k,i} y_{i,n} - D_k^{-1} v_{k,n} - d_{k,n} - r_k \right) \right) + b_{k,n} \\ v_{k,n+1} = v_{k,n} + \lambda_{m+k,n} (q_{k,n} - v_{k,n}), \end{array} \right. \end{array} \right. \end{aligned} \quad (4.4)$$

where, for every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , the following conditions hold

(i)  $(\forall n \in \mathbb{N}) U_{i,n+1} \succeq U_{i,n}$  and  $V_{k,n+1} \succeq V_{k,n}$ , and

$$\mu = \sup_{n \in \mathbb{N}} \{ \|U_{1,n}\|, \dots, \|U_{m,n}\|, \|V_{1,n}\|, \dots, \|V_{s,n}\| \} < +\infty. \quad (4.5)$$

(ii)  $(a_{i,n})_{n \in \mathbb{N}}$  and  $(c_{i,n})_{n \in \mathbb{N}}$  are absolutely summable sequences in  $\mathcal{H}_i$ .

(iii)  $(b_{k,n})_{n \in \mathbb{N}}$  and  $(d_{k,n})_{n \in \mathbb{N}}$  are absolutely summable sequences in  $\mathcal{G}_k$ .

(iv)  $(\lambda_{i,n})_{n \in \mathbb{N}}$  and  $(\lambda_{m+k,n})_{n \in \mathbb{N}}$  are in  $]0, 1]$  such that

$$\sum_{n \in \mathbb{N}} \left( |\lambda_{i,n} - \lambda_n| + |\lambda_{m+k,n} - \lambda_n| \right) < +\infty. \quad (4.6)$$

**Corollary 4.3** *Suppose that  $\mathcal{P} \neq \emptyset$  and there exists  $L_{k_0, i_0} \neq 0$  for some  $i_0 \in \{1, \dots, m\}$  and  $k_0 \in \{1, \dots, s\}$ , and (3.5) is satisfied. For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , let  $(x_{i,n})_{n \in \mathbb{N}}$  and  $(v_{k,n})_{n \in \mathbb{N}}$  be sequences generated by Algorithm 4.2. Then the following hold for some  $(\bar{x}_1, \dots, \bar{x}_m) \in \mathcal{P}$  and  $(\bar{v}_1, \dots, \bar{v}_s) \in \mathcal{D}$ .*

- (i)  $(\forall i \in \{1, \dots, m\}) \ x_{i,n} \rightarrow \bar{x}_i$  and  $(\forall k \in \{1, \dots, s\}) \ v_{k,n} \rightarrow \bar{v}_k$ .
- (ii) *Suppose that the operator  $(x_i)_{1 \leq i \leq m} \mapsto (C_j(x_i)_{1 \leq i \leq m})_{1 \leq j \leq m}$  is demiregular (see Lemma 2.2 for special cases) at  $(\bar{x}_1, \dots, \bar{x}_m)$ , then  $(\forall i \in \{1, \dots, m\}) \ x_{i,n} \rightarrow \bar{x}_i$ .*
- (iii) *Suppose that  $D_j^{-1}$  is demiregular at  $\bar{v}_j$ , for some  $j \in \{1, \dots, s\}$ , then  $v_{j,n} \rightarrow \bar{v}_j$ .*
- (iv) *Suppose that there exists  $j \in \{1, \dots, m\}$  and operator  $C: \mathcal{H}_j \rightarrow \mathcal{H}_j$  such that  $(\forall (x_i)_{1 \leq i \leq m} \in (\mathcal{H}_i)_{1 \leq i \leq m}) \ C_j(x_1, \dots, x_m) = Cx_j$  and  $C$  is demiregular (see Lemma 2.2 for special cases) at  $\bar{x}_j$ , then  $x_{j,n} \rightarrow \bar{x}_j$ .*

*Proof.* Set  $\mu_0 = \min\{\nu_1, \dots, \nu_s\}$  and define

$$(\forall k \in \{1, \dots, s\}) \ S_k: \mathcal{G}_1 \times \dots \times \mathcal{G}_s \rightarrow \mathcal{G}_k: (v_1, \dots, v_s) \mapsto D_k^{-1}v_k. \quad (4.7)$$

Then, for every  $(v_k)_{1 \leq k \leq s} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s$  and every  $(w_k)_{1 \leq k \leq s} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s$ , we obtain

$$\begin{aligned} \sum_{k=1}^s \langle v_k - w_k \mid S_k(v_1, \dots, v_s) - S_k(w_1, \dots, w_s) \rangle &= \sum_{k=1}^s \langle v_k - w_k \mid D_k^{-1}v_k - D_k^{-1}w_k \rangle \\ &\geq \sum_{k=1}^s \nu_k \|D_k^{-1}v_k - D_k^{-1}w_k\|^2 \\ &\geq \mu_0 \sum_{k=1}^s \|D_k^{-1}v_k - D_k^{-1}w_k\|^2 \\ &= \mu_0 \sum_{k=1}^s \|S_k(v_1, \dots, v_s) - S_k(w_1, \dots, w_s)\|^2, \end{aligned} \quad (4.8)$$

which shows that (1.2) is satisfied. Moreover, upon setting

$$(\forall n \in \mathbb{N}) \ \begin{cases} (\forall i \in \{1, \dots, m\}) \ C_{i,n} = C_i \\ (\forall k \in \{1, \dots, s\}) \ S_{k,n} = S_k, \end{cases} \quad (4.9)$$

the conditions (ii) and (iii) in Algorithm 3.1 are satisfied. Note that the conditions (i), (ii), (iii), and (iv) in Algorithm 4.2 are the same as in Algorithm 3.1. Moreover, the algorithm (3.1) reduces to (4.4) where  $B_k$  is replaced by  $B_k^{-1}$ . Next, since  $\mathcal{P} \neq \emptyset$ , we derive from (4.2) that, for every  $k \in \{1, \dots, s\}$ , there exists  $v_k \in \mathcal{G}_k$  such that

$$\bar{v}_k \in (D_k \square B_k) \left( \sum_{i=1}^m L_{k,i} \bar{x}_i - r_k \right) \Leftrightarrow \sum_{i=1}^m L_{k,i} \bar{x}_i - r_k \in B_k^{-1} \bar{v}_k + D_k^{-1} \bar{v}_k, \quad (4.10)$$

and

$$(\forall i \in \{1, \dots, m\}) \ z_i - \sum_{k=1}^s L_{k,i}^* \bar{v}_k \in A_i \bar{x}_i + C_i(\bar{x}_1, \dots, \bar{x}_m), \quad (4.11)$$

which shows that  $\Omega \neq \emptyset$  and  $\mathcal{D} \neq \emptyset$ . Inversely, if  $(\bar{x}_1, \dots, \bar{x}_m, \bar{v}_1, \dots, \bar{v}_s) \in \Omega$ , then the inclusions (4.10) and (4.11) are satisfied. Hence  $(\bar{v}_1, \dots, \bar{v}_s) \in \mathcal{D}$  and  $(\bar{x}_1, \dots, \bar{x}_m) \in \mathcal{P}$ . Therefore, the conclusions follow from Theorem 3.3.  $\square$

## 5 Application to minimization problems

We provide applications to minimization problems involving infimal convolutions, composite functions and coupling.

**Problem 5.1** Let  $m, s$  be strictly positive integers. For every  $i \in \{1, \dots, m\}$ , let  $\mathcal{H}_i$  be a real Hilbert space, let  $z_i \in \mathcal{H}_i$ , let  $f_i \in \Gamma_0(\mathcal{H}_i)$ . For every  $k \in \{1, \dots, s\}$ , let  $\mathcal{G}_k$  be a real Hilbert space, let  $r_k \in \mathcal{G}_k$ , let  $\ell_k \in \Gamma_0(\mathcal{G}_k)$  be  $\nu_k$ -strongly convex function, for some  $\nu_k \in ]0, +\infty[$ , let  $g_k \in \Gamma_0(\mathcal{G}_k)$ . For every  $i \in \{1, \dots, m\}$ , and every  $k \in \{1, \dots, s\}$ , let  $L_{k,i}: \mathcal{H}_i \rightarrow \mathcal{G}_k$  be a bounded linear operator. Let  $\varphi: \mathcal{H}_1 \times \dots \times \mathcal{H}_m \rightarrow \mathbb{R}$  be convex differentiable function with  $\nu_0^{-1}$ -Lipschitz continuous gradient. The primal problems is to

$$\underset{x_1 \in \mathcal{H}_1, \dots, x_m \in \mathcal{H}_m}{\text{minimize}} \sum_{i=1}^m (f_i(x_i) - \langle x_i | z_i \rangle) + \sum_{k=1}^s (\ell_k \square g_k) \left( \sum_{i=1}^m L_{k,i} x_i - r_k \right) + \varphi(x_1, \dots, x_m), \quad (5.1)$$

under the the assumption that,

$$\left( \forall i \in \{1, \dots, m\} \right) \quad z_i \in \text{ran} \left( \partial f_i + \sum_{k=1}^s L_{k,i}^* \circ \left( \partial \ell_k \square \partial g_k \right) \circ \left( \sum_{j=1}^m L_{k,j} \cdot -r_k \right) + \nabla_i \varphi \right), \quad (5.2)$$

where  $\nabla_i \varphi$  is the  $i$ th component of the gradient  $\nabla \varphi$ , and the dual problem is to

$$\underset{v_1 \in \mathcal{G}_1, \dots, v_s \in \mathcal{G}_s}{\text{minimize}} \left( \varphi^* \square \left( \sum_{i=1}^m f_i^* \right) \right) \left( \left( z_i - \sum_{k=1}^s L_{k,i}^* v_k \right)_{1 \leq i \leq m} \right) + \sum_{k=1}^s \left( \ell_k^*(v_k) + g_k^*(v_k) + \langle v_k | r_k \rangle \right). \quad (5.3)$$

In the case when the infimal convolutions are absent, Problem 5.1 often appears in the multi-components signal decomposition and recovery problems [2, 5, 4] and the references therein.

**Example 5.2** Some special cases of this problem are listed in the following:

- (i) In the case when  $\varphi: (x_1, \dots, x_m) \mapsto \sum_{i=1}^m h_i(x_i)$ , where for every  $i \in \{1, \dots, m\}$ ,  $h_i: \mathcal{H}_i \rightarrow \mathbb{R}$  is a convex differential function with  $\tau_i^{-1}$ -Lipschitz continuous gradient, for some  $\tau_i^{-1} \in ]0, +\infty[$ , Problem 5.1 reduces to the general minimization problem [19, Problem 5.1] which covers a wide class of the convex minimization problems in the literature.
- (ii) In the case when  $\varphi: (x_1, \dots, x_m) \mapsto 0$  and, for every  $k \in \{1, \dots, s\}$ ,  $\ell_k = \iota_{\{0\}}$  and  $g_k$  is differentiable with  $\tau_k^{-1}$ -Lipschitz continuous gradient, for some  $\tau_k \in ]0, +\infty[$ , Problem 5.1 reduces to [5, Problem 1.1].
- (iii) In the case when  $m = 1$ , Problem 5.1 reduces to [23, Problem 4.1] which is also studied in [26, 44].

**Algorithm 5.3** Let  $\alpha \in ]0, +\infty[$  and, for every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , let  $(U_{i,n})_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{H}_i)$  and let  $(V_{k,n})_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{G}_k)$ . Set  $\beta = \min\{\nu_0, \nu_1, \dots, \nu_s\}$ , and let  $\varepsilon \in ]0, \min\{1, \beta\}[$ , let  $(\lambda_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, 1]$ . Let  $(x_{i,0})_{1 \leq i \leq m} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m$  and  $(v_{k,0})_{1 \leq k \leq s} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s$ . Set

$$\begin{array}{l}
\text{For } n = 0, 1, \dots \\
\left[ \begin{array}{l}
\text{For } i = 1, \dots, m \\
\left[ \begin{array}{l}
p_{i,n} = \text{PROX}_{f_i}^{U_{i,n}^{-1}} \left( x_{i,n} - U_{i,n} \left( \sum_{k=1}^s L_{k,i}^* v_{k,n} + \nabla_i \varphi(x_{1,n}, \dots, x_{m,n}) + c_{i,n} - z_i \right) \right) + a_{i,n} \\
y_{i,n} = 2p_{i,n} - x_{i,n} \\
x_{i,n+1} = x_{i,n} + \lambda_{i,n}(p_{i,n} - x_{i,n})
\end{array} \right. \\
\text{For } k = 1, \dots, s \\
\left[ \begin{array}{l}
q_{k,n} = \text{PROX}_{g_k^*}^{V_{k,n}^{-1}} \left( v_{k,n} + V_{k,n} \left( \sum_{i=1}^m L_{k,i} y_{i,n} - \nabla \ell_k^*(v_{k,n}) - d_{k,n} - r_k \right) \right) + b_{k,n} \\
v_{k,n+1} = v_{k,n} + \lambda_{m+k,n}(q_{k,n} - v_{k,n}),
\end{array} \right.
\end{array} \right.
\end{array} \tag{5.4}$$

where, for every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , the following conditions hold

(i)  $(\forall n \in \mathbb{N}) U_{i,n+1} \succeq U_{i,n}$  and  $V_{k,n+1} \succeq V_{k,n}$ , and

$$\mu = \sup_{n \in \mathbb{N}} \{ \|U_{1,n}\|, \dots, \|U_{m,n}\|, \|V_{1,n}\|, \dots, \|V_{s,n}\| \} < +\infty. \tag{5.5}$$

(ii)  $(a_{i,n})_{n \in \mathbb{N}}$  and  $(c_{i,n})_{n \in \mathbb{N}}$  are absolutely summable sequences in  $\mathcal{H}_i$ .

(iii)  $(b_{k,n})_{n \in \mathbb{N}}$  and  $(d_{k,n})_{n \in \mathbb{N}}$  are absolutely summable sequences in  $\mathcal{G}_k$ .

(iv)  $(\lambda_{i,n})_{n \in \mathbb{N}}$  and  $(\lambda_{m+k,n})_{n \in \mathbb{N}}$  are in  $]0, 1]$  such that

$$\sum_{n \in \mathbb{N}} \left( |\lambda_{i,n} - \lambda_n| + |\lambda_{m+k,n} - \lambda_n| \right) < +\infty. \tag{5.6}$$

**Corollary 5.4** Suppose that there exists  $L_{k_0, i_0} \neq 0$  for some  $i_0 \in \{1, \dots, m\}$  and  $k_0 \in \{1, \dots, s\}$ , and (3.5) is satisfied. For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, s\}$ , let  $(x_{i,n})_{n \in \mathbb{N}}$  and  $(v_{k,n})_{n \in \mathbb{N}}$  be sequences generated by Algorithm 5.3. Then the following hold for some solution  $(\bar{x}_1, \dots, \bar{x}_m)$  to (5.1) and  $(\bar{v}_1, \dots, \bar{v}_s)$  to (5.3).

(i)  $(\forall i \in \{1, \dots, m\}) x_{i,n} \rightarrow \bar{x}_i$  and  $(\forall k \in \{1, \dots, s\}) v_{k,n} \rightarrow \bar{v}_k$ .

(ii) Suppose that  $\varphi$  is defined as in Example 5.2(i) and  $h_j$  is uniformly convex at  $\bar{x}_j$ , for some  $j \in \{1, \dots, m\}$ , then  $x_{j,n} \rightarrow \bar{x}_j$ .

(iii) Suppose that  $\ell_j^*$  is uniformly convex at  $\bar{v}_j$ , for some  $j \in \{1, \dots, s\}$ , then  $v_{j,n} \rightarrow \bar{v}_j$ .

*Proof.* Set

$$\begin{cases}
(\forall i \in \{1, \dots, m\}) & A_i = \partial f_i \quad \text{and} \quad C_i = \nabla_i \varphi, \\
(\forall k \in \{1, \dots, s\}) & B_k = \partial g_k \quad \text{and} \quad D_k = \partial \ell_k.
\end{cases} \tag{5.7}$$

Then it follows from [10, Theorem 20.40] that  $(A_i)_{1 \leq i \leq m}$ ,  $(B_k)_{1 \leq k \leq s}$ , and  $(D_k)_{1 \leq k \leq s}$  are maximally monotone. Moreover,  $(C_1, \dots, C_m) = \nabla \varphi$  is  $\nu_0$ -cocoercive [8, 9]. Moreover since, for every  $k \in$

$\{1, \dots, s\}$ ,  $\ell_k$  is  $\nu_k$ -strongly convex,  $\partial\ell_k$  is  $\nu_k$ -strongly monotone. Therefore, every conditions on the operators in Problem 1.1 are satisfied. Since, for every  $k \in \{1, \dots, s\}$ ,  $\text{dom } \ell_k^* = \mathcal{G}_k$ , we next derive from [10, Proposition 20.47] that

$$(\forall k \in \{1, \dots, s\}) \quad \partial(\ell_k \square g_k) = \partial g_k \square \partial\ell_k = B_k \square D_k. \quad (5.8)$$

Let  $\mathcal{H}$  and  $\mathcal{G}$  be defined as in the proof of Theorem 3.3, and let  $\mathbf{L}$ ,  $\mathbf{z}$  and  $\mathbf{r}$  be defined as in (3.13), and define

$$\begin{cases} f: \mathcal{H} \rightarrow ]-\infty, +\infty[ : \mathbf{x} \mapsto \sum_{i=1}^m f_i(x_i) \\ g: \mathcal{G} \rightarrow ]-\infty, +\infty[ : \mathbf{v} \mapsto \sum_{k=1}^s g_k(v_k) \\ \ell: \mathcal{G} \rightarrow ]-\infty, +\infty[ : \mathbf{v} \mapsto \sum_{k=1}^s \ell_k(v_k). \end{cases} \quad (5.9)$$

Observe that [10, Proposition 13.27],

$$f^*: \mathbf{y} \mapsto \sum_{i=1}^m f_i^*(y_i), \quad g^*: \mathbf{v} \mapsto \sum_{k=1}^s g_k^*(v_k), \quad \text{and} \quad \ell^*: \mathbf{v} \mapsto \sum_{k=1}^s \ell_k^*(v_k). \quad (5.10)$$

We also have

$$\ell \square g: \mathbf{v} \mapsto \sum_{k=1}^s (\ell_k \square g_k)(v_k). \quad (5.11)$$

Then the primal problem becomes

$$\underset{\mathbf{x} \in \mathcal{H}}{\text{minimize}} \quad f(\mathbf{x}) - \langle \mathbf{x} \mid \mathbf{z} \rangle + (\ell \square g)(\mathbf{L}\mathbf{x} - \mathbf{r}) + \varphi(\mathbf{x}), \quad (5.12)$$

and the dual problem becomes

$$\underset{\mathbf{v} \in \mathcal{G}}{\text{minimize}} \quad (\varphi^* \square f^*)(\mathbf{z} - \mathbf{L}^*\mathbf{v}) + \ell^*(\mathbf{v}) + g^*(\mathbf{v}) + \langle \mathbf{v} \mid \mathbf{r} \rangle. \quad (5.13)$$

Then, let  $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_m)$  be a solution to (4.2), i.e., for every  $i \in \{1, \dots, m\}$ ,

$$z_i \in \partial f_i(\bar{x}_i) + \sum_{k=1}^s L_{k,i}^* \left( (\partial\ell_k \square \partial g_k) \left( \sum_{j=1}^m L_{k,j} \bar{x}_j - r_k \right) \right) + \nabla_i \varphi(\bar{x}_1, \dots, \bar{x}_m). \quad (5.14)$$

Then, using (5.7), (5.8), [10, Corollary 16.38(iii)], [10, Proposition 16.8],

$$\mathbf{0} \in \partial(f + \langle \cdot \mid \mathbf{z} \rangle)(\bar{\mathbf{x}}) + \mathbf{L}^* \left( \partial(\ell \square g)(\mathbf{L}\bar{\mathbf{x}} - \mathbf{r}) \right) + \nabla \varphi(\bar{\mathbf{x}}). \quad (5.15)$$

Therefore, by [10, Proposition 16.5(ii)], we derive from (5.15) that

$$\mathbf{0} \in \partial \left( f + \langle \cdot \mid \mathbf{z} \rangle + (\ell \square g)(\mathbf{L} \cdot - \mathbf{r}) + \varphi \right) (\bar{\mathbf{x}}). \quad (5.16)$$

Hence, by Fermat's rule [10, Theorem 16.2] that  $\bar{\mathbf{x}}$  is a solution to (5.12), i.e,  $\bar{\mathbf{x}}$  is a solution to (5.1). We next let  $\bar{\mathbf{v}}$  be a solution to (4.3). Then using [10, Theorem 15.3] and (2.12),

$$\begin{aligned} -\mathbf{r} &\in -\mathbf{L} \left( (\partial f + \nabla \varphi)^{-1}(\mathbf{z} - \mathbf{L}^*\bar{\mathbf{v}}) \right) + (\partial g)^{-1}\bar{\mathbf{v}} + (\partial \ell)^{-1}\bar{\mathbf{v}} \\ &= -\mathbf{L} \left( \partial(f + \varphi)^*(\mathbf{z} - \mathbf{L}^*\bar{\mathbf{v}}) \right) + \partial g^*(\bar{\mathbf{v}}) + \partial \ell^*(\bar{\mathbf{v}}) \\ &= -\mathbf{L} \left( \partial(f^* \square \varphi^*)(\mathbf{z} - \mathbf{L}^*\bar{\mathbf{v}}) \right) + \partial g^*(\bar{\mathbf{v}}) + \partial \ell^*(\bar{\mathbf{v}}). \end{aligned} \quad (5.17)$$

Therefore, by [10, Proposition 16.5(ii)], we derive from (5.16) that

$$\mathbf{0} \in \partial\left((\varphi^* \square f^*)(\mathbf{z} - \mathbf{L}^*\cdot) + \ell^* + g^* + \langle \cdot | \mathbf{r} \rangle\right)(\bar{\mathbf{v}}). \quad (5.18)$$

Hence, by Fermat's rule [10, Theorem 16.2] that  $\bar{\mathbf{v}}$  is a solution to (5.13), i.e,  $\bar{\mathbf{v}}$  is a solution to (4.3).

Now, in view of (2.8), algorithm (5.4) is a special case of the algorithm (4.4). Moreover, every specific conditions in Corollary 4.3 are satisfied.

(i) It follows from Corollary 4.3(i) that  $(x_{1,n}, \dots, x_{m,n}) \rightarrow (\bar{x}_1, \dots, \bar{x}_m)$  which solves the primal problem (5.1), and  $(v_{1,n}, \dots, v_{s,n}) \rightarrow (\bar{v}_1, \dots, \bar{v}_s)$  which solves the dual problem (5.3).

(ii)(iii) The conclusions follow from Corollary 4.3(iii)(iv) and Lemma 2.2(vi).  $\square$

**Remark 5.5** Here are some remarks

- (i) Sufficient conditions which ensure that the condition (5.2) is satisfied are provided in [19, Proposition 5.3]. For instance, if (5.1) has at least one solution and  $(r_1, \dots, r_s)$  belongs to the strong relative interior of

$$E = \left\{ \left( \sum_{i=1}^m L_{k,i} x_i - v_k \right)_{1 \leq k \leq s} \mid \left\{ \begin{array}{l} (\forall i \in \{1, \dots, m\}) \quad x_i \in \text{dom } f_i \\ (\forall k \in \{1, \dots, s\}) \quad v_k \in \text{dom } g_k + \text{dom } \ell_k \end{array} \right. \right\}. \quad (5.19)$$

- (ii) In the case when  $m = 1$  and  $(\forall n \in \mathbb{N})(i \in \{1, \dots, m + s\}) \lambda_{i,n} = \lambda_n$ , the algorithm (5.4) is in [26, Eq.(5.26)] where the connections to existing work are available.

## 6 Multi-dictionary signal representation

Dictionary has been used in minimization problems in signal processing in [24, Section 4.3]. Let us recall that a sequence of unit norm vectors  $(o_k)_{k \in \mathbb{K}}$  ( $\emptyset \neq \mathbb{K} \subset \mathbb{N}$ ) in  $\mathcal{H}$  is a dictionary with dictionary constant  $\mu$  in  $]0, +\infty[$  if

$$(\forall x \in \mathcal{H}) \quad \sum_{k \in \mathbb{K}} |\langle x | o_k \rangle|^2 \leq \mu \|x\|^2. \quad (6.1)$$

Then the dictionary operator is defined by

$$F: \mathcal{H} \rightarrow \ell^2(\mathbb{K}): x \mapsto (\langle x | o_k \rangle)_{k \in \mathbb{K}} \quad (6.2)$$

and its adjoint is

$$F^*: \ell^2(\mathbb{K}) \rightarrow \mathcal{H}: (\omega_k)_{k \in \mathbb{K}} \mapsto \sum_{k \in \mathbb{K}} \omega_k o_k. \quad (6.3)$$

Dictionary extends the notion of orthonormal bases and frames which plays an important role in the theory of signal processing due to their ability to efficiently capture a wide range signal features [2, 15, 20, 21] and the references therein. The focus of this section is to explore the

information of the original signals  $(\bar{x}_i)_{1 \leq i \leq m}$  which are assumed to be available on the coefficients of dictionaries  $((\langle \bar{x} | o_{i,j} \rangle)_{1 \leq i \leq m})_{j \in \mathbb{K}}$  and close to (soft constraints) nonempty closed convex subsets  $(C_i)_{1 \leq i \leq m}$  modeling its prior information. The rest of the information available will be modeled by potential functions  $(f_i)_{1 \leq i \leq m}$  (hard constraints). Furthermore, the data-fitting terms are measured by non-smooth functions.

**Problem 6.1** Let  $\mathcal{H}$  be a real Hilbert space, let  $m, s$  be strictly positive integers such that  $s > m$ , let  $\gamma \in ]0, +\infty[$ , and let  $\mathbb{K}$  be a nonempty subset of  $\mathbb{N}$ . For every  $i \in \{1, \dots, m\}$ , let  $\mathcal{G}_i = \ell^2(\mathbb{K})$ , let  $f_i \in \Gamma_0(\mathcal{H})$ , let  $(o_{i,j})_{j \in \mathbb{K}}$  be a dictionary in  $\mathcal{H}$  with associated dictionary operator  $F_i$  and dictionary constant  $\mu_i$ , let  $(\phi_{i,j})_{j \in \mathbb{K}}$  be a sequence in  $\Gamma_0(\mathbb{R})$  such that  $(\forall j \in \mathbb{K}) \phi_{i,j} \geq \phi_{i,j}(0) = 0$ , let  $C_i$  be a nonempty closed convex subset of  $\mathcal{H}$ . For every  $k \in \{m+1, \dots, s\}$ , let  $\mathcal{Y}_k$  be a real Hilbert space, let  $r_k \in \mathcal{Y}_k$ , let  $\beta_k$  be in  $]0, +\infty[$ . For every  $i \in \{1, \dots, m\}$  and every  $k \in \{m+1, \dots, s\}$ , let  $R_{k,i}: \mathcal{H}_i \rightarrow \mathcal{Y}_k$  be a bounded linear operator. Set  $C = \times_{i=1}^m C_i$ . The primal problems is to

$$\begin{aligned} \text{minimize}_{x_1 \in \mathcal{H}, \dots, x_m \in \mathcal{H}} \quad & \sum_{i=1}^m f_i(x_i) + \sum_{i=1}^m \sum_{j \in \mathbb{K}} \phi_{i,j}(\langle x_i | o_{i,j} \rangle) \\ & + \sum_{k=m+1}^s \beta_k \left\| r_k - \sum_{i=1}^m R_{k,i} x_i \right\| + \gamma d_C(x_1, \dots, x_m)^2/2 \end{aligned} \quad (6.4)$$

and the dual problem is to

$$\begin{aligned} \text{minimize}_{\xi_1 \in \ell^2(\mathbb{K}), \dots, \xi_m \in \ell^2(\mathbb{K}), v_{m+1} \in \mathcal{G}_{m+1}, \dots, v_s \in \mathcal{G}_s} \quad & \sum_{i=1}^m \left( (\sigma_{C_i} + \frac{1}{2\gamma} \|\cdot\|^2) \square f_i^* \right) \left( -F_i^* \xi_i - \sum_{k=m+1}^s R_{k,i}^* v_k \right) \\ & + \sum_{k=1}^m \sum_{j \in \mathbb{K}} \phi_{i,j}^*(\xi_{i,j}) + \sum_{k=m+1}^s \langle r_k | v_k \rangle. \end{aligned} \quad (6.5)$$

**Lemma 6.2** *Problem 6.1 is a special case of Problem 5.1 with*

$$\begin{cases} (i \in \{1, \dots, m\}) & z_i = 0 \quad \text{and} \quad \varphi = \gamma d_C^2/2, \quad v_0 = \gamma, \\ (\forall k \in \{1, \dots, m\}) (i \in \{1, \dots, m\}) & \ell_k = \iota_{\{0\}} \quad \text{and} \quad L_{i,i} = F_i \quad \text{and} \quad L_{k,i} = 0 \quad \text{otherwise}, \\ (\forall k \in \{1, \dots, m\}) & r_k = 0, \quad \mathcal{G}_k = \ell^2(\mathbb{K}) \quad \text{and} \quad g_k: \ell^2(\mathbb{K}) \rightarrow ]-\infty, +\infty]: \xi_k \mapsto \sum_{j \in \mathbb{K}} \phi_{k,j}(\xi_{k,j}), \\ (\forall k \in \{m+1, \dots, s\}) & \mathcal{G}_k = \mathcal{Y}_k \quad \text{and} \quad g_k = \beta_k \|\cdot\|, \quad \text{and} \quad (\forall i \in \{1, \dots, m\}) L_{k,i} = R_{k,i}. \end{cases} \quad (6.6)$$

*Proof.* Let us note that, by [10, Corollary 12.30],  $\varphi$  is a convex differentiable function with

$$(\forall \mathbf{x} = (x_i)_{1 \leq i \leq m} \in (\mathcal{H}_i)_{1 \leq i \leq m}) \quad \nabla \varphi(\mathbf{x}) = \gamma(\mathbf{x} - P_C \mathbf{x}) = \gamma(x_i - P_{C_i} x_i)_{1 \leq i \leq m}. \quad (6.7)$$

Since  $\text{Id} - P_C$  is firmly nonexpansive [10, Proposition 4.8],  $\nabla \varphi$  is  $\gamma$ -cocoercive. Next for every  $k \in \{1, \dots, s\}$ ,  $\mathcal{G}_k$  is a real Hilbert space and  $\ell_k \in \Gamma_0(\mathcal{G}_k)$  and by [27, Example 2.19],  $g_k \in \Gamma_0(\mathcal{G}_k)$ . Hence the conditions imposed on the functions in Problem 5.1 are satisfied. Now we have

$$(\forall v \in \mathcal{G}_k) \quad (\ell_k \square g_k)(v) = \inf_{w \in \mathcal{G}_k} (\ell_k(w) + g_k(v - w)) = g_k(v). \quad (6.8)$$



Therefore, in view of (6.2) and (6.6), we have

$$\begin{aligned}
(\forall i \in \{1, \dots, m\})(\forall x_i \in \mathcal{H}) \quad & \sum_{k=1}^m (\ell_k \square g_k) \left( \sum_{i=1}^m L_{k,i} x_i - r_k \right) = \sum_{i=1}^m g_i(F_i x_i) \\
& = \sum_{i=1}^m \sum_{j \in \mathbb{K}} \phi_{i,j}(\langle x_i \mid o_{i,j} \rangle). \quad (6.9)
\end{aligned}$$

We derive from (6.9), (6.6) and (6.8) that (5.1) reduces to (6.4). For every  $k \in \{m+1, \dots, s\}$ , let  $B_k(0; \beta_k)$  be the closed ball of  $\mathcal{Y}_k$ , center at 0 with radius  $\beta_k$ . Using [10, Example 13.3(v)], [10, Proposition 13.27] and [10, Example 13.23], we obtain

$$g_k^* = (\beta_k \|\cdot\|)^* = \iota_{B_k(0; \beta_k)} \text{ and } (\forall i \in \{1, \dots, m\}) \quad g_i^* : (\xi_{i,j})_{j \in \mathbb{K}} \mapsto \sum_{j \in \mathbb{K}} \phi_i^*(\xi_{i,j}), \quad (6.10)$$

and

$$\varphi^* = \sigma_C + (\gamma|\cdot|^2/2)^* \circ \|\cdot\| = \sigma_C + \|\cdot\|^2/(2\gamma) = \sum_{i=1}^m (\sigma_{C_i} + \|\cdot\|^2/(2\gamma)). \quad (6.11)$$

Moreover,

$$\varphi^* \square \left( \sum_{i=1}^m f_i^* \right) = \sum_{i=1}^m \left( (\sigma_{C_i} + \|\cdot\|^2/(2\gamma)) \square f_i^* \right). \quad (6.12)$$

We derive from (6.9), (6.6), (6.10), (6.11) and (6.12) that (5.3) reduces to (6.5).  $\square$

Lemma 6.2 allows to solve Problem 6.1 by Algorithm 5.3. More precisely,

**Algorithm 6.3** Let  $\varepsilon \in ]0, \min\{1, \gamma\}[$ , let  $(\lambda_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, 1]$ , let  $(\gamma_i)_{1 \leq i \leq s+m}$  be a finite sequence in  $[\varepsilon, +\infty[$  such that

$$(2\gamma - \varepsilon) \left( 1 - \sqrt{\sum_{i=1}^m \gamma_i \mu_i \gamma_{m+i} + \sum_{i=1}^m \sum_{k=m+1}^s \gamma_i \gamma_{m+k} \|R_{k,i}\|^2} \right) \geq \max_{1 \leq i \leq m, 1 \leq k \leq s} \{\gamma_i, \gamma_{m+k}\}. \quad (6.13)$$

For every  $i \in \{1, \dots, m\}$ , let  $((\alpha_{i,n,j})_{j \in \mathbb{K}})_{n \in \mathbb{N}}$  be sequences in  $\mathbb{R}$  such that

$$\sum_{n \in \mathbb{N}} \sqrt{\sum_{j \in \mathbb{K}} |\alpha_{i,n,j}|^2} < +\infty, \quad (6.14)$$

let  $(a_{i,n})_{n \in \mathbb{N}}$  be a absolutely summable sequence in  $\mathcal{H}$ , let  $(\lambda_{i,n})_{n \in \mathbb{N}}$  be sequence in  $]0, 1[$ , and for every  $k \in \{1, \dots, s\}$ , let  $(\lambda_{m+k,n})_{n \in \mathbb{N}}$  be sequence in  $]0, 1[$  such that

$$\sum_{n \in \mathbb{N}} \left( |\lambda_{i,n} - \lambda_n| + |\lambda_{m+k,n} - \lambda_n| \right) < +\infty. \quad (6.15)$$

Let  $(x_{i,0})_{1 \leq i \leq m} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m$ , and for every  $i \in \{1, \dots, m\}$ , let  $(\xi_{i,0,j})_{j \in \mathbb{K}} \in \ell^2(\mathbb{K})$  and

$(v_{k,0})_{m+1 \leq k \leq s} \in \mathcal{G}_{m+1} \times \dots \times \mathcal{G}_s$ . Set

For  $n = 0, 1, \dots$

$$\begin{array}{l}
\text{For } i = 1, \dots, m \\
\left[ \begin{array}{l}
p_{i,n} = \text{prox}_{\gamma_i f_i} \left( x_{i,n} - \gamma_i \left( \sum_{j \in \mathbb{K}} \xi_{i,n,j} o_{i,j} + \sum_{k=m+1}^s R_{k,i}^* v_{k,n} + \gamma (x_{i,n} - P_{C_i} x_{i,n}) \right) \right) + a_{i,n} \\
y_{i,n} = 2p_{i,n} - x_{i,n} \\
x_{i,n+1} = x_{i,n} + \lambda_{i,n} (p_{i,n} - x_{i,n})
\end{array} \right. \\
\text{For } k = 1, \dots, m \\
\left[ \begin{array}{l}
\text{For every } j \in \mathbb{K} \\
\left[ \begin{array}{l}
\xi_{k,n+1,j} = \xi_{k,n,j} + \lambda_{m+k,n} \left( \text{prox}_{\gamma_{m+k} \phi_j^*} (\xi_{k,n,j} + \gamma_{m+k} \langle y_{k,n} | o_{k,j} \rangle) + \alpha_{k,n,j} - \xi_{k,n,j} \right)
\end{array} \right. \\
\text{For } k = m+1, \dots, s \\
\left[ \begin{array}{l}
v_{k,n+1} = v_{k,n} + \lambda_{m+k,n} \left( \beta_k \frac{v_{k,n} + \gamma_{m+k} \left( \sum_{i=1}^m R_{k,i} y_{i,n} - r_k \right)}{\max \left\{ \beta_k, \left\| v_{k,n} + \gamma_{m+k} \left( \sum_{i=1}^m R_{k,i} y_{i,n} - r_k \right) \right\| \right\}} - v_{k,n} \right).
\end{array} \right.
\end{array} \right.
\end{array} \tag{6.16}$$

**Corollary 6.4** Suppose that (6.4) has at least one solution and  $(0, \dots, 0, r_{m+1}, \dots, r_s)$  belongs to the strong relative interior of

$$E = \left\{ \left( \sum_{i=1}^m L_{k,i} x_i - v_k \right)_{1 \leq k \leq s} \mid \left\{ \begin{array}{l}
(\forall i \in \{1, \dots, m\}) \quad x_i \in \text{dom } f_i \\
(\forall k \in \{1, \dots, m\}) \quad v_k \in \ell^2(\mathbb{K}), \sum_{j \in \mathbb{K}} \phi_j(v_{k,j}) < +\infty \\
(\forall k \in \{m+1, \dots, s\}) \quad v_k \in \mathcal{Y}_k
\end{array} \right\} \right\}, \tag{6.17}$$

where  $L_{k,i}$  is defined as in (6.6). Let  $(x_{1,n}, \dots, x_{m,n})_{n \in \mathbb{N}}$  and  $(\xi_{1,n}, \dots, \xi_{m,n}, v_{m+1,n}, \dots, v_{s,n})_{n \in \mathbb{N}}$  be sequence generated by Algorithm 6.3. Then  $(x_{1,n}, \dots, x_{m,n}) \rightarrow (\bar{x}_1, \dots, \bar{x}_m)$  a solution to (6.4), and  $(\xi_{1,n}, \dots, \xi_{m,n}, v_{m+1,n}, \dots, v_{s,n}) \rightarrow (\bar{\xi}_1, \dots, \bar{\xi}_m, \bar{v}_{m+1}, \dots, \bar{v}_s)$  a solution to (6.5). Furthermore, if  $C_j = \{0\}$ , for some  $j \in \{1, \dots, m\}$ , then  $x_{j,n} \rightarrow \bar{x}_j$ .

*Proof.* For every  $i \in \{1, \dots, m\}$  and every  $j \in \mathbb{K}$ , we have  $\phi_{i,j}^* \geq \phi_{i,j}^*(0) = 0$ . Therefore, we derive from (6.10) and [10, Proposition 23.31] that

$$(\forall \xi = (\xi_j)_{j \in \mathbb{K}} \in \ell^2(\mathbb{K})) \quad \text{prox}_{g_i^*} \xi = (\text{prox}_{\phi_{i,j}^*} \xi_j)_{j \in \mathbb{K}} \tag{6.18}$$

Next, for every  $k \in \{m+1, \dots, s\}$ , using (6.10) again, we have

$$(\forall v \in \mathcal{G}_k) \quad \text{prox}_{g_k^*} v = P_{B(0; \beta_k)} v = \beta_k v / \max\{\beta_k, \|v\|\}. \tag{6.19}$$

In view of (6.18), (6.19), (6.7) and the definition of  $((L_{k,i})_{1 \leq k \leq s})_{1 \leq i \leq m}$  in (6.6), the algorithm (6.16) is a special case of (5.4) with

$$(\forall n \in \mathbb{N})(\forall i \in \{1, \dots, m\})(\forall k \in \{1, \dots, s\}) \quad \begin{cases} U_{i,n} = \gamma_i \text{Id} & \text{and} & V_{k,n} = \gamma_{m+k} \text{Id}, \\ c_{i,n} = 0 & \text{and} & d_{k,n} = 0, \\ b_{i,n} = (\alpha_{i,n,j})_{j \in \mathbb{K}}. \end{cases} \tag{6.20}$$

Moreover, we derive from (6.14) that the sequences  $((b_{i,n})_{n \in \mathbb{N}})_{1 \leq i \leq m}$  are absolutely summable, and from (6.13) that (3.5) holds. Finally, since (5.1) has at least one solution and  $(0, \dots, 0, r_{m+1}, \dots, r_s)$

belongs to the strong relative interior of  $E$ , as mentioned in Remark 5.5(i) that (5.2) holds. To sum up, every specific conditions of Algorithm 5.3 and Corollary 5.4 are satisfied. Therefore, the conclusions follow from Corollary 5.4(i)(ii).  $\square$

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