



# Commutative and non-commutative parallelogram geometry: en experimental approach

Wolfgang Bertram

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# Commutative and Non-commutative Parallelogram Geometry: an Experimental Approach

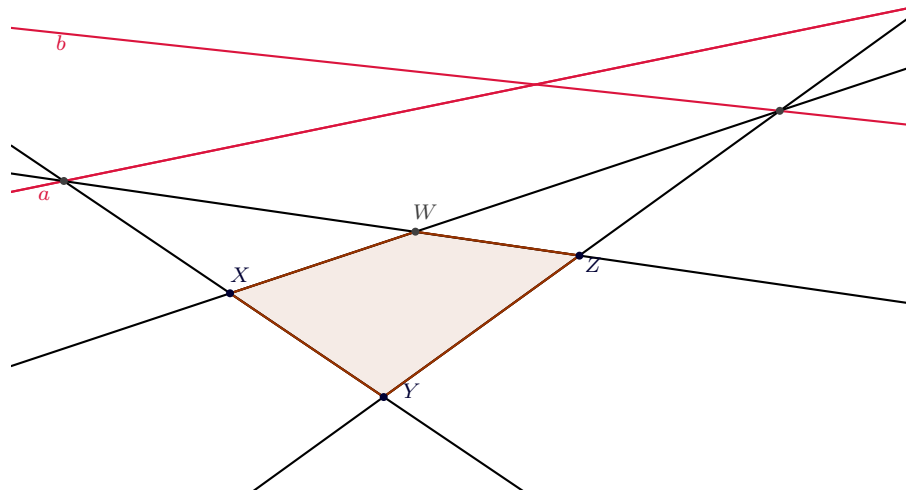
Wolfgang Bertram

## Abstract

By “parallelogram geometry” we mean the elementary, “commutative”, geometry corresponding to vector addition, and by “trapezoid geometry” a certain “non-commutative deformation” of the former. This text presents an elementary approach via exercises using dynamical software, hopefully accessible to a wide mathematical audience, from undergraduate students and high school teachers to researchers, proceeding in three steps:

- (1) *experimental geometry*,
- (2) *algebra (linear algebra and elementary group theory)*, and
- (3) *axiomatic geometry*.

**Introduction.** Sometimes, fundamental research leads to elementary results that can easily be explained to a wide audience of non-specialists, and which already bear the germ of the much more sophisticated mathematics lying at the background. In the present work I will try to show that our joint paper with M. Kinyon [BeKi12] is an example of this situation, and that even in such classical domains as plane geometry of points and lines, something new can be said and, maybe, taught. In a nutshell, everything is contained in the following figure:



The quadrangle  $XYZW$  is not a parallelogram, but its construction has something in common with the one of a parallelogram: we may call it a “non-commutative parallelogram”. This terminology is a wink towards “non-commutative geometry”: indeed, the

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2010 AMS classification: 06C05, 08A02, 51A05, 97G70, 97U70

picture illustrates the fundamental process of passing from a commutative, associative law (vector addition, corresponding to usual parallelograms) to a non-commutative, but still associative law: the key observation is that, when the lines  $a, b$  and the point  $Y$  are kept fixed, the law given by  $(X, Z) \mapsto W$  is associative and non-commutative; moreover it depends “nicely” on the parameters  $Y, a, b$ . Dynamical software, such as **geogebra**, permits to visualize such dependence in a very convincing way: we construct  $W$  as a *function* of  $X, Y, Z, a, b$ , and then directly *observe* its behaviour as the arguments move. In our opinion, this way of visualizing could give students a good intuitive idea of some basic notions from analysis in a geometric context, for instance, *continuity* and *singularity*. On the other hand, algebraic notions can also be visualized: *normal subgroups*, *semidirect products*, as well as their combination with notions from analysis: *deformation*, *contraction*.

This “experimental approach” (given in Chapter 1) automatically leads to formalization and to mathematical argumentation: to simplify the situation, we suppress the line  $b$  from our description by choosing it as “line at infinity”; then the function  $(X, Y, Z, a) \mapsto W$  may be studied from the point of view of affine analytic geometry – for vectors  $x, y, z, w \in \mathbb{R}^2$  and a linear form  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$  having the line  $a$  as kernel, this function is given by the formula (Chapter 2)

$$(A) \quad w = \alpha(z)\alpha(y)^{-1}(x - y) + z.$$

The same function can be studied from the point of view of “synthetic” geometry (Chapter 3), in terms of meets of lines ( $\wedge$ ) and joins of points ( $\vee$ )

$$(B) \quad w = \left( ((x \vee y) \wedge a) \vee z \right) \wedge \left( ((z \vee y) \wedge b) \vee x \right),$$

which places it into the context of *lattice theory*. In Chapter 2 we present, again in exercise form, the elementary linear algebra related to Formula (A), and in Chapter 3 some of the axiomatic aspects related to Formula (B). Since most of these exercises are non-standard, but rather easy, we hope that this presentation could be useful for teachers both at university or high-school looking for fresh and new material related to otherwise standard topics of the curriculum.

Finally, both formulas (A) and (B) lead to important generalizations and research topics: (A) generalizes to matrix or operator spaces (see section 2.5) and is closely related to the general framework of associative algebras (including  $C^*$ -algebras, and thus the framework of non-commutative geometry mentioned above); and formula (B) can be written for any lattice, and thus raises the question of the rôle that our function might play in the general theory of lattices. The case best understood so far is the case of lattices from *projective geometries*: for the specialist it will not be surprising that, axiomatically, *Desargues’ theorem* is at the root of associativity of the law  $(X, Z) \mapsto W$  (Chapter 3); but this law preserves many of its interesting properties (though not associativity) for certain *non-Desarguesian projective planes* (see [BeKi12]), and thus leads to the realm of geometries related to *non-associative algebras*, such as *alternative* or *Jordan algebras* (since *symmetric spaces* play an important rôle in this context, we take, in the experimental part, a closer look on *inversions* and *powers*). Some more remarks on these and other topics are given in the text, and the reader might agree that elementary questions can lead quickly to deep and non-trivial mathematical problems.

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# 1 Geometry: experimental approach

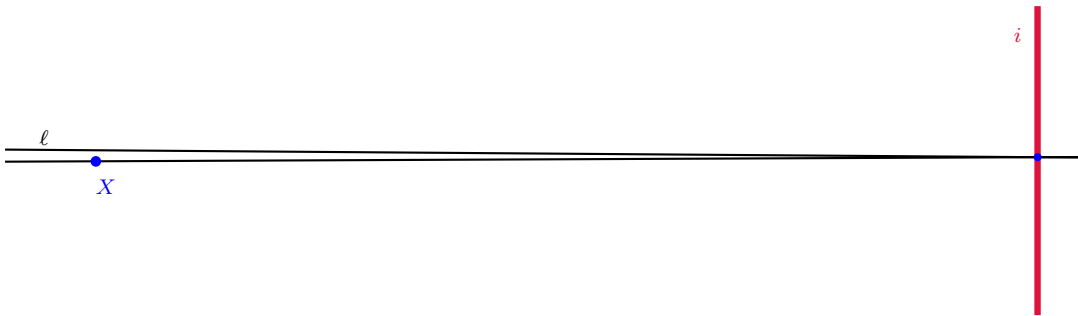
The following series of exercises is designed to give an “experimental approach to incidence geometry”: we draw figures only using points, lines joining points, intersections of lines, and parallels; we discover properties of these figures just by looking at dynamical images, but at this stage we do not aim at giving mathematical proofs. The first step of these exercises is to install on your computer some interactive geometry software – see [http://en.wikipedia.org/wiki/Interactive\\_geometry\\_software](http://en.wikipedia.org/wiki/Interactive_geometry_software) for an overview concerning existing software. For our purposes, we found that **geogebra**, freely available under the address <http://www.geogebra.org/cms/en>, is very convenient (and it has been used to create the images given in the sequel); for this reason we will use the word “geogebra” instead of the longer “interactive geometry software”.

## 1.1 Incidence geometry: preliminaries

Informally, *incidence geometry* may be defined as the geometry using only the following three kinds of the operations available on geogebra:

1. *line passing through two points*: for two distinct points  $x, y$  in the plane (or in space), we denote by  $x \vee y$  the affine line spanned by  $x$  and  $y$ ,
2. *intersection of two lines*: for two distinct and non-parallel lines  $u, v$  in the plane, we denote by  $u \wedge v$  their intersection point,
3. *parallel line*: for a given line  $\ell$  and a point  $x$ , there is exactly one line  $k$  through  $x$  that is parallel to  $\ell$ . We will use the notation  $k = x \vee (\ell \wedge i)$  to denote this unique parallel of  $\ell$  through  $x$ .

The notation used in the preceding point is simply a convention. The reason to use this convention is that we imagine a point “infinitely far” on the line  $\ell$ , given by intersecting  $\ell$  with an “infinitely far line”  $i$ ; then the parallel to  $x$  is the line joining this infinitely far point  $\ell \wedge i$  with  $x$  – the two horizontal lines in the following figure will become “true parallels”:



We don’t bother, for the moment, whether such a line  $i$  “really exists”: it is just a symbol in a mathematical formula which is executed by geogebra in the way just explained.

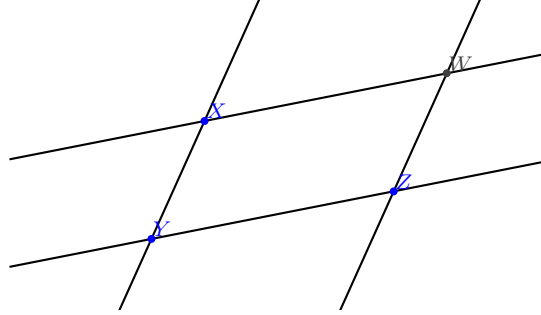
## 1.2 Parallelogram geometry

### 1.2.1 Usual view

Given three non-collinear points  $x, y, z$ , construct, using geogebra, a fourth point

$$w := \left( ((x \vee y) \wedge i) \vee z \right) \wedge \left( ((z \vee y) \wedge i) \vee x \right), \quad (1)$$

that is, according to our convention, the intersection of the parallel of  $x \vee y$  through  $z$  with the one of  $z \vee y$  through  $x$ . In other words, construct the last vertex  $w$  in a parallelogram having vertices  $x, y, z, w$ , like this:



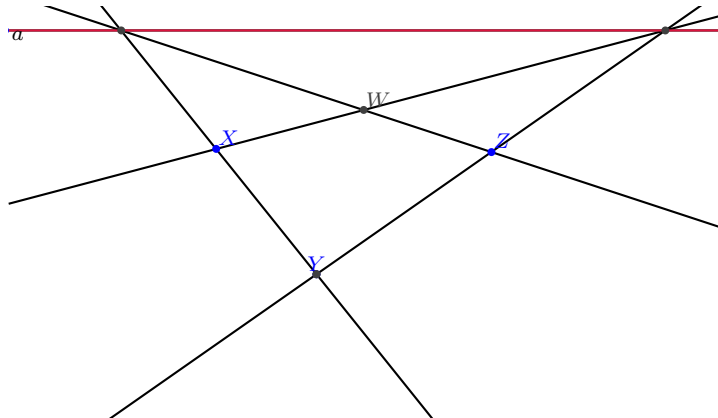
Now move one (or several) of the points  $x, y, z$ , and observe how the position of  $w$  changes when varying  $x, y$  or  $z$ . What happens to  $w$  if you drag the point  $x$  onto the line  $y \vee z$ : does it tend to some well-defined point? Note that the initial construction works well if we assume that  $x, y, z$  are not collinear, but it is not defined if  $x, y, z$  are on a common line, say  $\ell$ . Nevertheless, even if  $x, y, z$  are collinear, geogebra seems to attribute a well-defined position to the point  $w$  (in fact, a position on the line  $\ell$ )! In the language of analysis, this means that the map associating to the triple  $(x, y, z)$  the fourth point  $w$  is *continuous*, or, more precisely, that it admits a *continuous extension* from its initial domain of definition (non-collinear triples) to the bigger set of *all* triples. For geometry, the collinear case is of major importance – we will investigate below (see 1.2.5) in more detail what is going on in this case.

### 1.2.2 Perspective view

Let us have another view on a parallelogram: choose a line  $a$  in the plane (which you draw horizontally) and three non-collinear points  $x, y, z$  in the plane such that the lines  $x \vee y$  and  $z \vee y$  are not parallel to  $a$ . Using geogebra, construct the point

$$w := \left( ((x \vee y) \wedge a) \vee z \right) \wedge \left( ((z \vee y) \wedge a) \vee x \right) . \quad (2)$$

Your image may look like this:



You may imagine this drawing being a perspective view onto a plane in 3-dimensional space, where the red line  $a$  represents the “horizon”. Drag the line  $a$  in your drawing further and further up, away from  $x, y, z$ . Observe that the view becomes more and more

a view “from above”, and that the figure formed by  $x, y, z, w$  looks more and more like a “usual” parallelogram, which arises when  $a = i$  is the line at infinity. You may now move the point  $x$  (or  $z$ , or  $y$ ) and see what happens, in particular, in the collinear case, when  $x$  approaches the line  $y \vee z$ .

There is also another, rather special, kind of position for  $y$ , having the effect that the lines  $((x \vee y) \wedge a) \vee z$  and  $((z \vee y) \wedge a) \vee x$  are parallel, and hence the intersection point  $w$  does not exist in our drawing plane – find out this position! For simplicity, to avoid such complications, in the following subsections on parallelograms, we will stick to “usual” parallelograms, as in subsection 1.2.1 above (for perspective parallelograms, most of what we are going to say remains true, but the operations need not be defined for *all* triples  $(x, y, z)$ .)

### 1.2.3 Commutativity, associativity

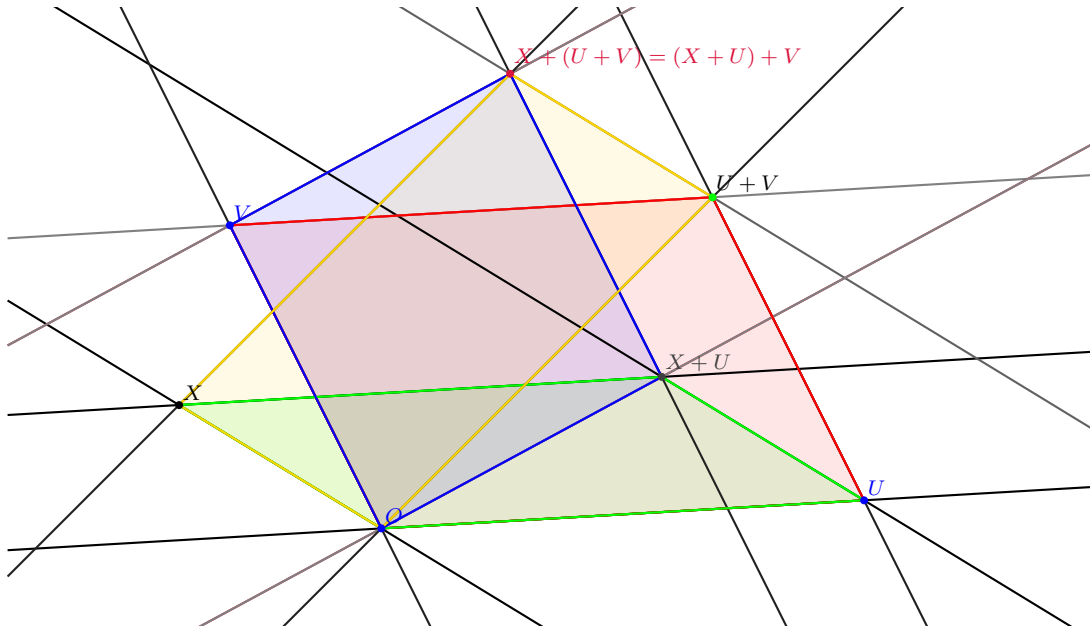
The fourth vertex  $w$  in a parallelogram depends on  $x, y, z$ , therefore we introduce the notation  $w := x +_y z$ , and sometimes we write  $w = (xyz)$ . In other words, we consider  $w$  as a function of  $(x, y, z)$ . If  $y$  is fixed as “origin”, we often write  $o$  instead of  $y$ , and let  $x + z := x +_o z$ .

The law  $(x, z) \mapsto x + z$  is commutative: indeed, this is obvious since the operations  $\vee$  and  $\wedge$  are symmetric in both arguments, and formula (1) is therefore symmetric in  $x$  and  $z$ , whence  $x + z = z + x$ .

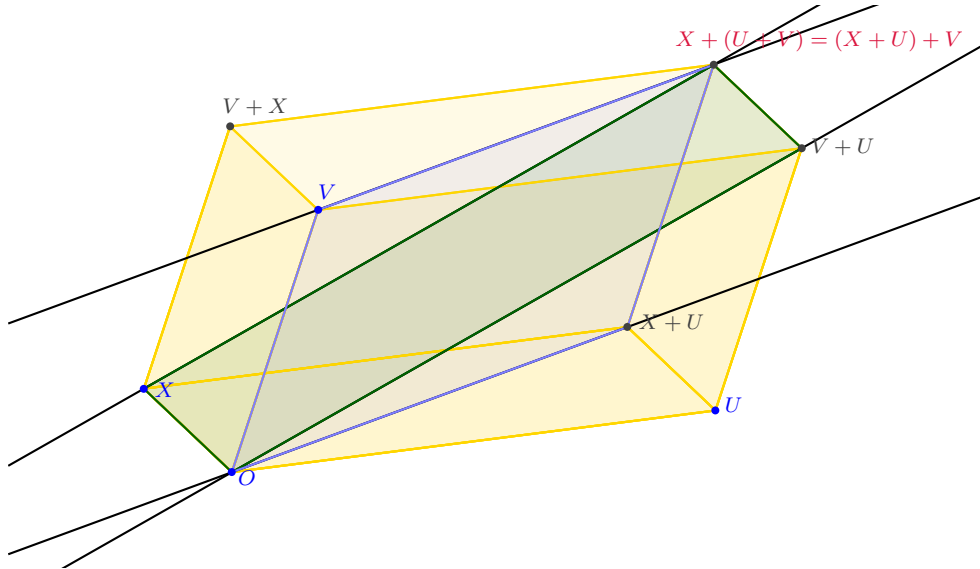
Exercise: show that  $+$  is associative:

$$(x + u) + v = x + (u + v). \quad (3)$$

In order to visualize this property, construct the points  $(x + u) + v$  and  $x + (u + v)$ , like in the following illustration; move the points  $x, u, v$  around and notice that the “last” vertex of the blue and of the yellow parallelogram always coincide. Convince yourself that this remains true even if some of these points are collinear!



Make another copy of the same file and add the point  $v + x$  to your image; mark parallelograms as in the following illustration such that a parallelepiped appears visually, having two of the parallelograms from the preceding illustration as “diagonal parallelograms”:



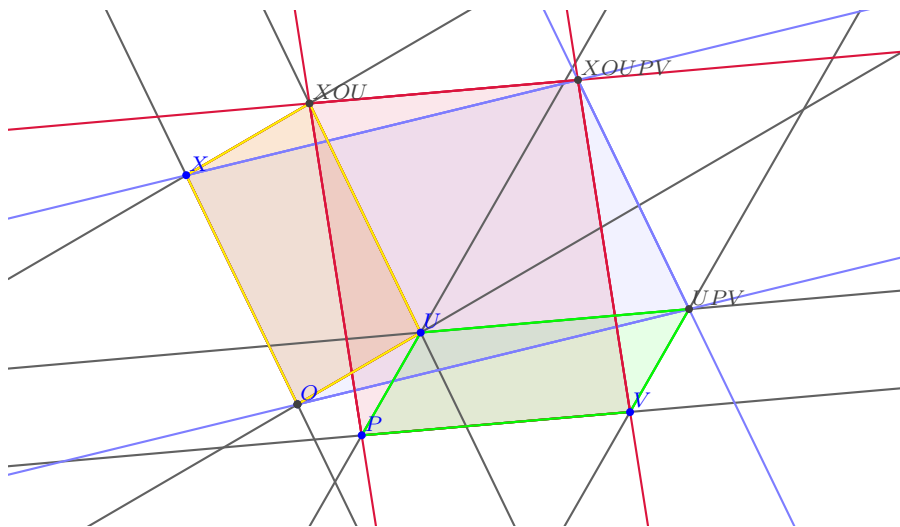
Using three-dimensional geometric intuition, you may find this image more convincing than the preceding one: it somehow “explains” why the associative law should be true. However, do not forget that our image really is two-dimensional, and three-dimensional imagination is just an artifice for understanding it. Note also that the point  $v + x$  is not really needed for the construction.

#### 1.2.4 Associativity: more general version

We have seen above that in all constructions the choice of the origin,  $o$  or  $y$ , was completely arbitrary: any point of the plane could play this rôle. Therefore it will be useful to be able to change the origin freely. This is facilitated by the following more general version of the associative law (called the *para-associative law*):

$$x +_o (u +_p v) = (x +_o u) +_p v \quad (4)$$

where  $o$  and  $p$  may be different points. Exercice: construct the points  $x +_o (u +_p v)$  and  $(x +_o u) +_p v$ , in a similar way as above:



(In the drawing we have used the notation  $UPV$ , etc., instead of  $U +_p V$ .) You will note that, wherever you move the points  $x, o, u, p, v$ , the “last” vertex of the blue and of the red parallelogram always coincide – this illustrates the para-associative law. If you drag  $o$  to  $p$ , this gives again the associative law from the preceding illustration.



### 1.2.5 Group law, and the collinear case revisited

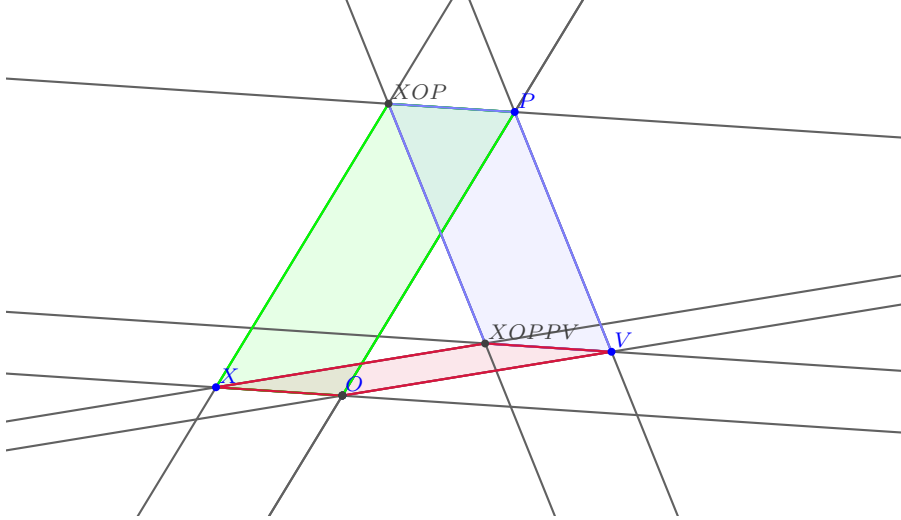
We want to show that (for any fixed origin  $o$ ) the plane  $P$  with  $x + z := x +_o z$  is a *commutative group* (with *neutral element*  $o$ ). All that remains to be shown is:

- (a)  $o$  is neutral:  $x + o = x = o + x$ , that is  $x +_y y = x = y +_y x$ ,
- (b) for each  $x$  there is an element  $z$  (called *inverse element*) such that  $x + z = o$ .

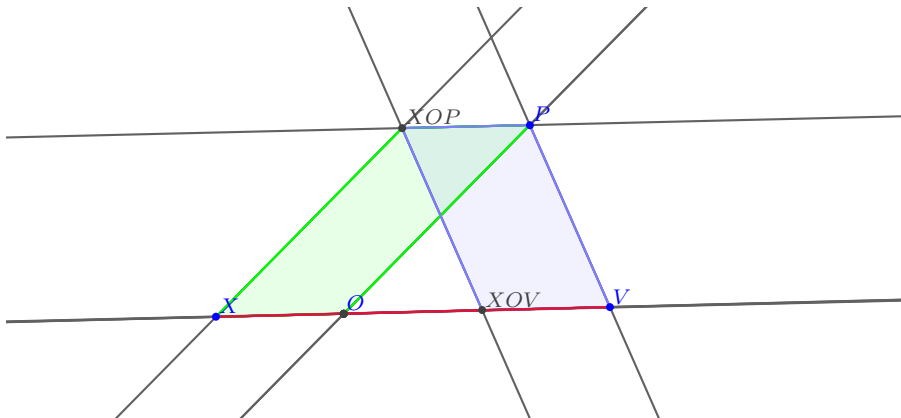
Using your dynamical image from subsection 1.2.1, convince yourself experimentally that (a) and (b) hold. As to (a), this seems to be rather obvious; as to (b), the problem arises that  $x, o, z$  will be collinear points, and hence our initial definition of  $x +_o z$  does not apply. Thus, for a better understanding of (b), we have to give, finally, a good definition of  $(xyz)$  for the “singular” case when  $x, y, z$  are collinear. With the help of the para-associative law, this can be done as follows: first of all, give an illustration of the special case  $u = p$  of the para-associative law:

$$(x +_o p) +_p v = x +_o (p +_p v) = x +_o v. \quad (5)$$

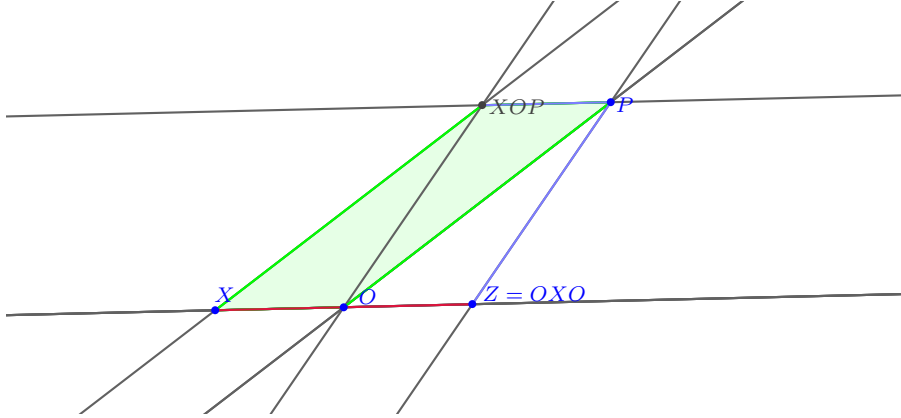
(We have used for the last equality that  $p +_p v = p$ .) Hence the vertex  $xoppv$  of the red triangle equals  $xov$  and thus does not depend on the choice of  $p$ :



Now flatten the red parallelogram more and more, until  $x, o, v$  become collinear. The fourth point  $xov$  is on the same line as  $x, o, v$ , and it still does not depend on  $p$ . Therefore we have found an incidence geometric construction of  $xov$  in the collinear case, namely: *choose any point  $p$  not on the same line as  $x, o, v$ , and then construct first  $xop$  and then  $(xop)p$ ; the result is  $xov$ :*



Using this interpretation, show that  $xoo = x$  (so  $x + o = x$ ), and that  $z = (oxo)$  is the inverse of  $x$ :



### 1.3 Trapezoid geometry

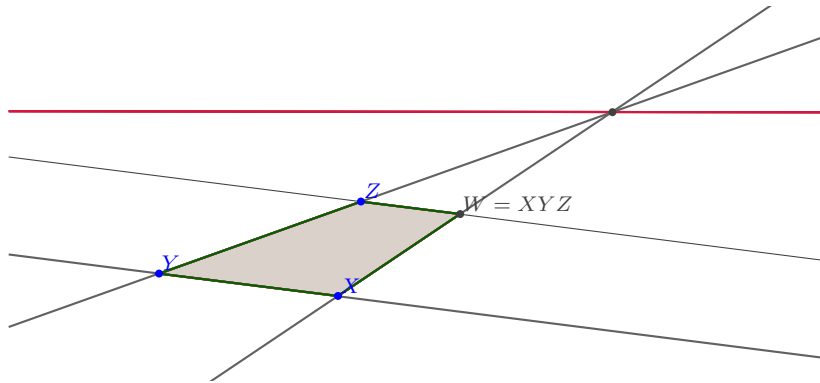
Now we are ready to introduce a kind of “twist” in the preceding constructions which will have the effect that we get *non-commutative* group laws. Instead of parallelograms we use trapezoids. The constructions will depend on the choice of some line  $a$  in the plane, to be considered as fixed for this section (drawn horizontally in the images). The underlying set of our constructions will be the set  $G := P \setminus a$  of all points of the plane  $P$  not on  $a$ .

#### 1.3.1 The generic construction: non-collinear case

We recall our convention on the “line at infinity”  $i$ . For the moment, let us fix also a point  $y$  not on  $a$ , and two other points  $x, z$  such that the line  $y \wedge z$  is not parallel to  $a$ . Construct the point

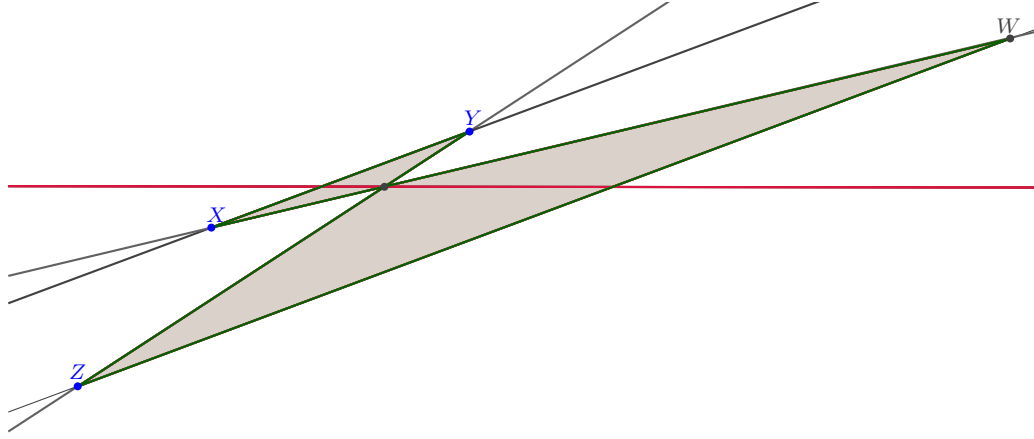
$$w := \left( ((x \vee y) \wedge i) \vee z \right) \wedge \left( ((z \vee y) \wedge a) \vee x \right). \quad (6)$$

Since the point  $w$  depends on  $x, y, z$ , in formulas will often write  $(xyz)$  for the point  $w$  given by (6) (and in drawings we will label it just by  $xyz$ ). The formula is a sort of hybrid between formulas (1) and (2): in these formulas, the same line appeared twice, whereas here *two different* lines appear. Thus the figure formed by  $x, y, z, w$  is no longer a parallelogram: it is a trapezoid. Keep the line  $a$  fixed and move the points  $x, y$  or  $z$  around. Observe. Your image may look like this:<sup>1</sup>



<sup>1</sup>A technical hint: use the “tool” command of geogebra to create a tool doing automatically this construction – this may be useful later on since we are going to use it a lot of times

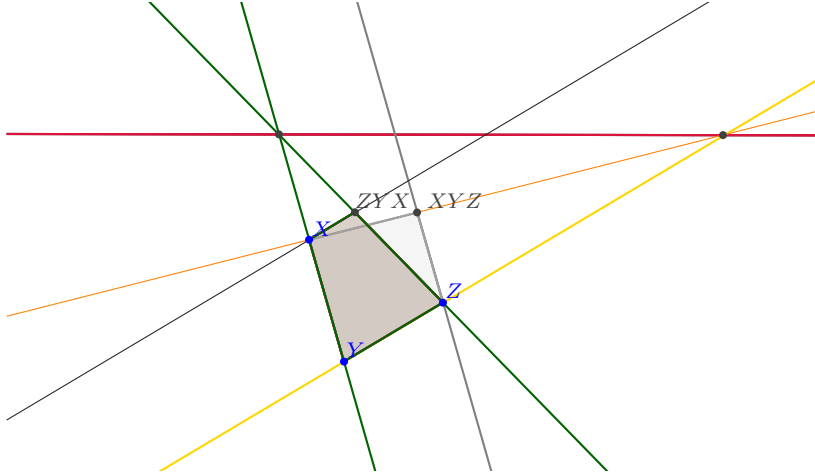
or like this (crossed trapezoid):



Note also that, if  $y \vee z$  happens to be parallel to  $a$ , then geogebra completes the construction in the appropriate way, namely by drawing a “true” parallelogram. Finally, you may also move the line  $a$ : observe that, if you move the line  $a$  “far away” from  $x, y, z$ , then the trapezoid converges to a “true” parallelogram; thus the parallelogram geometry from the preceding section can be seen as a “limit case” where  $a$  approaches the “line at infinity”  $i$ .

### 1.3.2 Non-commutativity

One observation that is probably not immediately visible is the fact that  $(xyz)$  is *not* symmetric in  $x$  and  $z$ : construct, by using another color, also the point  $(zyx)$  and observe that, except for some special situations (which ones?),  $(xyz)$  and  $(zyx)$  are different:



Although the product is not symmetric in  $x$  and  $z$ , there are, however, certain other “symmetry relations”: let  $R$  be the set of quadruples  $(x, y, z, w)$  of pairwise distinct points in  $G$  such that  $w = (xyz)$ . Show that  $(x, y, z, w) \in R$  if, and only if, the lines  $x \vee y$  and  $z \vee w$  are parallel and  $y \vee z$  and  $w \vee x$  meet on  $a$ . Conclude that the following four conditions are equivalent:

$$(xyz) = w, \quad (zwx) = y, \quad (yxw) = z, \quad (wzy) = x. \quad (7)$$

In other words,  $R$  is invariant under *double transpositions* (e.g., exchanging simultaneously  $(x$  and  $y)$  and  $(z$  and  $w)$ , and so on). We can rephrase this: *the relation  $R$*

is invariant under the Klein four group of double transpositions. In particular, letting  $w = (xyz)$  in the other three expressions from (7), deduce that

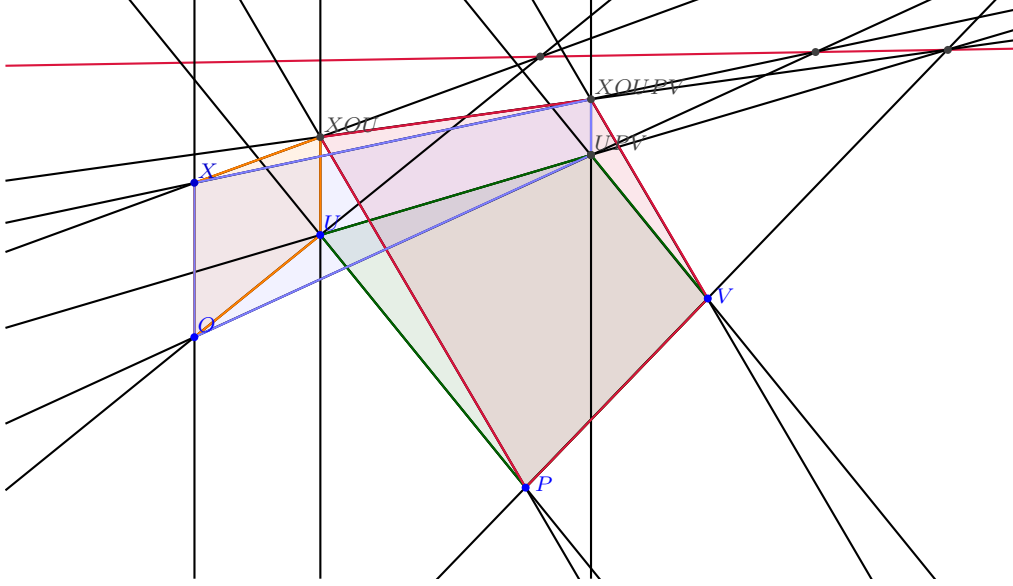
$$\boxed{(z(xyz)x) = y, \quad (yx(xyz)) = z, \quad ((xyz)zy) = x.} \quad (8)$$

### 1.3.3 Associativity

We have seen that the law  $(x, z) \mapsto w = (xyz)$  for fixed  $y$  is not commutative. It is all the more remarkable that it is *associative*, and even better: the generalized associativity law (4) holds, which we write in the new context in the form

$$\boxed{(xo(upv)) = ((xou)pv)}. \quad (9)$$

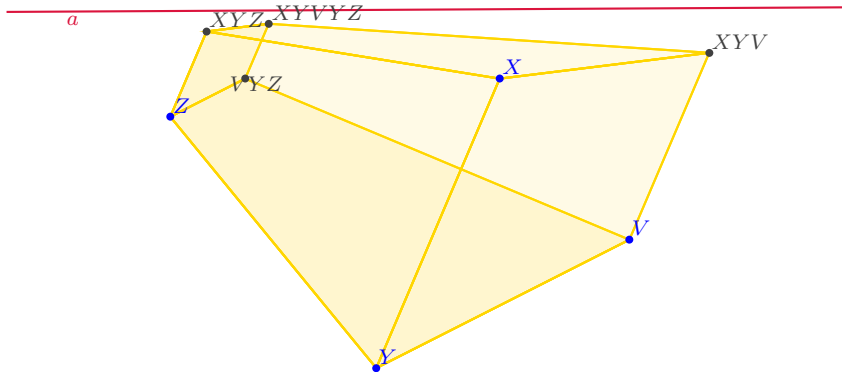
Exercise: construct the points  $(xo(upv))$  and  $((xou)pv)$ , as shown in the following illustration, and compare with the illustration in 1.2.4: the “last vertex” of the red trapezoid always coincides with the “last vertex” of the blue one.



If you drag  $o$  to  $p$ , you get the configuration of associativity of the law  $xz := (xoz)$ :  $x(uv) = (xu)v$ , that is,

$$(xo(uov)) = ((xou)ov). \quad (10)$$

In this case, there is a non-commutative analog of the “cubic” picture from 1.2.3 – we leave it to the reader to add the two “diagonal trapezoids of the non-commutative cube”:



### 1.3.4 Unit

In your drawing of the trapezoid  $(x, y, z, w)$ , drag the point  $x$  to  $y$  and observe that  $w = (xyz)$  tends to  $z$ , and similarly, if you drag  $z$  to  $y$ , then  $(xyz)$  tends to  $x$ . Since our “generic” construction of  $(xyz)$  is not defined for the cases  $x = y$  or  $x = z$ , we will now take as *definition* that

$$\boxed{(xyy) := (yyx) := x}. \quad (11)$$

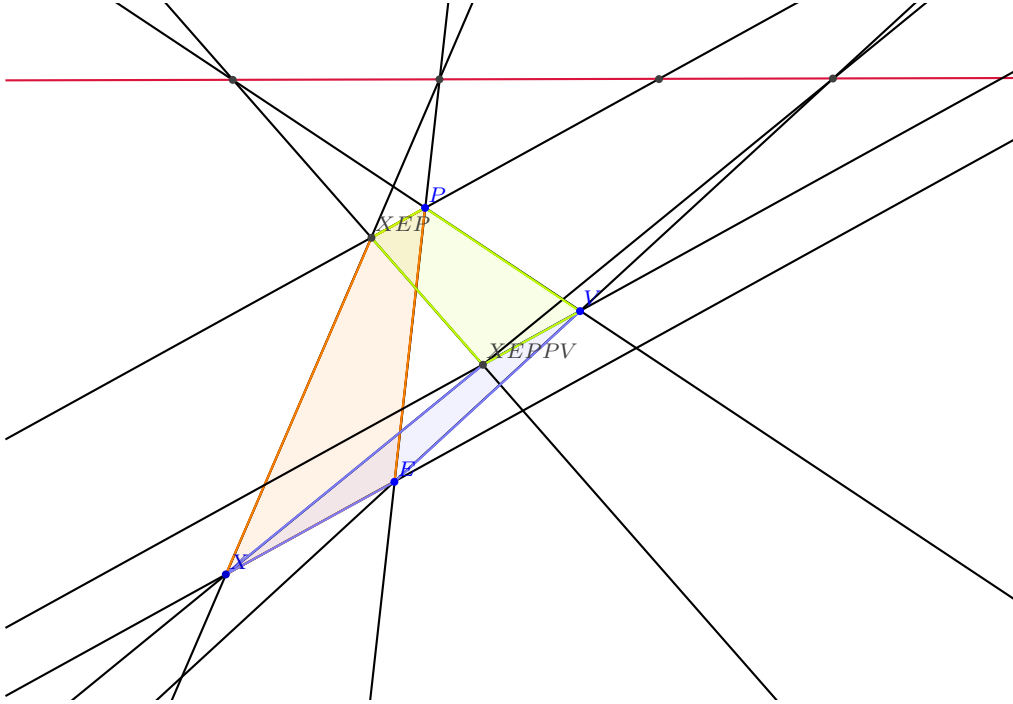
This identity is called the *idempotent law*. It means that, if we fix some element  $e \in G$ , then  $e$  is a *unit element* for the binary product  $xz = (xez)$ .

### 1.3.5 Trapezoid geometry: collinear case

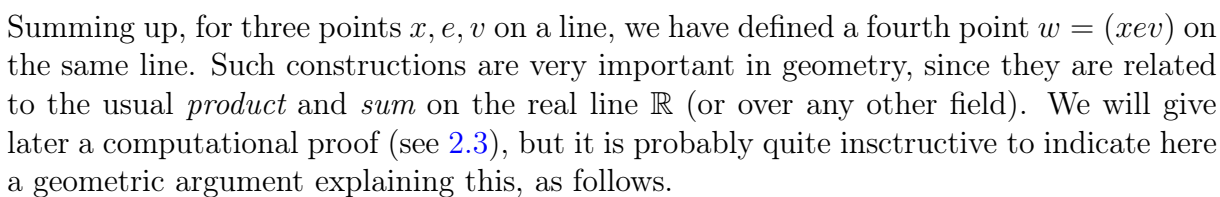
As in the case of parallelograms, using the generalized associativity law (9) together with the idempotent law (11), we can now better understand what is going on if  $x, y, z$  are collinear. Namely, assume that  $x, y, z$  are on a line  $\ell$  and choose any point  $p$  not on  $\ell$ . Associativity and idempotency give us

$$((xyp)pz) = (xy(ppz)) = (xyz),$$

hence in the collinear case  $(xyz)$  must be defined in the following way: *choose any point  $p$  not on the line spanned by  $x, y, z$ ; then define  $(xyz) := ((xyp)pz)$*  (note that the latter is defined by the generic construction). Illustrate this – reproduce first the illustration given above for the special case  $u = p$  (and writing  $e$  instead of  $o$ ):

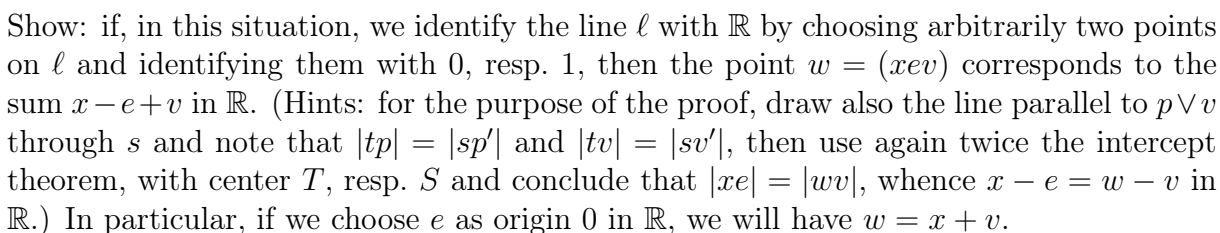


Next drag the point  $e$  to the line  $x \vee v$ . The point  $((xep)pz)$  then tends to a point on the same line, and observe that this point does not depend on  $p$ : it remains fixed if you move  $p$  around.



Using the preceding drawing, show: if we identify the line  $\ell = x \vee v$  with  $\mathbb{R}$  by choosing the intersection point of  $\ell$  with the horizontal line as origin  $0 \in \mathbb{R}$  (so  $x, e, v$  correspond to certain non-zero real numbers), then the point  $w = (xev)$  is identified with the element  $xe^{-1}v$  of  $\mathbb{R}$ . (Hints: let  $q := (xep)$ ; use the intercept (or Thales) theorem twice: using  $s$  as center, show that  $x/e = |q - u|/|p - u|$ , and using  $t$  as center, that  $w/v = |q - u|/|p - u|$ , whence  $w/v = x/e$ .) In particular, if we identify  $e$  with the element  $1 \in \mathbb{R}$ , this gives us  $w = xv$ .

There is, however, one special position of a different kind, namely when  $x, e, v$  all lie on a line that is parallel to  $a$ :



### 1.3.6 Group law

Let us put things together: convince yourself that, for any choice of origin  $e \in G$ , the set  $G = P \setminus a$ , is a group with product  $xz := (xez)$ . Indeed, associativity and property of the unit element have already been seen; and finally, every element  $x$  has an inverse element: check this first by direct inspection of your drawing of the trapezoid  $(x, e, z, w)$ : that is, check by trying that for each  $x$  there is some point  $z$  such that  $(xez) = e = (zex)$ .

By using the generalized associativity law, show that  $u = (exe)$  is the inverse of  $x$ . (Just check that  $(xeu) = e = (uex)$ .) Below we will inspect the geometric construction of the inverse.

Using associativity and idempotency, compute also  $((xe(eye))ez)$  and conclude that

$$\boxed{(xyz) = xy^{-1}z} \quad (12)$$

where the right hand side is computed in the group  $(G, e)$ . In other words, the ternary law  $(xyz)$  can be recovered from the binary product in the group with neutral element  $e$ .

### 1.3.7 One-parameter subgroups

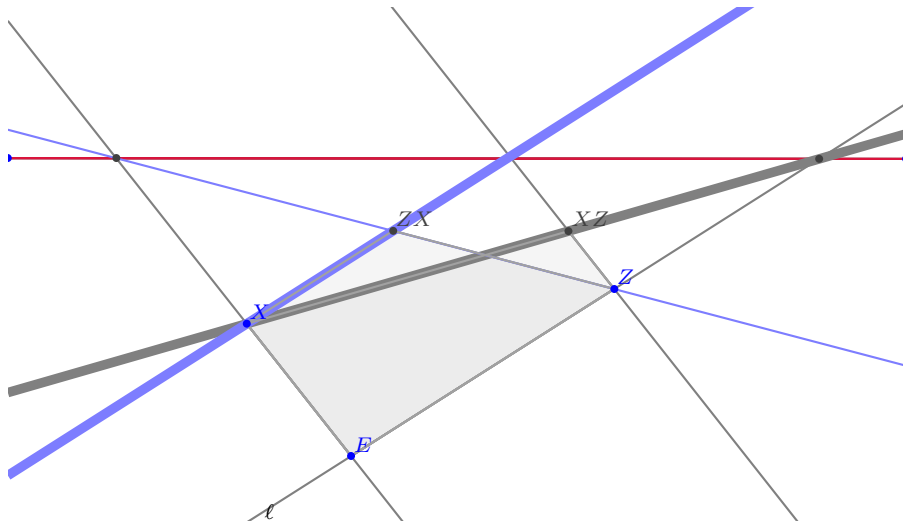
We wish to study the structure of the group  $(G, e)$  in more detail. First of all, note that  $G$  has *two connected components*: the “upper half plane” and the “lower half plane”; show that the half plane containing  $e$  (in our images: the lower one) is a subgroup, and the other is not (how are points from this other half plane multiplied?).

Show that every line  $\ell$  through  $e$  is a subgroup of  $G$  (it suffices to observe that, if  $x, e, z$  are on such a line, then  $(xez)$  is on the same line, and so are their inverses). Let us call a subgroup of this type a *one-parameter subgroup*. To be more precise: if  $\ell$  is not parallel to  $a$ , then the point  $\ell \wedge a$  has to be excluded, so the subgroup is really  $H := \ell \setminus a$  (as noted above, as a group, this is the multiplicative group  $\mathbb{R}^\times$ ). If  $\ell$  is parallel to  $a$ , then  $H := \ell$  is a subgroup (isomorphic to  $(\mathbb{R}, +)$ ).

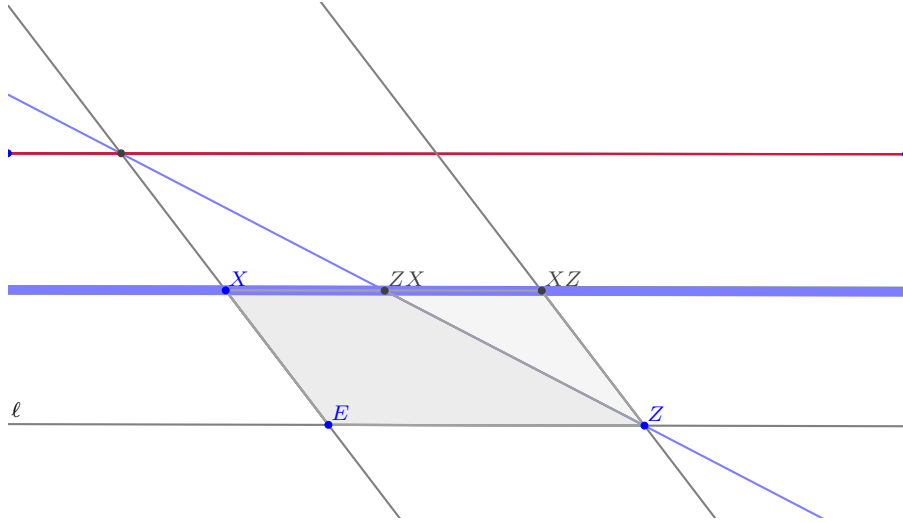
Fix an origin  $o$  and the line  $a$ , and let  $x, o, h$  not be collinear points. We have seen that the line  $H$  spanned by  $o$  and  $h$  is a subgroup of  $G$ . Draw the *left* and *right cosets* of  $H$  through  $x$ ,

$$xH := \{xz \mid z \in H\}, \quad Hx := \{zx \mid z \in H\}.$$

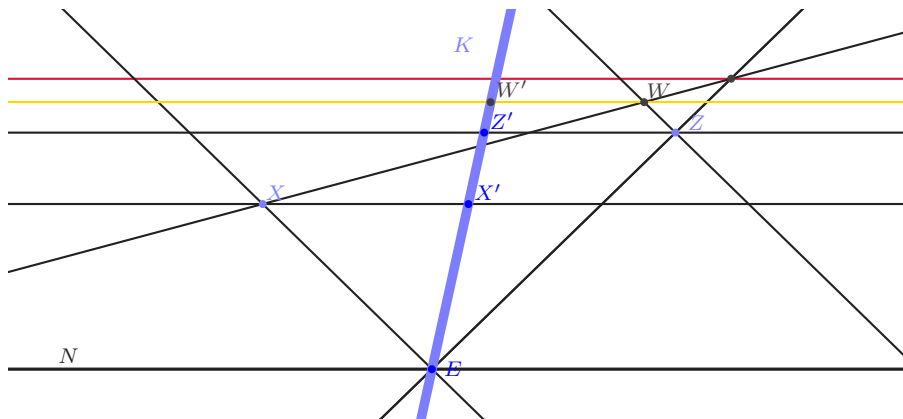
First of all, convince yourself that  $xH$  and  $Hx$  are lines passing through  $x$ . In your drawing, they may look as follows (the thick grey and blue lines, where  $z \in \ell$  is a typical element of the group  $H$ ):



Note that, in the drawing,  $xH$  and  $Hx$  are different lines. However, if you move the point  $z$  around (while  $e$  remains fixed), then there is one remarkable position for which we do have  $xH = Hx$ : show that this happens when the line  $\ell$  is parallel to  $a$  (this does not mean that the points  $xz$  and  $zx$  coincide, as you see here):

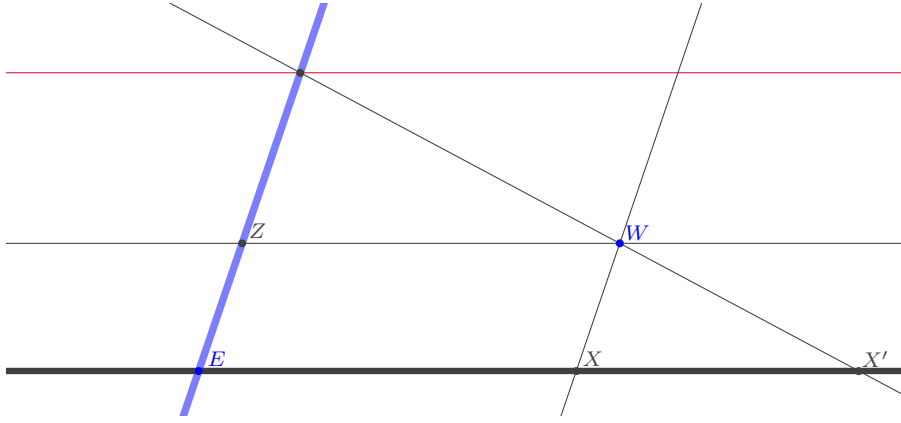


Recall from group theory: the subgroup  $H$  is called a *normal subgroup* if  $xH = Hx$  for all  $x \in G$ . We have seen:  $H$  is normal if, and only if,  $H$  is parallel to  $a$ . Thus there is precisely one one-parameter subgroup which is a normal subgroup; all others are not. Call this normal subgroup  $N$ , and fix some other one-parameter subgroup, say  $K$ . From group theory it is known that the quotient  $G/N$  is a group, and in fact this group is isomorphic to the complementary subgroup  $K$ . Illustrate this as follows: choose two lines parallel to  $N$  (and different from  $a$ ), and confine the variables  $x$  and  $z$  to these subgroups (that is, draw first the lines and then use the command “point on object”). Observe that the products  $xz$  then always lie on a third parallel to  $N$ , and hence we can “multiply” these parallels in a well-defined way. Since each such parallel intersects  $K$  in a unique point, the group  $K$  is indeed a group of representatives of these parallels, and the representatives  $x', z'$  are multiplied exactly as the parallels through them:



Finally, show that, for every element  $g \in G$ , there is exactly one  $z \in K$  and exactly one  $x \in N$  such that  $g = xz$  (we write then  $G = KN$ ), respectively such that  $g = z'x'$  (so  $G = NK$ ; show that then  $z = z'$  but in general  $x \neq x'$ ). Hint: given  $g = w$ , do the following construction:





Remark. In algebraic language,  $G$  is a *semidirect product* of  $N$  and  $K$ , see chapter 2. Thus  $G$  is isomorphic to a semidirect product of  $\mathbb{R}^\times$  with  $\mathbb{R}$ .

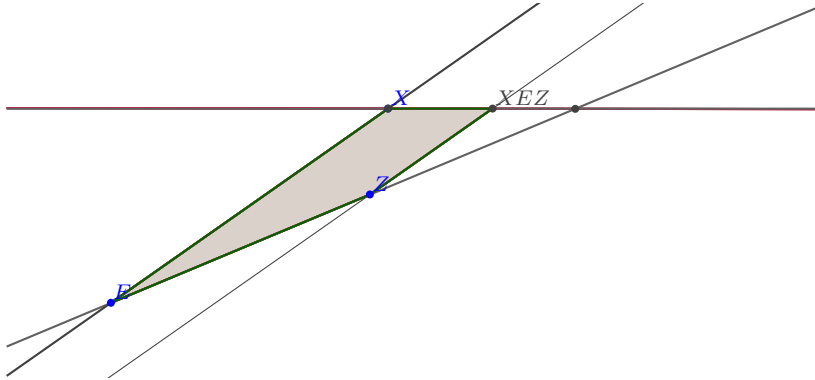
### 1.3.8 Boundary values and affine group of the line

So far we have excluded the line  $a$  from our space: we have assumed that  $x, y, z$  are not on  $a$ . However, the construction continues to make sense if one of the points  $x$  or  $z$  (but not  $y$ ) belongs to  $a$ : fix a point  $e \in G$  as origin and consider the product  $(xez)$  defined by formula (6). Using your drawing from section 1.3.1, observe what is going to happen:

- if  $z \in a$ , show that always  $(xez) = z$  (explain why!),
- if  $x \in a$ , show that  $(xez)$  always lies on  $a$ , and, more precisely, show that we get

$$(xez) = (((x \vee e) \wedge i) \vee z) \wedge a, \quad (13)$$

and this is the intersection of  $a$  with the parallel of  $x \vee e$  through  $z$ :



We have thus defined a map  $\rho(z) : a \rightarrow a, x \mapsto (xez)$ . Observe that the map  $\rho(z)$  described in the drawing has a *fixed point*: which one? Using the intercept theorem with respect to that fixed point, show that  $\rho(z)$  is a *dilation of the line a*. For which position of  $z$  does  $\rho(z)$  not admit any fixed points? Show that in this case  $\rho(z)$  is a *translation of a*. Summing up,  $\rho(z)$  is an *affine self-map of a*. Using the generalized associativity law, show that

$$\rho(zz') = \rho(z') \circ \rho(z)$$

(thus *the group  $(G, e)$  acts, via  $\rho$ , from the right on the line  $a$* ). Altogether, conclude that this establishes an isomorphism between the group  $(G, e)$  and the affine group of the line. Since the affine group of a line is a semidirect product of the additive group of a field with its multiplicative group, this is in keeping with what we observed above.

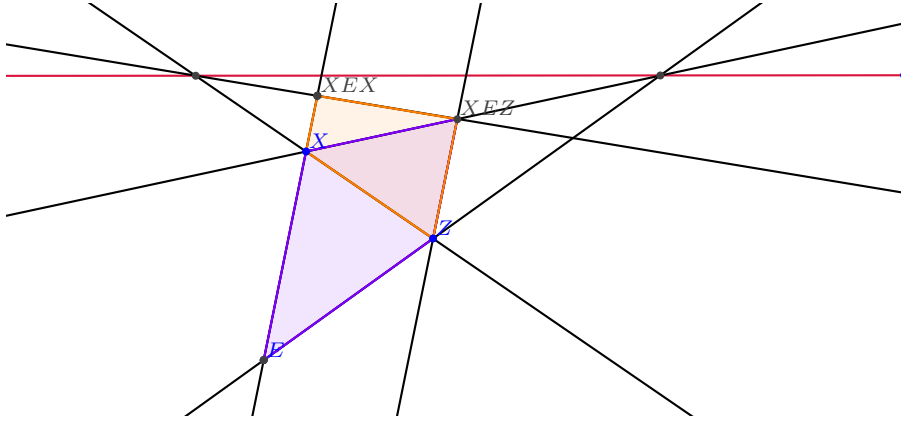
### 1.3.9 Transformations

In your drawing of the trapezoid configuration  $x, y, z, w$ , make all lines and auxiliary points invisible (in geogebra, click “object properties”, then turn off “show object”), so that only the four points remain (we won’t reproduce that image here since its “static” version is not interesting at all). Then move  $x$  and study the behaviour of  $w$ , and similarly for  $z$  and  $y$ : what can you say about the map  $x \mapsto w$ , resp.  $y \mapsto w$ , resp.  $z \mapsto w$ ? For instance, are there fixed points of these maps in  $G$ , or in  $P$ ? Are these maps bijections, and can they be of finite order? Can you reconstruct the line  $a$  from the behavior of these maps?

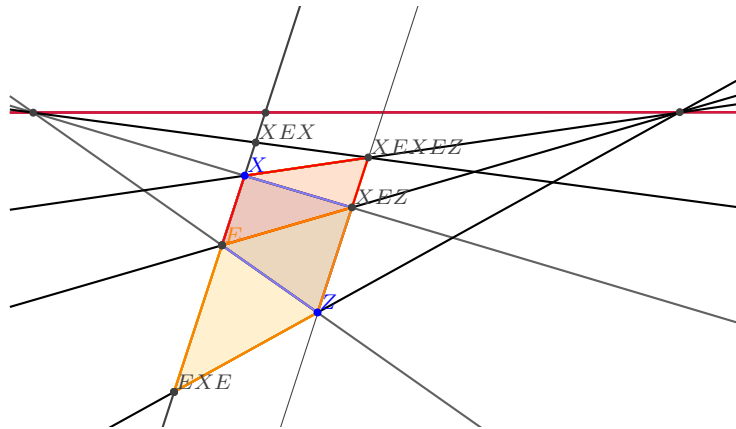
### 1.3.10 Powers and lattices

Let us come back to our group  $(G, e)$ . In a group with neutral element  $e$ , the *powers* of an element  $x$  are defined by  $x^{n+1} = xx^n$  and  $x^{-n} = (x^{-1})^n$ .

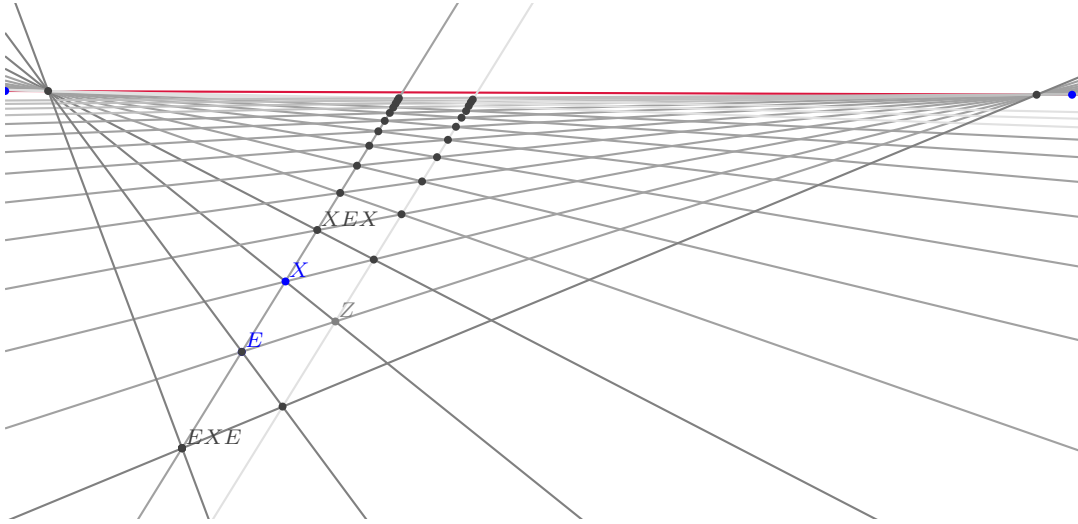
Exercise: fix an origin  $e$  and construct  $x^2 = (xex)$ . Note that this is a special case of the collinear construction from subsection 1.3.5: you have to choose an auxiliary element  $z$ . Hence  $x^2 = (xex) = (xe(zzx)) = ((xex)zx)$  is given by taking the product of  $(xex)$  and  $x$  with respect to  $z$ :



Move the point  $z$ : you will notice that the point  $x^2$  does not depend on the choice of  $z$ . Next construct  $x^{-1}$ . Since  $x^{-1} = (exe)$  is nothing but the square of  $e$  with respect to the origin  $x$ , we can use the same kind of construction as above:



Now you have seen the principle, construct at least ten powers of  $x$ , and then move again  $x, e, z$ :



The powers  $\{x^n \mid n \in \mathbb{Z}\}$  of an element  $x \in G$  form a *discrete subgroup* of  $G$ .

## 2 Computational approach: linear algebra

Geometry can be translated into “analytic formulas” by identifying the plane  $E$  with the real vector space  $\mathbb{R}^2$ . Vectors are written  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$ , and their sum is defined by  $x + y = (x_1 + y_1, x_2 + y_2)$ . In the following, the reader should avoid as much as possible to use the components (that is, avoid to use the standard basis of  $\mathbb{R}^2$ ), but just the vector space properties of  $E$ .

### 2.1 Some reminders

For two distinct points  $x, y \in E$ , recall that the *affine line spanned by  $x$  and  $y$*  is

$$x \vee y = \{tx + (1 - t)y \mid t \in \mathbb{R}\}. \quad (14)$$

Let  $z \vee w$  be another such line. Prove that the following are equivalent:

- (1) the vector  $z - w$  is a multiple of the vector  $x - y$ ,
- (2) either  $x \vee y = z \vee w$ , or  $(x \vee y) \cap (z \vee w) = \emptyset$ .

We then say that the lines  $x \vee y$  and  $z \vee w$  are *parallel*, and we write  $(x \vee y) \parallel (z \vee w)$ .

### 2.2 Parallelograms and vector addition

#### 2.2.1 Usual parallelograms

Let  $(x, y, z, w)$  a quadruple of points in the plane  $E$ , no two of them equal. Show that the following are equivalent:

- (1)  $(x \vee y) \parallel (z \vee w)$  and  $(x \vee w) \parallel (y \vee z)$ ,
- (2)  $w = x - y + z$ ,
- (3) the quadrangle with vertices  $xyzw$  (in this order) is a parallelogram.

Show that, for a fixed element  $y \in E$ , the law  $x +_y z := x - y + z$  defines a commutative group law on  $E$  with neutral element  $y$ . For  $y = 0$ , we get back usual vector addition.

*Some analysis.* Recall the notion of *open* subsets of  $\mathbb{R}^n$ . We may identify  $E^3$  with  $\mathbb{R}^6$ . Show that the set  $U := \{(x, y, z) \in E^3 \mid x \neq y \neq z, x \neq z\}$  is *open and dense* in  $E^3$  and that the map

$$U \rightarrow E, \quad (x, y, z) \mapsto x - y + z$$

is *continuous*. Show that this map admits a unique continuous extension to a map  $E^3 \rightarrow E$ , given by the same formula.

### 2.2.2 Perspective view

This exercise is optional (it is not needed in the sequel). Let  $\alpha : E \rightarrow \mathbb{R}$  to be a non-zero linear form and  $a := \ker(\alpha)$ . Recall that  $\dim a = \dim E - 1 = 1$ , so  $a$  is a line. (You may take  $\alpha(x) = x_2$  to get a horizontal line.) Let  $x, y, z$  in the plane such that the lines  $x \vee y$  and  $z \vee y$  are not parallel to  $a$ . Show that the point  $w$  defined by Equation (2) is given by the formula

$$w = \frac{\alpha(y)\alpha(z)}{d}x - \frac{\alpha(x)\alpha(z)}{d}y + \frac{\alpha(y)\alpha(x)}{d}z, \quad (15)$$

with denominator  $d = \alpha(y)\alpha(x) - \alpha(x)\alpha(z) + \alpha(y)\alpha(z)$ . (Hint: computations may be simplified by observing that the given expression of  $w$  is a certain *barycenter* of the three points  $x, y, z$ .) Note that this formula makes sense also if  $x, y, z$  are collinear, and also if one of these points belongs to the line  $a$  (what do you get if  $x \in a$ ? if  $z \in a$ ? if  $y \in a$ ?).

Describe the set of all  $(x, y, z) \in E^3 = \mathbb{R}^6$  such that  $d \neq 0$  and show that it is open and dense in  $E^3$ , and that the map  $(x, y, z) \mapsto w$  is continuous on this set. Compare with your image from section 1.2.2: what happens when  $d = 0$ ?

## 2.3 The linear algebra of trapezoid geometry

In this section, we fix a line  $a$  given by  $a = \{x \in \mathbb{R}^2 \mid \alpha(x) = 0\}$  for some non-zero linear form  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ ; to keep close to the drawings, you may choose  $\alpha(x) = -x_2$  (but try to avoid using this explicit formula). The set  $G$  is defined by

$$G := \{x \in E \mid \alpha(x) \neq 0\}. \quad (16)$$

### 2.3.1 Main formula

Show that the point  $w$  defined by Equation (6) is given by the formula

$$\boxed{(xyz) = w = \alpha(z)\alpha(y)^{-1}(x - y) + z}. \quad (17)$$

Hints: show first that  $w$  is of the form  $z + t(x - y)$  with some  $t \in \mathbb{R}$ ; then determine  $t$  by using the intercept theorem with center  $u$  being the intersection  $a \wedge (y \vee z)$  – show that the ratio  $|w - z|/|x - y|$  is equal to the ratio  $|u - z|/|u - y|$ , which in turn is equal to  $\alpha(z)/\alpha(y)$ , and conclude that  $t = \alpha(z)\alpha(y)^{-1}$ .

### 2.3.2 Group structure

Fix a point  $e \in G$  such that  $\alpha(e) = 1$ . Show that  $G$  is a group with neutral element  $e$  and product

$$x \cdot z = (xez) = \alpha(z)(x - e) + z. \quad (18)$$

Hint: start by proving the very useful formula

$$\boxed{\alpha((xyz)) = \alpha(x)\alpha(y)^{-1}\alpha(z)} \quad (19)$$

which gives you

$$\alpha(x \cdot z) = \alpha(x)\alpha(z), \quad (20)$$

and deduce from this that  $G$  is indeed stable under the product and next that the product is associative. Then show that  $e \cdot x = x = x \cdot e$ , and that the inverse of  $x$  is given by

$$x^{-1} = \frac{1}{\alpha(x)}(e - x) + x.$$

*Some analysis.* Show that  $G$  is open dense in  $E$  and that the group law  $G \times G \rightarrow G$  and the inversion map  $G \rightarrow G$ ,  $x \mapsto x^{-1}$  are continuous and even smooth of class  $C^\infty$ .

*Group structure: semidirect product.* Show that  $\alpha|_G : G \rightarrow \mathbb{R}^\times$  is a group homomorphism and that its kernel  $N = \{x \in G \mid \alpha(x) = 1\}$  is a subgroup isomorphic to  $(\mathbb{R}, +)$ . Show that the set

$$H := \mathbb{R}^\times e = \{re \mid r \in \mathbb{R}^\times\}$$

is a subgroup of  $G$ , and that  $\phi : \mathbb{R}^\times \rightarrow H$ ,  $r \mapsto re$  is a group isomorphism such that  $\alpha(\phi(t)) = t$ . This means that  $G$  is a semidirect product of  $N \cong (\mathbb{R}, +)$  with  $H \cong (\mathbb{R}^\times, \cdot)$ . For a complete description of  $G$  it remains to determine the action of  $H$  on the normal subgroup  $N$ : show that this action is the “standard action up to an inversion”, that is,

$$\mathbb{R}^\times \times \mathbb{R} \rightarrow \mathbb{R}, \quad (r, t) \mapsto r^{-1}t.$$

Hint: either use a direct computation to show that

$$re \cdot x \cdot (re)^{-1} = e + r^{-1}(x - e),$$

or change the origin as explained below (which is better adapted to the additive structure of the normal subgroup). In any case, you may now conclude that  $G$  is isomorphic to a semidirect product  $\mathbb{R} \rtimes \mathbb{R}^\times$ , and hence  $G$  is isomorphic to the “ $ax + b$ -group”, i.e., to the affine group of the real line, in matrix form:

$$\text{GA}(1, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}. \quad (21)$$

*Change of origin.* Sometimes it may be useful to present the group  $G$  in a different way: we translate the neutral element  $e$  to the origin 0 of the vector space  $E$ , that is, we translate  $G$  to the set  $G' := \{x \in E \mid \alpha(x) \neq 1\}$ , with the new product  $(x, y) \mapsto (x + e) \cdot (y + e) - e$ . Show that this new product is given by

$$G' \times G' \rightarrow G', \quad (x, z) \mapsto x + \alpha(x)z + z \quad (22)$$

and the new inverse by

$$x^{-1} = -\frac{1}{\alpha(x) + 1}x. \quad (23)$$

Compare with formula (18): in (18) it is not possible to take easily a “limit  $\alpha \rightarrow 0$ ”, whereas here this is possible: for  $\alpha = 0$  one gets vector addition as limit case!

### 2.3.3 Generalized associativity: torsors

Using (19), show that  $G$  is stable under the ternary product  $(x, y, z) \mapsto (xyz)$ , and that the ternary product  $(xyz)$  satisfies the generalized associativity law (9) and the idempotency law (11). In other words, show that  $G$  with product  $(xyz)$  given by formula (17) is a *torsor*:

**Definition.** A torsor<sup>2</sup> is a set  $G$  together with a ternary “product map”  $G^3 \rightarrow G$ ,  $(x, y, z) \mapsto (xyz)$  satisfying the idempotency law (11) and the generalized associative law (9). A homomorphism of torsors is a map between torsors respecting the ternary products. A torsor is called *commutative* if it satisfies the identity  $(xyz) = (zyx)$ .

In the following exercises let us collect some properties that hold for *all* torsors:

*From torsors to groups.* Let  $G$  be a torsor, and an arbitrary point  $e \in G$ . Show that  $(G, e)$  is a group with product  $x \cdot z := (xez)$ . (Hint: show that the inverse element of  $x$  is  $x^{-1} = (exe)$ .)

*From groups to torsors.* Let  $(G, e)$  be a group. Show that  $G$  with ternary law  $(xyz) = xy^{-1}z$  is a torsor. (Note: when  $G$  is commutative, the group law is often written additively, and then torsor law becomes  $x - y + z$ . Example: the torsors  $\mathbb{R}^n$ . On the other hand, for the commutative torsor  $\mathbb{R}^\times$  one uses of course a multiplicative notation.)

*Back and forth.* Show that the preceding two constructions are inverse to each other. Conclude that torsors are “the same as groups where one has forgotten the origin”. (If you prefer more formal statements: there is an equivalence of categories between the categories of groups and of torsors with a distinguished base point.)

*Para-associativity.* Show that in any torsor the *para-associative law*

$$(xw(vuz)) = (x(uvw)z) = ((xwv)uz) \quad (24)$$

holds. (Remark: the analog of a semigroup is a *semitorsor* – one just demands (24) to hold, but not necessarily the idempotency law.)

*Subtorsors.* A *subtorsor* is a subset  $H$  of a torsor  $G$  which is stable under the ternary law. Fix a point  $e \in G$ . Show that  $H \subset G$  is a subgroup iff  $H$  is a subtorsor containing  $e$ . (Remark: this characterization is simpler than the one for groups, since we do not have to investigate separately that inverses again belong to  $H$ .)

*Symmetry relations.* Show that for a quadruple of elements of a torsor the following four conditions are equivalent:

$$(xyz) = w, \quad (zwx) = y, \quad (yxw) = x, \quad (wzy) = x, . \quad (25)$$

In other words, the graph  $R$  of the ternary product map  $G^3 \rightarrow G$ ,

$$R = \{(x, y, z, w) \in G^4 \mid w = (xyz)\},$$

is invariant under double permutations, i.e., under the *Klein four group*. Deduce that in any torsor the identities (7) are satisfied. Assume that the torsor is commutative: which

---

<sup>2</sup>A remark on terminology is in order here: torsors are for groups what affine spaces are for vector spaces, and the concept named here “torsor” is at least 70 years old, see [Cer43]; however, it has not really become part of the mathematical mainstream, possibly because there is no universally accepted terminology – other terms such as *groud*, *heap*, *flock*, *principal homogeneous space* are also used, see [BeKi10a] for some more comments on the terminology we use.

subgroup of permutations of the four letters  $(x, y, z, w)$  leaves  $R$  invariant then? Compare with the proof given in section 1.3.2.

*Left- and Right translations.* In any torsor, define *left and right translation operators* by

$$L_{xy} : G \rightarrow G, \quad z \mapsto (xyz), \quad R_{zy} : G \rightarrow G, \quad x \mapsto (xyz). \quad (26)$$

Show that the left translations and the right translations form groups of bijections of  $G$ , both isomorphic to  $(G, e)$ , for any choice of origin  $e \in G$ .

### 2.3.4 Collinear case: one-parameter subgroups and powers

Now let again  $G$  be defined by (16) with ternary torsor law given by (17). Let  $e \in G$  be an arbitrary point. Show that  $G$  with product  $(x, z) \mapsto (xez)$  is a group, and that the inverse in this group is

$$x^{-1} = (exe) = \alpha(e)\alpha(x)^{-1}(e - x) + e. \quad (27)$$

Give a computational proof of the affirmations from subsection 1.3.5: let  $\ell$  be an affine line in the plane  $E$  and  $e \in \ell$ . Show that  $\ell$  is parallel to  $a$  iff  $\alpha(x) = \alpha(y)$  for all  $x, y \in \ell$ , and  $\ell$  is not parallel to  $a$  iff the restriction  $\phi : \ell \rightarrow \mathbb{R}, x \mapsto \alpha(x)$  is a bijection. Using this, show:

- (1) if  $\ell$  is parallel to  $a$ , then  $\ell$  is a subgroup of  $G$ , isomorphic to  $(\mathbb{R}, +)$
- (2) if  $\ell$  is not parallel to  $a$ , then  $\ell \cap G$  is a subgroup of  $G$ , isomorphic to  $(\mathbb{R}^\times, \cdot)$ . (Hint: use equation (19) in order to show that  $\phi$ , restricted to  $\ell \cap G$ , defines an isomorphism of groups.)

Recall that the powers in the group  $(G, e)$  are defined by  $x^2 = (xex)$ ,  $x^3 = (xe(xex))$ ,  $x^{n+1} = (xex^n)$ . Prove the following explicit formula for the powers:

$$x^n = \frac{\alpha(x)^n \alpha(e)^{-n+1} - \alpha(e)}{\alpha(x - e)} x + \frac{\alpha(x) - \alpha(x)^n \alpha(e)^{-n+1}}{\alpha(x - e)} e \quad (28)$$

Hints: we have seen that  $\alpha : G \rightarrow \mathbb{R}^\times$  is a torsor homomorphism, and it is an isomorphism when restricted to the line  $(x \vee e) \cap G$  (if it is not parallel to  $a$ ). Show that the map

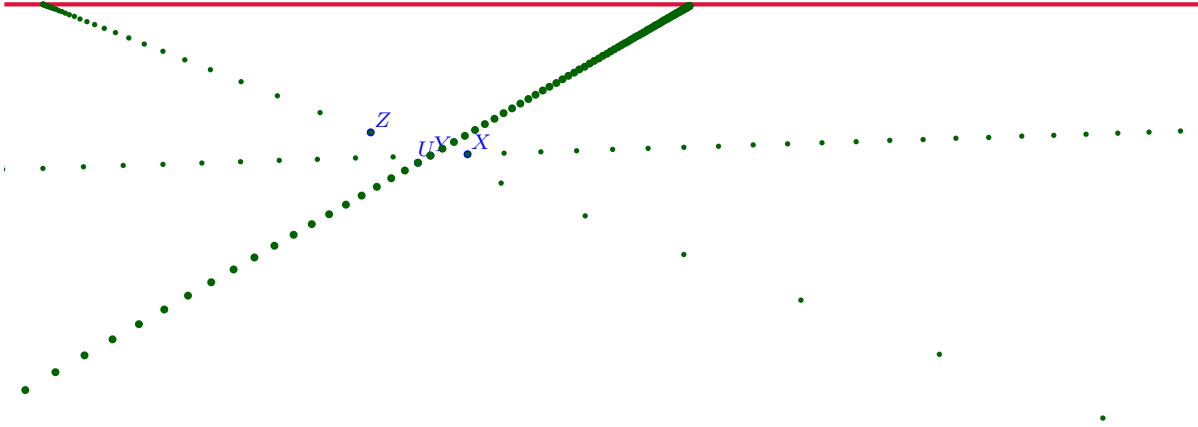
$$\phi : \mathbb{R}^\times \rightarrow G, \quad t \mapsto \frac{t - \alpha(e)}{\alpha(x - e)} x + \frac{\alpha(x) - t}{\alpha(x - e)} e \quad (29)$$

is a section of  $\alpha$ , i.e., it is a torsor homomorphism such that  $\alpha(\phi(t)) = t$ . It maps  $\alpha(x)$  to  $x$  and  $\alpha(e)$  to  $e$ , and hence defines an isomorphism  $\mathbb{R}^\times \rightarrow (x \vee e) \cap G$ . Deduce that  $x^n = \phi((\alpha(x)^n))$ .

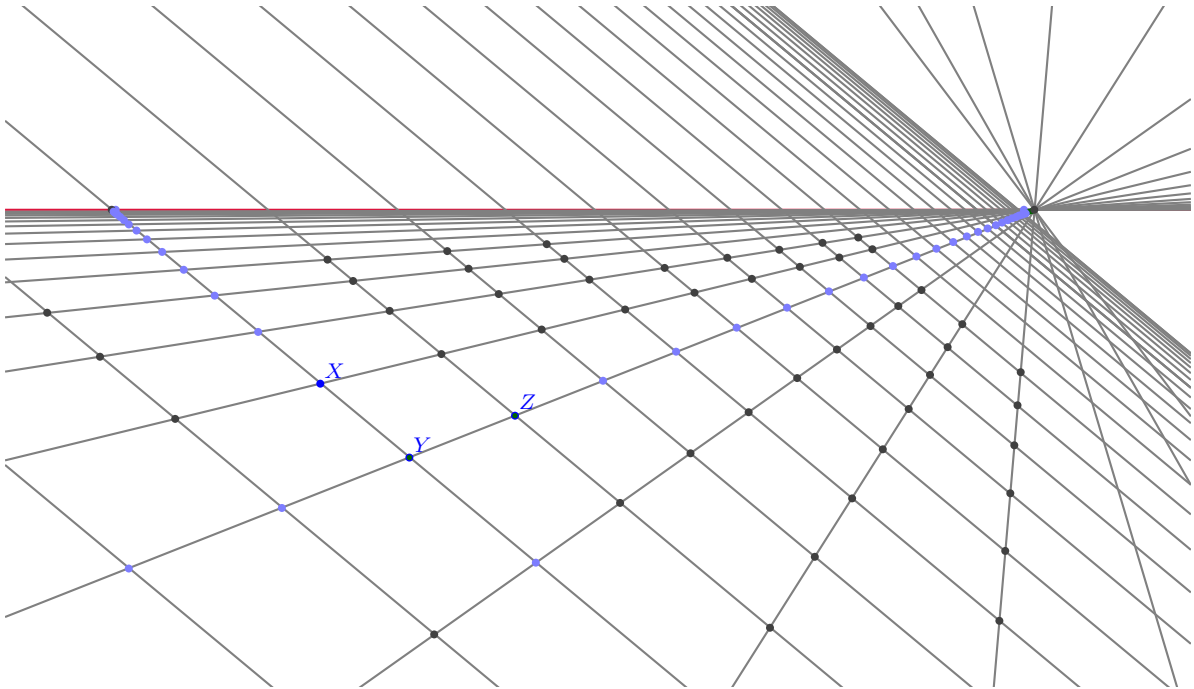
It is possible to plot the powers of  $x$  by programming (28) in geogebra: here is the geogebra command for drawing the powers  $X^{-50}, \dots, X^{50}$  with respect to the origin  $Y$ , and using the linear form  $\alpha(x) = -x_2$ :

Sequence[DynamicCoordinates[X, (x(X)y(Y) - y(X)x(Y) + y(X)<sup>(n+1)</sup>y(Y)<sup>(-n)</sup>(x(Y) - x(X)))(y(Y) - y(X))<sup>(-1)</sup>, y(X)<sup>(n+1)</sup>y(Y)<sup>(-n)</sup>], n, -50, 50]

The illustration shows the powers  $X^n, Z^n, U^n$  with respect to a common origin  $Y$ . The line  $X \vee Y$  is close to being parallel to  $a$ , and hence the powers look like an arithmetic sequence; the powers of the two other elements clearly form a geometric sequence converging to a point of  $a$  as  $n$  tends to  $+\infty$ :



Using the construction of the powers  $X^n$  and  $Z^m$  as above, we may complete the image and construct their products  $X^n Z^m$  (with respect to some origin  $Y$ ):



### 2.3.5 Boundary values

The line  $a$  does not belong to  $G$ : we may consider it as a sort of *boundary* of  $G$ . It is clear that the group  $G$  acts on itself from the left and from the right. Show that  $G$  also acts from the left and from the right on the boundary  $a$ , just by extending the domain of definition of our main formula (17):

- (1) show that, if  $z \in a$  and  $x, y \in G$ , we have  $(xyz) = z$ , and conclude that  $G$  acts trivially on  $a$  from the left,



(2) show that, if  $x \in a$  and  $y, z \in G$ , we have  $\alpha((xyz)) = 0$ , so  $w$  lies again on  $a$ .

Show that (2) defines an action of  $G$  on  $a$  from the right: fix an origin  $e \in G$  with  $\alpha(e) = 1$ , and show that the group  $(G, e)$  acts on the line  $a$  from the right by the formula

$$a \times G \rightarrow a, \quad (x, z) \mapsto (xez).$$

Moreover, show that the map  $\rho(z) : a \rightarrow a, x \mapsto (xez)$  is affine, and that every element of the group  $\text{GA}(1, \mathbb{R})$  of affine bijections of  $\mathbb{R}$  is obtained in this way. Show that the map

$$G \rightarrow \text{GA}(1, \mathbb{R}), \quad z \mapsto \rho(z)$$

is an isomorphism of groups.

From a slightly different viewpoint, show that  $(G, e)$  acts on  $E$  from the left and from the right, that both of these actions commute, and determine the orbits of both actions.

### 2.3.6 Connected components

The line  $a$  divides the set  $G$  into two half-planes which we call the two *connected components* of  $G$ . Let  $e \in G$ ; show that the connected component containing  $e$  is a subgroup of  $(G, e)$ , and that the other connected component is not a subgroup, but both components are subtorsors of  $G$ .

## 2.4 Generalization I: affine spaces, a hyperplane taken out

The constructions described above admit far-reaching generalizations. Unfortunately, it is no longer possible to illustrate them in a such a direct way since the most interesting new features need, to emerge, at least a four dimensional surrounding (the space of  $2 \times 2$ -matrices). In this sense experimental geometry has reached its limits – however, this does not mean that geometry disappears from the picture, see the last chapter.

The following is an immediate generalization of the preceding exercises. Let  $E = \mathbb{R}^n$  and  $a$  a hyperplane of  $E$ , which we write in the form  $a = \ker(\alpha)$ , for some non-zero linear form  $\alpha : E \rightarrow \mathbb{R}$ . Show that formula (17) defines a torsor structure on the set  $G = E \setminus a$ , and that  $\alpha : G \rightarrow \mathbb{R}^\times$  is a torsor homomorphism. Show that, after choice of an origin  $e \in G$ , the group  $(G, e)$  is isomorphic to a semidirect product  $\mathbb{R}^{n-1} \rtimes \mathbb{R}^\times$ . This group has the structure of the *group of dilations* of  $\mathbb{R}^{n+1}$  (subgroup of the affine group of  $\mathbb{R}^{n+1}$  generated by all dilations and translations). A geometric way to identify  $G$  with this group is by letting act  $G$  on the “boundary”  $a$  of  $G$ , as above. (The only difference with the case  $n = 2$  is that, for  $n > 2$ , the dilation group is strictly smaller than the full affine group, and that it is more difficult to make drawings.)

Show that formula (17) is related to incidence geometry in the same way as for  $n = 2$ : just replace the term “line” by “hyperplane”. As a consequence, show that we can project the  $n + 1$ -dimensional space homomorphically onto the  $n$ -dimensional one: let  $E_{n+1} = \mathbb{R}^{n+1}$  with the linear form  $\alpha(x) = x_1$ ; show that then the map  $E_{n+1} \rightarrow E_n, (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$  is a torsor-homomorphism. Its kernel is a one-dimensional subgroup. For instance, if  $n = 2$ , we get a homomorphic projection  $E_3 \rightarrow E_2$ , that is, a two-dimensional homomorphic image of a three-dimensional situation. This sheds some light on the impression that many of our drawings (and even more their dynamic versions) seem to have a “spacial interpretation”.

Finally, show that the field  $\mathbb{R}$  may everywhere be replaced by an arbitrary field  $\mathbb{K}$ .

## 2.5 Generalization II: spaces of rectangular matrices

The second generalization is most important: take as space  $E$  the space  $M(p, q; \mathbb{R})$  of  $q \times p$ -matrices, where  $p, q \in \mathbb{N}$ . Fix some  $p \times q$ -matrix  $A$ . For three  $q \times p$ -matrices  $X, Y, Z \in E$ , such that the  $p \times p$ -matrix  $AY$  is invertible, define another  $q \times p$ -matrix

$$W := (XYZ)_A := (X - Y)(AY)^{-1}AZ + Z = X(A Y)^{-1}AZ - Y(A Y)^{-1}AZ + Z. \quad (30)$$

(Convince yourself that this is well-defined, i.e., all matrix products are indeed defined!) Show that the ternary law  $(XYZ)_A$  defines a torsor structure on the set

$$G := \text{GL}(p, q; A, \mathbb{R}) := \{X \in M(p, q; \mathbb{R}) \mid \det(AY) \neq 0\}. \quad (31)$$

Hint: proceed as above (which is indeed the special case  $p = 1, q = n$ ): start by showing that  $A(XYZ)_A = (AX)(AY)^{-1}(AX)$ , i.e., the map

$$\alpha : \text{GL}(p, q; A, \mathbb{R}) \rightarrow \text{GL}(p, \mathbb{R}), \quad X \mapsto AX \quad (32)$$

is a homomorphism of ternary products.

A detailed study of the possible cases would lead us too far away from the topics we started with. Let us just mention that the most interesting and richest cases arise for  $p = q > 1$ : for instance, if  $p = q = 2$ , we may choose for  $A$  a matrix of rank 1 or 2. If the rank of  $A$  is 1, then the resulting geometry looks very much like the geometry of  $\mathbb{R}^4$  we mentioned in the preceding subsection; however, if the rank of  $A$  is 2, then it is quite different: in this case the homomorphism  $\alpha$  becomes an isomorphism, and hence our torsor  $\text{GL}(2, 2; A, \mathbb{R})$  is isomorphic to the general linear group  $\text{GL}(2, \mathbb{R})$ .

We add some short comments for the more expert reader: in the language of Lie group theory, we see that in the last mentioned case a *reductive* Lie group shows up, whereas in the preceding subsection only solvable Lie groups appeared. All other classical Lie groups can be realized in a similar way (but the constructions are more complicated, using the Jordan algebras of Hermitian or skew-Hermitian matrices), see [BeKi10a, BeKi10b, BeKi12, Be12].

## 3 Geometry revisited: axiomatic approach

In this final section, we say some words on the relation between *geometry* and *algebra*. The power of algebra consists, for instance, in the possibility of the vast generalizations mentioned in the preceding subsections. The power of geometry manifests itself in another kind of generalization: the various kinds of *non-Euclidean geometries*. In this section, which is far from being exhaustive, we would like to explain some of the challenges arising in this context.

### 3.1 Two-parallel planes

The most primitive structure appearing in an axiomatic approach to plane geometry is *incidence*: we have a set  $\mathcal{P}$  of *points* and a set  $\mathcal{L}$  of subsets of  $\mathcal{P}$  whose elements are called *lines*; a point  $p$  and a line  $\ell$  are *incident* if  $p \in \ell$ . Two lines  $\ell$  and  $m$  are called *parallel* (notation:  $m \parallel \ell$ ) if they are equal or disjoint. These data are called an *affine plane* if they satisfy the following axioms:

(A1) Any two distinct points lie on a unique line.

(A2) For a point  $p$  and a line  $m$ , there is a unique line  $\ell$  through  $p$  and parallel to  $m$ .

(A3) There are three points which are not collinear.

Show that parallelism is an *equivalence relation* on the set of lines, and show that, for any field  $\mathbb{K}$ , the plane  $\mathbb{K}^2$  with its usual incidence structure is an affine plane in the sense of the definition. There are interesting affine planes that are *not* of this form; we come to this item below. Non-Euclidean geometry arose from modifying some of Euclidean's axioms; in particular, it was questioned whether the uniqueness of parallels were a consequence of the other axioms or not – indeed, it turned out to be independent. In this subsection, we propose some exercises about a simple model of a geometry in which there are always *one or two parallels to a given line* through each point. Here are the axioms:

**Definition.** A two-parallel plane is a set  $U$  of “points” together with a set  $\mathcal{L}$  of subsets of  $U$ , called “lines”; we assume that there are two kinds of lines, that is, we have a partition  $\mathcal{L} = \mathcal{L}_1 \dot{\cup} \mathcal{L}_2$  into a set  $\mathcal{L}_1$  of lines of the first kind and a set  $\mathcal{L}_2$  of lines of the second kind, such that:

(TP1) Any two distinct points lie on a unique line.

(TP2) Given a line  $\ell \in \mathcal{L}_i$  ( $i = 1, 2$ ) and a point  $p$  not on  $\ell$ , there are exactly  $i$  different lines through  $p$  and parallel to  $\ell$ .

(TP3) (Consistency axiom.) The set  $P := \{(\ell, \ell') \in \mathcal{L}_2 \times \mathcal{L}_2 \mid \ell \parallel \ell', \ell \neq \ell'\}$  admits a partition  $P = P_a \dot{\cup} P_b$  (and call a pair of lines  $(\ell, \ell')$  *a-parallel*, resp. *b-parallel*, if  $(\ell, \ell') \in P_a$ , resp.  $(\ell, \ell') \in P_b$ ) such that: for a point  $p$  and a line  $\ell \in \mathcal{L}_2$ , there is exactly one *a-parallel* through  $p$  and exactly one *b-parallel* through  $p$ .

(TP4) There exist three non-collinear points (points not on a single line).

Show: if  $\mathcal{P}$  is an affine plane and  $a$  some line of this plane, then the set  $U := \mathcal{P} \setminus a$  is a two-parallel plane, where a line in  $U$  is the trace (=intersection with  $U$ ) of a line in  $\mathcal{P}$ . Hints: a line  $\ell$  is of the first kind if it is (in  $\mathcal{P}$ ) parallel to  $a$ , and of the second kind else;  $P_b$  is the set of pairs of lines that are in  $\mathcal{P}$  parallel, and  $P_a$  the set of pairs of lines that (in  $\mathcal{P}$ ) intersect on  $a$  – so in  $U$  they are parallel, too!

(Optional.) What about the converse: is every two-parallel plane obtained by the preceding construction? Hints: if you know how to construct a projective plane out of an affine plane, you may apply the same construction twice – so the answer will be positive.

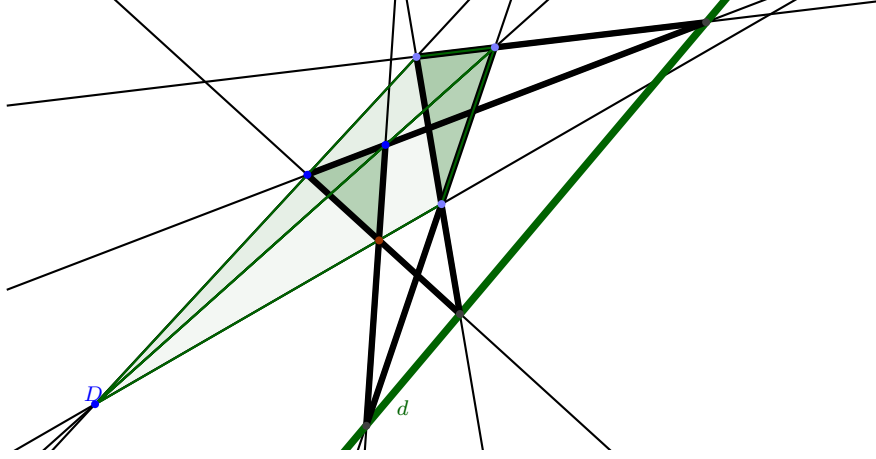
In a two-parallel plane, we may mimick our fundamental definition (6): for a generic triple of points  $(x, y, z) \in U^3$ , define a fourth point:

$$w := (xyz) := \text{intersection of [the } a\text{-parallel of } y \vee x \text{ through } z] \\ \text{with [the } b\text{-parallel of } y \vee z \text{ through } x].$$

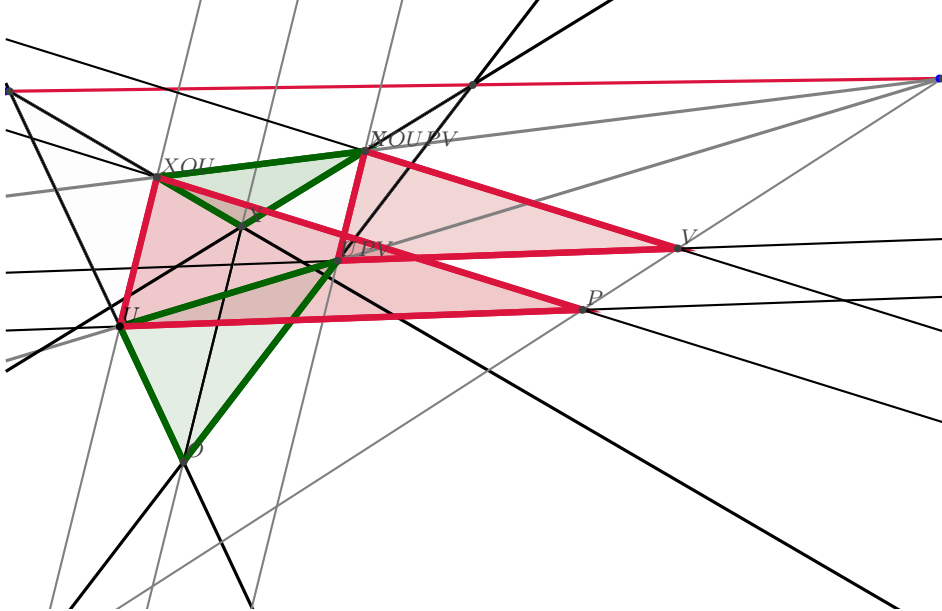
What can you say about the ternary law  $(x, y, z) \mapsto w$ : does it satisfy idempotency and general associativity, or other algebraic laws? And can we extend this definition in some “nice way” also to the case of collinear triples  $(x, y, z)$ ? These are difficult questions – in fact, they are essentially equivalent to similar questions from the theory of *projective planes* – but the viewpoint presented here seems to be new. In the following, let us explain that the answer to the first question is related to *Desargues' theorem*.

### 3.2 Associativity and Desargues theorem

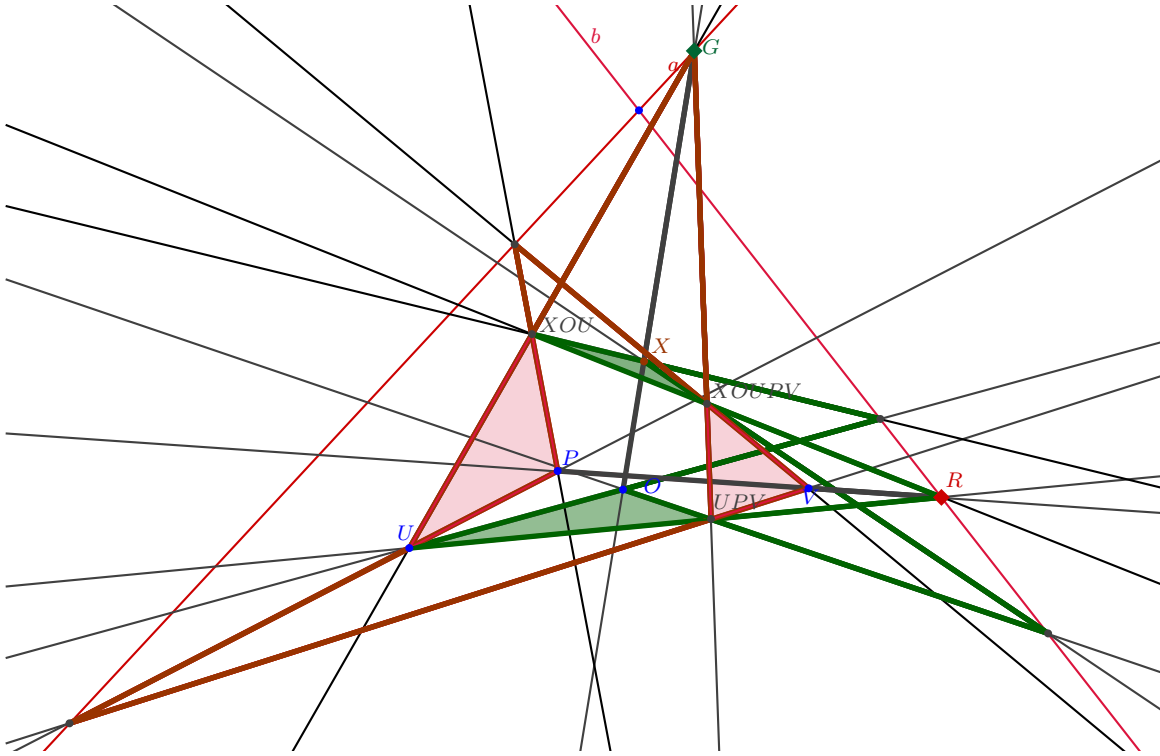
Preliminaries: inform yourself about *Desargues theorem*, either in its affine or projective version, saying that *two triangles are in perspective axially if and only if they are in perspective centrally*, and using geogebra, convince yourself experimentally of the validity of this theorem:



Now assume  $\mathcal{P}$  is an affine plane,  $a$  a line of  $\mathcal{P}$  and  $U := \mathcal{P} \setminus a$ . For a generic triple  $(x, y, z) \in U^3$ , we define  $w := (xyz)$  as explained above. Under which conditions does the identity  $((xou)pv) = (xo(upv))$  hold? To get an idea, reproduce, by using geogebra, the drawing illustrating this identity from subsection 1.3.3, by highlighting four triangles appearing in this construction, as follows:



Observe that the two green and the two red triangles are in a Desargues' configuration, respectively (more precisely: the two dual affine versions of the Desargues' configuration appear here). Use this observation to prove the following theorem: *The general associative law (4) holds in  $U = \mathcal{P} \setminus a$  if, and only if, Desargues' theorem holds in  $\mathcal{P}$* . For the reader familiar with the projective version of Desargues' theorem, we reproduce here the projective version of the preceding drawing. We have marked the Desargues point  $R$  of the "red triangles" (whose Desargues line is  $a$ ) and the Desargues point  $G$  of the "green triangles" (whose Desargues line is  $b$ ).



Summing up, a projective plane is Desarguesian if, and only if, all laws  $(xyz)_{ab}$  are associative. As mentioned in the introduction, the question arises then what can be said for general projective planes, or even for more general lattices.

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