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ROBUST UNBIASED TRANSFORMATIONS WITH CENSORING

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Abstract. In the literature, there have been several papers about unbiased estimation in presence of censoring. In this work, we generalise the doubly robust unbiased transformation with censoring proposed by Rubin and Van der Laan and Suzukawa. We show it is the uniformly minimum variance unbiased transformation with censoring. The new more general transformation we obtain is also doubly robust and asymptotically normal.

1. Introduction. The search for unbiased estimators from quantities of interest is a central problem in statistics. In the survival analysis field, the data are often censored. There are several types of censoring as described by the author in [7]. During the last decades, unbiased estimation in presence of censoring has received some attention in the literature. We can cite as an illustration the works from the authors in [2, 3, 6, 10, 13, 14, 15]. Here, we are especially interesting with the transformation given by the authors in [13, 14] under right random censoring.

In this work, we generalise a transformation proposed by Rubin and Van der Laan [13] and Suzukawa [14]. Our method is general in the context of a right random censoring scheme. First, we describe all possible unbiased transformations from some function which depends of the non observed random variable. Then, we fix this function in the procedure and we search the existence of a minimum variance unbiased transformation in this class. We find a solution. When the non observed random variable is real valued, this solution corresponds to the unbiased transformation proposed by the authors in [13, 14]. We show that this transformation is the uniformly minimum variance unbiased transformation. However, our method is different from the method used by the author in [14] and our construction is general. When the non observed random variable is vector valued, our approach is new. In this case also, we find a unique solution of minimum variance. As shown in the sequel, this is equivalent to minimise the quadratic distance between the general transformation and the objective function which depends of the non observed random variable. The transformation we obtain is doubly robust and generalises the previous for real valued random variables. Our motivation to generalise the original procedure is by example the following: how to estimate without bias the variance from a censored sample? To achieve this goal in the more general non censored case, we need at least two observations [9]. So that our generalisation allows to estimate without bias the variance in a censored sample. We also give a general result of doubly robust asymptotic normality. Indeed, we make a link between the optimal transformation we find and the general theory from U statistics [8].

This paper is organised as follows. In the Section 2, we treat the case of one observation, that is to say the case of real valued random variables. We do not any more contribution in this section. The results that are presented in this section are contained in [13] and [14]. However, our procedure is different. This section prepares the generalisation from the Section 3. In the Section 3, we present the general case for an arbitrary number of observations, that is to say for vector valued random variables. Our contribution is to find the general solution in this case. The transformation we find is constructed to be of minimum variance. Such it is the case. This transformation has another interesting property: it is doubly robust. In the Section 4, we give a result of asymptotic normality in the general case, by using the theory from U statistics. In the last section, we give some perspectives.

In the sequel, all the reasonings can be written conditionally to any covariates. For the sake of simplicity, we present them

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without covariates.

2. The case of one observation. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $Y, C : \Omega \rightarrow \mathbf{R}$ be independent random variables, $T = \min(Y, C)$, $D = 1_{Y \leq C}$, $\gamma : \mathbf{R} \rightarrow \mathbf{R}$ be a fixed borelian function, $V_1 = \{(t_1, t_2) \in \mathbf{R}^2 ; t_1 \leq t_2\}$ and $V_0 = \mathbf{R}^2 - V_1$. We denote by $S_C : \mathbf{R} \rightarrow [0, 1]$ the survival function of the random variable $C : \Omega \rightarrow \mathbf{R}$.

ASSUMPTION 2.1. *We suppose that for any real number c , we have,*

$$S_C(c) = \mathbb{P}(C > c) > 0$$

Under the Assumption 2.1, let $\theta_0 : \mathbf{R} \rightarrow \mathbf{R}$ be a borelian function such that for any real number y , the function $c \mapsto 1_{V_0}(y, c)\theta_0(c)$ is integrable with respect to the probability measure \mathbb{P}_C and for any real number y ,

$$\theta_1(y) = \frac{1}{S_C(y)}[\gamma(y) - \int_{\mathbf{R}} 1_{V_0}(y, c)\theta_0(c)d\mathbb{P}_C(c)]$$

and for any $(t, d) \in \mathbf{R} \times \{0, 1\}$,

$$U_{C, \gamma, \theta_0}(t, d) = (1 - d)\theta_0(t) + d\theta_1(t)$$

In the Proposition 2.1, we describe in this case all possible unbiased transformations with censoring. The parameters that are free are the borelian functions γ, θ_0 from \mathbf{R} to \mathbf{R} .

PROPOSITION 2.1. *We suppose that the functions $y \mapsto \int_{\mathbf{R}} 1_{V_0}(y, c)|\theta_0(c)|d\mathbb{P}_C(c)$ and $y \mapsto S_C(y)|\theta_1(y)|$ from \mathbf{R} to \mathbf{R} are integrable with respect to the probability measure \mathbb{P}_Y . Then, the random variables $U_{C, \gamma, \theta_0}(T, D) : \Omega \rightarrow \mathbf{R}$ and $\gamma \circ Y : \Omega \rightarrow \mathbf{R}$ are integrable with respect to the probability measure \mathbb{P} and we have,*

$$\mathbb{E}[U_{C, \gamma, \theta_0}(T, D)] = \mathbb{E}[\gamma \circ Y]$$

We denote by $C(\mathbf{R}, \mathbf{R})$ the set of all continuous functions from \mathbf{R} to \mathbf{R} and by Θ the set of all continuous functions $\theta_0 : \mathbf{R} \rightarrow \mathbf{R}$ such that, for any real number y , the function $c \mapsto 1_{V_0}(y, c)\theta_0(c)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C and,

$$\begin{aligned} \int_{\mathbf{R}^2} 1_{V_0}(y, c)\theta_0^2(c)d\mathbb{P}_{(Y, C)}(y, c) &\in \mathbf{R}_{\geq 0} \\ \int_{\mathbf{R}} \left\{ \frac{1}{\sqrt{S_C(y)}} \int_{\mathbf{R}} 1_{V_0}(y, c)\theta_0(c)d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) &\in \mathbf{R}_{\geq 0} \\ \int_{\mathbf{R}^2} 1_{V_1}(y, c)\theta_1^2(y)d\mathbb{P}_{(Y, C)}(y, c) &\in \mathbf{R}_{\geq 0} \end{aligned}$$

The random variable $U_{C, \gamma, \theta_0}(T, D) : \Omega \rightarrow \mathbf{R}$ is square integrable with respect to the probability measure \mathbb{P} , if and only if, we have,

$$\begin{aligned} \int_{\mathbf{R}^2} 1_{V_0}(y, c)\theta_0^2(c)d\mathbb{P}_{(Y, C)}(y, c) &\in \mathbf{R}_{\geq 0} \\ \int_{\mathbf{R}^2} 1_{V_1}(y, c)\theta_1^2(y)d\mathbb{P}_{(Y, C)}(y, c) &\in \mathbf{R}_{\geq 0} \end{aligned}$$

Therefore, if,

$$\theta_0 \in \Theta$$

we have,

$$\mathbb{E}[U_{C,\gamma,\theta_0}^2(T, D)] \in \mathbf{R}_{\geq 0}$$

For any $\theta_0 \in C(\mathbf{R}, \mathbf{R})$ such that for any real number y , the function $c \mapsto 1_{V_0}(y, c)\theta_0(c)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C and,

$$\int_{\mathbf{R}} \left\{ \frac{1}{\sqrt{S_C(y)}} \int_{\mathbf{R}} 1_{V_0}(y, c)\theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0}$$

if,

$$\int_{\mathbf{R}} \frac{\gamma^2(y)}{S_C(y)} d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0}$$

then we have,

$$\int_{\mathbf{R}^2} 1_{V_1}(y, c)\theta_1^2(y) d\mathbb{P}_{(Y,C)}(y, c) \in \mathbf{R}_{\geq 0}$$

ASSUMPTION 2.2. We suppose that the function $y \mapsto \frac{\gamma^2(y)}{S_C(y)}$ from \mathbf{R} to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure \mathbb{P}_Y , so that the zero function is in the set Θ .

PROPOSITION 2.2. Under the Assumption 2.2, the subset Θ from $C(\mathbf{R}, \mathbf{R})$ is a vector subspace from $C(\mathbf{R}, \mathbf{R})$.

Let $N : \Theta \rightarrow \mathbf{R}_{\geq 0}$ and $\chi : \Theta \rightarrow \mathbf{R}_{\geq 0}$ be the following functions,

$$\begin{aligned} N : \Theta &\rightarrow \mathbf{R}_{\geq 0} \\ \theta_0 &\mapsto \left[\int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c)\theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) + \int_{\mathbf{R}^2} 1_{V_0}(y, c)\theta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \right]^{\frac{1}{2}} \\ \chi : \Theta &\rightarrow \mathbf{R}_{\geq 0} \\ \theta_0 &\mapsto \mathbb{E}[U_{C,\gamma,\theta_0}^2(T, D)] \end{aligned}$$

ASSUMPTION 2.3. We suppose that there exist two continuous positive functions $f_Y : \mathbf{R} \rightarrow \mathbf{R}_{>0}$ and $f_C : \mathbf{R} \rightarrow \mathbf{R}_{>0}$ which are integrable with respect to the Lebesgue measure over \mathbf{R} and such that for any borelian $B \in B(\mathbf{R})$, we have,

$$\mathbb{P}_Y(B) = \int_{\mathbf{R}} 1_B(t) f_Y(t) d\lambda(t)$$

$$\mathbb{P}_C(B) = \int_{\mathbf{R}} 1_B(t) f_C(t) d\lambda(t)$$

Under the Assumption 2.3, we have for any real number t ,

$$S_Y(t) = \mathbb{P}(Y > t) > 0$$

$$S_C(t) = \mathbb{P}(C > t) > 0$$

In the Proposition 2.3, we show that the application N defines a norm over the vector space Θ .

PROPOSITION 2.3. Under the Assumption 2.3, the function $N : \Theta \rightarrow \mathbf{R}_{\geq 0}$ is a norm over the vector space Θ .

In the Proposition 2.4, we show that the application χ from Θ to $\mathbf{R}_{\geq 0}$ is differentiable over Θ , and we compute its differential.

PROPOSITION 2.4. *Under the Assumptions 2.2 and 2.3, the function $\chi : \Theta \rightarrow \mathbf{R}_{\geq 0}$ is differentiable over Θ , its differential is given by for any $(\theta_0, H_0) \in \Theta^2$,*

$$d_{\theta_0}\chi(H_0) = 2 \int_{\mathbf{R}^2} 1_{V_0}(y, c) [\{\theta_0(c) - \theta_1(y)\} H_0(c)] d\mathbb{P}_{(Y, C)}(y, c)$$

In the Proposition 2.5, we show that the application χ from Θ to $\mathbf{R}_{\geq 0}$ is strongly convex with modulus 2 over Θ , so that this application is strictly convex. Our aim here is to minimise this application over Θ .

PROPOSITION 2.5. *Under the Assumptions 2.2 and 2.3, the function $\chi : \Theta \rightarrow \mathbf{R}_{\geq 0}$ is strongly convex with modulus 2 over Θ .*

In the Proposition 2.6, we give a sufficient condition so that the application χ from Θ to $\mathbf{R}_{\geq 0}$ has a stationary point.

PROPOSITION 2.6. *Under the Assumptions 2.2 and 2.3, for any $\theta_0 \in \Theta$ such that the function $y \mapsto \theta_1^2(y)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y and,*

$$\theta_0(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \theta_1(y) d\mathbb{P}_Y(y)$$

we have,

$$d_{\theta_0}\chi = 0$$

Under the Assumption 2.2, for any real number c , the function $y \mapsto 1_{V_0}(y, c) \gamma(y)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y , let under the Assumption 2.3, $\eta : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number c ,

$$\eta(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \gamma(y) d\mathbb{P}_Y(y)$$

Under the Assumptions 2.2 and 2.3, for any real number y , the function $c \mapsto 1_{V_1}(c, y) \frac{\eta(c)}{S_C^2(c)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C , let $\phi : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number y ,

$$\phi(y) = \gamma(y) - S_C(y) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

ASSUMPTION 2.4. *Let for any real number y , $\gamma^+(y) = \max\{\gamma(y), 0\}$, $\gamma^-(y) = -\min\{\gamma(y), 0\}$ and for any real number c ,*

$$\eta_-(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \gamma^-(y) d\mathbb{P}_Y(y)$$

$$\eta_+(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \gamma^+(y) d\mathbb{P}_Y(y)$$

We suppose that for any real number c , the function $y \mapsto 1_{V_0}(y, c) \int_{\mathbf{R}} 1_{V_1}(c', y) \frac{\eta_-(c')}{S_C^2(c')} d\mathbb{P}_C(c')$ is integrable with respect to the probability measure \mathbb{P}_Y and the function $y \mapsto 1_{V_0}(y, c) \int_{\mathbf{R}} 1_{V_1}(c', y) \frac{\eta_+(c')}{S_C^2(c')} d\mathbb{P}_C(c')$ is integrable with respect to the probability measure \mathbb{P}_Y .

If the function γ from \mathbf{R} to \mathbf{R} is bounded and,

$$\int_{\mathbf{R}} \frac{d\mathbb{P}_Y(y)}{S_C(y)} \in \mathbf{R}_{\geq 0}$$

then, the Assumption 2.4 is verified.

PROPOSITION 2.7. *Under the Assumptions 2.2, 2.3 and 2.4, we have for any real number t ,*

$$\gamma(t) = \phi(t) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

Under the Assumptions 2.2 and 2.3, let $\mu : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number y ,

$$\mu(y) = \frac{\gamma(y)}{S_C(y)} - \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

ASSUMPTION 2.5. *We suppose that the function $y \mapsto \mu^2(y)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y .*

Under the Assumptions 2.2, 2.3 and 2.5, let $\theta_0^0 : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number c ,

$$\theta_0^0(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \mu(y) d\mathbb{P}_Y(y)$$

ASSUMPTION 2.6. *We suppose that,*

$$\int_{\mathbf{R}} \frac{d\mathbb{P}_Y(y)}{S_C(y)} \in \mathbf{R}_{\geq 0}$$

$$\lim_{t \rightarrow +\infty} \frac{S_Y(t)}{S_C(t)} = 0$$

PROPOSITION 2.8. *Under the Assumptions 2.2, 2.3, 2.4, 2.5 and 2.6, we have for any real number y ,*

$$\theta_1^0(y) = \mu(y)$$

and if the function $t \mapsto \gamma(t)$ from \mathbf{R} to \mathbf{R} is piecewise continuous over \mathbf{R} then, for any real number t ,

$$\theta_0^0(t) = \frac{\eta(t)}{S_C(t)} - \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

PROPOSITION 2.9. *Under the Assumption 2.2, we have,*

$$\mathbb{E}[\gamma^2(Y)] \in \mathbf{R}_{\geq 0}$$

and for any $\theta_0 \in \Theta$,

$$\mathbb{V}[\mathbf{U}_{C,\gamma,\theta_0}(T, D)] = \mathbb{E}[\{\mathbf{U}_{C,\gamma,\theta_0}(T, D) - \gamma(Y)\}^2] + \mathbb{V}[\gamma(Y)]$$

In the Theorem 2.1, we present the main result from this paragraph.

THEOREM 2.1. *Under the Assumptions 2.2, 2.3, 2.4, 2.5 and 2.6, if,*

$$\theta_0^0 \in \Theta$$

we have then for any $\theta_0 \in \Theta$ such that $\theta_0 \neq \theta_0^0$,

$$\mathbb{V}[\mathbf{U}_{C,\gamma,\theta_0}(T, D)] > \mathbb{V}[\mathbf{U}_{C,\gamma,\theta_0^0}(T, D)]$$

which is equivalent to,

$$\mathbb{E}[\{\mathbf{U}_{C,\gamma,\theta_0}(T, D) - \gamma(Y)\}^2] > \mathbb{E}[\{\mathbf{U}_{C,\gamma,\theta_0^0}(T, D) - \gamma(Y)\}^2]$$

In the Theorem 2.2, we show that the transformation which minimises the variance is also robust.

THEOREM 2.2. *Let $Y_0, C_0 : \Omega \rightarrow \mathbf{R}$ be independent random variables and,*

$$T_0 = \min(Y_0, C_0)$$

$$D_0 = 1_{Y_0 \leq C_0}$$

Then, under the Assumptions 2.2, 2.3, 2.4, 2.5 and 2.6, if $S_C = S_{C_0}$ or $S_Y = S_{Y_0}$, and,

$$\mathbb{E}[|U_{C, \gamma, \theta_0^0}(T_0, D_0)|] \in \mathbf{R}_{\geq 0}$$

we have always,

$$\mathbb{E}[U_{C, \gamma, \theta_0^0}(T_0, D_0)] = \mathbb{E}[\gamma(Y_0)]$$

When $\gamma = \text{id}_{\mathbf{R}}$, we retrieve the transformation proposed by the authors in [13].

Let $p \in \mathbf{N}_{\geq 1}$ and $n \in \mathbf{N}_{\geq 1}$ such that $p + 1 < n$.

Suppose that our model is the following linear regression model [11],

$$Y = X\beta + \epsilon$$

where the vector $Y = (Y_1, \dots, Y_n)$ from Ω to \mathbf{R}^n is the dependent variable, the fixed matrix of regressors $X = (X_{mj})_{1 \leq m \leq n, 0 \leq j \leq p}$ is such that for any integer $m \in \{1, \dots, n\}$, $X_{m0} = 1$, $\beta = (\beta_0, \dots, \beta_p) \in \mathbf{R}^{p+1}$ is a vector of fixed parameters to be estimated and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ is an unknown noise component, where the random variables $(\epsilon_j)_{1 \leq j \leq n}$ from Ω to \mathbf{R} are independent.

We make the following assumptions:

$$\mathbb{E}[\epsilon] = (0, \dots, 0)$$

there exists a positive real number σ such that,

$$\mathbb{V}[\epsilon] = \sigma^2 \mathbf{I}_n$$

and,

$$\text{rank}(X) = p + 1$$

Consequently, the matrix $X'X$ from $M_{p+1, p+1}(\mathbf{R})$ is an invertible matrix, where the notation $'$ refers to the transposed matrix.

The ordinary least squares estimator from $\beta = (\beta_0, \dots, \beta_p)$ when the vector $y = (y_1, \dots, y_n) \in \mathbf{R}^n$ is totally observed without censoring is given by $\hat{\beta}(y_1, \dots, y_n) = (X'X)^{-1}X'y$. From the Gauss and Markov theorem, it is the uniformly minimum variance unbiased linear estimator from the vector β , also we have,

$$\mathbb{V}[\hat{\beta}(Y_1, \dots, Y_n)] = \sigma^2 (X'X)^{-1}$$

As the noise component is unknown, we do not know the value of σ . Let $\hat{y} = X\hat{\beta}$ be the vector of predicted values and $\hat{\epsilon} = y - \hat{y}$ be the vector of the residuals. An unbiased estimator from σ^2 is given by,

$$\frac{1}{n - p - 1} \sum_{1 \leq j \leq n} \hat{\epsilon}_j^2$$

We are concerning with the unbiased estimation from the vector β from \mathbf{R}^{p+1} and the real number σ^2 , under random right independent censorship. We denote by $y = (y_1, \dots, y_n)$ the vector of the non observed values, by $c = (c_1, \dots, c_n)$ the vector from the censor, by $t = (t_1, \dots, t_n) = (\min(y_1, c_1), \dots, \min(y_n, c_n))$ the vector of eventually censored values and by $d = (d_1, \dots, d_n) = (1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n))$ the vector of the censoring indicators. We observe t and d and we want to estimate without bias the vector β and the real number σ^2 , from the data.

With the notations from this section, we have for any integer j such that $1 \leq j \leq n$,

$$\mathbb{E}[U_{C_j, \text{id}_{\mathbf{R}}, \theta_0}(T_j, D_j)] = \mathbb{E}[Y_j]$$

So that we have an unbiased transformation from the vector β , which is $\hat{\beta}\{U_{C_1, \text{id}_{\mathbf{R}}, \theta_0}(t_1, d_1), \dots, U_{C_n, \text{id}_{\mathbf{R}}, \theta_0}(t_n, d_n)\}$,

$$\mathbb{E}[\hat{\beta}\{U_{C_1, \text{id}_{\mathbf{R}}, \theta_0}(T_1, D_1), \dots, U_{C_n, \text{id}_{\mathbf{R}}, \theta_0}(T_n, D_n)\}] = \beta$$

Now, our problem is to estimate without bias the real number σ^2 . We suppose that for any borelian function $\gamma : \mathbf{R}^n \rightarrow \mathbf{R}$, we have found an unbiased estimator $\mu_\gamma(T_1, \dots, T_n, D_1, \dots, D_n)$ from the real number $\mathbb{E}[\gamma(Y_1, \dots, Y_n)]$, let $\gamma(y_1, \dots, y_n) = \frac{1}{n-p-1} \sum_{1 \leq j \leq n} \hat{\epsilon}_j^2$, we would obtain so,

$$\mathbb{E}[\mu_\gamma(T_1, \dots, T_n, D_1, \dots, D_n)] = \sigma^2$$

This problem motivates the construction below.

Another way to proceed would be to consider conditional expectation, however, it does not seem to lead to doubly robust transformations.

3. The general case. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $n \in \mathbf{N}_{\geq 1}$, $Y_1, \dots, Y_n, C_1, \dots, C_n : \Omega \rightarrow \mathbf{R}$ be random variables such that the random vector (Y_1, \dots, Y_n) is independent from the random vector (C_1, \dots, C_n) , $I = \{0, 1\}^n - \{(1, \dots, 1)\}$ and $\gamma, (\theta_i)_{i \in I} : \mathbf{R}^n \rightarrow \mathbf{R}$ be borelian functions. Let for any integer j such that $1 \leq j \leq n$,

$$T_j = \min(Y_j, C_j)$$

$$D_j = 1_{Y_j \leq C_j}$$

ASSUMPTION 3.1. *We suppose that for any $(c_1, \dots, c_n) \in \mathbf{R}^n$, we have,*

$$S_{(C_1, \dots, C_n)}(c_1, \dots, c_n) = \mathbb{P}(C_1 > c_1, \dots, C_n > c_n) > 0$$

Let $\gamma, (\theta_i)_{i \in I} : \mathbf{R}^n \rightarrow \mathbf{R}$ be borelian functions, such that for any $i \in I$ and for any $(y_1, \dots, y_n) \in \mathbf{R}^n$, the function $(c_1, \dots, c_n) \mapsto \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\}$ from \mathbf{R}^n to \mathbf{R} is integrable over \mathbf{R}^n with respect to the probability measure $\mathbb{P}_{(C_1, \dots, C_n)}$, under the Assumption 3.1, let for any $(y_1, \dots, y_n) \in \mathbf{R}^n$,

$$\begin{aligned} & \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \\ &= \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\gamma(y_1, \dots, y_n) \\ & - \sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)] \end{aligned}$$

and for any $(t_1, \dots, t_n, d_1, \dots, d_n) \in \mathbf{R}^n \times \{0, 1\}^n$,

$$U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(t_1, \dots, t_n, d_1, \dots, d_n) = \sum_{i \in \{0, 1\}^n} [\prod_{1 \leq j \leq n} \{d_j^{i_j} (1 - d_j)^{1 - i_j}\} \theta_i(t_1, \dots, t_n)]$$

In the Proposition 3.1, we describe all possible unbiased transformations with censoring. The parameters that are free are the borelian functions $\gamma, (\theta_i)_{i \in I}$ from \mathbf{R}^n to \mathbf{R} .

PROPOSITION 3.1. *We suppose that for any $i \in I$, the function*

$(y_1, \dots, y_n) \mapsto \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)$ from \mathbf{R}^n to \mathbf{R} is integrable over \mathbf{R}^n with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$. Then, the function γ from \mathbf{R}^n to \mathbf{R} is integrable over \mathbf{R}^n with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$ and we have,

$$\mathbb{E}[U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n)] = \mathbb{E}[\gamma(Y_1, \dots, Y_n)]$$

As an illustration, for estimating without bias the variance in a censored independent and identically distributed sample, we can take for any $n \in \mathbf{N}_{\geq 2}$ and any $(y_1, \dots, y_n) \in \mathbf{R}^n$,

$$\gamma(y_1, \dots, y_n) = \frac{1}{n-1} \left\{ \sum_{1 \leq j \leq n} (y_j - \frac{1}{n} \sum_{1 \leq k \leq n} y_k)^2 \right\}$$

The random variable $U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n)$ from Ω to \mathbf{R} is square integrable with respect to the probability measure \mathbb{P} , if and only if, we have for any $i \in \{0, 1\}^n$,

$$\int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}_{\geq 0}$$

We denote by $C(\mathbf{R}^n, \mathbf{R})$ the set of all functions from \mathbf{R}^n to \mathbf{R} that are continuous over \mathbf{R}^n and, by Θ the set of all elements $(\theta_i)_{i \in I}$ from $C(\mathbf{R}^n, \mathbf{R})^{|I|}$ such that, for any $(y_1, \dots, y_n) \in \mathbf{R}^n$ and any $i \in I$, the function $(c_1, \dots, c_n) \mapsto \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\}$ from \mathbf{R}^n to \mathbf{R} is integrable over \mathbf{R}^n with respect to the probability measure $\mathbb{P}_{(C_1, \dots, C_n)}$ and,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}_{\geq 0} \\ & \int_{\mathbf{R}^n} \left[\frac{1}{\sqrt{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)}} \right. \\ & \times \left. \left\{ \sum_{i \in I} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \right\}^2 d\mathbb{P}_{(Y_1, \dots, Y_n)} \right] \in \mathbf{R}_{\geq 0} \\ & \int_{\mathbf{R}^n} S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \in \mathbf{R}_{\geq 0} \end{aligned}$$

We have for any $(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}^{2n}$,

$$\begin{aligned} & U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n), 1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n) \} \\ & = \sum_{i \in \{0, 1\}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} \end{aligned}$$

If,

$$(\theta_i)_{i \in I} \in \Theta$$

then, the random variable $U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n)$ from Ω to \mathbf{R} is square integrable with respect to the probability measure \mathbb{P} .

For any $(\theta_i)_{i \in I} \in C(\mathbf{R}^n, \mathbf{R})^{|I|}$ such that for any $(y_1, \dots, y_n) \in \mathbf{R}^n$ and any $i \in I$, the function $(c_1, \dots, c_n) \mapsto \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\}$ from \mathbf{R}^n to \mathbf{R} is integrable over \mathbf{R}^n with respect to the probability measure $\mathbb{P}_{(C_1, \dots, C_n)}$ and,

$$\int_{\mathbf{R}^n} \left[\frac{1}{\sqrt{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)}} \times \left\{ \sum_{i \in I} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \right\}^2 d\mathbb{P}_{(Y_1, \dots, Y_n)} \right] \in \mathbf{R}_{\geq 0}$$

if,

$$\int_{\mathbf{R}^n} \frac{\gamma^2(y_1, \dots, y_n)}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \in \mathbf{R}_{\geq 0}$$

then we have,

$$\int_{\mathbf{R}^n} S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \in \mathbf{R}_{\geq 0}$$

ASSUMPTION 3.2. We suppose that $(0, \dots, 0) \in \Theta$, or equivalently, we suppose that the function $(y_1, \dots, y_n) \mapsto \frac{\gamma^2(y_1, \dots, y_n)}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)}$ from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$.

In the Proposition 3.2, we demonstrate under the Assumption 3.2, that the set Θ is a vector space.

PROPOSITION 3.2. Under the Assumption 3.2, the subset Θ from $C(\mathbf{R}^n, \mathbf{R})^{|I|}$ is a vector subspace from $C(\mathbf{R}^n, \mathbf{R})^{|I|}$.

Let $N : \Theta \rightarrow \mathbf{R}_{\geq 0}$ and $\chi : \Theta \rightarrow \mathbf{R}_{\geq 0}$ be the following functions,

$$\begin{aligned} N : \Theta &\rightarrow \mathbf{R}_{\geq 0} \\ (\theta_i)_{i \in I} &\mapsto \left\{ \int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} \left[\sum_{i \in I} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \right. \right. \\ &\quad \times \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \left. \right]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\ &\quad + \sum_{i \in I} \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \\ &\quad \times \theta_i^2 \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ &\quad \left. \right\}^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \chi : \Theta &\rightarrow \mathbf{R} \\ (\theta_i)_{i \in I} &\mapsto \mathbb{E}[U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2(T_1, \dots, T_n, D_1, \dots, D_n)] \end{aligned}$$

ASSUMPTION 3.3. We suppose that there exist some functions $f_{(Y_1, \dots, Y_n)} : \mathbf{R}^n \rightarrow \mathbf{R}_{>0}$ and $f_{(C_1, \dots, C_n)} : \mathbf{R}^n \rightarrow \mathbf{R}_{>0}$ that are continuous over \mathbf{R}^n and positive, integrable with respect to the Lebesgue measure over \mathbf{R}^n , and such that for any $B \in B(\mathbf{R})^{\otimes n}$, we have,

$$\begin{aligned} \mathbb{P}_{(Y_1, \dots, Y_n)}(B) &= \int_{\mathbf{R}^n} 1_B(t_1, \dots, t_n) f_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n) d\lambda_n(t_1, \dots, t_n) \\ \mathbb{P}_{(C_1, \dots, C_n)}(B) &= \int_{\mathbf{R}^n} 1_B(t_1, \dots, t_n) f_{(C_1, \dots, C_n)}(t_1, \dots, t_n) d\lambda_n(t_1, \dots, t_n) \end{aligned}$$

Under the Assumption 3.3, we have for any $(t_1, \dots, t_n) \in \mathbf{R}^n$,

$$S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n) = \mathbb{P}(Y_1 > t_1, \dots, Y_n > t_n) > 0$$

$$S_{(C_1, \dots, C_n)}(t_1, \dots, t_n) = \mathbb{P}(C_1 > t_1, \dots, C_n > t_n) > 0$$

In the Proposition 3.3, we show that the application N defines a norm over the vector space Θ .

PROPOSITION 3.3. *Under the Assumption 3.3, the application N from Θ to $\mathbf{R}_{\geq 0}$ is a norm over the vector space Θ .*

In the Proposition 3.4, we show that the application χ from Θ to $\mathbf{R}_{\geq 0}$ is differentiable over Θ , and we compute its differential.

PROPOSITION 3.4. *Under the Assumptions 3.2 and 3.3, the application χ from Θ to $\mathbf{R}_{\geq 0}$ is differentiable over Θ , for any $(\theta_i)_{i \in \mathbf{I}} \in \Theta$, we have,*

$$d_{(\theta_i)_{i \in \mathbf{I}}} \chi = L_{(\theta_i)_{i \in \mathbf{I}}}$$

where for any $(H_i)_{i \in \mathbf{I}} \in \Theta$,

$$\begin{aligned} & L_{(\theta_i)_{i \in \mathbf{I}}} \{ (H_i)_{i \in \mathbf{I}} \} \\ &= 2 \left\{ \sum_{i \in \mathbf{I}} \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \right. \\ & \quad \times \theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \} \\ & \quad - 2 \int_{\mathbf{R}^{2n}} \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \sum_{i \in \mathbf{I}} \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \\ & \quad \times H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \end{aligned}$$

In the Proposition 3.5, we show that the application χ from Θ to $\mathbf{R}_{\geq 0}$ is strongly convex with modulus 2 over Θ , so that this application is strictly convex. Our aim here is to minimise this application over Θ .

PROPOSITION 3.5. *Under the Assumptions 3.2 and 3.3, the application χ from Θ to $\mathbf{R}_{\geq 0}$ is strongly convex with modulus 2 over Θ .*

ASSUMPTION 3.4. *We suppose that the random variables $Y_1, \dots, Y_n, C_1, \dots, C_n$ from Ω to \mathbf{R} are all independent.*

In the Proposition 3.6, we give a sufficient condition so that the application χ from Θ to $\mathbf{R}_{\geq 0}$ has a stationary point.

PROPOSITION 3.6. *Under the Assumptions 3.2, 3.3 and 3.4, for any $(\theta_i)_{i \in \mathbf{I}} \in \Theta$ such that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$ and any $i \in \mathbf{I}$, the function $(y_1, \dots, y_n) \mapsto \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \theta_{(1, \dots, 1)}(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n})$ from \mathbf{R}^n to \mathbf{R} is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$ and the function $(y_1, \dots, y_n) \mapsto \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n)$ from \mathbf{R}^n to \mathbf{R} is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$ and if,*

$$\theta_i(t_1, \dots, t_n) = \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \theta_{(1, \dots, 1)}(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$$

then we have,

$$d_{(\theta_i)_{i \in \mathbf{I}}} \chi = 0$$

We have that for any $(H_i)_{i \in \mathbf{I}} \in \Theta$ and any $i \in \mathbf{I}$,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \quad \times H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ &= 0 \end{aligned}$$

PROPOSITION 3.7. *Under the Assumption 3.2, we have,*

$$\mathbb{E}[\gamma^2(Y_1, \dots, Y_n)] \in \mathbf{R}_{\geq 0}$$

and for any $(\theta_i)_{i \in \mathbf{I}} \in \Theta$,

$$\begin{aligned} & \mathbb{V}[\mathbf{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in \mathbf{I}}}(T_1, \dots, T_n, D_1, \dots, D_n)] \\ &= \mathbb{E}[\{\mathbf{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in \mathbf{I}}}(T_1, \dots, T_n, D_1, \dots, D_n) - \gamma(Y_1, \dots, Y_n)\}^2] + \mathbb{V}[\gamma(Y_1, \dots, Y_n)] \end{aligned}$$

ASSUMPTION 3.5. *We suppose that for any $i \in \{0, 1\}^n$ and any $(t_1, \dots, t_n) \in \mathbf{R}^n$, the function $(y_1, \dots, y_n) \mapsto \{\prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j)\} \gamma(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n})$ from \mathbf{R}^n to \mathbf{R} is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$, let for any $i \in \{0, 1\}^n$ and any $(t_1, \dots, t_n) \in \mathbf{R}^n$,*

$$\alpha_{i,-}(t_1, \dots, t_n) = \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \gamma^-(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$$

$$\alpha_{i,+}(t_1, \dots, t_n) = \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \gamma^+(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$$

where for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\gamma^-(t_1, \dots, t_n) = -\min\{\gamma(t_1, \dots, t_n), 0\}$$

$$\gamma^+(t_1, \dots, t_n) = \max\{\gamma(t_1, \dots, t_n), 0\}$$

We suppose that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$ and any $i \in \{0, 1\}^n$, the function

$(u_1, \dots, u_n) \mapsto [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1-i_j}] \frac{\alpha_{i,-}(t_1^{i_1} u_1^{1-i_1}, \dots, t_n^{i_n} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}}$ from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure $\mathbb{P}_{(C_1, \dots, C_n)}$ and, the function

$(u_1, \dots, u_n) \mapsto [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1-i_j}] \frac{\alpha_{i,+}(t_1^{i_1} u_1^{1-i_1}, \dots, t_n^{i_n} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}}$ from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure $\mathbb{P}_{(C_1, \dots, C_n)}$, let,

$$\begin{aligned} & \mu(t_1, \dots, t_n) \\ &= \sum_{i \in \{0, 1\}^n} \frac{(-1)^{n - \sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1-i_j}] \frac{\alpha_i(t_1^{i_1} u_1^{1-i_1}, \dots, t_n^{i_n} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}} d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) \end{aligned}$$

where for any $(u_1, \dots, u_n) \in \mathbf{R}^n$, we have,

$$\alpha_i(u_1, \dots, u_n) = \alpha_{i,+}(u_1, \dots, u_n) - \alpha_{i,-}(u_1, \dots, u_n)$$

We suppose that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, any $i \in \{0, 1\}^n$ and any $i' \in \mathbf{I}$, the function

$(y_1, \dots, y_n) \mapsto \frac{\{\prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j)\}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j^{i'_j} y_j^{1-i'_j})\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j^{i'_j} y_j^{1-i'_j})\}^{1-i_j}] \frac{\alpha_{i,-}(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}} d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n)$ from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$ and, the function

$(y_1, \dots, y_n) \mapsto \frac{\{\prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j)\}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j^{i'_j} y_j^{1-i'_j})\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j^{i'_j} y_j^{1-i'_j})\}^{1-i_j}]$

$\frac{\alpha_{i,+}(t_1^{i_1} y_1^{i_1(1-i_1')} u_1^{1-i_1}, \dots, t_n^{i_n} y_n^{i_n(1-i_n')} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}} d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n)$ from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$. We suppose also that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, any $i \in \{0, 1\}^n$ and any $i' \in \mathbf{I}$, the function $(c_1, \dots, c_n) \mapsto \frac{\prod_{1 \leq j \leq n} 1_{V_{i'_j}}(t_j, c_j)}{S_{(Y_1, \dots, Y_n)}\{\min(t_1, c_1), \dots, \min(t_n, c_n)\}} \int_{\mathbf{R}^n} \frac{\{\prod_{1 \leq j \leq n} 1_{V_0}(y_j, \min(t_j, c_j))\}}{\prod_{1 \leq j \leq n} \{S_{C_j}(\min(t_j, c_j))^{i'_j} y_j^{1-i'_j}\}^{i_j}}$

$\times \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, \min(t_j, c_j))^{i'_j} y_j^{1-i'_j}\}^{1-i_j}] \frac{\alpha_{i,-}(\min(t_1, c_1)^{i_1} y_1^{i_1(1-i_1')} u_1^{1-i_1}, \dots, \min(t_n, c_n)^{i_n} y_n^{i_n(1-i_n')} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}}$

$d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$ from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure $\mathbb{P}_{(C_1, \dots, C_n)}$ and, the function

$(c_1, \dots, c_n) \mapsto \frac{\prod_{1 \leq j \leq n} 1_{V_{i'_j}}(t_j, c_j)}{S_{(Y_1, \dots, Y_n)}\{\min(t_1, c_1), \dots, \min(t_n, c_n)\}} \int_{\mathbf{R}^n} \frac{\{\prod_{1 \leq j \leq n} 1_{V_0}(y_j, \min(t_j, c_j))\}}{\prod_{1 \leq j \leq n} \{S_{C_j}(\min(t_j, c_j))^{i'_j} y_j^{1-i'_j}\}^{i_j}}$

$\times \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, \min(t_j, c_j))^{i'_j} y_j^{1-i'_j}\}^{1-i_j}] \frac{\alpha_{i,+}(\min(t_1, c_1)^{i_1} y_1^{i_1(1-i_1')} u_1^{1-i_1}, \dots, \min(t_n, c_n)^{i_n} y_n^{i_n(1-i_n')} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}}$

$d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$ from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure $\mathbb{P}_{(C_1, \dots, C_n)}$.

Under the Assumption 3.4, if the function γ from \mathbf{R}^n to \mathbf{R} is bounded and for any integer j such that $1 \leq j \leq n$, we have,

$$\int_{\mathbf{R}} \frac{d\mathbb{P}_{Y_j}(y)}{S_{C_j}(y)} \in \mathbf{R}_{\geq 0}$$

then, the Assumption 3.5 is verified.

Under the Assumptions 3.3 and 3.5, let for any $i \in \mathbf{I}$, $\theta_i^0 : \mathbf{R}^n \rightarrow \mathbf{R}$ be the function such that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\theta_i^0(t_1, \dots, t_n) = \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \mu(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$$

ASSUMPTION 3.6. We suppose that the function $(y_1, \dots, y_n) \mapsto \mu^2(y_1, \dots, y_n)$ from \mathbf{R}^n to \mathbf{R} is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$.

ASSUMPTION 3.7. We suppose that for any integer j such that $1 \leq j \leq n$,

$$\int_{\mathbf{R}} \frac{d\mathbb{P}_{Y_j}(y)}{S_{C_j}(y)} \in \mathbf{R}_{\geq 0}$$

$$\lim_{t \rightarrow +\infty} \frac{S_{Y_j}(t)}{S_{C_j}(t)} = 0$$

We can give now in the Theorem 3.1, the main result from this paragraph.

THEOREM 3.1. Under the Assumptions 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7, if,

$$(\theta_i^0)_{i \in \mathbf{I}} \in \Theta$$

we have then for any $(\theta_i)_{i \in \mathbf{I}} \in \Theta$ such that $(\theta_i)_{i \in \mathbf{I}} \neq (\theta_i^0)_{i \in \mathbf{I}}$

$$\mathbb{V}[\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in \mathbf{I}}}(T_1, \dots, T_n, D_1, \dots, D_n)] > \mathbb{V}[\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i^0)_{i \in \mathbf{I}}}(T_1, \dots, T_n, D_1, \dots, D_n)]$$

this is equivalent to,

$$\begin{aligned} & \mathbb{E}[\{\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n) - \gamma(Y_1, \dots, Y_n)\}^2] \\ & > \mathbb{E}[\{\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i^0)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n) - \gamma(Y_1, \dots, Y_n)\}^2] \end{aligned}$$

We demonstrate that the transformation we have introduced in the Theorem 3.1 is robust. This result is presented in the Theorem 3.2.

THEOREM 3.2. *Let $Y_1^0, \dots, Y_n^0, C_1^0, \dots, C_n^0 : \Omega \rightarrow \mathbf{R}$ be independent random variables and for any integer j such that $1 \leq j \leq n$,*

$$\begin{aligned} T_j^0 &= \min(Y_j^0, C_j^0) \\ D_j^0 &= 1_{Y_j^0 \leq C_j^0} \end{aligned}$$

Then, under the Assumptions 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7, if for any integer j such that $1 \leq j \leq n$, $S_{C_j} = S_{C_j^0}$, or for any integer j such that $1 \leq j \leq n$, $S_{Y_j} = S_{Y_j^0}$, and,

$$\mathbb{E}[\|\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i^0)_{i \in I}}(T_1^0, \dots, T_n^0, D_1^0, \dots, D_n^0)\|] \in \mathbf{R}_{\geq 0}$$

we have always,

$$\mathbb{E}[\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i^0)_{i \in I}}(T_1^0, \dots, T_n^0, D_1^0, \dots, D_n^0)] = \mathbb{E}[\gamma(Y_1^0, \dots, Y_n^0)]$$

In the proposition 3.8, we demonstrate that for any $n_1, n_2 \in \mathbf{N}_{\geq 1}$ such that $n_1 \leq n_2$, the optimal transformation for $n = n_2$ generalises the optimal transformation for $n = n_1$.

PROPOSITION 3.8. *If it exist $q \in \mathbf{N}_{\geq 1}$ such that $q \leq n$, an application τ from $\{1, \dots, q\}$ to $\{1, \dots, n\}$ which is injective over $\{1, \dots, q\}$, and a function $\epsilon : \mathbf{R}^q \rightarrow \mathbf{R}$ such that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,*

$$\gamma(t_1, \dots, t_n) = \epsilon(t_{\tau(1)}, \dots, t_{\tau(q)})$$

then we have for any $(t_1, \dots, t_n) \in \mathbf{R}^n$ and any $(d_1, \dots, d_n) \in \{0, 1\}^n$,

$$\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i^0)_{i \in I}}(t_1, \dots, t_n, d_1, \dots, d_n) = \mathbb{U}_{C_{\tau(1)}, \dots, C_{\tau(q)}, \epsilon, (\theta_i^0)_{i \in I}}(t_{\tau(1)}, \dots, t_{\tau(q)}, d_{\tau(1)}, \dots, d_{\tau(q)})$$

4. Asymptotic normality. When the base function is a U statistic, the corresponding optimal transformation can be also viewed as a U statistic. Indeed, if the application γ is symmetric, then the corresponding optimal transformation is also symmetric and can be used as a kernel to construct the U statistic. This result is demonstrated in the Proposition 4.1. Therefore, many results from the theory of U statistics [8] apply to the optimal transformation.

Here, we present a result of doubly robust asymptotic normality for the optimal transformation we have introduced in the Section 3. This result is presented in the Theorem 4.1, it is derived from the theory of U statistics [4].

For any $r \in \mathbf{N}_{\geq 1}$, we denote by S_r the set of all permutations from $\{1, \dots, r\}$ to $\{1, \dots, r\}$.

Let $r \in \mathbf{N}_{\geq 1}$ and $(Y_1^j, \dots, Y_n^j)_{1 \leq j \leq r}$, $(C_1^j, \dots, C_n^j)_{1 \leq j \leq r}$ be independent random variables.

PROPOSITION 4.1. *If the application $\gamma : \mathbf{R}^{nr} \rightarrow \mathbf{R}$ is such that for any $\tau \in S_r$ and any $(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) \in \mathbf{R}^{nr}$,*

$$\gamma(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) = \gamma(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)})$$

and for any integer j such that $1 \leq j \leq r$,

$$\begin{aligned} S_{(Y_1^j, \dots, Y_n^j)} &= S_{(Y_1^1, \dots, Y_n^1)} \\ S_{(C_1^j, \dots, C_n^j)} &= S_{(C_1^1, \dots, C_n^1)} \end{aligned}$$

then we have,

$$\begin{aligned} &\mathbb{U}_{C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \gamma, (\theta_i^0)_{i \in \mathbb{I}}} (t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r, d_1^1, \dots, d_n^1, \dots, d_1^r, \dots, d_n^r) \\ &= \mathbb{U}_{C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \gamma, (\theta_i^0)_{i \in \mathbb{I}}} (t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}, d_1^{\tau(1)}, \dots, d_n^{\tau(1)}, \dots, d_1^{\tau(r)}, \dots, d_n^{\tau(r)}) \end{aligned}$$

The theorem 4.1 is a key result, we show the doubly robust asymptotic normality of the optimal transformation in the general case.

Let $\bar{\gamma} : \mathbf{R}^{nr} \rightarrow \mathbf{R}$ be the function such that for any $(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) \in \mathbf{R}^{nr}$, we have,

$$\begin{aligned} &\bar{\gamma}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) \\ &= \frac{1}{\Gamma(r+1)} \sum_{\tau \in S_r} \gamma(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}) \end{aligned}$$

Let $m \in \mathbf{N}_{\geq r}$ and $\psi^{(m, n, r)} : \mathbf{R}^{nm} \rightarrow \mathbf{R}$ be the function such that for any $(t_1^1, \dots, t_n^1, \dots, t_1^m, \dots, t_n^m) \in \mathbf{R}^{nm}$, we have,

$$\psi^{(m, n, r)}(t_1^1, \dots, t_n^1, \dots, t_1^m, \dots, t_n^m) = \binom{m}{r}^{-1} \sum_{1 \leq j_1 < \dots < j_r \leq m} \bar{\gamma}(t_1^{j_1}, \dots, t_n^{j_1}, \dots, t_1^{j_r}, \dots, t_n^{j_r})$$

Let $(Y_1^{j_0}, \dots, Y_n^{j_0})_{j \geq 1}, (C_1^{j_0}, \dots, C_n^{j_0})_{j \geq 1}$ be independent random variables such that for any $j \in \mathbf{N}_{\geq 1}$, we have,

$$\begin{aligned} S_{(Y_1^{j_0}, \dots, Y_n^{j_0})} &= S_{(Y_1^{1_0}, \dots, Y_n^{1_0})} \\ S_{(C_1^{j_0}, \dots, C_n^{j_0})} &= S_{(C_1^{1_0}, \dots, C_n^{1_0})} \end{aligned}$$

Let for any integer k such that $1 \leq k \leq n$ and any $j \in \mathbf{N}_{\geq 1}$,

$$\begin{aligned} T_k^{j_0} &= \min(Y_k^{j_0}, C_k^{j_0}) \\ D_k^{j_0} &= 1_{Y_k^{j_0} \leq C_k^{j_0}} \end{aligned}$$

We suppose that,

$$\mathbb{E}[\mathbb{U}_{C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}, (\theta_i^0)_{i \in \mathbb{I}}} (T_1^{1_0}, \dots, T_n^{1_0}, \dots, T_1^{r_0}, \dots, T_n^{r_0}, D_1^{1_0}, \dots, D_n^{1_0}, \dots, D_1^{r_0}, \dots, D_n^{r_0})] \in \mathbf{R}_{\geq 0}$$

Let $g_{Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r, C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}} : \mathbf{R}^n \times \{0, 1\}^n \rightarrow \mathbf{R}$ be the function such that for any $(t_1, \dots, t_n, d_1, \dots, d_n) \in \mathbf{R}^n \times \{0, 1\}^n$, we have,

$$\begin{aligned} &g_{Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r, C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}}(t_1, \dots, t_n, d_1, \dots, d_n) \\ &= \mathbb{E}[\mathbb{U}_{C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}, (\theta_i^0)_{i \in \mathbb{I}}} (t_1, \dots, t_n, T_1^{2_0}, \dots, T_n^{2_0}, \dots, T_1^{r_0}, \dots, T_n^{r_0}, \\ &d_1, \dots, d_n, D_1^{2_0}, \dots, D_n^{2_0}, \dots, D_1^{r_0}, \dots, D_n^{r_0})] \end{aligned}$$

Let $\sigma_{Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r, C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}}$ be the following non negative real number,

$$\sigma_{Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r, C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}}^2 = \mathbb{V}[g_{Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r, C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}}(T_1^{1_0}, \dots, T_n^{1_0}, D_1^{1_0}, \dots, D_n^{1_0})]$$

THEOREM 4.1. *Under the Assumptions 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7, if for any $j \in \mathbf{N}_{\geq 1}$, $S_{(C_1^j, \dots, C_n^j)} = S_{(C_1^{10}, \dots, C_n^{10})}$ and $S_{(Y_1^j, \dots, Y_n^j)} = S_{(Y_1^{10}, \dots, Y_n^{10})}$ or, for any $j \in \mathbf{N}_{\geq 1}$, $S_{(Y_1^j, \dots, Y_n^j)} = S_{(Y_1^{10}, \dots, Y_n^{10})}$ and $S_{(C_1^j, \dots, C_n^j)} = S_{(C_1^{10}, \dots, C_n^{10})}$, and $\sigma_{Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r, C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}} > 0$, we have always for any real number u ,*

$$\begin{aligned} & \lim_{m \rightarrow +\infty} \mathbb{P}\left(\frac{\sqrt{m}}{r\sigma_{Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r, C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}}}\right. \\ & \times \left\{ \mathbf{U}_{C_1^1, \dots, C_n^1, \dots, C_1^m, \dots, C_n^m, \psi(m, n, r), (\theta^0)_{i \in \mathbf{I}}} (T_1^{10}, \dots, T_n^{10}, \dots, T_1^{m0}, \dots, T_n^{m0}, D_1^{10}, \dots, D_n^{10}, \dots, D_1^{m0}, \dots, D_n^{m0}) \right. \\ & \left. \left. - \bar{\gamma}(Y_1^{10}, \dots, Y_n^{10}, \dots, Y_1^{r0}, \dots, Y_n^{r0}) \right\} \leq u\right) \\ & = \phi(u) \end{aligned}$$

where the function ϕ from \mathbf{R} to $[0, 1]$ is the cumulative distribution function of a random normal variable with mean zero and variance one.

5. Perspectives. In this paper, we have generalised the construction of a doubly robust unbiased transformation proposed by Rubin and Van der Laan [13], and of minimum variance, as shown by Suzukawa [14], from the case of real valued random variables to the case of vector valued random variables. Using a result from Hoeffding [4] about U statistics, we have proved the asymptotic normality of the optimal transformation we have introduced. From a theoretical point of view, our problem is solved and it can be viewed as a starting point for further more applied results. Indeed, this work raises new questions. How to use these transformations actually? What would be the impact of substituting the theoretical distribution by the Kaplan and Meier estimator [5], on the bias? Is the asymptotic normality of the resulting estimators is then preserved? What about others properties of U statistics? In a work in progress, we investigate these questions.

APPENDIX A: PROOFS FROM THE SECTION 2

Proof of Proposition 2.1

For any $(t, d) \in \mathbf{R} \times \{0, 1\}$, we have,

$$\begin{aligned} |U_{C, \gamma, \theta_0}(t, d)| &= (1-d)|\theta_0(t)| + d|\theta_1(t)| \\ &= \int_{\mathbf{R}^2} |U_{C, \gamma, \theta_0}\{\min(y, c), 1_{V_1}(y, c)\}| d\mathbb{P}_{(Y, C)}(y, c) \\ &= \int_{\mathbf{R}} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) |\theta_0(c)| d\mathbb{P}_C(c) \right\} d\mathbb{P}_Y(y) + \int_{\mathbf{R}} S_C(y) |\theta_1(y)| d\mathbb{P}_Y(y) \end{aligned}$$

Also we have,

$$\int_{\mathbf{R}^2} |U_{C, \gamma, \theta_0}\{\min(y, c), 1_{V_1}(y, c)\}| d\mathbb{P}_{(Y, C)}(y, c) \geq \int_{\mathbf{R}} |\gamma(y)| d\mathbb{P}_Y(y)$$

So that the random variable $U_{C, \gamma, \theta_0}(T, D)$ from Ω to \mathbf{R} is integrable with respect to the probability measure \mathbb{P} and the function γ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y . Moreover,

$$\mathbb{E}[U_{C, \gamma, \theta_0}(T, D)] = \mathbb{E}[\gamma(Y)]$$

Proof of Proposition 2.2

Let $\alpha \in \mathbf{R}$, $\theta_0 \in \Theta$ and $\mu_0 = \alpha\theta_0$, we have,

$$\begin{aligned} & \int_{\mathbf{R}} \frac{\gamma^2(y)}{S_C(y)} d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0} \\ & \int_{\mathbf{R}} \left\{ \frac{1}{\sqrt{S_C(y)}} \int_{\mathbf{R}} 1_{V_0}(y, c) \alpha \theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0} \end{aligned}$$

Therefore,

$$\int_{\mathbf{R}^2} 1_{V_1}(y, c) \mu_1^2(y) d\mathbb{P}_{(Y,C)}(y, c) \in \mathbf{R}_{\geq 0}$$

Hence,

$$\mu_0 \in \Theta$$

Let $\theta_0, \beta_0 \in \Theta$ and $\mu_0 = \theta_0 + \beta_0$, we have,

$$\begin{aligned} & \int_{\mathbf{R}} \frac{\gamma^2(y)}{S_C(y)} d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0} \\ & \int_{\mathbf{R}} \left\{ \frac{1}{\sqrt{S_C(y)}} \int_{\mathbf{R}} 1_{V_0}(y, c) \theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0} \\ & \int_{\mathbf{R}} \left\{ \frac{1}{\sqrt{S_C(y)}} \int_{\mathbf{R}} 1_{V_0}(y, c) \beta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0} \end{aligned}$$

So that,

$$\int_{\mathbf{R}} \left[\frac{1}{\sqrt{S_C(y)}} \int_{\mathbf{R}} 1_{V_0}(y, c) \{ \theta_0(c) + \beta_0(c) \} d\mathbb{P}_C(c) \right]^2 d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0}$$

Hence,

$$\mu_0 \in \Theta$$

As $0 \in \Theta$, this concludes the proof.

Proof of Proposition 2.3

We have for any $\lambda \in \mathbf{R}$ and any $\theta_0 \in \Theta$,

$$N(\lambda\theta_0) = |\lambda|N(\theta_0)$$

Let $\theta_0 \in \Theta$ such that,

$$N(\theta_0) = 0$$

As we suppose that the random variables Y and C from Ω to \mathbf{R} are absolutely continuous with respect to the Lebesgue measure with positive and continuous densities, we have,

$$\theta_0 = 0$$

Let $\theta_0, \beta_0 \in \Theta$, we have,

$$\begin{aligned} & \{N(\theta_0 + \beta_0)\}^2 - \{N(\theta_0) + N(\beta_0)\}^2 \\ &= \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \{ \theta_0(c) + \beta_0(c) \} d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) + \int_{\mathbf{R}^2} 1_{V_0}(y, c) \{ \theta_0(c) + \beta_0(c) \}^2 d\mathbb{P}_{(Y,C)}(y, c) \\ & - \left\{ \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) + \int_{\mathbf{R}^2} 1_{V_0}(y, c) \theta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \right\} \\ & - \left\{ \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \beta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) + \int_{\mathbf{R}^2} 1_{V_0}(y, c) \beta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \right\} \\ & - 2N(\theta_0)N(\beta_0) \\ &= 2 \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \theta_0(c) d\mathbb{P}_C(c) \int_{\mathbf{R}} 1_{V_0}(y, c) \beta_0(c) d\mathbb{P}_C(c) \right\} d\mathbb{P}_Y(y) \\ & + 2 \int_{\mathbf{R}^2} 1_{V_0}(y, c) \theta_0(c) \beta_0(c) d\mathbb{P}_{(Y,C)}(y, c) \\ & - 2N(\theta_0)N(\beta_0) \\ &\leq 2 \left[\int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \beta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \right]^{\frac{1}{2}} \\ & + 2 \left[\int_{\mathbf{R}^2} 1_{V_0}(y, c) \theta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \int_{\mathbf{R}^2} 1_{V_0}(y, c) \beta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \right]^{\frac{1}{2}} \\ & - 2N(\theta_0)N(\beta_0) \end{aligned}$$

Also we have,

$$\begin{aligned}
& \left\{ \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \beta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \right\}^{\frac{1}{2}} \\
& + \left[\int_{\mathbf{R}^2} 1_{V_0}(y, c) \theta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \int_{\mathbf{R}^2} 1_{V_0}(y, c) \beta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \right]^{\frac{1}{2}} \\
& - \{N(\theta_0)N(\beta_0)\}^2 \\
& = 2 \left[\int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \beta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \right. \\
& \times \int_{\mathbf{R}^2} 1_{V_0}(y, c) \theta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \int_{\mathbf{R}^2} 1_{V_0}(y, c) \beta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \left. \right]^{\frac{1}{2}} \\
& - \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \theta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \int_{\mathbf{R}^2} 1_{V_0}(y, c) \beta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \\
& - \int_{\mathbf{R}} \frac{1}{S_C(y)} \left\{ \int_{\mathbf{R}} 1_{V_0}(y, c) \beta_0(c) d\mathbb{P}_C(c) \right\}^2 d\mathbb{P}_Y(y) \int_{\mathbf{R}^2} 1_{V_0}(y, c) \theta_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) \\
& \leq 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\{N(\theta_0 + \beta_0)\}^2 & \leq \{N(\theta_0) + N(\beta_0)\}^2 \\
N(\theta_0 + \beta_0) & \leq N(\theta_0) + N(\beta_0)
\end{aligned}$$

This concludes the proof.

Proof of Proposition 2.4

Let $\theta_0, H_0 \in \Theta$, we have for any $(y, c) \in \mathbf{R}^2$,

$$\begin{aligned}
& U_{C, \gamma, \theta_0 + H_0}^2 \{\min(y, c), 1_{V_1}(y, c)\} - U_{C, \gamma, \theta_0}^2 \{\min(y, c), 1_{V_1}(y, c)\} \\
& = 1_{V_1}(y, c) \{\mu_1^2(y) - \theta_1^2(y)\} + 1_{V_0}(y, c) \{[\theta_0(c) + H_0(c)]^2 - \theta_0^2(c)\} \\
& = 1_{V_1}(y, c) \{\mu_1^2(y) - \theta_1^2(y)\} + 1_{V_0}(y, c) \{2\theta_0(c) + H_0(c)\}H_0(c)
\end{aligned}$$

where for any $y \in \mathbf{R}$,

$$\mu_1(y) = \frac{1}{S_C(y)} [\gamma(y) - \int_{\mathbf{R}} 1_{V_0}(y, c) \{\theta_0(c) + H_0(c)\} d\mathbb{P}_C(c)]$$

We have,

$$\begin{aligned}
& S_C(y) \{\mu_1^2(y) - \theta_1^2(y)\} + \int_{\mathbf{R}} [(-2)\theta_1(y) 1_{V_0}(y, c) H_0(c)] d\mathbb{P}_C(c) \\
& = S_C(y) \{[\mu_1(y) - \theta_1(y)] \{\mu_1(y) + \theta_1(y)\} + (-2)\theta_1(y) S_C(y) [\mu_1(y) - \theta_1(y)]\} \\
& = S_C(y) [\mu_1(y) - \theta_1(y)]^2
\end{aligned}$$

As the function $y \mapsto S_C(y) \{\mu_1^2(y) - \theta_1^2(y)\}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y and the function $y \mapsto S_C(y) [\mu_1(y) - \theta_1(y)]^2$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y , it follows that the function $(y, c) \mapsto 1_{V_0}(y, c) \theta_1(y) H_0(c)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure $\mathbb{P}_{(Y,C)}$, we have,

$$\begin{aligned}
& \int_{\mathbf{R}} S_C(y) \{\mu_1^2(y) - \theta_1^2(y)\} d\mathbb{P}_Y(y) + \int_{\mathbf{R}^2} [(-2)\theta_1(y) 1_{V_0}(y, c) H_0(c)] d\mathbb{P}_{(Y,C)}(y, c) \\
& = \int_{\mathbf{R}} S_C(y) [\mu_1(y) - \theta_1(y)]^2 d\mathbb{P}_Y(y)
\end{aligned}$$

We can put,

$$L_{\theta_0}(H_0) = 2 \int_{\mathbf{R}^2} 1_{V_0}(y, c) [\{\theta_0(c) - \theta_1(y)\} H_0(c)] d\mathbb{P}_{(Y,C)}(y, c)$$

We have,

$$\begin{aligned}
& \int_{\mathbf{R}^2} U_{C, \gamma, \theta_0 + H_0}^2 \{\min(y, c), 1_{V_1}(y, c)\} d\mathbb{P}_{(Y,C)}(y, c) - \int_{\mathbf{R}^2} U_{C, \gamma, \theta_0}^2 \{\min(y, c), 1_{V_1}(y, c)\} d\mathbb{P}_{(Y,C)}(y, c) - L_{\theta_0}(H_0) \\
& = \int_{\mathbf{R}^2} 1_{V_0}(y, c) H_0^2(c) d\mathbb{P}_{(Y,C)}(y, c) + \int_{\mathbf{R}} S_C(y) [\mu_1(y) - \theta_1(y)]^2 d\mathbb{P}_Y(y) \\
& = \{N(H_0)\}^2
\end{aligned}$$

The function $H_0 \mapsto L_{\theta_0}(H_0)$ from Θ to \mathbf{R} is linear. Finally, the function χ from Θ to $\mathbf{R}_{\geq 0}$ is differentiable over Θ and for any $\theta_0 \in \Theta$, we have,

$$d_{\theta_0}\chi = L_{\theta_0}$$

Proof of Proposition 2.5

Let $t \in [0, 1]$ and $\theta_{0,1}, \theta_{0,2} \in \Theta$, we have,

$$\begin{aligned} U_{C,\gamma,(1-t)\theta_{0,1}+t\theta_{0,2}} &= (1-t)U_{C,\gamma,\theta_{0,1}} + tU_{C,\gamma,\theta_{0,2}} \\ \{(1-t)U_{C,\gamma,\theta_{0,1}} + tU_{C,\gamma,\theta_{0,2}}\}^2 &= (1-t)U_{C,\gamma,\theta_{0,1}}^2 + tU_{C,\gamma,\theta_{0,2}}^2 - t(1-t)(U_{C,\gamma,\theta_{0,1}} - U_{C,\gamma,\theta_{0,2}})^2 \\ \{(1-t)U_{C,\gamma,\theta_{0,1}} + tU_{C,\gamma,\theta_{0,2}}\}^2 &= (1-t)U_{C,\gamma,\theta_{0,1}}^2 + tU_{C,\gamma,\theta_{0,2}}^2 - t(1-t)U_{C,0,\theta_{0,1}-\theta_{0,2}}^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \chi\{(1-t)\theta_{0,1} + t\theta_{0,2}\} &= (1-t)\chi(\theta_{0,1}) + t\chi(\theta_{0,2}) - t(1-t)\mathbb{E}[U_{C,0,\theta_{0,1}-\theta_{0,2}}^2(T, D)] \\ \chi\{(1-t)\theta_{0,1} + t\theta_{0,2}\} &= (1-t)\chi(\theta_{0,1}) + t\chi(\theta_{0,2}) - t(1-t)\{N(\theta_{0,1} - \theta_{0,2})\}^2 \end{aligned}$$

This concludes the proof.

Proof of Proposition 2.6

Let $\eta : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number u ,

$$\eta(u) = \int_{\mathbf{R}^2} 1_{]-\infty, u]}(c) 1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\} d\mathbb{P}_{(Y,C)}(y, c)$$

We have,

$$\eta'(u) = 0$$

So that there exists a real number k such that for any $u \in \mathbf{R}$, we have,

$$\eta(u) = k$$

As,

$$\lim_{u \rightarrow -\infty} \eta(u) = 0$$

we have,

$$k = 0$$

So that for any $u \in \mathbf{R}$,

$$\eta(u) = 0$$

Finally,

$$\eta = 0$$

For any $u \in \mathbf{R}$, we have,

$$\int_{\mathbf{R}^2} 1_{]-\infty, u]}(c) 1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\} d\mathbb{P}_{(Y,C)}(y, c) = 0$$

We denote by G_0 the set of all subsets $] -\infty, u]$ from \mathbf{R} , when the real number u is in \mathbf{R} and by G the set of all elements B from $B(\mathbf{R})$ such that,

$$\int_{\mathbf{R}^2} 1_B(c)1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}d\mathbb{P}_{(Y,C)}(y, c) = 0$$

We have,

$$\mathbf{R} \in G$$

Let $B_1, B_2 \in G$ such that $B_1 \subset B_2$ and $B = B_2 \cap (\mathbf{R} - B_1)$, we have,

$$\int_{\mathbf{R}^2} 1_{B_1}(c)1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}d\mathbb{P}_{(Y,C)}(y, c) = 0$$

$$\int_{\mathbf{R}^2} 1_{B_2}(c)1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}d\mathbb{P}_{(Y,C)}(y, c) = 0$$

Also we have,

$$1_B = 1_{B_2} - 1_{B_1}$$

Therefore,

$$\int_{\mathbf{R}^2} 1_B(c)1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}d\mathbb{P}_{(Y,C)}(y, c) = 0$$

$$B \in G$$

Let $(B_m)_{m \geq 0}$ be a sequence of elements from G such that for any non negative integer m , we have,

$$B_m \subset B_{m+1}$$

and let $B = \cup_{m \geq 0} B_m$, we have for any non negative integer m ,

$$\int_{\mathbf{R}^2} 1_{B_m}(c)1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}d\mathbb{P}_{(Y,C)}(y, c) = 0$$

For any fixed real number x , we have,

$$\lim_{m \rightarrow +\infty} 1_{B_m}(x) = 1_B(x)$$

Moreover, for any non negative integer m , we have for any $(y, c) \in \mathbf{R}^2$,

$$|1_{B_m}(c)1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}| \leq |1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}|$$

and,

$$\int_{\mathbf{R}^2} |1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}|d\mathbb{P}_{(Y,C)}(y, c) \in \mathbf{R}_{\geq 0}$$

With the dominated convergence theorem, we have,

$$\int_{\mathbf{R}^2} 1_B(c)1_{V_0}(y, c)\{\theta_0(c) - \theta_1(y)\}d\mathbb{P}_{(Y,C)}(y, c) = 0$$

Therefore,

$$B \in G$$

Finally, the subset G from $\mathcal{P}(\mathbf{R})$ is a λ system [1].

For any real number u , we have,

$$\int_{\mathbf{R}^2} 1_{]-\infty, u]}(c) 1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\} d\mathbb{P}_{(Y, C)}(y, c) = 0$$

$$G_0 \subset G$$

Moreover, the subset G_0 from $\mathcal{P}(\mathbf{R})$ is a π system [1], so that by the monotone class theorem [1], we have,

$$\sigma(G_0) \subset G$$

As $\sigma(G_0) = B(\mathbf{R})$, we have,

$$G = B(\mathbf{R})$$

For any real number x , let $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$, let $H_0 \in \Theta$, the function $(y, c) \mapsto \{1_{V_0}(y, c)H_0(c)\}^+$ from \mathbf{R}^2 to \mathbf{R} is non negative over \mathbf{R}^2 , we have for any $(y, c) \in \mathbf{R}^2$,

$$\{1_{V_0}(y, c)H_0(c)\}^+ = 1_{V_0}(y, c)H_0^+(c)$$

There exists a non decreasing sequence from non negative step functions $(w_m)_{m \geq 0}$ from \mathbf{R} to $\mathbf{R}_{\geq 0}$ such that for any real number c , we have,

$$H_0^+(c) = \lim_{m \rightarrow +\infty} w_m(c)$$

By the reasoning above, we have for any non negative integer m ,

$$\int_{\mathbf{R}^2} 1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\} w_m(c) d\mathbb{P}_{(Y, C)}(y, c) = 0$$

and for any $(y, c) \in \mathbf{R}^2$,

$$|1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\} w_m(c)| \leq 1_{V_0}(y, c) |\theta_0(c) - \theta_1(y)| H_0^+(c)$$

As the function $(y, c) \mapsto 1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\}$ from \mathbf{R}^2 to \mathbf{R} is square integrable with respect to the probability measure $\mathbb{P}_{(Y, C)}$, and the function $(y, c) \mapsto 1_{V_0}(y, c) H_0^+(c)$ from \mathbf{R}^2 to \mathbf{R} is square integrable with respect to the probability measure $\mathbb{P}_{(Y, C)}$, we have that the function $(y, c) \mapsto 1_{V_0}(y, c) |\theta_0(c) - \theta_1(y)| H_0^+(c)$ from \mathbf{R}^2 to \mathbf{R} is integrable with respect to the probability measure $\mathbb{P}_{(Y, C)}$. With the dominated convergence theorem, we have so,

$$\int_{\mathbf{R}^2} 1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\} H_0^+(c) d\mathbb{P}_{(Y, C)}(y, c) = 0$$

Similarly, we have,

$$\int_{\mathbf{R}^2} 1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\} H_0^-(c) d\mathbb{P}_{(Y, C)}(y, c) = 0$$

So that,

$$\int_{\mathbf{R}^2} 1_{V_0}(y, c) \{\theta_0(c) - \theta_1(y)\} H_0(c) d\mathbb{P}_{(Y, C)}(y, c) = 0$$

Therefore for any $H_0 \in \Theta$,

$$d_{\theta_0} \chi(H_0) = 0$$

$$d_{\theta_0} \chi = 0$$

This concludes the proof.

We require technical results, they are presented in the Lemmas [A.1](#) and [A.2](#).

LEMMA A.1. For any $a \in \mathbf{R}$, let $\gamma_a : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number t ,

$$\gamma_a(t) = 1_{]a, +\infty[}(t)$$

and $\eta_a : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number c ,

$$\eta_a(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \gamma_a(y) d\mathbb{P}_Y(y)$$

Then for any real number y , the function $c \mapsto 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C , let $\phi_a : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number y ,

$$\phi_a(y) = \gamma_a(y) - S_C(y) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

We suppose that the function $y \mapsto \frac{1}{S_C(y)}$ from \mathbf{R} to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure \mathbb{P}_Y , then for any real number c , the function $y \mapsto 1_{V_0}(y, c) \frac{\phi_a(y)}{S_C(y)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y and for any real number t , the function $c \mapsto \frac{1_{V_1}(c, t)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C , if,

$$\lim_{t \rightarrow +\infty} \frac{S_Y(t)}{S_C(t)} = 0$$

we have then,

$$\gamma_a(t) = \phi_a(t) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

Proof

We have for any $(c, t) \in \mathbf{R}^2$,

$$\begin{aligned} 0 &\leq \eta_a(c) \leq 1 \\ 0 &\leq 1_{V_1}(c, t) \frac{\eta_a(c)}{S_C^2(c)} \leq \frac{1}{S_C^2(t)} \end{aligned}$$

So that the function $c \mapsto 1_{V_1}(c, t) \frac{\eta_a(c)}{S_C^2(c)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C . We suppose that the function $y \mapsto \frac{1}{S_C(y)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y , we have for any real number y ,

$$\frac{\phi_a(y)}{S_C(y)} = \frac{\gamma_a(y)}{S_C(y)} - \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

for any real number u ,

$$1_{V_0}(y, u) \left| \frac{\phi_a(y)}{S_C(y)} \right| \leq \frac{1}{S_C(y)} + \left| \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) \right|$$

As $0 \leq \eta_a(u) \leq 1$, we have,

$$\begin{aligned} 1_{V_0}(y, u) \left| \frac{\phi_a(y)}{S_C(y)} \right| &\leq \frac{1}{S_C(y)} + \int_{\mathbf{R}} \frac{1_{V_1}(c, y)}{S_C^2(c)} d\mathbb{P}_C(c) \\ 1_{V_0}(y, u) \left| \frac{\phi_a(y)}{S_C(y)} \right| &\leq \frac{1}{S_C(y)} + \left\{ \frac{1}{S_C(y)} - 1 \right\} \end{aligned}$$

and for any real number t ,

$$\begin{aligned} \left| \frac{1_{V_1}(u, t)}{S_Y(u)} \int_{\mathbf{R}} 1_{V_0}(y, u) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y) \right| &\leq \frac{1_{V_1}(u, t)}{S_Y(u)} \int_{\mathbf{R}} 1_{V_0}(y, u) \left| \frac{\phi_a(y)}{S_C(y)} \right| d\mathbb{P}_Y(y) \\ \left| \frac{1_{V_1}(u, t)}{S_Y(u)} \int_{\mathbf{R}} 1_{V_0}(y, u) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y) \right| &\leq \frac{1}{S_Y(t)} \int_{\mathbf{R}} \left[\frac{1}{S_C(y)} + \left\{ \frac{1}{S_C(y)} - 1 \right\} \right] d\mathbb{P}_Y(y) \\ \left| \frac{1_{V_1}(u, t)}{S_Y(u)} \int_{\mathbf{R}} 1_{V_0}(y, u) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y) \right| &\leq \frac{1}{S_Y(t)} \left[\int_{\mathbf{R}} 2 \frac{d\mathbb{P}_Y(y)}{S_C(y)} \right] - 1 \end{aligned}$$

Thus for any real number t , the function $c \mapsto \frac{1_{V_1}(c, t)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C . Let $r_a : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number t ,

$$r_a(t) = -S_C(t) \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

For any real number c , we have,

$$\eta_a(c) = \frac{S_Y\{\max(a, c)\}}{S_Y(c)}$$

As the function $c \mapsto \eta_a(c)$ from \mathbf{R} to \mathbf{R} is continuous over \mathbf{R} , the function $t \mapsto \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c)$ from \mathbf{R} to \mathbf{R} is differentiable over \mathbf{R} . Moreover, the function $t \mapsto \phi_a(t)$ has only one point of discontinuity over \mathbf{R} , so that the function $t \mapsto \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y)$ from \mathbf{R} to \mathbf{R} is differentiable over \mathbf{R} . For any real number t , we have,

$$\begin{aligned} r'_a(t) &= f_C(t) \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) - S_C(t) \frac{\eta_a(t) f_C(t)}{S_C^2(t)} + \frac{f_C(t)}{S_Y(t)} \int_{\mathbf{R}} 1_{V_0}(y, t) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y) \\ \frac{S_Y(t) r'_a(t)}{f_C(t)} &= S_Y(t) \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) - \frac{S_Y(t) \eta_a(t)}{S_C(t)} + \int_{\mathbf{R}} 1_{V_0}(y, t) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y) \end{aligned}$$

Also we have,

$$\int_{\mathbf{R}} 1_{V_0}(y, t) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y) = \int_{\mathbf{R}} 1_{V_0}(y, t) \frac{1}{S_C(y)} \left\{ \gamma_a(y) - S_C(y) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) \right\} d\mathbb{P}_Y(y)$$

Therefore if $t \leq a$, we have,

$$\begin{aligned} \int_{\mathbf{R}} 1_{V_0}(y, t) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y) &= \int_{\mathbf{R}} 1_{V_0}(y, a) \frac{d\mathbb{P}_Y(y)}{S_C(y)} - \int_{\mathbf{R}} 1_{V_0}(y, t) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) \\ &= \frac{S_Y(t) r'_a(t)}{f_C(t)} \\ &= S_Y(t) S_Y(a) \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c) S_C^2(c)} - \frac{S_Y(a)}{S_C(t)} \\ &\quad + \int_{\mathbf{R}} 1_{V_0}(y, a) \frac{d\mathbb{P}_Y(y)}{S_C(y)} - \int_{\mathbf{R}} 1_{V_0}(y, t) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) \end{aligned}$$

Hence we have,

$$\begin{aligned} &\frac{d}{dt} \frac{S_Y(t) r'_a(t)}{f_C(t)} \\ &= -f_Y(t) S_Y(a) \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c) S_C^2(c)} + S_Y(t) S_Y(a) \frac{f_C(t)}{S_Y(t) S_C^2(t)} - \frac{S_Y(a) f_C(t)}{S_C^2(t)} + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) f_Y(t) \\ &= 0 \end{aligned}$$

Also we have,

$$\begin{aligned} \int_{\mathbf{R}^2} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) &= \int_{\mathbf{R}^2} 1_{V_1}(c, y) \frac{S_Y\{\max(a, c)\}}{S_Y(c)S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) = \int_{\mathbf{R}} \frac{S_Y\{\max(a, c)\}}{S_C^2(c)} d\mathbb{P}_C(c) \\ \int_{\mathbf{R}^2} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) &= \int_{\mathbf{R}} 1_{V_1}(c, a) \frac{S_Y\{\max(a, c)\}}{S_C^2(c)} d\mathbb{P}_C(c) + \int_{\mathbf{R}} 1_{V_0}(c, a) \frac{S_Y\{\max(a, c)\}}{S_C^2(c)} d\mathbb{P}_C(c) \\ \int_{\mathbf{R}^2} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) &= \int_{\mathbf{R}} 1_{V_1}(c, a) \frac{S_Y(a)}{S_C^2(c)} d\mathbb{P}_C(c) + \int_{\mathbf{R}} 1_{V_0}(c, a) \frac{S_Y(c)}{S_C^2(c)} d\mathbb{P}_C(c) \\ \int_{\mathbf{R}^2} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) &= \frac{S_Y(a)}{S_C(a)} - S_Y(a) + \int_{\mathbf{R}} 1_{V_0}(c, a) \frac{S_Y(c)}{S_C^2(c)} d\mathbb{P}_C(c) \end{aligned}$$

As we suppose,

$$\lim_{t \rightarrow +\infty} \frac{S_Y(t)}{S_C(t)} = 0$$

an integration by parts gives,

$$\int_{\mathbf{R}} 1_{V_0}(c, a) \frac{S_Y(c)}{S_C^2(c)} d\mathbb{P}_C(c) = 0 - \frac{S_Y(a)}{S_C(a)} + \int_{\mathbf{R}} 1_{V_0}(y, a) \frac{d\mathbb{P}_Y(y)}{S_C(y)}$$

Therefore,

$$\lim_{t \rightarrow -\infty} \frac{S_Y(t)r'_a(t)}{f_C(t)} = 0 - S_Y(a) + \int_{\mathbf{R}} 1_{V_0}(y, a) \frac{d\mathbb{P}_Y(y)}{S_C(y)} + S_Y(a) - \int_{\mathbf{R}} 1_{V_0}(y, a) \frac{d\mathbb{P}_Y(y)}{S_C(y)} = 0$$

Hence for any real number t such that $t \leq a$, we have,

$$r'_a(t) = 0$$

Moreover,

$$\lim_{t \rightarrow -\infty} r_a(t) = 0$$

So that for any real number t such that $t \leq a$, we have,

$$r_a(t) = 0$$

We have for any real number t such that $t \geq a$,

$$\begin{aligned} \int_{\mathbf{R}} 1_{V_0}(y, t) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y) &= \int_{\mathbf{R}} 1_{V_0}(y, t) \frac{d\mathbb{P}_Y(y)}{S_C(y)} - \int_{\mathbf{R}} 1_{V_0}(y, t) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) \\ &= \frac{S_Y(t)r'_a(t)}{f_C(t)} \\ &= S_Y(t)S_Y(a) \int_{\mathbf{R}} 1_{V_1}(c, a) \frac{d\mathbb{P}_C(c)}{S_Y(c)S_C^2(c)} + S_Y(t) \int_{\mathbf{R}} 1_{]a, t]}(c) \frac{d\mathbb{P}_C(c)}{S_C^2(c)} \\ &\quad - \frac{S_Y(t)}{S_C(t)} + \int_{\mathbf{R}} 1_{V_0}(y, t) \frac{d\mathbb{P}_Y(y)}{S_C(y)} - \int_{\mathbf{R}} 1_{V_0}(y, t) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) d\mathbb{P}_Y(y) \end{aligned}$$

Hence we have,

$$\begin{aligned}
& \frac{d}{dt} \frac{S_Y(t)r'_a(t)}{f_C(t)} \\
&= -f_Y(t)S_Y(a) \int_{\mathbf{R}} 1_{V_1}(c, a) \frac{d\mathbb{P}_C(c)}{S_Y(c)S_C^2(c)} + \frac{S_Y(t)f_C(t)}{S_C^2(t)} - f_Y(t) \int_{\mathbf{R}} 1_{]a, t]}(c) \frac{d\mathbb{P}_C(c)}{S_C^2(c)} - \frac{S_Y(t)f_C(t)}{S_C^2(t)} \\
&+ \frac{f_Y(t)}{S_C(t)} - \frac{f_Y(t)}{S_C(t)} + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) f_Y(t) \\
&= -f_Y(t)S_Y(a) \int_{\mathbf{R}} 1_{V_1}(c, a) \frac{d\mathbb{P}_C(c)}{S_Y(c)S_C^2(c)} + \int_{\mathbf{R}} 1_{V_1}(c, a) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) f_Y(t) \\
&- f_Y(t) \int_{\mathbf{R}} 1_{]a, t]}(c) \frac{d\mathbb{P}_C(c)}{S_C^2(c)} + \int_{\mathbf{R}} 1_{]a, t]}(c) \frac{\eta_a(c)}{S_C^2(c)} d\mathbb{P}_C(c) f_Y(t) \\
&= -f_Y(t)S_Y(a) \int_{\mathbf{R}} 1_{V_1}(c, a) \frac{d\mathbb{P}_C(c)}{S_Y(c)S_C^2(c)} + S_Y(a) \int_{\mathbf{R}} 1_{V_1}(c, a) \frac{d\mathbb{P}_C(c)}{S_Y(c)S_C^2(c)} f_Y(t) \\
&= 0
\end{aligned}$$

Also we have,

$$S_Y(t) \int_{\mathbf{R}} 1_{]a, t]}(c) \frac{d\mathbb{P}_C(c)}{S_C^2(c)} - \frac{S_Y(t)}{S_C(t)} = -\frac{S_Y(t)}{S_C(a)}$$

Therefore,

$$\lim_{t \rightarrow +\infty} \frac{S_Y(t)r'_a(t)}{f_C(t)} = 0$$

So that for any real number t such that $t \geq a$, we have,

$$r'_a(t) = 0$$

Moreover,

$$r_a(a) = 0$$

As the function $t \mapsto r_a(t)$ from \mathbf{R} to \mathbf{R} is continuous at the real point a , for any real number t such that $t \geq a$, we have,

$$r_a(t) = 0$$

Finally for any real number t , we have,

$$r_a(t) = 0$$

Therefore,

$$\gamma_a(t) = \phi_a(t) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_a(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

This concludes the proof.

LEMMA A.2. For any borelian $B \in B(\mathbf{R})$, let $\gamma_B : \mathbf{R} \rightarrow \mathbf{R}$ and $\eta_B : \mathbf{R} \rightarrow \mathbf{R}$ be the functions such that for any real number t ,

$$\begin{aligned}
\gamma_B(t) &= 1_B(t) \\
\eta_B(t) &= \frac{1}{S_Y(t)} \int_{\mathbf{R}} 1_{V_0}(y, t) \gamma_B(y) d\mathbb{P}_Y(y)
\end{aligned}$$

Then for any real number t , the function $c \mapsto 1_{V_1}(c, t) \frac{\eta_B(c)}{S_C^2(c)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C , let $\phi_B : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number t ,

$$\phi_B(t) = \gamma_B(t) - S_C(t) \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta_B(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

We suppose that the function $y \mapsto \frac{1}{S_C(y)}$ from \mathbf{R} to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure \mathbb{P}_Y , then for any real number c , the function $y \mapsto 1_{V_0}(y, c) \frac{\phi_B(y)}{S_C(y)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y , and for any real number t , the function $c \mapsto \frac{1_{V_1}(c, t)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_B(y)}{S_C(y)} d\mathbb{P}_Y(y)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C and if,

$$\lim_{t \rightarrow +\infty} \frac{S_Y(t)}{S_C(t)} = 0$$

we have then,

$$\gamma_B(t) = \phi_B(t) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_B(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

Proof

We have for any $(c, t) \in \mathbf{R}^2$,

$$\begin{aligned} 0 &\leq \eta_B(c) \leq 1 \\ 0 &\leq 1_{V_1}(c, t) \frac{\eta_B(c)}{S_C^2(c)} \leq \frac{1}{S_C^2(t)} \end{aligned}$$

So that the function $c \mapsto 1_{V_1}(c, t) \frac{\eta_B(c)}{S_C^2(c)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C , let $\phi_B : \mathbf{R} \rightarrow \mathbf{R}$ be the function such that for any real number t ,

$$\phi_B(t) = \gamma_B(t) - S_C(t) \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{\eta_B(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

We suppose that the function $y \mapsto \frac{1}{S_C(y)}$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_Y , we have for any real number y ,

$$\frac{\phi_B(y)}{S_C(y)} = \frac{\gamma_B(y)}{S_C(y)} - \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_B(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

for any real number u ,

$$1_{V_0}(y, u) \left| \frac{\phi_B(y)}{S_C(y)} \right| \leq \frac{1}{S_C(y)} + \left| \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_B(c)}{S_C^2(c)} d\mathbb{P}_C(c) \right|$$

As $0 \leq \eta_B(u) \leq 1$, we have,

$$1_{V_0}(y, u) \left| \frac{\phi_B(y)}{S_C(y)} \right| \leq \frac{1}{S_C(y)} + \int_{\mathbf{R}} \frac{1_{V_1}(c, y)}{S_C^2(c)} d\mathbb{P}_C(c)$$

$$1_{V_0}(y, u) \left| \frac{\phi_B(y)}{S_C(y)} \right| \leq \frac{1}{S_C(y)} + \left\{ \frac{1}{S_C(y)} - 1 \right\}$$

and for any real number t ,

$$\begin{aligned} \left| \frac{1_{V_1}(u, t)}{S_Y(u)} \int_{\mathbf{R}} 1_{V_0}(y, u) \frac{\phi_B(y)}{S_C(y)} d\mathbb{P}_Y(y) \right| &\leq \frac{1_{V_1}(u, t)}{S_Y(u)} \int_{\mathbf{R}} 1_{V_0}(y, u) \left| \frac{\phi_B(y)}{S_C(y)} \right| d\mathbb{P}_Y(y) \\ \left| \frac{1_{V_1}(u, t)}{S_Y(u)} \int_{\mathbf{R}} 1_{V_0}(y, u) \frac{\phi_B(y)}{S_C(y)} d\mathbb{P}_Y(y) \right| &\leq \frac{1}{S_Y(t)} \int_{\mathbf{R}} \left[\frac{1}{S_C(y)} + \left\{ \frac{1}{S_C(y)} - 1 \right\} \right] d\mathbb{P}_Y(y) \\ \left| \frac{1_{V_1}(u, t)}{S_Y(u)} \int_{\mathbf{R}} 1_{V_0}(y, u) \frac{\phi_B(y)}{S_C(y)} d\mathbb{P}_Y(y) \right| &\leq \frac{1}{S_Y(t)} \left[\int_{\mathbf{R}} 2 \frac{d\mathbb{P}_Y(y)}{S_C(y)} \right] - 1 \end{aligned}$$

Thus for any real number t , the function $c \mapsto \frac{1_{V_1}(c,t)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y,c) \frac{\phi_B(y)}{S_C(y)} d\mathbb{P}_Y(y)$ from \mathbf{R} to \mathbf{R} is integrable with respect to the probability measure \mathbb{P}_C . For any element $B \in B(\mathbf{R})$, we say that it verifies the property \mathcal{P} if we put for any real number t ,

$$\begin{aligned}\gamma_B(t) &= 1_B(t) \\ \eta_B(t) &= \frac{1}{S_Y(t)} \int_{\mathbf{R}} 1_{V_0}(u,t) \gamma_B(u) d\mathbb{P}_Y(u) \\ \phi_B(t) &= \gamma_B(t) - S_C(t) \int_{\mathbf{R}} 1_{V_1}(c,t) \frac{\eta_B(c)}{S_C^2(c)} d\mathbb{P}_C(c)\end{aligned}$$

we have,

$$\gamma_B(t) = \phi_B(t) + \int_{\mathbf{R}} 1_{V_1}(c,t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y,c) \frac{\phi_B(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

We denote by G_0 the set of all subsets $]u, +\infty[$ from \mathbf{R} when the real number u is in \mathbf{R} and by G the set of all borelian B from $B(\mathbf{R})$ that verify the property \mathcal{P} , following the Lemma A.1, we have,

$$G_0 \subset G$$

Let $B_1, B_2 \in G$ such that $B_1 \subset B_2$, we have,

$$B = B_2 \cap (\mathbf{R} - B_1)$$

We have for any real number t ,

$$1_B(t) = 1_{B_2}(t) - 1_{B_1}(t)$$

As the sum of two integrable functions is an integrable function and the integral is linear, we have,

$$B \in G$$

Let $(B_m)_{m \geq 0}$ be a sequence of elements from G such that for any non negative integer m ,

$$B_m \subset B_{m+1}$$

Let $B = \cup_{m \geq 0} B_m$, we have for any real number x ,

$$\lim_{m \rightarrow +\infty} 1_{B_m}(x) = 1_B(x)$$

so that by the dominated convergence theorem, we have,

$$B \in G$$

For any non negative integer m , let $B_m =]-m, +\infty[$, we have,

$$B_m \in G$$

$$B_m \subset B_{m+1}$$

$$\lim_{m \rightarrow +\infty} 1_{B_m}(x) = 1_{\mathbf{R}}(x) = 1$$

Hence,

$$\mathbf{R} \in G$$

Therefore, the subset G from $B(\mathbf{R})$ is a λ system [1]. Moreover, the subset G_0 from $B(\mathbf{R})$ is a subset from G , it is also a π system [1], so that by the monotone class theorem [1], we have,

$$\sigma(G_0) \subset G$$

As $\sigma(G_0) = B(\mathbf{R})$, we have,

$$G = B(\mathbf{R})$$

This concludes the proof.

Proof of Proposition 2.7

There exists $(w_m)_{m \geq 0}$ a non decreasing sequence from non negative step functions from \mathbf{R} to $\mathbf{R}_{\geq 0}$ such that for any real number y , we have,

$$\gamma^+(y) = \lim_{m \rightarrow +\infty} w_m(y)$$

By the Lemma A.2, we have for any non negative integer m ,

$$w_m(t) = \phi_m(t) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_m(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

where for any real number y , we have,

$$\phi_m(y) = w_m(y) - S_C(y) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_m(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

and any real number c ,

$$\eta_m(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) w_m(y) d\mathbb{P}_Y(y)$$

For any real number y and any non negative integer m , we have,

$$|1_{V_0}(y, c) w_m(y)| \leq 1_{V_0}(y, c) \gamma^+(y)$$

and by the Assumption 2.2, we have,

$$\eta_+(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \gamma^+(y) d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0}$$

The dominated convergence theorem gives, for any real number c ,

$$\eta_+(c) = \lim_{m \rightarrow +\infty} \eta_m(c)$$

Moreover, for any non negative integer m , we have,

$$\eta_m(c) \leq \eta_{m+1}(c)$$

Hence for any $(c, y) \in \mathbf{R}^2$, we have,

$$|1_{V_1}(c, y) \frac{\eta_m(c)}{S_C^2(c)}| \leq 1_{V_1}(c, y) \frac{\eta_+(c)}{S_C^2(c)}$$

As we have,

$$\int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_+(c)}{S_C^2(c)} d\mathbb{P}_C(c) \in \mathbf{R}_{\geq 0}$$

it follows by the dominated convergence theorem that,

$$\int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_+(c)}{S_C^2(c)} d\mathbb{P}_C(c) = \lim_{m \rightarrow +\infty} \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_m(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

Let for any real number y ,

$$\phi_+(y) = \gamma^+(y) - S_C(y) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_+(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

we have,

$$\phi_+(y) = \lim_{m \rightarrow +\infty} \phi_m(y)$$

Also, we have,

$$\int_{\mathbf{R}} \frac{d\mathbb{P}_Y(y)}{S_C(y)} \in \mathbf{R}_{\geq 0}$$

$$\int_{\mathbf{R}} \frac{\gamma^2(y)}{S_C(y)} d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0}$$

Hence,

$$\int_{\mathbf{R}} \frac{|\gamma(y)|}{S_C(y)} d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0}$$

$$\int_{\mathbf{R}} \frac{\gamma^+(y)}{S_C(y)} d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0}$$

For any non negative integer m , we have for any $(c, c', t, y) \in \mathbf{R}^2$,

$$|1_{V_0}(y, c) \frac{w_m(y)}{S_C(y)}| \leq \frac{\gamma^+(y)}{S_C(y)}$$

$$|1_{V_0}(y, c) \int_{\mathbf{R}} 1_{V_1}(c', y) \frac{\eta_m(c')}{S_C^2(c')} d\mathbb{P}_C(c')| \leq 1_{V_0}(y, c) \int_{\mathbf{R}} 1_{V_1}(c', y) \frac{\eta_+(c')}{S_C^2(c')} d\mathbb{P}_C(c')$$

As we have,

$$\int_{\mathbf{R}} 1_{V_0}(y, c) \int_{\mathbf{R}} 1_{V_1}(c', y) \frac{\eta_+(c')}{S_C^2(c')} d\mathbb{P}_C(c') d\mathbb{P}_Y(y) \in \mathbf{R}_{\geq 0}$$

and,

$$\frac{1_{V_1}(c, t)}{S_Y(c)} \leq \frac{1}{S_Y(t)}$$

we obtain,

$$\int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_+(y)}{S_C(y)} d\mathbb{P}_Y(y) = \lim_{m \rightarrow +\infty} \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_m(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

Hence, we have for any real number t ,

$$\gamma^+(t) = \phi_+(t) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_+(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

By a similar reasoning, we have,

$$\gamma^-(t) = \phi_-(t) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_-(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

where for any real number y , we have,

$$\phi_-(y) = \gamma^-(y) - S_C(y) \int_{\mathbf{R}} 1_{V_1}(c, y) \frac{\eta_-(c)}{S_C^2(c)} d\mathbb{P}_C(c)$$

and for any real number c ,

$$\eta_-(c) = \frac{1}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \gamma^-(y) d\mathbb{P}_Y(y)$$

Finally, we have,

$$\gamma(t) = \phi(t) + \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi(y)}{S_C(y)} d\mathbb{P}_Y(y)$$

This concludes the proof.

Proof of Proposition 2.8

For any real number t , we have,

$$\begin{aligned} \mu(t) &= \frac{\phi(t)}{S_C(t)} \\ \gamma(t) - S_C(t)\mu(t) - \int_{\mathbf{R}} 1_{V_1}(c, t) \theta_0^0(c) d\mathbb{P}_C(c) \\ &= \gamma(t) - \phi(t) - \int_{\mathbf{R}} 1_{V_1}(c, t) \frac{d\mathbb{P}_C(c)}{S_Y(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi(y)}{S_C(y)} d\mathbb{P}_Y(y) \\ &= \gamma(t) - \gamma(t) \\ &= 0 \end{aligned}$$

Therefore,

$$\theta_1^0(t) = \mu(t)$$

Also,

$$\begin{aligned} \gamma(t) - S_C(t)\theta_1^0(t) - \int_{\mathbf{R}} 1_{V_1}(c, t) \theta_0^0(c) d\mathbb{P}_C(c) \\ &= S_C(t) \int_{\mathbf{R}} 1_{V_1}(u, t) \frac{\eta(u)}{S_C^2(u)} d\mathbb{P}_C(u) - \int_{\mathbf{R}} 1_{V_1}(c, t) \theta_0^0(c) d\mathbb{P}_C(c) \\ &= 0 \end{aligned}$$

As the function $u \mapsto \gamma(u)$ from \mathbf{R} to \mathbf{R} is piecewise continuous over \mathbf{R} , the function $t \mapsto \eta(t)$ from \mathbf{R} to \mathbf{R} is continuous over \mathbf{R} , by differentiating the precedent equality with respect to the variable t , we obtain,

$$-f_C(t) \int_{\mathbf{R}} 1_{V_1}(u, t) \frac{\eta(u)}{S_C^2(u)} d\mathbb{P}_C(u) + S_C(t) \frac{\eta(t)}{S_C^2(t)} f_C(t) - \theta_0^0(t) f_C(t) = 0$$

Thus,

$$\theta_0^0(t) = \frac{\eta(t)}{S_C(t)} - \int_{\mathbf{R}} 1_{V_1}(u, t) \frac{\eta(u)}{S_C^2(u)} d\mathbb{P}_C(u)$$

Proof of Proposition 2.9

Let $\theta_0 \in \Theta$, we have,

$$\begin{aligned} &\mathbb{E}[\{U_{C, \gamma, \theta_0}(T, D) - \gamma(Y)\}^2] + \mathbb{V}[\gamma(Y)] \\ &= \mathbb{E}[U_{C, \gamma, \theta_0}^2(T, D)] + \mathbb{E}[\gamma^2(Y)] - 2\mathbb{E}[U_{C, \gamma, \theta_0}(T, D)\gamma(Y)] + \mathbb{E}[\gamma^2(Y)] - \mathbb{E}^2[\gamma(Y)] \\ &= \mathbb{E}[U_{C, \gamma, \theta_0}^2(T, D)] + \mathbb{E}[\gamma^2(Y)] - 2\mathbb{E}[\gamma^2(Y)] + \mathbb{E}[\gamma^2(Y)] - \mathbb{E}^2[\gamma(Y)] \\ &= \mathbb{E}[U_{C, \gamma, \theta_0}^2(T, D)] - \mathbb{E}^2[\gamma(Y)] \\ &= \mathbb{V}[U_{C, \gamma, \theta_0}(T, D)] \end{aligned}$$

Proof of Theorem 2.1

Indeed, the application χ from Θ to $\mathbf{R}_{\geq 0}$ is strictly convex over Θ and by the Proposition 2.8, if $\theta_0^0 \in \Theta$, it has a stationary point at $\theta_0 = \theta_0^0$, so that it is the unique minimum from the application χ over Θ [12].

Proof of Theorem 2.2

Let,

$$\phi_0 = \theta_1^0$$

and for any real number t ,

$$\mu(t) = S_Y(t)\theta_0^0(t) + \int_{\mathbf{R}} 1_{V_1}(y, t)\phi_0(y)d\mathbb{P}_Y(y)$$

We have,

$$\mu'(t) = -f_Y(t)\theta_1^0(t) + f_Y(t)\theta_1^0(t) = 0$$

So that there exists $x \in \mathbf{R}$ such that for any real number t , we have,

$$\mu(t) = x$$

When the real number t tends to minus infinity, we obtain,

$$x = \int_{\mathbf{R}} \theta_1^0(y)d\mathbb{P}_Y(y)$$

By the Proposition 2.6, with $H_0 = 1$, we have so,

$$\mu(t) = \mathbb{E}[\gamma(Y)]$$

For any $(t, d) \in \mathbf{R} \times \{0, 1\}$, we have,

$$U_{C, \gamma, \theta_0^0}(t, d) = U_{Y, \mu, \phi_0}(t, 1 - d)$$

This concludes the proof.

APPENDIX B: PROOFS FROM THE SECTION 3

In the general case from the Section 3, the reasoning for the proofs is the natural extension of the case for one observation from the Section 2.

Proof of Proposition 3.1

For any $(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}^{2n}$, we have,

$$\begin{aligned} & U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in \mathbf{I}}} \{\min(y_1, c_1), \dots, \min(y_n, c_n), 1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n)\} \\ &= \sum_{i \in \{0, 1\}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} \\ & |U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in \mathbf{I}}} \{\min(y_1, c_1), \dots, \min(y_n, c_n), 1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n)\}| \\ &= \sum_{i \in \{0, 1\}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} |\theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\}| \end{aligned}$$

$$\begin{aligned}
& \mathbb{E}[|U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n)|] \\
&= \int_{\mathbf{R}^{2n}} |U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}\{\min(y_1, c_1), \dots, \min(y_n, c_n), 1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n)\}| \\
& \quad d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\
&= \int_{\mathbf{R}^{2n}} [\sum_{i \in \{0, 1\}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} |\theta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\}|] \\
& \quad d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\
&= \int_{\mathbf{R}^n} [\sum_{i \in \{0, 1\}^n} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} |\theta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\}| d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)] \\
& \quad d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&= \int_{\mathbf{R}^n} (S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) |\theta_{(1, \dots, 1)}(y_1, \dots, y_n)| \\
& \quad + \sum_{i \in I} [\int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} |\theta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\}|] d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)) \\
& \quad d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&\geq \int_{\mathbf{R}^n} |S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \\
& \quad + \sum_{i \in I} [\int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\}] d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)| \\
& \quad d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&= \int_{\mathbf{R}^n} |\gamma(y_1, \dots, y_n)| d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)
\end{aligned}$$

Consequently, the function γ from \mathbf{R}^n to \mathbf{R} is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$ and we have,

$$\mathbb{E}[U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n)] = \mathbb{E}[\gamma(Y_1, \dots, Y_n)]$$

Proof of Proposition 3.2

Let $\alpha \in \mathbf{R}$, $(\theta_i)_{i \in I}, (\beta_i)_{i \in I} \in \Theta$ and $(\mu_i)_{i \in I} = \alpha(\theta_i)_{i \in I}$, we have,

$$\begin{aligned}
& \int_{\mathbf{R}^n} \frac{\gamma^2(y_1, \dots, y_n)}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \in \mathbf{R}_{\geq 0} \\
& \int_{\mathbf{R}^n} \left[\frac{\alpha}{\sqrt{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)}} (\sum_{i \in I} [\int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \right. \\
& \quad \times \theta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\}] d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n))]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \in \mathbf{R}_{\geq 0}
\end{aligned}$$

Therefore,

$$\int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_1}(y_j, c_j) \right\} \mu^2(y_1, \dots, y_n) d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}_{\geq 0}$$

Hence,

$$(\mu_i)_{i \in I} \in \Theta$$

Let $(\mu_i)_{i \in I} = (\theta_i)_{i \in I} + (\beta_i)_{i \in I}$, we have,

$$\begin{aligned}
& \int_{\mathbf{R}^n} \frac{\gamma^2(y_1, \dots, y_n)}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \in \mathbf{R}_{\geq 0} \\
& \int_{\mathbf{R}^n} \left[\frac{1}{\sqrt{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)}} (\sum_{i \in I} [\int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\} \right. \\
& \quad d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n))]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \in \mathbf{R}_{\geq 0} \\
& \int_{\mathbf{R}^n} \left[\frac{1}{\sqrt{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)}} (\sum_{i \in I} [\int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \beta_i\{\min(y_1, c_1), \dots, \min(y_n, c_n)\} \right. \\
& \quad d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n))]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \in \mathbf{R}_{\geq 0}
\end{aligned}$$

So that,

$$\int_{\mathbf{R}^n} \left[\frac{1}{\sqrt{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)}} \left(\sum_{i \in I} \left[\int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} + \beta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \}] d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \right]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \right) \right] \in \mathbf{R}_{\geq 0}$$

Therefore,

$$\int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_1}(y_j, c_j) \right\} \mu^2(y_1, \dots, y_n) d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}_{\geq 0}$$

Hence,

$$(\mu_i)_{i \in I} \in \Theta$$

As $0 \in \Theta$, this concludes the proof.

Proof of Proposition 3.3

We have for any $\lambda \in \mathbf{R}$ and any $(\theta_i)_{i \in I} \in \Theta$,

$$N\{\lambda(\theta_i)_{i \in I}\} = |\lambda| N\{(\theta_i)_{i \in I}\}$$

Let $(\theta_i)_{i \in I} \in \Theta$ such that,

$$N\{(\theta_i)_{i \in I}\} = 0$$

As the random vectors (Y_1, \dots, Y_n) and (C_1, \dots, C_n) from Ω to \mathbf{R}^n have a continuous positive density with respect to the Lebesgue measure over \mathbf{R}^n , it follows that,

$$(\theta_i)_{i \in I} = 0$$

Let $(\theta_i)_{i \in I}, (\beta_i)_{i \in I} \in \Theta$, we have,

$$\begin{aligned}
& N\{(\theta_i)_{i \in I} + (\beta_i)_{i \in I}\}^2 - [N\{(\theta_i)_{i \in I}\} + N\{(\beta_i)_{i \in I}\}]^2 \\
&= \int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times [\theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} + \beta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\}] d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)]^2 \\
& d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&+ \sum_{i \in I} \int_{\mathbf{R}^{2n}} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times [\theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} + \beta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\}]^2 \\
& d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\
&- (\int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&+ \sum_{i \in I} \int_{\mathbf{R}^{2n}} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\}^2 d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n)) \\
&- (\int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \beta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&+ \sum_{i \in I} \int_{\mathbf{R}^{2n}} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \beta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\}^2 d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n)) \\
&- 2N\{(\theta_i)_{i \in I}\}N\{(\beta_i)_{i \in I}\} \\
&= 2(\int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)] \\
&\times [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \beta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)] d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)) \\
&+ 2(\sum_{i \in I} \int_{\mathbf{R}^{2n}} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} \beta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} \\
& d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n)) \\
&- 2N\{(\theta_i)_{i \in I}\}N\{(\beta_i)_{i \in I}\}
\end{aligned}$$

By the Cauchy and Schwarz inequality, we have,

$$\begin{aligned}
& \int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)] \\
&\times [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \beta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)] d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&\leq \{ \int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&\times \int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\sum_{i \in I} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \\
&\times \beta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \}^{\frac{1}{2}}
\end{aligned}$$

and for any $i \in \mathbf{I}$,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} \beta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} \\ & d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & \leq \left[\int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} \right. \\ & d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & \times \left. \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \beta_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} \right. \\ & d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \left. \right]^{\frac{1}{2}} \end{aligned}$$

Since for any non negative real numbers $(a_m)_{m \geq 0}$ and $(b_m)_{m \geq 0}$, we have by the Cauchy and Schwarz inequality, for any $q \in \mathbf{N}$,

$$\left\{ \sum_{1 \leq m \leq q} \sqrt{a_m b_m} \right\}^2 \leq \left\{ \sum_{1 \leq m \leq q} a_m \right\} \left\{ \sum_{1 \leq m \leq q} b_m \right\}$$

we have so,

$$\begin{aligned} & \left(\int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} \left[\sum_{i \in \mathbf{I}} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \right. \right. \\ & \times \theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \left. \right]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\ & \times \int_{\mathbf{R}^n} \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} \left[\sum_{i \in \mathbf{I}} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \right. \\ & \times \beta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \left. \right]^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \left. \right)^{\frac{1}{2}} \\ & + \sum_{i \in \mathbf{I}} \left[\int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} \right. \\ & d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & \times \left. \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \beta_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} \right. \\ & d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \left. \right]^{\frac{1}{2}} \\ & \leq [N\{(\theta_i)_{i \in \mathbf{I}}\} N\{(\beta_i)_{i \in \mathbf{I}}\}]^2 \end{aligned}$$

Therefore,

$$\begin{aligned} & N\{(\theta_i)_{i \in \mathbf{I}} + (\beta_i)_{i \in \mathbf{I}}\}^2 - [N\{(\theta_i)_{i \in \mathbf{I}}\} + N\{(\beta_i)_{i \in \mathbf{I}}\}]^2 \leq 0 \\ & N\{(\theta_i)_{i \in \mathbf{I}} + (\beta_i)_{i \in \mathbf{I}}\} \leq N\{(\theta_i)_{i \in \mathbf{I}}\} + N\{(\beta_i)_{i \in \mathbf{I}}\} \end{aligned}$$

This concludes the proof.

Proof of Proposition 3.4

Let $(\theta_i)_{i \in \mathbf{I}}, (H_i)_{i \in \mathbf{I}} \in \Theta$, we have for any $(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}^{2n}$,

$$\begin{aligned} & U_{C_1, \dots, C_n, \gamma, (\theta_i + H_i)_{i \in \mathbf{I}}}^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n), 1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n) \} \\ & - U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in \mathbf{I}}}^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n), 1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n) \} \\ & = \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \{ \mu^2(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n) \} \\ & + \sum_{i \in \mathbf{I}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [(\theta_i + H_i)^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \}] \\ & = \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \{ \mu^2(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n) \} \\ & + \sum_{i \in \mathbf{I}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [(2\theta_i + H_i) \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \}] \end{aligned}$$

where for any $(y_1, \dots, y_n) \in \mathbf{R}^n$,

$$\begin{aligned} & \mu(y_1, \dots, y_n) \\ & = \frac{1}{S_{(C_1, \dots, C_n)}(y_1, \dots, y_n)} [\gamma(y_1, \dots, y_n) \\ & - \sum_{i \in \mathbf{I}} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} (\theta_i + H_i) \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n)] \end{aligned}$$

We have,

$$\begin{aligned}
& S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu^2(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n) \} \\
& - \int_{\mathbf{R}^n} (-2) \theta_{(1, \dots, 1)}(y_1, \dots, y_n) [\sum_{i \in I} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \} H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \}] \\
& d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\
& = S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu^2(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n) \} \\
& - 2 \theta_{(1, \dots, 1)}(y_1, \dots, y_n) S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu_{(1, \dots, 1)}(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \} \\
& = S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu_{(1, \dots, 1)}(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \} \\
& \times \{ \mu_{(1, \dots, 1)}(y_1, \dots, y_n) + \theta_{(1, \dots, 1)}(y_1, \dots, y_n) - 2 \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \} \\
& = S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu_{(1, \dots, 1)}(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \}^2
\end{aligned}$$

As the function $(y_1, \dots, y_n) \mapsto S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu^2(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n) \}$ from \mathbf{R}^n to \mathbf{R} is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$ and the function

$(y_1, \dots, y_n) \mapsto S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu_{(1, \dots, 1)}(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \}^2$ from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n)}$, it follows that the function

$(y_1, \dots, y_n, c_1, \dots, c_n) \mapsto \theta_{(1, \dots, 1)}(y_1, \dots, y_n) [\sum_{i \in I} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \} H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \}]$ is integrable with respect to the probability measure $\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}$, we have,

$$\begin{aligned}
& \int_{\mathbf{R}^{2n}} [\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \} \{ \mu^2(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}^2(y_1, \dots, y_n) \}] d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\
& - \int_{\mathbf{R}^{2n}} [(-2) \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \{ \sum_{i \in I} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \} H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} \}] \\
& d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\
& = \int_{\mathbf{R}^n} S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu_{(1, \dots, 1)}(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \}^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)
\end{aligned}$$

Let,

$$\begin{aligned}
& L_{(\theta_i)_{i \in I}} \{ (H_i)_{i \in I} \} \\
& = 2 \{ \sum_{i \in I} \int_{\mathbf{R}^{2n}} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \} \\
& \times \theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \} \\
& - 2 \int_{\mathbf{R}^{2n}} \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \sum_{i \in I} \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \\
& \times H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n)
\end{aligned}$$

We have,

$$\begin{aligned}
& \int_{\mathbf{R}^{2n}} U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n), 1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n) \} \\
& d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\
& - \int_{\mathbf{R}^{2n}} U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n), 1_{V_1}(y_1, c_1), \dots, 1_{V_1}(y_n, c_n) \} \\
& d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\
& - L_{(\theta_i)_{i \in I}} \{ (H_i)_{i \in I} \} \\
& = \int_{\mathbf{R}^{2n}} [\sum_{i \in I} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \} H_i^2 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \}] d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\
& + \int_{\mathbf{R}^n} S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \{ \mu_{(1, \dots, 1)}(y_1, \dots, y_n) - \theta_{(1, \dots, 1)}(y_1, \dots, y_n) \}^2 d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
& = [N \{ (H_i)_{i \in I} \}]^2
\end{aligned}$$

The application $(H_i)_{i \in I} \mapsto L_{(\theta_i)_{i \in I}} \{ (H_i)_{i \in I} \}$ from Θ to \mathbf{R} is linear. So that the application χ from Θ to $\mathbf{R}_{\geq 0}$ is differentiable over Θ , and for any $(\theta_i)_{i \in I} \in \Theta$, we have,

$$d_{(\theta_i)_{i \in I}} \chi = L_{(\theta_i)_{i \in I}}$$

Proof of Proposition 3.5

Let $t \in [0, 1]$, $(\theta_i)_{i \in I}, (\beta_i)_{i \in I} \in \Theta$ and,

$$(\mu_i)_{i \in I} = t\{(\theta_i)_{i \in I}\} + (1-t)\{(\beta_i)_{i \in I}\}$$

For any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\mu(t_1, \dots, t_n) = t\theta_{(1, \dots, 1)}(t_1, \dots, t_n) + (1-t)\beta_{(1, \dots, 1)}(t_1, \dots, t_n)$$

Therefore,

$$\begin{aligned} & U_{C_1, \dots, C_n, \gamma, (\mu_i)_{i \in I}}^2 \\ &= tU_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2 + (1-t)U_{C_1, \dots, C_n, \gamma, (\beta_i)_{i \in I}}^2 \\ &\quad - t(1-t)\{U_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}} - U_{C_1, \dots, C_n, \gamma, (\beta_i)_{i \in I}}\}^2 \\ &= tU_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2 + (1-t)U_{C_1, \dots, C_n, \gamma, (\beta_i)_{i \in I}}^2 - t(1-t)U_{C_1, \dots, C_n, 0, (\theta_i - \beta_i)_{i \in I}}^2 \end{aligned}$$

Consequently,

$$\chi[t\{(\theta_i)_{i \in I}\} + (1-t)\{(\beta_i)_{i \in I}\}] = t\chi\{(\theta_i)_{i \in I}\} + (1-t)\chi\{(\beta_i)_{i \in I}\} - t(1-t)[N\{(\theta_i - \beta_i)_{i \in I}\}]^2$$

This concludes the proof.

Proof of Proposition 3.6

Let $i \in I$ and $\eta : \mathbf{R}^n \rightarrow \mathbf{R}$ be the function such that for any $(u_1, \dots, u_n) \in \mathbf{R}^n$, we have,

$$\begin{aligned} & \eta(u_1, \dots, u_n) \\ &= \int_{\mathbf{R}^{2n}} \{\prod_{1 \leq j \leq n} 1_{V_j}(y_j, c_j)\} [\theta_i \{\min(y_1, c_1), \dots, \min(y_n, c_n)\} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times \prod_{1 \leq j \leq n} 1_{]-\infty, u_j]} \{\min(y_j, c_j)\} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \end{aligned}$$

We have,

$$\begin{aligned} & \eta(u_1, \dots, u_n) \\ &= \int_{\mathbf{R}^{2n}} \{\prod_{1 \leq j \leq n} 1_{V_j}(y_j, c_j)\} [\theta_i (y_1^{i_1} c_1^{1-i_1}, \dots, y_n^{i_n} c_n^{1-i_n}) - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times \prod_{1 \leq j \leq n} 1_{]-\infty, u_j]} (y_j^{i_j} c_j^{1-i_j}) d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & \quad \frac{\partial \eta}{\partial u_1 \dots \partial u_n}(u_1, \dots, u_n) \\ &= \{\prod_{1 \leq j \leq n} f_{Y_j}^{i_j}(u_j) f_{C_j}^{1-i_j}(u_j)\} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_j}(y_j^{1-i_j} u_j^{i_j}, c_j^{i_j} u_j^{1-i_j})\} \\ & \quad \times [\theta_i(u_1, \dots, u_n) - \theta_{(1, \dots, 1)}(y_1^{1-i_1} u_1^{i_1}, \dots, y_n^{1-i_n} u_n^{i_n})] \\ & \quad \prod_{1 \leq j \leq n} \{f_{Y_j}^{1-i_j}(y_j) f_{C_j}^{i_j}(c_j)\} d\lambda_n(y_1^{1-i_1} c_1^{i_1}, \dots, y_n^{1-i_n} c_n^{i_n}) \end{aligned}$$

We have,

$$\begin{aligned}
& \int_{\mathbf{R}^n} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j^{1-i_j} u_j^{i_j}, c_j^{i_j} u_j^{1-i_j}) \} \times [\theta_i(u_1, \dots, u_n) - \theta_{(1, \dots, 1)}(y_1^{1-i_1} u_1^{i_1}, \dots, y_n^{1-i_n} u_n^{i_n})] \\
& \prod_{1 \leq j \leq n} \{ f_{Y_j}^{1-i_j}(y_j) f_{C_j}^{i_j}(c_j) \} d\lambda_n(y_1^{1-i_1} c_1^{i_1}, \dots, y_n^{1-i_n} c_n^{i_n}) \\
& = \theta_i(u_1, \dots, u_n) \int_{\mathbf{R}^n} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j^{1-i_j} u_j^{i_j}, c_j^{i_j} u_j^{1-i_j}) \} \\
& \prod_{1 \leq j \leq n} \{ f_{Y_j}^{1-i_j}(y_j) f_{C_j}^{i_j}(c_j) \} d\lambda_n(y_1^{1-i_1} c_1^{i_1}, \dots, y_n^{1-i_n} c_n^{i_n}) \\
& - \int_{\mathbf{R}^n} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j^{1-i_j} u_j^{i_j}, c_j^{i_j} u_j^{1-i_j}) \} \theta_{(1, \dots, 1)}(y_1^{1-i_1} u_1^{i_1}, \dots, y_n^{1-i_n} u_n^{i_n}) \\
& \prod_{1 \leq j \leq n} \{ f_{Y_j}^{1-i_j}(y_j) f_{C_j}^{i_j}(c_j) \} d\lambda_n(y_1^{1-i_1} c_1^{i_1}, \dots, y_n^{1-i_n} c_n^{i_n}) \\
& = \theta_i(u_1, \dots, u_n) \prod_{1 \leq j \leq n} \{ S_{C_j}(u_j)^{i_j} S_{Y_j}(u_j)^{1-i_j} \} \\
& - \prod_{1 \leq j \leq n} \{ S_{C_j}(u_j)^{i_j} \} \int_{\mathbf{R}^n} \{ \prod_{1 \leq j \leq n, i_j=0} 1_{V_0}(y_j, u_j) \} \theta_{(1, \dots, 1)}(y_1^{1-i_1} u_1^{i_1}, \dots, y_n^{1-i_n} u_n^{i_n}) \\
& \prod_{1 \leq j \leq n, i_j=0} \{ f_{Y_j}(y_j) \} d\lambda_{n-\sum_{1 \leq j \leq n} i_j}(y_1, \dots, y_n) \\
& = \prod_{1 \leq j \leq n} \{ S_{C_j}(u_j)^{i_j} \} [\theta_i(u_1, \dots, u_n) \prod_{1 \leq j \leq n, i_j=0} \{ S_{Y_j}(u_j) \} \\
& - \int_{\mathbf{R}^n} \{ \prod_{1 \leq j \leq n, i_j=0} 1_{V_0}(y_j, u_j) \} \theta_{(1, \dots, 1)}(y_1^{1-i_1} u_1^{i_1}, \dots, y_n^{1-i_n} u_n^{i_n}) \\
& \prod_{1 \leq j \leq n, i_j=0} \{ f_{Y_j}(y_j) \} d\lambda_{n-\sum_{1 \leq j \leq n} i_j}(y_1, \dots, y_n)]
\end{aligned}$$

As we have,

$$\begin{aligned}
& \theta_i(u_1, \dots, u_n) \\
& = \frac{1}{S_{(Y_1, \dots, Y_n)}(u_1, \dots, u_n)} \int_{\mathbf{R}^n} \{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, u_j) \} \theta_{(1, \dots, 1)}(u_1^{i_1} y_1^{1-i_1}, \dots, u_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
& = \frac{\prod_{1 \leq j \leq n, i_j=1} S_{Y_j}(u_j)}{S_{(Y_1, \dots, Y_n)}(u_1, \dots, u_n)} \int_{\mathbf{R}^n} \{ \prod_{1 \leq j \leq n, i_j=0} 1_{V_0}(y_j, u_j) \} \theta_{(1, \dots, 1)}(u_1^{i_1} y_1^{1-i_1}, \dots, u_n^{i_n} y_n^{1-i_n}) \\
& \prod_{1 \leq j \leq n, i_j=0} f_{Y_j}(y_j) d\lambda_{n-\sum_{1 \leq j \leq n} i_j}(y_1, \dots, y_n) \\
& = \frac{1}{\prod_{1 \leq j \leq n, i_j=0} S_{Y_j}(u_j)} \int_{\mathbf{R}^n} \{ \prod_{1 \leq j \leq n, i_j=0} 1_{V_0}(y_j, u_j) \} \theta_{(1, \dots, 1)}(u_1^{i_1} y_1^{1-i_1}, \dots, u_n^{i_n} y_n^{1-i_n}) \\
& \prod_{1 \leq j \leq n, i_j=0} f_{Y_j}(y_j) d\lambda_{n-\sum_{1 \leq j \leq n} i_j}(y_1, \dots, y_n)
\end{aligned}$$

we obtain,

$$\frac{\partial \eta}{\partial u_1 \dots \partial u_n}(u_1, \dots, u_n) = 0$$

Therefore, there exists a function $k_1 : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ such that for any $(u_2, \dots, u_n) \in \mathbf{R}^{n-1}$, we have,

$$\frac{\partial \eta}{\partial u_2 \dots \partial u_n}(u_1, \dots, u_n) = k_1(u_2, \dots, u_n)$$

Also,

$$\begin{aligned}
& \frac{\partial \eta}{\partial u_2 \dots \partial u_n}(u_1, \dots, u_n) \\
& = \{ \prod_{2 \leq j \leq n} f_{Y_j}^{i_j}(u_j) f_{C_j}^{1-i_j}(u_j) \} \int_{\mathbf{R}} 1_{]-\infty, u_1]}(y_1^{i_1} c_1^{1-i_1}) [\int_{\mathbf{R}^n} 1_{V_{i_1}}(y_1, c_1) \{ \prod_{2 \leq j \leq n} 1_{V_{i_j}}(y_j^{1-i_j} u_j^{i_j}, c_j^{i_j} u_j^{1-i_j}) \} \\
& \times [\theta_i(y_1^{i_1} c_1^{1-i_1}, u_2, \dots, u_n) - \theta_{(1, \dots, 1)}(y_1, y_2^{1-i_2} u_2^{i_2}, \dots, y_n^{1-i_n} u_n^{i_n})] \\
& \prod_{1 \leq j \leq n} \{ f_{Y_j}^{1-i_j}(y_j) f_{C_j}^{i_j}(c_j) \} d\lambda_n(y_1^{1-i_1} c_1^{i_1}, \dots, y_n^{1-i_n} c_n^{i_n})] f_{Y_1}^{i_1}(y_1) f_{C_1}^{1-i_1}(c_1) d\lambda(y_1^{i_1} c_1^{1-i_1})
\end{aligned}$$

When the real number u_1 tends to minus infinity, we obtain,

$$k_1 = 0$$

So that for any $(u_1, \dots, u_n) \in \mathbf{R}^n$, we have,

$$\frac{\partial \eta}{\partial u_2 \dots \partial u_n}(u_1, \dots, u_n) = 0$$

Therefore, there exists a function $k_2 : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ such that for any $(u_1, u_3, \dots, u_n) \in \mathbf{R}^{n-1}$, we have,

$$\frac{\partial \eta}{\partial u_3 \cdots \partial u_n}(u_1, \dots, u_n) = k_2(u_1, u_3, \dots, u_n)$$

When the real number u_2 tends to minus infinity, we obtain,

$$k_2 = 0$$

So that for any $(u_1, \dots, u_n) \in \mathbf{R}^n$, we have,

$$\frac{\partial \eta}{\partial u_3 \cdots \partial u_n}(u_1, \dots, u_n) = 0$$

Finally, by continuing this reasoning, we obtain,

$$\frac{\partial \eta}{\partial u_n}(u_1, \dots, u_n) = 0$$

In the same way, for any integer j such that $1 \leq j \leq n-1$, we have,

$$\frac{\partial \eta}{\partial u_j}(u_1, \dots, u_n) = 0$$

Hence, there exists a real number k such that for any $(u_1, \dots, u_n) \in \mathbf{R}^n$, we have,

$$\eta(u_1, \dots, u_n) = k$$

When the real number u_1 tends to minus infinity, we obtain,

$$k = 0$$

Thus,

$$\eta = 0$$

For any $(u_1, \dots, u_n) \in \mathbf{R}^n$, we have,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times \prod_{1 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

Let u_2, \dots, u_n be real numbers, we denote by G_0 the set of all elements $] -\infty, u]$ from $B(\mathbf{R})$, when the real number u is in \mathbf{R} , and by G the set of all elements B from $B(\mathbf{R})$ such that,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_B \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

We have,

$$\mathbf{R} \in G$$

Let $B_1, B_2 \in G$ such that $B_1 \subset B_2$ and,

$$B = B_2 \cap (\mathbf{R} - B_1)$$

We have,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_{B_1} \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_{B_2} \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

Also,

$$1_B = 1_{B_2} - 1_{B_1}$$

So that,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_B \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

$$B \in G$$

Let $(B_m)_{m \geq 0}$ be a sequence of elements from G such that for any $m \in \mathbf{N}$, we have,

$$B_m \subset B_{m+1}$$

Let,

$$B = \cup_{m \geq 0} B_m$$

We have for any $m \in \mathbf{N}$,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_{B_m} \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

For any real number x , we have,

$$\lim_{m \rightarrow +\infty} 1_{B_m}(x) = 1_B(x)$$

Moreover, for any $m \in \mathbf{N}$, we have for any $(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}^{2n}$,

$$\begin{aligned} & \left| \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \right. \\ & \times 1_{B_m} \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} \left. \right| \\ & \leq \left| \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \right. \\ & \times \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} \left. \right| \end{aligned}$$

Also,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left| \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \right. \\ & \times \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} \left. \right| d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \in \mathbf{R}_{\geq 0} \end{aligned}$$

The dominated convergence theorem gives,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_B \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

Thus,

$$B \in G$$

Consequently, the subset G from $\mathcal{P}(\mathbf{R})$ is a λ system [1].

For any real number u_1 , we have,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_{]-\infty, u_1]} \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

$$G_0 \subset G$$

The subset G_0 from $\mathcal{P}(\mathbf{R})$ is a π system [1], so that by the monotone class theorem [1], we have,

$$\sigma(G_0) \subset G$$

As $\sigma(G_0) = B(\mathbf{R})$, we have,

$$G = B(\mathbf{R})$$

For any real numbers u_2, \dots, u_n and any $B \in B(\mathbf{R})$, we have therefore,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_B \{ \min(y_1, c_1) \} \prod_{2 \leq j \leq n} 1_{]-\infty, u_j]} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

By a similar reasoning, we have that for any $(B_1, \dots, B_n) \in B(\mathbf{R})^n$,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times \prod_{1 \leq j \leq n} 1_{B_j} \{ \min(y_j, c_j) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

Hence, by a similar reasoning, we have that for any $B \in B(\mathbf{R})^{\otimes n}$,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times 1_B \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

Given that any borelian non negative function from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$, is a non decreasing limit of a sequence from step functions from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$, and decomposing a function following its non positive part and its non negative part, it follows by the dominated convergence theorem, as in the Proposition 2.6, that for any $(H_i)_{i \in \mathbf{I}} \in \Theta$ and any $i \in \mathbf{I}$, we have,

$$\begin{aligned} & \int_{\mathbf{R}^{2n}} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} [\theta_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} - \theta_{(1, \dots, 1)}(y_1, \dots, y_n)] \\ & \times H_i \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n, C_1, \dots, C_n)}(y_1, \dots, y_n, c_1, \dots, c_n) \\ & = 0 \end{aligned}$$

Therefore, for any $(H_i)_{i \in \mathbf{I}} \in \Theta$,

$$\begin{aligned} d_{(\theta_i)_{i \in \mathbf{I}}} \chi \{ (H_i)_{i \in \mathbf{I}} \} & = 0 \\ d_{(\theta_i)_{i \in \mathbf{I}}} \chi & = 0 \end{aligned}$$

This concludes the proof.

Proof of Proposition 3.7

Let $(\theta_i)_{i \in I} \in \Theta$, we have,

$$\begin{aligned}
& \mathbb{E}[\{\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n) - \gamma(Y_1, \dots, Y_n)\}^2] + \mathbb{V}[\gamma(Y_1, \dots, Y_n)] \\
&= \mathbb{E}[\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2(T_1, \dots, T_n, D_1, \dots, D_n)] + \mathbb{E}[\gamma^2(Y_1, \dots, Y_n)] \\
&\quad - 2\mathbb{E}[\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n)\gamma(Y_1, \dots, Y_n)] + \mathbb{E}[\gamma^2(Y_1, \dots, Y_n)] - \mathbb{E}^2[\gamma(Y_1, \dots, Y_n)] \\
&= \mathbb{E}[\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2(T_1, \dots, T_n, D_1, \dots, D_n)] + \mathbb{E}[\gamma^2(Y_1, \dots, Y_n)] \\
&\quad - 2\mathbb{E}[\gamma^2(Y_1, \dots, Y_n)] + \mathbb{E}[\gamma^2(Y_1, \dots, Y_n)] - \mathbb{E}^2[\gamma(Y_1, \dots, Y_n)] \\
&= \mathbb{E}[\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}^2(T_1, \dots, T_n, D_1, \dots, D_n)] - \mathbb{E}^2[\gamma(Y_1, \dots, Y_n)] \\
&= \mathbb{V}[\mathbb{U}_{C_1, \dots, C_n, \gamma, (\theta_i)_{i \in I}}(T_1, \dots, T_n, D_1, \dots, D_n)]
\end{aligned}$$

Proof of Proposition 3.8

In this case, we have for any $i \in \{0, 1\}^n$ and any $(t_1, \dots, t_n) \in \mathbf{R}^n$,

$$\begin{aligned}
& \alpha_i^{(\gamma)}(t_1, \dots, t_n) \\
&= \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j)\} \epsilon(t_{\tau(1)}^{i_{\tau(1)}} y_{\tau(1)}^{1-i_{\tau(1)}}, \dots, t_{\tau(q)}^{i_{\tau(q)}} y_{\tau(q)}^{1-i_{\tau(q)}}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&= \frac{1}{S_{(Y_{\tau(1)}, \dots, Y_{\tau(q)})}(t_{\tau(1)}, \dots, t_{\tau(q)})} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq q} 1_{V_0}(y_j, t_{\tau(j)})\} \epsilon(t_{\tau(1)}^{i_{\tau(1)}} y_1^{1-i_{\tau(1)}}, \dots, t_{\tau(q)}^{i_{\tau(q)}} y_q^{1-i_{\tau(q)}}) \\
&\quad d\mathbb{P}_{(Y_{\tau(1)}, \dots, Y_{\tau(q)})}(y_1, \dots, y_q) \\
&= \alpha_{i_{\tau}}^{(\epsilon)}(t_{\tau(1)}, \dots, t_{\tau(q)})
\end{aligned}$$

where,

$$i_{\tau} = (i_{\tau(1)}, \dots, i_{\tau(q)}) \in \{0, 1\}^q$$

We have,

$$\begin{aligned}
& \mu^{(\gamma)}(t_1, \dots, t_n) \\
&= \sum_{i \in \{0,1\}^n} \frac{(-1)^{n-\sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1-i_j}] \frac{\alpha_i^{(\gamma)}(t_1^{i_1} u_1^{1-i_1}, \dots, t_n^{i_n} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}} d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) \\
&= \sum_{i \in \{0,1\}^n} \frac{(-1)^{n-\sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1-i_j}] \frac{\alpha_{i_\tau}^{(\epsilon)}(t_{\tau(1)}^{i_{\tau(1)}} u_{\tau(1)}^{1-i_{\tau(1)}}, \dots, t_{\tau(q)}^{i_{\tau(q)}} u_{\tau(q)}^{1-i_{\tau(q)}})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}} \\
& \quad d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) \\
&= \sum_{i \in \{0,1\}^n} \frac{(-1)^{n-\sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \prod_{1 \leq j \leq n, j \notin \tau(\{1, \dots, q\})} \left\{ \frac{1}{S_{C_j}(t_j)} - 1 \right\}^{1-i_j} \\
& \quad \times \int_{\mathbf{R}^q} [\prod_{1 \leq j \leq q} \{1_{V_1}(u_j, t_{\tau(j)})\}^{1-i_{\tau(j)}}] \frac{\alpha_{i_\tau}^{(\epsilon)}(t_{\tau(1)}^{i_{\tau(1)}} u_1^{1-i_{\tau(1)}}, \dots, t_{\tau(q)}^{i_{\tau(q)}} u_q^{1-i_{\tau(q)}})}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}^2(u_j)\}^{1-i_{\tau(j)}}} d\mathbb{P}_{(C_{\tau(1)}, \dots, C_{\tau(q)})}(u_1, \dots, u_q) \\
&= \sum_{i \in \{0,1\}^n} \frac{(-1)^{q-\sum_{1 \leq j \leq q} i_{\tau(j)}}}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}(t_{\tau(j)})\}^{i_{\tau(j)}}} \frac{(-1)^{n-q-\sum_{1 \leq j \leq n, j \notin \tau(\{1, \dots, q\})} i_j}}{\prod_{1 \leq j \leq n, j \notin \tau(\{1, \dots, q\})} \{S_{C_j}(t_j)\}^{i_j}} \prod_{1 \leq j \leq n, j \notin \tau(\{1, \dots, q\})} \left\{ \frac{1}{S_{C_j}(t_j)} - 1 \right\}^{1-i_j} \\
& \quad \times \int_{\mathbf{R}^q} [\prod_{1 \leq j \leq q} \{1_{V_1}(u_j, t_{\tau(j)})\}^{1-i_{\tau(j)}}] \frac{\alpha_{i_\tau}^{(\epsilon)}(t_{\tau(1)}^{i_{\tau(1)}} u_1^{1-i_{\tau(1)}}, \dots, t_{\tau(q)}^{i_{\tau(q)}} u_q^{1-i_{\tau(q)}})}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}^2(u_j)\}^{1-i_{\tau(j)}}} d\mathbb{P}_{(C_{\tau(1)}, \dots, C_{\tau(q)})}(u_1, \dots, u_q) \\
&= \sum_{i \in \{0,1\}^{\{1, \dots, n\} - \tau(\{1, \dots, q\})}} \left[\frac{(-1)^{n-q-\sum_{1 \leq j \leq n, j \notin \tau(\{1, \dots, q\})} i_j}}{\prod_{1 \leq j \leq n, j \notin \tau(\{1, \dots, q\})} \{S_{C_j}(t_j)\}^{i_j}} \prod_{1 \leq j \leq n, j \notin \tau(\{1, \dots, q\})} \left\{ \frac{1}{S_{C_j}(t_j)} - 1 \right\}^{1-i_j} \right] \\
& \quad \times \sum_{i \in \{0,1\}^q} \frac{(-1)^{q-\sum_{1 \leq j \leq q} i_j}}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}(t_{\tau(j)})\}^{i_j}} \int_{\mathbf{R}^q} [\prod_{1 \leq j \leq q} \{1_{V_1}(u_j, t_{\tau(j)})\}^{1-i_j}] \frac{\alpha_i^{(\epsilon)}(t_{\tau(1)}^{i_1} u_1^{1-i_1}, \dots, t_{\tau(q)}^{i_q} u_q^{1-i_q})}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}^2(u_j)\}^{1-i_j}} \\
& \quad d\mathbb{P}_{(C_{\tau(1)}, \dots, C_{\tau(q)})}(u_1, \dots, u_q) \\
&= \prod_{1 \leq j \leq n, j \notin \tau(\{1, \dots, q\})} \sum_{i \in \{0,1\}^i} \left[\frac{(-1)^{1-i}}{\{S_{C_j}(t_j)\}^i} \left\{ \frac{1}{S_{C_j}(t_j)} - 1 \right\}^{1-i} \right] \\
& \quad \times \sum_{i \in \{0,1\}^q} \frac{(-1)^{q-\sum_{1 \leq j \leq q} i_j}}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}(t_{\tau(j)})\}^{i_j}} \int_{\mathbf{R}^q} [\prod_{1 \leq j \leq q} \{1_{V_1}(u_j, t_{\tau(j)})\}^{1-i_j}] \frac{\alpha_i^{(\epsilon)}(t_{\tau(1)}^{i_1} u_1^{1-i_1}, \dots, t_{\tau(q)}^{i_q} u_q^{1-i_q})}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}^2(u_j)\}^{1-i_j}} \\
& \quad d\mathbb{P}_{(C_{\tau(1)}, \dots, C_{\tau(q)})}(u_1, \dots, u_q) \\
&= \sum_{i \in \{0,1\}^q} \frac{(-1)^{q-\sum_{1 \leq j \leq q} i_j}}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}(t_{\tau(j)})\}^{i_j}} \int_{\mathbf{R}^q} [\prod_{1 \leq j \leq q} \{1_{V_1}(u_j, t_{\tau(j)})\}^{1-i_j}] \frac{\alpha_i^{(\epsilon)}(t_{\tau(1)}^{i_1} u_1^{1-i_1}, \dots, t_{\tau(q)}^{i_q} u_q^{1-i_q})}{\prod_{1 \leq j \leq q} \{S_{C_{\tau(j)}}^2(u_j)\}^{1-i_j}} \\
& \quad d\mathbb{P}_{(C_{\tau(1)}, \dots, C_{\tau(q)})}(u_1, \dots, u_q) \\
&= \mu^{(\epsilon)}(t_{\tau(1)}, \dots, t_{\tau(q)})
\end{aligned}$$

$$\begin{aligned}
& \theta_i^{0(\gamma)}(t_1, \dots, t_n) \\
&= \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \mu^{(\gamma)}(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&= \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \mu^{(\epsilon)}(t_{\tau(1)}^{i_{\tau(1)}} y_{\tau(1)}^{1-i_{\tau(1)}}, \dots, t_{\tau(q)}^{i_{\tau(q)}} y_{\tau(q)}^{1-i_{\tau(q)}}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\
&= \frac{1}{S_{(Y_{\tau(1)}, \dots, Y_{\tau(q)})}(t_{\tau(1)}, \dots, t_{\tau(q)})} \\
& \quad \times \int_{\mathbf{R}^q} \left\{ \prod_{1 \leq j \leq q} 1_{V_0}(y_j, t_{\tau(j)}) \right\} \mu^{(\epsilon)}(t_{\tau(1)}^{i_{\tau(1)}} y_1^{1-i_{\tau(1)}}, \dots, t_{\tau(q)}^{i_{\tau(q)}} y_q^{1-i_{\tau(q)}}) d\mathbb{P}_{(Y_{\tau(1)}, \dots, Y_{\tau(q)})}(y_1, \dots, y_q) \\
&= \theta_{i_\tau}^{0(\epsilon)}(t_{\tau(1)}, \dots, t_{\tau(q)})
\end{aligned}$$

We have for any $(t_1, \dots, t_n) \in \mathbf{R}^n$ and any $(d_1, \dots, d_n) \in \{0, 1\}^n$,

$$\begin{aligned} & U_{C_1, \dots, C_n, \gamma, (\theta_i^0)_{i \in I}}(t_1, \dots, t_n, d_1, \dots, d_n) \\ &= \sum_{i \in \{0, 1\}^n} [\prod_{1 \leq j \leq n} \{d_j^{i_j} (1 - d_j)^{1 - i_j}\}] \theta_i^0(\gamma)(t_1, \dots, t_n) \\ &= \sum_{i \in \{0, 1\}^n} [\prod_{1 \leq j \leq n} \{d_j^{i_j} (1 - d_j)^{1 - i_j}\}] \theta_{i_\tau}^0(\epsilon)(t_{\tau(1)}, \dots, t_{\tau(q)}) \\ &= \sum_{i \in \{0, 1\}^q} [\prod_{1 \leq j \leq q} \{d_{\tau(j)}^{i_j} (1 - d_{\tau(j)})^{1 - i_j}\}] \theta_i^0(\epsilon)(t_{\tau(1)}, \dots, t_{\tau(q)}) \\ &= U_{C_{\tau(1)}, \dots, C_{\tau(q)}, \epsilon, (\theta_i^0)_{i \in I}}(t_{\tau(1)}, \dots, t_{\tau(q)}, d_{\tau(1)}, \dots, d_{\tau(q)}) \end{aligned}$$

This concludes the proof.

LEMMA B.1. *Let $B_1, \dots, B_n \in B(\mathbf{R})$ and $\gamma : \mathbf{R}^n \rightarrow \mathbf{R}$ be the function such that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,*

$$\gamma(t_1, \dots, t_n) = \prod_{1 \leq j \leq n} 1_{B_j}(t_j)$$

Then, the Assumption 3.5 is verified and under the Assumptions 3.4 and 3.7, for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\theta_{(1, \dots, 1)}^0(t_1, \dots, t_n) = \mu(t_1, \dots, t_n)$$

Proof of Lemma B.1

For any $i \in I$, we have so,

$$\alpha_i(t_1, \dots, t_n) = \prod_{1 \leq j \leq n} \left\{ \frac{1}{S_{Y_j}(t_j)} \int_{\mathbf{R}} 1_{V_0}(y, t_j) 1_{B_j}(t_j^{i_j} y^{1 - i_j}) d\mathbb{P}_{Y_j}(y) \right\}$$

Therefore,

$$\begin{aligned} & \mu(t_1, \dots, t_n) \\ &= \sum_{i \in \{0, 1\}^n} \frac{(-1)^{n - \sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1 - i_j}] \frac{\alpha_i(t_1^{i_1} u_1^{1 - i_1}, \dots, t_n^{i_n} u_n^{1 - i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1 - i_j}} d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) \\ &= \sum_{i \in \{0, 1\}^n} \frac{(-1)^{n - \sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \prod_{1 \leq j \leq n} \int_{\mathbf{R}} \{1_{V_1}(u, t_j)\}^{1 - i_j} \frac{\int_{\mathbf{R}} 1_{V_0}(y, t_j^{i_j} u^{1 - i_j}) 1_{B_j}(t_j^{i_j} y^{1 - i_j}) d\mathbb{P}_{Y_j}(y)}{S_{Y_j}(t_j^{i_j} u^{1 - i_j}) \prod_{1 \leq j \leq n} \{S_{C_j}^2(u)\}^{1 - i_j}} d\mathbb{P}_{C_j}(u) \\ &= \prod_{1 \leq j \leq n} \sum_{i \in \{0, 1\}} \frac{(-1)^{1 - i}}{\{S_{C_j}(t_j)\}^i} \int_{\mathbf{R}} \{1_{V_1}(u, t_j)\}^{1 - i} \frac{\int_{\mathbf{R}} 1_{V_0}(y, t_j^i u^{1 - i}) 1_{B_j}(t_j^i y^{1 - i}) d\mathbb{P}_{Y_j}(y)}{S_{Y_j}(t_j^i u^{1 - i}) \{S_{C_j}^2(u)\}^{1 - i}} d\mathbb{P}_{C_j}(u) \\ &= \prod_{1 \leq j \leq n} \left[\frac{1_{B_j}(t_j)}{S_{C_j}(t_j)} - \int_{\mathbf{R}} 1_{V_1}(u, t_j) \frac{\int_{\mathbf{R}} 1_{V_0}(y, u) 1_{B_j}(y) d\mathbb{P}_{Y_j}(y)}{S_{Y_j}(u) S_{C_j}^2(u)} d\mathbb{P}_{C_j}(u) \right] \\ &= \prod_{1 \leq j \leq n} \frac{\phi_j(t_j)}{S_{C_j}(t_j)} \end{aligned}$$

where for any integer j such that $1 \leq j \leq n$, we have for any real number t ,

$$\phi_j(t) = 1_{B_j}(t) - S_{C_j}(t) \int_{\mathbf{R}} 1_{V_1}(u, t) \frac{\int_{\mathbf{R}} 1_{V_0}(y, u) 1_{B_j}(y) d\mathbb{P}_{Y_j}(y)}{S_{Y_j}(u) S_{C_j}^2(u)} d\mathbb{P}_{C_j}(u)$$

Therefore, for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\theta_i^0(t_1, \dots, t_n) = \prod_{1 \leq j \leq n} \left\{ \frac{1}{S_{Y_j}(t_j)} \int_{\mathbf{R}} 1_{V_0}(y, t_j) \frac{\phi_j(t_j^{i_j} y^{1 - i_j})}{S_{C_j}(t_j^{i_j} y^{1 - i_j})} d\mathbb{P}_{Y_j}(y) \right\}$$

and for any $(y_1, \dots, y_n) \in \mathbf{R}^n$, we have,

$$\begin{aligned}
& \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_1}(y_j, c_j)\} \mu\{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\
& + \sum_{i \in \{0,1\}^n - \{(1, \dots, 1)\}} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i^0\{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\
& = \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_1}(y_j, c_j)\} \mu\{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\
& + \sum_{i \in \{0,1\}^n - \{(1, \dots, 1)\}} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i^0\{y_1^{i_1} c_1^{1-i_1}, \dots, y_n^{i_n} c_n^{1-i_n}\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\
& = \sum_{i \in \{0,1\}^n} \int_{\mathbf{R}^n} \prod_{1 \leq j \leq n} \left\{ \frac{1_{V_{i_j}}(y_j, c_j)}{S_{Y_j}(y_j^{i_j} c_j^{1-i_j})} \int_{\mathbf{R}} 1_{V_0}(y, y_j^{i_j} c_j^{1-i_j}) \frac{\phi_j\{(y_j^{i_j} c_j^{1-i_j})^{i_j} y^{1-i_j}\}}{S_{C_j}\{(y_j^{i_j} c_j^{1-i_j})^{i_j} y^{1-i_j}\}} d\mathbb{P}_{Y_j}(y) \right\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\
& = \sum_{i \in \{0,1\}^n} \prod_{1 \leq j \leq n} \int_{\mathbf{R}} \left\{ \frac{1_{V_{i_j}}(y_j, c)}{S_{Y_j}(y_j^{i_j} c^{1-i_j})} \int_{\mathbf{R}} 1_{V_0}(y, y_j^{i_j} c^{1-i_j}) \frac{\phi_j\{(y_j^{i_j} c^{1-i_j})^{i_j} y^{1-i_j}\}}{S_{C_j}\{(y_j^{i_j} c^{1-i_j})^{i_j} y^{1-i_j}\}} d\mathbb{P}_{Y_j}(y) \right\} d\mathbb{P}_{C_j}(c) \\
& = \prod_{1 \leq j \leq n} \sum_{i \in \{0,1\}} \int_{\mathbf{R}} \left\{ \frac{1_{V_i}(y_j, c)}{S_{Y_j}(y_j^i c^{1-i})} \int_{\mathbf{R}} 1_{V_0}(y, y_j^i c^{1-i}) \frac{\phi_j\{(y_j^i c^{1-i})^i y^{1-i}\}}{S_{C_j}\{(y_j^i c^{1-i})^i y^{1-i}\}} d\mathbb{P}_{Y_j}(y) \right\} d\mathbb{P}_{C_j}(c) \\
& = \prod_{1 \leq j \leq n} \left[\int_{\mathbf{R}} \left\{ \frac{1_{V_0}(y_j, c)}{S_{Y_j}(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_j(y)}{S_{C_j}(y)} d\mathbb{P}_{Y_j}(y) \right\} d\mathbb{P}_{C_j}(c) \right. \\
& \left. + \int_{\mathbf{R}} \left\{ \frac{1_{V_1}(y_j, c)}{S_{Y_j}(y_j)} \int_{\mathbf{R}} 1_{V_0}(y, y_j) \frac{\phi_j(y_j)}{S_{C_j}(y_j)} d\mathbb{P}_{Y_j}(y) \right\} d\mathbb{P}_{C_j}(c) \right] \\
& = \prod_{1 \leq j \leq n} \left[\int_{\mathbf{R}} \left\{ \frac{1_{V_0}(y_j, c)}{S_{Y_j}(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_j(y)}{S_{C_j}(y)} d\mathbb{P}_{Y_j}(y) \right\} d\mathbb{P}_{C_j}(c) + \phi_j(y_j) \right]
\end{aligned}$$

By the Lemma A.2, we have for any integer j such that $1 \leq j \leq n$,

$$1_{B_j}(y_j) = \int_{\mathbf{R}} \left\{ \frac{1_{V_0}(y_j, c)}{S_{Y_j}(c)} \int_{\mathbf{R}} 1_{V_0}(y, c) \frac{\phi_j(y)}{S_{C_j}(y)} d\mathbb{P}_{Y_j}(y) \right\} d\mathbb{P}_{C_j}(c) + \phi_j(y_j)$$

Hence,

$$\begin{aligned}
& \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_1}(y_j, c_j)\} \mu\{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\
& + \sum_{i \in \{0,1\}^n - \{(1, \dots, 1)\}} \int_{\mathbf{R}^n} \{\prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j)\} \theta_i^0\{\min(y_1, c_1), \dots, \min(y_n, c_n)\} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\
& = \gamma(y_1, \dots, y_n)
\end{aligned}$$

So that,

$$\theta_{(1, \dots, 1)}^0(y_1, \dots, y_n) = \mu(y_1, \dots, y_n)$$

LEMMA B.2. Let $B \in B(\mathbf{R})^{\otimes n}$ and $\gamma : \mathbf{R}^n \rightarrow \mathbf{R}$ be the function such that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\gamma(t_1, \dots, t_n) = 1_B(t_1, \dots, t_n)$$

Then, the Assumption 3.5 is verified and under the Assumptions 3.4 and 3.7, for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\theta_{(1, \dots, 1)}^0(t_1, \dots, t_n) = \mu(t_1, \dots, t_n)$$

Proof of Lemma B.2

We denote by G_0 the set of all elements $B_1 \times \dots \times B_n$ from $B(\mathbf{R})^{\otimes n}$, when the elements B_1, \dots, B_n are in $B(\mathbf{R})$, and by G the set of all elements B from $B(\mathbf{R})^{\otimes n}$ such that if for any $(t_1, \dots, t_n) \in \mathbf{R}^n$,

$$\gamma(t_1, \dots, t_n) = 1_B(t_1, \dots, t_n)$$

then we have,

$$\theta_{(1, \dots, 1)}^0(t_1, \dots, t_n) = \mu(t_1, \dots, t_n)$$

We have,

$$\mathbf{R}^n \in G$$

Let $B_1, B_2 \in G$ such that $B_1 \subset B_2$ and,

$$B = B_2 \cap (\mathbf{R} - B_1)$$

We have,

$$1_B = 1_{B_2} - 1_{B_1}$$

So that by the linearity of the integral,

$$B \in G$$

Let $(B_m)_{m \geq 0}$ be a sequence of elements from G such that for any $m \in \mathbf{N}$, we have,

$$B_m \subset B_{m+1}$$

Let,

$$B = \cup_{m \geq 0} B_m$$

For any non negative integer m and any $(x_1, \dots, x_n) \in \mathbf{R}^n$, we have,

$$\begin{aligned} 1_{B_m}(x_1, \dots, x_n) &\leq 1_B(x_1, \dots, x_n) \\ \lim_{m \rightarrow +\infty} 1_{B_m}(x_1, \dots, x_n) &= 1_B(x_1, \dots, x_n) \end{aligned}$$

The dominated convergence theorem gives,

$$B \in G$$

Consequently, the subset G from $\mathcal{P}(\mathbf{R}^n)$ is a λ system [1].

We have,

$$G_0 \subset G$$

The subset G_0 from $\mathcal{P}(\mathbf{R}^n)$ is a π system [1], so that by the monotone class theorem [1], we have,

$$\sigma(G_0) \subset G$$

As $\sigma(G_0) = B(\mathbf{R})^{\otimes n}$, we have,

$$G = B(\mathbf{R})^{\otimes n}$$

This concludes the proof.

LEMMA B.3. *Under the Assumptions 3.4, 3.5 and 3.7, we have then for any $(t_1, \dots, t_n) \in \mathbf{R}^n$,*

$$\theta_{(1, \dots, 1)}^0(t_1, \dots, t_n) = \mu(t_1, \dots, t_n)$$

Proof of Lemma B.3

There exists a non decreasing sequence $(w_m)_{m \geq 0}$ from non negative step functions from \mathbf{R}^n to $\mathbf{R}_{\geq 0}$ such that for any $(y_1, \dots, y_n) \in \mathbf{R}^n$, we have,

$$\gamma^+(y_1, \dots, y_n) = \lim_{m \rightarrow +\infty} w_m(y_1, \dots, y_n)$$

For any $(t_1, \dots, t_n) \in \mathbf{R}^n$, let for any non negative integer m ,

$$\begin{aligned} & \mu_m(t_1, \dots, t_n) \\ &= \sum_{i \in \{0,1\}^n} \frac{(-1)^{n-\sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1-i_j}] \frac{\alpha_{i,+m}(t_1^{i_1} u_1^{1-i_1}, \dots, t_n^{i_n} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}} d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) \end{aligned}$$

and,

$$\alpha_{i,+m}(t_1, \dots, t_n) = \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} w_m(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$$

$$\begin{aligned} & \mu_+(t_1, \dots, t_n) \\ &= \sum_{i \in \{0,1\}^n} \frac{(-1)^{n-\sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1-i_j}] \frac{\alpha_{i,+}(t_1^{i_1} u_1^{1-i_1}, \dots, t_n^{i_n} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}} d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) \end{aligned}$$

So that by the Lemma B.2, we have for any non negative integer m and any $(y_1, \dots, y_n) \in \mathbf{R}^n$,

$$\begin{aligned} & S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \mu_m(y_1, \dots, y_n) - w_m(y_1, \dots, y_n) \\ &+ \sum_{i \in I} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_{i,+m}^0 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\ &= 0 \end{aligned}$$

where for any $i \in I$ and any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\theta_{i,+m}^0(t_1, \dots, t_n) = \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \mu_m(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$$

By the Assumption 3.5, we have,

$$\begin{aligned} \mu_+(t_1, \dots, t_n) &= \lim_{m \rightarrow +\infty} \mu_m(t_1, \dots, t_n) \\ \theta_{i,+}^0(t_1, \dots, t_n) &= \lim_{m \rightarrow +\infty} \theta_{i,+m}^0(t_1, \dots, t_n) \end{aligned}$$

where for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\theta_{i,+}^0(t_1, \dots, t_n) = \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \mu_+(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$$

Also we have for any $(y_1, \dots, y_n) \in \mathbf{R}^n$,

$$\begin{aligned} & S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \mu_+(y_1, \dots, y_n) - \gamma^+(y_1, \dots, y_n) \\ &+ \sum_{i \in I} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_{i,+}^0 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\ &= 0 \end{aligned}$$

In the same way, we obtain,

$$\begin{aligned} & S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \mu_-(y_1, \dots, y_n) - \gamma^-(y_1, \dots, y_n) \\ &+ \sum_{i \in I} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_{i,-}^0 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\ &= 0 \end{aligned}$$

where for any $(t_1, \dots, t_n) \in \mathbf{R}^n$,

$$\begin{aligned} & \mu_{-}(t_1, \dots, t_n) \\ &= \sum_{i \in \{0,1\}^n} \frac{(-1)^{n-\sum_{1 \leq j \leq n} i_j}}{\prod_{1 \leq j \leq n} \{S_{C_j}(t_j)\}^{i_j}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n} \{1_{V_1}(u_j, t_j)\}^{1-i_j}] \frac{\alpha_{i,-}(t_1^{i_1} u_1^{1-i_1}, \dots, t_n^{i_n} u_n^{1-i_n})}{\prod_{1 \leq j \leq n} \{S_{C_j}^2(u_j)\}^{1-i_j}} d\mathbb{P}_{(C_1, \dots, C_n)}(u_1, \dots, u_n) \\ \theta_{i,-}^0(t_1, \dots, t_n) &= \frac{1}{S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n)} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_0}(y_j, t_j) \right\} \mu_{-}(t_1^{i_1} y_1^{1-i_1}, \dots, t_n^{i_n} y_n^{1-i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \end{aligned}$$

So that for any $(y_1, \dots, y_n) \in \mathbf{R}^n$,

$$\begin{aligned} & S_{(C_1, \dots, C_n)}(y_1, \dots, y_n) \mu(y_1, \dots, y_n) - \gamma(y_1, \dots, y_n) \\ &+ \sum_{i \in I} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{i_j}}(y_j, c_j) \right\} \theta_i^0 \{ \min(y_1, c_1), \dots, \min(y_n, c_n) \} d\mathbb{P}_{(C_1, \dots, C_n)}(c_1, \dots, c_n) \\ &= 0 \end{aligned}$$

Therefore for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\theta_{(1, \dots, 1)}^0(t_1, \dots, t_n) = \mu(t_1, \dots, t_n)$$

This concludes the proof.

Proof of Theorem 3.1

The proof results from the Lemma B.3, the application χ from Θ to $\mathbf{R}_{\geq 0}$ is strictly convex over Θ , differentiable over Θ , and the element $(\theta_i^0)_{i \in I}$ from Θ is a stationary point, so that it is the unique minimum from the application χ over Θ [12].

Proof of Theorem 3.2

Let for any $i \in I$,

$$\phi_i = \theta_{(1-i_1, \dots, 1-i_n)}^0$$

and for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, let,

$$\begin{aligned} & \psi(t_1, \dots, t_n) \\ &= S_{(Y_1, \dots, Y_n)}(t_1, \dots, t_n) \theta_{(0, \dots, 0)}^0(t_1, \dots, t_n) \\ &+ \sum_{i \in \{0,1\}^n - \{(1, \dots, 1)\}} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{1-i_j}}(y_j, t_j) \right\} \phi_i \{ \min(y_1, t_1), \dots, \min(y_n, t_n) \} d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \end{aligned}$$

We have for any integer k such that $1 \leq k \leq n$,

$$\begin{aligned} & \psi(t_1, \dots, t_n) \\ &= \sum_{i \in \{0,1\}^n} \int_{\mathbf{R}^n} \left\{ \prod_{1 \leq j \leq n} 1_{V_{1-i_j}}(y_j, t_j) \right\} \theta_{(1-i_1, \dots, 1-i_n)}^0(y_1^{1-i_1} t_1^{i_1}, \dots, y_n^{1-i_n} t_n^{i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\ &= \sum_{i \in \{0,1\}^n, i_k=0} \int_{\mathbf{R}^n} 1_{V_1}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\ &\times \theta_{(1-i_1, \dots, 1-i_n)}^0(y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, y_k, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \\ &+ \sum_{i \in \{0,1\}^n, i_k=1} \int_{\mathbf{R}^n} 1_{V_0}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\ &\times \theta_{(1-i_1, \dots, 1-i_n)}^0(y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, t_k, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n}) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) \end{aligned}$$

Let $i \in I$ such that $i_k = 0$, we have,

$$\begin{aligned}
& \int_{\mathbf{R}^n} 1_{V_1}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \theta_{(1-i_1, \dots, 1-i_n)}^0 (y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, y_k, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) d\lambda_n(y_1, \dots, y_n) \\
& = \int_{\mathbf{R}^{2n}} 1_{V_1}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \frac{1}{S_{(Y_1, \dots, Y_n)}(y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, y_k, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n})} \\
& \times 1_{V_0}(z_k, y_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_0}(z_j, y_j^{1-i_j} t_j^{i_j}) \right\} \\
& \times \mu(\{y_1^{1-i_1} t_1^{i_1}\}^{1-i_1} z_1^{i_1}, \dots, \{y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}\}^{1-i_{k-1}} z_{k-1}^{i_{k-1}}, y_k, \{y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}\}^{1-i_{k+1}} z_{k+1}^{i_{k+1}}, \dots, \{y_n^{1-i_n} t_n^{i_n}\}^{1-i_n} z_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) f_{(Y_1, \dots, Y_n)}(z_1, \dots, z_n) d\lambda_{2n}(y_1, \dots, y_n, z_1, \dots, z_n) \\
& = \int_{\mathbf{R}^{2n-1}} 1_{V_1}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \frac{1}{S_{(Y_1, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)}(y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n})} \\
& \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_0}(z_j, y_j^{1-i_j} t_j^{i_j}) \right\} \\
& \times \mu(\{y_1^{1-i_1} t_1^{i_1}\}^{1-i_1} z_1^{i_1}, \dots, \{y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}\}^{1-i_{k-1}} z_{k-1}^{i_{k-1}}, y_k, \{y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}\}^{1-i_{k+1}} z_{k+1}^{i_{k+1}}, \dots, \{y_n^{1-i_n} t_n^{i_n}\}^{1-i_n} z_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) f_{(Y_1, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) \\
& d\lambda_{2n-1}(y_1, \dots, y_n, z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) \\
& \\
& \frac{d}{dt_k} \int_{\mathbf{R}^n} 1_{V_1}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \theta_{(1-i_1, \dots, 1-i_n)}^0 (y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, y_k, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) d\lambda_n(y_1, \dots, y_n) \\
& = \int_{\mathbf{R}^{2n-2}} \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \frac{1}{S_{(Y_1, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)}(y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n})} \\
& \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_0}(z_j, y_j^{1-i_j} t_j^{i_j}) \right\} \\
& \times \mu(\{y_1^{1-i_1} t_1^{i_1}\}^{1-i_1} z_1^{i_1}, \dots, \{y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}\}^{1-i_{k-1}} z_{k-1}^{i_{k-1}}, t_k, \{y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}\}^{1-i_{k+1}} z_{k+1}^{i_{k+1}}, \dots, \{y_n^{1-i_n} t_n^{i_n}\}^{1-i_n} z_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(y_1, \dots, y_{k-1}, t_k, y_{k+1}, \dots, y_n) f_{(Y_1, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) \\
& d\lambda_{2n-2}(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n, z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n)
\end{aligned}$$

Let $i \in I$ such that $i_k = 1$, we have,

$$\begin{aligned}
& \int_{\mathbf{R}^n} 1_{V_0}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \theta_{(1-i_1, \dots, 1-i_n)}^0 (y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, t_k, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) d\lambda_n(y_1, \dots, y_n) \\
& = \int_{\mathbf{R}^{2n}} 1_{V_0}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \frac{1}{S_{(Y_1, \dots, Y_n)}(y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, t_k, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n})} \\
& \times 1_{V_0}(z_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_0}(z_j, y_j^{1-i_j} t_j^{i_j}) \right\} \\
& \times \mu(\{y_1^{1-i_1} t_1^{i_1}\}^{1-i_1} z_1^{i_1}, \dots, \{y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}\}^{1-i_{k-1}} z_{k-1}^{i_{k-1}}, z_k, \{y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}\}^{1-i_{k+1}} z_{k+1}^{i_{k+1}}, \dots, \{y_n^{1-i_n} t_n^{i_n}\}^{1-i_n} z_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) f_{(Y_1, \dots, Y_n)}(z_1, \dots, z_n) d\lambda_{2n}(y_1, \dots, y_n, z_1, \dots, z_n) \\
& = \int_{\mathbf{R}^{2n-1}} 1_{V_0}(z_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \frac{1}{S_{(Y_1, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)}(y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n})} \\
& \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_0}(z_j, y_j^{1-i_j} t_j^{i_j}) \right\} \\
& \times \mu(\{y_1^{1-i_1} t_1^{i_1}\}^{1-i_1} z_1^{i_1}, \dots, \{y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}\}^{1-i_{k-1}} z_{k-1}^{i_{k-1}}, z_k, \{y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}\}^{1-i_{k+1}} z_{k+1}^{i_{k+1}}, \dots, \{y_n^{1-i_n} t_n^{i_n}\}^{1-i_n} z_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(z_1, \dots, z_n) f_{(Y_1, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)}(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n) \\
& d\lambda_{2n-1}(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n, z_1, \dots, z_n) \\
& \frac{d}{dt_k} \int_{\mathbf{R}^n} 1_{V_0}(y_k, t_k) \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \theta_{(1-i_1, \dots, 1-i_n)}^0 (y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, t_k, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n) d\lambda_n(y_1, \dots, y_n) \\
& = - \int_{\mathbf{R}^{2n-2}} \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_{1-i_j}}(y_j, t_j) \right\} \\
& \times \frac{1}{S_{(Y_1, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)}(y_1^{1-i_1} t_1^{i_1}, \dots, y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}, y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}, \dots, y_n^{1-i_n} t_n^{i_n})} \\
& \left\{ \prod_{1 \leq j \leq n, j \neq k} 1_{V_0}(z_j, y_j^{1-i_j} t_j^{i_j}) \right\} \\
& \times \mu(\{y_1^{1-i_1} t_1^{i_1}\}^{1-i_1} z_1^{i_1}, \dots, \{y_{k-1}^{1-i_{k-1}} t_{k-1}^{i_{k-1}}\}^{1-i_{k-1}} z_{k-1}^{i_{k-1}}, t_k, \{y_{k+1}^{1-i_{k+1}} t_{k+1}^{i_{k+1}}\}^{1-i_{k+1}} z_{k+1}^{i_{k+1}}, \dots, \{y_n^{1-i_n} t_n^{i_n}\}^{1-i_n} z_n^{i_n}) \\
& \times f_{(Y_1, \dots, Y_n)}(z_1, \dots, z_{k-1}, t_k, z_{k+1}, \dots, z_n) f_{(Y_1, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)}(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n) \\
& d\lambda_{2n-2}(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n, z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n)
\end{aligned}$$

Consequently,

$$\frac{\partial \psi}{\partial t_k}(t_1, \dots, t_n) = 0$$

Therefore, there exists $x \in \mathbf{R}$ such that for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, we have,

$$\psi(t_1, \dots, t_n) = x$$

When the real numbers t_1, \dots, t_n tend to minus infinity, we have,

$$x = \int_{\mathbf{R}^n} \theta_{(1, \dots, 1)}^0(y_1, \dots, y_n) d\mathbb{P}_{(Y_1, \dots, Y_n)}(y_1, \dots, y_n)$$

Using the Proposition 3.6, with for any $i \in I$, $H_i = 1$, we have,

$$\psi(t_1, \dots, t_n) = \mathbb{E}[\gamma(Y_1, \dots, Y_n)]$$

Finally, we have for any $(t_1, \dots, t_n, d_1, \dots, d_n) \in \mathbf{R}^n \times \{0, 1\}^n$,

$$U_{C_1, \dots, C_n, \gamma, (\theta^0)_{i \in I}}(t_1, \dots, t_n, d_1, \dots, d_n) = U_{Y_1, \dots, Y_n, \psi, (\phi_i)_{i \in I}}(t_1, \dots, t_n, 1 - d_1, \dots, 1 - d_n)$$

This concludes the proof.

APPENDIX C: PROOFS FROM THE SECTION 4

Proof of Proposition 4.1

We have for any $(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) \in \mathbf{R}^{nr}$ and any $i = (i_1^1, \dots, i_n^1, \dots, i_1^r, \dots, i_n^r) \in \{0, 1\}^{nr}$,

$$\begin{aligned}
& \alpha_i(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^k, t_j^k) \} \\
& \times \gamma(t_1^1 i_1^1 y_1^1 1 - i_1^1, \dots, t_n^1 i_n^1 y_n^1 1 - i_n^1, \dots, t_1^r i_1^r y_1^r 1 - i_1^r, \dots, t_n^r i_n^r y_n^r 1 - i_n^r) d\mathbb{P}_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(y_1^1, \dots, y_n^1, \dots, y_1^r, \dots, y_n^r) \\
&= \sum_{i \in \{0, 1\}^{nr}} \frac{(-1)^{nr - \sum_{1 \leq j \leq n, 1 \leq k \leq r} i_j^k}}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^k}(t_j^k)\}^{i_j^k}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{1_{V_1}(u_j^k, t_j^k)\}^{1 - i_j^k}] \\
& \times \frac{\alpha_i(t_1^1 u_1^1 1 - i_1^1, \dots, t_n^1 u_n^1 1 - i_n^1, \dots, t_1^r u_1^r 1 - i_1^r, \dots, t_n^r u_n^r 1 - i_n^r)}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^k}^2(u_j^k)\}^{1 - i_j^k}} d\mathbb{P}_{(C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r)}(u_1^1, \dots, u_n^1, \dots, u_1^r, \dots, u_n^r)
\end{aligned}$$

Let $\tau \in S_r$, $i = (i_1^1, \dots, i_n^1, \dots, i_1^r, \dots, i_n^r) \in \{0, 1\}^{nr}$ and $i_\tau = (i_1^{\tau(1)}, \dots, i_n^{\tau(1)}, \dots, i_1^{\tau(r)}, \dots, i_n^{\tau(r)})$, we have,

$$\begin{aligned}
& \gamma(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) = \gamma(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}) \\
& \alpha_{i_\tau}(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)})} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^k, t_j^{\tau(k)}) \} \\
& \times \gamma(t_1^{\tau(1)} i_1^{\tau(1)} y_1^1 1 - i_1^{\tau(1)}, \dots, t_n^{\tau(1)} i_n^{\tau(1)} y_n^1 1 - i_n^{\tau(1)}, \dots, t_1^{\tau(r)} i_1^{\tau(r)} y_1^r 1 - i_1^{\tau(r)}, \dots, t_n^{\tau(r)} i_n^{\tau(r)} y_n^r 1 - i_n^{\tau(r)}) \\
& d\mathbb{P}_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(y_1^1, \dots, y_n^1, \dots, y_1^r, \dots, y_n^r) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^{\tau^{-1}(k)}, t_j^k) \} \\
& \times \gamma(t_1^1 i_1^1 y_1^{\tau^{-1}(1)} 1 - i_1^1, \dots, t_n^1 i_n^1 y_n^{\tau^{-1}(1)} 1 - i_n^1, \dots, t_1^r i_1^r y_1^{\tau^{-1}(r)} 1 - i_1^r, \dots, t_n^r i_n^r y_n^{\tau^{-1}(r)} 1 - i_n^r) \\
& d\mathbb{P}_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(y_1^1, \dots, y_n^1, \dots, y_1^r, \dots, y_n^r) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^{\tau^{-1}(k)}, t_j^k) \} \\
& \times \gamma(t_1^1 i_1^1 y_1^{\tau^{-1}(1)} 1 - i_1^1, \dots, t_n^1 i_n^1 y_n^{\tau^{-1}(1)} 1 - i_n^1, \dots, t_1^r i_1^r y_1^{\tau^{-1}(r)} 1 - i_1^r, \dots, t_n^r i_n^r y_n^{\tau^{-1}(r)} 1 - i_n^r) \\
& d\mathbb{P}_{(Y_1^{\tau^{-1}(1)}, \dots, Y_n^{\tau^{-1}(1)}, \dots, Y_1^{\tau^{-1}(r)}, \dots, Y_n^{\tau^{-1}(r)})}(y_1^{\tau^{-1}(1)}, \dots, y_n^{\tau^{-1}(1)}, \dots, y_1^{\tau^{-1}(r)}, \dots, y_n^{\tau^{-1}(r)}) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^{\tau^{-1}(k)}, t_j^k) \} \\
& \times \gamma(t_1^1 i_1^1 y_1^{\tau^{-1}(1)} 1 - i_1^1, \dots, t_n^1 i_n^1 y_n^{\tau^{-1}(1)} 1 - i_n^1, \dots, t_1^r i_1^r y_1^{\tau^{-1}(r)} 1 - i_1^r, \dots, t_n^r i_n^r y_n^{\tau^{-1}(r)} 1 - i_n^r) \\
& d\mathbb{P}_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(y_1^{\tau^{-1}(1)}, \dots, y_n^{\tau^{-1}(1)}, \dots, y_1^{\tau^{-1}(r)}, \dots, y_n^{\tau^{-1}(r)}) \\
&= \alpha_i(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)
\end{aligned}$$

$$\begin{aligned}
& \mu(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}) \\
&= \sum_{i \in \{0,1\}^{nr}} \frac{(-1)^{nr - \sum_{1 \leq j \leq n, 1 \leq k \leq r} i_j^k}}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^k}(t_j^{\tau(k)})\}^{i_j^k}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{1_{V_1}(u_j^k, t_j^{\tau(k)})\}^{1-i_j^k}] \\
&\times \frac{\alpha_i(t_1^{\tau(1) i_1^1} u_1^{1-i_1^1}, \dots, t_n^{\tau(1) i_n^1} u_n^{1-i_n^1}, \dots, t_1^{\tau(r) i_1^r} u_1^{r-1-i_1^r}, \dots, t_n^{\tau(r) i_n^r} u_n^{r-1-i_n^r})}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^k}^2(u_j^k)\}^{1-i_j^k}} d\mathbb{P}_{(C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r)}(u_1^1, \dots, u_n^1, \dots, u_1^r, \dots, u_n^r) \\
&= \sum_{i \in \{0,1\}^{nr}} \frac{(-1)^{nr - \sum_{1 \leq j \leq n, 1 \leq k \leq r} i_j^{\tau(k)}}}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^k}(t_j^{\tau(k)})\}^{i_j^{\tau(k)}}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{1_{V_1}(u_j^k, t_j^{\tau(k)})\}^{1-i_j^{\tau(k)}}] \\
&\times \frac{\alpha_{i_{\tau}}(t_1^{\tau(1) i_1^{\tau(1)}} u_1^{1-i_1^{\tau(1)}}, \dots, t_n^{\tau(1) i_n^{\tau(1)}} u_n^{1-i_n^{\tau(1)}}, \dots, t_1^{\tau(r) i_1^{\tau(r)}} u_1^{r-1-i_1^{\tau(r)}}, \dots, t_n^{\tau(r) i_n^{\tau(r)}} u_n^{r-1-i_n^{\tau(r)}})}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^k}^2(u_j^k)\}^{1-i_j^{\tau(k)}}} \\
&d\mathbb{P}_{(C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r)}(u_1^1, \dots, u_n^1, \dots, u_1^r, \dots, u_n^r) \\
&= \sum_{i \in \{0,1\}^{nr}} \frac{(-1)^{nr - \sum_{1 \leq j \leq n, 1 \leq k \leq r} i_j^k}}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^{\tau^{-1}(k)}}(t_j^k)\}^{i_j^k}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{1_{V_1}(u_j^{\tau^{-1}(k)}, t_j^k)\}^{1-i_j^k}] \\
&\times \frac{\alpha_i(t_1^{i_1^1} u_1^{\tau^{-1}(1) 1-i_1^1}, \dots, t_n^{i_n^1} u_n^{\tau^{-1}(1) 1-i_n^1}, \dots, t_1^{i_1^r} u_1^{\tau^{-1}(r) 1-i_1^r}, \dots, t_n^{i_n^r} u_n^{\tau^{-1}(r) 1-i_n^r})}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^{\tau^{-1}(k)}}^2(u_j^{\tau^{-1}(k)})\}^{1-i_j^k}} \\
&d\mathbb{P}_{(C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r)}(u_1^1, \dots, u_n^1, \dots, u_1^r, \dots, u_n^r) \\
&= \sum_{i \in \{0,1\}^{nr}} \frac{(-1)^{nr - \sum_{1 \leq j \leq n, 1 \leq k \leq r} i_j^k}}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^{\tau^{-1}(k)}}(t_j^k)\}^{i_j^k}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{1_{V_1}(u_j^{\tau^{-1}(k)}, t_j^k)\}^{1-i_j^k}] \\
&\times \frac{\alpha_i(t_1^{i_1^1} u_1^{\tau^{-1}(1) 1-i_1^1}, \dots, t_n^{i_n^1} u_n^{\tau^{-1}(1) 1-i_n^1}, \dots, t_1^{i_1^r} u_1^{\tau^{-1}(r) 1-i_1^r}, \dots, t_n^{i_n^r} u_n^{\tau^{-1}(r) 1-i_n^r})}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^{\tau^{-1}(k)}}^2(u_j^{\tau^{-1}(k)})\}^{1-i_j^k}} \\
&d\mathbb{P}_{(C_1^{\tau^{-1}(1)}, \dots, C_n^{\tau^{-1}(1)}, \dots, C_1^{\tau^{-1}(r)}, \dots, C_n^{\tau^{-1}(r)})}(u_1^{\tau^{-1}(1)}, \dots, u_n^{\tau^{-1}(1)}, \dots, u_1^{\tau^{-1}(r)}, \dots, u_n^{\tau^{-1}(r)}) \\
&= \sum_{i \in \{0,1\}^{nr}} \frac{(-1)^{nr - \sum_{1 \leq j \leq n, 1 \leq k \leq r} i_j^k}}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^k}(t_j^k)\}^{i_j^k}} \int_{\mathbf{R}^n} [\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{1_{V_1}(u_j^k, t_j^k)\}^{1-i_j^k}] \\
&\times \frac{\alpha_i(t_1^{i_1^1} u_1^{1-i_1^1}, \dots, t_n^{i_n^1} u_n^{1-i_n^1}, \dots, t_1^{i_1^r} u_1^{r-1-i_1^r}, \dots, t_n^{i_n^r} u_n^{r-1-i_n^r})}{\prod_{1 \leq j \leq n, 1 \leq k \leq r} \{S_{C_j^k}^2(u_j^k)\}^{1-i_j^k}} d\mathbb{P}_{(C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r)}(u_1^1, \dots, u_n^1, \dots, u_1^r, \dots, u_n^r) \\
&= \mu(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)
\end{aligned}$$

We have,

$$\begin{aligned}
& \theta_i^0(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^k, t_j^k) \} \\
&\times \mu(t_1^{i_1^1} y_1^{1-i_1^1}, \dots, t_n^{i_n^1} y_n^{1-i_n^1}, \dots, t_1^{i_1^r} y_1^{r-1-i_1^r}, \dots, t_n^{i_n^r} y_n^{r-1-i_n^r}) d\mathbb{P}_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(y_1^1, \dots, y_n^1, \dots, y_1^r, \dots, y_n^r)
\end{aligned}$$

$$\begin{aligned}
& \theta_{i_\tau}^0(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)})} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^k, t_j^{\tau(k)}) \} \\
& \times \mu(t_1^{\tau(1)} y_1^{1-i_1^{\tau(1)}}, \dots, t_n^{\tau(1)} y_n^{1-i_n^{\tau(1)}}, \dots, t_1^{\tau(r)} y_1^{1-i_1^{\tau(r)}}, \dots, t_n^{\tau(r)} y_n^{1-i_n^{\tau(r)}}) \\
& d\mathbb{P}_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(y_1^1, \dots, y_n^1, \dots, y_1^r, \dots, y_n^r) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^{\tau^{-1}(k)}, t_j^k) \} \\
& \times \mu(t_1^1 y_1^{\tau^{-1}(1)} 1^{-i_1^1}, \dots, t_n^1 y_n^{\tau^{-1}(1)} 1^{-i_n^1}, \dots, t_1^r y_1^{\tau^{-1}(r)} 1^{-i_1^r}, \dots, t_n^r y_n^{\tau^{-1}(r)} 1^{-i_n^r}) \\
& d\mathbb{P}_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(y_1^1, \dots, y_n^1, \dots, y_1^r, \dots, y_n^r) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^{\tau^{-1}(k)}, t_j^k) \} \\
& \times \mu(t_1^1 y_1^{\tau^{-1}(1)} 1^{-i_1^1}, \dots, t_n^1 y_n^{\tau^{-1}(1)} 1^{-i_n^1}, \dots, t_1^r y_1^{\tau^{-1}(r)} 1^{-i_1^r}, \dots, t_n^r y_n^{\tau^{-1}(r)} 1^{-i_n^r}) \\
& d\mathbb{P}_{(Y_1^{\tau^{-1}(1)}, \dots, Y_n^{\tau^{-1}(1)}, \dots, Y_1^{\tau^{-1}(r)}, \dots, Y_n^{\tau^{-1}(r)})}(y_1^{\tau^{-1}(1)}, \dots, y_n^{\tau^{-1}(1)}, \dots, y_1^{\tau^{-1}(r)}, \dots, y_n^{\tau^{-1}(r)}) \\
&= \frac{1}{S_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)} \int_{\mathbf{R}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} 1_{V_0}(y_j^k, t_j^k) \} \\
& \times \mu(t_1^1 y_1^{1-i_1^1}, \dots, t_n^1 y_n^{1-i_n^1}, \dots, t_1^r y_1^{1-i_1^r}, \dots, t_n^r y_n^{1-i_n^r}) \\
& d\mathbb{P}_{(Y_1^1, \dots, Y_n^1, \dots, Y_1^r, \dots, Y_n^r)}(y_1^1, \dots, y_n^1, \dots, y_1^r, \dots, y_n^r) \\
&= \theta_i^0(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r)
\end{aligned}$$

We have for any $(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) \in \mathbf{R}^{nr}$, any $(d_1^1, \dots, d_n^1, \dots, d_1^r, \dots, d_n^r) \in \{0, 1\}^{nr}$ and any $\tau \in S_r$,

$$\begin{aligned}
& \bigcup_{C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \gamma, (\theta_i^0)_{i \in I}} (t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}, d_1^{\tau(1)}, \dots, d_n^{\tau(1)}, \dots, d_1^{\tau(r)}, \dots, d_n^{\tau(r)}) \\
&= \sum_{i \in \{0, 1\}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} d_j^{k i_j^k} (1 - d_j^k)^{1-i_j^k} \} \theta_i^0(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}) \\
&= \sum_{i \in \{0, 1\}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} d_j^{\tau(k) i_j^{\tau(k)}} (1 - d_j^{\tau(k)})^{1-i_j^{\tau(k)}} \} \theta_{i_\tau}^0(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}) \\
&= \sum_{i \in \{0, 1\}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} d_j^{\tau(k) i_j^{\tau(k)}} (1 - d_j^{\tau(k)})^{1-i_j^{\tau(k)}} \} \theta_{i_\tau}^0(t_1^{\tau(1)}, \dots, t_n^{\tau(1)}, \dots, t_1^{\tau(r)}, \dots, t_n^{\tau(r)}) \\
&= \sum_{i \in \{0, 1\}^{nr}} \{ \prod_{1 \leq j \leq n, 1 \leq k \leq r} d_j^{k i_j^k} (1 - d_j^k)^{1-i_j^k} \} \theta_i^0(t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r) \\
&= \bigcup_{C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \gamma, (\theta_i^0)_{i \in I}} (t_1^1, \dots, t_n^1, \dots, t_1^r, \dots, t_n^r, d_1^1, \dots, d_n^1, \dots, d_1^r, \dots, d_n^r)
\end{aligned}$$

This concludes the proof.

Proof of Theorem 4.1

By the proposition 3.8, we have for any $(t_1^1, \dots, t_n^1, \dots, t_1^m, \dots, t_n^m) \in \mathbf{R}^{nm}$ and any $(d_1^1, \dots, d_n^1, \dots, d_1^m, \dots, d_n^m) \in \{0, 1\}^{nm}$,

$$\begin{aligned}
& \bigcup_{C_1^1, \dots, C_n^1, \dots, C_1^m, \dots, C_n^m, \psi^{(m, n, r)}, (\theta_i^0)_{i \in I}} (t_1^1, \dots, t_n^1, \dots, t_1^m, \dots, t_n^m, d_1^1, \dots, d_n^1, \dots, d_1^m, \dots, d_n^m) \\
&= \binom{m}{r}^{-1} \sum_{1 \leq j_1 < \dots < j_r \leq m} \\
& \bigcup_{C_1^1, \dots, C_n^1, \dots, C_1^r, \dots, C_n^r, \bar{\gamma}, (\theta_i^0)_{i \in I}} (t_1^{j_1}, \dots, t_n^{j_1}, \dots, t_1^{j_r}, \dots, t_n^{j_r}, d_1^{j_1}, \dots, d_n^{j_1}, \dots, d_1^{j_r}, \dots, d_n^{j_r})
\end{aligned}$$

The end of the proof is a direct consequence of the Theorem 3.2 and the Proposition 4.1, and of a result from Hoeffding [4] about U statistics.

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