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Additional material on local limit theorem for finite Additive Markov Processes [HL13]

Loïc HERVÉ and James LEDOUX *

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Abstract

This paper proposes additional material to the main statements of [HL13] which are recalled in Section 2. In particular an application of [HL13, Th 2.2] to Renewal Markov Processes is provided in Section 4 and a detailed checking of the assumptions of [HL13, Th 2.2] for the joint distribution of local times of a finite jump process is reported in Section 5. A uniform version of [HL13, Th 2.2] with respect to a compact set of transition matrices is given in Section 6 (see [HL13, Remark 2.4]). The basic material on the semigroup of Fourier matrices and the spectral approach used in [HL13] is recalled in Section 3 in order to obtain a good understanding of the properties involved in this uniform version.

Keywords: Gaussian approximation, Local time, Spectral method, Markov random walk.

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1 Notations

Any vector $v = (v_k)_{k \in \{1, \dots, N\}} \in \mathbb{C}^N$ is considered as a row-vector and v^\top is the corresponding column-vector. The vector with all components equal to 1 is denoted by $\mathbf{1}$. The euclidean product scalar and its associated norm on \mathbb{C}^N is denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. The set of $N \times N$ -matrices with complex entries is denoted by $\mathcal{M}_N(\mathbb{C})$. Let $\|\cdot\|_\infty$ denote the supremum norm on \mathbb{C}^N : $\forall v \in \mathbb{C}^N$, $\|v\|_\infty = \max_{k \in \{1, \dots, N\}} |v_k|$. For the sake of simplicity, $\|\cdot\|_\infty$ also stands for the associated matrix norm:

$$\forall A \in \mathcal{M}_N(\mathbb{C}), \quad \|A\|_\infty := \sup_{\|v\|_\infty=1} \|Av^\top\|_\infty.$$

We also use the following norm $\|\cdot\|_0$ on $\mathcal{M}_N(\mathbb{C})$:

$$\forall A = (A_{k,\ell})_{(k,\ell) \in \{1, \dots, N\}^2} \in \mathcal{M}_N(\mathbb{C}), \quad \|A\|_0 := \max_{(k,\ell) \in \{1, \dots, N\}^2} |A_{k,\ell}|.$$

It is easily seen that

$$\forall A \in \mathcal{M}_N(\mathbb{C}), \quad \|A\|_0 \leq \|A\|_\infty \leq N\|A\|_0. \quad (1)$$

For any bounded positive measure ν on \mathbb{R}^d , we define its Fourier transform as:

$$\forall \zeta \in \mathbb{R}^d, \quad \widehat{\nu}(\zeta) := \int_{\mathbb{R}^d} e^{i\langle \zeta, y \rangle} d\nu(y).$$

Let $\mathcal{A} = (\mathcal{A}_{k,\ell})_{(k,\ell) \in \{1, \dots, N\}^2}$ be a $N \times N$ -matrix with entries in the set of bounded positive measures on \mathbb{R}^d . We set

$$\forall B \in B(\mathbb{R}^d), \quad \mathcal{A}(1_B) := (\mathcal{A}_{k,\ell}(1_B))_{(k,\ell) \in \mathbb{X}^2} \quad (2a)$$

$$\forall \zeta \in \mathbb{R}^d, \quad \widehat{\mathcal{A}}(\zeta) := (\widehat{\mathcal{A}}_{k,\ell}(\zeta))_{(k,\ell) \in \mathbb{X}^2}. \quad (2b)$$

We denote by $[t]$ the integer part of any $t \in \mathbb{T}$.

2 The LLT for the density process

Let $\{(X_t, Z_t)\}_{t \in \mathbb{T}}$ be an MAP with state space $\mathbb{X} \times \mathbb{R}^d$, where $\mathbb{X} := \{1, \dots, N\}$ and the driving Markov process $\{X_t\}_{t \in \mathbb{T}}$ has transition semi-group $\{P_t\}_{t \in \mathbb{T}}$. We refer to [Asm03, Chap. XI] for the basic material on such MAPs. The conditional probability to $\{X_0 = k\}$ and its associated expectation are denoted by \mathbb{P}_k and \mathbb{E}_k respectively. Note that if $T : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is a linear transformation, then $\{X_t, T(Z_t)\}_{t \in \mathbb{T}}$ is still a MAP on $\mathbb{X} \times \mathbb{R}^m$ (see Lemma C.1). We suppose that $\{X_t\}_{t \in \mathbb{T}}$ has a unique invariant probability measure π . Set $m := \mathbb{E}_\pi[Z_1] = \sum_k \pi(k) \mathbb{E}_k[Z_1] \in \mathbb{R}^d$. Consider the centered MAP $\{(X_t, Y_t)\}_{t \in \mathbb{T}}$ where $Y_t := Z_t - t m$. The two next assumptions are involved in both CLT and LLT.

(I-A) : *The stochastic $N \times N$ -matrix $P := P_1$ is irreducible and aperiodic.*

(M α) : *The family of r.v. $\{Y_v\}_{v \in (0,1] \cap \mathbb{T}}$ satisfies the uniform moment condition of order α :*

$$M_\alpha := \max_{k \in \mathbb{X}} \sup_{v \in (0,1] \cap \mathbb{T}} \mathbb{E}_k[\|Y_v\|^\alpha] < \infty. \quad (3)$$

The next theorem provides a CLT for $t^{-1/2}Y_t$, proved for $d := 1$ in [KW64] for $\mathbb{T} = \mathbb{N}$ and in [FH67] for $\mathbb{T} = [0, \infty)$ (see [FHL12] for ρ -mixing driving Markov processes).

Theorem 2.1 *Under Assumptions (I-A) and (M2), $\{t^{-1/2}Y_t\}_{t \in \mathbb{T}}$ converges in distribution to a d -dimensional Gaussian law $\mathcal{N}(0, \Sigma)$ when $t \rightarrow +\infty$.*

Now let us specify the notations and assumptions involved in our LLT. First we assume that the MAP $\{(X_t, Y_t)\}_{t \in \mathbb{T}}$ satisfies the following usual non-lattice condition:

(NL) : *there is no $a \in \mathbb{R}^d$, no closed subgroup H in \mathbb{R}^d , $H \neq \mathbb{R}^d$, and finally no function $\beta : \mathbb{X} \rightarrow \mathbb{R}^d$ such that:*

$$\forall k \in \mathbb{X}, \quad Y_1 + \beta(X_1) - \beta(k) \in a + H \quad \mathbb{P}_k\text{-a.s.},$$

Second we introduce the assumptions on the density process of $t^{-1/2}Y_t$. For any $t \in \mathbb{T}$ and $(k, \ell) \in \mathbb{X}^2$, we define the bounded positive measure $\mathcal{Y}_{k, \ell, t}$ on \mathbb{R}^d :

$$\forall B \in B(\mathbb{R}^d), \quad \mathcal{Y}_{k, \ell, t}(1_B) := \mathbb{P}_k\{X_t = \ell, Y_t \in B\}. \quad (4)$$

Let ℓ_d denote the Lebesgue measure on \mathbb{R}^d . From the Lebesgue decomposition of $\mathcal{Y}_{k, \ell, t}$ w.r.t. ℓ_d , there are two bounded positive measures $\mathcal{G}_{k, \ell, t}$ and $\mu_{k, \ell, t}$ on \mathbb{R}^d such that

$$\forall B \in B(\mathbb{R}^d) \quad \mathcal{Y}_{k, \ell, t}(1_B) := \mathcal{G}_{k, \ell, t}(1_B) + \mu_{k, \ell, t}(1_B) \quad (5a)$$

$$\text{where} \quad \mathcal{G}_{k, \ell, t}(1_B) = \int_B g_{k, \ell, t}(y) dy \quad (5b)$$

for some measurable function $g_{k, \ell, t} : \mathbb{R}^d \rightarrow [0, +\infty)$, and such that $\mu_{k, \ell, t}$ and ℓ_d are mutually singular. The measure $\mathcal{G}_{k, \ell, t}$ is called the absolutely continuous (a.c.) part of $\mathcal{Y}_{k, \ell, t}$ with associated density $g_{k, \ell, t}$. For any $t \in \mathbb{T}$, we introduce the following $N \times N$ -matrices with entries in the set of bounded positive measures on \mathbb{R}^d

$$\mathcal{Y}_t := (\mathcal{Y}_{k, \ell, t})_{(k, \ell) \in \mathbb{X}^2}, \quad \mathcal{G}_t := (\mathcal{G}_{k, \ell, t})_{(k, \ell) \in \mathbb{X}^2}, \quad \mathcal{M}_t := (\mu_{k, \ell, t})_{(k, \ell) \in \mathbb{X}^2}, \quad (6a)$$

and for every $y \in \mathbb{R}^d$, the following real $N \times N$ -matrix:

$$G_t(y) := (g_{k, \ell, t}(y))_{(k, \ell) \in \mathbb{X}^2}. \quad (6b)$$

Then the component-wise equalities (5a)-(5b) read as follows in a matrix form: for any $t \in \mathbb{T}$

$$\forall B \in B(\mathbb{R}^d), \quad \mathcal{Y}_t(1_B) = \mathcal{G}_t(1_B) + \mathcal{M}_t(1_B) = \int_B G_t(y) dy + \mathcal{M}_t(1_B). \quad (6c)$$

The assumptions on the a.c. part \mathcal{G}_t and the singular part \mathcal{M}_t of \mathcal{Y}_t are the following ones.

AC1 : *There exist $c > 0$ and $\rho \in (0, 1)$ such that*

$$\forall t > 0, \quad \|\mathcal{M}_t(1_{\mathbb{R}^d})\|_0 \leq c\rho^t \quad (7)$$

and there exists $t_0 > 0$ such that

$$\rho^{t_0} \max(2, cN) \leq 1/4 \quad (8a)$$

$$\Gamma_{t_0}(\zeta) := \sup_{w \in [t_0, 2t_0]} \|\widehat{G}_w(\zeta)\|_0 \rightarrow 0 \quad \text{when } \|\zeta\| \rightarrow +\infty. \quad (8b)$$

AC2 : For any $t > 0$, there exists an open convex subset \mathcal{D}_t of \mathbb{R}^d such that G_t vanishes on $\mathbb{R}^d \setminus \overline{\mathcal{D}_t}$, where $\overline{\mathcal{D}_t}$ denotes the adherence of \mathcal{D}_t . Moreover G_t is continuous on $\overline{\mathcal{D}_t}$ and differentiable on \mathcal{D}_t , with in addition

$$\sup_{t>0} \sup_{y \in \overline{\mathcal{D}_t}} \|G_t(y)\|_0 < \infty \quad (9a)$$

$$\sup_{y \in \partial \mathcal{D}_t} \|G_t(y)\|_0 = O(t^{-(d+1)/2}) \quad \text{where } \partial \mathcal{D}_t := \overline{\mathcal{D}_t} \setminus \mathcal{D}_t \quad (9b)$$

$$j = 1, \dots, d : \sup_{t>0} \sup_{y \in \overline{\mathcal{D}_t}} \left\| \frac{\partial G_t}{\partial y_j}(y) \right\|_0 < \infty. \quad (9c)$$

The next theorem gives a LLT for the density of the a.c. part of the probability distribution of $t^{-1/2}Y_t$. This is the main contribution of [HL13].

Theorem 2.2 Let $\{(X_t, Y_t)\}_{t \in \mathbb{T}}$ be a centered MAP satisfying Assumptions **(I-A)**, **(M3)**, **(AC1)**-**(AC2)**. Moreover assume that the matrix Σ of Theorem 2.1 is invertible. Then, for every $k \in \mathbb{X}$, the density $f_{k,t}(\cdot)$ of the a.c. part of the probability distribution of $t^{-1/2}Y_t$ under \mathbb{P}_k satisfies the following property:

$$\sup_{y \in \mathbb{R}^d} |f_{k,t}(y) - \eta_\Sigma(y)| \leq O(t^{-1/2}) + O\left(\sup_{y \notin \mathcal{D}_t} \eta_\Sigma(t^{-1/2}y)\right) \quad (10)$$

where $\eta_\Sigma(\cdot)$ denotes the density of the non-degenerate d -dimensional Gaussian distribution $\mathcal{N}(0, \Sigma)$ involved in Theorem 2.1.

3 Semigroup of Fourier matrices and basic lemmas

We assume that the conditions of Theorem 2.2 hold, and for the sake of simplicity that Σ is the identity matrix. The proof of Theorem 2.2 involves Fourier analysis as in the i.d.d. case. In our Markov context, this study is based on the semi-group property of the matrix family $\{\widehat{\mathcal{Y}}_t(\zeta)\}_{t \in \mathbb{T}}$ for every $\zeta \in \mathbb{R}^d$ which allows to analyze the characteristic function of Y_t . In this section, we provide a collection of lemmas which highlights the connections between the assumptions of Section 2 and the behavior of this semi-group. This is the basic material for the derivation of Theorem 2.2 in [HL13, Section 3].

Recall that, for every $(k, \ell) \in \mathbb{X}^2$, the measures $\mathcal{G}_{k,\ell,t}$ (with density $g_{k,\ell,t}$) and $\mu_{k,\ell,t}$ denote the a.c. and singular parts of the bounded positive measure $\mathcal{Y}_{k,\ell,t}(1) = \mathbb{P}_k\{X_t = \ell, Y_t \in \cdot\}$ on \mathbb{R}^d . Using the notation of (5a)-(5b), the bounded positive measure $\sum_{\ell=1}^N \mathcal{G}_{k,\ell,t}$ with density

$$g_{k,t} := \sum_{\ell=1}^N g_{k,\ell,t}$$

is the a.c. part of the probability distribution of Y_t under \mathbb{P}_k , while $\mu_{k,t} := \sum_{\ell=1}^N \mu_{k,\ell,t}$ is its singular part. That is, we have for any $k \in \mathbb{X}$ and $t > 0$:

$$\forall B \in \mathcal{B}(\mathbb{R}^d), \quad \mathbb{P}_k\{Y_t \in B\} = \int_B g_{k,t}(y) dy + \mu_{k,t}(1_B). \quad (11)$$

The bounded positive measure \mathcal{Y}_t is defined in (6a) and its Fourier transform $\widehat{\mathcal{Y}}_t$ is (see (2b)):

$$\forall (t, \zeta) \in \mathbb{T} \times \mathbb{R}^d, \quad \forall (k, \ell) \in \mathbb{X}^2, \quad (\widehat{\mathcal{Y}}_t(\zeta))_{k,\ell} = \mathbb{E}_k[1_{\{X_t=\ell\}} e^{i\langle \zeta, Y_t \rangle}]. \quad (12)$$

Note that $\widehat{\mathcal{Y}}_t(0) = P_t$. From the additivity of the second component Y_t , we know that that $\{\widehat{\mathcal{Y}}_t(\zeta)\}_{t \in \mathbb{T}, \zeta \in \mathbb{R}^d}$ is a semi-group of matrices (e.g. see [FHL12] for details), that is using the usual product in the space $\mathcal{M}_N(\mathbb{C})$ of complex $N \times N$ -matrices:

$$\forall \zeta \in \mathbb{R}^d, \forall (s, t) \in \mathbb{T}^2, \quad \widehat{\mathcal{Y}}_{t+s}(\zeta) = \widehat{\mathcal{Y}}_t(\zeta) \widehat{\mathcal{Y}}_s(\zeta). \quad (\text{SG})$$

In particular the following property holds true

$$\forall \zeta \in \mathbb{R}^d, \forall n \in \mathbb{N}, \quad \widehat{\mathcal{Y}}_n(\zeta) := (\mathbb{E}_k[1_{\{X_n=\ell\}} e^{i\langle \zeta, Y_n \rangle}])_{(k, \ell) \in \mathbb{X}^2} = \widehat{\mathcal{Y}}_1(\zeta)^n. \quad (13)$$

For every $k \in \mathbb{X}$ and $t \in \mathbb{T}$, we denote by $\phi_{k,t}$ the characteristic function of Y_t under \mathbb{P}_k :

$$\forall \zeta \in \mathbb{R}^d, \quad \phi_{k,t}(\zeta) := \mathbb{E}_k[e^{i\langle \zeta, Y_t \rangle}] = e_k \widehat{\mathcal{Y}}_t(\zeta) \mathbf{1}^\top, \quad (14)$$

where e_k is the k -th vector of the canonical basis of \mathbb{R}^d .

The first lemma below provides the control of $\phi_{k,t}$ on a ball $B(0, \delta) := \{\zeta \in \mathbb{R}^d : \|\zeta\| < \delta\}$ for some $\delta > 0$, using a perturbation approach. The second one is on the control of $\phi_{k,t}$ on the annulus $\{\zeta \in \mathbb{R}^d : \delta \leq \|\zeta\| \leq A\}$ for any $A > \delta$. Finally the third lemma focuses on the Fourier transform of the density $g_{k,\ell,t}$ of the a.c. part $\mathcal{G}_{k,\ell,t}$ of $\mathcal{Y}_{k,\ell,t}$ in (5a) on the domain $\{\zeta \in \mathbb{R}^d : \|\zeta\| \geq A\}$ for some $A > 0$.

Lemma 3.1 ([HL13, Lem. 4.1]) *Under Assumptions (I-A) and (M3), there exists a real number $\delta > 0$ such that, for all $\zeta \in B(0, \delta)$, the characteristic function of Y_t satisfies*

$$\forall k \in \mathbb{X}, \forall t \in \mathbb{T}, \quad \phi_{k,t}(\zeta) = \lambda(\zeta)^{\lfloor t \rfloor} L_{k,t}(\zeta) + R_{k,t}(\zeta), \quad (15)$$

with \mathbb{C} -valued functions $\lambda(\cdot)$, $L_{k,t}(\cdot)$ and $R_{k,t}(\cdot)$ on $B(0, \delta)$ satisfying the next properties for $t \in \mathbb{T}$, $\zeta \in \mathbb{R}^d$ and $k \in \mathbb{X}$:

$$\|\zeta\| < \delta \Rightarrow \lambda(\zeta) = 1 - \frac{\|\zeta\|^2}{2} + O(\|\zeta\|^3) \quad (16a)$$

$$t \geq 2, \|\zeta\| < \delta \Rightarrow |\lambda(t^{-1/2}\zeta)^{\lfloor t \rfloor} - e^{-\|\zeta\|^2/2}| \leq C \frac{(1 + \|\zeta\|^3)e^{-\|\zeta\|^2/8}}{\sqrt{t}} \quad (16b)$$

$$\|\zeta\| < \delta \Rightarrow |L_{k,t}(\zeta) - 1| \leq C \|\zeta\| \quad (16c)$$

$$\exists r \in (0, 1), \sup_{\|\zeta\| \leq \delta} |R_{k,t}(\zeta)| \leq C r^{\lfloor t \rfloor} \quad (16d)$$

where the constant $C > 0$ in (16b)-(16d) only depends on δ and on M_3 in (M3) (see (3)).

Proof of (16b). From (16a) we obtain for $v \in \mathbb{R}^d$ such that $|v| \leq \delta$ (up to reduce δ)

$$|\lambda(v)| \leq 1 - \frac{\|v\|^2}{2} + \frac{\|v\|^2}{4} \leq e^{-\|v\|^2/4}.$$

Therefore we have for any $t \geq 2$ and any $\zeta \in \mathbb{R}^d$ such that $t^{-1/2}\|\zeta\| \leq \delta$,

$$|\lambda(t^{-1/2}\zeta)| \leq e^{-\frac{\|\zeta\|^2}{4t}}. \quad (17)$$

Now consider any $t \geq 2$ and any $\zeta \in \mathbb{R}^d$ such that $t^{-1/2}\|\zeta\| \leq \delta$. Set $n := \lfloor t \rfloor$, and write

$$\lambda(t^{-1/2}\zeta)^n - e^{-\frac{\|\zeta\|^2}{2}} = (\lambda(t^{-1/2}\zeta) - e^{-\frac{\|\zeta\|^2}{2n}}) \sum_{k=0}^{n-1} \lambda(t^{-1/2}\zeta)^{n-k-1} e^{-\frac{k\|\zeta\|^2}{2n}}.$$

We have

$$\begin{aligned}
\left| \sum_{k=0}^{n-1} \lambda(t^{-1/2}\zeta)^{n-k-1} e^{-\frac{k\|\zeta\|^2}{2n}} \right| &\leq \sum_{k=0}^{n-1} |\lambda(t^{-1/2}\zeta)|^{n-k-1} e^{-\frac{k\|\zeta\|^2}{2n}} \\
&\leq \sum_{k=0}^{n-1} e^{-\frac{\|\zeta\|^2(n-k-1)}{4t}} e^{-\frac{k\|\zeta\|^2}{4t}} \\
&\leq n e^{-\frac{n\|\zeta\|^2}{4t}} e^{\frac{\|\zeta\|^2}{4t}} \leq b t e^{-\frac{\|\zeta\|^2}{8}}
\end{aligned}$$

where $b := \sup_{|u| \leq \delta} \exp(u^2/4)$, since $n/t \geq (t-1)/t \geq 1/2$ for $t \geq 2$. Moreover, using (16a) and the fact that $|e^a - e^b| \leq |a - b|$ for any $a, b \in (-\infty, 0)$, we obtain that there exist positive constants D and D' such that

$$\begin{aligned}
|\lambda(t^{-1/2}\zeta) - e^{-\frac{\|\zeta\|^2}{2n}}| &\leq |\lambda(t^{-1/2}\zeta) - e^{-\frac{\|\zeta\|^2}{2t}}| + |e^{-\frac{\|\zeta\|^2}{2t}} - e^{-\frac{\|\zeta\|^2}{2n}}| \\
&\leq D t^{-\frac{3}{2}} \|\zeta\|^3 + \frac{\|\zeta\|^2}{2} \left| \frac{1}{t} - \frac{1}{n} \right| \\
&\leq D t^{-\frac{3}{2}} \|\zeta\|^3 + \frac{\|\zeta\|^2}{2} \frac{1}{t(t-1)} \\
&\leq D' (t^{-\frac{3}{2}} \|\zeta\|^3 + t^{-2} \|\zeta\|^2).
\end{aligned}$$

Thus

$$\begin{aligned}
\left| \lambda(t^{-1/2}\zeta)^n - e^{-\frac{\|\zeta\|^2}{2}} \right| &\leq b D' t e^{-\frac{\|\zeta\|^2}{8}} (t^{-\frac{3}{2}} \|\zeta\|^3 + t^{-2} \|\zeta\|^2) \\
&\leq b D' e^{-\frac{\|\zeta\|^2}{8}} \left(\frac{\|\zeta\|^3}{\sqrt{t}} + \frac{\|\zeta\|^2}{t} \right) \\
&\leq \frac{b D'}{\sqrt{t}} e^{-\frac{\|\zeta\|^2}{8}} (\|\zeta\|^3 + \|\zeta\|^2) \\
&\leq \frac{D''}{\sqrt{t}} e^{-\frac{\|\zeta\|^2}{8}} (1 + \|\zeta\|^3)
\end{aligned}$$

for some positive constant D'' . □

Lemma 3.2 ([HL13, Lem. 4.2]) *Assume that Conditions (I-A), (AC1) and (M α) for some $\alpha > 0$ hold. Let δ, A be any real numbers such that $0 < \delta < A$. There exist constants $D \equiv D(\delta, A) > 0$ and $\tau \equiv \tau(\delta, A) \in (0, 1)$ such that*

$$\forall k \in \mathbb{X}, \forall t \in \mathbb{T}, \quad \sup_{\delta \leq \|\zeta\| \leq A} |\phi_{k,t}(\zeta)| \leq D \tau^{\lfloor t \rfloor}. \quad (18)$$

Lemma 3.3 ([HL13, Th. 4.3]) *Under Condition (AC1), there exist positive constants A and C such that the following property holds:*

$$|\zeta| \geq A \implies \forall (k, \ell) \in \mathbb{X}^2, \forall t \in [t_0, +\infty[, \quad |\widehat{g}_{k,\ell,t}(\zeta)| \leq C \frac{t}{2t/t_0}.$$

4 Application to the Markov Renewal Processes

Let $\{X_n, Y_n\}_{n \in \mathbb{N}}$ be a discrete-time MAP with state space $\mathbb{X} \times \mathbb{R}$ and $\mathbb{X} := \{1, \dots, N\}$. When $\{\xi_n\}_{n \geq 0}$, with $\xi_0 := Y_0$ and $\xi_n := Y_n - Y_{n-1}$ for $n \geq 1$, is a sequence of non-negative random variables, then $\{X_n, Y_n\}_{n \in \mathbb{N}}$ is also known as a Markov Renewal Process (MRP). The so-called semi-Markov kernel $Q(\cdot; \{\cdot\} \times dy)$ is defined by (e.g. see [Asm03, VII.4])

$$\forall B \in B(\mathbb{R}), \forall (k, \ell) \in \mathbb{X}^2, \quad \mathbb{P}_k\{X_1 = \ell, Y_1 \in B\} = \int_B Q(k; \{\ell\} \times dy) \quad (19)$$

and is nothing else but the measure $\mathcal{Y}_{k, \ell, 1}$ in (4). The transition probability matrix P associated with the Markov chain $\{X_n\}_{n \in \mathbb{N}}$ is given by

$$\forall (k, \ell) \in \mathbb{X}^2, \quad P(k, \ell) = Q(k; \{\ell\} \times \mathbb{R}) = \int_{\mathbb{R}} Q(k; \{\ell\} \times dy).$$

For any $n \geq 2$, the bounded positive measure $\mathcal{Y}_{k, \ell, n}$ is defined by the convolution product of the semi-Markov kernel Q , that is

$$\forall B \in B(\mathbb{R}^d), \quad \mathcal{Y}_{k, \ell, n}(1_B) := \mathbb{P}_k\{X_n = \ell, Y_n \in B\} = Q^{*n}(k, \{\ell\} \times B) \quad (20)$$

Then, the Theorem 2.2 for the density process of Y_n/\sqrt{n} could be specified to this specific class of MAPs. Since we only have to replace time t by n in all the material developed in Section 2, we omit the details. Note that the only simplification in assumptions of Theorem 2.2 is on the uniform moment condition **(M α)** which reduces to: the r.v. Y_1 satisfies the following moment condition of order α :

$$M_\alpha := \max_{k \in \mathbb{X}} \mathbb{E}_k[\|Y_1\|^\alpha] < \infty. \quad (21)$$

We will only illustrate our main result on the MRP embedded in a Markovian Arrival Process (e.g. see [Asm03, XI.1]).

Recall that a N -state Markovian Arrival Process is a continuous-time MAP $\{(J_t, N_t)\}_{t \geq 0}$ on the state space $\{1, \dots, N\} \times \mathbb{N}$, where N_t represents the number of arrivals up to time t , while the states of the driving Markov process $\{J_t\}_{t \geq 0}$ are called phases. Let Y_n be the time at the n th arrival ($Y_0 = 0$ a.s.) and let X_n be the state of the driving process just after the n th arrival. Then $\{(X_n, Y_n)\}_{n \in \mathbb{N}}$ is known to be an MRP with the following semi-Markov kernel Q on $\{1, \dots, N\} \times \mathbb{R}$: for any $(k, \ell) \in \mathbb{X}^2$

$$Q(k; \{\ell\} \times dy) := (e^{yD_0} D_1)(k, \ell) 1_{(0, \infty)}(y) dy = e_k e^{yD_0} D_1 e_\ell^\top 1_{(0, \infty)}(y) dy \quad (22)$$

which is parametrized by a pair of $N \times N$ -matrices usually denoted by D_0 and D_1 . Such a process is also an instance of Markov Random Walk. The matrix $D_0 + D_1$ is the infinitesimal generator of the background Markov process $\{J_t\}_{t \geq 0}$. Matrix D_0 is always assumed to be stable and $D_0 + D_1$ to be irreducible. In this case, $\{J_t\}_{t \geq 0}$ has a unique invariant probability measure denoted by π . The process $\{X_n\}_{n \in \mathbb{N}}$ is a Markov chain with state space $\mathbb{X} := \{1, \dots, N\}$ and transition probability matrix P :

$$\forall (k, \ell) \in \mathbb{X}^2, \quad P(k, \ell) = Q(k; \{\ell\} \times \mathbb{R}) = ((-D_0)^{-1} D_1)(k, \ell). \quad (23)$$

This Markov chain has an invariant probability measure ϕ (different of π). From (22), the bounded positive measure $\mathcal{Y}_{k, \ell, 1}$ is absolutely continuous with respect to the Lebesgue measure

on \mathbb{R} with density $g_{k,\ell,1}(y) = e_k e^{yD_0} D_1 e^\top 1_{(0,\infty)}(y)$. Then matrix G_t defined in (6b) has the form

$$G_1(y) = e^{yD_0} D_1 1_{(0,\infty)}(y).$$

For any $n \geq 2$, the positive measure $\mathcal{Y}_{k,\ell,n}$ in (20) is absolutely continuous with respect to the Lebesgue measure with density given by

$$\forall y \in \mathbb{R}, \quad G_n(y) = (G_{n-1} \star G_1)(y) =: \left(\sum_{j \in \mathbb{X}} (G_{k,j,1} \star G_{j,\ell,n-1})(y) \right)_{k,\ell \in \mathbb{X}^2}. \quad (24)$$

Let us check the assumptions of Theorem 2.2. Assumption **(I-A)** on P in (23) is standard in the literature on the Markov Arrival Process. It is well known that Y_1 has a moment of order 3 given by $E_k[(Y_1)^3] = 3! e_k (-D_0)^{-3} \mathbf{1}^\top$, so that Condition (21) holds with $\alpha = 3$. Let us give some details for checking Conditions **(AC1)**-**(AC2)**.

(AC2) : The open convex \mathcal{D}_n involved in **(AC2)** is given for any $n \geq 1$ by

$$\mathcal{D}_n :=]0, \infty[, \quad \text{with } \overline{\mathcal{D}}_n = [0, \infty), \quad \partial \mathcal{D}_n = \{0\}.$$

It is clear that $G_1(y)$ is continuous on $\overline{\mathcal{D}}_1$ and differentiable on \mathcal{D}_1 with differential

$$\forall y > 0, \quad \frac{dG_1}{dy}(y) = D_0 e^{yD_0} D_1. \quad (25)$$

Since G_n is obtained from convolution product of G_1 , the same properties are also valid for G_n . Next, we prove by induction that

$$\sup_{n > 0} \sup_{y \in [0, +\infty)} \|G_n(y)\|_0 \leq N \|D_1\|_0. \quad (26)$$

For $n := 1$, we have from norm equivalence (1) and $\|e^{D_0}\|_\infty \leq 1$ that

$$\begin{aligned} \forall y \in [0, +\infty), \quad \|G_1(y)\|_0 &= \|e^{yD_0} D_1\|_0 \leq \|e^{yD_0} D_1\|_\infty \leq \|e^{yD_0}\|_\infty \|D_1\|_\infty \\ &\leq N \|D_1\|_0. \end{aligned}$$

Using the definition (24) of $G_n(y)$ and the induction hypothesis, we have

$$\begin{aligned} |G_{k,\ell,n}(y)| &= \sum_j \int_{\mathbb{R}} |G_{k,j,1}(u)| |G_{j,\ell,n-1}(t-u)| du \leq N \|D_1\|_0 \int_{\mathbb{R}} G_{k,j,1}(u) du \\ &\leq N \|D_1\|_0 \mathbb{P}_k\{X_1 = \ell\} \leq N \|D_1\|_0. \end{aligned}$$

The proof of estimate (26) is complete.

Second, we easily obtain from (25) that

$$\sup_{y \in (0, +\infty)} \left\| \frac{dG_1}{dy}(y) \right\|_0 \leq (N \max(\|D_0\|_0, \|D_1\|_0))^2$$

Next, using an induction, we can obtain from the following well known property of the convolution product

$$\frac{dG_n}{dy}(y) = \left(G_1 \star \frac{dG_{n-1}}{dy} \right)(y)$$

that

$$\sup_{n>0} \sup_{y \in (0, +\infty)} \left\| \frac{dG_n}{dy}(y) \right\|_0 \leq (N \max(\|D_0\|_0, \|D_1\|_0))^2.$$

Finally, $G_1(0) = D_1$ and $G_n(0) = 0$ for any $n \geq 2$.

(AC1) : First, observe that $\mu_{k,\ell,n}$ is 0 for any $(k, \ell) \in \mathbb{X}^2$ and $n \geq 1$. Thus (7) is satisfied for $\rho = 0$. In Condition (AC1), we only have to check that

$$\|\widehat{G}_{n_0}(\zeta)\|_0 \longrightarrow 0 \quad \text{when } |\zeta| \rightarrow +\infty$$

for n_0 large enough. It follows from the convolution definition (24) of G_n that

$$\forall n \in \mathbb{N}, \forall \zeta \in \mathbb{R}, \quad \widehat{G}_n(\zeta) := (\mathbb{E}_k[1_{\{X_n=\ell\}} e^{i(\zeta, Y_n)}])_{(k,\ell) \in \mathbb{X}^2} = \widehat{G}_1(\zeta)^n$$

where

$$\widehat{G}_1(\zeta) = \int_0^{+\infty} e^{i\zeta y} e^{D_0 y} dy D_1$$

is integrable. In fact, using an integration by parts, we obtain that

$$\forall \zeta \neq 0, \quad \widehat{G}_1(\zeta) = \frac{-1}{i\zeta} [D_1 + \int_0^{+\infty} e^{i\zeta y} D_0 e^{y D_0} D_1 dy]$$

so that

$$\forall \zeta \neq 0, \quad \|\widehat{G}_1(\zeta)\|_0 \leq \frac{2\|D_1\|_0}{|\zeta|}.$$

Therefore, we deduce from norm equivalence (1) that for any integer $n_0 \geq 1$,

$$\|\widehat{G}_{n_0}(\zeta)\|_0 \leq \frac{(2N\|D_1\|_0)^{n_0}}{|\zeta|^{n_0}} \longrightarrow 0 \quad \text{when } |\zeta| \rightarrow +\infty.$$

5 Application to the local times of a jump process

5.1 The local limit theorem for the density of local times

Let $\{X_t\}_{t \geq 0}$ be a Markov jump process with finite state space $\mathbb{X} := \{1, \dots, N\}$ and generator G . Its transition semi-group is given by

$$\forall t \geq 0, \quad P_t := e^{tG}.$$

The local time $L_t(i)$ at time t associated with state $i \in \mathbb{X}$, or the sojourn time in state i on the interval $[0, t]$, is defined by

$$\forall t \geq 0, \quad L_t(i) := \int_0^t 1_{\{X_s=i\}} ds.$$

It is well known that $L_t(i)$ is an additive functional of $\{X_t\}_{t \geq 0}$ and that $\{(X_t, L_t(i))\}_{t \geq 0}$ is an MAP. In this section, we consider the MAP $\{(X_t, L_t)\}_{t \geq 0}$ where L_t is the random vector of the local times

$$L_t := (L_t(1), \dots, L_t(N)).$$

Note that, for all $t > 0$, we have $\langle L_t, \mathbf{1} \rangle = t$, that is L_t is \mathcal{S}_t -valued where

$$\mathcal{S}_t := \{y \in [0, +\infty)^N : \langle y, \mathbf{1} \rangle = t\}.$$

Let us assume that $\{X_t\}_{t \geq 0}$ has an invariant probability measure π . This happens when the generator G is irreducible. Set $m = (m_1, \dots, m_N) := \mathbb{E}_\pi[L_1]$. We define the $\mathcal{S}_t^{(0)}$ -valued centered r.v.

$$Y_t = L_t - tm$$

where $\mathcal{S}_t^{(0)} := T_{-tm}(\mathcal{S}_t)$ with the translation T_{-tm} by vector $-tm$ in \mathbb{R}^N . Note that $\mathcal{S}_t^{(0)}$ is a subset of the hyperplane H of \mathbb{R}^N defined by

$$H := \{y \in \mathbb{R}^N : \langle y, \mathbf{1} \rangle = 0\}. \quad (27)$$

Let Λ be the bijective map from H into \mathbb{R}^{N-1} defined by $\Lambda(y) := (y_1, \dots, y_{N-1})$ for $y := (y_1, \dots, y_N) \in H$. We introduce the following $(N-1)$ -dimensional random vector

$$Y'_t := \Lambda(Y_t) = (Y_t(1), \dots, Y_t(N-1)).$$

The following lemma follows from Lemma C.1.

Lemma 5.1 *The process $\{(X_t, Y'_t)\}_{t \geq 0}$ is a $\mathbb{X} \times \overline{\mathcal{D}}_t$ -valued MAP where \mathcal{D}_t is the open convex of \mathbb{R}^{N-1} defined by*

$$\mathcal{D}_t := \{y' \in \mathbb{R}^{N-1} : j = 1, \dots, N-1, y'_j \in (-m_j t, (1-m_j)t), \langle y', \mathbf{1} \rangle < m_N t\}. \quad (28)$$

Proposition 5.1 *If G and the sub-generators $G_{i^c i} := (G(k, \ell))_{k, \ell \in \{i\}^c}$, $i = 1, \dots, N$ are irreducible then the MAP $\{(X_t, Y'_t)\}_{t \geq 0}$ satisfies the conditions **(M3)** and **(AC1)-(AC2)**.*

Note that the matrix Σ in the CLT for $\{t^{-1/2}Y'_t\}_{t > 0}$ is invertible from [HL13, Remark 2.3] and that $O(\sup_{y \notin \mathcal{D}_t} \eta_\Sigma(t^{-1/2}y)) = O(t^{-1/2})$ from (28). Under the assumptions of Proposition 5.1 on the generator of $\{X_t\}_{t \geq 0}$, Theorem 2.2 gives that

$$\sup_{y' \in \mathbb{R}^{N-1}} |f'_{k,t}(y') - (2\pi)^{-(N-1)/2} (\det \Sigma)^{-1/2} e^{-\frac{1}{2} \langle y', \Sigma y' \rangle}| = O(t^{-1/2}),$$

where $f'_{k,t}$ is the density of the a.c. part of the probability distribution of $t^{-1/2}Y'_t$. An explicit form of Σ is provided in [HL13, Remark 3.1].

In order to obtain a statement in terms of Y_t , we can use the following lemma. Recall that ℓ_{N-1} denotes the Lebesgue measure on \mathbb{R}^{N-1} and $B(H)$ stands for the Borelian σ -algebra on H . Let ν be the measure defined on $(H, B(H))$ as the image measure of ℓ_{N-1} under Λ^{-1} , that is: for any positive and $B(H)$ -measurable function $\psi : H \rightarrow \mathbb{R}$,

$$\int_H \psi(y) \nu(dy) := \int_{\mathbb{R}^{N-1}} \psi(\Lambda^{-1}y') dy'.$$

Using the bijection Λ , the following lemma can be easily proved (see Appendix A).

Lemma 5.2 *For any $(k, \ell) \in \mathbb{X}^2$, the a.c. and singular parts of the Lebesgue decomposition of the measure $\mathbb{P}_k\{X_t = \ell, Y_t \in \cdot\}$ on H with respect to ν are obtained from image measure under Λ of the a.c. and singular parts of the Lebesgue decomposition of the measure $\mathbb{P}_k\{X_t = \ell, Y'_t \in \cdot\}$ on \mathbb{R}^{N-1} with respect to ℓ_{N-1} .*

Therefore we know that the density, say $f_{k,t}$, of the a.c. part of the probability distribution of

$$t^{-1/2}Y_t = t^{-1/2}(L_t - tm)$$

with respect to the measure ν on the hyperplane H is given by

$$\forall h \in H, \quad f_{k,t}(h) := f'_{k,t}(\Lambda h).$$

Finally, we have obtained the following local limit theorem for $f_{k,t}$.

Proposition 5.2 ([HL13, Prop. 3.1]) *If G and the sub-generators $G_{i^c i^c} := (G(k, \ell))_{k, \ell \in \{i\}^c}$, $i = 1, \dots, N$ are irreducible, then the density $f_{k,t}$ of the a.c. part of the probability distribution of $t^{-1/2}(L_t - tm)$ under \mathbb{P}_k satisfies*

$$\sup_{h \in H} |f_{k,t}(h) - (2\pi)^{-(N-1)/2} (\det \Sigma)^{-1/2} e^{-\frac{1}{2} \langle \Lambda h^\top, \Sigma \Lambda h^\top \rangle}| = O(t^{-1/2}).$$

It remains to prove that Proposition 5.1 holds true.

5.2 The main lines of the derivation of Proposition 5.1

Let us introduce the randomized Markov chain $\{Z_n\}_{n \in \mathbb{N}}$ with state space \mathbb{X} and transition matrix

$$\tilde{P} := I + G/a \quad \text{with } a > \max(|G(j, j)|, j \in \mathbb{X}).$$

Since G is assumed to be irreducible, the transition matrix \tilde{P} is irreducible and aperiodic. Moreover, since $\tilde{P}(i, i) > 0$ for any $i \in \mathbb{X}$, the sub-stochastic matrix $\tilde{P}_{i^c i^c} := (\tilde{P}(k, \ell))_{k, \ell \in \{i\}^c}$ is aperiodic.

Let us define the following convex open set of \mathbb{R}^{N-1}

$$\forall t > 0, \quad \mathcal{C}_t := \{y \in]0, t[^{N-1}, \langle y, \mathbf{1} \rangle < t\}.$$

For any $t > 0$, the adherence of \mathcal{C}_t is denoted by $\bar{\mathcal{C}}_t$ and is $\bar{\mathcal{C}}_t := \{y \in [0, t]^{N-1}, \langle y, \mathbf{1} \rangle \leq t\}$ and the boundary of \mathcal{C}_t is $\partial \mathcal{C}_t := \bar{\mathcal{C}}_t \setminus \mathcal{C}_t$ given by

$$\partial \mathcal{C}_t = \bigcup_{i=1}^N \{y = (y_1, \dots, y_{N-1}) \in \bar{\mathcal{C}}_t \mid y_i := 0\} \cup \{y \in \bar{\mathcal{C}}_t \mid \langle y, \mathbf{1} \rangle = t\}. \quad (29)$$

The joint conditional distribution of $(X_t, L'_t) := (X_t, L_t(1), \dots, L_t(N-1))$ under \mathbb{P}_k is 0 on $\mathbb{R}^{N-1} \setminus \bar{\mathcal{C}}_t$ and its a.c. part has the following density density $\psi_{k, \ell, t}$ from [Ser99, Cor. 4.4]

$$\begin{aligned} \forall (k, \ell) \in \mathbb{X}^2, \forall y \in \mathbb{R}^{N-1}, \quad \psi_{k, \ell, t}(y) &:= 1_{\mathcal{C}_t}(y) a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \\ &\times \sum_{\substack{k_1 \geq 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n}} x_{n; k_1, \dots, k_{N-1}}^{t; y} p_{k, \ell}(n + N, k_1, \dots, k_{N-1}) \end{aligned} \quad (30)$$

where the non-negative coefficient $p_{k, \ell}(n + N, k_1, \dots, k_{N-1})$ satisfies $0 \leq \sum_{\ell=1}^{N-1} p_{k, \ell}(n + N, k_1, \dots, k_{N-1}) \leq 1$ and we have the following relation

$$\sum_{\substack{k_2 \geq 0, \dots, k_N \geq 0, \\ \sum_{j=2}^N k_j \leq n}} n! x_{n; k_2, \dots, k_N}^{t; y_2, \dots, y_N} = 1 \quad (31)$$

For every $(k, \ell) \in \mathbb{X}^2$, the Lebesgue decomposition of the positive measure $\mathcal{F}_{k, \ell, t}$ defined by $\mathbb{P}_k\{X_t = \ell, L'_t \in \cdot\}$ writes as follows

$$\forall B \in \mathcal{B}(\mathbb{R}^{N-1}), \quad \mathcal{F}_{k, \ell, t}(1_B) = \mathbb{P}_k\{X_t = \ell, L'_t \in B\} = \int_B \psi_{k, \ell, t}(y) dy + \alpha_{k, \ell, t}(1_B). \quad (32)$$

First note that Condition **(I-A)** is fulfilled since the matrix $P := P_1 = e^G$ is irreducible and aperiodic when the generator G of the driving jump process is irreducible. Second, the random variables $\|L_t\|$ (and so $\|Y_t\|$) are bounded, so that the moment condition **(M α)** is satisfied for any $\alpha > 0$. This allows us to apply Theorem 2.1 to derive a CLT for $\{t^{-1/2}Y'_t\}_{t \geq 0}$. Next, the main steps of the proof are to check that Conditions **(AC1)**-**(AC2)** hold under the assumptions of Proposition 5.1 on the generator G .

5.2.1 Checking Condition **(AC2)**

For every $y \in \mathbb{R}^{N-1}$, we introduce the following real $N \times N$ -matrix

$$\Psi_t(y) := (\psi_{k, \ell, t}(y))_{(k, \ell) \in \mathbb{X}^2}.$$

where $\psi_{k, \ell, t}$ is defined in (30). The useful properties of $\psi_{k, \ell, t}$ are given in the next lemma which is proved in Appendix B.

Lemma 5.3 *For any $t > 0$, Ψ_t vanishes on $\mathbb{R}^d \setminus \bar{\mathcal{C}}_t$, is continuous on $\bar{\mathcal{C}}_t$ and differentiable on \mathcal{C}_t . We have*

$$\sup_{t > 0} \sup_{y \in \mathbb{R}^{N-1}} \|\Psi_t(y)\|_0 \leq \sup_{t > 0} \sup_{y \in \mathbb{R}^{N-1}} \|\Psi_t(y) \mathbf{1}^\top\|_0 \leq a^{N-1}; \quad (33a)$$

$$j = 1, \dots, N-1 \quad \sup_{t > 0} \sup_{y \in \mathcal{C}_t} \left\| \frac{\partial \Psi_t}{\partial y_j}(y) \right\|_0 \leq 2a^N. \quad (33b)$$

There exists $\rho \in (0, 1)$ such that

$$\forall y \in \partial \mathcal{C}_t, \quad \|\Psi_t(y)\|_0 = O(e^{at(\rho-1)}(1+at)). \quad (33c)$$

Now, let us check that Conditions **(AC2)** hold for the function

$$G_t(y) = \Psi_t(y + m't) \quad (34)$$

associated with the absolutely continuous part of the probability distribution of the centered r.v. $(X_t, Y'_t) = (X_t, L'_t - m't)$ under \mathbb{P}_k and $m' := (\pi_1, \dots, \pi_{N-1})$. It vanishes on the open convex $\mathcal{D}_t := T_{-m't}(\mathcal{C}_t)$ which is the translate of \mathcal{C}_t by the translation $T_{-m't}$ of vector $-m't$. More precisely,

$$\mathcal{D}_t = \{y \in \mathbb{R}^{N-1} : j = 1, \dots, N-1, y_j \in (-m_j t, (1-m_j)t), \langle y, \mathbf{1} \rangle < m_N t\}.$$

with adherence

$$\bar{\mathcal{D}}_t = \{y \in \mathbb{R}^{N-1} : j = 1, \dots, N-1, y_j \in [-m_j t, (1-m_j)t], \langle y, \mathbf{1} \rangle \leq m_N t\}. \quad (35a)$$

and boundary

$$\partial \mathcal{D}_t = \bigcup_{j=1}^{N-1} \{y \in \bar{\mathcal{D}}_t \mid y_j = -m_j t\} \cup \{y \in \bar{\mathcal{D}}_t : \langle y, \mathbf{1} \rangle = m_N t\}. \quad (35b)$$

Then it follows from the basic properties of function Ψ_t on \mathcal{C}_t stated in Lemma 5.3 that G_t is continuous on $\overline{\mathcal{D}}_t$ and differentiable on \mathcal{D}_t with differential

$$\forall j = 1, \dots, N-1, \forall y \in \mathcal{D}_t, \quad \frac{\partial G_t}{\partial y_j}(y) = \frac{\partial \Psi_t}{\partial y_j}(y + m't).$$

Moreover, we obtain from (33a)-(33c)

$$\begin{aligned} \sup_{t>0} \sup_{y \in \mathbb{R}^{N-1}} \|G_t(y)\|_0 &\leq \sup_{t>0} \sup_{y \in \mathbb{R}^{N-1}} \|G_t(y)\mathbf{1}^\top\|_0 \leq \sup_{t>0} \sup_{y \in \mathbb{R}^{N-1}} \|\Psi_t(y)\mathbf{1}^\top\|_0 \leq a^{N-1}; \\ \forall y \in \partial\mathcal{D}_t, \quad \|G_t(y)\|_0 &= \|\Psi_t(y + m't)\|_0 = O\left(\frac{1}{t}\right); \\ \sup_{t>0} \sup_{y \in \mathcal{D}_t} \left\| \frac{\partial G_t}{\partial y}(y) \right\|_0 &\leq \sup_{t>0} \sup_{y \in \mathcal{C}_t} \left\| \frac{\partial \Psi_t}{\partial y}(y) \right\|_0 \leq 2a^N. \end{aligned}$$

5.2.2 Checking Condition (AC1)

For any $i \in \{1, \dots, N\}$, we introduce the following $(N-1) \times (N-1)$ -subgenerator of G , $G_{i^c i^c} := (G(k, \ell))_{k, \ell \in \{i\}^c}$. If $G_{i^c i^c}$ is irreducible then $\|e^{tG_{i^c i^c}}\|_0 = O(e^{-r_i t})$ where $-r_i$ is the Perron-Frobenius negative eigenvalue of $G_{i^c i^c}$. Thus, if $G_{i^c i^c}$ is irreducible for any $i \in \{1, \dots, N\}$

$$\max_{i \in \{1, \dots, N\}} \|e^{tG_{i^c i^c}}\|_0 = O(e^{-rt}) \quad \text{where } r := \min_i (r_i) > 0. \quad (36)$$

In a first step, we study the singular part \mathcal{A}_t of the Lebesgue decomposition (32) where \mathcal{A}_t is the $N \times N$ -matrix with entries in the set of bounded positive measures on \mathbb{R}^{N-1}

$$\mathcal{A}_t := (\alpha_{k, \ell, t})_{(k, \ell) \in \mathbb{X}^2}.$$

We show that $\|\mathcal{A}_t(1_{\mathbb{R}^{N-1}})\|_0$ goes to 0 at a geometric rate when t grows to infinity. Note that $\mathcal{A}_t(1_B) = 0$ for every $B \in B(\mathbb{R}^{N-1})$ such that $B \cap \partial\mathcal{C}_t = \emptyset$. Next, it remains to show that there exist $c > 0$ and $\rho \in (0, 1)$ such that

$$\forall t > 0, \quad \|\mathcal{A}_t(1_{\partial\mathcal{C}_t})\|_0 \leq c\rho^t.$$

First, let us consider, for any $i \in \{1, \dots, N-1\}$, the set $d_{i,t} := \{(y_1, \dots, y_{N-1}) \in \overline{\mathcal{C}}_t \mid y_i = 0\}$:

$$\alpha_{k, \ell, t}(1_{d_{i,t}}) \leq \mathbb{P}_k\{L_t(i) = 0\} = \begin{cases} 0 & \text{if } k = i \\ \sum_{\ell} e^{tG_{i^c i^c}}(k, \ell) & \text{if } k \neq i. \end{cases}$$

Thus, $\max_i \|\mathcal{A}_t(1_{d_{i,t}})\|_0 = O(e^{-rt})$ from (36). Second let us denote $s_t := \{(y_1, \dots, y_{N-1}) \in \overline{\mathcal{C}}_t \mid \sum_{j=1}^{N-1} y_j = t\}$. We can write for all $(k, \ell) \in \mathbb{X}^2$

$$\begin{aligned} \alpha_{k, \ell, t}(1_{s_t}) &\leq \mathbb{P}_k\{L_t(N) = 0, X_t = \ell\} \\ &\leq \mathbb{P}_k\{L_t(N) = 0\} = \begin{cases} 0 & \text{if } k = N \\ \sum_{\ell} e^{tG_{N^c N^c}}(k, \ell) & \text{if } k \neq N. \end{cases} \end{aligned}$$

Therefore, $\|\mathcal{A}_t(1_{s_t})\|_0 = O(e^{-rt})$. Combining the previous estimates, we obtain that there exist $c > 0$ and $\rho \in (0, 1)$ such that

$$\forall t > 0, \quad \|\mathcal{A}_t(1_{\mathbb{R}^{N-1}})\|_0 = \|\mathcal{A}_t(1_{\partial\mathcal{C}_t})\|_0 \leq c\rho^t.$$

It follows that there exist $c > 0$ and $\rho \in (0, 1)$ such that

$$\forall t > 0, \quad \|\mathcal{M}_t(1_{\mathbb{R}^{N-1}})\|_0 = \|\mathcal{A}_t(1_{\mathbb{R}^{N-1}})\|_0 \leq c\rho^t.$$

where \mathcal{M}_t is the matrix associated with the singular part of the probability distribution of (X_t, Y_t') .

In a second step, we prove that for every $t_0 > 0$

$$\sup_{t \in [t_0, 2t_0)} \|\widehat{\Psi}_t(\zeta)\|_0 \longrightarrow 0 \quad \text{when } \|\zeta\| \rightarrow +\infty.$$

Indeed, for every $t > 0$, for every $(k, \ell) \in \mathbb{X}^2$, the Fourier transform $\widehat{\psi}_{k, \ell, t}$ of $\psi_{k, \ell, t}$ has the following form (if $N = 2$ consider only the first integral)

$$\widehat{\psi}_{k, \ell, t}(\zeta) = \int_0^t \int_0^{t-y_1} \dots \int_0^{t-\sum_{j \leq N-2} y_j} \psi_{k, \ell, t}(y) e^{i\zeta_{N-1} y_{N-1}} dy_{N-1} \dots dy_2 dy_1.$$

Using an integration by part, we obtain that for every $\zeta_{N-1} \neq 0$

$$\begin{aligned} & \int_0^{t-\sum_{j \leq N-2} y_j} \psi_{k, \ell, t}(y) e^{i\zeta_{N-1} y_{N-1}} dy_{N-1} = \\ & \frac{1}{i\zeta_{N-1}} \left(\left[\psi_{k, \ell, t}(y) e^{i\zeta_{N-1} y_{N-1}} \right]_{y_{N-1}=0}^{y_{N-1}=t-\sum_{j \leq N-2} y_j} - \int_0^{t-\sum_{j \leq N-2} y_j} \frac{\partial \psi_{k, \ell, t}}{\partial y_{N-1}}(y) e^{iy_{N-1}\zeta_{N-1}} dy_{N-1} \right) \end{aligned}$$

so that, we obtain from the bounds (33a)-(33b) that

$$\begin{aligned} |\widehat{\psi}_{k, \ell, t}(\zeta)| & \leq \frac{1}{|\zeta_{N-1}|} \int_0^t \dots \int_0^{t-\sum_{j \leq N-3} y_j} (2a^{N-1} + 2a^N(t - \sum_{j \leq N-2} y_j)) dy_{N-2} \dots dy_1 \\ & \leq \frac{1}{|\zeta_{N-1}|} \left(2a^{N-1} \frac{t^{N-2}}{(N-2)!} + 2a^N \frac{t^{N-1}}{(N-1)!} \right). \end{aligned}$$

Finally, for every $t_0 > 0$

$$\sup_{t \in [t_0, 2t_0)} \|\widehat{\Psi}_t(\zeta)\|_0 \leq \frac{2a^{N-1}(1+a) \max(1, t_0^N)}{|\zeta_{N-1}|}.$$

For each $j \in \{1, \dots, y_{N-2}\}$, using similar computations from an integration by parts with respect to the variable y_j , the same inequality holds with ζ_j in place of ζ_{N-1} . Therefore, we obtain that

$$\forall \zeta \in \mathbb{R}^{N-1} \setminus \{0\}, \quad \sup_{t \in [t_0, 2t_0)} \|\widehat{\Psi}_t(\zeta)\|_0 \leq \frac{2a^{N-1}(1+a) \max(1, t_0^N)}{\|\zeta\|_0}.$$

and this quantity goes to 0 when $\|\zeta\| \rightarrow +\infty$. Since $G_t(y) = \Psi_t(y + tm)$ we have $\widehat{G}_t(\zeta) = e^{-i\langle m', \zeta \rangle t} \widehat{\Psi}_t(\zeta)$ for any $\zeta \in \mathbb{R}^{N-1}$ and

$$\forall t > 0, \quad \forall \zeta \in \mathbb{R}^{N-1}, \quad \|\widehat{G}_t(\zeta)\|_0 = \|\widehat{\Psi}_t(\zeta)\|_0.$$

It follows that

$$\Gamma(\zeta) \equiv \Gamma_{t_0}(\zeta) := \sup_{t \in [t_0, 2t_0)} \|\widehat{G}_t(\zeta)\|_0 \longrightarrow 0 \quad \text{when } \|\zeta\|_0 \rightarrow +\infty.$$

6 A uniform LLT with respect to transition matrix P

Let \mathcal{P} denote the set of irreducible and aperiodic stochastic $N \times N$ -matrices. The topology in \mathcal{P} is associated (for instance) with the distance $d(P, P') := \|P - P'\|_\infty$ ($P, P' \in \mathcal{P}$). Again $(X_t, Y_t)_{t \in \mathbb{T}}$ is a centered MAP with state space $\mathbb{X} \times \mathbb{R}^d$, where $\mathbb{X} := \{1, \dots, N\}$, and $\{P_t\}_{t \in \mathbb{T}}$ denotes the transition semi-group of the Markov process $\{X_t\}_{t \in \mathbb{T}}$. In this section, the stochastic matrix $P := P_1$ is assumed to belong to a compact subset \mathcal{P}_0 of \mathcal{P} , and we give assumptions for the LLT of Theorem 2.2 to hold uniformly in $P \in \mathcal{P}_0$.

For every $k \in \mathbb{X}$ the underlying probability measure \mathbb{P}_k and the associated expectation \mathbb{E}_k depend on P . To keep in mind this dependence, they are denoted by \mathbb{P}_k^P and \mathbb{E}_k^P respectively. Similarly we use the notations Σ^P , \mathcal{Y}_t^P , \mathcal{G}_t^P , \mathcal{M}_t^P and G_t^P for the covariance matrix of Theorem 2.1 and the matrices in (6a)-(6b) respectively. Let \mathfrak{M} denote the space of the probability measures on \mathbb{R}^d equipped with the total variation distance d_{TV} .

Let \mathcal{P}_0 be any compact subset of \mathcal{P} . Let us consider the following assumptions:

U1 : For any $(k, \ell) \in \mathbb{X}^2$, the map $P \mapsto \mathcal{Y}_{k, \ell, 1}^P$ is continuous from (\mathcal{P}_0, d) into (\mathfrak{M}, d_{TV}) .

U2 : There exist positive constants α and β such that

$$\forall P \in \mathcal{P}_0, \forall \zeta \in \mathbb{R}^d, \quad \alpha \|\zeta\|^2 \leq \langle \zeta, \Sigma^P \zeta \rangle \leq \beta \|\zeta\|^2. \quad (37)$$

U3 : The conditions **(M3)**, **(AC1)** and **(AC2)** hold uniformly in $P \in \mathcal{P}_0$.

Theorem 6.1 Under Assumptions (U1)-(U3), for every $k \in \mathbb{X}$, the density $f_{k,t}^P(\cdot)$ of the a.c. part of the probability distribution of $t^{-1/2}Y_t$ under \mathbb{P}_k^P satisfies the following asymptotic property when $t \rightarrow +\infty$:

$$\sup_{P \in \mathcal{P}_0} \sup_{y \in \mathbb{R}^d} |f_{k,t}^P(y) - \eta_{\Sigma^P}(y)| = O(t^{-1/2}) + O\left(\sup_{P \in \mathcal{P}_0} \sup_{y \notin \mathcal{D}_t} \eta_{\Sigma^P}(t^{-1/2}y)\right).$$

Assumptions (U3) read as follows:

U3-1 : The r.v. $\{Y_v\}_{v \in (0,1] \cap \mathbb{T}}$ satisfies the following moment condition:

$$M := \sup_{P \in \mathcal{P}_0} \max_{k \in \mathbb{X}} \sup_{v \in (0,1] \cap \mathbb{T}} \mathbb{E}_k^P[\|Y_v\|^3] < \infty. \quad (38)$$

U3-2 : There exist $c > 0$ and $\rho \in (0, 1)$ such that

$$\forall t > 0, \quad \sup_{P \in \mathcal{P}_0} \|\mathcal{M}_t^P(1_{\mathbb{R}^d})\|_0 \leq c\rho^t \quad (39)$$

and there exists $t_0 > 0$ such that

$$\rho^{t_0} \max(2, cN) \leq 1/4 \quad (40a)$$

$$\Gamma_{t_0}(\zeta) := \sup_{P \in \mathcal{P}_0} \sup_{w \in [t_0, 2t_0]} \|\widehat{G}_w^P(\zeta)\|_0 \rightarrow 0 \quad \text{when } \|\zeta\| \rightarrow +\infty. \quad (40b)$$

U3-3 : For any $t > 0$, there exists an open convex subset \mathcal{D}_t of \mathbb{R}^d such that G_t vanishes on $\mathbb{R}^d \setminus \overline{\mathcal{D}_t}$, where $\overline{\mathcal{D}_t}$ denotes the adherence of \mathcal{D}_t . Moreover G_t is continuous on $\overline{\mathcal{D}_t}$ and differentiable on \mathcal{D}_t , with in addition

$$\sup_{P \in \mathcal{P}_0} \sup_{t > 0} \sup_{y \in \overline{\mathcal{D}_t}} \|G_t^P(y)\|_0 < \infty \quad (41a)$$

$$\sup_{P \in \mathcal{P}_0} \sup_{y \in \partial \mathcal{D}_t} \|G_t^P(y)\|_0 = O\left(\frac{1}{t}\right) \quad \text{where } \partial \mathcal{D}_t := \overline{\mathcal{D}_t} \setminus \mathcal{D}_t \quad (41b)$$

$$j = 1, \dots, d : \quad \sup_{P \in \mathcal{P}_0} \sup_{t > 0} \sup_{y \in \mathcal{D}_t} \left\| \frac{\partial G_t^P}{\partial y_j}(y) \right\|_0 < \infty. \quad (41c)$$

Proof. The proof of Theorem 6.1 borrows the same way as for Theorem 2.2. What we only have to do is to prove that the bounds in Lemmas 3.1-3.3 are uniform in $P \in \mathcal{P}_0$ under Assumptions (U1)-(U2). For Lemma 3.3, this is obvious by using (U3-2). The proof of Theorem 6.1 will be complete if we establish the uniform version of Lemma 3.1 and 3.2. This is done below. \square

Lemma 6.1 Assume that Condition (U1) holds and that, for some $\alpha > 0$,

$$M_\alpha := \sup_{P \in \mathcal{P}_0} \max_{k \in \mathbb{X}} \sup_{v \in (0,1] \cap \mathbb{T}} \mathbb{E}_k^P[\|Y_v\|^\alpha] < \infty.$$

Then the map $(P, \zeta) \mapsto \widehat{\mathcal{Y}}_1^P(\zeta)$ is continuous from $\mathcal{P}_0 \times \mathbb{R}^d$ into $\mathcal{M}_N(\mathbb{C})$.

Proof. We may suppose that $\alpha \in (0, 1]$. Let $(k, \ell) \in \mathbb{X}^2$, $(P, P') \in \mathcal{P}_0^2$ and $(\zeta, \zeta') \in \mathbb{R}^d \times \mathbb{R}^d$. To simplify we write \mathcal{Y} and \mathcal{Y}' for $\mathcal{Y}_{k,\ell,1}^P$ and $\mathcal{Y}_{k,\ell,1}^{P'}$. Then

$$\begin{aligned} |(\widehat{\mathcal{Y}}_1^P(\zeta))_{k,\ell} - (\widehat{\mathcal{Y}}_1^{P'}(\zeta'))_{k,\ell}| &= \left| \int_{\mathbb{R}^d} e^{i\langle \zeta, y \rangle} \mathcal{Y}(dy) - \int_{\mathbb{R}^d} e^{i\langle \zeta', y \rangle} \mathcal{Y}'(dy) \right| \\ &\leq \int_{\mathbb{R}^d} |e^{i\langle \zeta, y \rangle} - e^{i\langle \zeta', y \rangle}| \mathcal{Y}(dy) \\ &\quad + \left| \int_{\mathbb{R}^d} e^{i\langle \zeta', y \rangle} \mathcal{Y}(dy) - \int_{\mathbb{R}^d} e^{i\langle \zeta', y \rangle} \mathcal{Y}'(dy) \right| \\ &\leq 2 \|\zeta - \zeta'\|^\alpha \int_{\mathbb{R}^d} \|y\|^\alpha \mathcal{Y}(dy) + d_{TV}(\mathcal{Y}, \mathcal{Y}') \\ &\leq 2M_\alpha \|\zeta - \zeta'\|^\alpha + d_{TV}(\mathcal{Y}, \mathcal{Y}') \end{aligned}$$

(use $|e^{iu} - 1| \leq 2|u|^\alpha$, $u \in \mathbb{R}$, and the Cauchy-Schwarz inequality to obtain the second inequality above). The desired continuity property then follows from (U1). \square

From now on, we sometimes omit the notational exponent P . The uniformity in Lemma 3.1 is obtained as follows. Recall that, for any $P \in \mathcal{P}_0$ fixed, Formula (15) with $t = n \in \mathbb{N}$ follows from the standard perturbation theory. Similarly, using Lemma 6.1, Formula (15) with $t = n \in \mathbb{N}$ can be obtained for every $P \in \mathcal{U}_0$ and for all $\zeta \in \mathbb{R}^d$ such that $\|\zeta\| \leq \delta$, with $\delta > 0$ independent from $P_0 \in \mathcal{P}_0$ since \mathcal{P}_0 is compact. Moreover the associated functions $\lambda(\cdot)$, $L_{k,t}(\cdot)$ and $R_{k,t}(\cdot)$ in (15) (depending on P) satisfy the properties (16a)-(16d) in a uniform way in $P \in \mathcal{P}_0$ from (U3-1). Note that Condition (U2) is useful to obtain (16b) uniformly in $P \in \mathcal{P}_0$. The passage from the discrete-time case to the continuous-time again follows from [FHL12, Prop. 4.4] since the derivatives of $\zeta \mapsto \widehat{\mathcal{Y}}_v(\zeta)$ are uniformly bounded in $(P, v) \in \mathcal{P}_0 \times (0, 1]$ from (U3-1).

From $\phi_{k,t}(\zeta) = e_k \widehat{\mathcal{Y}}_1(\zeta)^{\lfloor t \rfloor} \widehat{\mathcal{Y}}_v(\zeta) \mathbf{1}^\top$ with $v := t - \lfloor t \rfloor \in [0, 1)$ and $\|\widehat{\mathcal{Y}}_v(\zeta) \mathbf{1}\|_\infty \leq \|\widehat{\mathcal{Y}}_v(0)\|_\infty \leq 1$, the uniformity in Lemma 3.2 follows from the following lemma.

Lemma 6.2 *Let δ, A be such that $0 < \delta < A$. Then there exist $D \equiv D(\delta, A) > 0$ and $\tau \equiv \tau(\delta, A) \in (0, 1)$ such that*

$$\forall n \in \mathbb{N}, \quad \sup_{P \in \mathcal{P}_0} \sup_{\delta \leq \|\zeta\| \leq A} \|\widehat{\mathcal{Y}}_1^P(\zeta)^n\|_\infty \leq D \tau^n. \quad (42)$$

Proof. The spectral radius of any matrix $T \in \mathcal{M}_N(\mathbb{C})$ is denoted by $r(T)$. Suppose that

$$\rho_0 := \sup_{P \in \mathcal{P}_0} \sup_{\delta \leq \|\zeta\| \leq A} r(\widehat{\mathcal{Y}}_1^P(\zeta)) < 1. \quad (43)$$

Then (42) holds. Indeed, consider any $\tau \in (\rho_0, 1)$ and denote by Γ_τ the oriented circle in \mathbb{C} centered at 0 with radius τ . Let I denote the identity $N \times N$ -matrix. Property (43) and standard spectral calculus then give

$$\begin{aligned} \sup_{P \in \mathcal{P}_0} \sup_{\delta \leq \|\zeta\| \leq A} \|\widehat{\mathcal{Y}}_1^P(\zeta)^n\|_\infty &\leq \sup_{P \in \mathcal{P}_0} \sup_{\delta \leq \|\zeta\| \leq A} \left\| \frac{1}{2i\pi} \oint_{\Gamma_\tau} z^n (zI - \widehat{\mathcal{Y}}_1^P(\zeta))^{-1} dz \right\|_\infty \\ &\leq \tau^{n+1} \sup_{P \in \mathcal{P}_0} \sup_{\delta \leq \|\zeta\| \leq A} \|(zI - \widehat{\mathcal{Y}}_1^P(\zeta))^{-1}\|_\infty. \end{aligned}$$

This proves (42) since the last bound is finite from the continuity of $(P, \zeta) \mapsto \widehat{\mathcal{Y}}_1^P(\zeta)$ on the compact set $\mathcal{P}_0 \times \{\zeta \in \mathbb{R}^d : \delta \leq \|\zeta\| \leq A\}$ (Lemma 6.1).

It remains to prove (43). Assume that $\rho_0 \geq 1$. Then $\rho_0 = 1$ since $r(\widehat{\mathcal{Y}}_1(\zeta)) \leq \|\widehat{\mathcal{Y}}_1(\zeta)\|_\infty \leq \|\widehat{\mathcal{Y}}_1(0)\|_\infty \leq 1$. Thus there exists some sequences $(P_n)_n \in \mathcal{P}_0^{\mathbb{N}}$ and $(\zeta_n)_n \in (\mathbb{R}^d)^{\mathbb{N}}$ satisfying $\delta \leq \|\zeta_n\| \leq A$ such that

$$\lim_n r(\widehat{\mathcal{Y}}_1^{P_n}(\zeta_n)) = 1$$

By compactness one may suppose that $(P_n)_n$ and $(\zeta_n)_n$ respectively converge to some $P_\infty \in \mathcal{P}_0$ and some $\zeta_\infty \in \mathbb{R}^d$ such that $\delta \leq \|\zeta_\infty\| \leq A$. From Lemma 6.1 we obtain that

$$\lim_n \widehat{\mathcal{Y}}_1^{P_n}(\zeta_n) = \widehat{\mathcal{Y}}_1^{P_\infty}(\zeta_\infty).$$

Below, $\widehat{\mathcal{Y}}_1^{P_\infty}(\zeta_\infty)$ is simply denoted by $\mathcal{Y}_1(\zeta_\infty)$. From the upper semi-continuity of the map $T \mapsto r(T)$ on $\mathcal{M}_N(\mathbb{C})$ and from $r(\widehat{\mathcal{Y}}_1(\zeta_\infty)) \leq 1$, it follows that $r(\widehat{\mathcal{Y}}_1(\zeta_\infty)) = 1$. Write $S := \widehat{\mathcal{Y}}_1(\zeta_\infty)$ to simplify. Then S admits an eigenvalue λ of modulus one. Let $f = (f(k))_{k \in \mathbb{X}} \in \mathbb{C}^N$ be an associated nonzero eigenvector. We have

$$\forall k \in \mathbb{X}, \quad |\lambda f(k)| = |f(k)| \leq |(Sf)(k)| \leq (\widehat{\mathcal{Y}}_1(0)|f|)(k) = (P|f|)(k),$$

where $|f| := (|f(k)|)_{k \in \mathbb{X}}$. Recall that the P_∞ -invariant probability measure $\pi = (\pi(\ell))_{\ell \in \mathbb{X}}$ is such that $\forall \ell \in \mathbb{X}$, $\pi(\ell) > 0$ since P_∞ is irreducible. From $\pi(P_\infty|f|) = \pi(|f|)$ and the positivity of $P_\infty|f| - |f|$, it follows that $P_\infty|f| = |f|$. Thus $|f| = c\mathbf{1}$ with some constant c . We may assume that $|f| = \mathbf{1}$. Equality $\lambda f = Sf$ rewrites as

$$\forall k \in \mathbb{X}, \quad \lambda f(k) = \sum_{\ell=1}^N f(\ell) \widehat{\mathcal{Y}}_1(\zeta_\infty)_{k,\ell} = \sum_{\ell=1}^N f(\ell) \mathbb{E}_k^{P_\infty} [1_{\{X_1=\ell\}} e^{i\langle \zeta_\infty, Y_1 \rangle}] = \mathbb{E}_k^{P_\infty} [f(X_1) e^{i\langle \zeta_\infty, Y_1 \rangle}].$$

From $|f(k)| = 1$ for every $k \in \mathbb{X}$ and from standard convexity arguments, we obtain

$$\forall k \in \mathbb{X}, \quad \lambda f(k) = f(X_1) e^{i\langle \zeta_\infty, Y_1 \rangle} \quad \mathbb{P}_k^{P_\infty} - \text{a.s.}$$

Now writing $\lambda = e^{ib}$ with $b \in \mathbb{R}$ and $f(\ell) = e^{ig(\ell)}$ for every $\ell \in \mathbb{X}$, one can deduce from the previous equality that

$$\forall k \in \mathbb{X}, \quad \langle \zeta_\infty, Y_1 \rangle + g(X_1) - g(k) \in b + 2\pi\mathbb{Z} \quad \mathbb{P}_k^{P_\infty} - \text{a.s.}$$

Define $a := \frac{b}{\|\zeta_\infty\|^2} \zeta_\infty \in \mathbb{R}^d$ and $\theta : \mathbb{X} \rightarrow \mathbb{R}^d$ by $\theta(\ell) := \frac{g(\ell)}{\|\zeta_\infty\|^2} \zeta_\infty$. Consider the following closed subgroup $H := \frac{2\pi\mathbb{Z}}{\|\zeta_\infty\|^2} \zeta_\infty \oplus (\mathbb{R} \cdot \zeta_\infty)^\perp$ in \mathbb{R}^d . Then the previous property is equivalent to

$$\forall k \in \mathbb{X}, \quad Y_1 + \theta(X_1) - \theta(k) \in a + H \quad \mathbb{P}_k^{P_\infty} - \text{a.s.}$$

But Assumption (U3-2) obviously implies that P_∞ satisfies **(AC1)**, so that the last property is impossible as seen in the proof of Lemma 3.2. Property (43) is proved. \square

A Proof of Lemma 5.2

Assume that

$$\forall B' \in B(\mathbb{R}^{N-1}), \quad \mathbb{P}_k\{X_t = \ell, Y_t' \in B'\} = \int_{\mathbb{R}^{N-1}} g'(y) dy + \mu'(A)$$

avec $\mu' \perp \ell_{N-1}$. Let $B \in B(H)$. Then

$$\begin{aligned} \mathbb{P}_k\{X_t = \ell, Y_t \in B\} &= \mathbb{P}_k\{X_t = \ell, \Lambda(Y_t) \in \Lambda(B)\} \\ &= \int_{\Lambda(B)} g'(y) dy + \mu'(\Lambda(B)) \\ &= \int_{\mathbb{R}^{N-1}} 1_B(\Lambda^{-1}(y)) g'(\Lambda(\Lambda^{-1}y)) dy + \mu'(\Lambda(B)) \\ &= \int_B (g' \circ \Lambda)(x) d\eta(x) + \mu'(\Lambda(B)). \end{aligned}$$

Let $d\mu$ be the image measure of $d\mu'$ under Λ^{-1} , that is : $\forall B \in B(H)$, $\mu(B) := \mu'(\Lambda(B))$. Then we have $\mu \perp \eta$. Indeed, we know that there exist two disjoint sets $E', F' \in B(\mathbb{R}^{N-1})$ such that

$$\forall B' \in B(\mathbb{R}^{N-1}), \quad \ell_{N-1}(B') = \ell_{N-1}(B' \cap E') \quad \text{and} \quad \mu'(B') = \mu'(B' \cap F').$$

Then, introducing $E := \Lambda^{-1}(E')$ and $F := \Lambda^{-1}(F')$, we clearly have for any $B \in B(H)$

$$\eta(B) = \ell_{N-1}(\Lambda(B)) = \ell_{N-1}(\Lambda(B) \cap E') = \ell_{N-1}(\Lambda(B \cap E)) = \eta(B \cap E)$$

and

$$\mu(B) = \mu'(\Lambda(B)) = \mu'(\Lambda(B) \cap F') = \mu'(\Lambda(B \cap F)) = \mu(B \cap F),$$

so that η and μ are supported by the two disjoint sets E et F .

B Proof of Lemma 5.3

Let us recall that the density $\psi_{k,\ell,t}$ is given by (30). We can be a little bit more precise on the properties of the coefficients invoked in (30). Indeed, we know from [Ser99] that

$$p_{k,\ell}(n+N, k_1, \dots, k_{N-1}) := \mathbb{P}_k\{V_{n+N-1}^1 = k_1 + 1, \dots, V_{n+N-1}^{N-1} = k_{N-1} + 1, Z_{n+N-1} = \ell\}$$

with V_n^i the number of visits to state i at time n

$$\forall i = 1, \dots, N-1 \quad V_n^i = \sum_{j=0}^n 1_{\{Z_j=i\}}. \quad (44)$$

Moreover, if $y = (y_1, \dots, y_{N-1})$

$$x_{n;k_1, \dots, k_{N-1}}^{t;y} := \frac{n!}{k_1! \cdots k_{N-1}! (n - \sum_{j=1}^{N-1} k_j)!} \prod_{j=1}^{N-1} \binom{y_j}{t}^{k_j} \left(1 - \frac{\sum_{j=1}^{N-1} y_j}{t}\right)^{n - \sum_{j=1}^{N-1} k_j} \quad (45)$$

Recall that these coefficients satisfy the property (31).

We deduce (33a) from $\sum_{\ell \in \mathbb{X}} p_{k,\ell}(n+N, k_1, \dots, k_{N-1}) \leq 1$ and (31):

$$\forall y \in \mathbb{R}^{N-1}, \quad |\psi_{k,\ell,t}(y)| \leq \sum_{j \in \mathbb{X}} |\psi_{k,j,t}(y)| \leq 1_{\mathcal{C}_t}(y) a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \leq a^{N-1}.$$

Let us check (33b) for $j := 1$. We obtain

$$\begin{aligned} & \frac{\partial \psi_{k,\ell,t}}{\partial y_1} \\ &= a^{N-1} \sum_{n=1}^{\infty} e^{-at} \frac{(at)^n}{n!} \frac{1}{t} \\ & \quad \times \left[\sum_{\substack{k_1 \geq 1, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n}} n x_{n-1; k_1-1, \dots, k_{N-1}}^{t;y} p_{k,\ell}(n+N, k_1, \dots, k_{N-1}) \right. \\ & \quad \left. - \sum_{\substack{k_1 \geq 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n-1}} n x_{n-1; k_1, \dots, k_{N-1}}^{t;y} p_{k,\ell}(n+N, k_1, \dots, k_{N-1}) \right] \end{aligned}$$

so that

$$\begin{aligned} \frac{\partial \psi_{k,\ell,t}}{\partial y_1} &= a^N \sum_{n=1}^{\infty} e^{-at} \frac{(at)^{n-1}}{(n-1)!} \\ & \quad \times \left[\sum_{\substack{k_1 \geq 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n-1}} x_{n-1; k_1, \dots, k_{N-1}}^{t;y} p_{k,\ell}(n+N, k_1+1, k_2, \dots, k_{N-1}) \right. \\ & \quad \left. - \sum_{\substack{k_1 \geq 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n-1}} x_{n-1; k_1, \dots, k_{N-1}}^{t;y} p_{k,\ell}(n+N, k_1, \dots, k_{N-1}) \right]. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{\partial \psi_{k,\ell,t}}{\partial y_1}(y) &= a^N \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \sum_{\substack{k_1 \geq 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n}} x_{n;k_1, \dots, k_{N-1}}^{t;y} \\ &\quad \times [p_{k,\ell}(n+N+1, k_1+1, k_2, \dots, k_{N-1}) - p_{k,\ell}(n+N+1, k_1, \dots, k_{N-1})]. \end{aligned}$$

It follows that

$$\forall y \in \mathcal{C}_t, \quad \left| \frac{\partial \psi_{k,\ell,t}}{\partial y_1}(y) \right| \leq \sum_{\ell \in \mathbb{X}} \left| \frac{\partial \psi_{k,\ell,t}}{\partial y_1}(y) \right| \leq a^N 2 \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \leq 2a^N.$$

The same computation for each variable y_j $j = 2, \dots, N-1$ gives the bound in (33b).

For any $t > 0$, the definition of function Ψ_t can be extended on the boundary $\partial \mathcal{C}_t$ (see (29)) since the coefficients in (45) are well defined on $\bar{\mathcal{C}}_t$. Moreover, since these coefficients as function of y are continuous on $\bar{\mathcal{C}}_t$, it is easily seen that the extended version of Ψ_t is continuous on $\bar{\mathcal{C}}_t$ and is also denoted by Ψ_t in the sequel. Next, we study the behaviour of this function on the boundary $\partial \mathcal{C}_t$.

We have for any $y \in \{(y_1, \dots, y_{N-1}) \in \bar{\mathcal{C}}_t : y_i := 0\}$ that $x_{n;k_1, \dots, k_{N-1}}^{t;y} = 0$ if $k_i \geq 1$, so that

$$\begin{aligned} \psi_{k,\ell,t}(y) &:= a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \\ &\quad \times \sum_{\substack{k_1 \geq 0, \dots, k_i = 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n}} x_{n;k_1, \dots, 0, \dots, k_{N-1}}^{t;y} p_{k,\ell}(n+N, k_1, \dots, 0, \dots, k_{N-1}). \end{aligned}$$

Moreover, using $p_{k,\ell}(n+N, k_1, \dots, 0, \dots, k_{N-1}) \leq \mathbb{P}_k\{V_{n+N-1}^i = 1\}$ and (31), we obtain

$$\forall y \in \{(y_1, \dots, y_{N-1}) \in \bar{\mathcal{C}}_t \mid y_i := 0\}, \quad |\psi_{k,\ell,t}(y)| \leq a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \mathbb{P}_k\{V_{n+N-1}^i = 1\}. \quad (46a)$$

Finally, for any $y \in \bar{\mathcal{C}}_t$ such that $\langle y, \mathbf{1} \rangle = t$, we have $x_{n;k_1, \dots, k_{N-1}}^{t;y} = 0$ if $\sum_{j=1}^{N-1} k_j < n$, so that

$$\begin{aligned} \psi_{k,\ell,t}(y) &:= a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \\ &\quad \times \sum_{\substack{k_1 \geq 0, \dots, k_i \geq 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j = n}} x_{n;k_1, \dots, k_{N-1}}^{t;y} p_{k,\ell}(n+N, k_1, \dots, k_{N-1}). \end{aligned}$$

Since $n = \sum_{j=1}^{N-1} k_j$, we have $\sum_{j=1}^{N-1} (k_j + 1) = n + N - 1$ and for any $(k, \ell) \in \mathbb{X}^2$,

$$p_{k,\ell}(n+N, k_1, \dots, k_{N-1}) \leq \mathbb{P}_k\{V_{n+N-1}^N = 0\}.$$

It allows us to write that

$$y \in \left\{ (y_1, \dots, y_{N-1}) \in \bar{\mathcal{C}}_t \mid \sum_{j=1}^{N-1} y_j = t \right\}, \quad |\psi_{k,\ell,t}(y)| \leq a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \mathbb{P}_k\{V_{n+N-1}^N = 0\}. \quad (46b)$$

We deduce from the next Lemma B.1 and (46a)-(46b) that there is $\rho \in (0, 1)$ such that

$$\forall y \in \partial \mathcal{C}_t, \quad \|\Psi_t(y)\|_0 = O(e^{at(\rho-1)}(1+at)).$$

Lemma B.1 For a discrete time finite Markov chain $\{Z_n\}_{n \in \mathbb{N}}$ with transition matrix \tilde{P} , let V_n^i be the number of visits to state i with respect to n transitions of the Markov chain (see (44)). Let us introduce the $(N-1) \times (N-1)$ -matrix $\hat{P}_{i^c i^c} := (\hat{P}(k, \ell))_{k, \ell \in \mathbb{X} \setminus \{i\}}$ and assume that this matrix is irreducible and aperiodic. Let $0 < \rho_i < 1$ be the Perron-Frobenius eigenvalue of $\hat{P}_{i^c i^c}$. We have the following geometric estimates of the probabilities in (46a)-(46b) :

1. $\mathbb{P}_i\{V_n^i = 0\} = 0$ and $\mathbb{P}_k\{V_n^i = 0\} = O(\rho_i^n)$
2. $\mathbb{P}_k\{V_n^i = 1\} = O(\rho_i^n + n\rho_i^{n-1})$ for $k \neq i$ and $\mathbb{P}_i\{V_n^i = 1\} = O(\rho_i^n)$

Remark B.1 Note that the irreducibility of the sub-generators $G_{i^c i^c}$ allows us to derive exponential rate of convergence of $\|\Psi_t(y)\|_0$ on ∂C_t when t grows to infinity, given that only a rate in $1/t$ is required in Condition **(AC2)**.

Proof. Since $\hat{P}_{\{i\}^c \{i\}^c}$ is irreducible and aperiodic, $\|\hat{P}_{\{i\}^c \{i\}^c} \mathbf{1}^\top\|_\infty \leq \|\hat{P}_{\{i\}^c \{i\}^c}\|_\infty \leq C\rho_i$ for some constant C and $\rho_i \in (0, 1)$ is the Perron-Frobenius eigenvalue of $\hat{P}_{\{i\}^c \{i\}^c}$.

For the first assertion, note that $\mathbb{P}_i\{V_n^i = 0\} = 0$ since $V_n^i \geq 1$ and for $k \neq i$

$$\begin{aligned} \forall n \geq 0, \quad \mathbb{P}_k\{V_n = 0\} &= \mathbb{P}_k\left\{\sum_{\ell=1}^n 1_{\{Z_\ell=i\}} = 0\right\} = \mathbb{P}_k\{Z_1 \neq i, Z_2 \neq i, \dots, Z_n \neq i\} \\ &= \sum_{\ell \in \mathbb{X}} (\hat{P}_{\{i\}^c \{i\}^c})^n(k, \ell) = e_k(\hat{P}_{\{i\}^c \{i\}^c})^n \mathbf{1}^\top. \end{aligned} \quad (47)$$

Then, it follows that $\mathbb{P}_k\{V_n^i = 0\} = O(\rho_i^n)$.

For the second assertion, note that $\mathbb{P}_i\{V_n^i = 1\} = \mathbb{P}_i\{\sum_{j=1}^n 1_{\{Z_j=i\}} = 0\} = e_i(\hat{P}_{\{i\}^c \{i\}^c})^n \mathbf{1}^\top$ so that, we deduce that $\mathbb{P}_i\{V_n^i = 1\} = O(\rho_i^n)$. For $k \neq i$, we can write the following renewal equation

$$\begin{aligned} \forall n \geq 0, \quad \mathbb{P}_k\{V_{n+1} = 1\} &= \sum_{j \in \mathbb{X}} P(k, j) \mathbb{P}_j\{V_n = 1\} \\ &= \sum_{j \neq i} P(k, j) \mathbb{P}_j\{V_n = 1\} + P(k, i) \mathbb{P}_i\{V_n = 1\} \end{aligned}$$

with $\mathbb{P}_j\{V_0 = 1\} = 0$ if $j \neq i$ and $\mathbb{P}_i\{V_0 = 1\} = 1$. Let us introduce the following column vector $v_i(n+1) = (\mathbb{P}_k\{V_{n+1} = 1\})_{k \neq i}$. Then we have from the previous equation

$$\forall n \geq 0, \quad v_i(n+1) = P_{\{i\}^c \{i\}^c} v_i(n) + \mathbb{P}_i\{V_n = 1\} P_{\{i\}^c \{i\}}, \quad v_i(0) = 0, \mathbb{P}_i\{V_0 = 1\} = 1.$$

Then, we can obtain the following representation of vector $v_i(n+1)$

$$v_i(n+1) = \sum_{k=0}^n \mathbb{P}_i\{V_k = 1\} P_{\{i\}^c \{i\}^c}^{n-k} P_{\{i\}^c \{i\}}. \quad (48)$$

Note that $\mathbb{P}_i\{V_k = 1\}$ is known from (47). We obtain

$$v_i(n+1) = P_{\{i\}^c \{i\}^c}^n P_{\{i\}^c \{i\}} + \sum_{k=1}^n \left(P_{\{i\}^c \{i\}^c} P_{\{i\}^c \{i\}^c}^{k-1} \mathbf{1}^\top \right) \times P_{\{i\}^c \{i\}^c}^{n-k} P_{\{i\}^c \{i\}}.$$

The k th component of the vector is given by

$$\begin{aligned}
\mathbb{P}_k\{V_{n+1} = 1\} &= e_k^\top v_i(n+1) \\
&= e_k^\top P_{\{i\}^c\{i\}^c}^n P_{\{i\}^c\{i\}} \\
&\quad + \sum_{k=1}^n \left(P_{\{i\}\{i\}^c} P_{\{i\}^c\{i\}^c}^{k-1} \mathbf{1}^\top \right) \times e_k^\top P_{\{i\}^c\{i\}^c}^{n-k} P_{\{i\}^c\{i\}} \\
&\leq e_k^\top P_{\{i\}^c\{i\}^c}^n \mathbf{1}^\top + \sum_{k=1}^n \left(P_{\{i\}\{i\}^c} P_{\{i\}^c\{i\}^c}^{k-1} \mathbf{1}^\top \right) \times e_k^\top P_{\{i\}^c\{i\}^c}^{n-k} \mathbf{1}^\top.
\end{aligned}$$

Since $\|P_{\{i\}^c\{i\}^c}\| \leq C\rho_i$, there exists K such that

$$\mathbb{P}_k\{V_{n+1} = 1\} \leq K(\rho_i^n + \sum_{k=1}^n \rho_i^{k-1} \rho_i^{n-k}) = K(\rho_i^n + n\rho_i^{n-1}).$$

C Linear transformation of MAPs □

Let $\{(X_t, Y_t)\}_{t \in \mathbb{T}}$ be an MAP with state space $\mathbb{X} \times \mathbb{R}^d$, where $\mathbb{X} := \{1, \dots, N\}$ and $\mathbb{T} := \mathbb{N}$ or $\mathbb{T} := [0, \infty)$. Recall from [Asm03] that $\{(X_t, Y_t)\}_{t \in \mathbb{T}}$ is a Markov process with a transition semi-group, denoted by $\{Q_t\}_{t \in \mathbb{T}}$, which satisfies

$$\forall (k, y) \in \mathbb{X} \times \mathbb{R}^d, \forall (\ell, B) \in \mathbb{X} \times B(\mathbb{R}^d), \quad Q_t(k, y; \{\ell\} \times B) = Q_t(k, 0; \{\ell\} \times B - y). \quad (49)$$

Note that $Q_t(k, y; \{\ell\} \times B) := \mathbb{P}_k\{X_t = \ell, Y_t \in B\} = \mathcal{Y}_{k, \ell, t}(1_B)$. Let us consider any linear transformation $T: \mathbb{R}^d \rightarrow \mathbb{R}^m$ and introduce the $\mathbb{X} \times \mathbb{R}^m$ -valued process $\{(X_t, TY_t)\}_{t \in \mathbb{T}}$.

Lemma C.1 *The process $\{(X_t, TY_t)\}_{t \in \mathbb{T}}$ is an MAP with state space $\mathbb{X} \times \mathbb{R}^m$ and transition semi-group $Q_t^{(T)}$ defined by: $\forall (k, z) \in \mathbb{X} \times \mathbb{R}^m, \forall (\ell, B) \in \mathbb{X} \times B(\mathbb{R}^m)$,*

$$Q_t^{(T)}(k, z; \{\ell\} \times B) = Q_t(k, 0; \{\ell\} \times T^{-1}(B - z)). \quad (50)$$

Note that we only have to prove that $\{(X_t, TY_t)\}_{t \in \mathbb{T}}$ is a Markov process with transition semi-group defined by (50), since the additivity property for $\{(X_t, TY_t)\}_{t \in \mathbb{T}}$ is clearly satisfied from (50):

$$\begin{aligned}
Q_t^{(T)}(k, z; \{\ell\} \times B) &= Q_t(k, 0; \{\ell\} \times T^{-1}(B - z)) \\
&=: Q_t^{(T)}(k, 0; \{\ell\} \times B - z).
\end{aligned}$$

Proof. Let $\mathbb{F}_t^{(X, Y)} := \sigma(X_u, Y_u, u \leq t)$ and $\mathbb{F}_t^{(X, TY)} := \sigma(X_u, TY_u, u \leq t)$ be the filtration generated by the processes $\{(X_t, Y_t)\}_{t \in \mathbb{T}}$ and $\{(X_t, TY_t)\}_{t \in \mathbb{T}}$ respectively. Since $\{(X_t, Y_t)\}_{t \in \mathbb{T}}$ is an MAP with transition semi-group $\{Q_t\}_{t \in \mathbb{T}}$, we have by definition for any bounded function g on $\mathbb{X} \times \mathbb{R}^d$

$$\begin{aligned}
\mathbb{E} \left[g(X_{t+s}, Y_{t+s}) \mid \mathbb{F}_s^{(X, Y)} \right] &= \sum_{\ell \in \mathbb{X}} \int_{\mathbb{R}^d} g(\ell, y_1) Q_t(X_s, Y_s; \{\ell\} \times dy_1) \\
&= \sum_{\ell \in \mathbb{X}} \int_{\mathbb{R}^d} g(\ell, y_1 + Y_s) Q_t(X_s, 0; \{\ell\} \times dy_1).
\end{aligned}$$

Using the tower rule and the last representation of the condition expectation, we obtain for any $\ell \in \mathbb{X}$ and $B \in B(\mathbb{R}^m)$

$$\begin{aligned}
\mathbb{E} \left[1_{\{\ell\} \times B}(X_{t+s}, TY_{t+s}) \mid \mathbb{F}_s^{(X, TY)} \right] &= \mathbb{E} \left[\mathbb{E} \left[1_{\{\ell\} \times B}(X_{t+s}, TY_{t+s}) \mid \mathbb{F}_s^{(X, Y)} \right] \mid \mathbb{F}_s^{(X, TY)} \right] \\
&= \mathbb{E} \left[\int_{\mathbb{R}^d} 1_B(T(y_1 + Y_s)) Q_t(X_s, 0; \{\ell\} \times dy_1) \mid \mathbb{F}_s^{(X, TY)} \right] \\
&= \mathbb{E} \left[\int_{\mathbb{R}^d} 1_B(Ty_1 + TY_s) Q_t(X_s, 0; \{\ell\} \times dy_1) \mid \mathbb{F}_s^{(X, TY)} \right] \\
&= \int_{\mathbb{R}^d} 1_B(Ty_1 + TY_s) Q_t(X_s, 0; \{\ell\} \times dy_1) \\
&= \int_{\mathbb{R}^d} 1_{T^{-1}(B - TY_s)}(y_1) Q_t(X_s, 0; \{\ell\} \times dy_1) \\
&= Q_t(X_s, 0; \{\ell\} \times T^{-1}(B - TY_s)).
\end{aligned}$$

Then $\{(X_t, TY_t)\}_{t \in \mathbb{T}}$ is a Markov process with transition semi-group given by (50). \square

References

- [Asm03] S. Asmussen. *Applied probability and queues*, volume 51. Springer-Verlag, NY, 2nd edition, 2003.
- [FH67] M. Fukushima and M. Hitsuda. On a class of Markov processes taking values on lines and the central limit theorem. *Nagoya Math. J.*, 30:47–56, 1967.
- [FHL12] D. Ferré, L. Hervé, and J. Ledoux. Limit theorems for stationary Markov processes with L^2 -spectral gap. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48:396–423, 2012.
- [HL13] L. Hervé and J. Ledoux. Local limit theorem for densities of the additive component of a finite Markov additive process. *Statist. Probab. Lett.*, 83:2119–2128, 2013.
- [KW64] J. Keilson and D. M. G. Wishart. A central limit theorem for processes defined on a finite Markov chain. *Proc. Cambridge Philos. Soc.*, 60:547–567, 1964.
- [Ser99] B. Sericola. Occupation times in Markov processes. Technical Report 3806, INRIA Rennes, 1999.