



HAL
open science

Comment on higher derivative Lagrangians in relativistic theory

Mathieu Beau

► **To cite this version:**

Mathieu Beau. Comment on higher derivative Lagrangians in relativistic theory. 2013. hal-00825545v1

HAL Id: hal-00825545

<https://hal.science/hal-00825545v1>

Preprint submitted on 24 May 2013 (v1), last revised 16 Aug 2013 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Comment on the higher derivative Lagrangians in relativistic theory

Mathieu Beau

School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4*

(Dated: May 24, 2013)

We discuss the consequences of higher derivative Lagrangians of the form $\lambda_2 G_\mu \ddot{x}^\mu$, $\lambda_3 B_\mu \dot{x}^\mu$, $\lambda_4 G_\mu \dot{x}^\mu$ in relativistic theory.

I. HIGHER DERIVATIVE LAGRANGIANS AND DYNAMIC EQUATIONS

Ostrogradsky was the first who initiated the idea of higher derivative Lagrangians in classical mechanics [1], and more recently were published some articles about this topic[2],[3],[4]. But in my knowledge, there is no article dealing with the relativistic consequences if we consider higher derivative Lagrangians of this type:

$$\tilde{L}(\dot{x}, \ddot{x}, \dots, x^{(n)}) = \lambda_1 A_\mu(x) \dot{x}^\mu + \lambda_2 G_\mu \ddot{x}^\mu + \dots + \lambda_n U_{(n)\mu} x^{(n)\mu} \quad (1)$$

where $x^{(n)}(s) \equiv d^n x(s)/ds^n$ is the n -derivative of the position ($ds = cd\tau$, where τ is the proper time), and $U_{(n)\mu}(x)$, $n = 1, 2, 3, \dots$ are the generalised field coupling linearly with the derivatives. One can see that we denote the field $G_{(1)\mu}(x) = A_\mu(x)$ to refer to be the electromagnetic potential. For $n \geq 2$, we will see their meaning in the sequel. Now, we set the action:

$$S = \int ds L_0(\dot{x}) + \int ds L(\dot{x}, \ddot{x}, \dots, x^{(n)}) .$$

where $L_0(\dot{x}) \equiv \frac{mc^2}{2} \dot{x}_\mu \dot{x}^\mu$. We would not give an explicit general dynamic theory for a given n , we consider only $n = 2$ for the moment and we will discuss the general case later. By some obvious integral by part for $n = 2$ we get the equivalent action:

$$S = \lambda_1 \int ds A_\mu(x) \dot{x}^\mu + \lambda_2 \int ds \partial_\nu G_\mu \dot{x}^\mu \dot{x}^\nu ,$$

and one can see that the first part of the action is similar to the electromagnetic whereas the second part is similar to the gravitational action. Indeed, the generalised Euler-Lagrange equation is given by (we give the relativistic form of the equations in [1],[2][3][4]):

$$\frac{d^2}{ds^2} \left(\frac{\partial L}{\partial \ddot{x}^\mu} \right) - \frac{d}{ds} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) + \frac{\partial L}{\partial x^\mu} = 0 , \quad (2)$$

for $L = L_0 + \tilde{L}$, and so we get:

$$mc^2 \eta_{\mu\nu} \ddot{x}^\nu - \lambda_2 \varepsilon_{\mu\nu} \ddot{x}^\nu - \lambda_2 \Delta_{\mu\nu\sigma} \dot{x}^\nu \dot{x}^\sigma = -\lambda_1 F_{\mu\nu} \dot{x}^\nu \quad (3)$$

where $\varepsilon_{\mu\nu}$ and $\Delta_{\mu\nu\sigma}$ are defined as:

$$\begin{aligned} \varepsilon_{\mu\nu} &\equiv \partial_\mu G_\nu + \partial_\nu G_\mu \\ \Delta_{\mu\nu\sigma} &\equiv \partial_\nu \partial_\sigma G_\mu = \frac{1}{2} (\partial_\nu \varepsilon_{\mu\sigma} + \partial_\sigma \varepsilon_{\mu\nu} - \partial_\mu \varepsilon_{\nu\sigma}) \end{aligned} \quad (4)$$

and where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Of course one see the analogy with the geodesic equations for a charged particle in gravity field and in an electromagnetic field, but we can see also that the Minkowski metric is such a background and so that $G_\mu(x)$ is nothing but an additional field which can be seen as a deformation field and that $\varepsilon_{\mu\nu}$ can be viewed as an infinitesimal strain field by analogy with the deformation theory of a continuous medium [5].

We would like to go further than the rank 2. Let's take $n = 3$, we denote $G_{(3)\mu} = B_\mu$. Then one has

$$-\frac{d^3}{ds^3} \left(\frac{\partial L}{\partial \ddot{x}^\mu} \right) + \frac{d^2}{ds^2} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{d}{ds} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) + \frac{\partial L}{\partial x^\mu} = 0 \quad (5)$$

and so one can easily get:

$$\begin{aligned} mc^2 \eta_{\mu\nu} \ddot{x}^\nu + \lambda_3 H_{\mu\nu} \ddot{x}^\nu - \lambda_3 \Upsilon_{\mu\nu\sigma\rho} \dot{x}^\nu \dot{x}^\sigma \dot{x}^\rho - 3 \Sigma_{\mu\nu\sigma} \ddot{x}^\nu \dot{x}^\sigma \\ - \lambda_2 \varepsilon_{\mu\nu} \ddot{x}^\nu - \lambda_2 \Delta_{\mu\nu\sigma} \dot{x}^\nu \dot{x}^\sigma = -\lambda_1 F_{\mu\nu} \dot{x}^\nu \end{aligned} \quad (6)$$

where

$$\begin{aligned} H_{\mu\nu} &\equiv \partial_\mu B_\nu - \partial_\nu B_\mu \\ \Sigma_{\mu\nu\sigma} &\equiv \partial_\nu \partial_\sigma B_\mu \\ \Upsilon_{\mu\nu\sigma\rho} &\equiv \partial_\nu \partial_\sigma \partial_\rho B_\mu \end{aligned} \quad (7)$$

We can see that this field generalise the idea of electromagnetic field because of the asymmetry of $H_{\mu\nu}$. But there is some other gradient fields similar to the field $\Delta_{\mu\nu\sigma}$ coupling to the combination of the odd derivatives of $x^m u$, i.e. $\dot{x}^\nu \dot{x}^\sigma$ and $\dot{x}^\nu \dot{x}^\sigma \dot{x}^\rho$. One can easily consider that for $n = 4$ and even more. We denote the field $K_\mu(x) \equiv G_{(4)\mu}(x)$. It would give a similar structure in the dynamic equation than $G_\mu(x)$. Is is known indeed that in the non-relativistic theory, the Lagrangian $\frac{mc^2}{2} x \ddot{x}$ is equivalent to this Lagrangian $\frac{mc^2}{2} \dot{x}^2$ which could be interpreted as a *dynamic energy*. [2][3][4] Here the problem is similar: the Lagrangian $\lambda_4 K_\mu \ddot{x}^\mu$ is equivalent to $\lambda_4 \partial_\mu K_\nu \dot{x}^\mu \dot{x}^\nu + \lambda_4 \partial_\sigma \partial_\mu K_\nu \dot{x}^\mu \dot{x}^\nu \dot{x}^\sigma$ and hence for a more general field than the trivial one $K_\mu = x_\mu$, the equivalent Lagrangian seems to be more complicated and the dynamic equations have some similar structure that for $n = 2$.

More generally, for $G_{(n)\mu}(x)$, $n \geq 1$ we will get some term $\partial^1 \dots \partial^p G_{(n)}$, $p = 1, \dots, n$ multiplied by the combination of the derivatives $x^{(l_1)} x^{(l_2)} \dots x^{(l_r)}$, where $\sum_{j=1}^r l_j = n$. One can see that "even" n -field (for n even) could be associated with the "gravitational fields" and the "odd" n -field with the "electromagnetic field" since in the dynamic equations the n -derivative term of

* mbeau@stp.dias.ie

x^μ is multiplied symmetric (if n is even) / antisymmetric (if n is odd) first derivative of the field

$$(\partial_\mu U_\nu + (-1)^n \partial_\nu U_\mu) x^{(n)\nu}$$

where we denoted U_μ the n -field to remove the index n .

II. GENERAL FIELDS HYPOTHESIS

The point is to relate the possible existence of these field with some induction phenomena. Or in an other word, we will suppose some physical coupling, unknown yet but still relevant at least for future research.

As we noticed just above, the analogies with the electromagnetic field and gravity field are crucial to understand what is the physical meaning of these field. We give some example. Imagine an electrokinetic experiment where one set a constant intensity of electron current on a circuit (one can imagine a particle accelerator for a more modern discussion), whatever the shape of the circuit. Then one can expect that a serious student at the University could measure a magnetic field and check the well known Biot and Savart law. Now, imagine that we increase constantly the intensity such that $I = I_0 t$. This means that the electron on the circuit are uniformly accelerated. By analogy with the gravity field, we assume that the constant λ_2 in (1) is proportional to the energy mass of the electron, we set $\lambda_2 = \alpha m c^2$, where α is a universal constant. Then here is the fundamental question: what is the effect of the uniform acceleration of the current on the measurement of the electromagnetic field? Well, one can say that the student have to open a book [6] and he will see that following the usual law of Newton and Maxwell, the effect of the acceleration of an electric charge can be computed even if it could be laborious. Nevertheless, the imagination of the student can go further the academic answer and think about the consequences of a more general notion of stationary induction such that the one we gave. Indeed, Physics is always understood by the symmetry of phenomenon and so why a uniform accelerating current could not be seen as a symmetric transformation of a more general space-time structure? For example considering the existence of the coupling λ_4 in (1), the acceleration plays the rule of an energy, as we discussed above. Even for $n = 2$, one can see that the acceleration can play a rule in the motion of the bodies but of course all depend of the value of α . The question of the nature and of the reality of the acceleration in Physics was already discussed by famous Physicists and Philosopher like Mach and Einstein, see for example the reference [7].

A. Construction of the $n = 2$ -field equations by analogy with the vectorial electromagnetic field

In the Lagrangian (1) for $n = 2$ we notice that the field G_μ is coupled with the acceleration of the particles

similarly than the field A_μ is coupled with the velocity. Following the discussion we had above, by a analogous construction of the electromagnetic field theory, we can suggest those field equations:

$$\partial_\mu \varepsilon^{\mu\nu}(x) = -\kappa j^{(2)\nu}(x), \quad (8)$$

where the 4-acceleration density current $j^{(2)\nu}$ (generally non-conserved) is:

$$j^{(2)\nu} \equiv \rho(x) c^2 \frac{du^\nu}{ds}, \quad (9)$$

where $\rho(x)$ is the density of particles and $\frac{du^\nu}{ds}$ is the 4-acceleration. The constant κ is related to a coupling constant between the field G_μ and the current $j^{(2)\nu}$ and by dimensional analysis we can set that $\kappa = \frac{8\pi G \lambda^2}{c^2}$ where λ has the physical unit of a length.

To complete the system of field equations, we need 10 equations obviously given by

$$\partial_\sigma \partial^\sigma \varepsilon_{\mu\nu} + \partial_\mu \partial_\nu \varepsilon_\sigma^\sigma = \partial_\mu \partial^\sigma \varepsilon_{\sigma\nu} + \partial_\nu \partial^\sigma \varepsilon_{\sigma\mu}, \quad (10)$$

The equations (10) are analogous to the compatibility equations for the strain tensor in the three-dimensional non-relativistic theory of deformation of continuous media.[5]

Hence we get the following wave equations:

$$\square \varepsilon_{\nu\sigma}(x) + \partial_\nu \partial_\sigma \varepsilon_\mu^\mu(x) = -\kappa \xi_{\nu\sigma}^{(2)}(x) \quad (11)$$

where we get a stress tensor $\xi_{\nu\sigma}^{(2)}(x) \equiv \partial_\sigma j_\nu^{(2)}(x) + \partial_\nu j_\sigma^{(2)}(x)$.

B. Generalisation to the n -field equations

It comes naturally that for the $2n$ -field $G_{(2n)}$, so-called gravitational type field, the coupling is of the form:

$$\frac{8\pi G \lambda^{2n}}{c^2} G_{(2n)\mu}(x) j^{(2n)\mu}(x)$$

while the $2n + 1$ -field $G_{(2n+1)}$, so-called electromagnetic type field, the coupling is of the form:

$$\frac{\mu_0 \lambda^{2n}}{c^2} G_{(2n+1)\mu}(x) j^{(2n+1)\mu}(x)$$

where μ_0 is the vacuum permeability since we generalized for the $2n + 1$ -field the electromagnetic field theory (the λ^{2n} come from a dimensional analysis) as well as we did for the $2n$ -gravitational fields. The generalized currents are defined for $n = 1, 2, 3, \dots$ by:

$$j^{(n)\nu} \equiv \rho^{(n)}(x) c^2 \frac{d^n x^\nu}{ds^n}$$

where $\rho^{(n)}(x)$ give the density of mass for n even and the density of charge for n odd. From those rule we could

obtain similar wave equation than (11) but with higher order differential operator $\square\square\cdots\square$. For example, for the 4-field we give the wave equation for the trace of the tensor $\eta_{\mu\nu} \equiv \partial_\mu K_\nu + \partial_\nu H_\mu$:

$$(\square + 1)\square\eta_\mu^\mu(x) = -\kappa\lambda^2\partial_\mu j^{(4)\mu}(x) \quad (12)$$

and this means that the scalar field η_μ^μ is a massive field.

III. COMMENTS

The effect of the gravitation at the microscopic scale is not well known yet. We can expect that the current $J^{(2)}$ of acceleration of mass has a contribution in particle physics as well as the energy-impulsion tensor. We can suggest that the constant λ is constraint by the current observations. The recent cosmological observations show that there is a lack of energy due to the cosmological constant Λ that might be related to our constant λ (we could expect $\lambda = 1/\sqrt{\Lambda}$). The interpretation of this relation is not so obvious and again speculative, but remember that we gave above an analogy between the field $\varepsilon_{\mu\nu}$ with the strain-deformation tensor of a continuous media. This remark is not obsolete. Indeed, if we consider the covariant derivative for a Riemannian metric space ($\varepsilon_{\mu\nu} = D_\mu G_\nu + D_\nu G_\mu$), one can construct a stress field

$$\sigma_{\mu\nu} = \rho_G c^2 \varepsilon_{\mu\nu}$$

where $\rho_G c^2 = \frac{c^4}{8\pi G \lambda^2}$ is the density of energy constant analogous to the Young modulus. The stress tensor could be adding to the Einstein field equations of gravity and we get the relation:

$$D_\mu \sigma^{\mu\nu} + D_\nu T^{\mu\nu} = 0$$

where $T_{\mu\nu}$ is the energy-momentum tensor in the Einstein field equation. This last equation means that the

total energy in the Universe is conserved but that the “visible” energy can be accelerated and this variation of inertia is compensated by the stress energy variation due to the strain of the internal structure of the continuous medium. There is no contradiction with the Einstein theory of gravity field and moreover this gives a new perspective on the Mach principle revisiting the “absolute” acceleration concept as a natural motion in a space-time deformed by the matter-energy contained therein, we refer the reader on the paper of Einstein on this related topic [7]. The relativistic theory of an Aether was discussed several times, see for e.g. [8], [9]. In this paper, our hypothesis is different and gives a relativistic theory of the deformation of continuous media (for which the geometry is described by the metric field).

Now we come back to the microscopic scales. Nothing proves today that at those scales the gravitation can be viewed as a metric field. It might be a challenge to see if the higher derivative fields play a role for particle physics in the domain of astroparticle or for the future linear accelerator(s). The analogies between the $2n$ -fields (ex G_μ and K_μ) and between the $2n + 1$ -fields (ex A_μ and B_μ) can be viewed as a unity between those fields (where the constant λ plays a role again) and generalise the notion of gravitational and electromagnetic field for more general currents. Additionally, the unity between the odd and even field is another tempting idea, which is less obvious but still interesting. Indeed, we could see the field $G_{(2)\mu}$ and $G_{(1)\mu}$ as two components of the same field G_μ coupling to both currents $j^{(1)}$ and $j^{(2)\mu}$. This is rather speculative but also shows the wide perspective.

About the 4-field K_μ , in a more general theory of gravitational relativity, we can expect that his notion of “dynamic energy” would have a sense in a way that we could make an analogy between the tensor $\partial_\mu K_\nu$ with a more general tensor $k_{\mu\nu}$ coupling with $\ddot{x}^\mu \ddot{x}^\nu$ and its derivatives with the covariant form of $\ddot{x}^\mu \dot{x}^\nu \dot{x}^\sigma$. This could be a way to include the dynamic energy in general relativity theory.

-
- [1] M. Ostrogradsky, Mem. Acad. St. Petersburg **6** (4),385 (1850).
[2] M. Borneas, On a generalization of the Lagrange Function, American Journal of Physics, **27** (4), pp. 265-267 (1959).
[3] M. Borneas, Principle of the Action with Higher Derivatives, Phys.Rev **186** (5), 1299, (1969).
[4] D. Anderson, Equivalent Lagrangians in generalized mechanics, J. Math. Phys. **14**, 934 (1973).
[5] J. Salençon, Mécanique des milieux continus - Tome 1 : Concepts généraux, les Editions de l'Ecole Polytechnique (2005).
[6] L.D. Landau, E.M. Lifshitz, The Classical Theory of Fields. Vol. 2 (4th ed.) (1975).
[7] A. Einstein, “Ether and the Theory of Relativity”, in Collected Papers of Albert Einstein, Princeton University Press.
[8] C. Eling, T. Jacobson, D. Mattingly, “Einstein Aether Theory”, in Deserfest, eds. J. Liu, K. Stelle, R. P. Woodard (World Scientific, 2006).
[9] T. G. Zlosnik, P. G. Ferreira, G. D. Starkmann, Modifying gravity with the Aether: an alternative to Dark Matter, Phys. Rev. D **75**, 044017 (2007).