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Karine Beauchard, J.M. Coron

## To cite this version:

Karine Beauchard, J.M. Coron. Controllability of a quantum particle in a moving potential well. Journal of Functional Analysis, 2006, 232, pp.328-389. hal-00825517

## HAL Id: hal-00825517

## https://hal.science/hal-00825517

Submitted on 23 May 2013

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# Controllability of a quantum particle in a moving potential well 

K. Beauchard, O. Jean-Michel CORON $\dagger$


#### Abstract

Résumé We consider a non relativistic charged particle in a 1D moving potential well. This quantum system is subject to a control, which is the acceleration of the well. It is represented by a wave function solution of a Schrödinger equation. We prove the following controllability result for this bilinear control system : given $\psi_{0}$ closed enough to an eigenstate and $\psi_{f}$ closed enough to another eigenstate, the wave function can be moved exactly from $\psi_{0}$ to $\psi_{f}$ in finite time. Moreover, we can control the position and the velocity of the well. Our proof uses moment theory, a Nash-Moser implicit function theorem, the return method and expansion to the second order.


Keywords : Controllability, Schrödinger, Nash-Moser.

## 1 Introduction

Following P. Rouchon [14], we consider a quantum particle with a potential $V(z)$ in a non Galilean frame of absolute position $D(t)$, in a one dimension space. This system is represented by a complex valued wave function $(t, z) \mapsto \phi(t, z)$ solution of the Schrödinger equation

$$
\begin{equation*}
i \hbar \frac{\partial \phi}{\partial t}(t, z)=-\frac{\hbar^{2}}{2 m} \frac{\partial^{2} \phi}{\partial z^{2}}(t, z)+V(z-D(t)) \phi(t, z) . \tag{1}
\end{equation*}
$$

Up to a change of variables, we can assume $\hbar=1, m=1$. It was already noticed in [14] that the change of space variable $z \rightarrow q$ and function $\phi \rightarrow \psi$, defined by

$$
\begin{gathered}
q:=z-D \\
\psi(t, q):=e^{i\left(-z \dot{D}+D \dot{D}-\frac{1}{2} \int_{0}^{t} \dot{D}^{2}\right)} \phi(t, z),
\end{gathered}
$$

transforms (1) into

$$
\begin{equation*}
i \frac{\partial \psi}{\partial t}(t, q)=-\frac{1}{2} \frac{\partial^{2} \psi}{\partial q^{2}}(t, q)+(V(q)-u(t) q) \psi(t, q) \tag{2}
\end{equation*}
$$

where $u:=-\ddot{D}$. This equation also describes the non relativistic motion of a particle with a potential $V$ in a uniform electric field $t \mapsto u(t)$.

We study this quantum system in the case of a potential well (a box) :

$$
V(q)=0 \text { for } q \in I:=(-1 / 2,1 / 2) \text { and } V(q)=+\infty \text { for } q \notin I .
$$

Therefore, our system is

$$
\left(\Sigma_{0}\right)\left\{\begin{array}{l}
i \frac{\partial \psi}{\partial t}(t, q)=-\frac{1}{2} \frac{\partial^{2} \psi}{\partial q^{2}}(t, q)-u(t) q \psi(t, q), t \in \mathbb{R}_{+}, q \in I \\
\psi(t,-1 / 2)=\psi(t, 1 / 2)=0 \\
\dot{S}(t)=u(t) \\
\dot{D}(t)=S(t)
\end{array}\right.
$$

This is a control system, where

[^0]- the state is $(\psi, S, D)$ with $\int_{I}|\psi(t, q)|^{2} d q=1$ for every $t$,
- the control is the function $t \mapsto u(t) \in \mathbb{R}$.

It means that we want to control at the same time the wave function $\psi$ of the particle, the speed $S$ and the position $D$ of the box. The control is the acceleration of the box (with an easy change of variable, we could instead take the force applied to the box).
Definition 1 Let $T_{1}<T_{2}$ be two real numbers and $u \in C^{0}\left(\left[T_{1}, T_{2}\right], \mathbb{R}\right)$. A function $(\psi, S, D)$ is a solution of $\left(\Sigma_{0}\right)$ if

- $\psi$ belongs to $C^{0}\left(\left[T_{1}, T_{2}\right], H^{2} \cap H_{0}^{1}(I, \mathbb{C})\right) \cap C^{1}\left(\left[T_{1}, T_{2}\right], L^{2}(I, \mathbb{C})\right)$ and the first equality of $\left(\Sigma_{0}\right)$ holds in $L^{2}(I, \mathbb{C})$, for every $t \in\left[T_{1}, T_{2}\right]$,
$-S \in C^{1}\left(\left[T_{1}, T_{2}\right], \mathbb{R}\right)$ and satisfies the third equality of $\left(\Sigma_{0}\right)$, for every $t \in\left[T_{1}, T_{2}\right]$,
- $D \in C^{2}\left(\left[T_{1}, T_{2}\right], \mathbb{R}\right)$ and satisfies the fourth equality of $\left(\Sigma_{0}\right)$, for every $t \in\left[T_{1}, T_{2}\right]$.

Then, we say that $(\psi, S, D, u)$ is a trajectory of the control system $\left(\Sigma_{0}\right)$ (on $\left[T_{1}, T_{2}\right]$ ).
Note that the first equation of $\left(\Sigma_{0}\right)$ guarantees the conservation of the $L^{2}(I, \mathbb{C})$-norm of the wave function. Indeed, we have

$$
\frac{d}{d t}\|\psi(t)\|_{L^{2}(I, \mathbb{C})}^{2}=<\psi(t), \frac{\partial \psi}{\partial t}(t)>+<\frac{\partial \psi}{\partial t}(t), \psi(t)>=0
$$

where $<., .>$ denotes the usual scalar product on $L^{2}(I, \mathbb{C})$,

$$
<\psi, \varphi>:=\int_{I} \psi(q) \overline{\varphi(q)} d q
$$

and $\psi(t):=\psi(t,).$.
It has already been proved in [1] that the subsystem

$$
(\Sigma)\left\{\begin{array}{l}
i \frac{\partial \psi}{\partial t}(t, q)=-\frac{1}{2} \frac{\partial^{2} \psi}{\partial q^{2}}(t, q)-u(t) q \psi(t, q), t \in \mathbb{R}_{+}, q \in I \\
\psi(t,-1 / 2)=\psi(t, 1 / 2)=0
\end{array}\right.
$$

where the state is $\psi$ and the control is $u$, is locally controllable around any eigenstate state for $u \equiv 0$, which are the functions

$$
\psi_{n}(t, q):=\varphi_{n}(q) e^{-i \lambda_{n} t}, n \in \mathbb{N}^{*}
$$

Here $\lambda_{n}:=(n \pi)^{2} / 2$ are the eigenvalues of the operator $A$ defined on

$$
D(A):=H^{2} \cap H_{0}^{1}(I, \mathbb{C}) \quad \text { by } \quad A \varphi:=-\frac{1}{2} \varphi^{\prime \prime}
$$

and the functions $\varphi_{n}$ are the associated eigenvectors,

$$
\varphi_{n}(q):=\left\{\begin{array}{l}
\sqrt{2} \sin (n \pi q), \text { when } n \text { is even }  \tag{3}\\
\sqrt{2} \cos (n \pi q), \text { when } n \text { is odd }
\end{array}\right.
$$

Thus, we know that, for every eigenstate, the wave function can be moved arbitrarily in a neighbourhood of this eigenstate, in finite time.

The aim of this paper is to prove that we can also change the energy level. For example, we can move the wave function from any point in a neighbourhood of the ground state $\psi_{1}$ to any point in a neighbourhood of the first excited state $\psi_{2}$. We also prove that we can control the position $D$ and the speed $S$ of the box at the same time.

Let us introduce few notations in order to state this result,

$$
\begin{gathered}
\mathcal{S}:=\left\{\varphi \in L^{2}(I, \mathbb{C}) ;\|\varphi\|_{L^{2}(I, \mathbb{C})}=1\right\} \\
H_{(0)}^{7}(I, \mathbb{C}):=\left\{\varphi \in H^{7}(I, \mathbb{C}) ; A^{n} \varphi \in H_{0}^{1}(I, \mathbb{C}) \text { for } n=0,1,2,3\right\}
\end{gathered}
$$

Our main result is the following one.

Theorem 1 For every $n \in \mathbb{N}^{*}$, there exists $\eta_{n}>0$ such that, for every $n_{0}, n_{f} \in \mathbb{N}^{*}$, for every $\left(\psi_{0}, S_{0}, D_{0}\right),\left(\psi_{f}, S_{f}, D_{f}\right) \in\left[\mathcal{S} \cap H_{(0)}^{7}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}$ with

$$
\left\|\psi_{0}-\varphi_{n_{0}}\right\|_{H^{7}}+\left|S_{0}\right|+\left|D_{0}\right|<\eta_{n_{0}}, \quad\left\|\psi_{f}-\varphi_{n_{f}}\right\|_{H^{7}}+\left|S_{f}\right|+\left|D_{f}\right|<\eta_{n_{f}}
$$

there exists a time $\mathcal{T}>0$ and a trajectory $(\psi, S, D, u)$ of $\left(\Sigma_{0}\right)$ on $[0, \mathcal{T}]$, which satisfies $(\psi(0), S(0), D(0))=$ $\left(\psi_{0}, S_{0}, D_{0}\right),(\psi(\mathcal{T}), S(\mathcal{T}), D(\mathcal{T}))=\left(\psi_{f}, S_{f}, D_{f}\right)$ and $u \in H_{0}^{1}((0, \mathcal{T}), \mathbb{R})$.

Thus, we also have the following corollary.
Corollary 1 For every $n_{0}, n_{f} \in \mathbb{N}^{*}$, there exists a time $\mathcal{T}>0$ and a trajectory $(\psi, S, D, u)$ of $\left(\Sigma_{0}\right)$ on $[0, \mathcal{T}]$ such that $(\psi(0), S(0), D(0))=\left(\varphi_{n_{0}}, 0,0\right),(\psi(\mathcal{T}), S(\mathcal{T}), D(\mathcal{T}))=\left(\varphi_{n_{f}}, 0,0\right)$, and $u \in H_{0}^{1}((0, \mathcal{T})$.

For other results about the controllability of Schrödinger equations, we refer to the survey [16]

## 2 Sketch of the proof

### 2.1 Global strategy

Thanks to the reversibility of the control system $\left(\Sigma_{0}\right)$, in order to get Theorem 1 , it is sufficient to prove it with $n_{f}=n_{0}+1$. We prove it with $n_{0}=1$ and $n_{f}=2$ to simplify the notations.

First, we prove the local controllability of $\left(\Sigma_{0}\right)$ around the trajectory $\left(Y^{\theta, 0,0}, u \equiv 0\right)$ for every $\theta \in[0,1]$, where

$$
\begin{aligned}
& Y^{\theta, 0,0}(t):=\left(\psi_{\theta}(t), S(t) \equiv 0, D(t) \equiv 0\right) \\
& \psi_{\theta}(t):=\sqrt{1-\theta} \psi_{1}(t)+\sqrt{\theta} \psi_{2}(t) \text { for } \theta \in(0,1) \\
& Y^{k, 0,0}(t)=\left(\psi_{k-1}(t), S(t) \equiv 0, D(t) \equiv 0\right) \text { for } k=0,1
\end{aligned}
$$

Thus we know that

- there exists an open ball $V_{0}\left(\right.$ resp. $\left.V_{1}\right)$ centered at $Y^{0,0,0}(0)\left(\right.$ resp. $\left.Y^{1,0,0}(0)\right)$ such that $\left(\Sigma_{0}\right)$ can be moved in finite time between any two points in $V_{0}$ (resp. $V_{1}$ ),
- for every $\theta \in(0,1)$, there exists an open ball $V_{\theta}$ centered at $Y^{\theta, 0,0}(0)$ such that $\left(\Sigma_{0}\right)$ can be moved in finite time between any two points in $V_{\theta}$.
Then, we conclude thanks to a compactness argument : the segment

$$
\left[Y^{0,0,0}(0), Y^{1,0,0}(0)\right]:=\left\{\sqrt{\lambda} Y^{0,0,0}(0)+\sqrt{1-\lambda} Y^{1,0,0}(0) ; \lambda \in[0,1]\right\}
$$

is compact in $L^{2}(I, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ and covered by $\cup_{0 \leqslant \theta \leqslant 1} V_{\theta}$ thus there exists a increasing finite family $\left(\theta_{n}\right)_{1 \leqslant n \leqslant N}$ such that $\left[Y^{0,0,0}(0), Y^{1,0,0}(0)\right]$ is covered by $\cup_{1 \leqslant n \leqslant N} V_{\theta_{n}}$. We can assume $V_{n} \cap V_{n+1} \neq \emptyset$ for $n=1, \ldots, N-1$. Given $Y_{0} \in V_{1}$ and $Y_{f} \in V_{N}$, we move $\left(\Sigma_{0}\right)$ from $Y_{0}$ to a point $Y_{1} \in V_{\theta_{1}} \cap V_{\theta_{2}}$ in finite time, from $Y_{1}$ to a point $Y_{2} \in V_{\theta_{2}} \cap V_{\theta_{3}}$ in finite time...etc and we reach $Y_{f}$ in finite time.

Now, let us explain the proof of the local controllability of $\left(\Sigma_{0}\right)$ around $Y^{\theta, 0,0}$ for every $\theta \in[0,1]$. The strategy for $\theta \in(0,1)$ is different from the one for $\theta \in\{0,1\}$ but involves the same ideas. In the next sections, we details the two approaches. We start with the simplest case $\theta \in(0,1)$.

### 2.2 Local controllability of $\left(\Sigma_{0}\right)$ around $Y^{\theta, 0,0}$ for $\theta \in(0,1)$

A classical approach to prove the local controllability around a trajectory consists in proving the controllability of the linearized system around the trajectory studied and concluding with an inverse mapping theorem. This strategy does not work here because the linearized system around $\left(Y^{\theta, 0,0}(t), u \equiv 0\right)$ is not controllable. In section 3.1 , we justify that the linearized system misses
exactly two directions, which are $(\psi, S, D)=\left( \pm i \varphi_{1}, 0,0\right)$. We call this situation "controllability up to codimension one ".

First, we prove the local controllability up to codimension one of the non linear system $\left(\Sigma_{0}\right)$, in section 3.2. In the first paragraph of section 3.2.1, we explain that the situation is the same as in [1] : because of a loss of regularity in the controllability (up to codimension one) of the linearized system, the inverse mapping theorem cannot be applied. We deal with this difficulty by using a Nash-Moser theorem stated in the second paragraph of section 3.2.2. This theorem is an adaptation of L. Hörmander's one in [12], it is slightly different from the one used in [1]. The two last paragraphs of section 3.2.3 are dedicated to the application of this theorem.

Then, in section 3.3 , we justify that the nonlinear term in $\left(\Sigma_{0}\right)$ allows to move in the two directions which are missed by the linearized system. We fix the time, we perform a power series expansion and we prove that the second order term allows to move in the two directions $(\psi, S, D)=\left( \pm i \varphi_{1}, 0,0\right)$. This method is classical to study the local controllability of finite dimensional systems. It has already been used for an infinite dimensional one, the Korteweg-de Vries equation, in [6]. In this reference, an expansion to the second order was not sufficient and it was needed to compute the third order term.

In section 3.4 , we get the local controllability of $\left(\Sigma_{0}\right)$ around $Y^{\theta, 0,0}$ by applying the intermediate values theorem.

### 2.3 Local controllability of $\left(\Sigma_{0}\right)$ around $Y^{k, 0,0}$ for $k \in\{0,1\}$

Again, the classical approach does not work because the linearized system around ( $Y^{k, 0,0}, u \equiv 0$ ) is not controllable for $k \in\{0,1\}$. This result was proved by P. Rouchon in [14]. He proved this linearized system is steady-state controllable, but this result does not imply the same property for the nonlinear system. As noticed in section 4.1, the situation is even worse than the previous one because the linearized system misses an infinite number of directions (half of the projections).

The proof of the local controllability of $\left(\Sigma_{0}\right)$ around $Y^{k, 0,0}$ for $k \in\{0,1\}$ relies on the return method, a method introduced in [2] to solve a stabilisation problem, together with quasi-static transformations as in [5]. The return method has already been used for controllability problems by J.-M. Coron in [5], [3], [4], by A. V. Fursikov and O. Yu. Imanuvilov in [7], by O. Glass in [8], [9], by Th. Horsin in [11] and by E. Sontag in [15].

This strategy is divided in two steps. We explain it with $Y^{0,0,0}$ but everything works similarly with $Y^{1,0,0}$ instead of $Y^{0,0,0}$. First, in section 4.2 , we propose an other trajectory ( $Y^{\gamma, \alpha, \beta}, u \equiv \gamma$ ) such that $\left(\Sigma_{0}\right)$ is locally controllable around $Y^{\gamma, \alpha, \beta}$ in time $T^{*}$. Then, we deduce the local controllability around $Y^{0,0,0}$ in section 4.3 , by using quasi-static transformations, in the same way as in [5] and [1]. We fix $Y_{0}$ closed to $Y^{0,0,0}\left(t_{0}\right)$ and $Y_{f}$ closed to $Y^{0,0,0}\left(t_{f}\right)$ for some real constants $t_{0}$ and $t_{f}$. We use quasi-static transformations in order to move the system

- from $Y_{0}$ to a point $Y_{1}$, which is closed to $Y^{\gamma, \alpha, \beta}(0)$, for some real constants $\alpha, \beta, \gamma$,,
- from a point $Y_{2}$, which is closed to $Y^{\gamma, \alpha, \beta}\left(T^{*}\right)$, to $Y_{f}$.

Thanks to the local controllability around $Y^{\gamma, \alpha, \beta}$, we can move the system from $Y_{1}$ to $Y_{2}$ in finite time, it gives the conclusion. By "quasi-static transformations ", we mean that we use controls $u(t)$ which change slowly.

Finally, in section 5, we prove the local controllability of $\left(\Sigma_{0}\right)$ around $Y^{\gamma, \alpha, \beta}$. Again, this local controllability result cannot be proved by using the classical approach because the linearized system around $Y^{\gamma, \alpha, \beta}$ is not controllable. In section 5.1, we explain that this linearized system misses the two directions $(\psi, S, D)=(0, \pm 1,0)$. We conclude with the same strategy as in section 2.2.

In subsection 5.2 we prove that the same strategy as in [1] leads to the local controllability, in time T , of $\left(\Sigma_{0}\right)$, when the state is $(\psi, D)$ and the control is $u$, around $Y^{\gamma, \alpha, \beta}$. A loss of regularity in the controllability (up to codimension one) of the linearized system around ( $Y^{\gamma, \alpha, \beta}, u \equiv \gamma$ ) prevents us from applying the inverse mapping theorem. We use the Nash-Moser theorem stated in section 3.2.2, in the context given in section 5.2.1. The two last paragraphs of section 5.2 .2 are dedicated to the application of this theorem.

In section 5.3, we prove that the second order term allows to move in the two directions $(\psi=$ $0, S= \pm 1, D=0$ ) which are missed by the linearized system.

In section 5.4, we get the local controllability around $Y^{\gamma, \alpha, \beta}$ by applying the intermediate values theorem.

## 3 Local controllability of $\left(\Sigma_{0}\right)$ around $Y^{\theta, 0,0}$

In all the section $3, \theta \in(0,1)$ is fixed. The aim of this section is the proof of the following result
Theorem 2 Let $T:=4 / \pi$. There exists $\eta>0$ such that, for every $\left(\psi_{0}, S_{0}, D_{0}\right),\left(\psi_{f}, S_{f}, D_{f}\right) \in$ $\left[\mathcal{S} \cap H_{(0)}^{7}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}$ with

$$
\begin{array}{r}
\left\|\psi_{0}-\psi_{\theta}(0)\right\|_{H^{7}}+\left|S_{0}\right|+\left|D_{0}\right|<\eta \\
\left\|\psi_{f}-\psi_{\theta}(T)\right\|_{H^{7}}+\left|S_{f}\right|+\left|D_{f}\right|<\eta
\end{array}
$$

there exists a trajectory $(\psi, S, D)$ of $\left(\Sigma_{0}\right)$ on $[0,2 T]$ such that

$$
\begin{gathered}
(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right) \\
(\psi(2 T), S(2 T), D(2 T))=\left(\psi_{f}, S_{f}, D_{f}\right)
\end{gathered}
$$

and $u \in H_{0}^{1}((0,2 T), \mathbb{R})$.

### 3.1 Controllability up to codimension one of the linearized system around

 ( $Y^{\theta, 0,0}, u \equiv 0$ )Let us introduce, for $\psi \in \mathcal{S}$, the tangent space $T_{\mathcal{S}}(\psi)$ to the $L^{2}(I, \mathbb{C})$-sphere at the point $\psi$,

$$
T_{\mathcal{S}} \psi:=\left\{\varphi \in L^{2}(I, \mathbb{C}) ; \Re<\varphi, \psi>=0\right\}
$$

and for $k=2, \ldots, 9$, the following subspace of $H^{k}(I, \mathbb{C})$,

$$
H_{(0)}^{k}(I, \mathbb{C}):=\left\{\varphi \in H^{k}(I, \mathbb{C}) ; A^{n} \varphi \in H_{0}^{1}(I, \mathbb{C}) \text { for } n \in \mathbb{N}, n \leqslant(k-1) / 2\right\}
$$

The linearized control system around $\left(Y_{\theta}, u \equiv 0\right)$ is

$$
\left(\Sigma_{\theta}^{l}\right)\left\{\begin{array}{l}
i \frac{\partial \Psi}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \Psi}{\partial q^{2}}-w q \psi_{\theta} \\
\Psi(t, \pm 1 / 2)=0 \\
\dot{s}=w \\
\dot{d}=s
\end{array}\right.
$$

It is a control system where

- the state is $(\Psi, s, d)$ with $\Psi(t) \in T_{\mathcal{S}}\left(\psi_{\theta}(t)\right)$,
- the control is the real valued function $t \mapsto w(t)$.

Proposition 1 Let $T>0$ and $(\Psi, s, d)$ be a trajectory of $\left(\Sigma_{\theta}^{l}\right)$ on $[0, T]$. Then, the function

$$
t \mapsto \Im\left(<\Psi(t), \sqrt{1-\theta} \psi_{1}(t)-\sqrt{\theta} \psi_{2}(t)>\right)
$$

is constant on $[0, T]$. Thus, the control system $\left(\Sigma_{\theta}^{l}\right)$ is not controllable.
Proof : Let us consider the function $\xi_{\theta}(t):=\sqrt{1-\theta} \psi_{1}(t)-\sqrt{\theta} \psi_{2}(t)$. We have

$$
\begin{aligned}
i \frac{\partial \xi_{\theta}}{\partial t} & =-\frac{1}{2} \frac{\partial^{2} \xi_{\theta}}{\partial q^{2}} \\
\frac{d}{d t}\left(\Im<\Psi(t), \xi_{\theta}(t)>\right) & =\Im\left(i w<q \psi_{\theta}(t), \xi_{\theta}(t)>\right)
\end{aligned}
$$

The explicit expressions of $\psi_{\theta}$ and $\xi_{\theta}$ provide, for every $t$,

$$
<q \psi_{\theta}(t), \xi_{\theta}(t)>\in i \mathbb{R}
$$

which gives the conclusion.
Let $T>0$, and $\Psi_{0} \in T_{\mathcal{S}}\left(\psi_{\theta}(0)\right), \Psi_{f} \in T_{\mathcal{S}}\left(\psi_{\theta}(T)\right)$. A necessary condition for the existence of a trajectory of $\left(\Sigma_{\theta}^{l}\right)$ satisfying $\Psi(0)=\Psi_{0}$ and $\Psi(T)=\Psi_{f}$ is

$$
\Im\left(<\Psi_{f}, \sqrt{1-\theta} \psi_{1}(T)-\sqrt{\theta} \psi_{2}(T)>\right)=\Im\left(<\Psi_{0}, \sqrt{1-\theta} \varphi_{1}-\sqrt{\theta} \varphi_{2}>\right)
$$

This equality does not happen for an arbitrary choice of $\Psi_{0}$ and $\Psi_{f}$. Thus $\left(\Sigma_{\theta}^{l}\right)$ is not controllable.
Proposition 2 Let $T>0,\left(\Psi_{0}, s_{0}, d_{0}\right),\left(\Psi_{f}, s_{f}, d_{f}\right) \in H_{(0)}^{3}(I, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ be such that

$$
\begin{gather*}
\Re<\Psi_{0}, \psi_{\theta}(0)>=\Re<\Psi_{f}, \psi_{\theta}(T)>=0  \tag{4}\\
\Im<\Psi_{f}, \sqrt{1-\theta} \varphi_{1} e^{-i \lambda_{1} T}-\sqrt{\theta} \varphi_{2} e^{-i \lambda_{2} T}>=\Im<\Psi_{0}, \sqrt{1-\theta} \varphi_{1}-\sqrt{\theta} \varphi_{2}> \tag{5}
\end{gather*}
$$

There exists $w \in L^{2}((0, T), \mathbb{R})$ such that the solution of $\left(\Sigma_{\theta}^{l}\right)$ with control $w$ and such that $(\Psi(0), s(0), d(0))=$ $\left(\Psi_{0}, s_{0}, d_{0}\right)$ satisfies $(\Psi(T), s(T), d(T))=\left(\Psi_{f}, s_{f}, d_{f}\right)$.

Remark 1 The condition (5) means that we miss exactly two directions, which are $(\Psi, s, d)=$ $\left( \pm i \xi_{\theta}, 0,0\right)$. Thus, if we want to control the components $<\Psi, \varphi_{k}>$ for $k \geqslant 2$ and $\Re<\Psi, \varphi_{1}>$ then, we cannot control $\Im<\Psi, \varphi_{1}>$. This is why we say that we miss the two directions $(\Psi, s, d)=\left( \pm i \varphi_{1}, 0,0\right)$.

Proof : Let $\left(\Psi_{0}, s_{0}, d_{0}\right) \in L^{2}(I, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ with $\Psi_{0} \in T_{\mathcal{S}}\left(\psi_{\theta}(0)\right)$ and $T>0$. Let $(\Psi, s, d)$ be a solution of $\left(\Sigma_{\theta}^{l}\right)$ with $(\Psi(0), s(0), d(0))=\left(\Psi_{0}, s_{0}, d_{0}\right)$ and a control $w \in L^{2}((0, T), \mathbb{R})$. We have the following equality in $L^{2}(I, \mathbb{C})$

$$
\Psi(t)=\sum_{k=1}^{\infty} x_{k}(t) \varphi_{k} \text { where } x_{k}(t):=<\Psi(t), \varphi_{k}>\forall k \in \mathbb{N}
$$

Using the equation satisfied by $\Psi$, we get

$$
\begin{gather*}
x_{2 k}(t)=\left(<\Psi_{0}, \varphi_{2 k}>+i \sqrt{1-\theta} b_{2 k} \int_{0}^{t} w(\tau) e^{i\left(\lambda_{2 k}-\lambda_{1}\right) \tau} d \tau\right) e^{-i \lambda_{2 k} t},  \tag{6}\\
x_{2 k-1}(t)=\left(<\Psi_{0}, \varphi_{2 k-1}>+i \sqrt{\theta} c_{2 k-1} \int_{0}^{t} w(\tau) e^{i\left(\lambda_{2 k-1}-\lambda_{2}\right) \tau} d \tau\right) e^{-i \lambda_{2 k-1} t} \tag{7}
\end{gather*}
$$

where, for every $k \in \mathbb{N}^{*}, b_{k}:=<q \varphi_{k}, \varphi_{1}>$ and $c_{k}:=<q \varphi_{k}, \varphi_{2}>$. Thanks to the explicit expression of the functions $\varphi_{k}$ (see (3)), we get

$$
b_{k}=\left\{\begin{array}{l}
0 \text { if } k \text { is odd, }  \tag{8}\\
\frac{-8(-1)^{k / 2} k}{\pi^{2}(1+k)^{2}(1-k)^{2}} \text { if } k \text { is even },
\end{array} \quad c_{k}=\left\{\begin{array}{l}
\frac{16(-1)^{(k-1) / 2} k}{\pi^{2}(k+2)^{2}(k-2)^{2}} \text { if } k \text { is odd }, \\
0 \text { if } k \text { is even }
\end{array}\right.\right.
$$

Let $\left(\Psi_{f}, s_{f}, d_{f}\right) \in L^{2}(I, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ with $\Psi_{f} \in T_{\mathcal{S}}\left(\psi_{\theta}(T)\right)$. The equality $(\Psi(T), s(T), d(T))=$ $\left(\Psi_{f}, s_{f}, d_{f}\right)$ is equivalent to the following moment problem on $w$,

$$
\left\{\begin{array}{l}
\int_{0}^{T} w(t) e^{i\left(\lambda_{2 k}-\lambda_{1}\right) t} d t=\frac{-i}{\sqrt{1-\theta} b_{2 k}}\left(<\Psi_{f}, \varphi_{2 k}>e^{i \lambda_{2 k} T}-<\Psi_{0}, \varphi_{2 k}>\right), \forall k \in \mathbb{N}^{*}  \tag{9}\\
\int_{0}^{T} w(t) e^{i\left(\lambda_{2 k-1}-\lambda_{2}\right) t} d t=\frac{-i}{\sqrt{\theta} c_{2 k-1}}\left(<\Psi_{f}, \varphi_{2 k-1}>e^{i \lambda_{2 k-1} T}-<\Psi_{0}, \varphi_{2 k-1}>\right), \forall k \in \mathbb{N}^{*} \\
\int_{0}^{T} w(t) d t=s_{f}-s_{0} \\
\int_{0}^{T}(T-t) w(t) d t=d_{f}-d_{0}-s_{0} T
\end{array}\right.
$$

In the two first equalities of (9) with $k=1$, the left hand sides are complex conjugate numbers because $w$ is real valued. Thus a necessary condition on $\Psi_{0}$ and $\Psi_{f}$ for the existence of $w \in L^{2}((0, T), \mathbb{R})$ solution of (9) is

$$
\begin{equation*}
\frac{1}{\sqrt{1-\theta}}\left(\overline{<\Psi_{f}, \varphi_{2}>} e^{-i \lambda_{2} T}-\overline{<\Psi_{0}, \varphi_{2}>}\right)=\frac{-1}{\sqrt{\theta}}\left(<\Psi_{f}, \varphi_{1}>e^{i \lambda_{1} T}-<\Psi_{0}, \varphi_{1}>\right) \tag{10}
\end{equation*}
$$

The equality of the real parts of the two sides in (10) is guaranteed by (4). The equality of the imaginary parts of the two sides in (10) is equivalent to (??). Under the assumption $\Psi_{0}, \Psi_{f} \in H_{(0)}^{3}(I, \mathbb{C})$, the right hand side of (9) defines a sequence in $l^{2}$. Then, the existence, for every $T>0$, of $w \in L^{2}((0, T), \mathbb{R})$ solution of (9) is a classical result on trigonometric moment problems.

### 3.2 Local controllability up to codimension one of $\left(\Sigma_{0}\right)$ around $\left(Y^{\theta, 0,0}, u \equiv 0\right)$

Let us introduce the following closed subspace of $L^{2}(I, \mathbb{C})$

$$
V:=\overline{\operatorname{Span}\left\{\varphi_{k} ; k \geqslant 2\right\}}
$$

and the orthogonal projection $\mathcal{P}: L^{2}(I, \mathbb{C}) \rightarrow V$. The aim of this section is the proof of the following result.

Theorem 3 Let $T:=4 / \pi$. There exists $C>0, \delta>0$ and a continuous map

$$
\begin{array}{cccccc}
\Gamma: & \mathcal{V}(0) & \times & \mathcal{V}(T) & \rightarrow & H_{0}^{1}((0, T), \mathbb{R}) \\
& \left(\left(\psi_{0}, S_{0}, D_{0}\right)\right. & , & \left.\left(\widetilde{\psi_{f}}, S_{f}, D_{f}\right)\right) & \mapsto & u
\end{array}
$$

where

$$
\mathcal{V}(0):=\left\{\left(\psi_{0}, S_{0}, D_{0}\right) \in\left[\mathcal{S} \cap H_{(0)}^{7}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} ;\left\|\psi_{0}-\psi_{\theta}(0)\right\|_{H^{7}}+\left|S_{0}\right|+\left|D_{0}\right|<\delta\right\}
$$

$\mathcal{V}(T):=\left\{\left(\widetilde{\psi_{f}}, S_{f}, D_{f}\right) \in\left[H_{(0)}^{7}(I, \mathbb{C}) \cap V \cap B_{L^{2}}(0,1)\right] \times \mathbb{R} \times \mathbb{R} ;\left\|\widetilde{\psi_{f}}-\mathcal{P} \psi_{\theta}(T)\right\|_{H^{7}}+\left|S_{f}\right|+\left|D_{f}\right|<\delta\right\}$, such that, for every $\left(\left(\psi_{0}, S_{0}, D_{0}\right),\left(\widetilde{\psi_{f}}, S_{f}, D_{f}\right)\right) \in \mathcal{V}(0) \times \mathcal{V}(T)$, the trajectory of $\left(\Sigma_{0}\right)$ with control $\Gamma\left(\psi_{0}, S_{0}, D_{0}, \widetilde{\psi_{f}}, S_{f}, D_{f}\right)$ such that $(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right)$ satisfies

$$
(\mathcal{P} \psi(T), S(T), D(T))=\left(\widetilde{\psi_{f}}, S_{f}, D_{f}\right)
$$

and

$$
\begin{aligned}
\left\|\Gamma\left(\psi_{0}, S_{0}, D_{0}, \widetilde{\psi_{f}}, S_{f}, D_{f}\right)\right\|_{H_{0}^{1}((0, T), \mathbb{R})} \leqslant & C\left[\left\|\mathcal{P}\left(\psi_{0}-\psi_{\theta}(0)\right)\right\|_{H^{7}(I, \mathbb{C})}+\left|S_{0}\right|+\left|D_{0}\right|+\right. \\
& \left.\left\|\widetilde{\psi_{f}}-\mathcal{P} \psi_{\theta}(T)\right\|_{H^{7}(I, \mathbb{C})}+\left|S_{f}\right|+\left|D_{f}\right|\right]
\end{aligned}
$$

### 3.2.1 The inverse mapping theorem cannot be applied

In our situation, in order to prove the Theorem 3 with the classical approach, we would like to apply the inverse mapping theorem to the map

$$
\Phi:\left(\psi_{0}, S_{0}, D_{0}, u\right) \mapsto\left(\psi_{0}, S_{0}, D_{0}, \mathcal{P} \psi(T), S(T), D(T)\right)
$$

where $\psi$ solves

$$
\left\{\begin{array}{l}
i \dot{\psi}=-\frac{1}{2} \psi^{\prime \prime}-u q \psi \\
\dot{S}=w \\
\dot{D}=S
\end{array}\right.
$$

with $(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right)$.
The map $\Phi$ is $C^{1}$ between the following spaces
$\Phi:\left[\mathcal{S} \cap H_{(0)}^{2}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times L^{2}((0, T), \mathbb{R}) \rightarrow\left[\mathcal{S} \cap H_{(0)}^{2}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap B_{L^{2}}(0,1) \cap H_{(0)}^{2}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}$,
$\Phi:\left[\mathcal{S} \cap H_{(0)}^{3}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times H_{0}^{1}((0, T), \mathbb{R}) \rightarrow\left[\mathcal{S} \cap H_{(0)}^{3}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap B_{L^{2}}(0,1) \cap H_{(0)}^{3}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}$. Thus, in order to apply the inverse mapping theorem, we would need to construct a right inverse to the map $d \Phi\left(\psi_{\theta}(0), 0,0,0\right)$ which maps the following spaces

$$
\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{2}\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{2}\right] \times \mathbb{R} \times \mathbb{R} \rightarrow\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{2}\right] \times \mathbb{R} \times \mathbb{R} \times L^{2}((0, T), \mathbb{R})
$$

or

$$
\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{3}\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{3}\right] \times \mathbb{R} \times \mathbb{R} \rightarrow\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{3}\right] \times \mathbb{R} \times \mathbb{R} \times H_{0}^{1}((0, T), \mathbb{R})
$$

The controllability up to codimension one proved for the linearized system around ( $Y_{\theta}, u \equiv 0$ ) only provides a right inverse for $d \Phi\left(\psi_{\theta}(0), 0,0,0\right)$ which maps the following spaces

$$
\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{3}\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{3}\right] \times \mathbb{R} \times \mathbb{R} \rightarrow\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{3}\right] \times \mathbb{R} \times \mathbb{R} \times L^{2}((0, T), \mathbb{R})
$$

In order to deal with this loss of regularity in the controllability of the linearized system around $\left(Y^{\theta, 0,0}, u \equiv 0\right)$, we use a Nash-Moser implicit function theorem stated in the following section. It is an adaptation of L. Hörmander's one in [12], it is slightly different from the one proved in [1, section 3.2]. The use of the projection $\mathcal{P}$ introduce changes in the statement and the proof so we write them completely.

### 3.2.2 The Nash-Moser theorem used

As in [1], we consider a decreasing family of Hilbert spaces $\left(E_{a}\right)_{a \in\{1, \ldots, 9\}}$ with continuous injections $E_{b} \rightarrow E_{a}$ of norm $\leqslant 1$ when $b \geqslant a$. Suppose we have given linear operators $S_{\lambda}: E_{1} \rightarrow E_{9}$ for $\lambda \geqslant 1$. We assume there exists a constant $K>0$ such that for every $a \in\{1, \ldots, 9\}$, for every $\lambda \geqslant 1$ and for every $u \in E_{a}$ we have

$$
\begin{gather*}
\left\|S_{\lambda} u\right\|_{b} \leqslant K\|u\|_{a}, \forall b \in\{1, \ldots, a\},  \tag{11}\\
\left\|S_{\lambda} u\right\|_{b} \leqslant K \lambda^{b-a}\|u\|_{a}, \forall b \in\{a+1, \ldots, 9\}  \tag{12}\\
\left\|u-S_{\lambda} u\right\|_{b} \leqslant K \lambda^{b-a}\|u\|_{a}, \forall b \in\{1, \ldots, a-1\},  \tag{13}\\
\left\|\frac{d}{d \lambda} S_{\lambda} u\right\|_{b} \leqslant K \lambda^{b-a-1}\|u\|_{a}, \forall b \in\{1, \ldots, 9\} . \tag{14}
\end{gather*}
$$

Then, we have the convexity of the norms (see [12] for the proof) : there exists a constant $c \geqslant 1$ such that, for every $\lambda \in[0,1]$, for every $a, b \in\{1, \ldots, 9\}$ such that $a \leqslant b, \lambda a+(1-\lambda) b \in \mathbb{N}$ and for every $u \in E_{b}$,

$$
\|u\|_{\lambda a+(1-\lambda) b} \leqslant c\|u\|_{a}^{\lambda}\|u\|_{b}^{1-\lambda} .
$$

We fix a sequence $1=\theta_{0}<\theta_{1}<\ldots \rightarrow \infty$ of the form $\theta_{j}=(j+1)^{\delta}$ where $\delta>0$. We set $\Delta_{j}:=\theta_{j+1}-\theta_{j}$ and we introduce

$$
R_{j} u:=\frac{1}{\Delta_{j}}\left(S_{\theta_{j+1}}-S_{\theta_{j}}\right) u \text { if } j>0 \text { and } R_{0} u:=\frac{1}{\Delta_{0}} S_{\theta_{1}} u .
$$

Thanks to (13), we have

$$
u=\sum_{j=0}^{\infty} \Delta_{j} R_{j} u
$$

with convergence in $E_{b}$ when $u \in E_{a}$ and $a>b$. As noticed in [1], it follows from (14) that there exists $K^{\prime}>0$ such that, for every $a \in\{1, \ldots, 9\}$, for every $u \in E_{a}$, for every $b \in\{1, \ldots, 9\}$, for every $\delta \in(0,2]$ and for every $j \in \mathbb{N}^{*}$,

$$
\left\|R_{j} u\right\|_{b} \leqslant K^{\prime} \theta_{j}^{b-a-1}\|u\|_{a} .
$$

Let $a_{1}, a_{2} \in \mathbb{N}$ and $a \in \mathbb{R}$ be such that $1 \leqslant a_{1}<a<a_{2} \leqslant 9$. We define the space

$$
E_{a}^{\prime}:=\left\{\sum_{j=0}^{\infty} \Delta_{j} u_{j} ; u_{j} \in E_{a_{2}}, \exists M>0 / \forall j,\left\|u_{j}\right\|_{b} \leqslant M \theta_{j}^{b-a-1} \text { for } b=a_{1}, a_{2}\right\},
$$

with the norm $\|u\|_{a}^{\prime}$ given by the infimum of $M$ over all such decomposition of $u$. This space does not depend on the choice of $a_{1}$ and $a_{2}$ (see [12] for the proof). The norm $\|\cdot\|_{a}^{\prime}$ is stronger than the norm $\|\cdot\|_{b}$ when $b<a$,

$$
\begin{equation*}
\|u\|_{b} \leqslant M_{b, a}\|u\|_{a}^{\prime} \tag{15}
\end{equation*}
$$

and $\|\cdot\|_{a}^{\prime}$ is weaker than $\|\cdot\|_{a}$,

$$
\|u\|_{a}^{\prime} \leqslant K^{\prime}\|u\|_{a} .
$$

As noticed in [1] and [12], there exists a constant $K^{\prime \prime}$ such that, for every $a \in\{1, \ldots, 9\}$, for every $\theta \geqslant 1$, for every $b<a$ and for every $u \in E_{a}^{\prime}$ we have

$$
\begin{equation*}
\left\|u-S_{\theta} u\right\|_{b} \leqslant K^{\prime \prime} \theta^{b-a}\|u\|_{a}^{\prime} . \tag{16}
\end{equation*}
$$

We have another family $\left(F_{a}\right)_{a \in\{1, \ldots, 9\}}$ with the same properties as above, we use the same notations for the smoothing operators $S_{\lambda}$. Moreover, we assume the injection $F_{b} \rightarrow F_{a}$ is compact when $b>a$.

Theorem 4 Let $\alpha$ and $\beta$ be fixed positive real numbers such that

$$
\begin{equation*}
4<\alpha<\beta<7 \text { and } \beta-\alpha \geqslant 2 . \tag{17}
\end{equation*}
$$

Let $\mathcal{P}$ be a continuous linear operator from $F_{b}$ to $F_{b}$ of norm $\leqslant 1$, for $b=1, \ldots, 9$, such that $\mathcal{P} S_{\theta}=S_{\theta} \mathcal{P}$. Let $V$ be a convex $E_{\alpha}^{\prime}$-neighbourhood of 0 and $\Phi$ a map from $V \cap E_{7}$ to $F_{\beta}$ which is twice differentiable and satisfies

$$
\begin{equation*}
\left\|\Phi^{\prime \prime}(u ; v, w)\right\|_{7} \leqslant C \sum\left(1+\|u\|_{m_{j}^{\prime}}\right)\|v\|_{m_{j}^{\prime \prime}}\|w\|_{m_{j}^{\prime \prime \prime}} \tag{18}
\end{equation*}
$$

where the sum is finite, all the subscripts belong to $\{1,3,5,7\}$ and satisfy

$$
\begin{equation*}
\max \left(m_{j}^{\prime}-\alpha, 0\right)+\max \left(m_{j}^{\prime \prime}, 2\right)+m_{j}^{\prime \prime \prime}<2 \alpha, \forall j . \tag{19}
\end{equation*}
$$

We assume that $\Phi: E_{3} \rightarrow F_{3}$ is continuous. We also assume that $\Phi^{\prime}(v)$, for $v \in V \cap E_{9}$, has a right inverse $\psi(v)$ mapping $F_{9}$ into $E_{7}$, that $(v, g) \mapsto \psi(v) g$ is continuous from $\left(V \cap E_{9}\right) \times F_{9}$ to $E_{7}$ and that there exists a constant $C$ such that for every $(v, g) \in\left(V \cap E_{9}\right) \times F_{9}$,

$$
\begin{gather*}
\|\psi(v) g\|_{1} \leqslant C\left[\|\mathcal{P} g\|_{3}+\|v\|_{3}\|g\|_{3}\right],  \tag{20}\\
\|\psi(v) g\|_{3} \leqslant C\left[\|\mathcal{P} g\|_{5}+\|v\|_{3}\|g\|_{5}+\|v\|_{5}\|g\|_{3}\right],  \tag{21}\\
\|\psi(v) g\|_{5} \leqslant C\left[\|\mathcal{P} g\|_{7}+\|v\|_{3}\|g\|_{7}+\|v\|_{5}\|g\|_{5}+\left(\|v\|_{7}+\|v\|_{5}^{2}\right)\|g\|_{3}\right],  \tag{22}\\
\|\psi(v) g\|_{7} \leqslant C\left[\|\mathcal{P} g\|_{9}+\|v\|_{3}\|g\|_{9}+\|v\|_{5}\|g\|_{7}+\left(\|v\|_{7}+\|v\|_{5}^{2}\right)\|g\|_{5}+\left(\|v\|_{9}+\|v\|_{7}\|v\|_{5}+\|v\|_{5}^{3}\right)\|g\|_{3}\right] . \tag{23}
\end{gather*}
$$

For every $f \in F_{\beta}^{\prime}$ with sufficiently small norm, there exists $u \in E_{3}$ such that $\Phi(u)=\Phi(0)+f$.

Remark 2 The Nash-Moser theorem used in [1] corresponds to the case $\mathcal{P}=I d$. In what follows, we only emphasize where the projection $\mathcal{P}$ appears in the proof of [1, section 3.2].

Proof : Let $g \in F_{\beta}^{\prime}$. There exist decompositions (see [1, proof of Theorem 6])

$$
\begin{gather*}
g=\sum \Delta_{j} g_{j} \text { with }\left\|g_{j}\right\|_{b} \leqslant K^{\prime} \theta_{j}^{b-\beta-1}\|g\|_{\beta}^{\prime} \text { for every } b \in\{1, \ldots, 9\}  \tag{24}\\
\mathcal{P} g=\sum \Delta_{j} \mathcal{P} g_{j} \text { with }\left\|\mathcal{P} g_{j}\right\|_{b} \leqslant K^{\prime} \theta_{j}^{b-\beta-1}\|\mathcal{P} g\|_{\beta}^{\prime} \text { for every } b \in\{1, \ldots, 9\} . \tag{25}
\end{gather*}
$$

To get (25), we have used $\mathcal{P} S_{\theta}=S_{\theta} \mathcal{P}$. We claim that if $\|g\|_{\beta}^{\prime}$ is small enough, we can define a sequence $u_{j} \in E_{7} \cap V$ with $u_{0}=0$ by the recursive formula

$$
\begin{equation*}
u_{j+1}:=u_{j}+\Delta_{j} \dot{u}_{j}, \quad \dot{u}_{j}:=\psi\left(v_{j}\right) g_{j}, \quad v_{j}:=S_{\theta_{j}} u_{j} . \tag{26}
\end{equation*}
$$

We also claim that there exist constants $C_{1}, C_{2}, C_{3}$ such that for every $j \in \mathbb{N}$,

$$
\begin{gather*}
\left\|\dot{u}_{j}\right\|_{a} \leqslant C_{1}\|\mathcal{P} g\|_{\beta}^{\prime} \theta_{j}^{a-\alpha-1}, a \in\{1,3,5,7\}  \tag{27}\\
\left\|v_{j}\right\|_{a} \leqslant C_{2}\|\mathcal{P} g\|_{\beta}^{\prime} \theta_{j}^{a-\alpha}, a \in\{5,7,9\}  \tag{28}\\
\left\|v_{j}\right\|_{3} \leqslant C_{2}\|\mathcal{P} g\|_{\beta}^{\prime} \\
\left\|u_{j}-v_{j}\right\|_{a} \leqslant C_{3}\|\mathcal{P} g\|_{\beta}^{\prime} \theta_{j}^{a-\alpha}, a \in\{1,3,5,7\} . \tag{29}
\end{gather*}
$$

More precisely, we prove by induction on $k$ the following property

$$
\begin{aligned}
\left(P_{k}\right): & u_{j} \text { is well defined for } j=0, \ldots, k+1, \\
& (27) \text { is satisfied for } j=0, \ldots, k \\
& (28),(29) \text { are satisfied for } j=0, \ldots, k+1 .
\end{aligned}
$$

The property $\left(P_{0}\right)$ is easy to be checked. Let $k \in \mathbb{N}^{*}$. We suppose the property $\left(P_{k-1}\right)$ is true and we prove $\left(P_{k}\right)$.

Let us introduce a real number $\rho>0$ such that, for every $u \in E_{\alpha}^{\prime},\|u\|_{\alpha}^{\prime} \leqslant \rho$ implies $u \in V$. With the same kind of calculus as in [1], we get (27), (28), (29) with

$$
\begin{gathered}
C_{1}:=8 C K^{\prime} \\
C_{2}:=K C_{1} \max \left\{\frac{1}{7-\alpha}, \frac{2^{\delta(\alpha-4)}}{5-\alpha}, \frac{2^{\delta(\alpha-2}}{\alpha-1}\right\}, \\
C_{3}:=C_{1} \max \left\{\frac{1+K}{7-\alpha}, K^{\prime \prime}\right\},
\end{gathered}
$$

for every $g \in F_{\beta}^{\prime}$ with

$$
\|g\|_{\beta}^{\prime} \leqslant \min \left\{\frac{\rho}{K C_{1}}, \frac{1}{C_{2}},\right\}
$$

The inequality (27) proves that $\left(u_{k}\right)$ converges in $E_{3}$ to the vector $u:=\sum_{j=0}^{\infty} \Delta_{j} \dot{u}_{j}$ and

$$
\begin{equation*}
\|u\|_{3} \leqslant \widetilde{C_{1}}\|\mathcal{P} g\|_{\beta}^{\prime} \text { where } \widetilde{C_{1}}:=C_{1}\left(\sum_{j=0}^{\infty} \Delta_{j} \theta_{j}^{2-\alpha}\right) \tag{30}
\end{equation*}
$$

Now, let us consider the limit of $\left(\Phi\left(u_{k}\right)\right)_{k \in \mathbb{N}}$. We have

$$
\Phi\left(u_{j+1}\right)-\Phi\left(u_{j}\right)=\Phi\left(u_{j}+\Delta_{j} \dot{u}_{j}\right)-\Phi\left(u_{j}\right)=\Delta_{j}\left(e_{j}^{\prime}+e_{j}^{\prime \prime}+g_{j}\right)
$$

where

$$
\begin{gathered}
e_{j}^{\prime}:=\frac{1}{\Delta_{j}}\left(\Phi\left(u_{j}+\Delta_{j} \dot{u}_{j}\right)-\Phi\left(u_{j}\right)-\Phi^{\prime}\left(u_{j}\right) \Delta_{j} \dot{u}_{j}\right), \\
e_{j}^{\prime \prime}:=\left(\Phi^{\prime}\left(u_{j}\right)-\Phi^{\prime}\left(v_{j}\right)\right) \dot{u}_{j} .
\end{gathered}
$$

Thanks to (18), (19), (27), (28), (29) and the same calculus as in [1] we get the existence of $\epsilon, C_{4}, C_{5}>0$ such that, for every $j \in \mathbb{N}$,

$$
\begin{equation*}
\left\|e_{j}^{\prime}\right\|_{7} \leqslant C_{4}\|\mathcal{P} g\|_{\beta}^{2} \theta_{j}^{-1-\epsilon}, \quad\left\|e_{j}^{\prime \prime}\right\|_{7} \leqslant C_{5}\|\mathcal{P} g\|_{\beta}^{2} \theta_{j}^{-1-\epsilon} . \tag{31}
\end{equation*}
$$

Thus $\sum \Delta_{j}\left(e_{j}^{\prime}+e_{j}^{\prime \prime}\right)$ converges in $F_{7}$. Let us denote $T(g)$ its sum,

$$
T(g):=\sum_{j=0}^{\infty} \Delta_{j}\left(e_{j}^{\prime}+e_{j}^{\prime \prime}\right) .
$$

Thanks to (31), we get the existence of $C_{6}>0$ such that

$$
\|T g\|_{7} \leqslant C_{6}\|\mathcal{P} g\|_{\beta}^{\prime 2}
$$

The continuity of $\Phi$ gives $\Phi\left(u_{k}\right) \rightarrow \Phi(u)$ in $F_{3}$, thus we have the following equality in $F_{3}$

$$
\Phi(u)=\Phi(0)+T(g)+g .
$$

Let us fix $f \in F_{\beta}^{\prime}$. We search $u$ such that $\Phi(u)=\Phi(0)+f$. It is sufficient to find $g \in F_{\beta}^{\prime}$ such that $g+T g=f$. This is equivalent to prove the existence of a fixed point for the map

$$
\begin{array}{rllc}
F: \quad F_{\beta}^{\prime} & \rightarrow & F_{\beta}^{\prime} \\
g & \mapsto & & \mapsto-T(g) .
\end{array}
$$

We conclude by applying the Leray-Shauder fixed point theorem.
In our situation, we need the continuity of the map $f \mapsto u$ in order to apply the intermediate values theorem in section 3.4. This property can be proved by applying the Banach fixed point theorem instead of the Leray-Shauder fixed point theorem in the previous proof. In order to do this, we need more assumptions, which are given in the next theorem.

Theorem 5 Let us consider the same assumptions as in Theorem 4. We assume moreover that, for every $u, \tilde{u} \in V \cap E_{7}$,

$$
\begin{equation*}
\left\|\Phi^{\prime \prime}(u ; v, w)-\Phi^{\prime \prime}(\tilde{u} ; v, w)\right\|_{7} \leqslant C \sum\left(1+\|u-\tilde{u}\|_{n_{j}^{\prime}}\right)\|v\|_{n_{j}^{\prime \prime}}\|w\|_{n_{j}^{\prime \prime \prime}}, \tag{32}
\end{equation*}
$$

where the sum is finite, all the subscripts belong to $\{1,3,5,7\}$ and satisfy (19) with $m_{j} \leftarrow n_{j}$. We also assume that, for every $v, \tilde{v} \in V \cap E_{9}$,

$$
\begin{gather*}
\|(\psi(v)-\psi(\tilde{v})) g\|_{1} \leqslant C\|v-\tilde{v}\|_{3}\|g\|_{3},  \tag{33}\\
\|(\psi(v)-\psi(\tilde{v})) g\|_{3} \leqslant C\left[\|v-\tilde{v}\|_{3}\|g\|_{5}+\|v-\tilde{v}\|_{5}\|g\|_{3}\right],  \tag{34}\\
\|(\psi(v)-\psi(\tilde{v})) g\|_{5} \leqslant C\left[\|v-\tilde{v}\|_{3}\|g\|_{7}+\|v-\tilde{v}\|_{5}\|g\|_{5}+\left(\|v-\tilde{v}\|_{7}+\|v-\tilde{v}\|_{5}^{2}\right)\|g\|_{3}\right],  \tag{35}\\
\|(\psi(v)-\psi(\tilde{v})) g\|_{7} \leqslant \\
C\left[\|v-\tilde{v}\|_{3}\|g\|_{9}+\|v-\tilde{v}\|_{5}\|g\|_{7}+\right.  \tag{36}\\
\left(\|v-\tilde{v}\|_{7}+\|v-\tilde{v}\|_{5}^{2}\right)\|g\|_{5}+ \\
\\
\left.\left(\|v-\tilde{v}\|_{9}+\|v-\tilde{v}\|_{7}\|v-\tilde{v}\|_{5}+\|v-\tilde{v}\|_{5}^{3}\right)\|g\|_{3}\right] .
\end{gather*}
$$

Then, there exists $C^{\prime}>0, \epsilon>0$ and a continuous map

$$
\begin{array}{rlll}
\Pi: \quad V_{\beta}^{\prime} & \rightarrow E_{3} \\
f & \mapsto &
\end{array}
$$

where

$$
V_{\beta}^{\prime}:=\left\{f \in F_{\beta}^{\prime} ;\|f\|_{\beta}^{\prime}<\epsilon\right\}
$$

such that, for every $f \in V_{\beta}^{\prime}$,

$$
\begin{align*}
& \Phi(\Pi(f))=\Phi(0)+f \\
& \|\Pi(f)\|_{3} \leqslant C^{\prime}\|\mathcal{P} f\|_{\beta}^{\prime} \tag{37}
\end{align*}
$$

Proof : The first part of Theorem 5 has already been proved in [1, appendix C]. Here, we justify the bound (37). Let us recall that under the assumptions $(32),(33),(34),(35),(36)$, the map $T$ is a contraction on a small enough neighbourhood of zero in $F_{\beta}^{\prime}$ : there exists $\delta \in(0,1)$ such that

$$
\|T(g)-T(\tilde{g})\|_{\beta}^{\prime} \leqslant \delta\|g-\tilde{g}\|_{\beta}^{\prime}
$$

Thus, when $f=g+T(g)$ and $\tilde{f}=\tilde{g}+T(\tilde{g})$, we also have

$$
\|g-\tilde{g}\|_{\beta}^{\prime} \leqslant \frac{1}{1-\delta}\|f-\tilde{f}\|_{\beta}^{\prime}
$$

Let $f \in F_{\beta}^{\prime}$ small enough. Let $g \in F_{\beta}^{\prime}$ be the solution of $f=g+T(g)$ given by the Banach fixed point theorem. Using $\tilde{f}=0$, we have

$$
\|g\|_{\beta}^{\prime} \leqslant \frac{1}{1-\delta}\|f\|_{\beta}^{\prime}
$$

Let $u \in E_{3}$ be the vector built in the proof of Theorem 4. Using (30) and

$$
\mathcal{P} g=\mathcal{P} f-\mathcal{P} T(g), \quad\|\mathcal{P} T g\|_{\beta}^{\prime} \leqslant\|T g\|_{\beta}^{\prime} \leqslant C_{6}\|\mathcal{P} g\|_{\beta}^{\prime 2}
$$

we get

$$
\|\mathcal{P} g\|_{\beta}^{\prime} \leqslant 2\|\mathcal{P} f\|_{\beta}^{\prime} \text { when }\|f\|_{\beta}^{\prime} \leqslant \frac{1-\delta}{2 C_{6}}
$$

thus

$$
\|u\|_{3} \leqslant 2 \widetilde{C_{1}}\|\mathcal{P} f\|_{\beta}^{\prime}
$$

We apply Theorems 4 and 5 to the map $\Phi$ defined in section 3.2.1, in a neighbourhood of $\left(\psi_{\theta}(0), 0,0,0\right)$. Our spaces are defined, for $k=1,3,5,7,9$, by

$$
\begin{gathered}
E_{k}:=\left[\mathcal{S} \cap H_{(0)}^{k}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times H_{0}^{(k-1) / 2}((0, T), \mathbb{R}), \\
F_{k}:=\left[\mathcal{S} \cap H_{(0)}^{k}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{k}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}
\end{gathered}
$$

We work on the manifold $\mathcal{S}$ instead of a whole space. It does not matter because we can move the problem to an hyperplane of $L^{2}(I, \mathbb{C})$ by studying a new map

$$
\widetilde{\Phi}\left(\widetilde{\psi_{0}}, S_{0}, D_{0}, u\right):=\Phi\left(p^{-1}\left(\widetilde{\psi_{0}}\right), S_{0}, D_{0}, u\right)
$$

where $p$ is a suitable local diffeomorphism from a neighbourhood of the trajectory $\psi_{\theta}$ in the sphere $\mathcal{S}$ to an hyperplane of $L^{2}(I, \mathbb{C})$. For example, we can use the following one.

Proposition 3 Let $\mathcal{U}:=\left\{\psi \in \mathcal{S} ; \exists t \in[0,4 / \pi],\left\|\psi-\psi_{\theta}(t)\right\|_{L^{2}(I, \mathbb{C})}<\epsilon\right\}$ where $\epsilon>0$ is small enough, $\left.\mathcal{H}:=\left\{\psi \in L^{2}(I, \mathbb{C}) ; \Re<\psi, \varphi_{3}\right\rangle=0\right\}$ and $p: L^{2}(I, \mathbb{C}) \rightarrow \mathcal{H}$ defined by

$$
p(\psi):=\psi-\Re\left(<\psi, \varphi_{3}>\right) \varphi_{3}+\Re\left(<\psi, \varphi_{3}>\right)<\psi, \varphi_{1}>\varphi_{1} .
$$

The map $p$ is a $C^{1}$ diffeomorphism from $\mathcal{U}$ to an open subset of $\mathcal{H}$. Moreover, the norm of $d p(\psi)$ as linear operator from $\left(T_{\mathcal{S}}(\psi),\|.\|_{H^{s}}\right)$ to $\left(\mathcal{H},\|\cdot\|_{H^{s}}\right)$ is uniformly bounded on $\mathcal{U}$ for every integer $s \in[1,7]$.

The proof is similar to the one of [1, Proposition 2 section 3.2].
Now, we build smoothing operators. First, we smooth the wave function. Note that we need a smoothing operator preserving the space $\mathcal{H}$ defined in Proposition 3. Let $s \in C^{\infty}(\mathbb{R}, \mathbb{R})$ be such that

$$
s=1 \text { on }[0,1], 0 \leqslant s \leqslant 1, s=0 \text { on }[2,+\infty) .
$$

We define

$$
\widetilde{S}_{\lambda} \varphi:=\sum_{k=1}^{\infty} s\left(\frac{k}{\lambda}\right)<\varphi, \varphi_{k}>\varphi_{k} .
$$

The proof of the following proposition is easy.
Proposition 4 There exists a constant $K$ such that, for every $a \in\{1, \ldots, 9\}$, for every $\varphi \in H_{(0)}^{a}(I, \mathbb{C})$ and for every $\lambda \geqslant 1$, we have

$$
\begin{gathered}
\left\|\widetilde{S}_{\lambda} \varphi\right\|_{H^{b}} \leqslant K\|\varphi\|_{H^{a}}, b \in\{1, \ldots, a\}, \\
\left\|\widetilde{S}_{\lambda} \varphi\right\|_{H^{b}} \leqslant K \lambda^{b-a}\|\varphi\|_{H^{a}}, b \in\{a+1, \ldots, 9\}, \\
\left\|\varphi-\widetilde{S}_{\alpha} \varphi\right\|_{H^{b}} \leqslant K \lambda^{b-a}\|\varphi\|_{H^{a}}, b \in\{1, \ldots, a-1\}, \\
\left\|\frac{d}{d \lambda} \widetilde{S}_{\lambda} \varphi\right\|_{H^{b}} \leqslant K \lambda^{b-a-1}\|\varphi\|_{H^{a}}, b \in\{1, \ldots, 9\} .
\end{gathered}
$$

The suitable smoothing operators for the control, $\widehat{S}_{\lambda} u$, can be built with convolution products and truncations with a $C^{\infty}$-function with compact support as in [1, section 3.3.2]. This construction is inspired from [10].

Finally, we take on the spaces $E_{k}$

$$
S_{\lambda}\left(\psi_{0}, S_{0}, D_{0}, u\right):=\left(\widetilde{S}_{\lambda} \psi_{0}, S_{0}, D_{0}, \widehat{S}_{\lambda}(u)\right),
$$

and on the spaces $F_{k}$

$$
S_{\lambda}\left(\psi_{0}, S_{0}, D_{0}, \psi_{f}, S_{f}, D_{f}\right):=\left(\widetilde{S}_{\lambda} \psi_{0}, S_{0}, D_{0}, \widetilde{S}_{\lambda}\left(\psi_{f}\right), S_{f}, D_{f}\right)
$$

The bounds (18), (19), (32), (33), (34), (35), (36) can be checked in the same way as in [1]. In the following two sections, we focus on the most difficult part in the application of the Nash-Moser theorem, which is the proof of the existence of a right inverse for the differential, with the bounds (20), (21), (22), (23).

### 3.2.3 Controllability up to codimension one of the linearized system around ( $Y^{\theta, 0,0}, u \equiv 0$ ) and bounds (20), (21), (22), (23)

The aim of this section is the proof of the following proposition.
Proposition 5 Let $T:=4 / \pi$. There exists $C>0$ such that, for every

$$
\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right) \in\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R},
$$

there exists $w \in H_{0}^{3}((0, T), \mathbb{R})$ such that the solution of $\left(\Sigma_{\theta}^{l}\right)$ with control $w$ such that $(\Psi(0), s(0), d(0))=$ $\left(\Psi_{0}, s_{0}, d_{0}\right)$ satisfies $(\mathcal{P} \Psi(T), s(T), d(T))=\left(\widetilde{\Psi_{f}}, s_{f}, d_{f}\right)$ and

$$
\begin{array}{ll}
\|w\|_{L^{2}((0, T), \mathbb{R})} \leqslant C\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{F_{3}}, & \|w\|_{H_{0}^{1}((0, T), \mathbb{R})} \leqslant C\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{F_{5}}, \\
\|w\|_{H_{0}^{2}((0, T), \mathbb{R})} \leqslant C\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{F_{7}}, & \|w\|_{H_{0}^{3}((0, T), \mathbb{R})} \leqslant C\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{F_{9}} .
\end{array}
$$

Moreover, the map

$$
\begin{array}{ccccccccccccc}
{\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{9}\right]}
\end{array} \times \begin{array}{cccccccccc}
\left.V \cap H_{(0)}^{9}\right] & \times & \mathbb{R} & \times & \mathbb{R} & \rightarrow & H_{0}^{3}((0, T), \mathbb{R}) \\
\left(\Psi_{0}\right. & , & s_{0} & , & d_{0} & , & \widetilde{\Psi}_{f} & , & s_{f} & , \\
\left.d_{f}\right) & \mapsto & w
\end{array}
$$

is continuous.
Remark 3 The function $\Psi(T)$ is the unique function $\Psi_{f} \in T_{\mathcal{S}}\left(\psi_{\theta}(T)\right)$ which satisfies (??) and $\mathcal{P} \Psi_{f}=\widetilde{\Psi_{f}}$.

Let us introduce the notations, for $s \in\{0, \ldots, 6\}$

$$
\begin{aligned}
h^{s}(\mathbb{N}, \mathbb{C}):= & \left\{d=\left(d_{k}\right)_{k \in \mathbb{N}} ;\|d\|_{h^{s}(\mathbb{N}, \mathbb{C})}:=\left(\left|d_{0}\right|+\sum_{k=1}^{\infty}\left|k^{s} d_{k}\right|^{2}\right)^{1 / 2}<+\infty\right\}, \\
& h_{r}^{s}(\mathbb{N}, \mathbb{C}):=\left\{d=\left(d_{k}\right)_{k \in \mathbb{N}} \in h^{s}(\mathbb{N}, \mathbb{C}) ; d_{0}, d_{1} \in \mathbb{R}\right\},
\end{aligned}
$$

we write $l_{r}^{2}$ instead of $h_{r}^{0}$.
Proof : Let

$$
\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right) \in\left[T_{\mathcal{S}}\left(\psi_{\theta}(0)\right) \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R},
$$

and $(\Psi, s, d)$ be a solution of $\left(\Sigma_{\theta}^{l}\right)$ with $(\Psi(0), s(0), d(0))=\left(\Psi_{0}, s_{0}, d_{0}\right)$ and a control $w \in H_{0}^{3}((0, T), \mathbb{R})$. As noticed in section 3.1, the equality $(\mathcal{P} \Psi(T), s(T), d(T))=\left(\widetilde{\Psi_{f}}, s_{f}, d_{f}\right)$ is equivalent to

$$
Z(w)=D\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)
$$

where

$$
Z(w):=\left(Z(w)_{k}\right)_{k \in \mathbb{N}} \quad \text { and } \quad D\left(\Psi_{0}, s_{0}, d_{0}, \Psi_{f}, s_{f}, d_{f}\right):=\left(D_{k}\right)_{k \in \mathbb{N}}
$$

are defined by

$$
\begin{aligned}
\quad Z(w)_{0}:=\int_{0}^{T}(T-t) w(t) d t, & Z(w)_{1}:=\int_{0}^{T} w(t) d t, \\
Z(w)_{2 k}:=\int_{0}^{T} w(t) e^{i\left(\lambda_{2 k}-\lambda_{1}\right) t} d t, & Z(w)_{2 k+1}:=\int_{0}^{T} w(t) e^{i\left(\lambda_{2 k+1}-\lambda_{2}\right) t} d t, k \in \mathbb{N}^{*}, \\
D_{0}:=d_{f}-d_{0}-s_{0} T, & D_{1}:=s_{f}-s_{0}, \\
D_{2 k}:=\frac{-i}{\sqrt{1-\theta b_{2 k}}}<\Psi_{f}-\Psi_{0}, \varphi_{2 k}>, & D_{2 k+1}:=\frac{-i}{\sqrt{\theta} c_{2 k+1}}<\widetilde{\Psi_{f}}-\Psi_{0}, \varphi_{2 k+1}>, k \in \mathbb{N}^{*} .
\end{aligned}
$$

Using the behaviour of the coefficients $c_{k}$ and $b_{k}$ given by (8) and standard results about Fourier series, we get a constant $C>0$ such that, for every ( $\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}$ ), for $s=0,2,4,6$,

$$
\left\|D\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{h^{s}(\mathbb{N}, \mathrm{C})} \leqslant C\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{F_{s+3}} .
$$

Thus, it is sufficient to prove the following proposition to end this proof.

Proposition 6 The linear map $Z$ si continuous from $E$ to $F$ for every

$$
(E, F) \in\left\{\left(L^{2}, l_{r}^{2}\right),\left(H_{0}^{1}, h_{r}^{2}\right),\left(H_{0}^{2}, h_{r}^{4}\right),\left(H_{0}^{3}, h_{r}^{6}\right)\right\} .
$$

There exist $C>0$ and a continuous right inverse

$$
Z^{-1}: h_{r}^{6}(\mathbb{N}, \mathbb{C}) \rightarrow H_{0}^{3}((0, T), \mathbb{R})
$$

such that, for every $d \in h_{r}^{6}(\mathbb{N}, \mathbb{C})$,

$$
\begin{array}{ll}
\left\|Z^{-1}(d)\right\|_{L^{2}} \leqslant C\|d\|_{l^{2}}, & \left\|Z^{-1}(d)\right\|_{H_{0}^{1}} \leqslant C\|d\|_{h^{2}} \\
\left\|Z^{-1}(d)\right\|_{H_{0}^{2}} \leqslant C\|d\|_{h^{4}}, & \left\|Z^{-1}(d)\right\|_{H_{0}^{3}} \leqslant C\|d\|_{h^{6}}
\end{array}
$$

Proof : The first statement comes from integrations by parts and standard results about Fourier series. Let us introduce the notations

$$
\omega_{1}:=0, \quad \omega_{2 k}:=\lambda_{2 k}-\lambda_{1}, \quad \omega_{2 k+1}:=\lambda_{2 k+1}-\lambda_{2} \quad \text { for } k \in \mathbb{N}^{*}
$$

Let $d \in h_{r}^{6}(\mathbb{N}, \mathbb{C})$. A suitable candidate for $Z^{-1}(d)$ is the function

$$
\begin{aligned}
w(t):= & \frac{1}{T}\left[\frac{d_{1}}{6}+a_{2} e^{-i \omega_{2} t}+\overline{a_{2}} e^{i \omega_{2} t}+a_{3} e^{-i \omega_{3} t}+\overline{a_{3}} e^{i \omega_{3} t}+\right. \\
& \left.\sum_{k=4}^{\infty}\left(\frac{d_{k}}{6} e^{-i \omega_{k} t}+\frac{\overline{d_{k}}}{6} e^{i \omega_{k} t}\right)+\alpha e^{-i \omega t}+\bar{\alpha} e^{i \omega t}\right]\left(e^{i \frac{1}{2} \pi^{2} t}-1\right)^{2}\left(e^{-i \frac{1}{2} \pi^{2} t}-1\right)^{2},
\end{aligned}
$$

where

$$
\begin{gathered}
a_{2}:=\frac{6 d_{2}-d_{3}}{35}, \quad a_{3}:=\frac{6 d_{3}-d_{2}}{35} \\
\omega=\frac{1}{2} m \pi^{2} \text { with } m \in \mathbb{N} \text { and }\{m, m \pm 1, m \pm 2\} \cap\left\{\frac{2}{\pi^{2}} \omega_{k}, \frac{2}{\pi^{2}} \omega_{k} \pm 1, \frac{2}{\pi^{2}} \omega_{k} \pm 2 ; k \in \mathbb{N}^{*}\right\}=\emptyset
\end{gathered}
$$

and $\alpha \in \mathbb{C}$ is such that $\int_{0}^{T}(T-t) w(t) d t=d_{0}$.

### 3.2.4 Controllability up to codimension one of the linearized system around $(Y, u)$ and bounds (20), (21), (22), (23)

Let $\left(\psi_{0}, S_{0}, D_{0}, u\right) \in E_{9}$ The aim of this section is the proof of the existence of a right inverse to $d \Phi\left(\psi_{0}, S_{0}, D_{0}, u\right)$ with the estimates $(20),(21),(22),(23)$.

Let $(\psi, S, D)$ be the solution of $\left(\Sigma_{0}\right)$ with control $u$ such that $(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right)$. The linearized system around $(\psi, S, D, u)$ is

$$
\left\{\begin{array}{l}
i \dot{\Psi}=-\frac{1}{2} \Psi^{\prime \prime}-u q \Psi-w q \psi  \tag{38}\\
\Psi(t, \pm 1 / 2)=0 \\
\dot{s}=w \\
\dot{d}=s
\end{array}\right.
$$

It is a control system where

- the state is $(\Psi, s, d)$ with $\Psi(t) \in T_{\mathcal{S}}(\psi(t))$, for every $t$,
- the control is the real valued function $w$.

Let $T:=4 / \pi$ and

$$
\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right) \in\left[T_{\mathcal{S}}\left(\psi_{0}\right) \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}
$$

We look for $w \in H_{0}^{3}((0, T), \mathbb{R})$ such that the solution of (38) with $(\Psi(0), s(0), d(0))=\left(\Psi_{0}, s_{0}, d_{0}\right)$ satisfies

$$
\begin{equation*}
(\Psi(T), s(T), d(T))=\left(\widetilde{\mathcal{P}} \widetilde{\Psi_{f}}, s_{f}, d_{f}\right) \tag{39}
\end{equation*}
$$

and

$$
\begin{aligned}
\|w\|_{L^{2}} \leqslant & C\left[\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}+\Delta_{3}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}\right] \\
\|w\|_{H_{0}^{1}} \leqslant & C\left[\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{5}+\Delta_{3}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{5}\right. \\
& \left.+\Delta_{5}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}\right] \\
\|w\|_{H_{0}^{2}} \leqslant & C\left[\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{7}+\Delta_{3}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{7}\right. \\
& \left.+\Delta_{5}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{5}+\left(\Delta_{7}+\Delta_{5}^{2}\right)\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}\right] \\
\|w\|_{H_{0}^{3}} \leqslant & C\left[\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{9}+\Delta_{3}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{9}\right. \\
& +\Delta_{5}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{7}+\left(\Delta_{7}+\Delta_{5}^{2}\right)\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{5} \\
& \left.+\left(\Delta_{9}+\Delta_{7} \Delta_{5}+\Delta_{5}^{3}\right)\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}\right]
\end{aligned}
$$

where

$$
\Delta_{k}:=\left\|\left(\psi_{0}-\psi_{\theta}(0), S_{0}, D_{0}, u\right)\right\|_{E_{k}} \text { for } k=3,5,7,9
$$

Let us consider the decomposition $(\Psi, s, d)=\left(\Psi_{1}, s_{1}, d_{1}\right)+\left(\Psi_{2}, s_{2}, d_{2}\right)$ where

$$
\left\{\begin{array} { l } 
{ i \dot { \Psi } _ { 1 } = - \frac { 1 } { 2 } \Psi _ { 1 } ^ { \prime \prime } - u q \Psi _ { 1 } , } \\
{ \Psi _ { 1 } ( t , \pm 1 / 2 ) = 0 , } \\
{ \Psi _ { 1 } ( 0 ) = \Psi _ { 0 } , } \\
{ \dot { s } _ { 1 } = 0 , s _ { 1 } ( 0 ) = s _ { 0 } , } \\
{ \dot { d } _ { 1 } = s _ { 1 } , d _ { 1 } = d _ { 0 } }
\end{array} \quad \left\{\begin{array}{l}
i \dot{\Psi}_{2}=-\frac{1}{2} \Psi_{2}^{\prime \prime}-u q \Psi_{2}-w q \psi \\
\Psi_{2}(t, \pm 1 / 2)=0 \\
\Psi_{2}(0)=0 \\
\dot{s}_{2}=w, s_{2}(0)=0 \\
\dot{d}_{2}=s_{2}, d_{2}(0)=0
\end{array}\right.\right.
$$

The equality (39) is equivalent to

$$
\begin{equation*}
\left(\mathcal{P} \Psi_{2}(T), s_{2}(T), d_{2}(T)\right)=\left(\widetilde{\Psi_{f}}-\mathcal{P} \Psi_{1}(T), s_{f}-s_{0}, d_{f}-d_{0}-s_{0} T\right) \tag{41}
\end{equation*}
$$

Let us introduce, for $\gamma \in \mathbb{R}$ the operator $A_{\gamma}$ defined on

$$
D\left(A_{\gamma}\right):=H^{2} \cap H_{0}^{1}(I, \mathbb{C}) \text { by } A_{\gamma} \varphi:=-\frac{1}{2} \varphi^{\prime \prime}-\gamma q \varphi
$$

and $\left(\lambda_{k, \gamma}\right)_{k \in \mathbb{N}^{*}}$ the increasing sequence of eigenvalues for $A_{\gamma}$. We know from $[13$, chapter 7 , example 2.14] that $\lambda_{k, \gamma}$ are analytic functions of the parameter $\gamma$.

The equality (41) is equivalent to

$$
M_{\left(\psi_{0}, u\right)}(w)=D_{\left(\psi_{0}, u\right)}\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)
$$

where

$$
\begin{aligned}
M_{\left(\psi_{0}, u\right)}(w):=\left(d_{2}(T), s_{2}(T),<\right. & \left.\Psi_{2}(T), \varphi_{2}>e^{i \int_{0}^{T} \lambda_{2, u(s)} d s},<\Psi_{2}(T), \varphi_{3}>e^{i \int_{0}^{T} \lambda_{3, u(s)} d s}, \ldots\right) \\
D_{\left(\psi_{0}, u\right)}\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right):= & \left(d_{f}-d_{0}-s_{0} T, s_{f}-s_{0},<\widetilde{\Psi_{f}}-\Psi_{1}(T), \varphi_{2}>e^{i \int_{0}^{T} \lambda_{2, u(s)} d s}\right. \\
& \left.<\widetilde{\Psi_{f}}-\Psi_{1}(T), \varphi_{3}>e^{i \int_{0}^{T} \lambda_{3, u(s)} d s}, \ldots\right)
\end{aligned}
$$

Proposition 7 The linear map $M_{\left(\psi_{0}, u\right)}$ is continuous from $E$ to $F$ for every

$$
(E, F) \in\left\{\left(L^{2}, h_{r}^{3}\right),\left(H_{0}^{1}, h_{r}^{5}\right),\left(H_{0}^{2}, h_{r}^{7}\right),\left(H_{0}^{3}, h_{r}^{9}\right)\right\}
$$

There exist $C>0$ and $\eta>0$ such that, for every $\left(\psi_{0}, u\right) \in H_{(0)}^{9}(I, \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$ with

$$
\left\|\psi_{0}-\psi_{\theta}(0)\right\|_{H^{3}(I, \mathbb{C})}+\|u\|_{H_{0}^{1}((0, T), \mathbb{R})}<\eta
$$

there exists a continuous right inverse

$$
M_{\left(\psi_{0}, u\right)}^{-1}: h_{r}^{9}(\mathbb{N}, \mathbb{C}) \rightarrow H_{0}^{3}((0, T), \mathbb{R})
$$

such that, for every $d \in h_{r}^{9}(\mathbb{N}, \mathbb{C})$,

$$
\begin{aligned}
& \left\|M_{\left(\psi_{0}, u\right)}^{-1}(d)\right\|_{L^{2}} \leqslant C\|d\|_{h^{3}} \\
& \left\|M_{\left(\psi_{0}, u\right)}^{-1}(d)\right\|_{H_{0}^{1}} \leqslant C\left[\|d\|_{h^{5}}+\Delta_{5}\|d\|_{h^{3}}\right] \\
& \left\|M_{\left(\psi_{0}, u\right)}^{-1}(d)\right\|_{H_{0}^{2}} \leqslant C\left[\|d\|_{h^{7}}+\Delta_{5}\|d\|_{h^{5}}+\left(\Delta_{7}+\Delta_{5}^{2}\right)\|d\|_{h^{3}}\right] \\
& \left\|M_{\left(\psi_{0}, u\right)}^{-1}(d)\right\|_{H_{0}^{3}} \leqslant C\left[\|d\|_{h^{9}}+\Delta_{5}\|d\|_{h^{7}}+\left(\Delta_{7}+\Delta_{5}^{2}\right)\|d\|_{h^{5}}+\left(\Delta_{9}+\Delta_{7} \Delta_{5}+\Delta_{5}^{3}\right)\|d\|_{h^{3}}\right]
\end{aligned}
$$

where

$$
\Delta_{k}:=\left\|\left(\psi_{0}-\psi_{\theta}(0), S_{0}, D_{0}, u\right)\right\|_{E_{k}} \text { for } k=3,5,7,9
$$

In order to get this result, we prove that when $\left(\psi_{0}, u\right)$ is closed to $\left(\psi_{\theta}, 0\right)$ in $H_{(0)}^{3}(I, \mathbb{C}) \times H_{0}^{1}((0, T), \mathbb{R})$, the $\operatorname{map} M_{\left(\psi_{0}, u\right)}$ is closed enough to the $\operatorname{map} M_{\left(\psi_{\theta}(0), 0\right)}$, in a sense presented in the following proposition, so that

- the existence of a right inverse $M_{\left(\psi_{\theta}(0), 0\right)}^{-1}$ guarantees the existence of a right inverse $M_{\left(\psi_{0}, u\right)}^{-1}$,
- the bounds proved on $M_{\left(\psi_{\theta}(0), 0\right)}^{-1}$ give the same kind of bounds on $M_{\left(\psi_{0}, u\right)}^{-1}$.

More precisely, we apply the following proposition already proved in [1, Proposition 15 Section 3.6.1].
Proposition $8 \operatorname{Let} T:=4 / \pi, M$ and $M_{\theta}$ be bounded linear operators from $L^{2}((0, T), \mathbb{R})$ to $h^{3}(\mathbb{N}, \mathbb{C})$, from $H_{0}^{1}((0, T), \mathbb{R})$ to $h^{5}(\mathbb{N}, \mathbb{C})$, from $H_{0}^{2}((0, T), \mathbb{R})$ to $h^{7}(\mathbb{N}, \mathbb{C})$ and from $H_{0}^{3}((0, T), \mathbb{R})$ to $h^{9}(\mathbb{N}, \mathbb{C})$. We assume there exist a continuous linear operator $M_{\theta}^{-1}: h^{9}(\mathbb{N}, \mathbb{C}) \rightarrow H_{0}^{3}((0, T), \mathbb{R})$ and a positive constant $C_{0}$ such that for every $d \in h^{9}(\mathbb{N}, \mathbb{C}), M_{\theta} \circ M_{\theta}^{-1}(d)=d$ and $\left\|M_{\theta}^{-1}(d)\right\|_{E} \leqslant C_{0}\|d\|_{F}$ for every $(E, F) \in\left\{\left(L^{2}, h^{3}\right),\left(H_{0}^{1}, h^{5}\right),\left(H_{0}^{2}, h^{7}\right),\left(H_{0}^{3}, h^{9}\right)\right\}$. We also assume there exist positive constants $C_{1}$, $\Delta_{3}, \Delta_{5}, \Delta_{7}, \Delta_{9}$ with $C_{0} C_{1} \Delta_{3} \leqslant 1 / 2$, satisfying, for every $w \in H_{0}^{3}((0, T), \mathbb{R})$

$$
\begin{align*}
& \left\|\left(M-M_{\theta}\right)(w)\right\|_{h^{3}} \leqslant C_{1} \Delta_{3}\|w\|_{L^{2}} \\
& \left\|\left(M-M_{\theta}\right)(w)\right\|_{h^{5}} \leqslant C_{1}\left[\Delta_{3}\|w\|_{H_{0}^{1}}+\Delta_{5}\|w\|_{L^{2}}\right] \\
& \left\|\left(M-M_{\theta}\right)(w)\right\|_{h^{7}} \leqslant C_{1}\left[\Delta_{3}\|w\|_{H_{0}^{2}}+\Delta_{5}\|w\|_{H_{0}^{1}}+\Delta_{7}\|w\|_{L^{2}}\right]  \tag{42}\\
& \left\|\left(M-M_{\theta}\right)(w)\right\|_{h^{9}} \leqslant C_{1}\left[\Delta_{3}\|w\|_{H_{0}^{3}}+\Delta_{5}\|w\|_{H_{0}^{2}}+\Delta_{7}\|w\|_{H_{0}^{1}}+\Delta_{9}\|w\|_{L^{2}}\right]
\end{align*}
$$

Then, there exists a continuous linear operator $M^{-1}: h^{9}(\mathbb{N}, \mathbb{C}) \rightarrow H_{0}^{3}((0, T), \mathbb{R})$ such that for every $d \in h^{9}(\mathbb{N}, \mathbb{C}), M \circ M^{-1}(d)=d$ and the function $w:=M^{-1}(d)$ satisfies

$$
\begin{aligned}
\|w\|_{L^{2}} \leqslant & 2 C_{0}\|d\|_{h^{3}}, \\
\|w\|_{H_{0}^{1}} \leqslant & 2 C_{0}\left[\|d\|_{h^{5}}+2 C_{2} \Delta_{5}\|d\|_{h^{3}}\right] \\
\|w\|_{H_{0}^{2}} \leqslant & 2 C_{0}\left[\|d\|_{h^{7}}+2 C_{2} \Delta_{5}\|d\|_{h^{5}}+\left(2 C_{2} \Delta_{7}+8 C_{2}^{2} \Delta_{5}^{2}\right)\|d\|_{h^{3}}\right] \\
\|w\|_{H_{0}^{3}} \leqslant & 2 C_{0}\left[\|d\|_{h^{9}}+2 C_{2} \Delta_{5}\|d\|_{h^{7}}+\left(2 C_{2} \Delta_{7}+8 C_{2}^{2} \Delta_{5}^{2}\right)\|d\|_{h^{5}}+\right. \\
& \left.\left(2 C_{2} \Delta_{9}+16 C_{2}^{2} \Delta_{7} \Delta_{5}+48 C_{2}^{3} \Delta_{5}^{3}\right)\|d\|_{h^{3}}\right] .
\end{aligned}
$$

where $C_{2}:=C_{0} C_{1}$.
Let us recall that, for $\gamma \in \mathbb{R}$, the space $L^{2}(I, \mathbb{C})$ has a complete orthonormal system $\left(\varphi_{k, \gamma}\right)_{k \in \mathbb{N}^{*}}$ of eigenvectors for $A_{\gamma}$ :

$$
A_{\gamma} \varphi_{k, \gamma}=\lambda_{k, \gamma} \varphi_{k, \gamma}
$$

We know from [13, chapter 7 , example 2.14] that $\varphi_{k, \gamma}$ are analytic functions of the parameter $\gamma$. This result gives sense to the notation

$$
\left.\frac{d \varphi_{k, \gamma}}{d \gamma}\right]_{\gamma_{0}}
$$

which means the derivative of the map $\gamma \mapsto \varphi_{k, \gamma}$ with respect to $\gamma$ evaluated at the point $\gamma=\gamma_{0}$.

Proof of Proposition 7 : Let us consider the decomposition

$$
\Psi_{2}(t)=\sum_{k=1}^{\infty} x_{k}(t) \varphi_{k, u(t)} \text { where } x_{k}(t):=<\Psi_{2}(t), \varphi_{k, u(t)}>
$$

Using $u(T)=0$, we get

$$
M_{\left(\psi_{0}, u\right)}(w)=\left(\int_{0}^{T}(T-t) w(t) d t, \int_{0}^{T} w(t) d t, x_{2}(T) e^{i \int_{0}^{T} \lambda_{2, u(s)} d s}, x_{3}(T) e^{i \int_{0}^{T} \lambda_{3, u(s)} d s}, \ldots\right)
$$

The partial differential equation satisfied by $\Psi_{2}$ provides, for every $k \in \mathbb{N}^{*}$, an ordinary differential equation satisfied by the component $x_{k}$,

$$
\begin{gathered}
\left.\dot{x}_{k}(t)=<\frac{\partial \Psi_{2}}{\partial t}(t), \varphi_{k, u(t)}>+\dot{u}(t)<\Psi_{2}(t), \frac{d \varphi_{k, \gamma}}{d \gamma}\right]_{u(t)}> \\
<\frac{\partial \Psi_{2}}{\partial t}(t), \varphi_{k, u(t)}>=<-i A_{u(t)} \Psi_{2}(t)+i w(t) q \psi(t), \varphi_{k, u(t)}>=-i \lambda_{k, u(t)} x_{k}(t)+i w(t)<q \psi(t), \varphi_{k, u(t)}> \\
\left.\dot{x}_{k}(t)=-i \lambda_{k, u(t)} x_{k}(t)+i w(t)<q \psi(t), \varphi_{k, u(t)}>+\dot{u}(t)<\Psi_{2}(t), \frac{d \varphi_{k, \gamma}}{d \gamma}\right]_{u(t)}>
\end{gathered}
$$

Solving this equation, we get

$$
\left.M_{\left(\psi_{0}, u\right)}(w)_{k}=\int_{0}^{T}\left(i w(t)<q \psi(t), \varphi_{k, u(t)}>+\dot{u}(t)<\Psi_{2}(t), \frac{d \varphi_{k, \gamma}}{d \gamma}\right]_{u(t)}>\right) e^{i \int_{0}^{t} \lambda_{k, u(s) d s}} d t, k \geqslant 2
$$

We introduce the following decomposition

$$
\left(M_{\left(\psi_{0}, u\right)}-M_{\left(\psi_{\theta}(0), 0\right)}\right)(w)=\delta M(w)^{1}+\delta M(w)^{2}
$$

where

$$
\delta M(w)_{k}^{j}=0 \text { for } j=1,2 \text { and } k=0,1
$$

$$
\begin{aligned}
\delta M(w)_{k}^{1}= & i \int_{0}^{T} w(t)\left[<q \psi(t), \varphi_{k, u(t)}>e^{i \int_{0}^{t} \lambda_{k, u(s) d s}}-<q \psi_{\theta}(t), \varphi_{k}>e^{i \lambda_{k} t}\right] d t, k \geqslant 2 \\
& \left.\delta M(w)_{k}^{2}=\int_{0}^{T} \dot{u}(t)<\Psi_{2}(t), \frac{d \varphi_{k, \gamma}}{d \gamma}\right]_{u(t)}>e^{i \int_{0}^{t} \lambda_{k, u(s) d s}} d t, k \geqslant 2
\end{aligned}
$$

Let us justify the bounds (42) on the terms $\delta M(w)^{j}$ for $j=1,2$. The study of $\delta M(w)^{1}$ can be done in the same way as in the proof of [1, section 3.6.2 Proposition 27] (with $\gamma=0$ ). The study of $\delta M(w)^{2}$ can be done by applying [1, Propositions 18, 20, 23, 25 Section 3.6.2].

Proposition 9 We assume $\Delta_{3} \leqslant 1$. There exists $C>0$ such that, for every

$$
\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right) \in\left[T_{\mathcal{S}}\left(\psi_{0}\right) \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}
$$

we have

$$
\begin{aligned}
\left\|D\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{h^{3}} \leqslant & C\left[\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}+\Delta_{3}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}\right] \\
\left\|D\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{h^{5}} \leqslant & C\left[\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{5}+\Delta_{3}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \Psi_{f}, s_{f}, d_{f}\right)\right\|_{5}\right. \\
& \left.+\Delta_{5}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}\right] \\
\left\|D\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{h^{7}} \leqslant & C\left[\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{7}+\Delta_{3}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{7}\right. \\
& \left.+\Delta_{5}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{5}+\left(\Delta_{7}+\Delta_{5}^{2}\right)\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}\right] \\
\left\|D\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{h^{9}} \leqslant & C\left[\left\|\left(\mathcal{P} \Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{9}+\Delta_{3}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{9}\right. \\
& +\Delta_{5}\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{7}+\left(\Delta_{7}+\Delta_{5}^{2}\right)\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{5} \\
& \left.\left(\Delta_{9}+\Delta_{7} \Delta_{5}+\Delta_{5}^{3}\right)\left\|\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{3}\right] .
\end{aligned}
$$

Proof : Standard results about Fourier series provide the existence of $C>0$ such that, for every

$$
\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right) \in\left[T_{\mathcal{S}}\left(\psi_{0}\right) \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R} \times\left[V \cap H_{(0)}^{9}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}
$$

for every $s \in\{3,5,7,9\}$,

$$
\left\|D_{\left(\psi_{0}, u\right)}\left(\Psi_{0}, s_{0}, d_{0}, \widetilde{\Psi_{f}}, s_{f}, d_{f}\right)\right\|_{h^{s}(\mathbb{N}, \mathbb{C})} \leqslant C\left[\left\|\mathcal{P} \Psi_{1}(T)\right\|_{H^{s}}+\left\|\widetilde{\Psi_{f}}\right\|_{H^{s}}+\left|s_{0}\right|+\left|d_{0}\right|+\left|s_{f}\right|+\left|d_{f}\right|\right]
$$

Thus, it is sufficient to prove the existence of $C>0$ such that

$$
\begin{align*}
\left\|\mathcal{P} \Psi_{1}(T)\right\|_{H^{3}} \leqslant & C\left[\left\|\mathcal{P} \Psi_{0}\right\|_{H^{3}}+\Delta_{3}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{3} \times H_{0}^{1}}\right] \\
\left\|\mathcal{P} \Psi_{1}(T)\right\|_{H^{5}} \leqslant & C\left[\left\|\mathcal{P} \Psi_{0}\right\|_{H^{5}}+\Delta_{3}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{5} \times H_{0}^{2}}+\Delta_{5}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{3} \times H_{0}^{1}}\right] \\
\left\|\mathcal{P} \Psi_{1}(T)\right\|_{H^{7}} \leqslant & C\left[\left\|\mathcal{P} \Psi_{0}\right\|_{H^{7}}+\Delta_{3}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{7} \times H_{0}^{3}}+\Delta_{5}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{5} \times H_{0}^{2}}+\Delta_{7}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{3} \times H_{0}^{1}}\right] \\
\left\|\mathcal{P} \Psi_{1}(T)\right\|_{H^{9}} \leqslant & C\left[\left\|\mathcal{P} \Psi_{0}\right\|_{H^{9}}+\Delta_{3}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{9} \times H_{0}^{4}}+\Delta_{5}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{7} \times H_{0}^{3}}+\Delta_{7}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{5} \times H_{0}^{2}}\right. \\
& \left.+\Delta_{9}\left\|\left(\Psi_{0}, u\right)\right\|_{H^{3} \times H_{0}^{1}}\right] . \tag{43}
\end{align*}
$$

For every $s \in\{3,5,7,9\}$, we have

$$
\left\|\mathcal{P} \Psi_{1}(T)\right\|_{H^{s}} \leqslant C\left(\sum_{k=2}^{\infty}\left|k^{s} x_{k}(T)\right|^{2}\right)^{1 / 2} \text { where } x_{k}(t):=<\Psi_{1}(t), \varphi_{k, u(t)}>
$$

because $u(T)=0$. Thanks to the equation satisfied by $\Psi_{1}$, we get

$$
\begin{gathered}
\left.x_{k}(T)=\left(<\Psi_{0}, x_{k}>+\int_{0}^{T} \dot{u}(t)<\Psi_{1}(t), \frac{d \varphi_{k, \gamma}}{d \gamma}\right]_{u(t)}>e^{i \int_{0}^{t} \lambda_{k, u(s)} d s} d t\right) e^{-i \int_{0}^{T} \lambda_{k, u(s)} d s} \\
\left.\left(\sum_{k=2}^{\infty}\left|k^{s} x_{k}(T)\right|^{2}\right)^{1 / 2} \leqslant\left\|\mathcal{P} \Psi_{0}\right\|_{H^{s}}+\left(\sum_{k=2}^{\infty} \mid k^{s} \int_{0}^{T} \dot{u}(t)<\Psi_{1}(t), \frac{d \varphi_{k, \gamma}}{d \gamma}\right]_{u(t)}>\left.e^{i \int_{0}^{t} \lambda_{k, u(s)} d s} d t\right|^{2}\right)^{1 / 2} .
\end{gathered}
$$

Using [1, Propositions 17, 20, 23, 29], we get (43).
In conclusion, using Propositions 7 and 9, we get the bounds (40).

### 3.3 Motion in the directions $(\psi, S, D)=\left( \pm i \varphi_{1}, 0,0\right)$

The aim of this section is the proof of the following theorem.
Theorem 6 Let $T:=4 / \pi$. There exists $w_{ \pm} \in H^{4} \cap H_{0}^{3}((0, T), \mathbb{R}), \nu_{ \pm} \in H_{0}^{3}((0, T), \mathbb{R})$ such that the solutions of

$$
\begin{align*}
& \left\{\begin{array}{l}
i \dot{\Psi}_{ \pm}=-\frac{1}{2} \Psi_{ \pm}^{\prime \prime}-w_{ \pm} q \psi_{\theta}, \\
\Psi_{ \pm}(0)=0, \\
\Psi_{ \pm}(t,-1 / 2)=\Psi_{ \pm}(t, 1 / 2)=0, \\
\dot{s}_{ \pm}=w_{ \pm}, s_{ \pm}(0)=0, \\
\dot{d}_{ \pm}=s_{ \pm}, d_{ \pm}(0)=0,
\end{array}\right.  \tag{44}\\
& \left\{\begin{array}{l}
i \dot{\xi}_{ \pm}=-\frac{1}{2} \xi_{ \pm}^{\prime \prime}-w_{ \pm} q \Psi_{ \pm}-\nu_{ \pm} q \psi_{\theta}, \\
\xi_{ \pm}(0)=0, \\
\xi_{ \pm}(t,-1 / 2)=\xi_{ \pm}(t, 1 / 2)=0, \\
\dot{\sigma}_{ \pm}=\nu_{ \pm}, \sigma_{ \pm}(0)=0, \\
\dot{\delta}_{ \pm}=\sigma_{ \pm}, \delta_{ \pm}(0)=0,
\end{array}\right. \tag{45}
\end{align*}
$$

satisfy $\Psi_{ \pm}(T)=0, s_{ \pm}(T)=0, d_{ \pm}(T)=0, \xi_{ \pm}(T)= \pm i \varphi_{1}, \sigma_{ \pm}=0, \delta_{ \pm}=0$.

We introduce the following subspace of $L^{2}((0, T), \mathbb{C})$,

$$
\mathcal{X}:=\operatorname{Span}\left(1, t, e^{ \pm i\left(\lambda_{2 k}-\lambda_{1}\right) t}, e^{ \pm i\left(\lambda_{2 k+1}-\lambda_{2}\right) t} ; k \in \mathbb{N}^{*}\right)
$$

The symbol $\mathcal{X}^{\perp}$ denotes the orthogonal subspace to $\mathcal{X}$ in $L^{2}((0, T), \mathbb{C})$.
Proposition 10 There exists $w \in H^{4} \cap H_{0}^{3}((0, T), \mathbb{R}) \cap \mathcal{X}^{\perp}$ such that

$$
\begin{equation*}
\int_{0}^{T} w(t)<q \Psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t-\frac{\sqrt{\theta}}{\sqrt{1-\theta}} \int_{0}^{T} w(t)<q \varphi_{2}, \Psi(t)>e^{-i \lambda_{2} t} d t \in(0,+\infty)(\operatorname{resp} .(-\infty, 0)) \tag{46}
\end{equation*}
$$

where $\Psi$ is the solution of

$$
\left\{\begin{array}{l}
i \dot{\Psi}=-\frac{1}{2} \Psi^{\prime \prime}-w q \psi_{\theta}  \tag{47}\\
\Psi(0)=0 \\
\Psi(t, \pm 1 / 2)=0
\end{array}\right.
$$

Remark 4 If $w \in \mathcal{X}^{\perp}$ and $\Psi$ solves the previous system, then

$$
\int_{0}^{T} w(t)<q \Psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t-\frac{\sqrt{\theta}}{\sqrt{1-\theta}} \int_{0}^{T} w(t)<q \varphi_{2}, \Psi(t)>e^{-i \lambda_{2} t} d t \in \mathbb{R}
$$

Indeed, we have (see (6) and (7))

$$
\Psi(t)=\sum_{k=1}^{\infty} x_{k}(t) \varphi_{k}, \quad \text { where } \quad\left\{\begin{array}{l}
x_{2 k}(t)=i \sqrt{1-\theta} b_{2 k} e^{-i \lambda_{2 k} t} \int_{0}^{t} w(\tau) e^{i\left(\lambda_{2 k}-\lambda_{1}\right) \tau} d \tau  \tag{48}\\
x_{2 k-1}(t)=i \sqrt{\theta} c_{2 k-1} e^{-i \lambda_{2 k-1} t} \int_{0}^{t} w(\tau) e^{i\left(\lambda_{2 k-1}-\lambda_{2}\right) \tau} d \tau
\end{array}\right.
$$

where $b_{k}$ and $c_{k}$ are given by (8). Thus, we get

$$
\begin{gather*}
\int_{0}^{T} w(t)<q \Psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t=i \sqrt{1-\theta} \sum_{k=1}^{\infty} b_{2 k}^{2} f_{2 k}  \tag{49}\\
\int_{0}^{T} w(t)<q \varphi_{2}, \Psi(t)>e^{-i \lambda_{2} t} d t=-i \sqrt{\theta} \sum_{k=0}^{\infty} c_{2 k+1}^{2} f_{2 k+1} \tag{50}
\end{gather*}
$$

where

$$
\begin{aligned}
f_{2 k} & :=\int_{0}^{T} w(t) e^{i\left(\lambda_{1}-\lambda_{2 k}\right) t} \int_{0}^{t} w(\tau) e^{i\left(\lambda_{2 k}-\lambda_{1}\right) \tau} d \tau d t, \forall k \in \mathbb{N}^{*}, \\
f_{2 k+1} & :=\int_{0}^{T} w(t) e^{i\left(\lambda_{2 k+1}-\lambda_{2}\right) t} \int_{0}^{t} w(\tau) e^{i\left(\lambda_{2}-\lambda_{2 k+1}\right) \tau} d \tau d t, \forall k \in \mathbb{N} .
\end{aligned}
$$

Thanks to integrations by parts and the property $w \in \mathcal{X}^{\perp}$, we get, for every $k \in \mathbb{N}^{*}, f_{k} \in i \mathbb{R}$.
Proof of Proposition 10 : Let us consider functions of the form

$$
w(t)=a_{1} \sin \left(\frac{1}{2} n_{1} \pi^{2} t\right)+a_{2} \sin \left(\frac{1}{2} n_{2} \pi^{2} t\right)+\sin \left(\frac{1}{2} n_{3} \pi^{2} t\right)
$$

where $n_{1}, n_{2}, n_{3}$ are three distinct positive integers such that

$$
n_{1}, n_{2}, n_{3} \notin\left\{0, \pm\left[(2 k)^{2}-1\right], \pm\left[(2 k-1)^{2}-4\right] ; k \in \mathbb{N}^{*}\right\}
$$

and $a_{1}, a_{2}$ are defined by

$$
a_{1}:=\frac{n_{1}\left(n_{3}^{2}-n_{2}^{2}\right)}{n_{3}\left(n_{2}^{2}-n_{1}^{2}\right)}, \quad a_{2}:=\frac{n_{2}\left(n_{1}^{2}-n_{3}^{2}\right)}{n_{3}\left(n_{2}^{2}-n_{1}^{2}\right)} .
$$

Then,

$$
w \in H^{4} \cap H_{0}^{3}((0, T), \mathbb{R}) \cap \mathcal{X}^{\perp}
$$

Let $\Psi$ be the solution of (47). The condition (46) is equivalent to

$$
\begin{equation*}
i \sum_{k=1}^{\infty} b_{2 k}^{2} f_{2 k}+i \frac{\theta}{1-\theta} \sum_{k=0}^{\infty} c_{2 k+1}^{2} f_{2 k+1} \in(0,+\infty)(\operatorname{resp} .(-\infty, 0)) \tag{51}
\end{equation*}
$$

Using (8), the two previous infinite sums can be computed explicitly. We find

$$
\begin{gathered}
i \sum_{k=1}^{\infty} b_{2 k}^{2} f_{2 k}=\frac{32 T}{\pi^{6}}\left(a_{1}^{2} A_{n_{1}}+a_{2}^{2} A_{n_{2}}+A_{n_{3}}\right), \\
i \sum_{k=0}^{\infty} c_{2 k+1}^{2} f_{2 k+1}=\frac{32 T}{\pi^{6}}\left(a_{1}^{2} B_{n_{1}}+a_{2}^{2} B_{n_{2}}+B_{n_{3}}\right),
\end{gathered}
$$

where

$$
\begin{gathered}
A_{n}:=\sum_{k=1}^{\infty} \frac{(2 k)^{2}}{(1+2 k)^{4}(1-2 k)^{4}}\left(\frac{1}{n+4 k^{2}-1}+\frac{1}{-n+4 k^{2}-1}\right), \\
B_{n}:=\sum_{k=0}^{\infty} \frac{4(2 k+1)^{2}}{(3+2 k)^{4}(1-2 k)^{4}}\left(\frac{1}{-n+4-(2 k+1)^{2}}+\frac{1}{n+4-(2 k+1)^{2}}\right) .
\end{gathered}
$$

Let us choose $n_{1}=1, n_{2}=2, n_{3}=4$ (resp. $\left.n_{1}=1, n_{2}=4, n_{3}=6\right)$ then

$$
\begin{aligned}
& a_{1}^{2} A_{n_{1}}+a_{2}^{2} A_{n_{2}}+A_{n_{3}}>0(\text { resp. }<0) \\
& a_{1}^{2} B_{n_{1}}+a_{2}^{2} B_{n_{2}}+B_{n_{3}}>0(\text { resp. }<0)
\end{aligned}
$$

thus, for every $\theta \in(0,1)$, we have (51).
Proof of Theorem 6 : Let $w \in H^{4} \cap H_{0}^{3}((0, T), \mathbb{R}) \cap \mathcal{X}^{\perp}$ be such that

$$
\begin{equation*}
\int_{0}^{T} w(t)<q \Psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t-\frac{\sqrt{\theta}}{\sqrt{1-\theta}} \int_{0}^{T} w(t)<q \varphi_{2}, \Psi(t)>e^{-i \lambda_{2} t} d t=+1(\text { resp. }-1) \tag{52}
\end{equation*}
$$

Using (48) and the assumption $w \in \mathcal{X}^{\perp}$, we get $\Psi(T)=0, s(T)=0, d(T)=0$. Let us prove that there exists $\nu \in H_{0}^{3}((0, T), \mathbb{R})$ such that the solution $\xi$ of $(45)$ satisfies $\xi(T)=i \varphi_{1}\left(\right.$ resp. $\left.-i \varphi_{1}\right), \sigma(T)=0$, $\delta(T)=0$. We have

$$
\begin{gathered}
\xi(t)=\sum_{k=1}^{\infty} y_{k}(t) \varphi_{k} \\
y_{2 k}(t)=i\left(\int_{0}^{t}\left[w(\tau)<q \Psi(\tau), \varphi_{2 k}>+\nu(\tau) \sqrt{1-\theta} b_{2 k} e^{-i \lambda_{1} \tau}\right] e^{i \lambda_{2 k} \tau} d \tau\right) e^{-i \lambda_{2 k} t}, \\
y_{2 k+1}(t)=i\left(\int_{0}^{t}\left[w(\tau)<q \Psi(\tau), \varphi_{2 k+1}>+\nu(\tau) \sqrt{\theta} c_{2 k+1} e^{-i \lambda_{2} \tau}\right] e^{i \lambda_{2 k+1} \tau} d \tau\right) e^{-i \lambda_{2 k+1} t} .
\end{gathered}
$$

Thus the equality $(\xi(T), \sigma(T), \delta(T))=\left( \pm i \varphi_{1}, 0,0\right)$ is equivalent to

$$
\begin{gathered}
\int_{0}^{T} \nu(t) e^{i\left(\lambda_{1}-\lambda_{2}\right) t} d t=\frac{1}{\sqrt{\theta} c_{1}}\left( \pm 1-\int_{0}^{T} w(t)<q \Psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t\right) \\
\int_{0}^{T} \nu(t) e^{i\left(\lambda_{2 k}-\lambda_{1}\right) t} d t=\frac{-1}{\sqrt{1-\theta} b_{2 k}} \int_{0}^{T} w(t)<q \Psi(t), \varphi_{2 k}>e^{i \lambda_{2 k} t} d t, \forall k \in \mathbb{N}^{*}
\end{gathered}
$$

$$
\begin{gathered}
\int_{0}^{T} \nu(t) e^{i\left(\lambda_{2 k+1}-\lambda_{2}\right) t} d t=\frac{-1}{\sqrt{\theta} c_{2 k+1}} \int_{0}^{T} w(t)<q \Psi(t), \varphi_{2 k+1}>e^{i \lambda_{2 k+1} t} d t, \forall k \in \mathbb{N}^{*} \\
\int_{0}^{T} \nu(t) d t=0 \\
\int_{0}^{T}(T-t) \nu(t) d t=0
\end{gathered}
$$

The left hand sides of the two first equalities with $k=1$ are complex conjugate numbers when $\nu$ is real valued. Thus, a necessary condition for the existence of real-valued solution $\nu$ to this problem is

$$
\begin{gathered}
\quad \frac{1}{\sqrt{\theta}}\left(+1-\int_{0}^{T} w(t)<q \Psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t\right)=\frac{-1}{\sqrt{1-\theta}} \int_{0}^{T} w(t)<q \varphi_{2}, \Psi(t)>e^{-i \lambda_{2} t} d t \\
\text { (resp. } \left.\frac{1}{\sqrt{\theta}}\left(-1-\int_{0}^{T} w(t)<q \Psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t\right)=\frac{-1}{\sqrt{1-\theta}} \int_{0}^{T} w(t)<q \varphi_{2}, \Psi(t)>e^{-i \lambda_{2} t} d t\right) .
\end{gathered}
$$

This property is satisfied thanks to (52).
Let $d=\left(d_{k}\right)_{k \in \mathbb{N}}$ be the sequence defined by

$$
\begin{gathered}
d_{0}:=0, d_{1}=0 \\
d_{2 k}:=\frac{-1}{b_{2 k} \sqrt{1-\theta}} \int_{0}^{T} w(t)<q \Psi(t), \varphi_{2 k}>e^{i \lambda_{2 k} t} d t, \forall k \geqslant 1 \\
d_{2 k}:=\frac{-1}{c_{2 k+1} \sqrt{\theta}} \int_{0}^{T} w(t)<q \Psi(t), \varphi_{2 k+1}>e^{i \lambda_{2 k+1} t} d t, \forall k \geqslant 1
\end{gathered}
$$

The previous moment problem can be written $Z(\nu)=d$, where the map $Z$ has been defined in section 3.2.3. Thanks to (8) and Proposition 6, a sufficient condition for the existence of $\nu \in H_{0}^{3}((0, T), \mathbb{R})$ solution of this equation is $d \in h^{6}(\mathbb{N}, \mathbb{C})$. We can get this result by applying [1, Proposition 24 Section 3.6.2].

### 3.4 Proof of Theorem 2

In all this section $T:=4 / \pi$. Let $\rho \in \mathbb{R}, \psi_{0}, \psi_{f} \in H_{(0)}^{7}(I, \mathbb{C}), S_{0}, D_{0}, S_{f}, D_{f} \in \mathbb{R}$. Let us consider, for $t \in[0, T]$

$$
u(t):=\sqrt{|\rho|} w+|\rho| \nu
$$

where $w:=w_{+}, \nu:=\nu_{+}$if $\rho \geqslant 0$ and $w:=w_{-}, \nu:=\nu_{-}$if $\rho \leqslant 0$ and $w_{ \pm}, \nu_{ \pm}$are defined in Theorem 6. Let $(\psi, S, D)$ be the solution of $\left(\Sigma_{0}\right)$ on $[0, T]$ with control $u$ and such that

$$
(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right)
$$

Then, we have

$$
S(T)=S_{0}, \quad D(T)=D_{0}
$$

We have $u \in W^{3,1}((0, T), \mathbb{R})$ and $u(0)=u(T)=\dot{u}(0)=\dot{u}(T)=0$ thus (see [1, Appendix B Proposition 51]) the function $\psi(T)$ belongs to $H_{(0)}^{7}(I, \mathbb{C})$.

Proposition 11 There exists $\mathcal{C}>0$ such that, for every $\rho \in(-1,1)$, we have

$$
\left\|\psi(T)-\left(\psi_{\theta}(T)+i \rho \varphi_{1}\right)\right\|_{H^{7}(I, \mathbb{C})} \leqslant \mathcal{C}\left[\left\|\psi_{0}-\psi_{\theta}(0)\right\|_{H^{7}(I, \mathbb{C})}+|\rho|^{3 / 2}\right]
$$

Proof : We have $\psi(T)-\left(\psi_{\theta}(T)+i \rho \varphi_{1}\right)=(\psi-Z)(T)$ where $Z:=\psi_{\theta}+\Psi+\xi$ and $\Psi, \xi$ are the solutions of

$$
\begin{gathered}
\left\{\begin{array}{l}
i \dot{\Psi}=-\frac{1}{2} \Psi^{\prime \prime}-\sqrt{|\rho|} w q \psi_{\theta} \\
\Psi(t, \pm 1 / 2)=0 \\
\Psi(0)=0
\end{array}\right. \\
\left\{\begin{array}{l}
i \dot{\xi}=-\frac{1}{2} \xi^{\prime \prime}-\sqrt{|\rho|} w q \Psi-|\rho| \nu q \psi_{\theta} \\
\xi(t, \pm 1 / 2)=0 \\
\xi(0)=0
\end{array}\right.
\end{gathered}
$$

The function $\Delta:=\psi-Z$ solves

$$
\left\{\begin{array}{l}
i \dot{\Delta}=-\frac{1}{2} \Delta^{\prime \prime}-u q \Delta-|\rho| \nu q(\Psi+\xi)-\sqrt{|\rho|} w q \xi \\
\Delta(t, \pm 1 / 2)=0 \\
\Delta(0)=\psi_{0}-\psi_{\theta}(0)
\end{array}\right.
$$

We know from [1, proposition 51, Appendix B], that the following quantities

$$
\|\Psi\|_{C^{0}\left([0, T], H^{7}\right)},\|\Psi\|_{C^{1}\left([0, T], H^{5}\right)},\|\Psi\|_{C^{2}\left([0, T], H^{3}\right)},\|\Psi\|_{C^{3}\left([0, T], H^{1}\right)}
$$

are bounded by

$$
A_{7}(\Psi):=C\left[\|f\|_{C^{0}\left([0, T], H^{5}\right)}+\|f\|_{C^{1}\left([0, T], H^{3}\right)}+\|f\|_{W^{2,1}\left((0, T), H^{2}\right)}+\|f\|_{W^{3,1}\left((0, T), H^{1}\right)}\right]
$$

where $C$ is a positive constant and $f:=\sqrt{|\rho|} w q \psi_{\theta}$. Thus, there exists a constant $C_{1}$ such that

$$
A_{7}(\Psi) \leqslant C_{1} \sqrt{|\rho|}
$$

In the same way, we prove that there exists a constant $C_{2}$ such that

$$
\|\xi\|_{C^{0}\left([0, T], H^{7}\right)},\|\xi\|_{C^{1}\left([0, T], H^{5}\right)},\|\xi\|_{C^{2}\left([0, T], H^{3}\right)},\|\xi\|_{C^{3}\left([0, T], H^{1}\right)}
$$

are bounded by

$$
A_{7}(\xi) \leqslant C_{2}|\rho|
$$

Using [1, Appendix B, proposition 51] we get the existence of a constant $C_{3}>0$ such that

$$
\|\Delta(T)\|_{H^{7}} \leqslant C_{3}\left[\left\|\psi_{0}-\psi_{\theta}(0)\right\|_{H^{7}}+\sqrt{|\rho|} A_{7}(\xi)+|\rho| A_{7}(\Psi)\right] .[
$$

Now, we use the local controllability up to codimension one around $Y_{\theta}$. Let $\delta>0$ be as in Theorem 3. We assume

$$
\begin{gathered}
\left\|\psi_{0}-\psi_{\theta}(0)\right\|_{H^{7}(I, \mathbb{C})}<\frac{\delta}{4 \mathcal{C}}, \\
\left|S_{0}\right|+\left|D_{0}\right|<\frac{\delta}{2}, \\
\left\|\mathcal{P}\left[\psi_{f}-\psi_{\theta}(2 T)\right]\right\|_{H^{7}}+\left|S_{f}\right|+\left|D_{f}\right|<\delta .
\end{gathered}
$$

When $\rho$ satisfies

$$
|\rho|<\eta:=\min \left\{1 ; \frac{\delta}{4\left(\left\|\varphi_{1}\right\|_{H^{7}}+\mathcal{C}\right)}\right\}
$$

the previous proposition proves that

$$
\left\|\psi(T)-\psi_{\theta}(0)\right\|_{H^{7}} \leqslant\left(\left\|\varphi_{1}\right\|_{H^{7}}+\mathcal{C}\right)|\rho|^{3 / 2}+\frac{\delta}{4}<\frac{\delta}{2}
$$

Thus $\left(\psi(T), S_{0}, D_{0}\right) \in \mathcal{V}(0)$ and $\left(\mathcal{P} \psi_{f}, S_{f}, D_{f}\right) \in \mathcal{V}(T)$. Thanks to Theorem 3, there exists

$$
\tilde{u}:=\Gamma\left(\psi(T), S_{0}, D_{0}, \mathcal{P} \psi_{f}, S_{f}, D_{f}\right) \in H_{0}^{1}((T, 2 T), \mathbb{R})
$$

such that

$$
(\mathcal{P} \psi(2 T), S(2 T), D(2 T))=\left(\mathcal{P} \psi_{f}, S_{f}, D_{f}\right)
$$

where $(\psi, S, D)$ is the solution of $\left(\Sigma_{0}\right)$ with control $u$ on $[0,2 T]$, with $u$ extended to $[0,2 T]$ by $u:=\tilde{u}$ on $[T, 2 T]$. The Theorem 3 and the previous proposition give the existence of a constant $C$ such that

$$
\begin{equation*}
\|u\|_{H^{1}((T, 2 T), \mathbb{R})} \leqslant C\left[|\rho|^{3 / 2}+\left\|\psi_{0}-\psi_{\theta}(0)\right\|_{H^{7}}+\left|S_{0}\right|+\left|D_{0}\right|+\left\|\mathcal{P}\left(\psi_{f}-\psi_{\theta}(2 T)\right)\right\|_{H^{7}}+\left|S_{f}\right|+\left|D_{f}\right|\right] \tag{53}
\end{equation*}
$$

We define the map

$$
\begin{array}{ccc}
F: \quad(-\eta, \eta) & \rightarrow & \mathbb{R} \\
\rho & \mapsto & \Im\left(<\psi(2 T), \varphi_{1}>\right)
\end{array}
$$

Thanks to Theorem $3, F$ is continuous on $(-\eta, \eta)$. We can assume $\delta$ is small enough so that

$$
\Re\left(<\psi(2 T), \varphi_{1}>\right)>0
$$

because $\psi$ is closed enough to $\psi_{\theta}$. Since $\psi \in \mathcal{S}$ and $\Re\left(<\psi(2 T), \varphi_{1}>\right)$ is positive, we have

$$
\psi(2 T)=\psi_{f} \text { if and only if } F(\rho)=\Im\left(<\psi_{f}, \varphi_{1}>\right)
$$

Therefore, in order to get Theorem 2, it is sufficient to prove that $F$ is surjective on a neighbourhood of 0 .

Let $x(t):=<\psi(t), \varphi_{1}>$ on $[T, 2 T]$. We have

$$
x(2 T)=x(T)+i \int_{T}^{2 T} u(t)<q \psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t
$$

Thus

$$
F(\rho)=\rho+[\Im(x(T))-\rho]+\Im\left(i \int_{T}^{2 T} u(t)<q \psi(t), \varphi_{1}>e^{i \lambda_{1} t}\right)
$$

where

$$
\begin{gathered}
|\Im(x(T))-\rho| \leqslant\left\|\psi(T)-\left(\psi_{\theta}(T)+i \rho\right)\right\|_{L^{2}} \leqslant C\left[|\rho|^{3 / 2}+\left\|\psi_{0}-\psi_{\theta}(0)\right\|_{H^{7}}\right] \\
\left|\int_{T}^{2 T} u(t)<q \psi(t), \varphi_{1}>e^{i \lambda_{1} t} d t\right| \leqslant T\|u\|_{L^{\infty}((T, 2 T), \mathbb{R})}
\end{gathered}
$$

Using (53), we get the existence of a constant $K$ such that

$$
\left.|F(\rho)-\rho| \leqslant K\left[|\rho|^{3 / 2}+\| \psi_{0}-\psi_{\theta}(0)\right]\left\|_{H^{7}}+\right\| \mathcal{P}\left[\psi_{f}-\psi_{\theta}(2 T)\right] \|_{H^{7}}+\left|S_{f}\right|+\left|D_{f}\right|+\left|S_{0}\right|+\left|D_{0}\right|\right]
$$

There exists $\tau \in(0, \eta)$ such that

$$
K|\tau|^{3 / 2}<\frac{\tau}{3}
$$

Let us assume that

$$
\left.K\left[\| \psi_{0}-\psi_{\theta}(0)\right]\left\|_{H^{7}}+\right\| \mathcal{P}\left[\psi_{f}-\psi_{\theta}(2 T)\right] \|_{H^{7}}+\left|S_{f}\right|+\left|D_{f}\right|+\left|S_{0}\right|+\left|D_{0}\right|\right]<\frac{\tau}{3}
$$

Then

$$
F(\tau)>\frac{\tau}{3} \text { and } F(-\tau)<-\frac{\tau}{3}
$$

thus the intermediate values theorem guarantees that $F$ is surjective on a neighbourhood of zero, this ends the proof of Theorem 2.

## 4 Local controllability of $\left(\Sigma_{0}\right)$ around $Y^{0,0,0}$

The aim of this section is the proof of the following theorem.
Theorem 7 Let $\phi_{0}, \phi_{1} \in \mathbb{R}$. There exist $\mathcal{T}>0$ and $\eta>0$ such that, for every $\left(\psi_{0}, S_{0}, D_{0}\right)$, $\left(\psi_{f}, S_{f}, D_{f}\right) \in\left[\mathcal{S} \cap H_{(0)}^{7}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}$ with

$$
\begin{aligned}
& \left\|\psi_{0}-\varphi_{1} e^{i \phi_{0}}\right\|_{H^{7}(I, \mathbb{C})}+\left|S_{0}\right|+\left|D_{0}\right|<\eta \\
& \left\|\psi_{f}-\varphi_{1} e^{i \phi_{1}}\right\|_{H^{7}(I, \mathbb{C})}+\left|S_{f}\right|+\left|D_{f}\right|<\eta
\end{aligned}
$$

there exists a trajectory $(\psi, S, D, u)$ of $\left(\Sigma_{0}\right)$ on $[0, \mathcal{T}]$ such that

$$
\begin{gathered}
(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right) \\
(\psi(\mathcal{T}), S(\mathcal{T}), D(\mathcal{T}))=\left(\psi_{f}, S_{f}, D_{f}\right)
\end{gathered}
$$

and $u \in H_{0}^{1}((0, \mathcal{T}), \mathbb{R})$.

### 4.1 Non controllability of the linearized system around ( $Y^{0,0,0}, u \equiv 0$ )

The linearized system around $\left(Y^{0,0,0}, u \equiv 0\right)$ is

$$
\left(\Sigma_{0}^{l}\right)\left\{\begin{array}{l}
i \dot{\Psi}=-\frac{1}{2} \Psi^{\prime \prime}-w q \psi_{1} \\
\Psi(t, \pm 1 / 2)=0 \\
\dot{s}=w \\
\dot{d}=s
\end{array}\right.
$$

It is a control system where

- the state is $(\Psi, s, d)$ with $\Psi(t) \in T_{\mathcal{S}}\left(\psi_{1}(t)\right)$ for every $t$,
- the control is the real valued function $w$.

Let $\left(\Psi_{0}, s_{0}, d_{0}\right) \in T_{\mathcal{S}}\left(\psi_{1}(0)\right) \times \mathbb{R} \times \mathbb{R}$ and $(\Psi, s, d)$ be the solution of $\left(\Sigma_{0}^{l}\right)$ such that $(\Psi(0), s(0), d(0))=$ $\left(\Psi_{0}, s_{0}, d_{0}\right)$, with some control $w \in L^{2}((0, T), \mathbb{R})$. We have the following equality in $L^{2}(I, \mathbb{C})$

$$
\Psi(t)=\sum_{k=1}^{\infty} x_{k}(t) \varphi_{k} \quad \text { where } \quad x_{k}(t):=<\Psi(t), \varphi_{k}>\forall k \in \mathbb{N}^{*}
$$

Using the parity of the functions $\varphi_{k}$ and the equation solved by $\Psi$, we get

$$
i \dot{x}_{2 k+1}=\lambda_{2 k+1} x_{2 k+1}, \forall k \in \mathbb{N}
$$

Half of the components have a dynamic independent of the control $w$. Thus the control system ( $\Sigma_{0}^{l}$ ) is not controllable.

### 4.2 Local controllability of $\left(\Sigma_{0}\right)$ around $Y^{\gamma, 0,0}$ for $\gamma \neq 0$

Let $\gamma \in \mathbb{R}^{*}$. The ground state for $u \equiv \gamma$ is the function

$$
\psi_{1, \gamma}(t, q):=\varphi_{1, \gamma}(q) e^{-i \lambda_{1, \gamma} t}
$$

where $\lambda_{1, \gamma}$ is the first eigenvalue and $\varphi_{1, \gamma}$ the associated normalised eigenvector of the operator $A_{\gamma}$ defined on

$$
D\left(A_{\gamma}\right):=H^{2} \cap H_{0}^{1}(I, \mathbb{C}) \text { by } A_{\gamma} \varphi:=-\frac{1}{2} \varphi^{\prime \prime}-\gamma q \varphi
$$

When $\alpha, \beta \in \mathbb{R}$, the function

$$
Y^{\gamma, \alpha, \beta}(t):=\left(\psi_{1, \gamma}(t), \alpha+\gamma t, \beta+\alpha t+\gamma t^{2} / 2\right)
$$

solves $\left(\Sigma_{0}\right)$ with $u \equiv \gamma$. We define $T:=4 / \pi, T^{*}:=2 T$ and, for $s=1,3,5,7,9$ the space

$$
H_{(\gamma)}^{s}(I, \mathbb{C}):=\left\{\varphi \in H^{s}(I, \mathbb{C}) ; A_{\gamma}^{n} \varphi \in H_{0}^{1}(I, \mathbb{C}) \text { for } n=0, \ldots,(s-1) / 2\right\}
$$

We admit the following result which will be proved in section 5 .
Theorem 8 There exists $\gamma_{0}>0$ such that, for every $\gamma \in\left(0, \gamma_{0}\right)$, there exists $\delta=\delta(\gamma)>0$, such that, for every $\left(\psi_{0}, S_{0}, D_{0}\right),\left(\psi_{f}, S_{f}, D_{f}\right) \in\left[\mathcal{S} \cap H_{(\gamma)}^{7}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}$ with

$$
\begin{gathered}
\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}}+\left|S_{0}-\alpha\right|+\left|D_{0}-\beta\right|<\delta \\
\left\|\psi_{f}-\psi_{1, \gamma}\left(T^{*}\right)\right\|_{H^{7}(I, \mathbb{C})}+\left|S_{f}-\alpha-\gamma T^{*}\right|+\left|D_{f}-\beta-\alpha T^{*}-\gamma T^{* 2} / 2\right|<\delta
\end{gathered}
$$

for some real constants $\alpha, \beta$, there exists $v \in H_{0}^{1}\left(\left(0, T^{*}\right), \mathbb{R}\right)$ such that, the unique solution of $\left(\Sigma_{0}\right)$ on $\left[0, T^{*}\right]$, with control $u:=\gamma+v$, such that $(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right)$ satisfies

$$
\left(\psi\left(T^{*}\right), S\left(T^{*}\right), D\left(T^{*}\right)\right)=\left(\psi_{f}, S_{f}, D_{f}\right)
$$

### 4.3 Quasi-static transformations

Let $\gamma \in\left(0, \gamma_{0}\right)$ with $\gamma_{0}$ as in Theorem 8. Let $f \in C^{4}([0,3], \mathbb{R})$ be such that

$$
\begin{gather*}
f \equiv 0 \text { on }[0,1 / 2] \cup[5 / 2,3]  \tag{54}\\
f(t)=t \text { for } t \in[1,3 / 2]  \tag{55}\\
\int_{0}^{3} f(t) d t=0 \tag{56}
\end{gather*}
$$

For $\epsilon>0$, we define

$$
\begin{array}{ccc}
u_{\epsilon}:[0,3 / \epsilon] & \rightarrow & \mathbb{R} \\
t & \mapsto & \gamma f^{\prime}(\epsilon t)
\end{array}
$$

Let $\phi_{0}, \phi_{1} \in \mathbb{R}$. Let $\left.\psi_{\epsilon}, S_{\epsilon}, D_{\epsilon}\right)$ be the solution on $[0,1 / \epsilon]$ of

$$
\left\{\begin{array}{l}
i \frac{\partial \psi_{\epsilon}}{\partial t}(t, q)=-\frac{1}{2} \frac{\partial^{2} \psi_{\epsilon}}{\partial q^{2}}(t, q)-u_{\epsilon}(t) q \psi_{\epsilon} \\
\psi_{\epsilon}(0, q)=\varphi_{1}(q) e^{i \phi_{0}} \\
\psi_{\epsilon}(t,-1 / 2)=\psi_{\epsilon}(t, 1 / 2)=0 \\
\dot{S}_{\epsilon}(t)=u_{\epsilon}(t), S_{\epsilon}(0)=0 \\
\dot{D}_{\epsilon}(t)=S_{\epsilon}(t), D_{\epsilon}(0)=0
\end{array}\right.
$$

The following result has been proved in [1, section 4].
Proposition 12 There exist $\epsilon_{0}>0, C_{0}>0$ such that, for every $\epsilon \in\left(0, \epsilon_{0}\right]$,

$$
\left\|\psi_{\epsilon}(1 / \epsilon)-\varphi_{1, \gamma} e^{i\left(\phi_{0}-\int_{0}^{1 / \epsilon} \lambda_{1, \gamma f^{\prime}(\epsilon s)} d s\right)}\right\|_{H^{7}(I, \mathbb{C})} \leqslant C_{0} \gamma^{1 / 8} \epsilon^{1 / 32}
$$

The continuity with respect to initial conditions gives the following proposition.

Proposition 13 Let $\epsilon \in\left(0, \epsilon_{0}\right)$. There exists $\eta_{0}=\eta_{0}(\epsilon)>0$ such that, for every $\left(\psi_{0}, S_{0}, D_{0}\right) \in$ $H_{(0)}^{7}(I, \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$, with

$$
\left\|\psi_{0}-\varphi_{1} e^{i \phi_{0}}\right\|_{H^{7}(I, \mathbb{C})} \leqslant \eta_{0}
$$

the solution $(\psi, S, D)$ of $\left(\Sigma_{0}\right)$ on $[0,1 / \epsilon]$ with initial condition $(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right)$ and control $u_{\epsilon}$ satisfies

$$
\begin{gathered}
\left\|\psi(1 / \epsilon)-\varphi_{1, \gamma} e^{i\left(\phi_{0}-\int_{0}^{1 / \epsilon} \lambda_{1, \gamma f^{\prime}(\epsilon s)} d s\right)}\right\|_{H^{7}(I, \mathbb{C})} \leqslant 2 C_{0} \gamma^{1 / 8} \epsilon^{1 / 32} \\
S(1 / \epsilon)=S_{0}+\frac{\gamma}{\epsilon}, D(1 / \epsilon)=D_{0}+\frac{S_{0}}{\epsilon}+\frac{\gamma}{\epsilon^{2}} \int_{0}^{1} f
\end{gathered}
$$

Let $\left(\xi_{\epsilon}, s_{\epsilon}, d_{\epsilon}\right)$ be the solution on $\left[(1 / \epsilon)+T^{*},(3 / \epsilon)\right]$ of

$$
\left\{\begin{array}{l}
i \frac{\partial \xi_{\epsilon}}{\partial t}(t, q)=-\frac{1}{2} \frac{\partial^{2} \xi_{\epsilon}}{\partial q^{2}}(t, q)-u_{\epsilon}(t) q \xi_{\epsilon} \\
\xi_{\epsilon}(3 / \epsilon, q)=\varphi_{1}(q) e^{i \phi_{\epsilon}} \\
\xi_{\epsilon}(t,-1 / 2)=\xi_{\epsilon}(t, 1 / 2)=0 \\
\dot{s}_{\epsilon}(t)=u_{\epsilon}(t), s_{\epsilon}(3 / \epsilon)=0 \\
\dot{d}_{\epsilon}(t)=s_{\epsilon}(t), d_{\epsilon}(3 / \epsilon)=0
\end{array}\right.
$$

where $\phi_{\epsilon}$ is the unique solution in $\left[\phi_{1}, \phi_{1}+2 \pi\right)$ of

$$
\begin{equation*}
\phi_{\epsilon}+\int_{1 / \epsilon+T^{*}}^{3 / \epsilon} \lambda_{1, \gamma f^{\prime}(\epsilon t)} d t=\phi_{0}-\int_{0}^{1 / \epsilon} \lambda_{1, \gamma f^{\prime}(\epsilon t)} d t-\lambda_{1, \gamma} T^{*},(\bmod 2 \pi) \tag{57}
\end{equation*}
$$

In the same way as in $[1$, section 4] and thanks to (57), we get the following proposition.
Proposition 14 There exist $\epsilon_{f}>0, C_{f}>0$ such that, for every $\epsilon \in\left(0, \epsilon_{f}\right]$,

$$
\left\|\xi_{\epsilon}\left((1 / \epsilon)+T^{*}\right)-\varphi_{1, \gamma} e^{i\left(\phi_{0}-\int_{0}^{1 / \epsilon} \lambda_{1, \gamma f^{\prime}(\epsilon s)} d s-\lambda_{1, \gamma} T^{*}\right)}\right\|_{H^{7}(I, \mathbb{C})} \leqslant C_{f} \gamma^{1 / 8} \epsilon^{1 / 32}
$$

Let us extend $\xi_{\epsilon}$ to $\left[(1 / \epsilon)+T^{*},(3 / \epsilon)+\tau_{\epsilon}\right]$ in such way that $\xi_{\epsilon}\left((3 / \epsilon)+\tau_{\epsilon}\right)=\varphi_{1} e^{i \phi_{1}}$. Let $\tau_{\epsilon}$ be the unique solution in $\left[0,2 \pi / \lambda_{1}\right)$ of

$$
\phi_{\epsilon}-\lambda_{1} \tau_{\epsilon}=\phi_{1}(\bmod 2 \pi)
$$

We extend $u_{\epsilon}$ to $\left[(1 / \epsilon)+T^{*},(3 / \epsilon)+\tau_{\epsilon}\right]$ by zero :

$$
u_{\epsilon}(t):=0, \text { for every } t \in\left[3 / \epsilon,(3 / \epsilon)+\tau_{\epsilon}\right]
$$

We still denote by $\left(\xi_{\epsilon}, s_{\epsilon}, d_{\epsilon}\right)$ the solution of the last system on $\left[(1 / \epsilon)+T^{*},(3 / \epsilon)+\tau_{\epsilon}\right]$. Then,

$$
\xi_{\epsilon}\left((3 / \epsilon)+\tau_{\epsilon}\right)=\varphi_{1} e^{i \phi_{1}}, S_{\epsilon}\left((3 / \epsilon)+\tau_{\epsilon}\right)=0, D_{\epsilon}\left((3 / \epsilon)+\tau_{\epsilon}\right)=0
$$

Again, the continuity with respect to initial conditions gives the following proposition.
Proposition 15 Let $\epsilon \in\left(0, \epsilon_{f}\right)$ such that $\epsilon<1 /\left(2 T^{*}\right)$. There exists $\eta_{f}=\eta_{f}(\epsilon)>0$ such that, for every $\left(\psi_{f}, S_{f}, D_{f}\right) \in H_{(0)}^{7}(I, \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$, with

$$
\left\|\psi_{f}-\varphi_{1} e^{i \phi_{1}}\right\|_{H^{7}(I, \mathbb{C})} \leqslant \eta_{f}
$$

the solution $(\psi, S, D)$ of $\left(\Sigma_{0}\right)$ on $\left[(1 / \epsilon)+T^{*},(3 / \epsilon)+\tau_{\epsilon}\right]$ with initial condition $\left(\left(\psi(3 / \epsilon)+\tau_{\epsilon}\right), S((3 / \epsilon)+\right.$ $\left.\left.\tau_{\epsilon}\right), D\left((3 / \epsilon)+\tau_{\epsilon}\right)\right)=\left(\psi_{f}, S_{f}, D_{f}\right)$ and control $u_{\epsilon}$ satisfies

$$
\begin{gathered}
\left\|\psi\left((1 / \epsilon)+T^{*}\right)-\varphi_{1, \gamma} e^{i\left(\phi_{0}-\int_{0}^{1 / \epsilon} \lambda_{1, \gamma f^{\prime}(\epsilon s)} d s-\lambda_{1, \gamma} T^{*}\right)}\right\|_{H^{7}(I, \mathbb{C})} \leqslant 2 C_{f} \gamma^{1 / 8} \epsilon^{1 / 32} \\
S\left((1 / \epsilon)+T^{*}\right)=S_{f}+\frac{\gamma}{\epsilon}+\gamma T^{*} \\
D\left((1 / \epsilon)+T^{*}\right)=D_{f}+S_{f}\left(T^{*}-\frac{2}{\epsilon}-\tau_{\epsilon}\right)+\frac{\gamma}{\epsilon^{2}} \int_{0}^{1} f+\frac{\gamma}{\epsilon} T^{*}+\frac{1}{2} \gamma T^{* 2}
\end{gathered}
$$

Proof of Theorem 7: We fix $\epsilon \in\left(0, \epsilon_{0}\right)$ such that

$$
\epsilon<1 /\left(2 T^{*}\right) \text { and } 2 \max \left(C_{0}, C_{f}\right) \gamma^{1 / 8} \epsilon^{1 / 32}<\frac{\delta}{3},
$$

where $\delta$ is given by Theorem 8. Let $\left(\psi_{0}, S_{0}, D_{0}\right),\left(\psi_{f}, S_{f}, D_{f}\right) \in\left[\mathcal{S} \cap H_{(0)}^{7}(I, \mathbb{C})\right] \times \mathbb{R} \times \mathbb{R}$ be such that

$$
\begin{gather*}
\left\|\psi_{0}-\varphi_{1, \gamma} e^{i \phi_{0}}\right\|_{H^{7}(I, \mathrm{C})}<\eta_{0}(\epsilon),  \tag{58}\\
\left\|\psi_{f}-\varphi_{1, \gamma} e^{i \phi_{1}}\right\|_{H^{7}(I, \mathrm{C})}<\min \left(\eta_{f}(\epsilon), \delta / 3\right),  \tag{59}\\
\left|S_{0}\right|<\delta / 6,\left|S_{f}\right|<\delta / 6,  \tag{60}\\
\left|D_{0}\right|+\left|D_{f}\right|+\left|S_{0}\right|\left(\frac{1}{\epsilon}+T^{*}\right)+\left|S_{f}\right|\left(\frac{2}{\epsilon}+T^{*}+2 \pi\right)<\delta / 3 . \tag{61}
\end{gather*}
$$

Then, the solution $(\psi, S, D)$ of $\left(\Sigma_{0}\right)$ on $[0,1 / \epsilon]$ with control $u_{\epsilon}$ such that $(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right)$ satisfies

$$
\begin{gathered}
\left\|\psi(1 / \epsilon)-\psi_{1, \gamma}\left(\theta_{\epsilon}\right)\right\|_{H^{7}(I, \mathbb{C})} \leqslant 2 C_{0} \gamma^{1 / 8} \epsilon^{1 / 32}<\delta \\
S(1 / \epsilon)=S_{0}+\frac{\gamma}{\epsilon}, D(1 / \epsilon)=D_{0}+\frac{S_{0}}{\epsilon}+\frac{\gamma}{\epsilon^{2}} \int_{0}^{1} f,
\end{gathered}
$$

where $\theta_{\epsilon}$ is such that

$$
-\lambda_{1, \gamma} \theta_{\epsilon}=\phi_{0}-\int_{0}^{1 / \epsilon} \lambda_{1, \gamma f^{\prime}(\epsilon s)} d s
$$

The solution $(\psi, S, D)$ of $\left(\Sigma_{0}\right)$ on $\left[(1 / \epsilon)+T^{*},(3 / \epsilon)+\tau_{\epsilon}\right]$ with control $u_{\epsilon}$ such that

$$
\left(\psi\left((3 / \epsilon)+\tau_{\epsilon}\right), S\left((3 / \epsilon)+\tau_{\epsilon}\right), D\left((3 / \epsilon)+\tau_{\epsilon}\right)\right)=\left(\psi_{f}, S_{f}, D_{f}\right),
$$

satisfies

$$
\begin{gathered}
\left\|\psi\left((1 / \epsilon)+T^{*}\right)-\psi_{1, \gamma}\left(\theta_{\epsilon}+T^{*}\right)\right\|_{H^{7}(I, \mathbb{C})} \leqslant 2 C_{f} \gamma^{1 / 8} \epsilon^{1 / 32}<\delta / 3, \\
S\left((1 / \epsilon)+T^{*}\right)=S_{f}+\frac{\gamma}{\epsilon}+\gamma T^{*}, \\
D\left((1 / \epsilon)+T^{*}\right)=D_{f}+S_{f}\left(T^{*}-(2 / \epsilon)-\tau_{\epsilon}\right)+\frac{\gamma}{\epsilon^{2}} \int_{0}^{1} f+\frac{\gamma}{\epsilon} T^{*}+\frac{1}{2} \gamma T^{* 2} .
\end{gathered}
$$

We apply Theorem 8 with

$$
\alpha:=S(1 / \epsilon), \beta:=D(1 / \epsilon) .
$$

The assumptions (60) and (61) give

$$
\begin{gathered}
\left|S\left((1 / \epsilon)+T^{*}\right)-\alpha-\gamma T^{*}\right|<\delta / 3 \\
\mid D\left((1 / \epsilon)+T^{*}\right)-\beta-\alpha T^{*}-\gamma T^{* 2} / 2<\delta / 3
\end{gathered}
$$

Thus, there exists $v \in H_{0}^{1}\left(\left(0, T^{*}\right), \mathbb{R}\right)$ such that the solution $(\tilde{\psi}, \tilde{S}, \tilde{D})$ of $\left(\Sigma_{0}\right)$ on $\left[0, T^{*}\right]$ with control $u:=\gamma+v$ such that $(\tilde{\psi}(0), \tilde{S}(0), \tilde{D}(0))=(\psi(1 / \epsilon), S(1 / \epsilon), D(1 / \epsilon))$ satisfies

$$
\left(\tilde{\psi}\left(T^{*}\right), \tilde{S}\left(T^{*}\right), \tilde{D}\left(T^{*}\right)\right)=\left(\psi\left((1 / \epsilon)+T^{*}\right), S\left((1 / \epsilon)+T^{*}\right), D\left((1 / \epsilon)+T^{*}\right)\right) .
$$

Thus, the control $u:\left[0,(3 / \epsilon)+\tau_{\epsilon}\right] \rightarrow \mathbb{R}$ defined by

$$
\begin{gathered}
u=u_{\epsilon} \text { on }[0,1 / \epsilon] \cup\left[(1 / \epsilon)+T^{*},(3 / \epsilon)+\tau_{\epsilon}\right], \\
u(t)=\gamma+v(t-1 / \epsilon) \text { for every } t \in\left[1 / \epsilon,(1 / \epsilon)+T^{*}\right]
\end{gathered}
$$

gives the result.

## 5 Local controllability of $\left(\Sigma_{0}\right)$ around $Y^{\gamma, \alpha, \beta}$

The aim of this section is the proof of Theorem 8. In [1] a similar local controllability result has been proved for the subsystem $(\Sigma)$ defined in the Introduction. It is the following one.

Theorem 9 There exists $\gamma_{0}>0$ such that, for every $\gamma \in\left(0, \gamma_{0}\right)$, there exist $\delta>0, C>0$ and a continuous map

$$
\begin{aligned}
\Gamma_{\gamma}: \begin{array}{ccccc}
\mathcal{V}_{\gamma}(0) & \times & \mathcal{V}_{\gamma}(T) & \rightarrow & H_{0}^{1}((0, T), \mathbb{R}) \\
\left(\psi_{0}\right. & , & \left.\psi_{f}\right) & \mapsto & v
\end{array}, ~\left(\begin{array}{ll} 
& \mapsto
\end{array}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{V}_{\gamma}(0) & :=\left\{\psi_{0} \in \mathcal{S} \cap H_{(\gamma)}^{7}(I, \mathbb{C}) ;\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}(I, \mathbb{C})}<\delta\right\} \\
\mathcal{V}_{\gamma}(T) & :=\left\{\psi_{f} \in \mathcal{S} \cap H_{(\gamma)}^{7}(I, \mathbb{C}) ;\left\|\psi_{f}-\psi_{1, \gamma}(T)\right\|_{H^{7}(I, \mathbb{C})}<\delta\right\}
\end{aligned}
$$

such that, for every $\psi_{0} \in \mathcal{V}_{\gamma}(0), \psi_{f} \in \mathcal{V}_{\gamma}(T)$, the unique solution of $(\Sigma)$ with control $u:=\gamma+v$ such that $\psi(0)=\psi_{0}$ satisfies $\psi(T)=\psi_{f}$ and

$$
\left\|\Gamma_{\gamma}\left(\psi_{0}, \psi_{f}\right)\right\|_{H_{0}^{1}((0, T), \mathbb{R})} \leqslant C\left[\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}(I, \mathbb{C})}+\left\|\psi_{f}-\psi_{1, \gamma}(T)\right\|_{H^{7}(I, \mathbb{C})}\right]
$$

Let us recall the main ideas of the proof of this Theorem in order to emphasize the difficulty of Theorem 8. We proved that the linearized system of $(\Sigma)$ around $\left(\psi_{1, \gamma}, u \equiv \gamma\right)$ is controllable and we concluded by applying an implicit function theorem of Nash-Moser type.

This strategy does not work with $\left(\Sigma_{0}\right)$ because the linearized system of $\left(\Sigma_{0}\right)$ around $Y^{\gamma, \alpha, \beta}$ is not controllable.

### 5.1 Controllability up to codimension one of the linearized system around

( $Y^{\gamma, \alpha, \beta}, u \equiv \gamma$ )
In this section, we fix $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{*}$. The linearized control system around $\left(Y^{\gamma, \alpha, \beta}, u \equiv \gamma\right)$ is

$$
\left(\Sigma_{\gamma}^{l}\right)\left\{\begin{array}{l}
i \frac{\partial \Psi}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \Psi}{\partial q^{2}}-\gamma q \Psi-w q \psi_{1, \gamma} \\
\Psi(t,-1 / 2)=\Psi(t, 1 / 2)=0 \\
\dot{s}=w \\
\dot{d}=s
\end{array}\right.
$$

It is a control system where

- the state is $(\Psi, s, d)$ with $\Psi(t) \in T_{\mathcal{S}}\left(\psi_{1, \gamma}(t)\right)$,
- the control is the real valued function $w$.

Let us recall that the space $L^{2}(I, \mathbb{C})$ has a complete orthonormal system $\left(\varphi_{k, \gamma}\right)_{k \in \mathbb{N}^{*}}$ of eigenfunctions for the operator $A_{\gamma}$ defined on

$$
\begin{gathered}
D\left(A_{\gamma}\right):=H^{2} \cap H_{0}^{1}(I, \mathbb{C}) \quad \text { by } \quad A_{\gamma} \varphi:=-\frac{1}{2} \varphi^{\prime \prime}-\gamma q \varphi \\
A_{\gamma} \varphi_{k, \gamma}=\lambda_{k, \gamma} \varphi_{k, \gamma}
\end{gathered}
$$

where $\left(\lambda_{k, \gamma}\right)_{k \in \mathbb{N}^{*}}$ is an increasing sequence of positive real numbers. For technical reasons, we introduce the notation

$$
b_{k, \gamma}:=<\varphi_{k, \gamma}, q \varphi_{1, \gamma}>
$$

It has already been proved in [1, Proposition 1, Section 3.1] that, for $\gamma$ small enough and different from zero, $b_{k, \gamma}$ is different from zero for every $k \in \mathbb{N}^{*}$ and, roughly speaking, behaves like $1 / k^{3}$ when $k \rightarrow+\infty$. In all this section, we assume we are in this situation.

Proposition 16 Let $T>0$ and $(\Psi, s, d)$ be a trajectory of $\left(\Sigma_{\gamma}^{l}\right)$ on $[0, T]$. Then, for every $t \in[0, T]$, we have

$$
\begin{equation*}
s(t)=s(0)+\frac{1}{i b_{1, \gamma}}\left(<\Psi(t), \varphi_{1, \gamma}>e^{i \lambda_{1, \gamma} t}-<\Psi(0), \varphi_{1, \gamma}>\right) \tag{62}
\end{equation*}
$$

Thus, the control system $\left(\Sigma_{\gamma}^{l}\right)$ is not controllable.
Proof : Let $x_{1}(t):=<\Psi(t), \varphi_{1, \gamma}>$. We have

$$
\begin{gathered}
\dot{x}_{1}(t)=<\frac{\partial \Psi}{\partial t}(t), \varphi_{1, \gamma}>=<-i A_{\gamma} \Psi(t)+i w(t) q \psi_{1, \gamma}(t), \varphi_{1, \gamma}> \\
\dot{x}_{1}(t)=-i \lambda_{1, \gamma} x_{1}(t)+i b_{1, \gamma} w(t) e^{-i \lambda_{1 \gamma} t} \\
x_{1}(t)=\left(x_{1}(0)+i b_{1, \gamma} \int_{0}^{t} w(\tau) d \tau\right) e^{-i \lambda_{1, \gamma} t}
\end{gathered}
$$

We get (62) by using

$$
s(t)=s(0)+\int_{0}^{t} w(\tau) d \tau
$$

Let $T>0, \Psi_{0} \in T_{\mathcal{S}}\left(\psi_{1, \gamma}(0)\right), \Psi_{f} \in T_{\mathcal{S}}\left(\psi_{1, \gamma}(T)\right), s_{0}, s_{f} \in \mathbb{R}$. A necessary condition for the existence of a trajectory of $\left(\Sigma_{\gamma}^{l}\right)$ such that $\Psi(0)=\Psi_{0}, s(0)=s_{0}, \Psi(T)=\Psi_{f}, s(T)=s_{f}$ is

$$
s_{f}-s_{0}=\frac{1}{i b_{1, \gamma}}\left(<\Psi_{f}, \varphi_{1, \gamma}>e^{i \lambda_{1, \gamma} T}-<\Psi_{0}, \varphi_{1, \gamma}>\right)
$$

This equality does not happen for an arbitrary choice of $\Psi_{0}, \Psi_{f}, s_{0}, s_{f}$. Thus $\left(\Sigma_{\gamma}^{l}\right)$ is not controllable. $\square$
Proposition 17 Let $T>0,\left(\Psi_{0}, s_{0}, d_{0}\right),\left(\Psi_{f}, s_{f}, d_{f}\right) \in H_{(0)}^{3}(I, \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$ be such that

$$
\begin{gather*}
\Re<\Psi_{0}, \psi_{1, \gamma}(0)>=\Re<\Psi_{f}, \psi_{1, \gamma}(T)>=0  \tag{63}\\
s_{f}-s_{0}=\frac{i}{b_{1, \gamma}}\left(<\Psi_{0}, \varphi_{1, \gamma}>-<\Psi_{f}, \varphi_{1, \gamma}>e^{i \lambda_{1, \gamma} T}\right) . \tag{64}
\end{gather*}
$$

Then there exists $w \in L^{2}((0, T), \mathbb{R})$ such that the solution of $\left(\Sigma_{\gamma}^{l}\right)$ with control $w$ and such that $(\Psi(0), s(0), d(0))=\left(\Psi_{0}, s_{0}, d_{0}\right)$ satisfies $(\Psi(T), s(T), d(T))=\left(\Psi_{f}, s_{f}, d_{f}\right)$.

Remark 5 We can control $\Psi$ and d but we cannot control s. We miss only two directions which are $(\Psi, s, d)=(0, \pm 1,0)$.

Proof : Let $\left(\Psi_{0}, s_{0}, d_{0}\right) \in T_{\mathcal{S}}\left(\psi_{1, \gamma}(0)\right) \times \mathbb{R} \times \mathbb{R}$ and $T>0$. Let $(\Psi, s, d)$ be a solution of $\left(\Sigma_{\gamma}^{l}\right)$ with $(\Psi(0), s(0), d(0))=\left(\Psi_{0}, s_{0}, d_{0}\right)$ and a control $w \in L^{2}((0, T), \mathbb{R})$. Let $\left(\Psi_{f}, s_{f}, d_{f}\right) \in T_{\mathcal{S}}\left(\psi_{1, \gamma}(T)\right) \times \mathbb{R} \times \mathbb{R}$. The equality $(\Psi(T), s(T), d(T))=\left(\Psi_{f}, s_{f}, d_{f}\right)$ is equivalent to the following moment problem on $w$,

$$
\begin{align*}
& \int_{0}^{T} w(t) e^{i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) t} d t=\frac{i}{b_{k, \gamma}}\left(<\Psi_{0}, \varphi_{k, \gamma}>-<\Psi_{f}, \varphi_{k, \gamma}>e^{i \lambda_{k, \gamma} T}\right), \forall k \in \mathbb{N}^{*} \\
& \int_{0}^{T} w(t) d t=s_{f}-s_{0}  \tag{65}\\
& \int_{0}^{T}(T-t) w(t) d t=d_{f}-d_{0}-s_{0} T
\end{align*}
$$

The left hand sides of the two first equalities with $k=1$ are equal, the equality of the right hand sides is guaranteed by (64). Under the assumption $\Psi_{0}, \Psi_{f} \in H_{(0)}^{3}(I, \mathbb{C})$, the right hand side of (65) defines a sequence in $l^{2}$. Thus, under the assumptions (64), and $\Psi_{0}, \Psi_{f} \in H_{(0)}^{3}(I, \mathbb{C})$, the existence of a solution $w \in L^{2}((0, T), \mathbb{R})$ of $(65)$ can be proved in the same way as in [1, Theorem 5].

### 5.2 Local controllability up to codimension one of $\left(\Sigma_{0}\right)$ around $Y^{\gamma, \alpha, \beta}$

In this section, we fix $\alpha, \beta \in \mathbb{R}$. The aim of this section is the proof of the following result.
Theorem 10 There exists $\gamma_{0}>0$ such that, for every $\gamma \in\left(0, \gamma_{0}\right)$, for every $S_{0} \in \mathbb{R}$, there exist $\delta>0$, $C>0$ and a continuous map

$$
\begin{array}{cccccc}
\Gamma_{\gamma, S_{0}}: & \mathcal{V}_{\gamma}(0) & \times & \mathcal{V}_{\gamma, S_{0}}(T) & \rightarrow & H_{0}^{1}((0, T), \mathbb{R}) \\
\left(\left(\psi_{0}, D_{0}\right)\right. & , & \left.\left(\psi_{f}, D_{f}\right)\right) & \mapsto & v
\end{array}
$$

where

$$
\begin{aligned}
& \mathcal{V}_{\gamma}(0):=\left\{\left(\psi_{0}, D_{0}\right) \in\left[\mathcal{S} \cap H_{(\gamma)}^{7}(I, \mathbb{C})\right] \times \mathbb{R} ;\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}(I, \mathbb{C})}+\left|D_{0}-\beta\right|<\delta\right\}, \\
& \mathcal{V}_{\gamma, S_{0}}(T):=\left\{\left(\psi_{f}, D_{f}\right) \in\left[\mathcal{S} \cap H_{(\gamma)}^{7}(I, \mathbb{C})\right] \times \mathbb{R} ;\right. \\
&\left.\left\|\psi_{f}-\psi_{1, \gamma}(T)\right\|_{H^{7}(I, \mathbb{C})}+\left|D_{f}-\beta-S_{0} T-\gamma T^{2} / 2\right|<\delta\right\},
\end{aligned}
$$

such that, for every $\left(\psi_{0}, D_{0}\right) \in \mathcal{V}_{\gamma}(0),\left(\psi_{f}, D_{f}\right) \in \mathcal{V}_{\gamma, S_{0}}(T)$, the unique solution of $\left(\Sigma_{0}\right)$ with control $u:=\gamma+v$, such that

$$
(\psi(0), S(0), D(0))=\left(\psi_{0}, S_{0}, D_{0}\right)
$$

satisfies $\psi(T)=\psi_{f}, D(T)=D_{f}$ and

$$
\begin{aligned}
\left\|\Gamma_{\gamma, S_{0}}\left(\psi_{0}, \psi_{f}\right)\right\|_{H_{0}^{1}((0, T), \mathbb{R})} \leqslant & C\left[\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}(I, \mathbb{C})}+\left|D_{0}-\beta\right|+\right. \\
& \left.\left\|\psi_{f}-\psi_{1, \gamma}(T)\right\|_{H^{7}(I, \mathrm{C})}+\left|D_{f}-\beta-S_{0} T-\gamma T^{2} / 2\right|\right] .
\end{aligned}
$$

Remark 6 The same result is true if one replaces, $\psi_{1, \gamma}(0)$ by $\psi_{1, \gamma}(\theta)$ and $\psi_{1, \gamma}(T)$ by $\psi_{1, \gamma}(\theta+T)$ for some $\theta \in \mathbb{R}$. Indeed, if $(\psi, S, D)$ solves $\left(\Sigma_{0}\right)$ on $[0, T]$ with initial condition $(\psi(0), S(0), D(0))=$ $\left(\psi_{0}, S_{0}, D_{0}\right)$ and control $u$, then, $\left(\tilde{\psi}:=\psi e^{-i \lambda_{1}, \gamma}, S, D\right)$ solves $\left(\Sigma_{0}\right)$ on $[0, T]$ with initial condition $(\tilde{\psi}(0), S(0), D(0))=\left(\psi_{0} e^{-i \lambda_{1, \gamma} \theta}, S_{0}, D_{0}\right)$ and control $u$.

The same loss of regularity as in section 3.2.1 prevents us from using the inverse mapping theorem. We use exactly the same strategy as in [1]. We expose in the next sections the few differences in the proof.

In [1], the local controllability of ( $\Sigma$ ) in a neighbourhood of ( $\psi_{1, \gamma}, u \equiv \gamma$ ) was got by proving a local surjectivity result on the map $\Phi_{\gamma}:\left(\psi_{0}, v\right) \mapsto\left(\psi_{0}, \psi(T)\right)$, where $\psi$ is the solution of $(\Sigma)$ with $u:=\gamma+v$ such that $\psi(0)=\psi_{0}$. Thus, in order to prove the local controllability of $\left(\Sigma_{0}\right)$ in a neighbourhood of $\left(\psi_{1, \gamma}, \beta+\alpha t+\gamma t^{2} / 2\right)$ we consider the map

$$
\widetilde{\Phi}_{\gamma, S_{0}}:\left(\psi_{0}, D_{0}, v\right) \mapsto\left(\psi_{0}, D_{0}, \psi(T), D(T)\right),
$$

where $(\psi, S, D)$ is the solution of $\left(\Sigma_{0}\right)$ with control $u:=\gamma+v$ such that $\psi(0)=\psi_{0}, S(0)=S_{0}$, $D(0)=D_{0}$. As in [1] we get a local surjectivity result on this map by applying a Nash-Moser theorem, stated in section 3.2.2.

### 5.2.1 Context for the Nash-Moser theorem

We apply the Theorem 4 to the map $\widetilde{\Phi}_{\gamma}$ with $\mathcal{P}=I d$ and the spaces defined, for $k=1,3,5,7,9$, by

$$
\begin{aligned}
& \widetilde{E}_{k}^{\gamma}:=\left[\mathcal{S} \cap H_{(\gamma)}^{k}(I, \mathbb{C})\right] \times \mathbb{R} \times H_{0}^{(k-1) / 2}((0, T), \mathbb{R}), \\
& \widetilde{F}_{k}^{\gamma}:=\left[\mathcal{S} \cap H_{(\gamma)}^{k}(I, \mathbb{C})\right] \times \mathbb{R} \times\left[\mathcal{S} \cap H_{(\gamma)}^{k}(I, \mathbb{C})\right] \times \mathbb{R}
\end{aligned}
$$

The smoothing operators defined on the spaces

$$
\begin{gathered}
E_{k}^{\gamma}:=\left[\mathcal{S} \cap H_{(\gamma)}^{k}(I, \mathbb{C})\right] \times H_{0}^{(k-1) / 2}((0, T), \mathbb{R}) \text { and } \\
F_{k}^{\gamma}:=\left[\mathcal{S} \cap H_{(\gamma)}^{k}(I, \mathbb{C}) \times\left[\mathcal{S} \cap H_{(\gamma)}^{k}(I, \mathbb{C})\right] \times \mathbb{R}\right.
\end{gathered}
$$

in [1, section 3.3], give easily suitable smoothing operators on the spaces $\widetilde{E}_{k}^{\gamma}$ and $\widetilde{F}_{k}^{\gamma}$ : we don't do anything on the constants in $\mathbb{R}$.

As in [1, section 3.4], the map $\widetilde{\Phi}_{\gamma, S_{0}}: \widetilde{E}_{7}^{\gamma} \rightarrow \widetilde{F}_{\beta}^{\gamma}$ is twice differentiable. The maps $\widetilde{\Phi}_{\gamma, S_{0}}^{\prime}$ and $\widetilde{\Phi}_{\gamma, S_{0}}^{\prime \prime}$ do not depend on $S_{0}$, thus, we just write $\widetilde{\Phi}_{\gamma}^{\prime}$ and $\widetilde{\Phi}_{\gamma}^{\prime \prime}$. The map $\widetilde{\Phi}_{\gamma}^{\prime \prime}$ satisfies the inequality (18). Indeed, if we write

$$
\Phi_{\gamma}^{\prime \prime}\left(\psi_{0}, v\right) \cdot\left(\left(\phi_{0}, \nu\right),\left(\xi_{0}, \mu\right)\right)=(0, h(T)),
$$

then, we have

$$
\widetilde{\Phi}_{\gamma}^{\prime \prime}\left(\psi_{0}, D_{0}, v\right) \cdot\left(\left(\phi_{0}, d_{0}, \nu\right),\left(\xi_{0}, \delta_{0}, \mu\right)\right)=(0,0, h(T), 0)
$$

and the inequality (18) was already proved for $\Phi_{\gamma}$ in [1, section 3.4].
The assumptions of Theorem 5 can be checked in the same way as in [1, appendix C]. In the following two sections, we focus on the most difficult part in the application of the Nash-Moser theorem, which is the existence of a right inverse to the differential with the bounds (20),(21),(22),(23).
5.2.2 Controllability up to codimension one of the linearized system around ( $\psi_{1, \gamma}(t), D(t)=$ $\beta+S_{0} t+\gamma t^{2} / 2, u \equiv \gamma$ ) and bounds (20), (21), (22), (23)
In [1, Section 3.5], in order to study the controllability of $\Psi$, we introduced the map

$$
Z_{\gamma}: w \mapsto\left(\int_{0}^{T} w(t) e^{i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) t} d t\right)_{k \in \mathbb{N}^{*}} .
$$

Thus, in order to study the controllability of $(\Psi, d)$, it is natural to introduce the map $\widetilde{Z}_{\gamma}$, defined by

$$
\begin{aligned}
& \widetilde{Z}_{\gamma}(w)_{0}:=\int_{0}^{T}(T-t) w(t) d t, \\
& \widetilde{Z}_{\gamma}(w)_{k}:=Z_{\gamma}(w)_{k}, \forall k \in \mathbb{N}^{*} .
\end{aligned}
$$

Let $\Psi_{0}, \Psi_{f} \in L^{2}(I, \mathbb{C}), s_{0}, d_{0}, d_{f} \in \mathbb{R}, T>0$ and $(\Psi, s, d)$ the solution of $\left(\Sigma_{\gamma}^{l}\right)$ such that $(\Psi(0), s(0), d(0))=\left(\Psi_{0}, s_{0}, d_{0}\right)$ with some control $w \in L^{2}((0, T), \mathbb{R})$. As noticed in section 5.1, the equality $(\Psi(T), s(T), d(T))=\left(\Psi_{f}, s_{f}, d_{f}\right)$ is equivalent to $Z_{\gamma}(w)=D$ where $D=\left(D_{k}\right)_{k \in \mathbb{N}}$ is defined by

$$
\begin{gathered}
D_{0}:=d_{f}-d_{0}-s_{0} T, \\
D_{k}:=\frac{i}{b_{k, \gamma}}\left(<\Psi_{0}, \varphi_{k, \gamma}>-<\Psi_{f}, \varphi_{k, \gamma}>e^{i \lambda_{k, \gamma} T}\right), \forall k \in \mathbb{N}^{*} .
\end{gathered}
$$

Proposition 18 Let $T=4 / \pi$. There exists $\gamma_{0}>0, C_{1}>0$ such that, for every $\gamma \in\left(-\gamma_{0}, \gamma_{0}\right)$,

1. the linear map $\widetilde{Z}_{\gamma}$ is continuous from $L^{2}((0, T), \mathbb{R})$ to $l_{r}^{2}(\mathbb{N}, \mathbb{C})$, from $H_{0}^{1}((0, T), \mathbb{R})$ to $h_{r}^{2}(\mathbb{N}, \mathbb{C})$, from $H_{0}^{2}((0, T), \mathbb{R})$ to $h_{r}^{4}(\mathbb{N}, \mathbb{C})$, from $H_{0}^{3}((0, T), \mathbb{R})$ to $h_{r}^{6}(\mathbb{N}, \mathbb{C})$.
2. for every $w \in H_{0}^{3}((0, T), \mathbb{R})$,

$$
\left\|\left(\widetilde{Z}_{\gamma}-\widetilde{Z}_{0}\right)(w)\right\|_{F} \leqslant C_{1} \gamma^{2}\|w\|_{E}
$$

for $(E, F)=\left(L^{2}, l_{r}^{2}\right),\left(H_{0}^{1}, h_{r}^{2}\right),\left(H_{0}^{2}, h_{r}^{4}\right),\left(H_{0}^{3}, h_{r}^{6}\right)$.
The same results have already been proved for the maps $Z_{\gamma}$ in $[1$, Propositions 11,13$]$. The new term in $\widetilde{Z}_{\gamma}$ has no influence.
Proposition 19 Let $T=4 / \pi$. There exists a continuous linear map

$$
\widetilde{Z}_{0}^{-1}: h_{r}^{6}(\mathbb{N}, \mathbb{C}) \rightarrow H_{0}^{3}((0, T), \mathbb{R})
$$

such that, for every $d \in h_{r}^{6}(\mathbb{N}, \mathbb{C}), \widetilde{Z}_{0} \circ \widetilde{Z}_{0}^{-1}(d)=d$. Moreover, there exists a constant $C_{0}$ such that, for every $d \in h_{r}^{6}(\mathbb{N}, \mathbb{C})$, the function $w:=\widetilde{Z}_{0}^{-1}(d)$ satisfies

$$
\|w\|_{L^{2}} \leqslant C_{0}\|d\|_{l^{2}},\|w\|_{H_{0}^{1}} \leqslant C_{0}\|d\|_{h^{2}},\|w\|_{H_{0}^{2}} \leqslant C_{0}\|d\|_{h^{4}},\|w\|_{H_{0}^{3}} \leqslant C_{0}\|d\|_{h^{6}}
$$

Proof : As in [1, Proof of Proposition 12], we introduce the notations, for $k \in \mathbb{N}^{*}$,

$$
\omega_{k}:=\lambda_{k+1}-\lambda_{1}, \omega_{-k}:=-\omega_{k}
$$

Let $d \in h_{r}^{6}(\mathbb{N}, \mathbb{C})$. We define $\tilde{d} \in h^{6}(\mathbb{Z}, \mathbb{C})$ by

$$
\widetilde{d}_{k}:=d_{k+1}, \widetilde{d}_{-k}:=\widetilde{\widetilde{d}_{k}}, \text { for every } k \in \mathbb{N}
$$

A candidate for $\widetilde{Z}^{-1}(d)$ is

$$
w(t):=\left(\frac{1}{T} \sum_{k \in \mathbb{Z}} \tilde{d}_{k} e^{i \omega_{k} t}+\alpha\left(e^{i \frac{1}{2} n \pi^{2} t}+e^{-i \frac{1}{2} n \pi^{2} t}\right)\right)\left(1-e^{i \frac{1}{2} \pi^{2} t}\right)^{2}\left(1-e^{-i \frac{1}{2} \pi^{2} t}\right)^{2}
$$

where $n \in \mathbb{N}$ with $\{n, n \pm 1, n \pm 2\} \cap\left\{ \pm\left(k^{2}-1\right) ; k \in \mathbb{N}^{*}\right\}=\emptyset$ and $\alpha \in \mathbb{R}$ is such that $\int_{0}^{T}(T-t) w(t) d t=$ $d_{0}$. There exists a constant $C=C(n)$ such that $|\alpha| \leqslant C\|d\|_{l^{2}(\mathbb{N}, \mathbb{C})}$.

Finally, we get the following proposition, which poof is the same as the one of [1, Proposition 14] Proposition 20 Let $T=4 / \pi$. There exists $\gamma_{0}>0, C_{2}>0$ such that, for every $\gamma \in\left(-\gamma_{0}, \gamma_{0}\right)$, there exists a linear map

$$
\widetilde{Z}_{\gamma}^{-1}: h_{r}^{6}(\mathbb{N}, \mathbb{C}) \rightarrow H_{0}^{3}((0, T), \mathbb{R})
$$

such that, for every $d \in h_{r}^{6}(\mathbb{N}, \mathbb{C}), \widetilde{Z}_{\gamma} \circ \widetilde{Z}_{\gamma}^{-1}(d)=d$. Moreover, for every $d \in h_{r}^{6}(\mathbb{N}, \mathbb{C})$, the function $w:=\widetilde{Z}_{\gamma}^{-1}(d)$ satisfies

$$
\|w\|_{L^{2}} \leqslant C_{2}\|d\|_{l^{2}},\|w\|_{H_{0}^{1}} \leqslant C_{2}\|d\|_{h^{2}},\|w\|_{H_{0}^{2}} \leqslant C_{2}\|d\|_{h^{4}},\|w\|_{H_{0}^{3}} \leqslant C_{2}\|d\|_{h^{6}}
$$

Thanks to the behaviour of the coefficients $b_{k, \gamma}$, we get the following controllability result for the linearized system around $\left(\psi_{1, \gamma}, \beta+\alpha t+\gamma t^{2} / 2, u \equiv \gamma\right)$.
Theorem 11 There exists $\gamma_{0}>0$ such that, for every $\gamma \in\left(-\gamma_{0}, \gamma_{0}\right)$ different from zero, there exist $C>0$ and a continuous map

$$
\begin{aligned}
& \Pi_{\gamma}:\left[\mathcal{T}_{\gamma, 0} \cap H_{(\gamma)}^{9}\right] \times \mathbb{R} \times\left[\mathcal{T}_{\gamma, T} \cap H_{(\gamma)}^{9}\right] \times \quad \mathbb{R} \quad \rightarrow \quad \widetilde{E}_{7}^{\gamma} \\
& \left(\Psi_{0} \quad, \quad d_{0} \quad, \quad \Psi_{f} \quad, \quad d_{f}\right) \mapsto\left(\Psi_{0}, d_{0}, w\right)
\end{aligned}
$$

where, for every $t \in \mathbb{R}$,

$$
\mathcal{T}_{\gamma, t}:=\left\{\varphi \in L^{2}(I, \mathbb{C}) ; \Re\left(<\varphi, \psi_{1, \gamma}(t)>\right)=0\right\}
$$

such that, for every $\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right) \in \widetilde{F}_{9}^{\gamma}$ with $\Psi_{0} \in \mathcal{T}_{\gamma, 0}$ and $\Psi_{f} \in \mathcal{T}_{\gamma, T}$, we have

$$
\begin{gathered}
\widetilde{\Phi}_{\gamma}^{\prime}\left(\varphi_{1, \gamma}, 0,0\right) \cdot \Pi_{\gamma}\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right)=\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right) \\
\|w\|_{E} \leqslant C\left\|\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right)\right\|_{F}
\end{gathered}
$$

for any $(E, F) \in\left\{\left(L^{2}, \widetilde{F}_{3}^{\gamma}\right),\left(H_{0}^{1}, \widetilde{F}_{5}^{\gamma}\right),\left(H_{0}^{2}, \widetilde{F}_{7}^{\gamma}\right),\left(H_{0}^{3}, \widetilde{F}_{9}^{\gamma}\right)\right\}$.

### 5.2.3 Controllability up to codimension one of the linearized system around $(Y(t), u(t))$

Let $\gamma \in\left(-\gamma_{0}, \gamma_{0}\right)$ different from zero, where $\gamma_{0}$ is as in Theorem 11. Let $T:=4 / \pi,\left(\psi_{0}, D_{0}, v\right) \in \widetilde{E}_{9}^{\gamma}$ and $S_{0} \in \mathbb{R}$. As in [1, Section 3.6.3] we introduce

$$
\begin{array}{ll}
\Delta_{3}:=\gamma+\delta_{3}, & \Delta_{5}:=\gamma+\delta_{5}, \\
\Delta_{7}:=\gamma+\delta_{7}+\delta_{5}^{2}, & \Delta_{9}:=\gamma+\delta_{9}+\delta_{7} \delta_{5}+\delta_{5}^{3},
\end{array}
$$

where $\delta_{i}:=\left\|\left(\psi_{0}, d_{0}, v\right)-\left(\varphi_{1, \gamma}, 0,0\right)\right\|_{\widetilde{E}_{i}^{0}}$.
Let $Y(t):=(\psi(t), S(t), D(t))$ be the solution of $\left(\Sigma_{0}\right)$ with control $u:=\gamma+v$ such that $\psi(0)=\psi_{0}$, $S(0)=S_{0}$ and $D(0)=D_{0}$. The linearized system around $(\psi(t), D(t), u(t))$ is

$$
\left\{\begin{array}{l}
i \frac{\partial \Psi}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \Psi}{\partial q^{2}}-u q \Psi-w q \psi \\
\Psi(t, \pm 1 / 2)=0 \\
\dot{s}=w \\
\dot{d}=s
\end{array}\right.
$$

If $(\Psi(0), s(0), d(0))=\left(\Psi_{0}, 0, d_{0}\right)$, the equality $(\Psi(T), d(T))=\left(\Psi_{f}, d_{f}\right)$ is equivalent to $\widetilde{M}_{\left(\psi_{0}, u\right)}(w)=$ $\widetilde{d}\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right)$ where

$$
\begin{array}{ll}
\widetilde{M}_{\left(\psi_{0}, u\right)}(w)_{0}:=\int_{0}^{T}(T-t) w(t) d t, & \widetilde{d}\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right)_{0}:=d_{f}-d_{0}, \\
\widetilde{M}_{\left(\psi_{0}, u\right)}(w)_{k}:=M_{\psi_{0}, u}(w)_{k}, & \widetilde{d}\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right)_{k}:=d\left(\Psi_{0}, \Psi_{f}\right)_{k}, \forall k \in \mathbb{N}^{*}
\end{array}
$$

and the map $M_{\left(\psi_{0}, u\right)}$ is defined in [1, Section3.6.1].
As in [1], we prove a surjectivity result on $\widetilde{M}_{\left(\psi_{0}, u\right)}$ when $\Delta_{3}$ is small enough. The argument is the following one : we know a right inverse for $\widetilde{M}_{\left(\varphi_{1}, \gamma, \gamma\right)}$, built in the previous subsection, and we prove that, when $\Delta_{3}$ is small, $\widetilde{M}_{\left(\psi_{0}, u\right)}$ and $\widetilde{M}_{\left(\varphi_{1, \gamma}, \gamma\right)}$ are close enough, in order to get a right inverse of $\widetilde{M}_{\left(\psi_{0}, u\right)}$.

The study of $\left(\widetilde{M}_{\left(\psi_{0}, u\right)}-\widetilde{M}_{\left(\varphi_{1, \gamma}, \gamma\right)}\right)(w)$ reduces to the study of $\left(M_{\left(\psi_{0}, u\right)}-M_{\left(\varphi_{1, \gamma}, \gamma\right)}\right)(w)$ [1, section 3.6.3], because the new terms are equal. The study of the right hand side $\widetilde{d}\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right)$ is the same as in [ 1 , section 3.6.4]. In this way, we get the following theorem.

Theorem 12 Let $\gamma \in\left(-\gamma_{0}, \gamma_{0}\right)$ different from zero. Let $\left(\psi_{0}, D_{0}, v\right) \in \widetilde{E}_{9}^{\gamma}$ and $(\psi, S, D)$ be the associated solution of $\left(\Sigma_{0}\right)$ with $u:=\gamma+v$. If $\Delta_{3}:=\gamma+\left\|\left(\psi_{0}, D_{0}, v\right)-\left(\varphi_{1, \gamma}, 0,0\right)\right\|_{\tilde{E}_{3}^{0}}$ is small enough, then there exists a constant $C>0$ and a continuous map
where

$$
\mathcal{T}_{\psi, t}:=\left\{\varphi \in L^{2}(I, \mathbb{C}) ; \Re(<\varphi, \psi(t)>)=0\right\},
$$

such that, for every $\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right) \in \widetilde{F}_{9}^{\gamma}$ with $\Psi_{0} \in \mathcal{T}_{\psi, 0}, \Psi_{f} \in \mathcal{T}_{\psi, T}$, we have

$$
\widetilde{\Phi}_{\gamma, S_{0}}^{\prime}\left(\psi_{0}, D_{0}, v\right) \cdot \Pi_{\psi, v}\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right)=\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right),
$$

and the same bounds as in [1, Theorem 9], with everywhere $\left\|\left(\Psi_{0}, \Psi_{T}\right)\right\|_{F_{k}^{\gamma}}$ replaced by $\left\|\left(\Psi_{0}, d_{0}, \Psi_{f}, d_{f}\right)\right\|_{\tilde{F}_{k}^{\gamma}}$.
Now we can apply the Nash-Moser implicit function theorem stated in section 3.2.2 and we get Theorem 10.

### 5.3 Motion in the directions $(\psi, S, D)=(0, \pm 1,0)$

The aim of this section is to prove of the following theorem.
Theorem 13 There exists $\gamma_{0}>0$ such that, for every $\gamma \in\left(0, \gamma_{0}\right)$, there exist $w_{ \pm}, \nu_{ \pm} \in H_{0}^{3}((0, T), \mathbb{R})$ such that the solutions of

$$
\begin{gathered}
\left\{\begin{array}{l}
i \frac{\partial \Psi_{ \pm}}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \Psi_{ \pm}}{\partial q^{2}}-\gamma q \Psi_{ \pm}-w_{ \pm} q \psi_{1, \gamma} \\
\Psi_{ \pm}(0)=0 \\
\Psi_{ \pm}(t,-1 / 2)=\Psi_{ \pm}(t, 1 / 2)=0 \\
\dot{s}_{ \pm}=w_{ \pm}, s_{ \pm}(0)=0 \\
\dot{d}_{ \pm}=s_{ \pm}, d_{ \pm}(0)=0
\end{array}\right. \\
\left\{\begin{array}{l}
i \frac{\partial \xi_{ \pm}}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \xi_{ \pm}}{\partial q^{2}}-\gamma q \xi_{ \pm}-w_{ \pm} q \Psi_{ \pm}-\nu_{ \pm} q \psi_{1, \gamma} \\
\xi_{ \pm}(0)=0 \\
\xi_{ \pm}(t,-1 / 2)=\xi_{ \pm}(t, 1 / 2)=0 \\
\dot{\sigma}_{ \pm}=\nu_{ \pm}, \sigma_{ \pm}(0)=0 \\
\dot{\delta}_{ \pm}=\sigma_{ \pm}, \delta_{ \pm}(0)=0
\end{array}\right.
\end{gathered}
$$

satisfy $\Psi_{ \pm}(T)=0, s_{ \pm}(T)=0, d_{ \pm}(T)=0, \xi_{ \pm}(T)=0, \sigma_{ \pm}(T)= \pm 1, \delta_{ \pm}(T)=0$.
Let us introduce new notations. Let $\gamma \in \mathbb{R}^{*}$. We define the subspace of $L^{2}((0, T), \mathbb{C})$

$$
\mathcal{X}_{\gamma}:=\operatorname{Span}\left(t, e^{i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) t}, e^{-i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) t} ; k \in \mathbb{N}^{*}\right) .
$$

The symbol $\mathcal{X}_{\gamma}^{\perp}$ denotes the orthogonal subspace to $\mathcal{X}_{\gamma}$ in $L^{2}((0, T), \mathbb{C})$. We recall that we have

$$
\lambda_{k}:=\lambda_{k, 0}=\frac{1}{2}(k \pi)^{2}, \quad \varphi_{k}:=\varphi_{k, 0}=\left\{\begin{array}{l}
\sqrt{2} \sin (k \pi q), \text { when } k \text { in even, } \\
\sqrt{2} \cos (k \pi q), \text { when } k \text { in odd. }
\end{array}\right.
$$

The parity of the functions $\varphi_{k}$ gives $b_{2 k+1}:=b_{2 k+1,0}=0$ for every $k \in \mathbb{N}$.
One has the following proposition.
Proposition 21 There exists $\gamma_{0}>0$ such that, for every $\gamma \in\left(0, \gamma_{0}\right)$, there exists $w_{\gamma} \in H_{0}^{4}((0, T), \mathbb{R}) \cap$ $\mathcal{X}_{\gamma}^{\perp}$ such that

$$
\int_{0}^{T} w_{\gamma}(t)<q \Psi_{\gamma}(t), \psi_{1, \gamma}(t)>d t \in(0,+\infty),(\text { resp. } \in(-\infty, 0))
$$

where $\Psi_{\gamma}$ is the solution of

$$
\left\{\begin{array}{l}
i \frac{\partial \Psi_{\gamma}}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \Psi_{\gamma}}{\partial q^{2}}-\gamma q \Psi_{\gamma}-w_{\gamma} q \psi_{1, \gamma} \\
\Psi_{\gamma}(0)=0, \\
\Psi_{\gamma}(t,-1 / 2)=\Psi_{\gamma}(t, 1 / 2)=0
\end{array}\right.
$$

Remark 7 Let $\gamma \in \mathbb{R}$. If $w_{\gamma} \in \mathcal{X}_{\gamma}^{\perp}$, and $\Psi_{\gamma}$ is the solution of the previous system, then

$$
\int_{0}^{T} w_{\gamma}(t)<q \Psi_{\gamma}(t), \psi_{1, \gamma}(t)>d t \in \mathbb{R}
$$

Indeed, we have

$$
\Psi_{\gamma}(t)=\sum_{k=1}^{+\infty} x_{k}(t) \varphi_{k, \gamma}, \text { where } x_{k}(t)=i b_{k, \gamma} e^{-i \lambda_{k, \gamma} t} \int_{0}^{t} w_{\gamma}(\tau) e^{i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) \tau} d \tau
$$

Thus

$$
\int_{0}^{T} w_{\gamma}(t)<q \Psi_{\gamma}(t), \psi_{1, \gamma}(t)>d t=\sum_{k=1}^{+\infty} i b_{k, \gamma}^{2} f_{k, \gamma}
$$

where

$$
f_{k, \gamma}:=\int_{0}^{T} w_{\gamma}(t) e^{-i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) t} \int_{0}^{t} w_{\gamma}(\tau) e^{i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) \tau} d \tau d t
$$

Thanks to an integration by parts in the definition of $f_{k, \gamma}$ and the property $w_{\gamma} \in \mathcal{X}_{\gamma}^{\perp}$, we get: for every $k \in \mathbb{N}^{*}, f_{k, \gamma} \in i \mathbb{R}$.

Proof of Proposition 21 : First, we study the case $\gamma=0$. Let us consider functions of the form

$$
\begin{equation*}
w(t):=\sin \left(\frac{1}{2} n_{0} \pi^{2} t\right)+a_{1} \sin \left(\frac{1}{2} n_{1} \pi^{2} t\right)+a_{2} \sin \left(\frac{1}{2} n_{2} \pi^{2} t\right)+a_{3} \sin \left(\frac{1}{2} n_{3} \pi^{2} t\right) \tag{66}
\end{equation*}
$$

where $n_{0}, n_{1}, n_{2}, n_{3}$ are four different positive integers such that

$$
n_{0}, n_{1}, n_{2}, n_{3} \notin\left\{ \pm\left(k^{2}-1\right) ; k \in \mathbb{N}^{*}\right\}
$$

and $a_{1}, a_{2}, a_{3} \in \mathbb{R}$ solve

$$
\left(\begin{array}{ccc}
\frac{1}{n_{1}} & \frac{1}{n_{2}} & \frac{1}{n_{3}} \\
n_{1} & n_{2} & n_{3} \\
n_{1}^{3} & n_{2}^{3} & n_{3}^{3}
\end{array}\right)\left(\begin{array}{l}
a_{1} \\
a_{2} \\
a_{3}
\end{array}\right)=\left(\begin{array}{c}
-\frac{1}{n_{0}} \\
-n_{0} \\
-n_{0}^{3}
\end{array}\right) .
$$

Then,

$$
w \in H_{0}^{4}((0, T), \mathbb{R}) \cap \mathcal{X}_{0}^{\perp}
$$

Let $\Psi$ be the solution of

$$
\left\{\begin{array}{l}
i \frac{\partial \Psi}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \Psi}{\partial q^{2}}-w q \psi_{1} \\
\Psi(0)=0, \\
\Psi(t,-1 / 2)=\Psi(t, 1 / 2)=0
\end{array}\right.
$$

We have

$$
\begin{gather*}
\Psi(t)=\sum_{k=1}^{+\infty} x_{k}(t) \varphi_{k} \text { where } x_{k}(t):=<\Psi(t), \varphi_{k}>, \forall k \in \mathbb{N}^{*} \\
\int_{0}^{T} w(t)<q \Psi(t), \psi_{1}(t)>d t=\sum_{k=1}^{+\infty} b_{2 k} \int_{0}^{T} w(t) x_{2 k}(t) e^{i \lambda_{1} t} d t  \tag{67}\\
x_{2 k}(t)=i b_{2 k}\left(\int_{0}^{t} w(\tau) e^{i\left(\lambda_{2 k}-\lambda_{1}\right) \tau} d \tau\right) e^{-i \lambda_{2 k} t} \\
b_{2 k}=-\frac{(-1)^{k} 16 k}{\pi^{2}(1+2 k)^{2}(1-2 k)^{2}} .
\end{gather*}
$$

Thus, the right hand side of (67) can be explicitly computed. We find

$$
\int_{0}^{T} w(t)<q \Psi(t), \psi_{1}(t)>d t=\frac{32 T}{\pi^{6}}\left(S_{n_{0}}+a_{1}^{2} S_{n_{1}}+a_{2}^{2} S_{n_{2}}+a_{3}^{2} S_{n_{3}}\right)
$$

where, for every $p \in \mathbb{N}$ with $p \notin\left\{ \pm\left(k^{2}-1\right) ; k \in \mathbb{N}^{*}\right\}, S_{p}$ is defined by

$$
S_{p}:=\sum_{k=1}^{+\infty} \frac{(2 k)^{2}}{(1+2 k)^{4}(1-2 k)^{4}}\left(\frac{1}{-p+4 k^{2}-1}+\frac{1}{p+4 k^{2}-1}\right) .
$$

Let us choose $n_{0}=1, n_{1}=2, n_{2}=4, n_{3}=5$ (resp. $n_{0}=4, n_{1}=5, n_{2}=6, n_{3}=7$ ), we get

$$
\begin{equation*}
\int_{0}^{T} w(t)<q \Psi(t), \psi_{1}(t)>d t \in(0,+\infty)(\text { resp. }(-\infty, 0)) \tag{68}
\end{equation*}
$$

Now, we study the case $\gamma \neq 0$. We use the following proposition, which will be proved later on.
Proposition 22 Let $T=4 / \pi$. There exists $\gamma_{*}>0, C_{1}, C_{2}>0$ such that, for every $\gamma \in\left(-\gamma_{*}, \gamma_{*}\right)$,

1. the linear map $\widetilde{Z}_{\gamma}$ is continuous from $H_{0}^{4}((0, T), \mathbb{R})$ to $h_{r}^{8}(\mathbb{N}, \mathbb{C})$,
2. for every $w \in H_{0}^{4}((0, T), \mathbb{R})$,

$$
\left\|\left(\widetilde{Z}_{\gamma}-\widetilde{Z}_{0}\right)(w)\right\|_{h^{8}} \leqslant C_{1} \gamma^{2}\|w\|_{H_{0}^{4}}
$$

3. there exists a linear map

$$
\widetilde{Z}_{\gamma}^{-1}: h_{r}^{8}(\mathbb{N}, \mathbb{C}) \rightarrow H_{0}^{4}((0, T), \mathbb{R})
$$

such that, for every $d \in h_{r}^{8}(\mathbb{N}, \mathbb{C}), \widetilde{Z}_{\gamma} \circ \widetilde{Z}_{\gamma}^{-1}(d)=d$ and the function $w:=\widetilde{Z}_{\gamma}^{-1}(d)$ satisfies

$$
\|w\|_{H_{0}^{4}} \leqslant C_{2}\|d\|_{h^{8}}
$$

Let $\gamma \in\left(-\gamma_{*}, \gamma_{*}\right)$ different from zero. We define

$$
w_{\gamma}:=w-\widetilde{Z}_{\gamma}^{-1}\left(\widetilde{Z}_{\gamma}(w)\right)
$$

where $\widetilde{Z}_{\gamma}^{-1}$ is defined in Proposition 22 and $w$ is defined in (66). We have $w \in H_{0}^{4}((0, T)$, $\mathbb{R})$, so $\widetilde{Z}_{\gamma}(w) \in h_{r}^{8}(\mathbb{N}, \mathbb{C})$ and $\widetilde{Z}_{\gamma}^{-1}\left(\widetilde{Z}_{\gamma}(w)\right) \in H_{0}^{4}((0, T), \mathbb{R})$, thus

$$
w_{\gamma} \in H_{0}^{4}((0, T), \mathbb{R}) \cap \mathcal{X}_{0}^{\perp}
$$

We have

$$
\begin{equation*}
\left\|w-w_{\gamma}\right\|_{H_{0}^{4}}=\left\|\widetilde{Z}_{\gamma}^{-1}\left(\left(\widetilde{Z}_{\gamma}-\widetilde{Z}_{0}\right)(w)\right)\right\|_{H_{0}^{4}} \leqslant C_{2} C_{1} \gamma^{2}\|w\|_{H_{0}^{4}} \tag{69}
\end{equation*}
$$

Let us consider the map

$$
\begin{aligned}
G:\left(-\gamma_{*}, \gamma_{*}\right) & \rightarrow \mathbb{R} \\
\gamma & \mapsto \int_{0}^{T} w_{\gamma}(t)<q \Psi_{\gamma}(t), \psi_{1, \gamma}(t)>d t
\end{aligned}
$$

where, for every $\gamma \in\left(-\gamma_{*}, \gamma_{*}\right), \Psi_{\gamma}$ is the solution of the system written in Proposition 21. The bound (69) proves that $G$ is continuous at $\gamma=0$. We know from (68), that $G(0)>0$ (resp. $<0$ ). Thus, there exists $\gamma_{0}>0$ such that, for every $\gamma \in\left(-\gamma_{0}, \gamma_{0}\right), G(\gamma)>0($ resp. $<0)$.

Proof of Proposition 22 : The strategy is the same as in Section 3.1.2. We just need to build a right inverse for $\widetilde{Z}_{0}$ which maps $h_{r}^{8}(\mathbb{N}, \mathbb{C})$ into $H_{0}^{4}((0, T), \mathbb{R})$. With the same notations as in the proof of Proposition 19, a suitable candidate for $\widetilde{Z}_{0}^{-1}(d)$ is

$$
w(t):=\left(\frac{1}{T} \sum_{k \in \mathbb{Z}} \tilde{d}_{k} e^{i \omega_{k} t}+\alpha\left(e^{i \frac{1}{2} n \pi^{2} t}+e^{-i \frac{1}{2} n \pi^{2} t}\right)\right)\left(1-e^{i \frac{1}{2} \pi^{2} t}\right)^{3}\left(1-e^{-i \frac{1}{2} \pi^{2} t}\right)^{3}
$$

where $n \in \mathbb{N}$ with $\{n, n \pm 1, n \pm 2, n \pm 3\} \cap\left\{ \pm\left(k^{2}-1\right) ; k \in \mathbb{N}^{*}\right\}=\emptyset$ and $\alpha \in \mathbb{R}$ is such that $\int_{0}^{T}(T-t) w(t) d t=d_{0}$.

Proof of Theorem 13 : Let $\gamma \in\left(0, \gamma_{0}\right)$, where $\gamma_{0}$ is given in Proposition 21. Let $w \in H_{0}^{4}((0, T), \mathbb{R}) \cap$ $\mathcal{X}_{\gamma}^{\perp}$ be such that

$$
\int_{0}^{T} w(t)<q \Psi(t), \psi_{1, \gamma}(t)>d t=-b_{1, \gamma}\left(\text { resp. }=+b_{1, \gamma}\right)
$$

We have

$$
\Psi(t)=\sum_{k=1}^{+\infty} x_{k}(t) \varphi_{k, \gamma} \text { where } x_{k}(t)=i b_{k, \gamma} e^{-i \lambda_{k, \gamma} t} \int_{0}^{t} w(\tau) e^{i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) \tau} d \tau
$$

The assumption $w \in \mathcal{X}_{\gamma}^{\perp}$ gives $\Psi(T)=0, s(T)=0$ and $d(T)=0$. Let us prove that there exists $\nu \in L^{2}((0, T), \mathbb{R})$ such that the solution of

$$
\left\{\begin{array}{l}
i \frac{\partial \xi}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \xi}{\partial q^{2}}-\gamma q \xi-w q \Psi-\nu q \psi_{1, \gamma}, \\
\xi(0)=0 \\
\xi(t,-1 / 2)=\xi(t, 1 / 2)=0 \\
\dot{\sigma}=\nu, \sigma(0)=0 \\
\dot{\delta}=\sigma, \delta(0)=0
\end{array}\right.
$$

satisfies $\xi(T)=0, \sigma(T)=1$ (resp. $=-1$ ) , $\delta(T)=0$. We have

$$
\begin{gathered}
\xi(t)=\sum_{k=1}^{+\infty} y_{k}(t) \varphi_{k, \gamma} \\
y_{k}(t)=i e^{-i \lambda_{k, \gamma} t} \int_{0}^{t}\left(w(\tau)<q \Psi(\tau), \varphi_{k, \gamma}>+\nu(\tau) b_{k, \gamma} e^{-i \lambda_{1, \gamma} \tau}\right) e^{i \lambda_{k, \gamma} \tau} d \tau
\end{gathered}
$$

The equality $(\xi(T), \sigma(T), \delta(T))=(0,1,0)($ resp. $=(0,-1,0))$ is equivalent to

$$
\begin{gathered}
\int_{0}^{T} \nu(t) e^{i\left(\lambda_{k, \gamma}-\lambda_{1, \gamma}\right) t} d t=-\frac{1}{b_{k, \gamma}} \int_{0}^{T} w(t)<q \Psi(t), \varphi_{k, \gamma}>e^{i \lambda_{k, \gamma} t} d t, \forall k \in \mathbb{N}^{*} \\
\int_{0}^{T} \nu(t) d t=1(\text { resp. }=-1) \\
\int_{0}^{T}(T-t) \nu(t) d t=0
\end{gathered}
$$

A necessary condition for the existence of a solution $\nu$ to this moment problem is

$$
-\frac{1}{b_{1, \gamma}} \int_{0}^{T} w(t)<q \Psi(t), \varphi_{1, \gamma}>e^{i \lambda_{1, \gamma}} d t=+1(\text { resp. }=-1) .
$$

The choice of $w$ has been done in order to satisfy this condition.
Then, a sufficient condition for the existence of a solution $\nu \in H_{0}^{3}((0, T), \mathbb{R})$ is

$$
\begin{equation*}
\left(\int_{0}^{T} w(t)<q \Psi(t), \varphi_{k, \gamma}>e^{i \lambda_{k, \gamma} t} d t\right)_{k \in \mathbb{N}^{*}} \in h_{r}^{9}\left(\mathbb{N}^{*}, \mathbb{C}\right) \tag{70}
\end{equation*}
$$

(see [1, Section 3.1], for the behaviour of $b_{k, \gamma}$ and Proposition 20 for the existence of $\widetilde{Z}_{\gamma}^{-1}$ between the suitable spaces).

The assumption $w \in H_{0}^{4}((0, T), \mathbb{R})$ implies (70). Indeed, integrations by parts lead to

$$
\begin{aligned}
\int_{0}^{T} w<q \Psi, \varphi_{k, \gamma}>e^{i \lambda_{k, \gamma} t} d t= & \frac{1}{\lambda_{k}^{4}, \gamma} \int_{0}^{T}\left(\frac{\partial^{4} w}{\partial t^{4}}<q \Psi, \varphi_{k, \gamma}>+\right. \\
& 4 \frac{\partial^{3} w}{\partial t^{3}}<q \dot{\Psi}, \varphi_{k, \gamma}>+6 \ddot{w}<q \ddot{\Psi}, \varphi_{k, \gamma}>+ \\
& \left.4 \dot{w}<q \frac{\partial^{3} \Psi}{\partial t^{3}}, \varphi_{k, \gamma}>+w<q \frac{\partial^{4} \Psi}{\partial t^{4}}, \varphi_{k, \gamma}>\right) e^{i \lambda_{k, \gamma} t} d t .
\end{aligned}
$$

Moreover, when $v \in L^{2}((0, T), \mathbb{R})$ and $f \in C^{0}\left([0, T], L^{2}(I, \mathbb{C})\right)$, we have

$$
\left|\int_{0}^{T} v(t)<f(t), \varphi_{k, \gamma}>e^{i \lambda_{k, \gamma} t} d t\right|^{2} \leqslant\|v\|_{L^{2}((0, T), \mathbb{R})}^{2} \int_{0}^{T}\left|<f(t), \varphi_{k, \gamma}>\right|^{2} d t
$$

Therefore, since the family $\left(\varphi_{k, \gamma}\right)_{k \in \mathbb{N}^{*}}$ is orthonormal in $L^{2}(I, \mathbb{C})$, the sequence

$$
\left(\int_{0}^{T} v(t)<f(t), \varphi_{k, \gamma}>e^{i \lambda_{k, \gamma} t} d t\right)_{k \in \mathbb{N}^{*}}
$$

belongs to $l^{2}\left(\mathbb{N}^{*}, \mathbb{C}\right)$.

### 5.4 Proof of Theorem 8

In all this section, we fix $\gamma \in\left(-\gamma_{0}, \gamma_{0}\right)$ different from zero, where $\gamma_{0}$ is as in Theorem 13.
Let $\rho \in \mathbb{R}, \psi_{0}, \psi_{f} \in H_{(\gamma)}^{7}(I, \mathbb{C}), S_{0}, D_{0}, D_{f} \in \mathbb{R}$. Let us consider, for $t \in[0, T]$,

$$
v(t):=\sqrt{|\rho|} w+|\rho| \nu
$$

where $w:=w_{+}, \nu:=\nu_{+}$if $\rho \geqslant 0$ and $w:=w_{-}, \nu:=\nu_{-}$if $\rho<0$. Let $(\psi, S, D)$ be the solution of $\left(\Sigma_{0}\right)$ on $[0, T]$ with $u:=\gamma+v$. Then,

$$
S(T)=S_{0}+\gamma T+\rho \text { and } D(T)=D_{0}+S_{0} T+\gamma T^{2} / 2
$$

We have $v \in W^{3,1}((0, T), \mathbb{R}), v(0)=v(T)=\dot{v}(0)=\dot{v}(T)=0$, so [1, Appendix B, proposition 51], $\psi \in C^{0}\left([0, T], H^{7}(I, \mathbb{C})\right)$ and $\psi(T) \in H_{(\gamma)}^{7}(I, \mathbb{C})$.

Proposition 23 There exists a constant $\mathcal{C}$ such that, for every $\rho \in(-1,1)$, we have

$$
\left\|\left(\psi-\psi_{1, \gamma}\right)(T)\right\|_{H^{7}(I, \mathbb{C})} \leqslant \mathcal{C}\left[\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}(I, \mathbb{C})}+|\rho|^{3 / 2}\right]
$$

Proof : We have $\left(\psi-\psi_{1, \gamma}\right)(T)=(\psi-Z)(T)$ where $Z:=\psi_{1, \gamma}+\Psi+\xi$ and $\Psi, \xi$ are the solutions of the following systems

$$
\begin{gathered}
\left\{\begin{array}{l}
i \frac{\partial \Psi}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \Psi}{\partial q^{2}}-\gamma q \Psi-\sqrt{|\rho|} w q \psi_{1, \gamma} \\
\Psi(0)=0 \\
\Psi(t,-1 / 2)=\Psi(t, 1 / 2)=0
\end{array}\right. \\
\left\{\begin{array}{l}
i \frac{\partial \xi}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \xi}{\partial q^{2}}-\gamma q \xi-\sqrt{|\rho|} w q \Psi-|\rho| \nu q \psi_{1, \gamma} \\
\xi(0)=0 \\
\xi(t,-1 / 2)=\xi(t, 1 / 2)=0
\end{array}\right.
\end{gathered}
$$

The function $\Delta:=\psi-Z$ solves

$$
\left\{\begin{array}{l}
i \frac{\partial \Delta}{\partial t}=-\frac{1}{2} \frac{\partial^{2} \Delta}{\partial q^{2}}-(\gamma+v) q \Delta-\sqrt{|\rho|} w q \xi-|\rho| \nu q(\Psi+\xi)  \tag{71}\\
\Delta(0)=\psi_{0}-\psi_{1, \gamma}(0) \\
\Delta(t,-1 / 2)=\Delta(t, 1 / 2)=0
\end{array}\right.
$$

We know from [1, proposition 51, Appendix B], that the following quantities

$$
\|\Psi\|_{C^{0}\left([0, T], H^{7}\right)},\|\Psi\|_{C^{1}\left([0, T], H^{5}\right)},\|\Psi\|_{C^{2}\left([0, T], H^{3}\right)},\|\Psi\|_{C^{3}\left([0, T], H^{1}\right)}
$$

are bounded by

$$
A_{7}(\Psi):=C\left[\|f\|_{C^{0}\left([0, T], H^{5}\right)}+\|f\|_{C^{1}\left([0, T], H^{3}\right)}+\|f\|_{W^{2,1}\left((0, T), H^{2}\right)}+\|f\|_{W^{3,1}\left((0, T), H^{1}\right)}\right]
$$

where $C$ is a positive constant and $f:=\sqrt{|\rho|} w q \psi_{1, \gamma}$. Thus, there exists a constant $C_{1}$ such that

$$
A_{7}(\Psi) \leqslant C_{1} \sqrt{|\rho|} .
$$

In the same way, we prove that there exists a constant $C_{2}$ such that

$$
\|\xi\|_{C^{0}\left([0, T], H^{7}\right)},\|\xi\|_{C^{1}\left([0, T], H^{5}\right)},\|\xi\|_{C^{2}\left([0, T], H^{3}\right)},\|\xi\|_{C^{3}\left([0, T], H^{1}\right)}
$$

are bounded by

$$
A_{7}(\xi) \leqslant C_{2}|\rho|
$$

Using (71) and [1, Appendix B, proposition 51] we get the existence of a constant $C_{3}>0$ such that

$$
\|\Delta(T)\|_{H^{7}} \leqslant C_{3}\left[\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}}+\sqrt{|\rho|} A_{7}(\xi)+|\rho| A_{7}(\Psi)\right] . \text {. }
$$

Now, we apply the local controllability of ( $\psi, D$ ) on $[0, T]$ around $\left(\psi_{1, \gamma}(t), \beta+\alpha t+\gamma t^{2} / 2\right)$, with

$$
\alpha:=S_{0} \text { and } \beta:=D_{0} .
$$

Let $\delta>0$ as in Theorem 10. We assume

$$
\begin{gather*}
\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}(I, \mathbb{C})}<\frac{\delta}{2 \mathcal{C}}  \tag{72}\\
\left\|\psi_{f}-\psi_{1, \gamma}(2 T)\right\|_{H^{7}(I, \mathbb{C})}+\left|D_{f}-D_{0}-2 S_{0} T-2 \gamma T^{2} / 2\right|<\delta,  \tag{73}\\
|\rho|<\eta:=\left(\frac{\delta}{2 \mathcal{C}}\right)^{2 / 3}
\end{gather*}
$$

Then we have

$$
\begin{gathered}
\left\|\psi(T)-\psi_{1, \gamma}(T)\right\|_{H^{7}(I, \mathbb{C})}+\left|D(T)-\left(\beta+\alpha T+\gamma T^{2} / 2\right)\right|<\mathcal{C}\left[\frac{\delta}{2 \mathcal{C}}+\eta^{3 / 2}\right]<\delta, \\
\left\|\psi_{f}-\psi_{1, \gamma}(2 T)\right\|_{H^{7}(I, \mathbb{C})}+\left|D_{f}-\left(\beta+2 \alpha T+2 \gamma T^{2}\right)\right|<\delta .
\end{gathered}
$$

So there exists $\tilde{v} \in H_{0}^{1}((T, 2 T), \mathbb{R})$ such that

$$
\psi(2 T)=\psi_{f} \text { and } D(2 T)=D_{f}
$$

where $(\psi, S, D)$ still the solution of $\left(\Sigma_{0}\right)$ with control $u:=\gamma+v$ on $[0,2 T]$, with $v$ extended to $[0,2 T]$ by $v:=\tilde{v}$ on $[T, 2 T]$. We know that $\tilde{v}$ can be chosen so that there exists a constant $K$ such that

$$
\begin{aligned}
\|v\|_{L^{2}((T, 2 T), \mathbb{R})} \leqslant & K\left[\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}(I, \mathbb{C})}+|\rho|^{3 / 2}+\left\|\psi_{f}-\psi_{1, \gamma}(2 T)\right\|_{H^{7}(I, \mathbb{C})}+\right. \\
& \left.\left|D_{f}-\left(\beta+2 \alpha T+2 \gamma T^{2} / 2\right)\right|\right]
\end{aligned}
$$

We used Proposition 23 in order to get this bound.
Moreover, we have

$$
S(2 T)=S_{0}+2 \gamma T+\rho+\int_{T}^{2 T} v(t) d t
$$

We define the map

$$
\begin{aligned}
F: \quad(-\eta, \eta) & \rightarrow \mathbb{R} \\
\rho & \mapsto
\end{aligned}
$$

There exist $\tau \in(0, \eta)$ such that

$$
\sqrt{T} K \tau^{3 / 2}<\tau / 3 .
$$

Let us assume

$$
\begin{gathered}
\sqrt{T} K\left\|\psi_{0}-\psi_{1, \gamma}(0)\right\|_{H^{7}(I, \mathbb{C})}<\tau / 6 \\
\sqrt{T} K\left(\left\|\psi_{f}-\psi_{1, \gamma}(2 T)\right\|_{H^{7}(I, \mathrm{C})}+\left|D_{f}-\left(\beta+2 \alpha T+2 \gamma T^{2} / 2\right)\right|\right)<\tau / 6 .
\end{gathered}
$$

Then,

$$
F(\tau)-\left(S_{0}+2 \gamma T\right)>\tau / 3>0 \text { and } F(-\tau)-\left(S_{0}+2 \gamma T\right)<-\tau / 3<0 .
$$

The map $F$ is continuous, thus, $F$ is surjective on a neighborhood of $S_{0}+2 \gamma T$, this ends the proof of Theorem 2.

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[^0]:    *Département de Mathématique, Bât. 425, Université Paris-Sud, 91405 Orsay, France
    ${ }^{\dagger}$ Département de Mathématique, Bât. 425, Université Paris-Sud, 91405 Orsay, France and Institut Universitaire de France, Bât. 425, Université Paris-Sud, 91405 Orsay, France

